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FROM APPROXIMATING TO INTERPOLATORY NON-STATIONARY SUBDIVISION SCHEMES WITH THE SAME GENERATION PROPERTIES

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Abstract

In this paper we describe a general, computationally feasible strategy to deduce a family of interpolatory non-stationary subdivision schemes from a symmetric non-stationary, non-interpolatory one satisfying quite mild assumptions. To achieve this result we extend our previous work [C. Conti, L. Gemignani, L. Romani, Linear Algebra Appl. 431 (2009), no. 10, 1971–1987] to full generality by removing additional assumptions on the input symbols. For the so obtained interpolatory schemes we prove that they are capable of reproducing the same exponential polynomial space as the one generated by the original approximating scheme. Moreover, we specialize the computational methods for the case of symbols obtained by shifted non-stationary affine combinations of exponential B-splines, that are at the basis of most non-stationary subdivision schemes. In this case we find that the associated family of interpolatory symbols can be determined to satisfy a suitable set of generalized interpolating conditions at the set of the zeros (with reversed signs) of the input symbol. Finally, we discuss some computational examples by showing that the proposed approach can yield novel smooth non-stationary interpolatory subdivision schemes possessing very interesting reproduction properties.

Keywords: Subdivision schemes, Structured matrices, Polynomials.

1 Introduction

Binary interpolatory subdivision schemes are efficient iterative procedures for the generation of interpolatory curves: starting with the set of points to be interpolated, at each recursion step a new point is inserted in between any two given points so that the limit curve, whenever exists, not only interpolates the initial set of points but also all the points generated through the whole process. Taking into account that a curve is displayed on the screen by visualizing a discrete set of its points, from a computational viewpoint interpolatory subdivision schemes turn out to be more efficient than classical interpolating methods in several situations. In fact the limit points obtained within five or six subdivision iterations are in general enough for a good discrete representation of the limit shape. This is one of the reasons why interpolatory subdivision schemes are widely used in applications and often preferred to standard methods.

Two important areas where interpolatory subdivision schemes play a crucial role are Computer Aided Geometric Design (CAGD) and wavelets construction (see [10] and [18], respectively). In these fields a fundamental issue that recently emerged is concerned with the study of numerical algorithms for converting known approximating schemes into new interpolatory ones. Starting from the works [17] and [22], where the conversion is obtained for a specific approximating scheme by means of a *push-back* or a *tweak* operator, geometric approaches based on the idea that an interpolatory refinement can be interpreted as an averaging step on the control points followed by a further adjustment of some of them to fit the interpolation constraints were presented [15, 16]. Very recently a completely different technique relying upon the interplay between polynomial and structured matrix computations has been proposed in [5]. In that work for a given *symmetric Hurwitz* approximating symbol an associated family of interpolatory symbols is determined in such a way to satisfy an auxiliary polynomial equation. As it clearly appears, although the latter strategy turns out to be more general than the previous ones, it is limited to the context of stationary subdivision schemes. Being non-stationary subdivision schemes more powerful than stationary ones and very attractive in several applications such as in CAGD (because of their ability to reproduce conic sections, spirals or widely used trigonometric curves) it is of fundamental importance to provide a general and efficient method to convert a given non-stationary, non-interpolatory scheme into a family of interpolatory ones. To our knowledge, there exists only a new paper [1] addressing this problem, which presents a strategy that is restricted to the case of symmetric subdivision masks of odd width, namely symmetric subdivision symbols of even degree.

The goal of this paper is to elaborate on our recent work [5] to progress along different directions. In particular, (i) we extend the applicability of the proposed construction, (ii) we investigate the reproduction properties of the so-obtained interpolatory schemes and (iii) we design algorithms specifically suited for the case of approximating symbols generated from exponential B-splines, that are at the basis of most non-stationary subdivision schemes. More specifically, in this paper we prove that the strategy described in [5] can still be pursued under very relaxed conditions

on the approximating symbols we deal with, say $\{a^{(k)}(z), \ k \geq 0\}$. If, for a given fixed $k \geq 0$, $a^{(k)}(z)$, $a^{(k)}(-z)$ are relatively prime polynomials, then a double family of interpolatory symbols associated with $a^{(k)}(z)$ can be generated by solving two different Bezout-like polynomial equations. In the symmetric case where $a^{(k)}(z)$ is a symmetric polynomial, the double family reduces to one single family since the solutions of these two equations are suitably related. In the Hurwitz case where $a^{(k)}(z)$ is a Hurwitz polynomial, the distribution of the roots implies the primality condition. Whenever such a condition is satisfied for any $k \geq 0$ then the correspondence of $a^{(k)}(z)$ with any member of the associated double family allows one to define a family of interpolatory subdivision schemes derived from the given non-stationary approximating one. The computation of the interpolatory symbol amounts to solve the corresponding polynomial equation. If the approximating symbol is specified by spectral information, as it is generally the case of exponential B-splines, then it is shown that the equation can be efficiently solved by using the tool of (incomplete) partial fraction decomposition. This gives a representation of the associated interpolatory symbol in terms of a set of generalized interpolating conditions attained at the zeros (with reversed signs) of the approximating symbol. For the newly generated interpolating schemes we prove an important reproduction result: the exponential polynomial space reproduced by the interpolatory scheme is the same function space generated by the approximating one it is originated from. On the contrary, a general result concerning convergence and/or smoothness of a non-stationary interpolatory subdivision scheme induced by a non-stationary approximating one is not yet available. However, in many specific examples we have considered, the analysis can be performed by using ad-hoc techniques. In this way, by starting with approximating schemes suitably generated by five term affine combinations of exponential B-splines, we are able to find novel smooth non-stationary interpolatory subdivision schemes possessing very interesting reproduction properties.

The paper is organized as follows. In Section 2 the needed background on non-stationary subdivision schemes is given. In Subsection 3.1 we review and generalize the basic strategy proposed in [5] for the construction of an interpolatory subdivision mask from a given approximating one. Effective computational procedures for implementing this strategy are discussed in Subsection 3.2. These procedures are the key ingredients of our algorithm, named *Appint* and stated in Subsection 3.3, to move from a non-stationary approximating subdivision scheme to a family of non-stationary interpolatory ones. The reproduction properties of these schemes are studied in Section 4 whereas in Section 5 the application of the algorithm to several instances of non-stationary approximating subdivision schemes generating exponential polynomials is considered. Finally, conclusions and further work are drawn in Section 6.

2 Background

In this section we briefly recall some needed background on stationary and non-stationary subdivision schemes. For more material on subdivision schemes we refer the reader to the seminal work by Cavaretta, Dahmen and Micchelli [4], to the more recent survey by Dyn and Levin [10] and to the well-known book by Warren and Weimer [24].

Subdivision schemes are simple iterative algorithms to efficiently generate curves and surfaces. Any subdivision scheme is defined by an infinite sequence of coefficients collected in the so called *refinement masks* $\{\mathbf{a}^{(k)}, \ k \geq 0\}$. We assume that any mask $\mathbf{a}^{(k)} := \left(a_i^{(k)} \in \mathbb{R}, \ i \in \mathbb{Z} \right)$ is of real numbers and has finite support for all $k \geq 0$ *i.e.* $a_i^{(k)} = 0$ for $i \notin [-n(k), n(k)]$ for suitable $n(k) \geq 0$. The *k-level subdivision operator* associated with the k-level mask $\mathbf{a}^{(k)}$ is

$$S_{\mathbf{a}^{(k)}} \ : \ \ell(\mathbb{Z}) \to \ell(\mathbb{Z}) \ , \qquad (S_{\mathbf{a}^{(k)}} \ \mathbf{q})_i := \sum_{j \in \mathbb{Z}} a_{i-2j}^{(k)} \ q_j, \quad i \in \mathbb{Z} \ , \qquad (2.1)$$

where $\ell(\mathbb{Z})$ denotes the linear space of real sequences indexed by $\mathbb{Z}$ whose elements will be denoted by boldface letter, $\mathbf{q} := (q_i \in \mathbb{R}, i \in \mathbb{Z})$. The *subdivision scheme* consists of the subsequent application of $S_{\mathbf{a}^{(0)}}, \cdots, S_{\mathbf{a}^{(k)}}$ from a given starting sequence, say $\mathbf{q}$, generating the scalar sequences

$$\mathbf{q}^{(0)} := \mathbf{q} \ , \quad \mathbf{q}^{(k+1)} := S_{\mathbf{a}^{(k)}} \ \mathbf{q}^{(k)} \ \text{ for } \ k \geq 0. \qquad (2.2)$$

In case the masks $\{\mathbf{a}^{(k)}, \ k \geq 0\}$ are kept fixed over the iterations, that is $\mathbf{a}^{(k)} = \mathbf{a}$ for all $k \geq 0$, the subdivision scheme is said to be *stationary*, otherwise *non-stationary*. Attaching the data $q_i^{(k)}$ generated at the k-th step to the parameter values $t_i^{(k)}$ with

$$t_i^{(k)} < t_{i+1}^{(k)}, \quad \text{and} \quad t_{i+1}^{(k)} - t_i^{(k)} = 2^{-k}, \quad k \geq 0$$

(these are usually set as $t_i^{(k)} := \frac{i}{2^k}$) we see that the subdivision process generates denser and denser sequences of data so that a notion of convergence can be established by taking into account the piecewise linear function $Q^{(k)}$ that interpolates the data, namely

$$Q^{(k)}(t_i^{(k)}) = q_i^{(k)}, \qquad Q^{(k)}\big|_{[t_i^{(k)}, t_{i+1}^{(k)}]} \in \Pi_1, \qquad i \in \mathbb{Z}, \quad k \geq 0,$$

where Π_1 is the space of linear polynomials. If the sequence $\{Q^{(k)}, \ k \geq 0\}$ converges, then we denote its limit by

$$f_{\mathbf{q}} := \lim_{k \to \infty} Q^{(k)}$$

and say that $f_{\mathbf{q}}$ is the *limit function* of the subdivision scheme based on the rule (2.2) for the data $\mathbf{q}$. Several subdivision properties can be read off from the *symbols*

$$a^{(k)}(z) = \sum_{i \in \mathbb{Z}} a_i^{(k)} \ z^i, \quad k \geq 0, \qquad z \in \mathbb{C} \setminus \{0\}$$

associated to the masks $\{\mathbf{a}^{(k)}, \ k \geq 0\}$. Also, the corresponding *sub–symbols*

$$a^{(k)}_{\text{even}}(z) = \sum_{i \in \mathbb{Z}} a^{(k)}_{2i} z^i, \quad a^{(k)}_{\text{odd}}(z) = \sum_{i \in \mathbb{Z}} a^{(k)}_{2i+1} z^i, \quad z \in \mathbb{C} \setminus \{0\},$$

related to the symbols by the relation

$$a^{(k)}_{\text{even}}(z^2) + z \cdot a^{(k)}_{\text{odd}}(z^2) = a^{(k)}(z),$$

are useful tools for subdivision analysis. Note that since the masks are always supposed to be finitely supported, all symbols are Laurent polynomials. Nevertheless, for the analysis of subdivision properties of our concern we can always assume to work with polynomial symbols, at least after the application of a suitable shift at each iteration.

A celebrated class of stationary subdivision schemes is given by degree-n *polynomial B-spline* subdivision schemes, whose (unique) symbol is

$$B_n(z) = \frac{(1+z)^{n+1}}{2^n}, \quad k \geq 0. \tag{2.3}$$

The non-stationary counterpart of (2.3) is the symbol of the so-called *exponential B-splines*. They are piecewise functions whose pieces are exponential polynomials (the latter ones will be recalled in the next definition). These are defined in terms of a linear differential operator and turn out to be of great interest in geometric modeling for the design of important analytical shapes like conic sections, spirals and classical trigonometric curves.

Definition 1. *(Space of exponential polynomials) Let $T \in \mathbb{Z}_+$ and $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \cdots, \gamma_T)$ with $\gamma_T \neq 0$ a finite set of real or imaginary numbers and let D^n the n-th order differentiation operator. The space of exponential polynomials $V_{T,\boldsymbol{\gamma}}$ is the subspace*

$$V_{T,\boldsymbol{\gamma}} := \{f : \mathbb{R} \to \mathbb{C}, f \in C^T(\mathbb{R}) : \sum_{j=0}^{T} \gamma_j D^j f = 0\}. \tag{2.4}$$

A characterization of the space $V_{T,\boldsymbol{\gamma}}$ is provided by the following:

Lemma 1. *[3] Let $\gamma(z) = \sum_{j=0}^{T} \gamma_j z^j$ and denote by $\{\theta_\ell, \tau_\ell\}_{\ell=1,\cdots,N}$ the set of zeros with multiplicity of $\gamma(z)$ satisfying*

$$\gamma^{(r)}(\theta_\ell) = 0, \quad r = 0, \cdots, \tau_\ell - 1, \quad \ell = 1, \cdots, N.$$

It results

$$T = \sum_{\ell=1}^{N} \tau_\ell, \qquad V_{T,\boldsymbol{\gamma}} := Span\{x^r e^{\theta_\ell x}, \ r = 0, \cdots, \tau_\ell - 1, \ \ell = 1, \cdots, N\}.$$

As proved in [19] (see also [24]) exponential B-splines can be generated via a non-stationary subdivision scheme based on the symbols

$$B_n^{(k)}(z) = 2 \prod_{\ell=1}^{N} \left(\frac{e^{\frac{\theta_\ell}{2^{k+1}}} z + 1}{e^{\frac{\theta_\ell}{2^{k+1}}} + 1} \right)^{\tau_\ell}, \quad k \geq 0. \tag{2.5}$$

Its limit function belongs to the subclass of C^{T-2} degree-n L-splines [23] (with $n = T - 1$) whose pieces are exponentials of the space $V_{T,\boldsymbol{\gamma}}$. Notice that, when $\theta_1 = 0$ with $\tau_1 = n + 1$, then $B_n^{(k)}(z)$ in (2.5) does not depend on k being the symbol of a degree-n B-spline given in (2.3). An important aspect of subdivision schemes is their convergence capability to specific classes of functions. In particular, a subdivision scheme is said to possess the property of *generating exponential polynomials* if, for any initial data uniformly sampled from some exponential polynomial function, the scheme yields a function belonging to the same space in the limit. Even more, the subdivision scheme is *reproducing exponential polynomials* if, for any initial data uniformly sampled from some exponential polynomial function, the scheme yields the *same* function in the limit. To this purpose, we recall the following two important definitions (see, for example, [7] and [25]).

Definition 2 ($V_{T,\boldsymbol{\gamma}}$-Generation). *Let $\{a^{(k)}(z), \ k \geq 0\}$ be a set of subdivision symbols. The subdivision scheme associated with the set of symbols $\{a^{(k)}(z), \ k \geq 0\}$ is said to be $V_{T,\boldsymbol{\gamma}}$-generating if it is convergent and for $f \in V_{T,\boldsymbol{\gamma}}$ and for the initial sequence $\mathbf{f}^0 := \{f(t_i^0), \ i \in \mathbb{Z}\}$, it results*

$$\lim_{k \to \infty} S_{\mathbf{a}^{(k)}} \cdots S_{\mathbf{a}^{(0)}} \mathbf{f}^0 = \tilde{f}, \quad \tilde{f} \in V_{T,\boldsymbol{\gamma}}.$$

Definition 3 ($V_{T,\boldsymbol{\gamma}}$-Reproduction). *Let $\{a^{(k)}(z), \ k \geq 0\}$ be a set of subdivision symbols. The subdivision scheme associated with the symbols $\{a^k(z), \ k \geq 0\}$ is said to be $V_{T,\boldsymbol{\gamma}}$-reproducing if it is convergent and for $f \in V_{T,\boldsymbol{\gamma}}$ and for the initial sequence $\mathbf{f}^0 := \{f(t_i^0), \ i \in \mathbb{Z}\}$, it results*

$$\lim_{k \to \infty} S_{\mathbf{a}^{(k)}} \cdots S_{\mathbf{a}^{(0)}} \mathbf{f}^0 = f.$$

Since the space of exponential polynomials trivially includes standard polynomials, Definitions 2 and 3 include, as special cases, the notion of polynomial generation and polynomial reproduction, respectively. For a complete analysis of the latter concepts in the stationary situation –which are very much related to the approximation order of the subdivision scheme– the interested reader can see [11].

We conclude by recalling that a subdivision scheme is said to be interpolatory if the refinement masks $\{\mathbf{a}^{(k)}, \ k \geq 0\}$ satisfy

$$a_{2i}^{(k)} = \delta_{i,0}, \quad \text{or equivalently,} \quad a_{\text{even}}^{(k)}(z) = 1, \quad k \geq 0, \tag{2.6}$$

meaning that all points generated by the subdivision process at a given level k will be kept in the next level $k + 1$. We also mention that from (2.6) it follows that a mask $\mathbf{a}^{(k)}$ is interpolatory if and only if all its symbols $a^{(k)}(z)$ satisfy the algebraic condition

$$a^{(k)}(z) + a^{(k)}(-z) = 2, \quad \forall k \geq 0. \tag{2.7}$$

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3 From approximating to interpolatory subdivision schemes

In this section we introduce the key ingredients of our proposed algorithm termed *Appint* to generate a family of non-stationary interpolatory subdivision schemes starting from an initial non-stationary approximating one. At the core of this algorithm there is a procedure which, for a given fixed non-interpolatory subdivision symbol $a^{(k)}(z)$, $k \geq 0$, effectively constructs a corresponding interpolatory symbol denoted by $m^{(k)}(z)$. The procedure is applied step-by-step for $k = 0, 1, \ldots$. For the sake of notational simplicity we can therefore omit the superscript k by denoting $a^{(k)}(z) = a(z)$ and $m^{(k)}(z) = m(z)$. The construction stems from a theoretical result presented in [5, Theorem 2] which describes the conditions being satisfied for the associated interpolatory symbol $m(z)$. In Subsection 3.1 this result is reviewed and generalized to some extent by removing unnecessary restrictions on the input symbol $a(z)$. In the case where $a(z)$ is of the form (2.5) and it is known in factorized form by means of the set of zeros $\{\theta_\ell, \tau_\ell\}_{\ell=1,\cdots,N}$, then an efficient method for computing a suitable representation of $m(z)$ is described in Subsection 3.2. Finally, by putting all these ingredients together, *Appint* is formally stated in Subsection 3.3.

3.1 From approximating to interpolatory subdivision symbols

In the matrix environment the linear operator $S_{\mathbf{a}}$ defined in (2.1) and associated with the symbol $a(z) = \sum_{i \in \mathbb{Z}} a_i z^i$, $z \in \mathbb{C} \setminus \{0\}$ is represented by a bi-infinite Toeplitz-like matrix $S_{\mathbf{a}} = (a_{i-2j})$, $i, j \in \mathbb{Z}$. Since $a(z)$ is a Laurent polynomial, say $a(z) = \sum_{j=-\kappa}^{\kappa} a_j z^j$, $\max\{|a_{-\kappa}|, |a_\kappa|\} > 0$, it follows that $S_{\mathbf{a}}$ is banded with bandwidth $\lceil \frac{\kappa}{2} \rceil$ at most. Let $p(z) = \sum_{j=-h}^{h} p_j z^j$, $\max\{|p_{-h}|, |p_h|\} > 0$, be another Laurent polynomial and denote by $\mathcal{P}$ the bi-infinite Toeplitz matrix associated with $p(z)$, namely, $\mathcal{P} = (p_{i-j})$. Observe that $\mathcal{P}$ is again banded with bandwidth h. For the product operator

$$\mathcal{S} := \mathcal{P} \cdot S_{\mathbf{a}} = (s_{i,j}), \quad i, j \in \mathbb{Z},$$

we have

$$s_{i,j} = \sum_{r=i-h}^{i+h} p_{i-r} a_{r-2j} = \sum_{\ell=-h}^{h} p_\ell a_{i-2j-\ell} = s_{i+2,j+1}, \quad i, j \in \mathbb{Z}.$$

This means that the product operator $\mathcal{S}$ is a bi-infinite Toeplitz-like matrix of the same form as the subdivision operator $S_{\mathbf{a}}$ with entries $s_{i,j} = s_{i-2j}$, $i, j \in \mathbb{Z}$. By setting

$$q(z) = a(z) \cdot p(z) = \sum_{j=-h-\kappa}^{h+\kappa} q_j z^j, \quad (q_j = 0 \text{ if } |j| > h + \kappa),$$

we find that

$$q_j = \sum_{i=-h}^{h} p_i a_{j-i}, \quad -(h+\kappa) \leq j \leq h + \kappa,$$

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and, therefore,

$$q_{i-2j} = s_{i,j} = s_{i-2j}, \quad i, j \in \mathbb{Z}.$$

There follows that the product operator $\mathcal{S}$ can be seen as the subdivision operator associated with the Laurent polynomial $q(z)$, i.e.,

$$\mathcal{S} = S_{\mathbf{q}}, \quad q(z) = a(z) \cdot p(z),$$

where $a(z)$ is the symbol of $S_{\mathbf{a}}$ and $p(z)$ can be suitably chosen in such a way to satisfy the interpolation condition. By expressing $q(z)$ in terms of its sub–symbols

$$q(z) = q_{\text{even}}(z^2) + z \cdot q_{\text{odd}}(z^2) \qquad z \in \mathbb{C} \setminus \{0\},$$

we find that

$$q(z) + q(-z) = 2 \cdot q_{\text{even}}(z^2).$$

Then by imposing the interpolation condition (2.6), i.e., $q_{\text{even}}(z) = 1$, we arrive at the relation

$$a(z) \cdot p(z) + a(-z) \cdot p(-z) = 2 \tag{3.8}$$

which is a generalized Bezout equation providing necessary and sufficient conditions for a Laurent polynomial $p(z)$ to convert the subdivision operator associated with $a(z)$ into the interpolating subdivision operator generated by $q(z) = a(z) \cdot p(z)$.

Suitable coefficient-wise representations of $p(z)$ are introduced to investigate conditions under which the (generalized) Bezout equation is solvable as well as to develop effective computational methods for its solution. Observe that if $p(z)$ is of the form

$$p(z) = p_\kappa z^\kappa + p_{\kappa+1} z^{\kappa+1} + \ldots + p_{\kappa+m} z^{\kappa+m}, \tag{3.9}$$

with $m = 2\kappa - 1$, and, moreover, it satisfies

$$a(z) \cdot p(z) + (-1)^j a(-z) \cdot p(-z) = 2z^j, \quad 0 \le j \le 2m + 1, \tag{3.10}$$

then $z^{-j} p(z)$ solves (3.8). Computing polynomial solutions of (3.10) of the form (3.9) reduces in a matrix setting to solving a structured linear system whose coefficient matrix is Sylvester-like. Let $\boldsymbol{a}_0 = [a_{-\kappa}, \ldots, a_0, \ldots, a_\kappa]^T \in \mathbb{R}^{2\kappa+1}$ denote the coefficient vector of the Laurent polynomial $a(z)$. The associated extended coefficient vector $\widehat{\boldsymbol{a}}_+ \in \mathbb{R}^{2\kappa+m+1}$ is defined by $\widehat{\boldsymbol{a}}_+^T = [\boldsymbol{a}_0^T, 0, \ldots, 0]$. Similarly let us introduce the extended coefficient vector $\widehat{\boldsymbol{a}}_- \in \mathbb{R}^{2\kappa+m+1}$ associated with the polynomial $a(-z)$. Moreover let $Z = (z_{i,j}) \in \mathbb{R}^{2(m+1)\times 2(m+1)}$ be the down-shift matrix given by $z_{i,j} = \delta_{i-1,j}$, where $\delta_{i,j}$ is the Kronecker delta symbol. Set $\mathcal{R}_+ \in \mathbb{R}^{2(m+1)\times(m+1)}$ the striped Toeplitz matrix

$$\mathcal{R}_+ = [\widehat{\boldsymbol{a}}_+ | Z\widehat{\boldsymbol{a}}_+ | \ldots | Z^m \widehat{\boldsymbol{a}}_+],$$

and, similarly, define

$$\mathcal{R}_- = [\widehat{\boldsymbol{a}}_- | Z\widehat{\boldsymbol{a}}_- | \ldots | Z^m \widehat{\boldsymbol{a}}_-].$$

The coefficient matrix of the linear system (3.10) is $\mathcal{R}^+ = [\mathcal{R}_+ | \mathcal{R}_-] \in \mathbb{R}^{2(m+1)\times 2(m+1)}$ or $\mathcal{R}^- = [\mathcal{R}_+ | -\mathcal{R}_-] \in \mathbb{R}^{2(m+1)\times 2(m+1)}$ depending on the parity of j. It is well known

that $\mathcal{R}^+$ and $\mathcal{R}^-$ are resultant matrices and, therefore, they are invertible if and only if $a(z)$ and $a(-z)$ are relatively prime polynomials.

Due to the special structure of the polynomial pair $(a(z), a(-z))$ it is shown that both linear systems can be reduced to smaller systems of half the size. Let $P_{m+1} \in \mathbb{R}^{2(m+1)\times 2(m+1)}$, $P_{m+1} = (\delta_{i,\sigma(j)})$ be the permutation matrix associated with the "perfect shuffle" permutation given by

$$\sigma \ : \ \{1,\ldots,2m+2\} \to \{1,\ldots,2m+2\}, \quad \sigma(j) = \begin{cases} (j+1)/2 + m + 1, & \text{if } j \text{ is odd}; \\[2mm] j/2, & \text{if } j \text{ is even}. \end{cases}$$

Furthermore, let $G_{m+1} \in \mathbb{R}^{2k\times 2k}$ be the matrix defined by

$$G_{m+1} = \left(\begin{array}{c|c} I_{m+1} & -D_{m+1} \\ \hline D_{m+1} & I_{m+1} \end{array} \right),$$

where $D_{m+1} = \text{diag}[-1, (-1)^2, \ldots, (-1)^{k-1}, (-1)^{m+1}]$. There follows that

$$P_{m+1} \cdot \mathcal{R}^- \cdot G_{m+1}^{-1} = \mathcal{H}^- \oplus \mathcal{H}, \tag{3.11}$$

where $\mathcal{H} \in \mathbb{R}^{(m+1)\times(m+1)}$ is a certain matrix and

$$\mathcal{H}^- = \begin{bmatrix} a_{-\kappa+1} & a_{-\kappa} & 0 & \ldots & \ldots & \ldots \\ a_{-\kappa+3} & a_{-\kappa+2} & a_{-\kappa+1} & a_{-\kappa} & 0 & \ldots \\ a_{-\kappa+5} & a_{-\kappa+4} & a_{-\kappa+3} & \ldots & & \ldots \\ \vdots & \vdots & \vdots & \vdots & & \ldots \\ \vdots & \vdots & \vdots & \vdots & & \ldots \\ a_{-\kappa+2m+1} & a_{-\kappa+2m} & a_{-\kappa+2m-1} & \ldots & & \ldots \end{bmatrix}.$$

Similarly we find that

$$P_{m+1} \cdot \mathcal{R}^+ \cdot G_{m+1}^{-1} = \widehat{\mathcal{H}} \,\widehat{\oplus}\, \mathcal{H}^+, \tag{3.12}$$

where $\widehat{\mathcal{H}} \in \mathbb{R}^{(m+1)\times(m+1)}$ is a certain matrix, $\widehat{\oplus}$ denotes the direct sum with respect to the main anti-diagonal, and, moreover,

$$\mathcal{H}^+ = \begin{bmatrix} a_{-\kappa} & 0 & \ldots & \ldots & \ldots \\ a_{-\kappa+2} & a_{-\kappa+1} & a_{-\kappa} & 0 & \ldots \\ a_{-\kappa+4} & a_{-\kappa+3} & a_{-\kappa+2} & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ldots \\ a_{-\kappa+2m} & a_{-\kappa+2m-1} & a_{-\kappa+2m-2} & \ldots & \ldots \end{bmatrix}.$$

In this way we arrive at the following generalization of [5, Theorem 2].

Proposition 1. *Let $\widehat{a}(z) = z^\kappa a(z)$ be a degree-n polynomial, $n = m + 1$, relatively prime with $\widehat{a}(-z)$. Then $\mathcal{H}^-$ and $\mathcal{H}^+$ are invertible and, moreover, the polynomial*

$p_i^\star(z)$, $\star \in \{+, -\}$, with coefficients given by the entries of the i-th column of $(\mathcal{H}^\star)^{-1}$, $1 \le i \le n$, is the unique polynomial of degree less than n such that

$$\widehat{a}(z)p_i^\star(z) \;\star\; \widehat{a}(-z)p_i^\star(-z) = 2\,z^{2i-\ell^\star}, \qquad 1 \le i \le n, \qquad \star \in \{+, -\} \qquad (3.13)$$

where

$$\ell^\star = \left\{ \begin{array}{l} 2 \text{ if } \star = +; \\ 1, \text{elsewhere.} \end{array} \right.$$

As an immediate consequence of Proposition 1 we obtain the following.

Proposition 2. *Given a degree-n polynomial $\widehat{a}(z)$ relatively prime with $\widehat{a}(-z)$ and such that $\widehat{a}(1) = 2$, $\widehat{a}(-1) = 0$, then the Laurent polynomials*

$$m_i^\star(z) := \frac{\widehat{a}(z)p_i^\star(z)}{z^{2i-1}}, \quad 1 \le i \le n, \qquad (3.14)$$

where $p_i^\star(z)$ solves (3.13), $\star \in \{+, -\}$, are the associated interpolatory symbols and satisfy

$$m_i^\star(1) = 2, \quad m_i^\star(-1) = 0, \quad 1 \le i \le n.$$

Remark 1. *It is worth noting that Proposition 1 defines a double family of associated interpolatory symbols depending on the sign of $\star$. In the symmetric case where $\widehat{a}(z)$ is a symmetric polynomial, that is, $a_j = a_{-j}$, $0 \le j \le \kappa$, the number of associated symbols halves since all the matrices $\mathcal{H}$, $\widehat{\mathcal{H}}$, $\mathcal{H}^+$ and $\mathcal{H}^-$ are suitably related and, in particular, $\mathcal{H}^+$ can be obtained from $\mathcal{H}^-$ by reversion of rows and columns.*

These results provide a practical way to construct a family of finitely supported interpolatory masks from a given approximating one consisting in computing the matrix $(\mathcal{H}^\star)^{-1}$ and reading its entries. This approach seems to be especially tailored for symmetric Hurwitz subdivision symbols which result into computations with totally positive (TP) Hurwitz matrices. The procedures described in [20] can be adjusted for the efficient and stable computations of the coefficients of the interpolatory masks generated in the B-spline case and "shifted" affine combinations of them (see [5, Section 4]). However, in the case of exponential B-splines and their affine combinations the approximating symbol is generally known by assigning the spectrum of the symbol, that is, its zeros with their multiplicity. It is therefore interesting to design a completely different machinery for solving (3.13) using the information on the roots.

3.2 A root-based polynomial equation solver

Let us suppose that

$$\widehat{a}(z) = \widehat{a}_0 + \widehat{a}_1 z + \ldots + \widehat{a}_n z^n = \widehat{a}_n \prod_{j=0}^{m} (z - z_j)^{k_j},$$

with $z_i \neq z_j$ if $i \neq j$ and $k_0 + \ldots + k_m = n$. Then it is shown that the unique solution $p_i(z)$ of (3.13) can be obtained by imposing certain interpolation conditions at the zeros of $\widehat{a}(z)$ and $\widehat{a}(-z)$.

Let us start by recalling the concept of *Hermite-Lagrange interpolation polynomial* of a given differentiable function $f(z)$ on the set of nodes $\eta_0, \ldots, \eta_\ell$ with multiplicities $h_0, \ldots, h_\ell$, $h_0 + \ldots + h_\ell = r + 1$, respectively. Suppose that the function $f(z)$ possesses derivatives $f^{(j)}(\eta_i)$, $0 \leq j \leq h_i - 1$, $0 \leq i \leq \ell$. Then there exists a unique polynomial $H_f(z)$ of degree at most r satisfying the interpolation conditions

$$H_f^{(j)}(\eta_i) = f^{(j)}(\eta_i), \quad 0 \leq j \leq h_i - 1, \ 0 \leq i \leq \ell.$$

This polynomial is generally referred to as the Hermite-Lagrange interpolation polynomial of $f(z)$ on the prescribed set of nodes. By setting $\omega(z) := (z - \eta_0)^{h_0} \cdots (z - \eta_\ell)^{h_\ell}$ we find the Lagrange-type representation

$$H_f(z) = \sum_{i=0}^{\ell} \sum_{j=0}^{h_i-1} \sum_{h=0}^{h_i-j-1} f^{(j)}(\eta_i) \frac{1}{h!j!} \left(\frac{(z-\eta_i)^{h_i}}{\omega(z)} \right)^{(h)}_{z=\eta_i} \frac{\omega(z)}{(z-\eta_i)^{h_i-j-h}}$$

and, equivalently, the partial-fraction representation

$$H_f(z) = \omega(z) \sum_{i=0}^{\ell} \sum_{s=1}^{h_i} \frac{1}{(z-\eta_i)^s} \left(\sum_{j=0}^{h_i-s} \mathcal{S}(h_i - j - s, j, i) \right) = \omega(z) \sum_{i=0}^{\ell} \sum_{s=1}^{h_i} \frac{c_{i,h_i-s}}{(z-\eta_i)^s},$$

where

$$\mathcal{S}(h, j, i) = f^{(j)}(\eta_i) \frac{1}{h!j!} \left(\frac{1}{\omega_i(z)} \right)^{(h)}_{z=\eta_i}, \quad w_i(z) = \frac{\omega(z)}{(z-\eta_i)^{k_i}},$$

and, moreover, by Leibniz's rule

$$c_{i,j} = \sum_{\ell=0}^{j} \mathcal{S}(j - \ell, \ell, i) = \frac{1}{j!} \left(\frac{H_f(z)}{\omega_i(z)} \right)^{(j)}_{z=\eta_i}.$$

Let $\ell = 2m + 1$ and $\eta_0 = z_0, \ldots, \eta_{(\ell-1)/2} = z_m$, $\eta_{(\ell+1)/2} = -z_0, \ldots, \eta_\ell = -z_m$ with multiplicities $h_0 = h_{(\ell+1)/2} = k_0, \ldots, h_\ell = h_{(\ell-1)/2} = k_m$. Observe that

$$r + 1 = h_0 + \ldots + h_\ell = 2k_0 + \ldots + 2k_m = 2n$$

and

$$\omega(z) = \prod_{i=0}^{m} (z - z_i)^{k_i} \prod_{i=0}^{m} (z + z_i)^{k_i} = \widehat{a}_n^{-2}(-1)^n \widehat{a}(z)\widehat{a}(-z).$$

By replacing the right hand side $f(z) = 2z^{2t-\ell^\star}$ of (3.13), where t is fixed and $1 \leq t \leq n$, with its Hermite-Lagrange form we find that

$$(-1)^n \widehat{a}_n^2 \left(\frac{p_t^\star(z)}{\widehat{a}(-z)} \ \star \ \frac{p_t^\star(-z)}{\widehat{a}(z)} \right) = \sum_{i=0}^{\ell} \sum_{s=1}^{h_i} \frac{c_{i,h_i-s}}{(z-\eta_i)^s}, \qquad \star \in \{+, -\}.$$

Since $\widehat{a}(z)$ and $\widehat{a}(-z)$ are relatively prime we can separate the partial fraction decompositions of the two rational functions on the left-hand side. This gives the following

Proposition 3. *Let $\widehat{a}(z) = \widehat{a}_n \prod_{j=0}^{m}(z - z_j)^{k_j}$ be a polynomial of degree n, where $z_i \neq z_j$ if $i \neq j$, $k_0 + \ldots + k_m = n$ and $\widehat{a}(z)$ and $\widehat{a}(-z)$ are relatively prime. Then, the unique polynomial solution $p_t^\star(z)$, $1 \leq t \leq n$, $\star \in \{+, -\}$, of (3.13) satisfies*

$$p_t(z) = (-1)^{\ell^\star} \widehat{a}_n^{-1} \prod_{j=0}^{m}(z + z_j)^{k_j} \sum_{i=0}^{m} \sum_{s=1}^{k_i} \frac{(-1)^s c_{i,k_i-s}}{(z + z_i)^s},$$

where

$$c_{i,j} = \frac{1}{j!} \left(\frac{2z^{2t-\ell^\star}}{\omega_i(z)} \right)^{(j)}_{z=z_i}, \quad 0 \leq j \leq k_i - 1, \ 0 \leq i \leq m,$$

$\omega(z)$ is the monic polynomial associated with $\widehat{a}(z)\widehat{a}(-z)$ and $w_i(z) = \dfrac{\omega(z)}{(z - z_i)^{k_i}}$.

Example 1. *To illustrate the computational meaning of the previous result let us consider the interpolatory symbols associated with the cubic exponential B-spline with k-level symbol*

$$B_3^{(k)}(z) = \frac{1}{2}(z + 1)^2 \frac{z^2 + 2v^{(k)}z + 1}{2(v^{(k)} + 1)},$$

where the parameter $v^{(k)} \in (0, +\infty)$ is defined through the expression

$$v^{(k)} = \frac{1}{2} \left(e^{\theta/2^{k+1}} + e^{-\theta/2^{k+1}} \right)$$

with $\theta \in \{\theta_\ell, \ \ell = 1, ..., N\}$, as in Lemma 1. As shown in [2] this means that $B_3^{(k)}(z)$ corresponds to (2.5) with $N = 3$, $\theta_1 = 0$, $\theta_2 = t$, $\theta_3 = -t$ and $\tau_1 = 2$, $\tau_2 = \tau_3 = 1$, and, moreover, once assigned the starting value $v^{(-1)} \in (-1, +\infty)$, the parameter $v^{(k)}$ can be recursively updated at each successive iteration through the formula

$$v^{(k)} = \sqrt{\frac{v^{(k-1)} + 1}{2}}, \ k \geq 0. \tag{3.15}$$

For any fixed $k \geq 0$, the symmetric interpolatory scheme of smallest support associated with $B_3^{(k)}(z)$ is obtained from the choice $i = 2$ and $\star = -$ in (3.13). By using Proposition 3 we find that the corresponding solution $p_2^k(z)$ is given by

$$p_2^{(k)}(z) = \frac{(1-z)^2}{2v^k(v^k - 1)} - \frac{z^2 - 2v^k z + 1}{2(v^k - 1)} = \frac{1}{2v^k} \left(-z^2 + 2(v^k + 1)z - 1 \right),$$

which from Proposition 2 defines the interpolatory symbol

$$m_{3,2}^{(k)}(z) := \frac{B_3^{(k)}(z) p_2^{(k)}(z)}{z^3}, \quad k \geq 0.$$

The partial fraction decomposition is not a flexible computational tool and several difficulties arise in order to find efficient updating procedures for computing the solutions of (3.13) associated with slightly modified symbols (as usually it is the case in non-stationary subdivision schemes depending on a parameter, see Section

5). In this respect the tool of *incomplete partial fraction decomposition* [14] is much more suited. The general strategy proceeds as follows. From the partial fraction decomposition we get two polynomials $h(z)$ and $k(z)$ of degree less than n such that $\frac{1}{\widehat{a}(z)\widehat{a}(-z)} = \frac{h(z)}{\widehat{a}(z)} + \frac{k(z)}{\widehat{a}(-z)}$. Since $\widehat{a}(z)$ is given in factored form we can determine $k(z)$ as the Hermite-Lagrange polynomial interpolating the function $g(z) = 1/\widehat{a}(z)$ on the zeros of $\widehat{a}(-z)$. Then the polynomial $p_t^\star(z)$ which solves (3.13) can be obtained by means of the polynomial division between $2z^{2t-\ell^\star}k(z)$ and $\widehat{a}(-z)$. Again this operation reduces to computing the Hermite-Lagrange polynomial interpolating $2z^{2t-\ell^\star} \cdot k(z)$ on the zeros of $\widehat{a}(-z)$. In the case where the initial symbol $\widehat{a}(z)$ is modified by a linear or a quadratic factor, both the two steps in the above procedure can be modified accordingly. For instance the polynomial $k(z)$ can be specified in the form $k(z) = k_1(z) + \widehat{a}(-z)\psi(z)$, where $k_1(z)$ is the Hermite-Lagrange polynomial interpolating the function $g(z) = 1/\widehat{a}(z)$ on the zeros of $\widehat{a}(-z)$ and $\psi(z)$ is a linear factor whose coefficients are determined so that $k(z)$ satisfies the modified equation. This approach has been implemented and used for computing the interpolatory symbols associated with certain affine combinations of exponential B-splines. Some computational results are shown in Section 5.

3.3 The Appint algorithm for the non-stationary case

So far we have introduced a quite general strategy for deriving a family of interpolatory symbols from a given approximating symbol based on the solution of equation (3.13). In the non-stationary setting, we compute a family of non-stationary interpolatory subdivision schemes associated with a non-stationary approximating one via the solution of (3.13) at each recursion step. Therefore, the procedure we consider turns out to be as follows: assuming $\{\widehat{a}^{(k)}(z), \ k \geq 0\}$ are the degree-$n(k)$ symbols of an approximating non-stationary scheme with $\widehat{a}^{(k)}(z)$ and $\widehat{a}^{(k)}(-z)$ relatively prime for all $k \geq 0$, we construct the non-stationary interpolatory subdivision scheme based on the symbols $\{m_{i(k)}^{(k)}(z), \ k \geq 0\}$ where, for each k, $m_{i(k)}^{(k)}(z)$, $1 \leq i(k) \leq n(k)$, is one of the interpolatory symbols satisfying (3.13). Here and hereafter for the sake of simplicity we omit the superscript $\star \in \{+, -\}$ since we assume that the sequence $(i(k), \star)$, $k \geq 0$, is given in input and, therefore, $m_{i(k)}^{(k)}(z)$ denotes the unique solution of (3.13) for the given pair $(i(k), \star)$. Surely, the performance of the non-stationary subdivision scheme will depend on the selection of the sequence $(i(k), \star)$, $k \geq 0$. The computational kernel consists of finding the solution of (3.13) for the input symbol $\widehat{a}^{(k)}(z)$ and the fixed pair $(i(k), \star)$. This task can be accomplished by the inversion of the corresponding matrices $\mathcal{H}^\star$ or, alternatively, by means of the procedure described in the previous section based on computing the incomplete partial fraction decomposition. The auxiliary routine *Solve* takes in input a suitable representation of $\widehat{a}^{(k)}(z)$ together with the pair $(i(k), \star)$ and returns as output the corresponding solution $p_{i(k)}^{(k)}(z)$ of (3.13). For clarity we describe the overall procedure in algorithmic form.

<div style="border:1px solid black; padding:10px;">

Appint Algorithm
Input: $\{\widehat{a}^{(k)}(z), \; k \geq 0\}$, degree-$n(k)$ symbols;
$\qquad \{(i(k), \star), \; k \geq 0\}$, with $1 \leq i(k) \leq n(k)$
For $k = 0, 1, \ldots$
$\qquad$ Check whether $\widehat{a}^{(k)}(z)$ is relatively prime with $\widehat{a}^{(k)}(-z)$
$\qquad$ Set $p_{i(k)}^{(k)}(z) := \mathbf{Solve}[\widehat{a}^{(k)}(z), \; (i(k), \star)]$
$\qquad$ Construct the interpolatory symbol $m_{i(k)}^{(k)}(z) := \frac{\widehat{a}^{(k)}(z) p_{i(k)}^{(k)}(z)}{z^{2\,i(k)-1}}$
Output: $\{m_{i(k)}^{(k)}(z), \; k \geq 0\}$

</div>

Some theoretical properties of the computed sequence $\{m_{i(k)}^{(k)}(z), \; k \geq 0\}$ are discussed in Section 4 whereas computational examples are reported in Section 5.

4 Properties of non-stationary interpolatory subdivision schemes derived from their approximating counterparts

For the family of non-stationary interpolatory subdivision schemes generated by symbols $\{m_i^{(k)}(z), k \geq 0\}$, $1 \leq i \leq n(k)$, we can prove an important reproduction result: the exponential polynomial space reproduced by the interpolatory scheme is the same function space generated by the approximating scheme it is originated from. To prove it, we first need a preliminary result given in [12]. Within the rest of this section $V_{T,\boldsymbol{\gamma}}$ is the space given in Definition 1 and $z_\ell^{(k)} := e^{-\frac{\theta_\ell}{2^{k+1}}}$, $\quad \ell = 1, \cdots, N, \quad k \geq 0$.

Proposition 4. *Let $\{m^{(k)}(z), \; k \geq 0\}$ be a sequence of interpolatory symbols. The subdivision scheme associated with such a sequence reproduces $V_{T,\boldsymbol{\gamma}}$ if and only if for each $k \geq 0$*

$$m^{(k)}(z_\ell^{(k)}) = 2, \qquad m^{(k)}(-z_\ell^{(k)}) = 0, \quad \ell = 1, \cdots, N$$

$$\frac{d^r}{dz^r} m^{(k)}(\pm z_\ell^{(k)}) = 0, \quad r = 1, \cdots, \tau_\ell - 1, \quad \ell = 1, \cdots, N.$$

(4.16)

We are now in a position to state the reproduction result.

Proposition 5. *Let $\{\widehat{a}^{(k)}(z), \; k \geq 0\}$ be a sequence of symbols with $\widehat{a}^{(k)}(z)$ relatively prime with $\widehat{a}^{(k)}(-z)$ for all $k \geq 0$. If the non-stationary approximating subdivision scheme based on the symbols $\{\widehat{a}^{(k)}(z), \; k \geq 0\}$ generates the space $V_{T,\boldsymbol{\gamma}}$, then for all $1 \leq i \leq n(k)$ the non-stationary interpolatory subdivision scheme based on the symbols*

$$m_i^{(k)}(z) = \frac{\widehat{a}^k(z) p_i^k(z)}{z^{2i-1}}, \quad k \geq 0,$$

whenever convergent, reproduces the same space $V_{T,\boldsymbol{\gamma}}$.

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Proof: Due to [25, Theorem 1] the symbols $\widehat{a}^{(k)}(z)$ satisfy

$$\widehat{a}^{(k)}(-z_\ell^{(k)}) = 0, \quad \frac{d^r}{dz^r}\,\widehat{a}^{(k)}(-z_\ell^{(k)}) = 0, \quad r = 1,\cdots,\tau_\ell-1,\ \ell = 1,\cdots,N.$$

By the Leibnitz's differentiation rule, we easily get an analogous relation to be satisfied by all $m_i^{(k)}(z)$ (for any $1 \le i \le n(k)$) that is

$$m_i^{(k)}(-z_\ell^{(k)}) = 0, \quad \frac{d^r}{dz^r}\,m_i^{(k)}(-z_\ell^{(k)}) = 0, \quad r = 1,\cdots,\tau_\ell-1,\ \ell = 1,\cdots,N.$$

It remains to consider the behavior of $m_i^{(k)}(z)$ and its derivatives at the points $z_\ell^{(k)}$. Now, since for each k

$$m_i^{(k)}(z) + m_i^{(k)}(-z) = 2, \quad 1 \le i \le n(k),$$

it follows that

$$m_i^{(k)}(z_\ell^{(k)}) = 2$$

as well as

$$\frac{d^r}{dz^r}m_i^{(k)}(z_\ell^{(k)}) = (-1)^{r+1}\frac{d^r}{dz^r}m_i^{(k)}(-z_\ell^{(k)}) = 0,\ r = 1,\cdots,\tau_\ell-1,\ \ell = 1,\cdots,N.$$

The use of Proposition 4 concludes the proof. $\qquad\square$

Remark 2. *We notice that, if an interpolatory subdivision scheme is $V_{T,\boldsymbol{\gamma}}$-generating, then due to the interpolatory nature (that is due to the fulfillment of equation (2.7)), it is also $V_{T,\boldsymbol{\gamma}}$-reproducing.*

Remark 3. *Unfortunately, contrary to the result in Proposition 5, a general result concerning convergence and/or smoothness of a non-stationary interpolatory subdivision scheme induced by a non-stationary approximating one is not available. However, in all specific examples discussed in Section 5 and many others we tested, convergence and smoothness analysis of the induced non-stationary interpolatory subdivision schemes is provided. From the examples we see that the smoothness order of the interpolatory scheme is the half of that of the approximating one it is originated from. This observation gives us a hint for a theoretical result to be investigated in future researches.*

5 Interpolatory exponential reproducing non-stationary subdivision schemes

Aim of this section is to show the application of our strategy to a family of approximating schemes depending on free parameters. This leads to a parameter-dependent family of corresponding interpolatory schemes that can be used to design interesting new non-stationary interpolatory schemes. In particular, we show that by means of a five term affine combination of exponential B-splines, we can generate novel

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smooth non-stationary interpolatory subdivision schemes possessing very interesting reproduction properties.

Let us consider the interpolatory scheme based on the symbols $m_{3,2}^{(k)}(z)$ introduced in Example 1. The C^2 approximating scheme with symbols $\{B_3^{(k)}(z), \ k \geq 0\}$ was originally introduced in [19] where the authors also showed its capability generation of the function space $V_{4,\boldsymbol{\gamma}} = \{1, x, e^{tx}, e^{-tx}\}$ (see also [24]). According to the results in Section 4, the associated interpolatory scheme turns out to be the C^1 4-point interpolatory scheme reproducing the function space $V_{4,\boldsymbol{\gamma}} = \{1, x, e^{tx}, e^{-tx}\}$ (see also [21]). The reproduction properties of this scheme can be improved by considering the family of approximating subdivision schemes given by a 5-term affine combination of $B_3^{(k)}(z)$ of the form

$$
\begin{aligned}
\widehat{a}^{(k)}(z) &= B_3^{(k)}(z)\left(\alpha^{(k)} + \beta^{(k)}z + (1 - 2\alpha^{(k)} - 2\beta^{(k)})z^2 + \beta^{(k)}z^3 + \alpha^{(k)}z^4\right) \\
&= B_3^{(k)}(z)\left(\alpha^{(k)} + \frac{\beta^{(k)} + \sqrt{(4\alpha^{(k)} + \beta^{(k)})^2 - 4\alpha^{(k)}}}{2}z + \alpha^{(k)}z^2\right)\left(1 + \frac{2(1 - 2\beta^{(k)} - 4\alpha^{(k)})}{\beta^{(k)} + \sqrt{(4\alpha^{(k)} + \beta^{(k)})^2 - 4\alpha^{(k)}}}z + z^2\right)
\end{aligned}
$$

where $\alpha^{(k)}, \beta^{(k)} \in \mathbb{R}$ are free parameters. By imposing the primality conditions for $\widehat{a}^{(k)}(z), \widehat{a}^{(k)}(-z)$ it turns out that (3.13) can be solved whenever $\alpha^{(k)} \neq 0$ and $\beta^{(k)} \notin \{0, \frac{1}{2} - 2\alpha^{(k)}, \frac{4(v^{(k)})^2\alpha^{(k)} - 4\alpha^{(k)} + 1}{2(1 - v^{(k)})}\}$. In the case $\alpha^{(k)} = 0$ the equation can be degree-reduced in such a way that a polynomial solution can still be found.

By applying the procedure described in Subsection 3.2 we have computed the polynomial $p^{(k)}(z)$ corresponding with the pair $(i(k), \star) = (4, -)$, $k \geq 0$, and set

$$
m^{(k)}(z) = \widehat{a}^{(k)}(z)p^{(k)}(z)z^{-7}.
$$

By accurately choosing the free parameters $\alpha^{(k)}$ and $\beta^{(k)}$, we can obtain an interpolatory scheme $m^{(k)}(z)$ that improves the properties of the interpolatory scheme $m_{3,2}^{(k)}(z)$ associated with the combined symbol $B_3^{(k)}(z)$. Improvements can concern with its reproduction capabilities and/or its smoothness order. In particular:

1. When $\alpha^{(k)} = 0$ and $\beta^{(k)} = \frac{1}{4}$, $\widehat{a}^{(k)}(z) = \frac{(z+1)^4(z^2 + 2v^{(k)}z + 1)}{16(v^{(k)} + 1)}$, namely it is the C^4 exponential B-spline that generates $V_{6,\boldsymbol{\gamma}} = \{1, x, x^2, x^3, e^{tx}, e^{-tx}\}$. The symbol $m^{(k)}(z)$ is the C^2 interpolatory 6-point scheme that reproduces the same space (as previously shown in [21]).

2. When $\alpha^{(k)} = 0$ and $\beta^{(k)} = \frac{1}{4(v^{(k)})^2}$, then

$$
\widehat{a}^{(k)}(z) = \frac{(z+1)^2(z^2 + 2v^{(k)}z + 1)(z^2 + 2(2(v^{(k)})^2 - 1)z + 1)}{16(v^{(k)})^2(v^{(k)} + 1)},
$$

namely it is the C^4 exponential B-spline that generates $V_{6,\boldsymbol{\gamma}} = \{1, x, e^{tx}, e^{-tx}, e^{2tx}, e^{-2tx}\}$, while $m^{(k)}(z)$ is the C^2 interpolatory 6-point scheme that reproduces the same space (see, again, [21]).

3. When $\alpha^{(k)} = 0$ and $\beta^{(k)} = \frac{1}{2(1 + v^{(k)})}$, then

$$
\widehat{a}^{(k)}(z) = \frac{(z+1)^2(z^2 + 2v^{(k)}z + 1)^2}{8(v^{(k)} + 1)^2},
$$

namely it is the C^4 exponential B-spline that generates $V_{6,\boldsymbol{\gamma}} = \{1, x, e^{tx}, e^{-tx},$ $xe^{tx}, xe^{-tx}\}$ and $m^{(k)}(z)$ is the C^2 interpolatory 6-point scheme that reproduces the same space [21].

4. When $\alpha^{(k)} = \frac{1}{8(v^{(k)})^2(v^{(k)}+1)(2v^{(k)}-1)^2}$ and $\beta^{(k)} = \frac{4(v^{(k)})^2-2v^{(k)}-1}{4(v^{(k)})^2(2v^{(k)}-1)^2}$, then

$$\widehat{a}^{(k)}(z) = \frac{(z+1)^2(z^2+2v^{(k)}z+1)(z^2+2(4(v^{(k)})^3-3v^{(k)})z+1)(z^2+2(2(v^{(k)})^2-1)z+1)}{32(v^{(k)})^2(v^{(k)}+1)^2(2v^{(k)}-1)^2},$$

   i.e. it is the C^6 exponential B-spline generating $V_{8,\boldsymbol{\gamma}} = \{1, x, e^{tx}, e^{-tx}, e^{2tx}, e^{-2tx},$ $e^{3tx}, e^{-3tx}\}$, while $m^{(k)}(z)$ defines the C^3 interpolatory 8-point scheme that reproduces the same space (see [6]).

5. When $\alpha^{(k)} = \frac{1}{8(v^{(k)})^2(v^{(k)}+1)}$ and $\beta^{(k)} = \frac{2v^{(k)}-1}{4(v^{(k)})^2}$, we deal with the C^6 exponential B-spline

$$\widehat{a}^{(k)}(z) = \frac{(z+1)^2(z^2+2v^{(k)}z+1)^2(z^2+2(2(v^{(k)})^2-1)z+1)}{32(v^{(k)})^2(v^{(k)}+1)^2}$$

   generating the function space $V_{8,\boldsymbol{\gamma}} = \{1, x, e^{tx}, e^{-tx}, e^{2tx}, e^{-2tx}, xe^{tx}, xe^{-tx}\}$. The symbols $m^{(k)}(z)$ define a C^3 interpolatory 8-point scheme that reproduces the same space (see Proposition 5). The smoothness of the subdivision scheme $\{m^{(k)}(z), \ k \geq 0\}$ can be obtained through asymptotical equivalence [9] with the C^3 Dubuc-Deslauriers 8-point interpolatory scheme [8, 13].

The last non-stationary interpolatory subdivision scheme corresponds to a new proposal never presented in the literature. Other interesting proposals can be obtained by assigning different suitable values to the free parameters $\alpha^{(k)}$ and $\beta^{(k)}$. In all these kinds of interpolatory schemes, by making the parameter $v^{(-1)}$ local, namely by assuming a different parameter $v_i^{(-1)}$ in correspondence of each edge $\overline{q_i\,q_{i+1}}$ of the starting polyline, we can combine the two important issues of local shape control and special functions reproduction. This means that, in the same limit curve, we can include an alternation of exponential polynomial pieces in those regions where the starting samples belong to one of these curves and smooth limit segments with local tension otherwise. Also, due to the recurrence relation (3.15), the shape parameter $v^{(k)}$ turns out to be independent of the parametric values $\mathbf{t}^{(k)}$, thus reducing computational costs of the algorithm. In addition to the general reasons discussed in the introduction, these properties contribute to make these interpolatory subdivision schemes more convenient with respect to the corresponding classical interpolatory methods.

6 Conclusions and future work

A novel approach has been presented for the computation of a family of interpolatory non-stationary subdivision schemes from a non-stationary, non-interpolatory

one. The approach reduces the updating problem either to the inversion of certain structured matrices (which can be of Hurwitz type or Sylvester resultant matrices) or to the solution of certain Bezout-like polynomial equations. If the approximating symbols are defined in terms of spectral information it is shown that the partial fraction decomposition provides an effective tool for solving these equations by yielding a representation of the associated interpolatory symbols in terms of generalized interpolating conditions. The newly constructed interpolatory schemes are capable of reproducing the same exponential polynomial space as the one generated by the original approximating scheme. Although a general result concerning the relationship between convergence and/or smoothness orders of the approximating and interpolatory schemes is not yet available, ad hoc techniques can be used by showing that in many cases the proposed approach leads to novel smooth non-stationary interpolatory subdivision schemes possessing very interesting reproduction properties. The analysis of more general convergence properties of the subdivision schemes generated by our techniques is an ongoing research.

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