ASYMPTOTIC EXPANSIONS AND UNIQUE CONTINUATION AT DIRICHLET-NEUMANN BOUNDARY JUNCTIONS FOR PLANAR ELLIPTIC EQUATIONS

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Abstract. We consider elliptic equations in planar domains with mixed boundary conditions of Dirichlet-Neumann type. Sharp asymptotic expansions of the solutions and unique continuation properties from the Dirichlet-Neumann junction are proved.

1. Introduction

The present paper deals with elliptic equations in planar domains with mixed boundary conditions and aims at proving asymptotic expansions and unique continuation properties for solutions near boundary points where a transition from Dirichlet to Neumann boundary conditions occurs.

A great attention has been devoted to the problem of unique continuation for solutions to partial differential equations starting from the paper by Carleman [5], whose approach was based on some weighted a priori inequalities. An alternative approach to unique continuation was developed by Garofalo and Lin [14] for elliptic equations in divergence form with variable coefficients, via local doubling properties and Almgren monotonicity formula; we also quote [18] for quantitative uniqueness obtained by monotonicity methods.

The monotonicity approach has the advantage of giving not only unique continuation but also precise asymptotics of solutions near a fixed point, via a suitable combination of monotonicity methods with blow-up analysis, as done in [9–13]. The method based on doubling properties and Almgren monotonicity formula has also been successfully applied to treat the problem of unique continuation from the boundary in [1, 2, 9, 19, 27] under homogeneous Dirichlet conditions and in [26] under homogeneous Neumann conditions. Furthermore, in [9] a sharp asymptotic description of the behaviour of solutions at conical boundary points was given through a fine blow-up analysis. In the present paper, we extend the procedure developed in [9–13] to the case of mixed Dirichlet/Neumann boundary conditions, providing sharp asymptotic estimates for solutions near the Dirichlet-Neumann junction and, as a consequence, unique continuation properties. In addition, comparing our result with the aforementioned papers, here we also provide an estimate of the remainder term in the difference between the solution and its asymptotic profile.

Let Ω be an open subset of $\mathbb{R}^2$ with Lipschitz boundary. Let $\Gamma_n \subset \partial\Omega$ and $\Gamma_d \subset \partial\Omega$ be two nonconstant curves (open in $\partial\Omega$) such that $\overline{\Gamma_n} \cap \overline{\Gamma_d} = \{P\}$ for some $P \in \partial\Omega$. We are interested in

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regularity of weak solutions $u \in H^1(\Omega)$ to the mixed boundary value problem

$$\begin{cases} -\Delta u = f(x)u, & \text{in } \Omega, \\ \partial_\nu u = g(x)u, & \text{on } \Gamma_n, \\ u = 0, & \text{on } \Gamma_d, \end{cases} \tag{1.1}$$

with $f \in L^\infty(\Omega)$ and $g \in C^1(\overline{\Gamma_n})$, see Section 2 for the weak formulation. Our aim is to prove unique continuation properties from the Dirichlet-Neumann junction $\{P\} = \overline{\Gamma_n} \cap \overline{\Gamma_d}$ and sharp asymptotics of nontrivial solutions near P provided $\partial\Omega$ is of class $C^{2,\gamma}$ in a neighborhood of P. We mention that some regularity results for solutions to second-order elliptic problems with mixed Dirichlet–Neumann type boundary conditions were obtained in [16, 25], see also the references therein.

Some interest in the derivation of asymptotic expansions for solutions to planar mixed boundary value problems at Dirichlet-Neumann junctions arises in the study of crack problems, see e.g. [6,20]. Indeed, if we consider an elliptic equation in a planar domain with a crack and prescribe Neumann conditions on the crack and Dirichlet conditions on the rest of the boundary, in the case of the crack end-point belonging to the boundary of the domain we are lead to consider a problem of the type described above in a neighborhood of the crack's tip (which corresponds to the Dirichlet-Neumann junction). We recall (see e.g. [6]) that, in crack problems, the coefficients of the asymptotic expansion of solutions near the crack's tip are related to the so called *stress intensity factor*.

In order to get a precise asymptotic expansion of u at point $P \in \overline{\Gamma_n} \cap \overline{\Gamma_d}$, we will need to assume that $\partial\Omega$ is of class $C^{2,\delta}$ near P. The asymptotic profile of the solution will be given by the function

$$F_k(r\cos\theta, r\sin\theta) = r^{\frac{2k-1}{2}} \cos\left(\frac{2k-1}{2}\theta\right), \quad r > 0,\ \theta \in (0,\pi), \tag{1.2}$$

for some $k \in \mathbb{N} \setminus \{0\}$. We note that $F_k \in H^1_{\rm loc}(\mathbb{R}^2)$ and solves the equation

$$\begin{cases} \Delta F_k = 0, & \text{in } \mathbb{R}^2_+, \\ F_k(x_1, 0) = 0, & \text{for } x_1 < 0, \\ \partial_{x_2} F_k(x_1, 0) = 0, & \text{for } x_1 > 0, \end{cases} \tag{1.3}$$

where here and in the following $\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 > 0\}$.

The main result of the present paper provides an evaluation of the behavior of weak solutions $u \in H^1(\Omega)$ to (1.1) at the boundary point where the boundary conditions change. In order to simplify the statement and without losing generality, we can fix the cartesian axes in such a way that the following assumptions on $\Omega \subset \mathbb{R}^2$ are satisfied. Here and in the remaining of this paper, $\Gamma_n, \Gamma_d \subset \partial\Omega$ are nonconstant curves (open as subsets of $\partial\Omega$) such that $\overline{\Gamma_n} \cap \overline{\Gamma_d} = \{0\}$ with $0 \in \partial\Omega$.

(i) The domain Ω is of class $C^{2,\delta}$ in a neighborhood of 0, for some $\delta > 0$.
(ii) The unit vector $e_1 := (1, 0)$ is tangent to $\partial\Omega$ at 0 and pointed towards Γ_n. Moreover, the exterior unit normal vector to $\partial\Omega$ at 0 is $(0, -1)$.

We are now in position to state the main result of the present paper.

Theorem 1.1. *We assume that Ω satisfies the assumption (i)-(ii) above. Let $u \in H^1(\Omega)$ be a nontrivial weak solution to (1.1), with $f \in L^\infty(\Omega)$ and $g \in C^1(\overline{\Gamma_n})$. Then, there exist $k_0 \in \mathbb{N} \setminus \{0\}$,*

$\beta \in \mathbb{R} \setminus \{0\}$ and $r > 0$ such that, for every $\varrho \in (0, 1/2)$, there exists $C > 0$ such that

$$|u(x) - \beta F_{k_0}(\varphi(x))| \leqslant C|x|^{\frac{2k_0-1}{2}+\varrho}, \quad \text{for every } x \in \Omega \cap \overline{B_r^+}. \tag{1.4}$$

Here, the function $\varphi : \overline{\Omega \cap B_{r_0}} \to \overline{\mathbb{R}^2_+}$ is a conformal map of class C^2, for some $r_0 > 0$ only depending on Ω.

Remark 1.1. *Here and in the sequel, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$; hence, by a* conformal map *on an open set $U \subset \mathbb{R}^2$ we mean a holomorphic function with complex derivative everywhere non-zero on U. We notice that, if Ω satisfies (i)-(ii) and $\varphi : \overline{\Omega \cap B_{r_0}} \to \overline{\mathbb{R}^2_+}$ is conformal, then $D\varphi(0) = \alpha\,\mathrm{Id}$ and $\varphi'(0) = \alpha$ for some real $\alpha > 0$, where $D\varphi$ denotes the jacobian matrix of φ and φ' denotes the complex derivative of φ.*

As a direct consequence of Theorem 1.1, we derive the following Hopf-type lemma.

Corollary 1.2. *Under the same assumptions as in Theorem 1.1, let $u \in H^1(\Omega)$ be a non-trivial weak solution to (1.1), with $u \geqslant 0$. Then*

(i) *for every $t \in [0, \pi)$,*

$$\lim_{r\to 0} \frac{u(r\cos t, r\sin t)}{r^{1/2}} = \beta\alpha^{1/2}\cos\left(\frac{t}{2}\right) > 0,$$

where $\alpha = \varphi'(0) > 0$ and φ is as in Theorem 1.1;

(ii) *for every cone $\mathcal{C} \subset \mathbb{R}^2$ satisfying $(1, 0) \in \mathcal{C}$ and $(-1, 0) \in \mathbb{R}^2 \setminus \overline{\mathcal{C}}$, we have*

$$\liminf_{\substack{x\to 0\\ x\in\overline{\Omega}\cap\mathcal{C}}} \frac{u(x)}{|x|^{1/2}} > 0. \tag{1.5}$$

A further relevant byproduct of our asymptotic analysis is the following unique continuation principle, whose proof follows directly from Theorem 1.1.

Corollary 1.3. *Under the same assumptions as in Theorem 1.1, let $u \in H^1(\Omega)$ be a weak solution to (1.1) such that $u(x) = O(|x|^n)$ as $x \in \Omega$, $|x| \to 0$, for any $n \in \mathbb{N}$. Then $u \equiv 0$.*

We observe that Theorem 1.1 provides a sharp asymptotic expansion (and consequently a unique continuation principle) at the boundary for $\frac{1}{2}$-fractional elliptic equations in dimension 1. Indeed, if $v \in H^{1/2}(\mathbb{R})$ weakly solves

$$\begin{cases} (-\Delta)^{1/2} v = g(x)v, & \text{in } (0, R), \\ v = 0, & \text{in } \mathbb{R} \setminus (0, R), \end{cases}$$

for some $g \in C^1([0, R])$, then its harmonic extension $V \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ weakly solves

$$\begin{cases} -\Delta V = 0, & \text{in } \mathbb{R}^2_+, \\ \partial_\nu V = g(x)V, & \text{on } (0, R) \times \{0\}, \\ V = 0, & \text{on } (\mathbb{R} \setminus (0, R)) \times \{0\}, \end{cases} \tag{1.6}$$

see [4]. Theorem 1.1 and Corollary 1.3 apply to (1.6). Hence, V (and in particular its restriction v) satisfies expansion (1.4) and a strong unique continuation principle from 0 (i.e. from a boundary point of the domain of v). We mention that unique continuation principles from interior points for fractional elliptic equations were established in [8].

We do not know if the $C^{2,\delta}$ regularity on Ω and C^1 regularity of the boundary potential g in Theorem 1.1 can be weakened in order to obtain a unique continuation property. On the other hand, we can conclude that a regularity assumption on the boundary is crucial for excluding the presence of logarithms in the asymptotic expansion at the junction. Indeed, in Section 8 we produce an example of a harmonic function on a domain with a C^1-boundary which is not of class $C^{2,\delta}$, satisfying null Dirichlet boundary conditions on a portion of the boundary and null Neumann boundary conditions on the other portion, but exhibiting dominant logarithmic terms in its asymptotic expansion.

The proof of Theorem 1.1 combines the use of an Almgren type monotonicity formula, blow-up analysis and sharp regularity estimates. Indeed regularity estimates yield the expansion of u near zero as follows:

$$\left\| u - \sum_{k=1}^{k_0} a_k(r) F_k \circ \varphi \right\|_{L^\infty(B_r)} \leqslant C r^{\frac{2k_0-1}{2}+\varrho}, \tag{1.7}$$

for every $\varrho \in (0, 1/2)$, for some $C > 0$, $k_0 \geqslant 1$ and where $a_k = \frac{\langle u, F_k \circ \varphi \rangle_{L^2(B_r)}}{\|F_j \circ \varphi\|^2_{L^2(B_r)}}$. Now, if u is nontrivial, a blow-up analysis combined with Almgren type monotonicity formula allows to depict a $k_0 \geqslant 1$ for which $a_{k_0}(r) \to \beta \neq 0$ and $a_k(r) \to 0$ for every $k < k_0$ as $r \to 0$. The proof of (1.7) uses also a blow-up analysis argument inspired by Serra [24], see also [22, 23].

Remark 1.2. *The extension of our results to higher dimensions are the object of current investigation. First of all, the implementation of the monotonicity argument for Dirichlet-Neumann problems exhibits substantial additional difficulties due to the positive dimension of the junction set and some role played by the geometry of the domain. Moreover, further technical difficulties appear in higher dimension since, in such a situation, we can no more make use of conformal transformations like the ones employed in Section 2 which are based on the Riemann mapping Theorem.*

Remark 1.3. *For the sake of simplicity of the exposition, in the present paper we considered an elliptic problem with the Laplacian and a linear term with a bounded potential; a possible extension to more general elliptic problems with variable coefficients and first order terms could be obtained with a more sophisticated monotonicity approach like in [9].*

The paper is organized as follows. In Section 2 we introduce an auxiliary equivalent problem obtained by a conformal diffeomorphic deformation straightening $B_1 \cap \partial\Omega$ near 0 and state Theorem 2.1 giving the sharp asymptotic behaviour of its solutions. Section 3 contains some Hardy-Poincaré type inequalities for H^1-functions vanishing on a portion of the boundary of half-balls. In Section 4 we develop an Almgren type monotonicity formula for the auxiliary problem which yields good energy estimates for rescaled solutions thus allowing the fine blow-up analysis performed in Section 5 and hence the proof of Theorem 2.1. Section 7 contains the proof of the main Theorem 1.1, which is based on Theorem 2.1 and on some regularity and approximation results established in Section 6. Finally, Section 8 is devoted to the construction of an example of a solution with logarithmic dominant term in a domain violating the $C^{2,\delta}$-regularity assumption.

2. The auxiliary problem

For every $R > 0$ let $B_R = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < R^2\}$ and $B_R^+ = \{(x_1, x_2) \in B_R : x_2 > 0\}$. Since $\partial\Omega$ is of class $C^{2,\delta}$ near zero, we can find $r_0 > 0$ such that $\Gamma := \partial\Omega \cap B_{r_0}$ is a $C^{2,\delta}$ curve.

Here and in the following, we let $\mathcal{B}$ be a $C^{2,\delta}$ simply connected open bounded set such that $\mathcal{B} \subset \Omega$ and $\partial\mathcal{B} \cap \partial\Omega = \Gamma$. For some functions

$$f \in L^\infty(\mathcal{B}) \quad \text{and} \quad g \in C^1(\overline{\Gamma_n}), \tag{2.1}$$

let $u \in H^1(\mathcal{B})$ be a solution to

$$\begin{cases} -\Delta u = f(x)u, & \text{in } \mathcal{B}, \\ \partial_\nu u = g(x)u, & \text{on } \Gamma_n, \\ u = 0, & \text{on } \Gamma_d. \end{cases} \tag{2.2}$$

We introduce the space $H^1_{0,\Gamma_d}(\mathcal{B})$ as the closure in $H^1(\mathcal{B})$ of the subspace

$$C^\infty_{0,\Gamma_d}(\mathcal{B}) := \{u \in C^\infty(\overline{\mathcal{B}}) : u = 0 \text{ on } \Gamma_d \cap \partial\mathcal{B}\}.$$

We say that $u \in H^1(\mathcal{B})$ is a weak solution to (2.2) if

$$\begin{cases} u \in H^1_{0,\Gamma_d}(\mathcal{B}), \\ \displaystyle\int_{\mathcal{B}} \nabla u(x)\nabla v(x)\,dx = \int_{\mathcal{B}} f(x)u(x)v(x)\,dx + \int_{\Gamma_n} guv\,ds \quad \text{for any } v \in C^\infty_{0,\partial\mathcal{B}\setminus\Gamma_n}(\mathcal{B}) \end{cases}$$

where $C^\infty_{0,\partial\mathcal{B}\setminus\Gamma_n}(\mathcal{B}) = \{u \in C^\infty(\overline{\mathcal{B}}) : u = 0 \text{ on } \partial\mathcal{B} \setminus \Gamma_n\}$. Since $\mathcal{B}$ is of class $C^{2,\delta}$, in view of the Riemann mapping Theorem and [17, Theorem 5.2.4], there exists a conformal map $\hat\varphi : \overline{\mathcal{B}} \to \overline{B}_1$ which is of class C^2. Let $N = \hat\varphi(0) \in \partial B_1$ and let S be its antipodal. We then consider the map $\widetilde\varphi : \mathbb{R}^2 \setminus \{S\} \to \mathbb{R}^2 \setminus \{\overline{S}\}$ given by $\widetilde\varphi(z) := 2\frac{\overline{z-S}}{|z-S|^2} + \overline{S}$, where, for every $z \in \mathbb{R}^2 \simeq \mathbb{C}$, $\overline{z}$ denotes the complex conjugate of z. This map is conformal and $\widetilde\varphi(N) = 0$. In addition $\widetilde\varphi(\overline{B_1} \setminus \{S\}) \subset \overline{\mathcal{P}}$ where $\mathcal{P}$ is the half plane not containing $\overline{S}$ whose boundary is the line passing through the origin orthogonal to $\overline{S}$.

Then the map $\widetilde\varphi \circ \hat\varphi$ is a conformal map which is of class C^2 from a neighborhood of the origin $\overline{\mathcal{B} \cap B_r}$ into $\overline{\mathcal{P}}$ for some $r > 0$. It is now clear that there exists a rotation $\mathcal{R}$ and a real number $R > 0$ such that, letting $\mathcal{U}_R := \varphi^{-1}(B^+_R)$, the map $\varphi := \mathcal{R} \circ \widetilde\varphi \circ \hat\varphi : \overline{\mathcal{U}_R} \to \overline{B^+_R}$ is an invertible conformal map of class C^2 with inverse $\varphi^{-1} : \overline{B^+_R} \to \overline{\mathcal{U}_R}$ of class C^2. Moreover $\varphi(0) = 0$.

Since φ is a conformal diffeomorphism, in view of Remark 1.1 we have that, under the assumptions of Theorem 1.1,

$$D\,\varphi(0) = \alpha\,\mathrm{Id}, \quad \text{with } \alpha = \varphi'(0) > 0, \tag{2.3}$$

being $\varphi'(0)$ the complex derivative of φ at 0, which turns out to be real because of the assumption that $(1,0)$ is tangent to $\partial\Omega$ at 0 and strictly positive because of the assumption that the exterior unit normal vector to $\partial\Omega$ at 0 is $(0,-1)$. In addition, (2.3) implies that, if R is chosen sufficiently small, $\varphi^{-1}((-R,0) \times \{0\}) \subset \Gamma_d$ and $\varphi^{-1}((0,R) \times \{0\}) \subset \Gamma_n$.

Therefore letting $w = u \circ \varphi^{-1} : B^+_R \to \mathbb{R}$ and $\Psi := \varphi^{-1}$, we then have that $w \in H^1(B^+_R)$ solves

$$\begin{cases} -\Delta w(z) = p(z)w(z), & \text{in } B^+_R, \\ \partial_\nu w(x_1,0) = q(x_1)w(x_1,0), & x_1 \in (0,R), \\ w = 0, & \text{on } (-R,0) \times \{0\}, \end{cases} \tag{2.4}$$

with

$$p(z) = |\Psi'(z)|^2 f(\Psi(z)), \qquad q(x_1) = (g(\Psi(x_1,0))|\Psi'(x_1,0)|.$$

It is plain that $p \in L^\infty(B_R^+)$ and $q \in C^1([0,R))$. Here and in the following, for every $r>0$, we define

$$\Gamma_n^r := (0,r)\times\{0\} \qquad \text{and} \qquad \Gamma_d^r := (-r,0)\times\{0\}. \tag{2.5}$$

The following theorem describes the behaviour of w at 0 in terms of the limit of the Almgren quotient associated to w, which is defined as

$$\mathcal{N}(r) = \frac{\int_{B_r^+} |\nabla w|^2 dz - \int_{B_r^+} p w^2 dz - \int_0^r q(x) w^2(x,0)\,dx}{\int_0^\pi w^2(r\cos t, r\sin t)\,dt}.$$

In Section 4 we will prove that $\mathcal{N}$ is well defined in the interval $(0,R_0)$ for some $R_0>0$.

Theorem 2.1. *Let w be a nontrivial solution to (2.4). Then there exists $k_0 \in \mathbb{N}$, $k_0 \geqslant 1$, such that*

$$\lim_{r\to 0^+} \mathcal{N}(r) = \frac{2k_0-1}{2}. \tag{2.6}$$

Furthermore

$$\tau^{-\frac{2k_0-1}{2}} w(\tau z) \to \beta |z|^{\frac{2k_0-1}{2}} \cos\left(\tfrac{2k_0-1}{2}\operatorname{Arg} z\right) \quad \text{as } \tau\to 0^+$$

strongly in $H^1(B_r^+)$ for all $r>0$ and in $C^{0,\mu}_{\rm loc}(\overline{\mathbb{R}^2_+}\setminus\{0\})$ for every $\mu\in(0,1)$, where $\beta\neq 0$ and

$$\begin{aligned}\beta =& \frac{2}{\pi}\int_0^\pi R^{-\frac{2k_0-1}{2}} w(R\cos s, R\sin s)\cos\left(\tfrac{2k_0-1}{2}s\right) ds \\ &+ \frac{2}{\pi}\int_0^\pi \left[\int_0^R \frac{t^{-k_0+3/2} - R^{1-2k_0}t^{k_0+1/2}}{2k_0-1} p(t\cos s, t\sin s) w(t\cos s, t\sin s)\,dt\right]\cos\left(\tfrac{2k_0-1}{2}s\right) ds \\ &+ \frac{2}{\pi}\int_0^R \frac{t^{1/2-k_0} - R^{1-2k_0}t^{k_0-1/2}}{2k_0-1} q(t) w(t,0)\,dt.\end{aligned} \tag{2.7}$$

In particular

$$\tau^{-\frac{2k_0-1}{2}} w(\tau\cos t, \tau\sin t) \to \beta\cos\left(\tfrac{2k_0-1}{2}t\right) \quad \text{in } C^{0,\mu}([0,\pi]) \quad \text{as } \tau\to 0^+. \tag{2.8}$$

The proof of Theorem 2.1 is based on the study of the monotonicity properties of the Almgren function $\mathcal{N}$ and on a fine blow-up analysis which will be performed in Sections 4 and 5.

3. Hardy-Poincaré type inequalities

In the description of the asymptotic behavior at the Dirichlet-Neumann junction of solutions to equation (2.4) a crucial role is played by eigenvalues and eigenfunctions of the angular component of the principal part of the operator.

Let us consider the eigenvalue problem

$$\begin{cases} -\psi'' = \lambda\psi, & \text{in } [0,\pi],\\ \psi'(0) = 0,\\ \psi(\pi) = 0. \end{cases} \tag{3.1}$$

It is easy to verify that (3.1) admits the sequence of (all simple) eigenvalues

$$\lambda_k = \frac{1}{4}(2k-1)^2, \quad k\in\mathbb{N},\ k\geqslant 1,$$

with corresponding eigenfunctions

$$\psi_k(t) = \cos\left(\tfrac{2k-1}{2}t\right), \quad k \in \mathbb{N},\ k \geqslant 1.$$

It is well known that the normalized eigenfunctions

$$\left\{\sqrt{\tfrac{2}{\pi}}\cos\left(\tfrac{2k-1}{2}t\right)\right\}_{k\geqslant 1} \tag{3.2}$$

form an orthonormal basis of the space $L^2(0,\pi)$. Furthermore, the first eigenvalue $\lambda_1 = \frac{1}{4}$ can be characterized as

$$\lambda_1 = \frac{1}{4} = \min_{\substack{\psi\in H^1(0,\pi)\setminus\{0\}\\ \psi(\pi)=0}} \frac{\int_0^\pi |\psi'(t)|^2\,dt}{\int_0^\pi |\psi(t)|^2\,dt}. \tag{3.3}$$

For every $r > 0$, we let (recall (2.5) for the definition of Γ_d^r)

$$\mathcal{H}_r = \{w \in H^1(B_r^+) : w = 0 \text{ on } \Gamma_d^r\ \}.$$

As a consequence of (3.3) we obtain the following Hardy-Poincaré inequality in $\mathcal{H}_r$.

Lemma 3.1. *For every $r > 0$ and $w \in \mathcal{H}_r$, we have that*

$$\int_{B_r^+} |\nabla w(z)|^2\,dz \geqslant \frac{1}{4}\int_{B_r^+} \frac{|w(z)|^2}{|z|^2}\,dz.$$

Proof. Let $w \in C^\infty(\overline{B_r^+})$ with $w = 0$ on $\overline{\Gamma_d^r} = [-r,0]\times\{0\}$. Then, in view of (3.3),

$$\begin{aligned}
&\int_{B_r^+} |\nabla w(z)|^2\,dz\\
&\quad= \int_0^r\int_0^\pi \rho\left(\left|\tfrac{\partial}{\partial\rho}(w(\rho\cos t,\rho\sin t))\right|^2 + \frac{1}{\rho^2}\left|\tfrac{\partial}{\partial t}(w(\rho\cos t,\rho\sin t))\right|^2\right)dt\,d\rho\\
&\quad\geqslant \int_0^r \frac{1}{\rho}\left(\int_0^\pi \left|\tfrac{\partial}{\partial t}(w(\rho\cos t,\rho\sin t))\right|^2 dt\right)d\rho\\
&\quad\geqslant \frac{1}{4}\int_0^r \frac{1}{\rho}\left(\int_0^\pi |w(\rho\cos t,\rho\sin t)|^2 dt\right)d\rho = \frac{1}{4}\int_{B_r^+}\frac{|w(z)|^2}{|z|^2}\,dz
\end{aligned}$$

We conclude by density, recalling that the space of smooth functions vanishing on $[-r,0]\times\{0\}$ is dense in $\mathcal{H}_r$, see e.g. [7]. □

Lemma 3.2. *For every $r > 0$ and $w \in \mathcal{H}_r$, we have that $x_1^{-1}w^2(x_1,0) \in L^1(0,r)$ and*

$$\int_0^r \frac{w^2(x_1,0)}{x_1}\,dx_1 \leqslant \pi\int_{B_r^+}|\nabla w(z)|^2\,dz.$$

Proof. Let $w \in C^\infty(\overline{B_r^+})$ with $w = 0$ on $[-r,0]\times\{0\}$. Then for any $0 < x_1 < r$

$$\begin{aligned}
|w(x_1,0)| &= \left|\int_0^\pi \frac{d}{dt}w(x_1\cos t, x_1\sin t)\,dt\right| = \left|\int_0^\pi x_1\nabla w(x_1\cos t,x_1\sin t)\cdot(-\sin t,\cos t)\,dt\right|\\
&\leqslant \sqrt{\pi}\sqrt{\int_0^\pi x_1^2|\nabla w(x_1\cos t,x_1\sin t)|^2\,dt}.
\end{aligned}$$

It follows that

$$\int_0^r \frac{w^2(x_1,0)}{x_1}\,dx_1 \leqslant \pi \int_0^r \int_0^\pi x_1 |\nabla w(x_1\cos t, x_1 \sin t)|^2\,dt\,dx_1 = \pi \int_{B_r^+} |\nabla w(z)|^2\,dz.$$

We conclude by density. □

4. The monotonicity formula

Let $w \in H^1(B_R^+)$ be a non trivial solution to (2.4). For every $r \in (0,R]$ we define

$$D(r) = \int_{B_r^+} |\nabla w|^2 dz - \int_{B_r^+} p w^2 dz - \int_0^r q(x_1) w^2(x_1,0)\,dx_1 \tag{4.1}$$

and

$$H(r) = \frac{1}{r}\int_{S_r^+} w^2\,ds = \int_0^\pi w^2(r\cos t, r\sin t)\,dt, \tag{4.2}$$

where $S_r^+ := \{(x_1,x_2) : x_1^2 + x_2^2 = r^2 \text{ and } x_2 > 0\}$.

In order to differentiate the functions D and H, the following Pohozaev type identity is needed.

Theorem 4.1. *Let w solve (2.4). Then for a.e. $r \in (0,R)$ we have*

$$\begin{aligned}\frac{r}{2}\int_{S_r^+} |\nabla w|^2 ds &= r\int_{S_r^+} \left|\frac{\partial w}{\partial \nu}\right|^2 ds \\ &\quad - \frac{1}{2}\int_0^r \big(q(x_1) + x_1 q'(x_1)\big) w^2(x_1,0)\,dx_1 + \frac{r}{2} q(r) w^2(r,0) + \int_{B_r^+} pwz\cdot\nabla w\,dz\end{aligned} \tag{4.3}$$

and

$$\int_{B_r^+} |\nabla w|^2 dz = \int_{B_r^+} p w^2 dz + \int_{S_r^+} \frac{\partial w}{\partial \nu} w\,ds + \int_0^r q(x_1) w^2(x_1,0)\,dx_1. \tag{4.4}$$

Proof. We observe that, by elliptic regularity theory, $w \in H^2(B_r^+ \setminus B_\varepsilon^+)$ for all $0 < \varepsilon < r < R$. Furthermore, the fact that w has null trace on Γ_d^R implies that $\frac{\partial w}{\partial x_1}$ has null trace on Γ_d^R. Then, testing (2.4) with $z\cdot\nabla w$ and integrating over $B_r^+ \setminus B_\varepsilon^+$, we obtain that

$$\begin{aligned}\frac{r}{2}\int_{S_r^+} |\nabla w|^2 ds - \frac{\varepsilon}{2}\int_{S_\varepsilon^+} |\nabla w|^2 ds &= \int_{B_r^+\setminus B_\varepsilon^+} pwz\cdot\nabla w\,dz \\ &\quad + r\int_{S_r^+} \left|\frac{\partial w}{\partial \nu}\right|^2 ds - \varepsilon\int_{S_\varepsilon^+} \left|\frac{\partial w}{\partial \nu}\right|^2 ds + \int_\varepsilon^r q(x_1) w(x_1,0) x_1 \frac{\partial w}{\partial x_1}(x_1,0)\,dx_1.\end{aligned} \tag{4.5}$$

An integration by parts, which can be easily justified by an approximation argument, yields that

$$\begin{aligned}\int_\varepsilon^r q(x_1) w(x_1,0) x_1 \frac{\partial w}{\partial x_1}(x_1,0)\,dx_1 &= \frac{r}{2} q(r) w^2(r,0) \\ &\quad - \frac{\varepsilon}{2} q(\varepsilon) w^2(\varepsilon,0) - \frac{1}{2}\int_\varepsilon^r (q + x_1 q') w^2(x_1,0)\,dx_1.\end{aligned} \tag{4.6}$$

We observe that there exists a sequence $\varepsilon_n \to 0^+$ such that

$$\lim_{n\to\infty}\left[\varepsilon_n w^2(\varepsilon_n,0) + \varepsilon_n \int_{S_{\varepsilon_n}^+} |\nabla w|^2 ds\right] = 0.$$

Indeed, if no such sequence exists, there would exist $\varepsilon_0 > 0$ such that

$$w^2(r,0) + \int_{S_r^+} |\nabla w|^2 ds \geqslant \frac{C}{r} \quad \text{for all } r \in (0,\varepsilon_0), \quad \text{for some } C > 0;$$

integration of the above inequality on $(0,\varepsilon_0)$ would then contradict the fact that $w \in H^1(B_R^+)$ and, by trace embedding, $w \in L^2(\Gamma_n^{\varepsilon_0})$. Then, passing to the limit in (4.5) and (4.6) with $\varepsilon = \varepsilon_n$ yields (4.3). Finally (4.4) follows by testing (2.4) with w and integrating by parts in B_r^+. □

In the following lemma we compute the derivative of the function H.

Lemma 4.2. *$H \in W^{1,1}_{\rm loc}(0,R)$ and*

$$H'(r) = 2\int_0^\pi w(r\cos t, r\sin t)\tfrac{\partial w}{\partial \nu}(r\cos t, r\sin t)\,dt = \frac{2}{r}\int_{S_r^+} w\tfrac{\partial w}{\partial \nu}\,ds, \tag{4.7}$$

in a distributional sense and for a.e. $r \in (0,R)$, and

$$H'(r) = \frac{2}{r}D(r), \quad \text{for a.e. } r \in (0,R). \tag{4.8}$$

Proof. Let $\varphi \in C_{\rm c}^\infty(0,R)$. Since $w, \nabla w \in L^2(B_R^+)$ and $w \in C^1(B_R^+)$, using twice Fubini's Theorem we obtain that

$$\begin{aligned}
&\int_0^R H(r)\varphi'(r)\,dr = \int_0^R \Big(\int_0^\pi w^2(r\cos t, r\sin t)\,dt\Big)\varphi'(r)\,dr\\
&= \int_0^\pi \Big(\int_0^R w^2(r\cos t, r\sin t)\varphi'(r)\,dr\Big)\,dt = -\int_0^\pi \Big(\int_0^R \frac{d}{dr}\Big(w^2(r\cos t, r\sin t)\Big)\varphi(r)\,dr\Big)\,dt\\
&= -\int_0^\pi \Big(\int_0^R \Big(2w(r\cos t, r\sin t)\tfrac{\partial w}{\partial \nu}(r\cos t, r\sin t)\Big)\varphi(r)\,dr\Big)\,dt\\
&= -\int_0^R \Big(\int_0^\pi \Big(2w(r\cos t, r\sin t)\tfrac{\partial w}{\partial \nu}(r\cos t, r\sin t)\Big)\,dt\Big)\varphi(r)\,dr
\end{aligned}$$

thus proving (4.7). Identity (4.8) follows directly from (4.7) and (4.4). □

Let us now study the regularity of the function D.

Lemma 4.3. *The function D defined in (4.1) belongs to $W^{1,1}(0,R)$ and*

$$\begin{aligned}
D'(r) = 2\int_{S_r^+}\Big|\frac{\partial w}{\partial \nu}\Big|^2 ds\\
- \frac{1}{r}\int_0^r \big(q(x_1) + x_1 q'(x_1)\big)w^2(x_1,0)\,dx_1 + \frac{2}{r}\int_{B_r^+} pwz\cdot\nabla w\,dz - \int_{S_r^+} pw^2\,ds
\end{aligned} \tag{4.9}$$

in a distributional sense and for a.e. $r \in (0,R)$.

Proof. From the fact that $w \in H^1(B_R^+)$ and $w\big|_{\Gamma_n^R} \in L^2(\Gamma_n^R)$, we deduce that D belongs to $W^{1,1}(0,R)$ and

$$D'(r) = \int_{S_r^+} |\nabla w|^2 ds - \int_{S_r^+} pw^2 ds - q(r)w^2(r,0) \tag{4.10}$$

for a.e. $r \in (0,R)$ and in the distributional sense.

The conclusion follows combining (4.10) and (4.3). □

Lemma 4.4. *There exists $R_0 \in (0,R)$ such that $H(r)>0$ for any $r\in(0,R_0)$.*

Proof. Let $R_0\in(0,R)$ be such that

$$4\|p\|_{L^\infty(B_R^+)}R_0^2+\pi\|q\|_{L^\infty(\Gamma_n^R)}R_0<1. \tag{4.11}$$

Assume by contradiction that there exists $r_0\in(0,R_0)$ such that $H(r_0)=0$, so that $w=0$ a.e. on $S_{r_0}^+$. From (4.4) it follows that

$$\int_{B_{r_0}^+}|\nabla w|^2dz-\int_{B_{r_0}^+}pw^2dz-\int_0^{r_0}q(x_1)w^2(x_1,0)\,dx_1=0.$$

From Lemmas 3.1 and 3.2, we get

$$\begin{aligned}0&=\int_{B_{r_0}^+}|\nabla w|^2dz-\int_{B_{r_0}^+}pw^2dz-\int_0^{r_0}q(x_1)w^2(x_1,0)\,dx_1\\&\geqslant\Big[1-4\|p\|_{L^\infty(B_R^+)}r_0^2-\pi\|q\|_{L^\infty(\Gamma_n^R)}r_0\Big]\int_{B_{r_0}^+}|\nabla w|^2dz,\end{aligned}$$

which, together with (4.11) and Lemma 3.1, implies $w\equiv0$ in $B_{r_0}^+$. From classical unique continuation principles for second order elliptic equations with locally bounded coefficients (see e.g. [28]) we can conclude that $w=0$ a.e. in B_R^+, a contradiction. □

Thanks to Lemma 4.4, the frequency function

$$\mathcal{N}:(0,R_0)\to\mathbb{R},\quad \mathcal{N}(r)=\frac{D(r)}{H(r)}, \tag{4.12}$$

is well defined. Using Lemmas 4.2 and 4.3, we now compute the derivative of $\mathcal{N}$.

Lemma 4.5. *The function $\mathcal{N}$ defined in* (4.12) *belongs to $W^{1,1}_{\rm loc}(0,R_0)$ and*

$$\mathcal{N}'(r)=\nu_1(r)+\nu_2(r) \tag{4.13}$$

in a distributional sense and for a.e. $r\in(0,R_0)$, where

$$\nu_1(r)=\frac{2r\Big[\Big(\int_{S_r^+}\big|\frac{\partial w}{\partial\nu}\big|^2\,ds\Big)\cdot\Big(\int_{S_r^+}w^2\,ds\Big)-\Big(\int_{S_r^+}w\frac{\partial w}{\partial\nu}\,ds\Big)^2\Big]}{\Big(\int_{S_r^+}w^2\,ds\Big)^2} \tag{4.14}$$

and

$$\nu_2(r)=-\frac{\int_0^r\big(q(x)+xq'(x)\big)w^2(x,0)\,dx}{\int_{S_r^+}w^2\,ds}+2\frac{\int_{B_r^+}pwz\cdot\nabla w\,dz}{\int_{S_r^+}w^2\,ds}-\frac{r\int_{S_r^+}pw^2\,ds}{\int_{S_r^+}w^2ds}. \tag{4.15}$$

Proof. From Lemmas 4.2, 4.4, and 4.3, it follows that $\mathcal{N}\in W^{1,1}_{\rm loc}(0,R_0)$. From (4.8) we deduce that

$$\mathcal{N}'(r)=\frac{D'(r)H(r)-D(r)H'(r)}{(H(r))^2}=\frac{D'(r)H(r)-\frac12r(H'(r))^2}{(H(r))^2}$$

and the proof of the lemma easily follows from (4.7) and (4.9). □

We now prove that $\mathcal{N}(r)$ admits a finite limit as $r\to0^+$.

Lemma 4.6. *There exists $\gamma \in [0, +\infty)$ such that $\lim_{r\to 0^+} \mathcal{N}(r) = \gamma$.*

Proof. From Lemmas 3.1 and 3.2 it follows that

$$D(r) \geqslant \Big[1 - 4\|p\|_{L^\infty(B_R^+)} r^2 - \pi \|q\|_{L^\infty(\Gamma_n^R)} r\Big] \int_{B_r^+} |\nabla w|^2 dz,$$

hence there exist $\bar r \in (0, R_0)$ and $C_1 > 0$ such that

$$D(r) \geqslant C_1 \int_{B_r^+} |\nabla w|^2 dz, \quad \text{for all } r \in (0, \bar r).$$

In particular

$$\mathcal{N}(r) \geqslant 0, \quad \text{for all } r \in (0, \bar r). \tag{4.16}$$

Moreover, using again Lemmas 3.1 and 3.2 we can estimate ν_2 in $(0, \bar r)$ as follows

$$\begin{aligned} |\nu_2(r)| &\leqslant \frac{\|q + xq'\|_{L^\infty(\Gamma_n^R)} \pi r \int_{B_r^+} |\nabla w|^2 dz}{\int_{S_r^+} w^2 \, ds} \\ &\quad + \frac{\|p\|_{L^\infty(B_R^+)} r(1 + 4r^2) \int_{B_r^+} |\nabla w(z)|^2 \, dz}{\int_{S_r^+} w^2 \, ds} + r\|p\|_{L^\infty(B_R^+)} \\ &\leqslant \frac{1}{C_1} \Big(\|q + xq'\|_{L^\infty(\Gamma_n^R)} \pi + \|p\|_{L^\infty(B_R^+)} (1 + 4\bar r^2) \Big) \mathcal{N}(r) + \bar r \|p\|_{L^\infty(B_R^+)}. \end{aligned} \tag{4.17}$$

Since $\nu_1 \geqslant 0$ by Schwarz's inequality, from Lemma 4.5 and the above estimate it follows that there exists $C_2 > 0$ such that

$$\mathcal{N}'(r) \geqslant -C_2(\mathcal{N}(r) + 1) \quad \text{for all } r \in (0, \bar r), \tag{4.18}$$

which implies that

$$\frac{d}{dr} \left(e^{C_2 r} (1 + \mathcal{N}(r)) \right) \geqslant 0.$$

It follows that the limit of $r \mapsto e^{C_2 r}(1 + \mathcal{N}(r))$ as $r \to 0^+$ exists and is finite; hence the function $\mathcal{N}$ has a finite limit γ as $r \to 0^+$. From (4.16) we deduce that $\gamma \geqslant 0$. □

The function H defined in (4.2) can be estimated as follows.

Lemma 4.7. *Let $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ be as in Lemma 4.6. Then*

$$H(r) = O(r^{2\gamma}) \quad \text{as } r \to 0^+. \tag{4.19}$$

Moreover, for any $\sigma > 0$,

$$r^{2\gamma+\sigma} = O(H(r)) \quad \text{as } r \to 0^+. \tag{4.20}$$

Proof. From Lemma 4.6 we have that

$$\mathcal{N} \text{ is bounded in a neighborhood of } 0, \tag{4.21}$$

hence from (4.18) it follows that $\mathcal{N}' \geqslant -C_3$ for some positive constant C_3 in a neighborhood of 0. Then

$$\mathcal{N}(r) - \gamma = \int_0^r \mathcal{N}'(\rho) \, d\rho \geqslant -C_3 r \tag{4.22}$$

in a neighborhood of 0. From (4.8), (4.12), and (4.22) we deduce that, in a neighborhood of 0,

$$\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} \geqslant \frac{2\gamma}{r} - 2C_3,$$

which, after integration, yields (4.19).

Since $\gamma = \lim_{r\to 0^+} \mathcal{N}(r)$, for any $\sigma > 0$ there exists $r_\sigma > 0$ such that $\mathcal{N}(r) < \gamma + \sigma/2$ for any $r \in (0, r_\sigma)$ and hence $\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} < \frac{2\gamma+\sigma}{r}$ for all $r \in (0, r_\sigma)$. By integration we obtain (4.20). □

5. Blow-up analysis for the auxiliary problem

Lemma 5.1. *Let $w \in H^1(B_R^+)$ be a non trivial solution to (2.4). Let $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ be as in Lemma 4.6. Then there exists $k_0 \in \mathbb{N}$, $k_0 \geqslant 1$, such that*

$$\gamma = \frac{2k_0 - 1}{2}.$$

Furthermore, for every sequence $\tau_n \to 0^+$, there exist a subsequence $\{\tau_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\frac{w(\tau_{n_k} z)}{\sqrt{H(\tau_{n_k})}} \to \widetilde{w}(z) \tag{5.1}$$

strongly in $H^1(B_r^+)$ and in $C^{0,\mu}_{\rm loc}(\overline{B_r^+} \setminus \{0\})$ for every $\mu \in (0,1)$ and all $r \in (0,1)$, where

$$\widetilde{w}(r\cos t, r\sin t) = \pm\sqrt{\frac{2}{\pi}}\, r^{\frac{2k_0-1}{2}} \cos\left(\frac{2k_0-1}{2}t\right), \quad \text{for all } r \in (0,1) \text{ and } t \in [0,\pi]. \tag{5.2}$$

Proof. Let us set

$$w^\tau(z) = \frac{w(\tau z)}{\sqrt{H(\tau)}}. \tag{5.3}$$

We notice that, for all $\tau \in (0,R)$, $w^\tau \in \mathcal{H}_1$ and $\int_{S_1^+} |w^\tau|^2 ds = \int_0^\pi |w^\tau(\cos t, \sin t)|^2\, dt = 1$. Moreover, by scaling and (4.21),

$$\int_{B_1^+} \left(|\nabla w^\tau(z)|^2 - \tau^2 p(\tau z)|w^\tau(z)|^2\right) dz - \tau \int_0^1 q(\tau x)|w^\tau(x,0)|^2\, dx = \mathcal{N}(\tau) = O(1) \tag{5.4}$$

as $\tau \to 0^+$, whereas from Lemmas 3.1 and 3.2 it follows that

$$\begin{aligned} \mathcal{N}(\tau) &\geqslant \frac{1}{H(\tau)}\Big[1 - 4\|p\|_{L^\infty(B_R^+)}\tau^2 - \pi\|q\|_{L^\infty(\Gamma_n^R)}\tau\Big] \int_{B_\tau^+} |\nabla w|^2 dz \\ &= \Big[1 - 4\|p\|_{L^\infty(B_R^+)}\tau^2 - \pi\|q\|_{L^\infty(\Gamma_n^R)}\tau\Big] \int_{B_1^+} |\nabla w^\tau|^2 dz \end{aligned} \tag{5.5}$$

for every $\tau \in (0, R_0)$, being R_0 as in (4.11). From (5.4), (5.5), and Lemma 3.1 we deduce that

$$\{w^\tau\}_{\tau\in(0,R_0)} \quad \text{is bounded in } H^1(B_1^+). \tag{5.6}$$

Therefore, for any given sequence $\tau_n \to 0^+$, there exists a subsequence $\tau_{n_k} \to 0^+$ such that $w^{\tau_{n_k}} \rightharpoonup \widetilde{w}$ weakly in $H^1(B_1^+)$ for some $\widetilde{w} \in H^1(B_1^+)$. Due to compactness of trace embeddings, we have that $\widetilde{w} = 0$ on Γ_d^1 and

$$\int_{S_1^+} |\widetilde{w}|^2 ds = 1. \tag{5.7}$$

In particular $\widetilde{w} \not\equiv 0$. For every small $\tau \in (0, R_0)$, w^τ satisfies

$$\begin{cases} -\Delta w^\tau = \tau^2 p(\tau z) w^\tau, & \text{in } B_1^+, \\ \partial_\nu w^\tau = \tau q(\tau x_1, 0) w^\tau, & \text{on } \Gamma_n^1, \\ w^\tau = 0, & \text{on } \Gamma_d^1, \end{cases} \tag{5.8}$$

in a weak sense, i.e.

$$\int_{B_1^+} \nabla w^\tau(z) \cdot \nabla \varphi(z)\, dz = \tau^2 \int_{B_1^+} p(\tau z) w^\tau(z) \varphi(z)\, dz + \tau \int_0^1 q(\tau x) w^\tau(x,0) \varphi(x,0)\, dx$$

for all $\varphi \in H^1(B_1^+)$ s.t. $\varphi = 0$ on $S_1^+ \cup \Gamma_d^1$. From weak convergence $w^{\tau_{n_k}} \rightharpoonup \widetilde{w}$ in $H^1(B_1^+)$, we can pass to the limit in (5.8) along the sequence τ_{n_k} and obtain that $\widetilde{w}$ weakly solves

$$\begin{cases} -\Delta \widetilde{w} = 0, & \text{in } B_1^+, \\ \partial_\nu \widetilde{w} = 0, & \text{on } \Gamma_n^1, \\ \widetilde{w} = 0, & \text{on } \Gamma_d^1. \end{cases} \tag{5.9}$$

From (5.6) it follows that $\{\tau q(\tau x) w^\tau(x,0)\}_{\tau \in (0,R_0)}$ is bounded in $H^{1/2}(\Gamma_n^1)$. Then, by elliptic regularity theory, for every $0 < r_1 < r_2 < 1$ we have that $\{w^\tau\}_{\tau \in (0,R_0)}$ is bounded in $H^2(B_{r_2}^+ \setminus \overline{B_{r_2}^+})$. From compactness of trace embeddings we have that, up to passing to a further subsequence, $\frac{\partial w^{\tau_{n_k}}}{\partial \nu} \to \frac{\partial \widetilde{w}}{\partial \nu}$ in $L^2(S_r^+)$ for every $r \in (0,1)$. Testing equation (5.8) for $\tau = \tau_{n_k}$ with w^τ on B_r^+ we obtain that

$$\begin{aligned} \int_{B_r^+} |\nabla w^{\tau_{n_k}}(z)|^2\, dz &= \int_{S_r^+} \frac{\partial w^{\tau_{n_k}}}{\partial \nu} w^{\tau_{n_k}}\, ds \\ &\quad + \tau_{n_k}^2 \int_{B_r^+} p(\tau_{n_k} z) |w^{\tau_{n_k}}(z)|^2\, dz + \tau_{n_k} \int_0^r q(\tau_{n_k} x) |w^{\tau_{n_k}}(x,0)|^2\, dx \\ &\underset{k \to +\infty}{\to} \int_{S_r^+} \frac{\partial \widetilde{w}}{\partial \nu} \widetilde{w}\, ds = \int_{B_r^+} |\nabla \widetilde{w}(z)|^2\, dz, \end{aligned}$$

thus proving that $\|w^{\tau_{n_k}}\|_{H^1(B_r^+)} \to \|\widetilde{w}\|_{H^1(B_r^+)}$ for all $r \in (0,1)$, and hence

$$w^{\tau_{n_k}} \to \widetilde{w} \quad \text{in } H^1(B_r^+) \tag{5.10}$$

for every $r \in (0,1)$. Furthermore, by compact Sobolev embeddings, we also have that, up to extracting a further subsequence,

$$w^{\tau_{n_k}} \to \widetilde{w} \quad \text{in } C^{0,\mu}_{\mathrm{loc}}(\overline{B_r^+} \setminus \{0\}),$$

for every $r \in (0,1)$ and $\mu \in (0,1)$.

For any $r \in (0,1)$ and $k \in \mathbb{N}$, let us define the functions

$$\begin{aligned} D_k(r) &= \int_{B_r^+} |\nabla w^{\tau_{n_k}}|^2\, dz - \tau_{n_k}^2 \int_{B_r^+} p(\tau_{n_k} z) |w^{\tau_{n_k}}(z)|^2 dz - \tau_{n_k} \int_0^r q(\tau_{n_k} x) |w^{\tau_{n_k}}(x,0)|^2\, dx, \\ H_k(r) &= \frac{1}{r} \int_{S_r^+} |w^{\tau_{n_k}}|^2\, ds, \end{aligned}$$

and $\mathcal{N}_k(r) := \frac{D_k(r)}{H_k(r)}$. Direct calculations yield that $\mathcal{N}_k(r) = \mathcal{N}(\tau_{n_k} r)$ for all $r \in (0,1)$. From (5.10) it follows that, for any fixed $r \in (0,1)$,

$$D_k(r) \to \widetilde{D}(r) := \int_{B_r^+} |\nabla \widetilde{w}|^2 \, dz \quad \text{and} \quad H_k(r) \to \widetilde{D}(r) := \frac{1}{r} \int_{S_r^+} |\widetilde{w}|^2 \, ds.$$

From classical unique continuation principles for harmonic functions it follows that $\widetilde{D}(r) > 0$ and $\widetilde{H}(r) > 0$ for all $r \in (0,1)$ (indeed $\widetilde{D}(r) = 0$ or $\widetilde{H}(r) = 0$ for some $r \in (0,1)$ would imply that $\widetilde{w} \equiv 0$ in B_r^+ and, by unique continuation, $\widetilde{w} \equiv 0$ in B_1^+, a contradiction). Hence, by Lemma 4.6,

$$\widetilde{\mathcal{N}}(r) = \frac{\widetilde{D}(r)}{\widetilde{H}(r)} = \lim_{k\to\infty} \mathcal{N}_k(r) = \lim_{k\to\infty} \mathcal{N}(\tau_{n_k} r) = \gamma \tag{5.11}$$

for all $r \in (0,1)$. Therefore $\widetilde{\mathcal{N}}$ is constant in $(0,1)$ and hence $\widetilde{\mathcal{N}}'(r) = 0$ for any $r \in (0,1)$. By (5.9) and Lemma 4.5 with $p \equiv 0$ and $q \equiv 0$, we obtain

$$\left(\int_{S_r^+} \left| \frac{\partial \widetilde{w}}{\partial \nu} \right|^2 ds \right) \cdot \left(\int_{S_r^+} \widetilde{w}^2 \, ds \right) - \left(\int_{S_r^+} \widetilde{w} \frac{\partial \widetilde{w}}{\partial \nu} \, ds \right)^2 = 0 \quad \text{for all } r \in (0,1),$$

which implies that $\widetilde{w}$ and $\frac{\partial \widetilde{w}}{\partial \nu}$ are parallel as vectors in $L^2(S_r^+)$. Hence there exists $\eta = \eta(r)$ such that $\frac{\partial \widetilde{w}}{\partial \nu}(r\cos t, r \sin t) = \eta(r) \widetilde{w}(r\cos t, r \sin t)$ for all $r \in (0,1)$ and $t \in [0,\pi]$. It follows that

$$\widetilde{w}(r\cos t, r\sin t) = \varphi(r)\psi(t), \quad r \in (0,1), \ t \in [0,\pi], \tag{5.12}$$

where $\varphi(r) = e^{\int_1^r \eta(s) ds}$ and $\psi(t) = \widetilde{w}(\cos t, \sin t)$. From (5.7) we have that $\int_0^\pi \psi^2 = 1$. From (5.9) and (5.12) we can conclude that

$$\begin{cases} \varphi''(r)\psi(t) + \frac{1}{r}\varphi'(r)\psi(t) + \frac{1}{r^2}\varphi(r)\psi''(t) = 0, \quad r \in (0,1), \quad t \in [0,1], \\ \psi(\pi) = 0, \\ \psi'(0) = 0. \end{cases}$$

Taking r fixed, we deduce that ψ is necessarily an eigenfunction of the eigenvalue problem (3.1). Then there exists $k_0 \in \mathbb{N} \setminus \{0\}$ such that $\psi(t) = \pm\sqrt{\frac{2}{\pi}} \cos(\frac{2k_0-1}{2} t)$ and $\varphi(r)$ solves the equation

$$\varphi''(r) + \frac{1}{r}\varphi' - \frac{(2k_0-1)^2}{4r^2}\varphi(r) = 0.$$

Hence $\varphi(r)$ is of the form

$$\varphi(r) = c_1 r^{\frac{2k_0-1}{2}} + c_2 r^{-\frac{2k_0-1}{2}}$$

for some $c_1, c_2 \in \mathbb{R}$. Since the function $r^{-\frac{2k_0-1}{2}}\psi(t) \notin H^1(B_1^+)$, we deduce that necessarily $c_2 = 0$ and $\varphi(r) = c_1 r^{\frac{2k_0-1}{2}}$. Moreover, from $\varphi(1) = 1$, we obtain that $c_1 = 1$ and then

$$\widetilde{w}(r\cos t, r\sin t) = \pm\sqrt{\frac{2}{\pi}}\, r^{\frac{2k_0-1}{2}} \cos\left(\frac{2k_0-1}{2} t\right), \quad \text{for all } r \in (0,1) \text{ and } t \in [0,\pi]. \tag{5.13}$$

From (5.13) it follows that

$$\widetilde{H}(r) = \int_0^\pi \widetilde{w}^2(r\cos t, r\sin t) \, dt = r^{2k_0-1}.$$

Hence, in view of (4.8),

$$\gamma = \widetilde{\mathcal{N}}(r) = \frac{r}{2}\frac{\widetilde{H}'(r)}{\widetilde{H}(r)} = \frac{r}{2}(2k_0-1)\frac{r^{2k_0-2}}{r^{2k_0-1}} = \frac{2k_0-1}{2}.$$

The proof of the lemma is thereby complete. □

We observe that at this stage of our analysis we cannot exclude that the limit function $\widetilde{w}$ found in Lemma 5.1 depends on the subsequence. In order to prove that the convergence in (5.1) actually holds as $\tau \to 0^+$ we need to univocally identify the limit profile $\widetilde{w}$.

Lemma 5.2. *Let $w \not\equiv 0$ satisfy (2.4), H be defined in (4.2), and $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ be as in Lemma 4.6. Then the limit $\lim_{r\to 0^+} r^{-2\gamma}H(r)$ exists and it is finite.*

Proof. In view of (4.19) it is sufficient to prove that the limit exists. By (4.2), (4.8), and Lemma 4.6 we have that

$$\frac{d}{dr}\frac{H(r)}{r^{2\gamma}} = 2r^{-2\gamma-1}(D(r)-\gamma H(r)) = 2r^{-2\gamma-1}H(r)\int_0^r \mathcal{N}'(\rho)d\rho,$$

and then, by integration over (r, R_0),

$$\frac{H(R_0)}{R_0^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} = 2\int_r^{R_0}\frac{H(\rho)}{\rho^{2\gamma+1}}\left(\int_0^\rho \nu_1(t)dt\right)d\rho + 2\int_r^{R_0}\frac{H(\rho)}{\rho^{2\gamma+1}}\left(\int_0^\rho \nu_2(t)dt\right)d\rho \tag{5.14}$$

where ν_1 and ν_2 are as in (4.14) and (4.15). Since, by Schwarz's inequality, $\nu_1 \geqslant 0$, we have that $\lim_{r\to 0^+}\int_r^{R_0}\rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho \nu_1(t)dt\right)d\rho$ exists. On the other hand, from Lemma 4.6 $\mathcal{N}$ is bounded and hence from (4.17) we deduce that ν_2 is bounded close to 0^+. Hence, in view of (4.19), the function $\rho \mapsto \rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho \nu_2(t)dt\right)$ is bounded and hence integrable near 0. We conclude that both terms at the right hand side of (5.14) admit a limit as $r \to 0^+$ thus completing the proof. □

The following lemma provides some pointwise estimate for solutions to (2.4).

Lemma 5.3. *Let $w \in H^1(B_R^+)$ be a nontrivial solution to (2.4). Then there exist $C_4, C_5 > 0$ and $\bar{r} \in (0, R_0)$ such that*

(i) $\sup_{S_r^+}|w|^2 \leqslant \frac{C_4}{r}\int_{S_r^+}|w(z)|^2\,ds$ *for every $0 < r < \bar{r}$,*
(ii) $|w(z)| \leqslant C_5|z|^\gamma$ *for all $z \in B_{\bar{r}}^+$, with γ as in Lemma 4.6.*

Proof. We first notice that (ii) follows directly from (i) and (4.19). In order to prove (i), we argue by contradiction and assume that there exists a sequence $\tau_n \to 0^+$ such that

$$\sup_{t\in[0,\pi]}\left|w\Big(\frac{\tau_n}{2}\cos t, \frac{\tau_n}{2}\sin t\Big)\right|^2 > nH\Big(\frac{\tau_n}{2}\Big)$$

with H as in (4.2), i.e., defining w^τ as in (5.3)

$$\sup_{x\in S_{1/2}^+}|w^{\tau_n}(z)|^2 > 2n\int_{S_{1/2}^+}|w^{\tau_n}(z)|^2 ds. \tag{5.15}$$

From Lemma 5.1, there exists a subsequence τ_{n_k} such that $w^{\tau_{n_k}} \to \widetilde{w}$ in $C^0(S_{1/2}^+)$ with $\widetilde{w}$ being as in (5.2), hence passing to the limit in (5.15) a contradiction arises. □

To obtain a sharp asymptotics of $H(r)$ as $r \to 0^+$, it remains to prove that $\lim_{r\to 0^+} r^{-2\gamma} H(r)$ is strictly positive.

Lemma 5.4. *Under the same assumptions as in Lemmas 5.2 and 5.3, we have that*

$$\lim_{r\to 0^+} r^{-2\gamma} H(r) > 0.$$

Proof. From Lemma 5.1 there exists $k_0 \in \mathbb{N}$, $k_0 \geqslant 1$ such that $\gamma = \frac{2k_0-1}{2}$. Let us expand w as

$$w(r\cos t, r\sin t) = \sum_{k=1}^{\infty} \varphi_k(r) \cos\left(\tfrac{2k-1}{2} t\right) \tag{5.16}$$

where

$$\varphi_k(r) = \frac{2}{\pi} \int_0^{\pi} w(r\cos t, r\sin t) \cos\left(\tfrac{2k-1}{2} t\right)\, dt. \tag{5.17}$$

The Parseval identity yields

$$H(r) = \frac{\pi}{2} \sum_{k=1}^{\infty} \varphi_k^2(r), \quad \text{for all } 0 < r \leqslant R. \tag{5.18}$$

From (4.19) and (5.18) it follows that, for all $k \geqslant 1$,

$$\varphi_k(r) = O(r^{\gamma}) \quad \text{as } r \to 0^+. \tag{5.19}$$

Let $\eta \in C_c^{\infty}(0,R)$. Testing (2.4) with the function $\eta(r)\cos\left(\frac{2k-1}{2}t\right)$, by (5.16) we obtain

$$\begin{aligned} \frac{\pi}{2} \int_0^R r\varphi_k'(r)\eta'(r)\, dr + \frac{\pi}{2} \int_0^R \tfrac{(2k-1)^2}{4} \frac{1}{r} \varphi_k(r)\eta(r)\, dr = \int_0^R q(r) w(r,0)\eta(r)\, dr \\ + \int_0^R r\eta(r) \left(\int_0^{\pi} p(r\cos t, r\sin t) w(r\cos t, r\sin t) \cos\left(\tfrac{2k-1}{2}\, t\right)\, dt \right) dr\,. \end{aligned} \tag{5.20}$$

Integrating by parts in the first in integral on the left hand side of (5.20) and exploiting the fact that $\eta \in C_c^{\infty}(0,R)$ is an arbitrary test function, we infer

$$-\varphi_k''(r) - \frac{1}{r}\varphi_k'(r) + \frac{1}{4}(2k-1)^2 \frac{\varphi_k(r)}{r^2} = \zeta_k(r), \quad \text{in } (0,R),$$

where

$$\zeta_k(r) = \frac{2}{\pi r}\, q(r) w(r,0) + \frac{2}{\pi} \int_0^{\pi} p(r\cos t, r\sin t) w(r\cos t, r\sin t) \cos\left(\tfrac{2k-1}{2} t\right)\, dt. \tag{5.21}$$

Then, by a direct calculation, there exist $c_1^k, c_2^k \in \mathbb{R}$ such that

$$\varphi_k(r) = r^{\frac{2k-1}{2}} \left(c_1^k + \int_r^R \frac{t^{\frac{1-2k}{2}+1}}{2k-1} \zeta_k(t)\, dt \right) + r^{\frac{1-2k}{2}} \left(c_2^k + \int_r^R \frac{t^{\frac{2k-1}{2}+1}}{1-2k} \zeta_k(t)\, dt \right). \tag{5.22}$$

From Lemma 5.3 it follows that

$$\zeta_{k_0}(r) = O\left(r^{\frac{2k_0-1}{2}-1} \right) \quad \text{as } r \to 0^+, \tag{5.23}$$

and hence the functions

$$t \mapsto t^{\frac{1-2k_0}{2}+1} \zeta_{k_0}(t) \quad \text{and} \quad t \mapsto t^{\frac{2k_0-1}{2}+1} \zeta_{k_0}(t)$$

belong to $L^1(0,R)$. Hence

$$r^{\frac{2k_0-1}{2}}\left(c_1^{k_0}+\int_r^R \frac{t^{\frac{1-2k_0}{2}+1}}{2k_0-1}\zeta_{k_0}(t)\,dt\right)=o(r^{\frac{1-2k_0}{2}}) \quad \text{as } r\to 0^+,$$

and then, by (5.19), there must be

$$c_2^{k_0}=\int_0^R \frac{t^{\frac{2k_0-1}{2}+1}}{2k_0-1}\zeta_{k_0}(t)\,dt.$$

From (5.23), we then deduce that

$$r^{\frac{1-2k_0}{2}}\left(c_2^{k_0}+\int_r^R \frac{t^{\frac{2k_0-1}{2}+1}}{1-2k_0}\zeta_{k_0}(t)\,dt\right)=r^{\frac{1-2k_0}{2}}\int_0^r \frac{t^{\frac{2k_0-1}{2}+1}}{2k_0-1}\zeta_{k_0}(t)\,dt=O(r^{k_0+\frac{1}{2}}) \tag{5.24}$$

as $r\to 0^+$. From (5.22) and (5.24), we obtain that

$$\varphi_{k_0}(r)=r^{\frac{2k_0-1}{2}}\left(c_1^{k_0}+\int_r^R \frac{t^{\frac{1-2k_0}{2}+1}}{2k_0-1}\zeta_{k_0}(t)\,dt+O(r)\right) \quad \text{as } r\to 0^+. \tag{5.25}$$

Let us assume by contradiction that $\lim_{r\to 0^+} r^{-2\gamma}H(r)=0$. Then (5.18) would imply that

$$\lim_{r\to 0^+} r^{-\frac{2k_0-1}{2}}\varphi_{k_0}(r)=0,$$

and hence, in view of (5.25), we would have that

$$c_1^{k_0}+\int_0^R \frac{t^{\frac{1-2k_0}{2}+1}}{2k_0-1}\zeta_{k_0}(t)\,dt=0,$$

which, together with (5.23), implies

$$r^{\frac{2k_0-1}{2}}\left(c_1^{k_0}+\int_r^R \frac{t^{\frac{1-2k_0}{2}+1}}{2k_0-1}\zeta_{k_0}(t)\,dt\right)=r^{\frac{2k_0-1}{2}}\int_0^r \frac{t^{\frac{1-2k_0}{2}+1}}{1-2k_0}\zeta_{k_0}(t)\,dt=O(r^{\frac{1}{2}+k_0}) \tag{5.26}$$

as $r\to 0^+$. From (5.25) and (5.26), we conclude that $\varphi_{k_0}(r)=O(r^{\frac{1}{2}+k_0})$ as $r\to 0^+$, namely,

$$\sqrt{H(\tau)}\int_0^\pi w^\tau(\cos t,\sin t)\cos\left(\tfrac{2k_0-1}{2}t\right)\,dt=O(\tau^{\frac{1}{2}+k_0}) \quad \text{as } \tau\to 0^+,$$

where w^τ is defined in (5.3). From (4.20), there exists $C>0$ such that $\sqrt{H(\tau)}\geqslant C\tau^{\gamma+\frac{1}{2}}$ for τ small, and therefore

$$\int_0^\pi w^\tau(\cos t,\sin t)\cos\left(\tfrac{2k_0-1}{2}t\right)\,dt=O(\tau^{\frac{1}{2}}) \quad \text{as } \tau\to 0^+. \tag{5.27}$$

From Lemma 5.1, for every sequence $\tau_n\to 0^+$, there exist a subsequence $\{\tau_{n_k}\}_{k\in\mathbb{N}}$ such that

$$w^{\tau_{n_k}}(\cos t,\sin t)\to \pm\sqrt{\frac{2}{\pi}}\cos\left(\frac{2k_0-1}{2}t\right) \quad \text{in } L^2(0,\pi). \tag{5.28}$$

From (5.27) and (5.28), we infer that

$$0=\lim_{k\to+\infty}\int_0^\pi w^{\tau_{n_k}}(\cos t,\sin t)\cos\left(\tfrac{2k_0-1}{2}t\right)\,dt=\pm\sqrt{\frac{2}{\pi}}\int_0^\pi \cos^2\left(\tfrac{2k_0-1}{2}t\right)\,dt=\pm\sqrt{\frac{\pi}{2}},$$

thus reaching a contradiction. □

Proof of Theorem 2.1. Identity (2.6) follows from Lemma 5.1, thus there exists $k_0 \in \mathbb{N}$, $k_0 \geqslant 1$, such that $\gamma = \lim_{r\to 0^+} \mathcal{N}(r) = \frac{2k_0-1}{2}$.

Let $\{\tau_n\}_{n\in\mathbb{N}} \subset (0,+\infty)$ be such that $\lim_{n\to+\infty} \tau_n = 0$. Then, from Lemmas 5.1 and 5.4, scaling and a diagonal argument, there exists a subsequence $\{\tau_{n_k}\}_{k\in\mathbb{N}}$ and $\beta \neq 0$ such that

$$\frac{w(\tau_{n_k} z)}{\tau_{n_k}^{\gamma}} \to \beta |z|^{\frac{2k_0-1}{2}} \cos\left(\tfrac{2k_0-1}{2} \operatorname{Arg} z\right) \tag{5.29}$$

strongly in $H^1(B_r^+)$ for all $r > 0$ and in $C^{0,\mu}_{\rm loc}(\overline{\mathbb{R}^2_+} \setminus \{0\})$ for every $\mu \in (0,1)$. In particular

$$\tau_{n_k}^{-\gamma} w(\tau_{n_k}(\cos t, \sin t)) \to \beta \cos\left(\frac{2k_0-1}{2} t\right) \tag{5.30}$$

in $C^{0,\mu}([0,\pi])$. To prove that the above converge occurs as $\tau \to 0^+$ and not only along subsequences, we are going to show that β depends neither on the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\tau_{n_k}\}_{k\in\mathbb{N}}$.

Defining φ_{k_0} and ζ_{k_0} as in (5.17) and (5.21), from (5.30) it follows that

$$\begin{aligned} \frac{\varphi_{k_0}(\tau_{n_k})}{\tau_{n_k}^{\gamma}} &= \frac{2}{\pi} \int_0^{\pi} \frac{w(\tau_{n_k}\cos t, \tau_{n_k} \sin t)}{\tau_{n_k}^{\gamma}} \cos\left(\tfrac{2k_0-1}{2} t\right)\, dt \\ &\to \frac{2}{\pi} \beta \int_0^{\pi} \cos^2\left(\tfrac{2k_0-1}{2} t\right)\, dt = \beta \end{aligned} \tag{5.31}$$

as $k \to +\infty$. On the other hand, from (5.22), (5.24) , and (5.25) we know that that

$$\begin{aligned} \varphi_{k_0}(\tau) &= \tau^{\frac{2k_0-1}{2}} \left(c_1^{k_0} + \int_{\tau}^{R} \frac{t^{\frac{1-2k_0}{2}+1}}{2k_0-1} \zeta_{k_0}(t)\, dt \right) + \tau^{\frac{1-2k_0}{2}} \int_0^{\tau} \frac{t^{\frac{2k_0-1}{2}+1}}{2k_0-1} \zeta_{k_0}(t)\, dt \\ &= \tau^{\frac{2k_0-1}{2}} \left(c_1^{k_0} + \int_{\tau}^{R} \frac{t^{\frac{1-2k_0}{2}+1}}{2k_0-1} \zeta_{k_0}(t)\, dt + O(\tau) \right) \quad \text{as } \tau \to 0^+. \end{aligned} \tag{5.32}$$

Choosing $\tau = R$ in the first line of (5.32), we obtain

$$c_1^{k_0} = R^{-\frac{2k_0-1}{2}} \varphi_{k_0}(R) - R^{1-2k_0} \int_0^R \frac{t^{\frac{2k_0-1}{2}+1}}{2k_0-1} \zeta_{k_0}(t)\, dt.$$

Hence, from the second line of (5.32), we obtain that

$$\tau^{-\gamma} \varphi_{k_0}(\tau) \to R^{-\frac{2k_0-1}{2}} \varphi_{k_0}(R) - R^{1-2k_0} \int_0^R \frac{t^{\frac{2k_0-1}{2}+1}}{2k_0-1} \zeta_{k_0}(t)\, dt + \int_0^R \frac{t^{\frac{1-2k_0}{2}+1}}{2k_0-1} \zeta_{k_0}(t)\, dt,$$

as $\tau \to 0^+$. Then, from (5.31) we deduce that

$$\beta = R^{-\frac{2k_0-1}{2}} \varphi_{k_0}(R) - R^{1-2k_0} \int_0^R \frac{t^{\frac{2k_0-1}{2}+1}}{2k_0-1} \zeta_{k_0}(t)\, dt + \int_0^R \frac{t^{\frac{1-2k_0}{2}+1}}{2k_0-1} \zeta_{k_0}(t)\, dt. \tag{5.33}$$

In particular β depends neither on the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\tau_{n_k}\}_{k\in\mathbb{N}}$, thus implying that the convergence in (5.29) actually holds as $\tau \to 0^+$ and proving the theorem. We observe that (2.7) follows by replacing (5.17) and (5.21) into (5.33). □

6. Some regularity estimates

In this section, we prove some regularity and approximation results, which will be used to estimate the Hölder norm of the difference between a solution u to (1.1) and its asymptotic profile βF_{k_0}.

Proposition 6.1. *Let $f \in L^\infty(B_4^+)$, $g \in L^\infty(\Gamma_n^4)$ and let $v \in H^1(B_4) \cap L^\infty(B_4^+)$ solve*

$$\begin{cases} -\Delta v = f, & \text{in } B_4^+, \\ \partial_\nu v = g, & \text{on } \Gamma_n^4, \\ v = 0, & \text{on } \Gamma_d^4. \end{cases} \tag{6.1}$$

Then, for every $\varepsilon > 0$, there exists a constant $C > 0$ (independent of v, f, and g) such that

$$\|v\|_{C^{1/2-\varepsilon}(\overline{B_2^+})} \leqslant C \left(\|f\|_{L^\infty(B_4^+)} + \|g\|_{L^\infty(\Gamma_n^4)} + \|v\|_{L^\infty(B_4^+)} \right). \tag{6.2}$$

Proof. In the sequel we denote as $C > 0$ a positive constant independent of v, f, and g which may vary from line to line. We consider a C^2 domain Ω' such that $B_3^+ \subset \Omega' \subset B_4^+$ and $\Gamma_n^3 \cup \Gamma_d^3 \subset \partial\Omega'$. We define the functions (obtained uniquely by minimization arguments) $v_1 \in H^1(\Omega')$ satisfying

$$\begin{cases} -\Delta v_1 = f, & \text{in } \Omega', \\ \partial_\nu v_1 = 0, & \text{on } \Gamma_n^3, \\ v_1 = 0, & \text{on } \partial\Omega' \setminus \Gamma_n^3, \end{cases} \tag{6.3}$$

and $\widetilde{v}_2 \in H^{1/2}(\mathbb{R})$ satisfying

$$\begin{cases} (-\Delta)^{\frac{1}{2}} \widetilde{v}_2 = g, & \text{in } (0,4), \\ \widetilde{v}_2 = 0, & \text{on } \mathbb{R} \setminus (0,4). \end{cases}$$

Therefore by (fractional) elliptic regularity theory (see e.g. [21, Proposition 1.1]), we deduce that

$$\|\widetilde{v}_2\|_{C^{1/2}(\mathbb{R})} \leqslant C \|g\|_{L^\infty(\Gamma_n^4)}. \tag{6.4}$$

Consider the Poisson kernel $P(x_1, x_2) = \frac{1}{\pi} x_2 |x|^{-2}$ with respect to the half-space $\mathbb{R}^2_+$, see [4, Section 2.4]. We define

$$v_2(x_1, x_2) = (P(\cdot, x_2) \star \widetilde{v}_2)(x_1) = \frac{1}{\pi} x_2 \int_{\mathbb{R}} \frac{\widetilde{v}_2(t)}{x_2^2 + (x_1 - t)^2} \, dt = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\widetilde{v}_2(x_1 - r x_2)}{1 + r^2} \, dr$$

where with the symbol $\star$ we denoted the convolution product with respect to the first variable. One can check that $v_2 \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ (see for example [3, Subsection 2.1]) and

$$\begin{cases} -\Delta v_2 = 0, & \text{in } \mathbb{R}^2_+, \\ \partial_\nu v_2 = g, & \text{on } \Gamma_n^4, \\ v_2 = 0, & \text{on } \mathbb{R} \setminus (0,4). \end{cases} \tag{6.5}$$

It is easy to see that

$$\|v_2\|_{L^\infty(\mathbb{R}^2_+)} \leqslant C \|\widetilde{v}_2\|_{L^\infty(\mathbb{R})}.$$

Moreover by (6.4), for $x, y \in \overline{\mathbb{R}^2_+}$ we get

$$\begin{aligned}|v_2(x) - v_2(y)| &\leqslant C\|g\|_{L^\infty(\Gamma_n^4)}|x-y|^{1/2}\int_{\mathbb{R}} \frac{\max(1,|r|^{1/2})}{1+r^2}\,dr\\ &\leqslant C\|g\|_{L^\infty(\Gamma_n^4)}|x-y|^{1/2}.\end{aligned}$$

It follows that

$$\|v_2\|_{C^{1/2}(\overline{\mathbb{R}^2_+})} \leqslant C\|g\|_{L^\infty(\Gamma_n^4)}. \tag{6.6}$$

By [25, Theorem 1] and continuous embeddings of Besov spaces into Hölder spaces, we get

$$\|v_1\|^2_{C^{1/2-\varepsilon}(\overline{\Omega'})} \leqslant C\|v_1\|_{H^1(\Omega')}\left(\|f\|_{L^\infty(B_4^+)} + \|v_1\|_{H^1(\Omega')}\right).$$

Multiplying (6.3) by v_1, integrating by parts and using Young's inequality, we get

$$C\|v_1\|^2_{L^2(\Omega')} \leqslant \|\nabla v_1\|^2_{L^2(\Omega')} \leqslant \|v_1\|_{L^2(\Omega')}\|f\|_{L^2(B_4^+)} \leqslant \varepsilon\|v_1\|^2_{L^2(\Omega')} + C_\varepsilon\|f\|^2_{L^\infty(B_4^+)},$$

where in the first estimate we have used the Poincaré inequality for functions vanishing on a portion of the boundary. We then conclude that

$$\|v_1\|_{C^{1/2-\varepsilon}(\overline{\Omega'})} \leqslant C\|f\|_{L^\infty(B_4^+)}. \tag{6.7}$$

Now, thanks to (6.1), (6.3) and (6.5), the function $V := v - (v_1 + v_2) \in H^1(\Omega')$ solves the equation

$$\begin{cases} -\Delta V = 0, & \text{in } \Omega',\\ \partial_\nu V = 0, & \text{on } \Gamma_n^3,\\ V = 0, & \text{on } \Gamma_d^3. \end{cases} \tag{6.8}$$

By elliptic regularity theory, we have that

$$\|V\|_{C^2(\overline{B^+_{5/2}\setminus B^+_1})} \leqslant C\|V\|_{H^1(B_r^+)} \tag{6.9}$$

where r is a fixed radius satisfying $\frac{5}{2} < r < 3$ and $C > 0$ is independent of V. Let η a radial cutoff function compactly supported in B_3 satisfying $\eta \equiv 1$ in B_r; testing (6.8) with ηV, we infer that $\|V\|_{H^1(B_r^+)} \leqslant C\|V\|_{L^2(\Omega')}$ for some constant $C > 0$ independent of V. Hence by (6.9) we obtain

$$\|V\|_{C^2(\overline{B^+_{5/2}\setminus B^+_1})} \leqslant C\|V\|_{L^\infty(\Omega')}. \tag{6.10}$$

Let $\widetilde{\eta} \in C_c^\infty(B_{5/2})$ be a radial function, with $\widetilde{\eta} \equiv 1$ on B_2. Then the function $\widetilde{V} := \widetilde{\eta}V \in H^1(\mathbb{R}^2_+)$ solves

$$\begin{cases} -\Delta\widetilde{V} = -V\Delta\widetilde{\eta} - 2\nabla V\cdot\nabla\widetilde{\eta}, & \text{in } \mathbb{R}^2_+,\\ \partial_\nu\widetilde{V}(x_1,0) = 0, & x_1 \in (0,+\infty),\\ \widetilde{V}(x_1,0) = 0, & x_1 \in (-\infty,0). \end{cases}$$

Then by [25, Theorem 1], the arguments above, (6.10), (6.6) and (6.7), we deduce that

$$\begin{aligned}\|v - (v_1+v_2)\|_{C^{1/2-\varepsilon}(\overline{B_2^+})} &\leqslant \|\widetilde{V}\|_{C^{1/2-\varepsilon}(\overline{\mathbb{R}^2_+})} \leqslant C\|V\|_{L^\infty(\Omega')}\\ &\leqslant C\left(\|f\|_{L^\infty(B_4^+)} + \|g\|_{L^\infty(\Gamma_n^4)} + \|v\|_{L^\infty(B_4^+)}\right).\end{aligned}$$

This, combined again with (6.6) and (6.7) completes the proof. □

Recalling (1.2), for every $k \in \mathbb{N}$ with $k \geqslant 1$, we consider the finite dimensional linear subspace of $L^2(B_r^+)$, given by

$$\mathcal{S}_k := \left\{ \sum_{j=1}^{k} a_j F_j \,:\, (a_1, \dots, a_k) \in \mathbb{R}^k \right\}.$$

For every $r > 0$, $k \geqslant 1$, and $u \in L^2(B_r^+)$, we let

$$F_{k,r}^u := \mathrm{Argmin}_{F \in \mathcal{S}_k} \int_{B_r^+} (u(x) - F(x))^2 \, dx$$

be the $L^2(B_r^+)$-projection of u on $\mathcal{S}_k$, so that

$$\min_{F \in \mathcal{S}_k} \int_{B_r^+} (u(x) - F(x))^2 \, dx = \int_{B_r^+} (u(x) - F_{k,r}^u(x))^2 \, dx$$

and

$$\int_{B_r^+} (u(x) - F_{k,r}^u(x)) F(x) \, dx = 0, \quad \text{for all } F \in \mathcal{S}_k. \tag{6.11}$$

Next, we estimate the L^∞ norm of the difference between a solution of a mixed boundary value problem on B_1^+ and its projection on $\mathcal{S}_k$.

Proposition 6.2. *Let $u \in H^1(B_1^+) \cap L^\infty(\mathbb{R}^2_+)$ solve*

$$\begin{cases} -\Delta u = f, & \text{in } B_1^+, \\ \partial_\nu u = g, & \text{on } \Gamma_n^1, \\ u = 0, & \text{on } \Gamma_d^1, \end{cases} \tag{6.12}$$

where, for some $k \in \mathbb{N} \setminus \{0\}$ and $\overline{C} > 0$,

$$|f(x)| \leqslant \overline{C} |x|^{\max(\gamma_k - \frac{3}{2}, 0)}, \quad \text{for every } x \in B_1^+,$$
$$|g(x_1)| \leqslant \overline{C} |x_1|^{\max(\gamma_k - \frac{1}{2}, 0)}, \quad \text{for every } x_1 \in (0,1),$$

and $\gamma_k = \frac{2k-1}{2}$. Then, for every $\alpha \in (0, 1/2)$, we have that

$$\sup_{r>0} r^{-\gamma_k - \alpha} \|u - F_{k,r}^u\|_{L^\infty(B_r^+)} < \infty. \tag{6.13}$$

Proof. In the sequel, $C > 0$ stands for a positive constant, only depending on $\alpha, \overline{C}$ and k, which may vary from line to line. Assume by contradiction that, there exists $\alpha \in (0, 1/2)$ such that

$$\sup_{r>0} r^{-\gamma_k - \alpha} \|u - F_{k,r}^u\|_{L^\infty(B_r^+)} = \infty. \tag{6.14}$$

We consider the nonincreasing function

$$\Theta(r) := \sup_{\overline{r} > r} \overline{r}^{-\gamma_k - \alpha} \|u - F_{k,\overline{r}}^u\|_{L^\infty(B_{\overline{r}}^+)}. \tag{6.15}$$

It is clear from our assumption that

$$\Theta(r) \nearrow +\infty \qquad \text{as } r \to 0.$$

Then there exists a sequence $r_n \to 0$ such that

$$r_n^{-\gamma_k - \alpha} \|u - F_{k,r_n}^u\|_{L^\infty(B_{r_n}^+)} \geqslant \frac{\Theta(r_n)}{2}.$$

We define

$$v_n(x) := r_n^{-\gamma_k-\alpha}\frac{u(r_nx)-F^u_{k,r_n}(r_nx)}{\Theta(r_n)},$$

so that

$$\|v_n\|_{L^\infty(B_1^+)} \geqslant \frac{1}{2}. \tag{6.16}$$

Moreover, by a change of variable in (6.11), we get

$$\int_{B_1^+} v_n(x)F(x)\,dx = 0 \qquad \text{for every } F\in\mathcal{S}_k. \tag{6.17}$$

Claim: For $R=2^m$ and $r>0$, we have

$$\frac{1}{r^{\gamma_k+\alpha}\Theta(r)}\|F^u_{k,rR}-F^u_{k,r}\|_{L^\infty(B^+_{rR})} \leqslant CR^{\gamma_k+\alpha}. \tag{6.18}$$

Indeed, by definition, for every $\overline{r}>r>0$, we have

$$\|u-F^u_{k,\overline{r}}\|_{L^\infty(B^+_{\overline{r}})} \leqslant \overline{r}^{\gamma_k+\alpha}\Theta(r)$$

and thus, using the monotonicity of Θ, for every $x\in B_r^+$ we get

$$\begin{aligned}|F^u_{k,2r}(x)-F^u_{k,r}(x)| &\leqslant \|u-F^u_{k,2r}\|_{L^\infty(B^+_{2r})}+\|u-F^u_{k,r}\|_{L^\infty(B^+_r)}\\ &\leqslant 2^{1+\gamma_k+\alpha}r^{\gamma_k+\alpha}\Theta(r)\leqslant Cr^{\gamma_k+\alpha}\Theta(r).\end{aligned} \tag{6.19}$$

Letting $F^u_{k,r}=\sum_{j=1}^k a_j(r)F_j$ and $\gamma_j=\frac{2j-1}{2}$, by taking the $L^2(B_r^+)$-norm in (6.19), we get

$$|a_j(2r)-a_j(r)|r^{\gamma_j} \leqslant Cr^{\gamma_k+\alpha}\Theta(r) \qquad \text{for every } r>0. \tag{6.20}$$

Then

$$\begin{aligned}\frac{1}{r^{\gamma_k+\alpha}\Theta(r)}\|F^u_{k,r2^m}-F^u_{k,r}\|_{L^\infty(B^+_{r2^m})} &\leqslant \frac{1}{r^{\gamma_k+\alpha}\Theta(r)}\sum_{j=1}^k|a_j(r2^m)-a_j(r)|(r2^m)^{\gamma_j}\\ &\leqslant \frac{1}{r^{\gamma_k+\alpha}\Theta(r)}\sum_{j=1}^k\sum_{i=1}^m|a_j(2^ir)-a_j(2^{i-1}r)|(r2^m)^{\gamma_j}\\ &\leqslant \frac{C}{r^{\gamma_k+\alpha}\Theta(r)}\sum_{j=1}^k\sum_{i=1}^m 2^{\gamma_jm}2^{(\gamma_k-\gamma_j+\alpha)(i-1)}r^{\gamma_k+\alpha}\Theta(2^{i-1}r)\\ &\leqslant C\sum_{j=1}^k\sum_{i=1}^m 2^{\gamma_jm}2^{(\gamma_k-\gamma_j+\alpha)(i-1)} \leqslant C\sum_{j=1}^k 2^{\gamma_jm}2^{(\gamma_k-\gamma_j+\alpha)m}\\ &\leqslant C2^{m(\gamma_k+\alpha)}.\end{aligned}$$

This proves the **claim**.

From the definition of Θ and (6.18), for $R = 2^m \geqslant 1$, we have

$$\begin{aligned}\sup_{x\in B_R^+}|v_n(x)| &= \frac{1}{r_n^{\gamma_k+\alpha}\Theta(r_n)}\|u - F^u_{k,r_n}\|_{L^\infty(B^+_{r_nR})}\\ &\leqslant \frac{1}{r_n^{\gamma_k+\alpha}\Theta(r_n)}\|u - F^u_{k,r_nR}\|_{L^\infty(B^+_{r_nR})} + \frac{1}{r_n^{\gamma_k+\alpha}\Theta(r_n)}\|F^u_{k,r_nR} - F^u_{k,r_n}\|_{L^\infty(B^+_{r_nR})}\\ &\leqslant \frac{1}{r_n^{\gamma_k+\alpha}\Theta(r_n)}(r_nR)^{\gamma_k+\alpha}\Theta(r_n) + CR^{\gamma_k+\alpha}\\ &\leqslant CR^{\gamma_k+\alpha}.\end{aligned}$$

Consequently, letting $R \geqslant 1$ and $m_0 \in \mathbb{N}$ be the smallest integer such that $2^{m_0} \geqslant R$, we obtain that

$$\begin{aligned}\sup_{x\in B_R^+}|v_n(x)| &\leqslant \sup_{x\in B^+_{2^{m_0}}}|v_n(x)| \leqslant C2^{m_0(\gamma_k+\alpha)} \leqslant C(2R)^{\gamma_k+\alpha}\\ &\leqslant CR^{\gamma_k+\alpha},\end{aligned} \tag{6.21}$$

with C being a positive constant independent of R. Thanks to (1.3) and (6.12), it is plain that

$$\begin{cases}-\Delta v_n = \frac{r_n^{2-\gamma_k-\alpha}}{\Theta(r_n)}f(r_n\cdot), & \text{in } B^+_{1/r_n},\\ \partial_\nu v_n = \frac{r_n^{1-\gamma_k-\alpha}}{\Theta(r_n)}g(r_n\cdot), & \text{on } \Gamma_n^{1/r_n},\\ v_n = 0, & \text{on } \Gamma_d^{1/r_n}.\end{cases}$$

By assumption, we have that $\frac{r_n^{2-\gamma_k-\alpha}}{\Theta(r_n)}f(r_nx)$ and $\frac{r_n^{1-\gamma_k-\alpha}}{\Theta(r_n)}g(r_nx_1)$ are bounded in $L^\infty(B_M^+)$ and $L^\infty(\Gamma_n^M)$ respectively, for every $M > 0$. Hence, by Proposition 6.1 and (6.21), we have that v_n is bounded in $C^\delta(\overline{B_M^+})$ for every $M > 0$ and $\delta \in (0, 1/2)$. Furthermore, it is easy to verify that v_n is bounded in $H^1(B_M^+)$ for every $M > 0$. Then, for every $M > 0$ and $\delta \in (0, 1/2)$, v_n converges in $C^\delta(\overline{B_M^+})$ (and weakly in $H^1(B_M^+)$) to some $v \in C^\delta_{\rm loc}(\overline{\mathbb{R}^2_+}) \cap H^1_{\rm loc}(\mathbb{R}^2_+)$ satisfying

$$\begin{cases}-\Delta v = 0, & \text{in } \mathbb{R}^2_+,\\ \partial_\nu v = 0, & \text{on } \Gamma_n^\infty,\\ v = 0, & \text{on } \Gamma_d^\infty,\end{cases}$$

and by (6.21), for every $R > 1$,

$$\|v\|_{L^\infty(B_R^+)} \leqslant CR^{\gamma_k+\alpha}.$$

By Lemma 6.3 (below), we deduce that necessarily

$$v \in \mathcal{S}_k.$$

This clearly yields a contradiction when passing to the limit in (6.16) and (6.17). □

The following Liouville type result was used in the proof of Proposition 6.2.

Lemma 6.3 (Liouville theorem)**.** *Let $v \in C(\overline{\mathbb{R}^2_+}) \cap H^1_{\rm loc}(\mathbb{R}^2_+)$ satisfy*

$$\begin{cases} -\Delta v = 0, & \text{in } \mathbb{R}^2_+, \\ \partial_\nu v = 0, & \text{on } \Gamma_n^\infty, \\ v = 0, & \text{on } \Gamma_d^\infty, \end{cases}$$

and, for some $\alpha \in (0,1/2)$ and $C > 0$,

$$\|v\|_{L^\infty(B_R^+)} \leqslant C\, R^{\gamma_k+\alpha} \quad \text{for every } R > 1, \tag{6.22}$$

where $\gamma_k = \frac{2k-1}{2}$, $k \in \mathbb{N} \setminus \{0\}$. Then

$$v \in \mathcal{S}_k. \tag{6.23}$$

Proof. Arguing as in the proof of Lemma 5.4, we expand v in Fourier series with respect to the orthonormal basis of $L^2(0,\pi)$ given in (3.2) as

$$v(r\cos t, r\sin t) = \sum_{j=1}^{\infty} \varphi_j(r) \cos\left(\tfrac{2j-1}{2} t\right)$$

where $\varphi_j(r) = \frac{2}{\pi}\int_0^\pi v(r\cos t, r\sin t)\cos\left(\frac{2j-1}{2}t\right)\, dt$. From assumption (6.22) and the Parseval identity we have that

$$\frac{\pi}{2}\sum_{j=1}^{\infty} \varphi_j^2(r) = \int_0^\pi v^2(r\cos t, r\sin t)\, dt \leqslant \pi C^2 r^{2(\gamma_k+\alpha)}, \quad \text{for all } r > 1.$$

It follows that

$$|\varphi_j(r)| \leqslant \text{const}\, r^{\gamma_k+\alpha} \quad \text{for all } j \geqslant 1 \text{ and } r > 1, \tag{6.24}$$

for some const > 0 independent of j and r.

From the equation satisfied by v it follows that the functions φ_j satisfy

$$-\varphi_j''(r) - \frac{1}{r}\varphi_j'(r) + \frac{1}{4}(2j-1)^2 \frac{\varphi_j(r)}{r^2} = 0, \quad \text{in } (0,+\infty),$$

and then, for all $j \geqslant 1$, there exist $c_1^j, c_2^j \in \mathbb{R}$ such that

$$\varphi_j(r) = c_1^j r^{\frac{2j-1}{2}} + c_2^j r^{\frac{1-2j}{2}} \quad \text{for all } r > 0.$$

The fact v is continous and $v(0) = 0$ implies that $\varphi_j(r) = o(1)$ as $r \to 0^+$. As a consequence we have that $c_2^j = 0$ for all $j \geqslant 1$. On the other hand (6.24) implies that $c_1^j = 0$ for all $j > k$. Therefore we conclude that

$$v(r\cos t, r\sin t) = \sum_{j=1}^{k} c_1^j r^{\frac{2j-1}{2}} \cos\left(\tfrac{2j-1}{2}t\right) = \sum_{j=1}^{k} c_1^j F_j(r\cos t, r\sin t),$$

i.e. $v \in \mathcal{S}_k$. □

7. Asymptotics for u

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $w = u \circ \varphi^{-1}$, with $\varphi : \overline{\mathcal{U}_R} \to \overline{B_R^+}$ being the conformal map constructed in Section 2. Let $\gamma = \frac{2k_0-1}{2}$, with k_0 being as in Theorem 2.1. We define (recalling (5.3))

$$\widetilde{w}^\tau(z) := \tau^{-\gamma} w(\tau z) = \tau^{-\gamma}\sqrt{H(\tau)}w^\tau(z).$$

From Theorem 2.1 we have that there exists $\beta \neq 0$ such that $\widetilde{w}^\tau \to \beta F_{k_0}$ in $H^1(B_r^+)$ for all $r > 0$ and in $C^{0,\mu}_{\mathrm{loc}}(\overline{\mathbb{R}^2_+} \setminus \{0\})$ for every $\mu \in (0,1)$.

Claim 1: We have

$$w(y) = \beta F_{k_0}(y) + o(|y|^\gamma) \qquad \text{as } |y| \to 0 \text{ and } y \in B_R^+. \tag{7.1}$$

If this does not hold true then there exists a sequence of points $y_m \in (B_R^+ \cup \Gamma_n^R) \setminus \{0\}$ and $C > 0$ such that $y_m \to 0$ and

$$|y_m|^{-\gamma}|w(y_m) - \beta F_{k_0}(y_m)| = |\widetilde{w}^{\tau_m}(z_m) - \beta F_{k_0}(z_m)| \geqslant C > 0,$$

where $\tau_m = |y_m|$ and $z_m = \frac{y_m}{|y_m|}$. If m is large enough, we get a contradiction with (2.8). This proves (7.1) as claimed.

Let $\varrho \in (0, 1/2)$ and let p and q be the functions introduced in (2.4). By (7.1), by the fact that $p \in L^\infty(B_R^+)$ and $q \in C^1([0,R))$, and by Proposition 6.2 applied to w, we have that, for every $r \in (0,R)$,

$$|w(x) - F^w_{k_0,r}(x)| \leqslant Cr^{\gamma+\varrho}, \quad \text{for every } x \in B_r^+, \tag{7.2}$$

for some positive constant $C > 0$ independent of r, which could vary from line to line in the sequel.

From (7.1) and (7.2) we deduce that

$$\sup_{x \in B_r^+} r^{-\gamma}|\beta F_{k_0}(x) - F^w_{k_0,r}(x)| \to 0, \quad \text{as } r \to 0^+. \tag{7.3}$$

Claim 2: We have

$$|\beta F_{k_0}(x) - F^w_{k_0,r}(x)| \leqslant Cr^{\gamma+\varrho}, \quad \text{for every } x \in B_r^+. \tag{7.4}$$

Once this claim is proved, then according to (7.2), we can easily deduce that for any $r \in (0,R)$

$$|w(x) - \beta F_{k_0}(x)| \leqslant |w(x) - F^w_{k_0,r}(x)| + |F^w_{k_0,r} - \beta F_{k_0}(x)| \leqslant Cr^{\gamma+\varrho}, \quad \text{for every } x \in B_r^+.$$

In particular,

$$|w(x) - \beta F_{k_0}(x)| \leqslant C|x|^{\gamma+\varrho}, \quad \text{for every } x \in B_R^+$$

which finishes the proof of Theorem 1.1.

Let us now prove **Claim 2**. Writing $F^w_{k_0,r}(x) = \sum_{j=1}^{k_0} a_j(r)F_j(x)$, by (7.3) we have that

$$|\beta - a_{k_0}(r)| \to 0, \quad \text{as } r \to 0^+. \tag{7.5}$$

Moreover by taking the $L^2(B_r^+)$-norms in (7.3), we find that

$$(a_{k_0}(r) - \beta)^2 r^{2\gamma+2} + \sum_{j=1}^{k_0-1} a_j^2(r) r^{2\gamma_j+2} \leqslant Cr^{2\gamma+2}, \quad \text{for every } R > r > 0,$$

with $\gamma_j = \frac{2j-1}{2}$. This yields, for $j = 1, \dots, k_0 - 1$,

$$|a_j(r)| \leqslant Cr^{\gamma-\gamma_j} \to 0 \qquad \text{as } r \to 0. \tag{7.6}$$

From (7.2), we get, for every $x \in B_r^+$ and $R > r > 0$,

$$\left| w(x) - \sum_{j=1}^{k_0} a_j(r) F_j(x) \right| \leqslant Cr^{\varrho+\gamma}.$$

Hence, for every $x \in B_{r/2}^+$, we have that

$$\left| \sum_{j=1}^{k_0} (a_j(r) - a_j(2^{-1}r)) F_j(x) \right| \leqslant |F_{k_0,r}^w(x) - w(x)| + |F_{k_0,2^{-1}r}^w(x) - w(x)| \leqslant Cr^{\varrho+\gamma}.$$

Taking the $L^2(B_{r/2}^+)$-norms in the previous inequality, we find that, for every $r \in (0, R)$

$$\sum_{j=1}^{k_0} |a_j(r) - a_j(2^{-1}r)| r^{\gamma_j} \leqslant Cr^{\gamma+\varrho}.$$

This implies that

$$|a_j(r) - a_j(2^{-1}r)| \leqslant Cr^{\varrho+\gamma-\gamma_j} \qquad \text{for all } 1 \leqslant j \leqslant k_0 \text{ and } r \in (0, R).$$

From this, (7.5) and (7.6), we obtain

$$\begin{aligned} |\beta - a_{k_0}(r)| r^{-\varrho} + \sum_{j=1}^{k_0-1} |a_j(r)| r^{-\varrho-\gamma+\gamma_j} &\leqslant \sum_{j=1}^{k_0} \sum_{i=0}^{\infty} |a_j(r2^{-i-1}) - a_j(r2^{-i})| r^{-\varrho-\gamma+\gamma_j} \\ &\leqslant C \sum_{i=0}^{\infty} 2^{-i\varrho}. \end{aligned}$$

This implies that, for every $x \in B_r^+$,

$$|\beta F_{k_0}(x) - F_{k_0,r}^w(x)| \leqslant |\beta - a_{k_0}(r)| r^{\gamma} + \sum_{j=1}^{k_0-1} |a_j(r)| r^{\gamma_j} \leqslant Cr^{\gamma+\varrho}.$$

That is (7.4) as claimed. □

Remark 7.1. *(i) Since φ is conformal, we have that $\widetilde{F} := F_{k_0} \circ \varphi$ satisfies $\widetilde{F} \in H^1(\mathcal{U}_R)$ and solves the homogeneous equation*

$$\begin{cases} \Delta \widetilde{F} = 0, & \text{in } \mathcal{U}_R, \\ \widetilde{F} = 0, & \text{on } \Gamma_d \cap \partial \mathcal{U}_R \\ \partial_\nu \widetilde{F} = 0, & \text{on } \Gamma_n \cap \partial \mathcal{U}_R. \end{cases} \tag{7.7}$$

(ii) Let $\Upsilon : U^+ := \mathcal{B} \cap U \to B_\rho^+$ define a C^2 parametrization (e.g. given by a system of Fermi coordinates), for some open neighborhood U of 0, with $\Upsilon(0) = 0$, $D\Upsilon(0) = Id$, $\Upsilon(\Gamma_n \cap U) \subset \Gamma_n^\rho$ and $\Upsilon(\Gamma_d \cap U) \subset \Gamma_d^\rho$. By Theorem 1.1, for every $\varrho \in (0, 1/2)$, there exist $C, \rho_0 > 0$ such that

$$|u(\Upsilon^{-1}(y)) - \beta \alpha^{\frac{2k_0-1}{2}} F_{k_0}(y)| \leqslant C|y|^{\frac{2k_0-1}{2}+\varrho}, \quad \text{for every } y \in B_{\rho_0}^+, \tag{7.8}$$

with $\alpha > 0$ as in (2.3). *Indeed, to see this, we first observe that* (7.8) *is equivalent to*

$$|u(x) - \beta F_{k_0}(\alpha\Upsilon(x))| \leqslant c|x|^{\frac{2k_0-1}{2}+\varrho}, \quad \text{for every } x \in \Upsilon^{-1}(B^+_{\rho_0}), \tag{7.9}$$

for some constant $c > 0$. We then further note that

$$|DF_{k_0}(x)| \leqslant c|x|^{\frac{2k_0-1}{2}-1}$$

and thus

$$\begin{aligned} |F_{k_0}(\alpha\Upsilon(x)) - F_{k_0}(\varphi(x))| &\leqslant c|x|^{\frac{2k_0-1}{2}-1}|\alpha\Upsilon(x) - \varphi(x)| \\ &\leqslant c|x|^{\frac{2k_0-1}{2}-1}|x|^2 \\ &\leqslant c|x|^{\frac{2k_0-1}{2}+1}, \end{aligned}$$

in a neighborhood of 0, *where $c > 0$ is a positive constant independent of x possibly varying from line to line. This, together with* (1.4) *and the triangular inequality, gives* (7.9).

Proof of Corollary 1.2. From Theorem 1.1 and (7.8) it follows that, if $u \in H^1(\Omega)$ is a non-trivial solution to (1.1), then there exist $k_0 \in \mathbb{N} \setminus \{0\}$ and $\beta \in \mathbb{R} \setminus \{0\}$ such that, for every $t \in [0, \pi)$,

$$\lim_{r\to 0} r^{-\frac{2k_0-1}{2}} u(r\cos t, r\sin t) = \beta\alpha^{\frac{2k_0-1}{2}} \cos\left(\tfrac{2k_0-1}{2}t\right). \tag{7.10}$$

Therefore, if $u \geqslant 0$, we have that necessarily $k_0 = 1$ so that statement (i) follows. Moreover, (7.8) implies that

$$u(r\cos t, r\sin t) \geqslant \beta\alpha^{1/2} r^{1/2} \cos\left(\tfrac{t}{2}\right) - Cr^{1/2+\varrho},$$

which easily provides statement (ii). □

Proof of Corollary 1.3. Let us assume by contradiction that $u \not\equiv 0$. Then, Theorem 1.1 and (7.8) imply that (7.10) holds for every $t \in [0, \pi)$ and for some $k_0 \in \mathbb{N} \setminus \{0\}$ and $\beta \in \mathbb{R} \setminus \{0\}$. Taking $n > \frac{2k_0-1}{2}$, (7.10) contradicts the assumption that $u(x) = O(|x|^n)$ as $|x| \to 0$. □

8. An example

In this section we show that the presence of a logarithmic term in the asymptotic expansion cannot be excluded without assuming enough regularity of the boundary.

Let us define in the Gauss plane the set

$$A := \mathbb{C} \setminus \{x_1 \in \mathbb{R} \subset \mathbb{C} : x_1 \leqslant 0\}$$

and the holomorphic function $\eta : A \to \mathbb{C}$ defined as follows:

$$\eta(z) := \log r + i\theta \quad \text{for any } z = re^{i\theta} \in A,\ r > 0,\ \theta \in (-\pi, \pi).$$

Let us consider the holomorphic function

$$v(z) := e^{2\eta(-iz)}\eta(-iz) \qquad \text{for any } z \in \mathbb{C} \setminus \{ix_2 : x_2 \leqslant 0\}$$

and the set

$$\mathcal{Z} := \{z \in \mathbb{C} \setminus \{ix_2 : x_2 \leqslant 0\} : \Im(v(z)) = 0\}. \tag{8.1}$$

If $z = re^{i\theta}$ with $r > 0$, $\theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus \{-\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}\}$, then $z \in \mathcal{Z}$ if and only

$$r = \rho(\theta) := \exp\left[-\left(\theta - \frac{\pi}{2}\right)\cot(2\theta)\right]. \tag{8.2}$$

For some fixed $\sigma \in \left(0, \frac{\pi}{2}\right)$, we define the curve $\Gamma_+ \subset \mathcal{Z}$ parametrized by

$$\Gamma_+ : \begin{cases} x_1(\theta) = \rho(\theta)\cos\theta \\ x_2(\theta) = \rho(\theta)\sin\theta \end{cases} \qquad \theta \in (-\sigma, 0)\,. \tag{8.3}$$

If we choose $\sigma > 0$ sufficiently small then Γ_+ is the graph of a function h_+ defined in a open right neighborhood U_+ of 0. Moreover h_+ is a Lipschitz function in U_+, $h_+ \in C^2(U_+)$ and

$$\lim_{x_1 \to 0^+} \frac{h_+(x_1)}{x_1} = 0\,, \quad \lim_{x_1 \to 0^+} h_+'(x_1) = 0\,. \tag{8.4}$$

Then we define the harmonic function

$$u(x_1, x_2) := -\Im(v(z)) \qquad \text{for any } z = x_1 + ix_2 \in \mathbb{C} \setminus \{iy : y \leqslant 0\}\,. \tag{8.5}$$

In polar coordinates the function u reads

$$u(r, \theta) = r^2 \left[(\log r)\sin(2\theta) + \left(\theta - \frac{\pi}{2}\right)\cos(2\theta)\right]. \tag{8.6}$$

From (8.1–8.2) and (8.6) we deduce that u vanishes on Γ_+.

The next step is to find a curve Γ_- on which $\frac{\partial u}{\partial \nu} = 0$ where $\nu = (\nu_1, \nu_2)$ is the unit normal to Γ_- satisfying $\nu_2 \leqslant 0$. We observe that

$$u(x_1, x_2) = x_1 x_2 \log(x_1^2 + x_2^2) + \left[\arctan\left(\frac{x_2}{x_1}\right) + \frac{\pi}{2}\right](x_1^2 - x_2^2) \quad \text{for any } x_1 < 0,\ x_2 \in \mathbb{R}\,.$$

From direct computation we obtain

$$\frac{\partial u}{\partial x_1}(x_1, x_2) = x_2 \log(x_1^2 + x_2^2) + x_2 + 2\left[\arctan\left(\frac{x_2}{x_1}\right) + \frac{\pi}{2}\right] x_1\,,$$

$$\frac{\partial u}{\partial x_2}(x_1, x_2) = x_1 \log(x_1^2 + x_2^2) + x_1 - 2\left[\arctan\left(\frac{x_2}{x_1}\right) + \frac{\pi}{2}\right] x_2\,.$$

We now define

$$H_1(x_1, x_2) = \frac{2\left[\arctan\left(\frac{x_2}{x_1}\right) + \frac{\pi}{2}\right] x_1}{\log(x_1^2 + x_2^2)} \quad \text{and} \quad H_2(x_1, x_2) = \frac{2\left[\arctan\left(\frac{x_2}{x_1}\right) + \frac{\pi}{2}\right] x_2}{\log(x_1^2 + x_2^2)}$$

on the set $B_1 \cap \Pi_-$ where $\Pi_- := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0\}$. One can easily check that H_1, H_2 admit continuous extensions defined on $B_1 \cap \overline{\Pi}_-$ which we still denote by H_1 and H_2 respectively. We also observe that $H_1, H_2 \in C^1(B_1 \cap \overline{\Pi}_-)$. Therefore H_1, H_2 may be extended also on the right of the x_2-axis up to restrict them to a disk of smaller radius. For example one may define

$$H_1(x_1, x_2) := 3H_1(-x_1, x_2) - 2H_1(-2x_1, x_2) \quad \text{and} \quad H_2(x_1, x_2) := 3H_2(-x_1, x_2) - 2H_2(-2x_1, x_2)$$

for any $(x_1, x_2) \in B_{1/2} \cap \Pi_+$ where we put $\Pi_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$. One may check that the new functions H_1, H_2 belong to $C^1(B_{1/2})$.

We can now define the functions $V_1, V_2 : B_{1/2} \to \mathbb{R}$ by

$$V_1(x_1,x_2) := \begin{cases} x_2 + \frac{x_2}{\log(x_1^2+x_2^2)} + H_1(x_1,x_2) & \text{if } (x_1,x_2) \neq (0,0) \\ 0 & \text{if } (x_1,x_2) = (0,0)\,, \end{cases}$$
$$V_2(x_1,x_2) := \begin{cases} x_1 + \frac{x_1}{\log(x_1^2+x_2^2)} - H_2(x_1,x_2) & \text{if } (x_1,x_2) \neq (0,0) \\ 0 & \text{if } (x_1,x_2) = (0,0)\,. \end{cases}$$

One may verify that $V_1, V_2 \in C^1(B_{1/2})$. Moreover we have

$$\frac{\partial V_1}{\partial x_1}(0,0) = 0\,, \quad \frac{\partial V_1}{\partial x_2}(0,0) = 1\,, \quad \frac{\partial V_2}{\partial x_1}(0,0) = 1\,, \quad \frac{\partial V_2}{\partial x_2}(0,0) = 0\,.$$

Then we consider the dynamical system

$$\begin{cases} x_1'(t) = V_1(x_1(t),x_2(t)) \\ x_2'(t) = V_2(x_1(t),x_2(t))\,. \end{cases} \tag{8.7}$$

After linearization at $(0,0)$, by [15, Theorem IX.6.2] we deduce that the stable and unstable manifolds corresponding to the stationary point $(0,0)$ of (8.7), are respectively tangent to the eigenvectors $(1,-1)$ and $(1,1)$ of the matrix $DV(0,0)$ where V is the vector field (V_1,V_2).

We define the curve Γ_- as the stable manifold of (8.7) at $(0,0)$ intersected with $B_\varepsilon \cap \Pi_-$ where $\varepsilon \in (0,\frac{1}{2})$ can be chosen sufficiently small in such a way that Γ_- becomes the graph of a function h_- defined in a open left neighborhood U_- of 0. Combining the definitions of h_+ and h_- we can introduce a function $h : U_+ \cup U_- \cup \{0\} \to \mathbb{R}$ such that $h \equiv h_+$ on U_+, $h \equiv h_-$ on U_- and $h(0) = 0$.

Then we introduce a positive number R sufficiently small and a domain $\Omega \subseteq B_R$ such that $\Omega = \{(x_1,x_2) \in B_R : x_2 > h(x_1)\}$. One can easily check that the function u defined in (8.5) belongs to $H^1(\Omega)$. From the above construction, we deduce that $u = 0$ on $\Gamma_+ \cap \partial\Omega$ and $\frac{\partial u}{\partial \nu} = 0$ on $\Gamma_- \cap \partial\Omega$. We observe that $\partial\Omega$ admits a corner at 0 of amplitude $\frac{3\pi}{4}$.

The presence of a logarithmic term in u can be explained since the $C^{2,\delta}$-regularity assumption is not satisfied from the right, i.e. $h_{|U_+\cup\{0\}} \notin C^{2,\delta}(U_+ \cup \{0\})$ for any $\delta \in (0,1)$. To see this, it is sufficient to study the behavior of $h(x_1) - x_1 h'(x_1)$ in a right neighborhood of zero.

By (8.3) we know that $\theta \in \big(-\frac{\pi}{2}, 0\big)$ and hence, if x_1 belongs to a sufficiently small right neighborhood of 0, by (8.2) we have

$$\frac{1}{2}\log\left(x_1^2 + (h_+(x_1))^2\right)\tan\left[2\arctan\left(\frac{h_+(x_1)}{x_1}\right)\right] + \arctan\left(\frac{h_+(x_1)}{x_1}\right) - \frac{\pi}{2} = 0. \tag{8.8}$$

By (8.4) and (8.8) we have that, as $x_1 \to 0^+$,

$$\tan\left[2\arctan\left(\frac{h_+(x_1)}{x_1}\right)\right] = -\frac{2\arctan\left(\frac{h_+(x_1)}{x_1}\right) - \pi}{\log\left(x_1^2 + (h_+(x_1))^2\right)} = \frac{\pi}{2}\frac{1}{\log x_1} + o\left(\frac{1}{\log x_1}\right). \tag{8.9}$$

Differentiating both sides of (8.8) and multiplying by $x_1^2 + (h_+(x_1))^2$ we obtain the identity

$$\begin{aligned}(8.10)\quad &\big(x_1 + h_+(x_1)h'_+(x_1)\big)\,\tan\left[2\arctan\left(\frac{h_+(x_1)}{x_1}\right)\right]\\ &+\left\{1+\frac{\log\left(x_1^2+(h_+(x_1))^2\right)}{\cos^2\left[2\arctan\left(\frac{h_+(x_1)}{x_1}\right)\right]}\right\}\big(x_1h'_+(x_1)-h_+(x_1)\big)=0\end{aligned}$$

and hence (8.4) and (8.9) yield

$$x_1h'_+(x_1)-h_+(x_1)\sim-\frac{\pi}{4}\,\frac{x_1}{\log^2 x_1}\qquad\text{as } x_1\to0^+. \tag{8.11}$$

This shows that $h_+\notin C^2(U_+\cup\{0\})$ (and a fortiori cannot be extended to be of class $C^{2,\delta}$).

We observe that the reason of the appearance of a logarithmic term is not due to the presence of a corner at 0; indeed we are going to construct a domain with C^1-boundary for which the same phenomenon occurs. In order to do this, it is sufficient to take the domain Ω and the function u defined above and to apply a suitable deformation in order to remove the angle. We recall that Ω exhibits a corner at 0 whose amplitude is $\frac{3\pi}{4}$.

For this reason, we define $F:\mathbb{C}\setminus\{ix_2 : x_2\leqslant 0\}\to\mathbb{C}$ by

$$F(z):=r^{\frac43}\,e^{i\frac43\theta}\quad\text{for any } z=re^{i\theta}\,,\ r>0\,,\theta\in\left(-\frac{\pi}{2},\frac{3\pi}{2}\right).$$

We observe that, up to shrink R if necessary, the map $F:\Omega\to F(\Omega)$ is invertible so that we may define $\widetilde\Omega:=F(\Omega)$ and $\widetilde u:\widetilde\Omega\to\mathbb{R}$, $\widetilde u(y_1,y_2):=u(F^{-1}(y_1,y_2))$ for any $(y_1,y_2)\in\widetilde\Omega$.

We also define the curves $\widetilde\Gamma_+:=F(\Gamma_+)$ and $\widetilde\Gamma_-:=F(\Gamma_-)$. Up to shrink R if necessary, we may assume that $\widetilde\Gamma_+$ and $\widetilde\Gamma_-$ are respectively the graphs of two functions $\widetilde h_+$ and $\widetilde h_-$.

It is immediate to verify that $\widetilde u=0$ on $\widetilde\Gamma_+$. We also prove that $\frac{\partial\widetilde u}{\partial\nu}=0$ on $\widetilde\Gamma_-$. To avoid confusion with the notion of normal unit vectors to Γ_- and $\widetilde\Gamma_-$ we denote them respectively with ν_{Γ_-} and $\nu_{\widetilde\Gamma_-}$. Since $\widetilde u$ is still harmonic, $\frac{\partial u}{\partial\nu_{\Gamma_-}}=0$ on Γ_- and F is a conformal mapping, for any $\widetilde\varphi\in C_c^\infty(\widetilde\Omega\cup\widetilde\Gamma_-)$, we have

$$\begin{aligned}\int_{\widetilde\Gamma_-}\frac{\partial\widetilde u}{\partial\nu_{\widetilde\Gamma_-}}\widetilde\varphi\,ds&=\int_{\widetilde\Omega}\nabla\widetilde u(y)\nabla\widetilde\varphi(y)\,dy=\int_{\widetilde\Omega}[\nabla u(F^{-1}(y))(DF(F^{-1}(y)))^{-1}]\nabla\widetilde\varphi(y)\,dy\\ &=\int_\Omega\left[\nabla u(x)(DF(x))^{-1}\right]\nabla\widetilde\varphi(F(x))\,|\det(DF(x))|\,dx\\ &=\int_\Omega\left[\nabla u(x)(DF(x))^{-1}\right]\left[\nabla\varphi(x)(DF(x))^{-1}\right]|\det(DF(x))|\,dx\\ &=\int_\Omega\nabla u(x)\nabla\varphi(x)\,dx=\int_{\Gamma_-}\frac{\partial u}{\partial\nu_{\Gamma_-}}\varphi\,ds=0\end{aligned}$$

where we put $\varphi(x)=\widetilde\varphi(F(x))$. This proves that $\frac{\partial\widetilde u}{\partial\nu_{\widetilde\Gamma_-}}=0$ on $\widetilde\Gamma_-$.

Finally we prove for $\widetilde h_+$ an estimate similar to (8.11).

From the definition of F it follows that $\widetilde{\Gamma}_+$ admits a representation in polar coordinates of the type

$$r = \widetilde{\rho}(\theta) := \exp\left[-\left(\theta - \frac{2\pi}{3}\right)\cot\left(\frac{3\theta}{2}\right)\right]. \tag{8.12}$$

Proceeding exactly as for (8.8)-(8.9) one can prove that

$$\frac{1}{2}\log\left(x_1^2 + (\widetilde{h}_+(x_1))^2\right)\tan\left[\frac{3}{2}\arctan\left(\frac{\widetilde{h}_+(x_1)}{x_1}\right)\right] + \arctan\left(\frac{\widetilde{h}_+(x_1)}{x_1}\right) - \frac{2\pi}{3} = 0\,. \tag{8.13}$$

As we did for h_+, also for the function $\widetilde{h}_+$ one can prove that

$$\lim_{x_1\to 0}\frac{\widetilde{h}_+(x_1)}{x_1} = 0\,, \quad \lim_{x_1\to 0^+}\widetilde{h}'_+(x_1) = 0\,. \tag{8.14}$$

By (8.14) we have

$$\begin{aligned}\tan\left[\frac{3}{2}\arctan\left(\frac{\widetilde{h}_+(x_1)}{x_1}\right)\right] &= -\frac{2\arctan\left(\frac{\widetilde{h}_+(x_1)}{x_1}\right) - \frac{4\pi}{3}}{\log\left(x_1^2 + (\widetilde{h}_+(x_1))^2\right)} \\ &= \frac{2\pi}{3}\frac{1}{\log x_1} + o\left(\frac{1}{\log x_1}\right) \quad \text{as } x_1 \to 0^+\,.\end{aligned} \tag{8.15}$$

Differentiating both sides of (8.13) and multiplying by $x_1^2 + (\widetilde{h}_+(x_1))^2$ we obtain the identity

$$\begin{aligned}&\left(x_1 + \widetilde{h}_+(x_1)\widetilde{h}'_+(x_1)\right)\tan\left[\frac{3}{2}\arctan\left(\frac{\widetilde{h}_+(x_1)}{x_1}\right)\right] \\ &\qquad + \left\{1 + \frac{3\log\left(x_1^2 + (\widetilde{h}_+(x_1))^2\right)}{4\cos^2\left[\frac{3}{2}\arctan\left(\frac{\widetilde{h}_+(x_1)}{x_1}\right)\right]}\right\}\left(x_1\widetilde{h}'_+(x_1) - \widetilde{h}_+(x_1)\right) = 0\end{aligned} \tag{8.16}$$

By (8.14), (8.15) and (8.16), we obtain

$$x_1\widetilde{h}'_+(x_1) - \widetilde{h}_+(x_1) \sim -\frac{4\pi}{9}\frac{x_1}{\log^2 x_1} \qquad \text{as } x_1 \to 0^+. \tag{8.17}$$

The above arguments show that $\partial\widetilde{\Omega}$ is of class C^1 but not of class $C^{1,\delta}$ (and a fortiori not of class $C^{2,\delta}$).

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