Anisotropic Scaling Matrices and Subdivision Schemes

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Abstract

Unlike wavelet, shearlets have the capability to detect directional discontinuities together with their directions. To achieve this, the considered scaling matrices have to be not only expanding, but also anisotropic. Shearlets allow for the definition of a directional multiple multiresolution analysis, where perfect reconstruction of the filterbank can be easily ensured by choosing an appropriate interpolatory multiple subdivision scheme. A drawback of shearlets is their relative large determinant that leads to a substantial complexity. The aim of the paper is to find scaling matrices in $\mathbb{Z}^{d \times d}$ which share the properties of shearlet matrices, i.e. anisotropic expanding matrices with the so-called *slope resolution property*, but with a smaller determinant. The proposed matrices provide a directional multiple multiresolution analysis whose behaviour is illustrated by some numerical tests on images.

1 Introduction

Shearlets were introduced by [6] as a continuous transformation and quite immediately became a popular tool in signal processing (see e.g. [7]) due to the possibility of detecting directional features in the analysis process. In the shearlet context, the scaling is performed by matrices of the form

$$M = D_2 S, \quad D_2 = \begin{pmatrix} 4I_p & 0 \\ 0 & 2I_{d-p} \end{pmatrix} \in \mathbb{R}^{d \times d}, \qquad 0 < p < d, \qquad (1)$$

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and

$$S = \begin{pmatrix} I_p & * \\ 0 & I_{d-p} \end{pmatrix}. \tag{2}$$

The diagonal matrix D_2 results in the the *anisotropy* of the resulting shearlets while the so-called *shear matrix S*, is responsible for the directionality of the shearlets. With a proper mix of these two properties, shearlets give a wavelet-like construction that is suited to deal with anisotropic problems such as detection of the singularities along curves or wavefront resolution [7, 8, 3].

In contrast to other approaches, such as curvelets or ridgelets, shearlets admit, besides a transform, also a suitably extended version of the multiresolution analysis and therefore an efficient implementation in terms of filterbanks. This is the concept of Multiple Multiresolution Analysis (MMRA) [8]. As for the classical multiresolution analysis, also the MMRA can be connected to a convergent subdivision scheme, or more precisely to a *multiple subdivision scheme* [13]. The refinement in a multiple subdivision scheme is controlled by choosing different scaling matrices from a finite dictionary $(M_j : j \in \mathbb{Z}_s)$, where we use the convenient abbreviation $\mathbb{Z}_s := \{0, \ldots, s-1\}$. If those scaling matrices are products of a common anisotropic dilation matrix with different shears, the resulting *shearlet matrices* yield a directionally adapted analysis of a signal. In particular, if we set $M_j = D_2 S_j$, $j \in \mathbb{Z}_s$, an n–fold product of such matrices,

$$M_\epsilon := M_{\epsilon_n} \cdots M_{\epsilon_1}, \qquad \epsilon \in \mathbb{Z}_s^n,$$

allows to explore singularities along a direction directly connected to the sequence ϵ. In the shearlet case, this connection is essentially a binary expansion of the slope [8] while for more general matrices the relationship can be more complicated [2].

The key ingredient for the construction of a multiple subdivision scheme and its associated filterbanks [12] is the dictionary of scaling matrices. It turns out that, in order to have a well–defined (interpolatory) multiple subdivision scheme and perfect reconstruction filterbanks, we need to consider matrices $(M_j : j \in \mathbb{Z}_s)$ that are expanding individually and *jointly expanding* considering them all together. Shearlet scaling matrices quite automatically satisfy these properties and they are also able to catch all the directions in the space due to the so-called *slope resolution* property. This means that any possible directions can be generated by M_ϵ applied to a given *reference direction*. Therefore, any transform that analyses singularities across this reference direction in the unsheared case, analyses singularities across arbitrary lines for an appropriate ϵ.

The drawback of the discrete shearlets is the huge determinant of the scaling matrix which is 2^{p+d}, as minimum, and increases dramatically with the dimension d. The determinant of the scaling matrix gives the number of analysis and synthesis filters needed in the critically sampled filterbank, so it is related to the efficiency and the computational cost of any implementation. Therefore, it is worthwhile to try to reduce, or even better minimize it.

The aim of this paper is to present a family of matrices with lower determinant that in any dimension d enjoys all the valuable properties of shearlets by essentially just giving up the requirement that the scaling has to be parabolic.

This was also the aim of [2], where the authors present a set of matrices in $\mathbb{Z}^{2\times 2}$ with minimum determinant among anisotropic integer valued matrices. In contrast to that, here we consider matrices $M \in \mathbb{Z}^{d\times d}$ which are the product of an anisotropic diagonal matrix of the form

$$D = \begin{pmatrix} aI_p & 0 \\ 0 & bI_{d-p} \end{pmatrix}, \qquad a \neq b > 1,\ 0 < p < d, \tag{3}$$

with a shear matrix. The choice to work with matrices of this form is motivated by the desire to preserve another feature of the shearlet matrices: the so-called *pseudo-commuting* property [1]. Even if this property is not necessary to construct a directional multiresolution analysis, it is very useful to give an explicit expression for the iterated matrices M_ϵ, $\epsilon \in \mathbb{Z}_s^n$ and the associated refined grid $M_\epsilon^{-1}\mathbb{Z}^d$.

In this setting, the matrices with minimal determinant are

$$D = \begin{pmatrix} 3I_p & 0 \\ 0 & 2I_{d-p} \end{pmatrix}. \tag{4}$$

In this paper, we consider exactly this type of matrices and we prove that this new family has all the desired properties: they are expanding, jointly expanding, and provide the slope resolution property as well as the pseudo commuting property. Furthermore, it is possible to generate convergent multiple subdivision schemes and filterbanks related to these matrices.

The paper is organized as follows. In Section 2, we recall some background material. Sections 2.2 and 2.3 introduce the concepts of MMRA and the related construction of multiple subdivision schemes and filterbanks for a general set of jointly expanding matrices. In Sections 2.4 we study the properties of shearlet scaling matrices and in Section 2.5 we recall some theorems by [1] that show which unimodular matrices pseudo commute with an anisotropic diagonal matrix in dimension d. This allows us to find pseudo-commuting matrices. Then, in Section 3, we introduce our choice of scaling

matrices for a general dimension d. These matrices are pseudo commuting matrices and we prove that they satisfy all the requested properties. Finally, in Section 4 few examples illustrate how our bidimensional matrices work on images.

2 Background material

In this section, we recall some of the fundamental concepts needed for MM-RAs.

2.1 Basic notation and definitions

All concepts that follow are based on expanding matrices which are the key ingredient and fix the way to refine the given data, hence to manipulate them. A matrix $M \in \mathbb{Z}^{d \times d}$ is called a *dilation* or *expanding matrix* for $\mathbb{Z}^d$ if all its eigenvalues are greater than one in absolute value, or, equivalently if $\|M^{-n}\| \to 0$ for some matrix norm as n increases. These properties suggest that $|\det M|$ is an integer ≥ 2. A matrix is said to be *isotropic* if all its eigenvalues have the same modulus and is called *anisotropic* if this is not the case. In this paper we consider a set of s expanding matrices, denoted by $(M_j : j \in \mathbb{Z}_s)$ where $\mathbb{Z}_s := \{0, \ldots, s-1\}$.
In order to recover the lattice $\mathbb{Z}^d$ from the shifts of the sublattice $M\mathbb{Z}^d$, we have to consider the cosets

$$\mathbb{Z}^d_\xi := \xi + M\mathbb{Z}^d := \{\xi + Mk : k \in \mathbb{Z}^d\}, \qquad \xi \in M[0,1)^d \cap \mathbb{Z}^d,$$

where ξ is called the *representer* of the coset. Any two cosets are either identical or disjoint, and the union of all cosets gives $\mathbb{Z}^d$; the number of cosets is $|\det M|$.
For the definition of a subdivision scheme, we have to fix some notation for infinite sequences. The space of real valued bi-infinite sequences indexed by $\mathbb{Z}^d$ or, equivalently, the space of all functions $\mathbb{Z}^d \to \mathbb{R}$ is denoted by $\ell(\mathbb{Z}^d)$. By $\ell_p(\mathbb{Z}^d)$, $1 \leq p < \infty$, we denote the sequences such that

$$\|c\|_{\ell_p(\mathbb{Z}^d)} := \left(\sum_{\alpha \in \mathbb{Z}^d} |c(\alpha)|^p \right)^{1/p} < \infty,$$

and use $\ell_\infty(\mathbb{Z}^d)$ for all bounded sequences. Finally, $\ell_{00}(\mathbb{Z}^d)$ is set of sequences from $\mathbb{Z}^d$ to $\mathbb{R}$ with finite support.

2.2 Multiple subdivision schemes and MMRA

The concept of multiresolution analysis provides the theoretical background needed in the discrete wavelets context. Unlike curvelets and ridgelets, shearlets allow to define and use a fully discrete transform based on cascaded filterbanks using the concept of MMRA. Here we recall the basic aspects of MMRA and show when and how it is possible to generate a multiple multiresolution analysis with different families of scaling matrices.
A *subdivision operator* $S : \ell(\mathbb{Z}^d) \to \ell(\mathbb{Z}^d)$ based on a *mask* $a \in \ell_{00}(\mathbb{Z}^d)$ maps a sequence $c \in \ell(Z^d)$ to the sequence given by

$$Sc = \sum_{\alpha \in \mathbb{Z}^d} a(\cdot - M\alpha) \, c(\alpha),$$

which is then interpreted as a function on the finer grid $M^{-1}\mathbb{Z}^d$. An algebraic tool in studying subdivision operators is its *symbol*, defined as the Laurent polynomial whose coefficients are the values of the mask a,

$$a^*(z) = \sum_{\alpha \in \mathbb{Z}^d} a(\alpha)z^{\alpha}, \quad z \in (\mathbb{C} \setminus \{0\})^d.$$

As shown by [13], it is always possible to generate MMRA for an arbitrary family of jointly expanding scaling matrices $(M_j : j \in \mathbb{Z}_s)$ in the shearlet context for an arbitrary dimension d, based on a generalization of subdivision schemes: multiple subdivision.

Definition 1 *A multiple subdivision scheme S_ϵ consists of a dictionary $(S_j : j \in \mathbb{Z}_s)$ of s subdivision schemes with respect to dilatation matrices $(M_j : j \in \mathbb{Z}_s)$ and masks $a_j \in \ell_{00}(\mathbb{Z}^d)$, $j \in \mathbb{Z}_s$, yielding the subdivision operators*

$$S_j c = \sum_{\alpha \in \mathbb{Z}^d} a_j(\cdot - M_j\alpha) \, c(\alpha), \qquad j \in \mathbb{Z}_s.$$

Given $n \in \mathbb{N}$ and $\epsilon \in \mathbb{Z}_s^n$, $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, the multiple subdivision operator takes the form

$$S_\epsilon c = S_{\epsilon_n} \cdots S_{\epsilon_1} c = \sum_{\alpha \in \mathbb{Z}^d} a_\epsilon(\cdot - M_\epsilon\alpha) \, c(\alpha), \qquad a_\epsilon \in \ell_{00}(\mathbb{Z}^d),$$

where $M_\epsilon := M_{\epsilon_n} \cdots M_{\epsilon_1}$.

In other words, the main idea of a multiple subdivision scheme is that in each step of the iterative process one can choose an arbitrary subdivision scheme from the family $(S_j : j \in \mathbb{Z}_s)$.

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Definition 2 *The multiple subdivision scheme is called* (uniformly) *convergent if for any infinite sequence* $\epsilon : \mathbb{N} \to \mathbb{Z}_s$ *of* digits *and any* $c : \mathbb{Z}^d \to \mathbb{R}$, $c \in C^\infty(\mathbb{Z}^d)$ *there exists a uniformly continuous function* $f_{\epsilon,c} : \mathbb{R}^d \to \mathbb{R}$ *such that*

$$\lim_{n \to \infty} \sup_{\alpha \in \mathbb{Z}^d} \left| S_{\epsilon_n} \cdots S_{\epsilon_1} c - f_{\epsilon,c} \left(M_{\epsilon_1}^{-1} \cdots M_{\epsilon_n}^{-1} \alpha \right) \right| = 0,$$

and $f_{\epsilon,c} \neq 0$ *for at least one choice of* c. *Similar notions exist for convergence in* $L_p(\mathbb{R}^d)$, $1 \leq p < \infty$ *[13].*

In multiple subdivision one considers *all* S_ϵ simultaneously and a subdivision scheme is convergent if *all* possible combinations of the subdivision schemes converge (see for details [13, 12]. As a consequence, not only the individual matrix M_j, $j \in \mathbb{Z}_s$ has to be expanding, but we need that they are *jointly expanding*, i.e. M_ϵ is an expanding matrix for any choice $\epsilon \in \mathbb{Z}_s^n$, $n \in \mathbb{N}$. While in the shearlets context this property follows directly from the particular choice of the M_j, $j \in \mathbb{Z}_s$, made there, for general families $(M_j : j \in \mathbb{Z}_s)$ is an additional assumption that has to be verified in each concrete case.

From now on, we assume that $(M_j : j \in \mathbb{Z}_s)$ is a family of jointly expanding matrices in $\mathbb{Z}^{d \times d}$. We recall the following generic construction for convergent multiple subdivision schemes that relies on the algebraic properties of the matrices M_j: given a scaling matrix M_j, $j \in \mathbb{Z}_s$, we follow [2] and consider the *Smith factorization*

$$M_j = \Theta_j \Sigma_j \Theta_j', \tag{5}$$

where $\Theta_j, \Theta_j' \in \mathbb{Z}^d$ are *unimodular* matrices, i.e., $|\det \Theta_j| = |\det \Theta_j'| = 1$, and Σ_j is a diagonal matrix with diagonal values σ_{jk}, $k = 1, \ldots, d$. Ordering the diagonal values such as one values is divisible for the previous one, this can also be turned into a *Smith normal form*, see [9] for details.

To construct a convergent subdivision scheme for M_j, we use a convergent scheme for Σ_j by means of a tensor product of univariate interpolatory schemes with *arity* equal to the respective diagonal values σ_{jk} and masks b_{jk}. The automatically convergent tensor product mask $b_{\Sigma_j} : \mathbb{Z}^d \to \mathbb{R}$ is then of the form

$$b_{\Sigma_j} := \bigotimes_{k=1}^d b_{jk}, \qquad b_{\Sigma_j}(\alpha) = \prod_{k=1}^d b_{jk}(\alpha_k), \qquad \alpha \in \mathbb{Z}^d, \tag{6}$$

so that the corresponding *symbol* can be written as

$$b_{\Sigma_j}^*(z) = \sum_{\alpha \in \mathbb{Z}^d} b_{\Sigma_j}(\alpha) z^\alpha = \prod_{k=1}^d b_{jk}^*(z_i), \qquad z \in (\mathbb{C} \setminus \{0\})^d. \tag{7}$$

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Thus, the mask of the scheme related to M_j is

$$a_j := b_{\Sigma_j}(\Theta_j^{-1}\cdot),$$

and its symbol becomes

$$a_j^*(z) = \sum_{\alpha \in \mathbb{Z}^d} b_{\Sigma_j}(\Theta_j^{-1}\alpha)z^\alpha = \sum_{\alpha \in \mathbb{Z}^d} b_{\Sigma_j}(\alpha)z^{\Theta_j\alpha} = b_{\Sigma_j}^*(z^{\Theta_j}). \qquad (8)$$

In this way, the unimodular matrix Θ_j takes the role of a resampling operator. Indeed, if $\Theta \in \mathbb{Z}^{d \times d}$ is any unimodular matrix, then, by the above reasoning, the resampled sequence $c(\Theta^{-1}\cdot)$ has symbol

$$\left(c(\Theta^{-1}\cdot)\right)^*(z) = c^*(z^\Theta), \qquad z^\Theta := \left(z^{\theta_1}, \ldots, z^{\theta_d}\right), \qquad \Theta = [\theta_1, \ldots, \theta_d],$$

where θ_i, $i = 1, \ldots, d$ are the column vector of Θ.

The following concept by [13] is a generalization of the well–known $(z + 1)$–*factor* of univariate subdivision that works as a multivariate smoothing operator.

Definition 3 *If* $M = \Theta\Sigma\Theta'$ *is a Smith factorization, the associated* canonical factor *with respect to* M *is the Laurent polynomial*

$$\psi_M(z) := \prod_{i=1}^d \sum_{k=0}^{\sigma_i - 1} z^{\theta_i k} = \prod_{i=1}^d \frac{z^{\sigma_i \theta_i} - 1}{z^{\theta_i} - 1}, \qquad (9)$$

where again θ_i *are the column vectors of* Θ *and* σ_i *are the diagonal elements of* Σ, $i = 1, \ldots, d$.

Theorem 4 ([13]) *The multiple subdivision scheme generated by* (a_j, M_j), *with Laurent polynomial* $a_j^* = \psi_{M_j}$, $j \in \mathbb{Z}_s$, *converges in* $L_1(\mathbb{R}^d)$.

Since the autoconvolution of a compactly supported L_1 function is continuous, the following conclusion can be done directly from Theorem 4.

Corollary 5 *The multiple subdivision scheme generated by* $\{(a_j, M_j)\}$ *with* $a_j^* = \frac{1}{\det M_j}\psi_{M_j}^2$, $j \in \mathbb{Z}_s$, *converges to a continuous limit function.*

To each M_j, $j \in \mathbb{Z}_s$, we can associate a subdivision scheme by means of (8) by choosing a collection of univariate interpolatory subdivision schemes with arities σ_{jk}, $k = 1, \ldots, d$, where the latter ones are the diagonal values of Σ_j in the Smith factorization of M_j. Considering the tensor product of

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such schemes we get a the multiple subdivision scheme generated by M_j, $j \in \mathbb{Z}_s$. As an example, if b_{jk} is the piecewise linear interpolatory scheme with arity σ_{jk}, it has mask

$$b_{jk} = \frac{1}{\sigma_{jk}} \left(0, 1, 2, \ldots, \sigma_{jk} - 1, \sigma_{jk}, \sigma_{jk} - 1, \ldots, 2, 1, 0 \right),$$

and its symbol is

$$b_{jk}^*(z) = \frac{z^{-(\sigma_{jk}-1)}}{\sigma_{jk}} (1 + z + z^2 + \ldots + z^{\sigma_{jk}-1})^2, \qquad k = 1, \ldots, d.$$

According to (7), the symbol of (6) takes the form

$$b_{\Sigma_j}^*(z) = \prod_{k=1}^{d} \frac{z_k^{-(\sigma_{jk}-1)}}{\sigma_{jk}} \left(\sum_{\ell=0}^{\sigma_{jk}-1} z_i^\ell \right)^2, \tag{10}$$

hence, the symbol of the scheme associated to M_j (8) is

$$a_j^*(z) = b_{\Sigma_j}^* \left(z^{\Theta_j} \right) = \prod_{k=1}^{d} \frac{z^{-(\sigma_{jk}-1)\theta_k}}{\sigma_{jk}} \left(\sum_{\ell=0}^{\sigma_{jk}-1} z^{\ell \theta_k} \right)^2. \tag{11}$$

In this way we obtain a convergent multiple subdivision scheme.

Proposition 6 *The scheme S_ϵ, $\epsilon \in \mathbb{Z}_s^n$ and $n \in \mathbb{N}$, as constructed above, is a convergent multiple subdivision scheme that converges to a continuous function.*

Proof. For each matrix M_j, $j \in \mathbb{Z}_s$, the corresponding canonical factor is

$$\psi_{M_j}(z) = \prod_{k=1}^{d} \frac{z^{\sigma_{jk}\theta_k} - 1}{z^{\sigma_{jk}} - 1} = \prod_{k=1}^{d} (1 + z^{\theta_k} + \ldots + z^{(\sigma_{jk}-1)\theta_k}) = \prod_{k=1}^{d} \left(\sum_{\ell=0}^{\sigma_{jk}-1} z^{\ell\theta_i} \right)$$

and since

$$\frac{1}{\det M_j} \psi_{M_j}^2(z) = \prod_{k=1}^{d} \frac{1}{\sigma_{jk}} \left(\sum_{\ell=0}^{\sigma_{jk}-1} z^{\ell\theta_k} \right)^2$$

is equal to a_j^* up to the shift factors $z^{-(\sigma_{jk}-1)\theta_k}$, Corollary 5 yields that S_ϵ, $\epsilon \in \mathbb{Z}_s^n$, $n \in \mathbb{N}$, is convergent to a continuous limit function. ∎

2.3 Filterbanks

Filters, more precisely *Linear and Time Invariant (LTI) filters* [4], are linear operators on discrete signals that commute with translations and are basic building blocks of signal processing. Commuting with translations yields that they are *stationary processes* and can be represented by convolutions,

$$Fc = f * c = \sum_{\alpha \in \mathbb{Z}^d} f(\cdot - \alpha) \, c(\alpha), \qquad c : \mathbb{Z}^d \to \mathbb{R}.$$

A *filterbank* consists of a finite set of analysis and synthesis filters, named f_k and g_k, $k \in \mathbb{Z}_m$, respectively, and operations to split a given signal into m subband components by means of filtering and downsampling, the so called *analysis* process, and to recombine these components into an output signal by upsampling and filtering, called the *synthesis* process (see [14]). The *downsampling* operator $\downarrow_M$, respect to an expanding matrix M, is defined as

$$\downarrow_M c = c(M \cdot), \qquad c \in \ell(\mathbb{Z}^d).$$

To invert downsampling one introduces the *upsampling* operator $\uparrow_M$

$$\uparrow_M c(\alpha) = \begin{cases} c(M^{-1}\alpha), & \alpha \in M\mathbb{Z}^d, \\ 0, & \alpha \notin M\mathbb{Z}^d, \end{cases} \qquad c \in \ell(\mathbb{Z}^d).$$

It is worthwhile to remark that only $\downarrow_M \uparrow_M$ is an identity while $\uparrow_M \downarrow_M$ is a lossy operator due to the decimation involved in the upsampling

$$\uparrow_M \downarrow_M c(\alpha) = \begin{cases} c(\alpha), & \alpha \in M\mathbb{Z}^d, \\ 0, & \alpha \notin M\mathbb{Z}^d. \end{cases}$$

The expanding matrix M defines the decimation which is performed in the downsampling after the convolution with the analysis filters. In the usual case of critically sampled filterbanks, the decimation rate $|\det M|$ coincides with the number m of filters in the filterbank in order to have the same amount of information after decomposition. The filters f_0 and g_0 are normally low-pass filters and all the other filters f_k, g_k with $k \neq 0$ are chosen as high-pass filters. Thus, the decomposition of a signal yields a coarse component, convolving with f_0 and downsampling, and $m - 1$ details components given by the convolution of the signal with f_k, $k \neq 0$, followed by downsampling. Hence, analysis computes the vector sequences

$$c \mapsto (c_k := \downarrow_M \ f_k * c : k \in \mathbb{Z}_m)$$

while synthesis combines them into

$$\sum_{k \in \mathbb{Z}_m} g_k * (\uparrow_M c_k).$$

In order to achieve perfect reconstruction, the usual standard assumption for a reasonable filterbank, the analysis and synthesis parts have to be the inverse of one another. This is equivalent to

$$c = \sum_{k \in \mathbb{Z}_m} g_k * (\uparrow_M \downarrow_M f_k * c), \qquad c : \mathbb{Z}^d \to \mathbb{R}.$$

It was shown, for example by [12], that starting with an interpolatory convergent subdivision scheme it allows us to define filterbanks that achieve the perfect reconstruction by the prediction-correction method. A convergent interpolatory subdivision scheme with mask a takes coarse data and predicts how to refine them, thus it can be used as a synthesis filters $g_0 = a$. In this case, the best way to define the low-pass analysis filter is to set $f_0 = \delta$, so the analysis process samples the initial data. Since, in general, prediction from decimation cannot reconstruct the signal, we introduce a correction which defines the high-pass analysis filters as $f_k^*(z) = z^{\xi_k} - a_{-\xi_k}^*(z^M)$, where $\xi_k \in E_M := M[0,1)^d \cap \mathbb{Z}^d$ are the coset representers, $k \in \mathbb{Z}_m \setminus \{0\}$ and $a_{-\xi_k}$ the corresponding submask. This analysis can be complemented by a canonical synthesis process yielding the filters

$$\begin{aligned}
f_0^*(z) &= 1, & f_k^*(z) &= z^{\xi_k} - a_{-\xi_k}(z^M), \\
g_0^*(z) &= a^*(z), & g_k^*(z) &= z^{-\xi_k},
\end{aligned} \tag{12}$$

with $\xi_k \in E_M$, $k \neq 0$. If we have not a single expanding matrix M but a dictionary $(M_j : j \in \mathbb{Z}_s)$ of expanding matrices, we consider s filterbanks, one for each M_j, $j \in \mathbb{Z}_s$, defined in the same way as (12), hence

$$\begin{aligned}
f_{j,0}^*(z) &= 1, & f_{j,k}^*(z) &= z^{\xi_k} - a_{j,-\xi_k}^*(z^{M_j}), \\
g_{j,0}^*(z) &= a_j^*(z), & g_{j,k}^*(z) &= z^{-\xi_k}.
\end{aligned} \tag{13}$$

with $\xi_k \in E_{M_j}$, $k \neq 0$. In each of the filterbanks the number of filters is equal to $|\det M_j| = m_j$, $j \in \mathbb{Z}_s$, that is, the filterbanks are *critically sampled*. At each step of decompositions we choose a matrix M_j, $j \in \mathbb{Z}_s$, and we apply the corresponding analysis filters $f_{j,k}$, $k \in \mathbb{Z}_{m_j}$.

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2.4   Shear scaling matrices

In the context of shearlets, the dilatation matrices are of the form

$$M = D_a S_W, \tag{14}$$

where D_a is a *parabolic* expanding anisotropic diagonal matrix

$$D_a = \begin{pmatrix} a^2 \mathrm{I}_p & 0 \\ 0 & a\,\mathrm{I}_{d-p} \end{pmatrix}, \qquad a \in \mathbb{N},\ a \geq 2,\ p < d, \tag{15}$$

and S_W a shear matrix

$$S_W = \begin{pmatrix} \mathrm{I}_p & W \\ 0 & \mathrm{I}_{d-p} \end{pmatrix}, \qquad W \in \mathbb{Z}^{p \times (d-p)}. \tag{16}$$

The shear matrix takes the role of a rotation which would be the desirable transformation which, unfortunately, neither offers a group structure in the continuous case [**?**, see]]furh2015 nor keeps the grid $\mathbb{Z}^d$ invariant in the discrete case. Shears, on the other hand, both work in the continuous and the discrete case.

The matrix D_a is called a *parabolic scaling matrix* because its eigenvalues are squares of each other. This implies

$$\det D_a = a^{d+p},$$

which quickly leads to huge values when d increases. Since the determinant of D_a also corresponds to the number of filters for decomposition level, a high value of this determinant is not at all desirable in applications where a reasonable depth of decomposition could only be reached starting with a very huge amount of input data.

Shear matrices (16) are unimodular matrices in $\mathbb{Z}^{d \times d}$, i.e. $\det S_W = \pm 1$. Their inverses are again shear matrices $S_W^{-1} = S_{-W}$. Moreover, they satisfy

$$S_W^j = S_{jW} \qquad \text{and} \qquad S_W\, S_{W'} = S_{W+W'}.$$

Due to these properties we have a *pseudo-commuting* property, namely

$$D_a S_W = S_W^a D_a$$

which is very useful when dealing with powers of M and different scaling matrices as in the discrete shearlet transform [8]. In fact

$$D_a S_W = \begin{pmatrix} a^2 \mathrm{I}_p & 0 \\ 0 & a\,\mathrm{I}_{d-p} \end{pmatrix} \begin{pmatrix} \mathrm{I}_p & W \\ 0 & \mathrm{I}_{d-p} \end{pmatrix} = \begin{pmatrix} a^2 \mathrm{I}_p & a^2 W \\ 0 & a\mathrm{I}_{d-p} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{I}_p & aW \\ 0 & \mathrm{I}_{d-p} \end{pmatrix} \begin{pmatrix} a^2 \mathrm{I}_p & 0 \\ 0 & a\mathrm{I}_{d-p} \end{pmatrix} = S_{aW} D_a = S_W^a D_a.$$

The minimum determinant of (14) is given by $a = 2$

$$D_2 = \begin{pmatrix} 4\mathrm{I}_p & 0 \\ 0 & 2\mathrm{I}_{d-p} \end{pmatrix}, \tag{17}$$

and $\det D_2 = 2^{p+d}$.

For the discrete shearlet transform it was proved by [8, 12, 13] that it is possible to generate a MMRA and achieve perfect reconstruction if the accumulated scaling matrices

$$M_\epsilon = M_{\epsilon_n} \dots M_{\epsilon_1}, \qquad \epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0, \dots, s-1\}^n =: \mathbb{Z}_s^n, \quad n = |\epsilon|,$$

satisfy the following properties: all M_ϵ are expanding matrices, the lattice $M_\epsilon^{-1} \mathbb{Z}^d$ is a refinement of $\mathbb{Z}^d$ and the family $(M_j^{-1} : j \in \mathbb{Z}_s)$ has the *slope resolution* property.

Definition 7 *A family $(M_j : j \in \mathbb{Z}_s)$ provides the* slope resolution *if there exists a reference line ℓ_0 through the origin such that for any line ℓ there exists a sequence $\epsilon \in \mathbb{Z}_s^n$ such that $M_\epsilon \ell_0 \to \ell$.*

[12] has verified these three properties for the parabolic matrix (17) in a general dimension d. There, the *pseudo commuting* property

$$D_2 S_W = S_W^2 D_2$$

for parabolic matrices turned out to be useful because it allows to write explicitly the formulas for the iterated matrices

$$M_\epsilon = \prod_{j=1}^n D_2 S_{W_j} = S_{W'} D_2^n, \quad W' = \sum_{j=1}^n 2^j W_j.$$

For example, recalling that shear matrices are unimodular, the identity

$$M_\epsilon^{-1} \mathbb{Z}^d = D_2^{-n} S_{-W'} \mathbb{Z}^d = D_2^{-n} \mathbb{Z}^d,$$

allows us to conclude that $M_\epsilon^{-1} \mathbb{Z}^d$ is a refinement of $\mathbb{Z}^d$ for any $n \in \mathbb{N}$ and $\epsilon \in \mathbb{Z}_s^n$.

2.5 Pseudo-commuting matrices

Motivated by a simple and very useful property of the scaling matrices in bivariate discrete shearlets (see [8]), the idea by [1] was to look for anisotropic

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diagonal expanding matrices $D \in \mathbb{Z}^{d \times d}$ and unimodular matrices A, not necessarily shear matrices, such that there exist $m, n \in \mathbb{N}$ for which the *commuting* relationship

$$DA^m = A^n D \qquad (18)$$

is true. If two matrices A, B satisfy the requirement (18), i.e., $BA^m = A^n B$, we say that they are *pseudo commuting*. Of course, any pair of commuting matrices has this property. It turns out (see [1]), that this depends strongly on the dimension d. For $d = 2$ we have the following result.

Proposition 8 ([1]) *Let D an anisotropic diagonal expanding matrix in $\mathbb{Z}^{2 \times 2}$ and A a non-diagonal unimodular matrix in $\mathbb{Z}^{2 \times 2}$. The identity*

$$DA^m = A^n D$$

is satisfied if and only if one of the following three cases holds:

1. $A^q = I$, and m, n such that $m = \ell_1 q$, $n = \ell_2 q$, for some ℓ_1, $\ell_2 \in \mathbb{Z}$;

2. $D = \begin{pmatrix} rk & 0 \\ 0 & k \end{pmatrix}$, m, n such that $r = \frac{n}{m}$ and $A = \pm \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$;

3. $D = \begin{pmatrix} rk & 0 \\ 0 & k \end{pmatrix}$, m, n such that $r = \frac{m}{n}$ and $A = \pm \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}^T$.

In particular, any 2×2 matrix that pseudo commutes with an anisotropic scaling matrix has to be a shear matrix. For $d = 3$, the situation is different.

Proposition 9 ([1]) *Let*

$$D = \begin{pmatrix} krs & 0 & 0 \\ 0 & kr & 0 \\ 0 & 0 & k \end{pmatrix}, \qquad r, s \in \mathbb{Q}_+, \ r, s \neq 1,$$

and

$$A = \left(\begin{array}{c|c} B & v \\ \hline 0 & \lambda \end{array} \right),$$

where $B \in \mathbb{Z}^{2 \times 2}$ is unimodular and $v \in \mathbb{Z}^2$. Then $DA = A^n D$ holds for some $n \in \mathbb{N}$ if and only A has one of the following forms

1. $v = 0$, $\lambda = \pm 1$, B a shear matrix and $s = n$ or $\lambda = -1$, $B^n = B$, $s = 1$ and n is odd;

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2. $v \neq 0$, $\lambda = 1$ and either $B = \pm \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$, $s = n$ and $r \in \{1, n\}$ or

$B = \pm \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}^T$, $s = \frac{1}{n}$ and $r \in \{1, \frac{1}{n}\}$.

3. $\lambda = -1$, n odd and either $B = \pm \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$, $s = n$, and $r \in \{1, \frac{1}{n}\}$ or

$B = \pm \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}^T$, $s = \frac{1}{n}$, and $r \in \{1, n\}$.

4. $\lambda = 1$, $B^n = B$, $Bv = v$, $s = 1$ and $r = n$.

A matrix B will be called *n-periodic* if $B^n = B$. Proposition 9 shows that, in contrast to $d = 2$, there exists non-shear matrices in $\mathbb{Z}^{3 \times 3}$ that pseudo–commute with anisotropic diagonal matrices. The same extends to higher dimensions.

Proposition 10 *For $d > 3$ consider*

$$D := \left(\begin{array}{c|c} krI_p & 0 \\ \hline 0 & kI_{d-p} \end{array} \right),$$

and let

$$A := \left(\begin{array}{c|c} B & W \\ \hline 0 & I_{d-p} \end{array} \right),$$

where $B \in \mathbb{Z}^{p \times p}$ is a unimodular n-periodic matrix, $W \in \mathbb{Z}^{p \times d-p}$, and $BW = W$. Then the relation $DA = A^n D$ holds if $r = n$.

3 Anisotropic diagonal scaling matrices and shears

From the previous section we show that in order to generate a directional multiple multiresolution analysis we have to consider a family of jointly expanding matrices that satisfies the slope resolution property. Unfortunately, parabolic matrices in form (17) have a huge determinant that grows rapidly with the dimension d, so we choose here to work with an anisotropic diagonal dilatation matrix with determinant as small as possible. This is obviously the case for

$$D = \begin{pmatrix} 3I_p & 0 \\ 0 & 2I_{d-p} \end{pmatrix}. \tag{19}$$

One of the most important application in the shearlet context is the detection of tangential hyperplanes in a point, because this permits us to catch

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faces of a given contour. For this reason we choose $p = d - 1$ and we fix on a dilatation matrix of the form

$$D = \begin{pmatrix} 3I_{d-1} & 0 \\ 0 & 2 \end{pmatrix}. \tag{20}$$

As for the bivariate shearlets (see [8]), this choice is arbitrary and creates an asymmetry between the variables where the last variable is treated differently from the first $d-1$ ones with respect to which the shearing is symmetric. We will also see that this prohibits the resolution of certain directions. To obtain a system that resolves all possible directions, d multiresolution systems have to be used where the *special* role is taken by all variables in turn.

In two dimensions the difference between the parabolic matrix D_2 and D from (20) is still small, $\det D_2 = 8$ and $\det D = 6$, but when d increase the difference increases rapidly:

$$\det D_2 = 2^{2d-1}, \quad \det D = 2^d \left(1 + \frac{1}{2}\right)^{d-1},$$
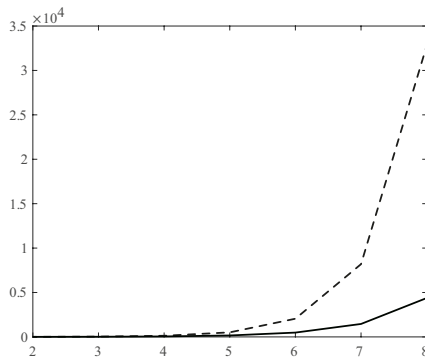
see also Figure 1.

Figure 1: Comparison between $\det D_2 = 2^{2d-1}$ (dash line) and $\det D = 2^d \left(1 + \frac{1}{2}\right)^{d-1}$ (solid line) for $d = 2, \ldots, 8$.

The special case $d = 2$ is discussed by [11], here we want to generalize to dimension d.

We consider *shear matrices* of the form

$$S_j = \begin{pmatrix} I_{d-1} & -e_j \\ 0 & 1 \end{pmatrix}, \qquad j \in \mathbb{Z}_d, \tag{21}$$

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where e_j, $j \in \mathbb{Z}_d$, are the vertices of the standard simplex in $\mathbb{R}^{d-1}$, i.e., $e_0 = 0$ and $e_1, \ldots, e_{d-1}$ defined by $(e_j)_k = \delta_{jk}$, $k = 1, \ldots, d-1$. By Proposition 10, these matrices interact with the dilation matrix D in the following way:

$$DS_j^2 = S_j^3 D, \qquad j \in \mathbb{Z}_d. \tag{22}$$

In analogy with the shearlet case in [8], we define the refinement matrices M_j, $j \in \mathbb{Z}_d$, as

$$M_j := DS_j^2 = \begin{pmatrix} 3\mathrm{I}_{d-1} & -6e_j \\ 0 & 2 \end{pmatrix} = S_j^3 D, \tag{23}$$

whose inverses can be easily computed as

$$M_j^{-1} = S_j^{-2} D^{-1} = \begin{pmatrix} \frac{1}{3}\mathrm{I}_{d-1} & e_j \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{24}$$

The multiple subdivision scheme is now governed by the matrices

$$M_\epsilon := M_{\epsilon_n} \ldots M_{\epsilon_1}, \qquad \epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{Z}_d^n, \qquad n \in \mathbb{N},$$

for which we can can give an explicit expression.

Lemma 11 *For $n \in \mathbb{N}$ and $\epsilon \in \mathbb{Z}_d^n$ we have*

$$M_\epsilon^{-1} = \begin{pmatrix} 3^{-n}\mathrm{I}_{d-1} & 2^{1-n}p_\epsilon\left(\frac{2}{3}\right) \\ 0 & 2^{-n} \end{pmatrix}, \tag{25}$$

with the $d-1$ multivariate polynomials

$$p_\epsilon(x) = \sum_{j=1}^n x^{j-1} e_{\epsilon_j}, \qquad x \in [0,1].$$

Proof. Proceeding by induction on n, we first note that in the case $n = 1$ we obtain exactly the matrices M_ϵ^{-1}, $\epsilon \in \mathbb{Z}_d$ in (24).
We now suppose that the claim is true for $n > 1$ and verify it for $n+1$ and $\epsilon = (\epsilon', \epsilon_{n+1}) \in \mathbb{Z}_d^{n+1}$. The matrix M_ϵ^{-1} becomes

$$M_\epsilon^{-1} = M_{\epsilon'}^{-1} M_{\epsilon_{n+1}}^{-1} = \begin{pmatrix} 3^{-n}\mathrm{I}_{d-1} & 2^{1-n}p_{\epsilon'}(2/3) \\ 0 & 2^{-n} \end{pmatrix} \begin{pmatrix} 3^{-1} & e_{\epsilon_{n+1}} \\ 0 & 2^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 3^{-(n+1)}\mathrm{I}_{d-1} & 2^{-n}p_{\epsilon'}(2/3) + 3^{-n}e_{\epsilon_{n+1}} \\ 0 & 2^{-(n+1)} \end{pmatrix},$$

and taking into account that

$$2^{-n} p_{\epsilon'} \left(\frac{2}{3} \right) + 3^{-n} e_{\epsilon_{n+1}} = 2^{-n} \left(p_{\epsilon'} \left(\frac{2}{3} \right) + \left(\frac{2}{3} \right)^n e_{\epsilon_{n+1}} \right) = 2^{-n} p_\epsilon \left(\frac{2}{3} \right)$$

we get (25). ∎

Inverting (25), we also obtain the general expression for M_ϵ, namely

$$M_\epsilon = \begin{pmatrix} 3^n \mathrm{I}_{d-1} & 3^{n+1} q_\epsilon(2/3) \\ 0 & 2^n \end{pmatrix}, \qquad q_\epsilon(x) := -\sum_{j=1}^{n} x^j e_{\epsilon_j}, \qquad \epsilon \in \mathbb{Z}_d^n, \quad n \in \mathbb{N}. \tag{26}$$

Now we verify some properties of M_ϵ that imply that these family of scaling matrices is useful for a directional multiresolution analysis based on an MMRA: they are jointly expanding, i.e., the grid $M_\epsilon^{-1} \mathbb{Z}^d$ tends to $\mathbb{R}^d$ uniformly and they satisfy a slope resolution property.

Proposition 12 $(M_j : j \in \mathbb{Z}_d)$ *are jointly expanding matrices.*

Proof. We have to show that the joint spectral radius of the scaling matrices

$$\rho(M_j : j \in \mathbb{Z}_d) = \lim_{n \to \infty} \max_{\epsilon \in \mathbb{Z}_d^n} \left\| M_\epsilon^{-1} \right\|^{1/n},$$

defined by [10], is strictly less than one. By the equivalence of norms on finite dimensional spaces, we can choose the underlying matrix norm arbitrarily. Using $\|\cdot\|_1$, we obtain for $n \in \mathbb{N}$ that

$$\max_{\epsilon \in \mathbb{Z}_d^n} \left\| M_\epsilon^{-1} \right\|_1^{1/n} = \max_{\epsilon \in \mathbb{Z}_d^n} \left\| \begin{pmatrix} 3^{-n} & 2^{1-n} p_\epsilon(2/3) \\ 0 & 2^{-n} \end{pmatrix} \right\|_1^{1/n}$$

$$= \max_{\epsilon \in \mathbb{Z}_d^n} \left(2^{1-n} \left\| p_\epsilon \left(\frac{2}{3} \right) \right\|_1 + 2^{-n} \right)^{1/n}$$

$$\leq \frac{1}{2} \left(2 \sum_{j=0}^{n-1} \left(\frac{2}{3} \right)^j + 1 \right)^{1/n} \leq \frac{1}{2} \left(2 \sum_{j=0}^{\infty} \left(\frac{2}{3} \right)^j + 1 \right)^{1/n}$$

$$= \frac{7^{1/n}}{2}$$

and therefore $\max_{\epsilon \in \mathbb{Z}_d^n} \left\| M_\epsilon^{-1} \right\|_1^{1/n} < 1$ for $n \geq 3$ independently of ϵ. ∎

For each $\epsilon \in \mathbb{Z}_d^n$, we can rewrite (26) as

$$M_\epsilon = \begin{pmatrix} 3^n \mathrm{I}_{d-1} & 3^{n+1} q_\epsilon(2/3) \\ 0 & 2^n \end{pmatrix} = D^n \begin{pmatrix} \mathrm{I}_{d-1} & 3 q_\epsilon(2/3) \\ 0 & 1 \end{pmatrix} =: D^n S_\epsilon \tag{27}$$

and (25) as

$$M_\epsilon^{-1} = \begin{pmatrix} 3^{-n}\mathrm{I}_{d-1} & 2^{1-n}p_\epsilon(2/3) \\ 0 & 2^{-n} \end{pmatrix} = D^{-n}\begin{pmatrix} \mathrm{I}_{d-1} & 3\sum_{j=1}^n \left(\frac{3}{2}\right)^{n-j} e_{\epsilon_j} \\ 0 & 1 \end{pmatrix}$$

$$=: D^{-n}\widehat{S}_\epsilon. \tag{28}$$

Taking inverses of (27) and (28) also yields

$$M_\epsilon = \widehat{S}_\epsilon^{-1}D^n, \qquad M_\epsilon^{-1} = S_\epsilon^{-1}D^{-n}. \tag{29}$$

These formulations of M_ϵ and M_ϵ^{-1} allow us to compute efficiently the grids $M_\epsilon\mathbb{Z}^d$ and $M_\epsilon^{-1}\mathbb{Z}^d$ as transformations of the D^n or D^{-n} refined grid. These transformations are still shears, however, in contrast to the shearlet case with the parabolic matrix D_2, they have no integer translations any more. Given a hyperplane $H = \{x \in \mathbb{R}^d : v^T x = 0\} \subset \mathbb{R}^d$ whose normal has the property $v_d \neq 0$, we can renormalize v such that $v_d = 1$. Writing $v = (t,1)^T$, we call $t \in \mathbb{R}^{d-1}$ the *slope* of the hyperplane H.

The next result shows that any such hyperplane can be obtained, up to arbitrary precision, from a single reference hyperplane by applying a proper matrix M_ϵ. This property is called *slope resolution* property and ensures that an anisotropic transform which captures singularities across the reference hyperplane can capture directional singularities across arbitrary hyperplanes in a MMRA, where the modified direction can be directly read off from the index ϵ. The slope of the reference hyperplane can even be chosen arbitrarily in a scaled version of the standard simplex

$$\Delta_{d-1} := \left\{ t \in \mathbb{R}^{d-1} : t_i \geq 0, \sum_{i=1}^{d-1} t_i \leq 1 \right\}.$$

Theorem 13 *For any reference hyperplane H with slope $t \in 6\Delta_{d-1}$, the matrix family $(M_j : j \in \mathbb{Z}_d)$ has the slope resolution property: for any hyperplane H' with slope $t' \in \mathbb{R}^{d-1}$ and any $\delta > 0$ there exists $\epsilon : \mathbb{N} \to \mathbb{Z}_d$ such that*

$$\left\| \begin{pmatrix} t' \\ 1 \end{pmatrix} - 2^{-n}M_\epsilon \begin{pmatrix} t \\ 1 \end{pmatrix} \right\| < \delta.$$

Proof. Let H' be any hyperplane with slope $t' \in \mathbb{R}^{d-1}$. A multiplication with M_j^{-1}, $j \in \mathbb{Z}_d$, changes the slope in the following way:

$$M_j^{-1}\begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\mathrm{I}_{d-1} & e_j \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{2}{3}t + 2e_j \\ 1 \end{pmatrix}, \qquad t \in \mathbb{R}^{d-1},$$
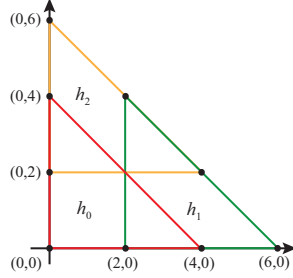
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Figure 2: In two dimension $(d-1=2)$, we have three different functions h_j, $j \in \{0,1,2\}$ and the union of $h_0(6\Delta_2)$ (red), $h_1(6\Delta_2)$ (green), $h_2(6\Delta_2)$ (orange) gives $6\Delta_2$.

due to which we define the affine contractions $h_j : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ by

$$h_j(t) := \frac{2}{3}t + 2e_j, \qquad j \in \mathbb{Z}_d, \tag{30}$$

which satisfy

$$h_j(6\Delta_{d-1}) = \frac{2}{3}6\Delta_{d-1} + 2e_j = 4\Delta_{d-1} + 2e_j \subset 6\Delta_{d-1} \tag{31}$$

and

$$\bigcup_{j\in\mathbb{Z}_d} h_j(6\Delta_{d-1}) = 6\Delta_{d-1}, \tag{32}$$

as visualized in Figure 2 for $d=3$. As pointed out by [5], the compact set $6\Delta_{d-1}$ is an invariant space with respect to the contractions h_j, $j \in \mathbb{Z}_d$ and for any compact subset $X \subset \mathbb{R}^{d-1}$ the generating property

$$6\Delta_{d-1} = \lim_{n\to\infty} \bigcup_{\epsilon\in\mathbb{Z}_d^n} h_\epsilon(X) \tag{33}$$

holds true in the Hausdorff metric. For general $\epsilon \in \mathbb{Z}_d^n$ we have

$$M_\epsilon^{-1}\begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} 3^{-n}\mathrm{I}_{d-1} & 2^{1-n}p_\epsilon\left(\frac{2}{3}\right) \\ 0 & 2^{-n} \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix} = 2^{-n}\begin{pmatrix} \left(\frac{2}{3}\right)^n t + 2p_\epsilon\left(\frac{2}{3}\right) \\ 1 \end{pmatrix},$$

which involves the affine contractions

$$h_\epsilon(t) = h_{\epsilon_n}\left(h_{(\epsilon_1,\dots,\epsilon_{n-1})}(t)\right) = \left(\frac{2}{3}\right)^n t + 2p_\epsilon\left(\frac{2}{3}\right), \qquad \epsilon \in \mathbb{Z}_d^n, \qquad n \in \mathbb{N}.$$

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This iterative definition and (31) yield that

$$h_\epsilon(6\Delta_{d-1}) \subset 6\Delta_{d-1}, \qquad \epsilon \in \mathbb{Z}_d^n, \quad n \in \mathbb{N}. \tag{34}$$

For given $t' \in \mathbb{R}^{d-1}$, and $0 < \delta' < \delta$, we first consider the compact set $R_{\delta'}(t') = \{u \in \mathbb{R}^{d-1} : \|t' - u\|_1 \leq \delta'\}$, for which the generating property (33) implies that there exist $n_1 \in \mathbb{N}$ and $\epsilon^1 \in \mathbb{Z}_d^{n_1}$ such that

$$I := h_{\epsilon^1}\left(R_{\delta'}(t')\right) \cap 6\Delta \neq \emptyset,$$

Again by (33), this time applied to the closure of I, there exist $n_2 \in \mathbb{N}$ and $\epsilon^2 \in \mathbb{Z}_d^{n_2}$ such that

$$t \in h_{\epsilon^2}(I) \subset h_{\epsilon^2}\left(h_{\epsilon^1}\left(R_{\delta'}(t')\right)\right) =: h_\epsilon\left(R_{\delta'}(t')\right), \qquad \epsilon = (\epsilon^2, \epsilon^1) \in \mathbb{Z}_d^n, \tag{35}$$

hence $h_\epsilon^{-1}(t) \in R_{\delta'}(t')$ or, equivalently,

$$\left\| \binom{t'}{1} - 2^{-n} M_\epsilon \binom{t}{1} \right\| \leq \delta' < \delta$$

as claimed. ∎

4   Some numerical examples

Finally, we illustrate how these matrices work as a MMRA, see Section 2.2, by providing some explicit examples and applying them to images. To that end, we focus on the two dimensional case, where we consider the family of matrices

$$M_0 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 3 & -6 \\ 0 & 2 \end{pmatrix}, \tag{36}$$

with $\det(M_j) = 6$, for $j \in \mathbb{Z}_2$. This means that we have to provide 6 analysis filters and 6 synthesis filters for each M_j, $j \in \mathbb{Z}_2$.
The Smith factorization form of the matrices is

$$M_0 = IDI, \qquad M_1 = \begin{pmatrix} 3 & -6 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = S^3 DI. \tag{37}$$

Following the construction presented in Section 2.2, we start with two univariate interpolatory subdivision schemes with scaling factors $\sigma_1 = 3$ and $\sigma_2 = 2$, respectively, like the ternary and dyadic piecewise linear interpolatory schemes with masks

$$b_1 = \left(\ldots, 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{2}{3}, \frac{1}{3}, 0, \ldots\right), \quad b_2 = \left(\ldots, 0, \frac{1}{2}, 1, \frac{1}{2}, 0, \ldots\right).$$

Using (11), the symbols of the subdivision schemes related to the matrices M_0 and M_1 are

$$a_0^*(z) = b_D^*(z) = \frac{z_1^{-2}z_2^{-1}}{6}(1 + z_1 + z_1^2)^2(1 + z_2)^2, \tag{38}$$

$$a_1^*(z) = b_D^*(z^{S^3}) = \frac{z_1 z_2^{-1}}{6}(1 + z_1 + z_1^2)^2(1 + z_1^{-3}z_2)^2. \tag{39}$$

Once we have the subdivision schemes for both the matrices we can compute the filters using the prediction-correction method and (13). Then, the analysis and synthesis filters for the diagonal matrix M_0 are

$$f_{0,0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{0,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{0,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f_{0,3} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{0,4} = \begin{pmatrix} 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{0,5} = \begin{pmatrix} 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$g_{0,0} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} & \mathbf{1} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}, \quad g_{0,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad g_{0,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$g_{0,3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad g_{0,4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad g_{0,5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

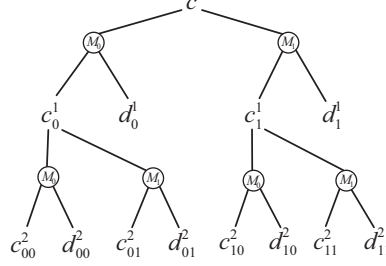where the boldface entry highlights the $(0,0)$ element. In the same way we

Figure 3: Multiple decomposition tree up to level 2.

obtain the analysis and synthesis filters for the matrix M_1,

$$f_{1,0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad f_{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathbf{2}}{\mathbf{3}} & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f_{1,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathbf{1}}{\mathbf{3}} & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad f_{1,3} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\mathbf{1}}{\mathbf{2}} \end{pmatrix},$$

$$f_{1,4} = \begin{pmatrix} -\frac{1}{3} & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\mathbf{1}}{\mathbf{3}} & 0 & 0 & -\frac{1}{6} \end{pmatrix},$$

$$f_{1,5} = \begin{pmatrix} -\frac{1}{6} & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\mathbf{1}}{\mathbf{6}} & 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

$$g_{1,0} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & \mathbf{1} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}, \qquad g_{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$g_{1,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad g_{1,3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_{1,4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad g_{1,5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In each step of the analysis process we can now choose either M_0 or M_1. After two decomposition levels we thus have 4 different decompositions, as depicted by the tree in Figure 3. If we apply two levels of decompositions on
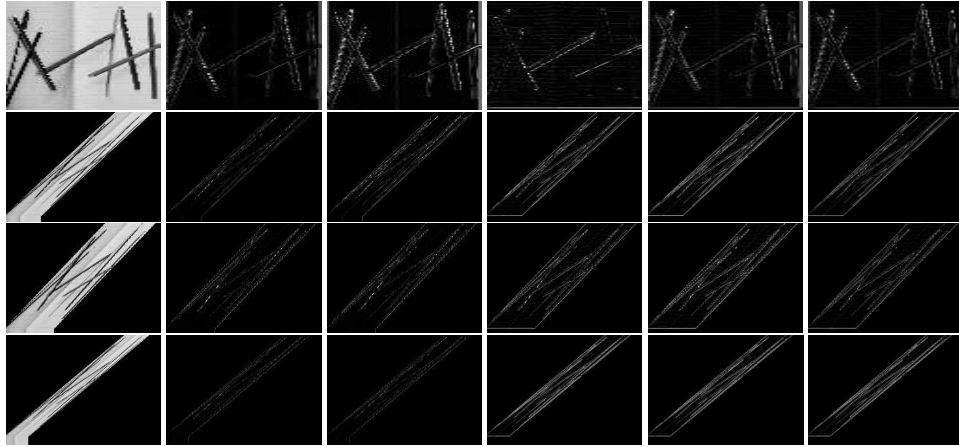
Figure 4: Two level decomposition of the image in Figure 6a. The six images in each row are the scaling coefficients c_ϵ^2 and details coefficients $d_{\epsilon,k}^2$ with $\epsilon \in \mathbb{Z}_2$ and $k = 1, \ldots, 5$. Each row represents a branch of the tree in Figure 3 from $\epsilon = (0,0)$ to $(1,1)$.

Figure 5: Two level decomposition of the image in Figure 6a using Daubechies wavelet of order 2. The 4 images are the scaling coefficients c^2 and details coefficients along the horizontal, vertical, diagonal direction, from left to right.

the test image in Figure 6a, then the resulting scaling and details coefficients are displayed in Figure 4. The effect of the matrix M_1, or better of the shear, on the decomposition coefficients, is to shift the dominant directions for edges in the image. The more applications of M_1 we do, the more the resulting image is "rotated" and stretched, as shown in the last row of Figure 4.

As for wavelets, each detail coefficient detects features along a certain direction. In the tensor product wavelet case these relevant directions are only the horizontal, the vertical and the diagonal one, see Figure 5. In our case the combination of the details directions ξ_k with the shearing of the images allows us to detect different directional features in different branches of the tree. One can analyze this behavior looking at Figures 6 and 7 where, after

two levels and following all the different branches separately, we reconstruct only the details while removing the scaling coefficients. We recall that the detail coefficients are higher along the directions considered, this clearly shows which directions are considered for each different branch.

Due to the tree structure, the analysis with shearlets is highly redundant. This could be a disadvantage from the point of view of memory consumption but keeping only the well approximated parts from each branch will eventually have to potential for very efficient compression. In fact once we have computed the full decomposition tree, we can reconstruct the original image either following a single branch or combining together the branches that emerge from the same knot. In the second case, every time we reconstruct one level we take some average between the reconstructed images that refer to the same knot in the tree. In Figures 6 and 7, we compare the reconstructed details following a specific branch and combining all the branches, see Figures 6b and 7b. The reconstructed details as mean of all the branches detects more features than the reconstruction along a single branch. Moreover, taking the mean of all the branches mitigates the artifacts that occur in the single branch. For example, the double application of M_1 shows some blur on the edges of the image, see the bottom right images in Figures 6 and 7. This effect is less visible if we reconstruct averaging all the branches of the tree, Figures 6b and 7b.

5 Conclusion

We proposed a shearlet-like decomposition family of matrices that allow to define a directional transform, based on shears and a nonparabolic scaling matrix. These matrices are jointly expanding, thus suitable for a multiple multiresolution process. In particular, we show how to define a convergent multiple subdivision scheme and the respective filterbanks using this family of matrices. In any dimension d, we have also proved that they satisfy the slope resolution property, crucial for having a tool capable to capture the directional features of a signal in arbitrary directions. Moreover, the proposed matrices satisfy the pseudo commuting property that allows a faster computation of the refined (coarse) lattice.

In contrast to classical shearlets, the matrices considered here have smaller determinant. This is a useful feature because the determinant gives the number of filters to be considered and so it is strictly related to the computational complexity of the implementation.

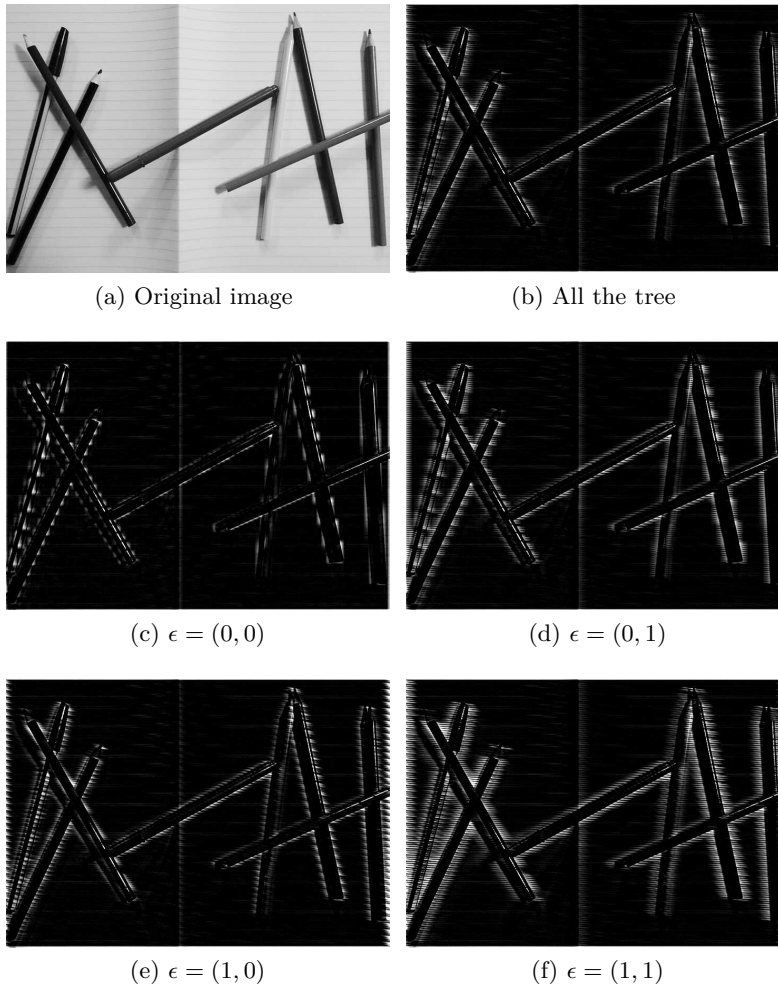We have also illustrated by means of a few examples how this family of ma-

(a) Original image

(b) All the tree

(c) $\epsilon = (0, 0)$

(d) $\epsilon = (0, 1)$

(e) $\epsilon = (1, 0)$

(f) $\epsilon = (1, 1)$

Figure 6: Reconstructed details following different branches of the tree (Fig. 3) after two levels of decomposition.

(a) Original image

(b) All the tree

(c) $\epsilon = (0, 0)$

(d) $\epsilon = (0, 1)$

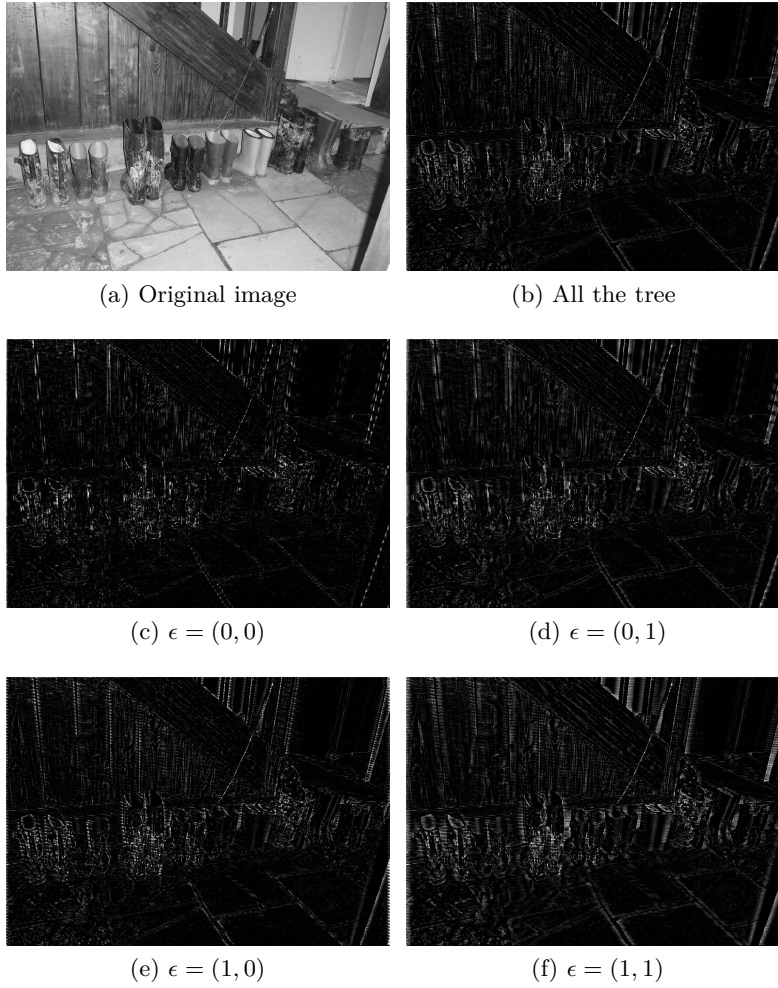(e) $\epsilon = (1, 0)$

(f) $\epsilon = (1, 1)$

Figure 7: Reconstructed details, after two levels of decompositions, following different branches of the tree or considering all the tree together.

trices acts on images and the potential of using it. In fact, as expected, the details of the images capture the discontinuities along different directions.

Acknowledgements

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