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From the Classical Gini Index of Income Inequality to a New Zenga-Type Relative Measure of Risk: A Modeller's Perspective

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Abstract: The underlying idea behind the construction of indices of economic inequality is based on measuring deviations of various portions of low incomes from certain references or benchmarks, which could be point measures like the population mean or median, or curves like the hypotenuse of the right triangle into which every Lorenz curve falls. In this paper, we argue that, by appropriately choosing population-based references (called societal references) and distributions of personal positions (called gambles, which are random), we can meaningfully unify classical and contemporary indices of economic inequality, and various measures of risk. To illustrate the herein proposed approach, we put forward and explore a risk measure that takes into account the relativity of large risks with respect to small ones.

Keywords: economic inequality; reference measure; personal gamble; inequality index; risk measure; relativity

JEL Classification: D63; D81; C46

1. Introduction

The Gini mean difference and its normalized version, known as the Gini index, have aided decision makers since their introduction by Corrado Gini more than a hundred years ago (Gini 1912, 1914, 1921); see also (Giorgi 1990, 1993, 1921); Ceriani and Verme 2012; and references therein). In particular, the Gini index has been widely used by economists and sociologists to measure economic inequality. Measures inspired by the index have been employed to assess the equality of opportunity (e.g., Weymark 2003; Kovacevic 2010; Roemer 2013) and estimate income mobility (e.g., Shorrocks 1978). Policymakers have used the Gini index in quantitative development policy analysis (e.g., Sadoulet and de Janvry 1995) and in particular for assessing the impact of carbon tax on income distribution (e.g., Oladosu and Rose 2007). The index has been employed for analysing inequality in the use of natural resources (e.g., Thompson 1976) and for developing informed policies for sustainable consumption and social justice (e.g., Druckman and Jackson 2008). Various extensions and generalizations of the index have been used to evaluate social welfare programs (e.g., Duclos 2000; Kenworthy and Pontusson 2005; Korpi and Palme 1998; Ostry et al. 2014) and to improve the knowledge of tax-base and tax-rate effects, as well as of temporal repercussions of distinct patterns of taxation and public finance on the society (e.g., Pfähler 1990; Slemrod 1992; Yitzhaki 1994; Van De Ven et al. 2001). Furthermore, Denneberg (1990) has advocated the use of the Gini mean difference as a safety loading for insurance premiums, with recent developments in the area by Furman and Zitikis (2017), and Furman et al. (2017).

Naturally, a multitude of interpretations, mathematical expressions, and generalizations of the index have manifested in the literature. As noted by Ceriani and Verme (2012), Corrado Gini himself proposed no less than thirteen formulations of his original index. Yitzhaki (1998, 2003), and Yitzhaki and Schechtman (2013) have discussed a great variety of interpretations of the Gini index. Many monographs and handbooks have been written on measuring economic inequality, where the Gini index and its various extensions and generalizations have played prominent roles: Amiel and Cowell (1999), Atkinson and Bourguignon (2000, 2015), Atkinson and Piketty (2007), Banerjee and Duflo (2011), Champernowne and Cowell (1998), Cowell (2011), Kakwani (1980a), Lambert (2001), Nygård and Sandström (1981), Ostry et al. (2014), Piketty (2014), Sen (1997), Silber (1999), Yitzhaki and Schechtman (2013), to name a few.

Given the diversity, one naturally wonders if there is one underlying thread that unifies all these indices. The population Lorenz function, as well as its various distances from the hypotenuse of the right triangle into which every Lorenz function falls, have traditionally provided such a thread. However, recent developments in the area of measuring economic inequality (e.g., Palma 2006; Zenga 2007; Greselin 2014; Gastwirth 2014; Kośny and Yalonetzky 2015) have highlighted the need for departure from the population mean, which is inherent in the definition of the Lorenz function as the benchmark, or reference point, for measuring economic inequality. The newly developed indices have deviated from the aforementioned unifying thread and thus initiated a fresh rethinking of the problem of measuring inequality.

Bennett and Zitikis (2015) ventured in this direction by suggesting a way to bridge the Harsanyi (1953) and Rawls (1971) conceptual frameworks via a spectrum of random societal positions. In this paper, we make a further step by developing a mathematically rigorous approach for unifying and interpreting numerous classical and contemporary indices of economic inequality, as well as those of risk. Briefly, the approach we have developed is based on appropriately chosen

1. societal references such as the population mean, median, or some population distribution-tail based measures, and
2. distributions of random personal positions, or gambles, that determine person's position on a certain population-based function.

Certainly, the literature is permeated by discussions related to points 1 and 2. Relativity issues have been explored in virtually every work, empirical and theoretical, due to the simple reason that they are a fact of life (e.g., Amiel and Cowell 1997, 1999). Naturally, fundamental measures of inequality, such as the Lorenz function, are also relative quantities, e.g., with respect to the population mean income. For discussions of various choices of reference measures and inherent relativity issues, we refer to, e.g., Sen (1983, 1998); Amiel and Cowell (1997, 1999); Zoli (1999, 2012); Duclos (2000); and references therein. To illustrate the point, which will become pivotal in our following deliberations, we recall a remark by Claudio Zoli, who wrote:

> In particular, Amiel and Cowell (1997, 1999) find evidence that "the appropriate inequality equivalence concept depends on the income levels at which inequality comparisons are made." Moreover, they show that, as income increases, the equivalence concept moves from the relative attitude to the absolute one, a pattern consistent with our intuition (Zoli 2012, p. 4).

This remark leads us towards the use of what we call relative-value functions, which, as we shall see later in this paper, offer a flexible way for coupling fundamental measures of economic inequality, or risk, with appropriate reference points, such as the mean (e.g., Equation (7) below). This is very much in the spirit of Definition 3 by Cowell (2003). We shall come back to the latter work in the second half of Section 4.

Finally, we note that the construction of distributions that govern personal random positions on population-based functions have been explored within the dual or rank-dependent utility theory (Quiggin 1982, 1993; Schmeidler 1986, 1989; Yaari 1987), other non-expected utility theories (e.g., Puppe 1991; Machina 1987, 2008; and references therein), distortion risk measures (Wang 1995, 1998), and weighted insurance premium calculation principles (Furman and Zitikis 2008, 2009).

The rest of the paper is organized as follows. In Section 2, we revisit the classical Gini index and, in particular, express it in two ways—absolute and relative—within the framework of expected utility theory using appropriately chosen gambles and societal functions (i.e., Lorenz and Bonferroni). In Section 3, we step aside from the Lorenz and Bonferroni functions and, crucially for this paper, suggest using a (financial) average value at risk as the underlying societal function on which various personal gambles are played; however, the reference measure remains the mean income μ_F. In Section 4, we depart from the latter reference and introduce a general index that accommodates any population-based reference measure. In Sections 5 and 6, we show how the Donaldson-Weymark-Kakwani index and the Wang (or distortion) risk measure, as well as their generalizations, fall into the expected utility framework with collective mean-income references and appropriately chosen personal gambles. In Section 7, we argue for the need for incorporating personal preferences into reference measures, and, in Section 8, we demonstrate how this yields a new measure of risk that takes into account the relativity of large risks with respect to smaller ones. Section 9 finishes the paper with a general index of inequality and risk.

2. The Classical Gini Index Revisited

Naturally, we begin our arguments with the classical index of Gini (1914). Let X be a random variable (think of 'income') with non-negatively supported cdf $F(x)$ and finite mean $\mu_F = \mathbf{E}[X]$. The Gini index, which we denote by G_F, is usually interpreted as twice the area between the actual population Lorenz function (Lorenz 1905; Pietra 1915; Gastwirth 1971)

$$\mathrm{L}_F(p) = \frac{1}{\mu_F} \int_0^p F^{-1}(t)dt$$

and the egalitarian Lorenz function $\mathrm{L}_E(p) = p$, $0 \le p \le 1$, which is the hypotenuse of the right triangle that we have alluded to in the abstract. For parametric expressions of $\mathrm{L}_F(p)$, we refer to Gastwirth (1971), Kakwani and Podder (1973), as well as to more recent works of Sarabia (2008), Sarabia et al. (2010), and references therein. Hence, the Gini index is

$$\begin{aligned} \mathrm{G}_F &= 2\int_0^1 \left(\mathrm{L}_E(p) - \mathrm{L}_F(p)\right) dp \\ &= 2\mathbf{E}\left[\mathrm{L}_E(\pi) - \mathrm{L}_F(\pi)\right], \end{aligned} \tag{1}$$

where the gamble π follows the uniform density on the unit interval $[0,1]$, that is, $f(p) = 1$ for all $p \in [0,1]$. Intuitively, π governs person's position in terms of income percentiles, and we thus call it *personal* gamble. In other words, barring the normalizing constant 2, the Gini index G_F is the expected *absolute*-deviation of person's position π on the actual Lorenz function $\mathrm{L}_F(p)$ from his/her position on the reference (egalitarian) Lorenz function $\mathrm{L}_E(p)$. Naturally, the position π is random, and we have already seen in the case of the Gini index that it follows the uniform on $[0,1]$ distribution. This means that the person has an equal chance of receiving any income among all the available incomes which are, in terms of percentiles, identified with the unit interval $[0,1]$.

In general, the personal gamble π can follow various distributions on $[0,1]$, and we shall see a variety of examples throughout this paper. The choice of distribution of π carries information about person's probable positions and is thus inevitably subjective, but many of the examples that we have encountered in the literature follow the beta distribution

$$f_{\mathrm{Beta}}(p \mid \alpha, \beta) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)} \quad \text{for} \quad 0 < p < 1,$$

which we have visualized in Figure 1. We succinctly write $\pi \sim \mathrm{Beta}(\alpha,\beta)$ and so, for example, the Gini index (cf. Equation (1)) is based on $\pi \sim \mathrm{Beta}(1,1)$. For illuminating statistical and historical notes on the beta and other related distributions in the context of measuring economic inequality, we refer to Kleiber and Kotz (2003). For very general yet remarkably tractable beta-generated families of distributions for greater modeling flexibility, we refer to Alexander et al. (2012), and references therein.

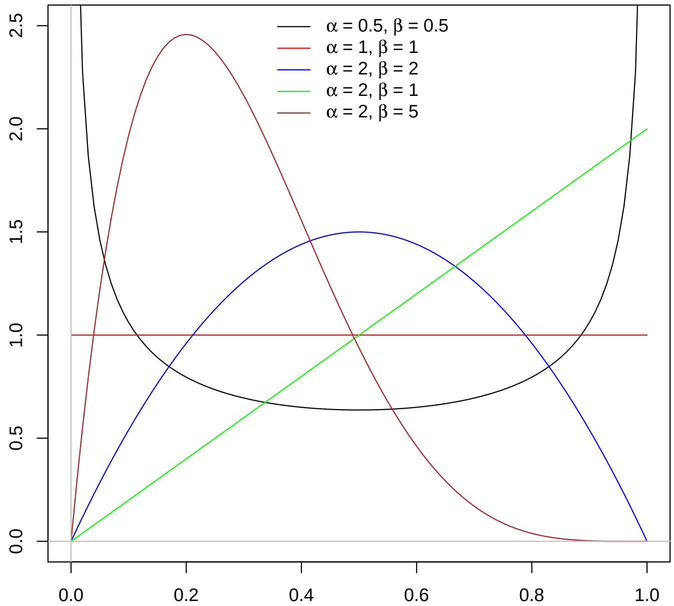

Figure 1. Beta densities of gambles π for various values of α and β.

Importantly for our following discussion, the Gini index G_F can also be viewed as the expected *relative*-deviation of person's position π on the actual Lorenz function $L_F(p)$ from his/her position on the reference Lorenz function $L_E(p)$, as seen from the equations:

$$\begin{aligned} G_F &= \int_0^1 \left(1 - \frac{L_F(p)}{L_E(p)}\right) 2p\, dp \\ &= \mathbf{E}\left[1 - \frac{L_F(\pi)}{L_E(\pi)}\right], \end{aligned} \tag{2}$$

where $\pi \sim \text{Beta}(2,1)$, which is a considerable change from $\pi \sim \text{Beta}(1,1)$ used in the absolute-deviation based representation (1) of the Gini index. Note that the right-hand side of Equation (2) can be succinctly written as $\mathbf{E}[B_F(\pi)]$, where

$$B_F(p) = 1 - \frac{L_F(p)}{L_E(p)} = 1 - \frac{L_F(p)}{p} \tag{3}$$

is the Bonferroni function of inequality (cf. Bonferroni 1930), which is also known in the literature as the Gini function of inequality because it appeared in Gini (1914). For details on the Bonferroni function and the corresponding Bonferroni index, we refer to Tarsitano (1990) and references therein.

In addition to its role when studying income and poverty, the Bonferroni function $B_F(p)$ has also found many uses in other fields such as reliability, demography, insurance, and medicine (e.g., Giorgi and Crescenzi 2001; and references wherein). For detailed historical notes and references with explicit expressions of the Lorenz and Bonferroni functions, as well as of the Gini and Bonferroni indices, for many parametric distributions, we refer to Giorgi and Nadarajah (2010). The role of the Bonferroni function within the framework of L-functions for measuring economic inequality and actuarial risks can be found in Tarsitano (2004), and Greselin et al. (2009).

3. From Egalitarian Lorenz to the Mean Reference

Not only the classical Gini index but also a multitude of other indices of economic inequality can be viewed as deviation measures (e.g., functional distances) between the actual and egalitarian Lorenz functions (cf., e.g., Zitikis 2002). Note, however, that the actual Lorenz function $\mathrm{L}_F(p)$ itself is a relative measure that compares $p \times 100\%$ lowest incomes with the population mean income μ_F. This two-stage relativity—first with respect to the egalitarian Lorenz function and then with the mean income—warrants a rethinking of the inequality measurement.

Toward this end, we next rephrase the definition of the Gini index G_F by first rewriting the Bonferroni function $\mathrm{B}_F(p)$ as follows:

$$\mathrm{B}_F(p) = 1 - \frac{\mathrm{AV@R}_F(p)}{\mu_F}, \tag{4}$$

where

$$\mathrm{AV@R}_F(p) = \frac{1}{p}\int_0^p F^{-1}(t)dt$$

is the (financial) average value at risk of X. Indeed, with a little mathematical caveat, $\mathrm{AV@R}_F(p)$ is the conditional expectation $\mathbf{E}[X \mid X \le F^{-1}(p)]$, which is the mean income of those who are below the 'poverty line' $F^{-1}(p)$. In summary, Equation (2) becomes

$$\mathrm{G}_F = \mathbf{E}\left[1 - \frac{\mathrm{AV@R}_F(\pi)}{\mu_F}\right] \tag{5}$$

with the gamble $\pi \sim \mathrm{Beta}(2,1)$. If, instead of the latter gamble, we use $\pi \sim \mathrm{Beta}(1,1)$ on the right-hand side of Equation (5), then the expectation turns into the Bonferroni index

$$\mathrm{B}_F = \int_0^1 \left(1 - \frac{\mathrm{AV@R}_F(p)}{\mu_F}\right)dp. \tag{6}$$

For details on the Bonferroni index, we refer to Tarsitano (1990) and references therein. For a comparison of the two weighting schemes, that is, of the gambles π employed in the Gini and Bonferroni cases, we refer to De Vergottini (1940). Implications of using the Bonferroni index on welfare measurement have been studied by, e.g., (Benedetti 1986; Aaberge 2000; Chakravarty 2007). Nygård and Sandström (1981) give a wide-ranging discussion of the use of Bonferroni-type concepts in the measurement of economic inequality. Giorgi and Crescenzi (2001), and Chakravarty and Muliere (2004) propose poverty measures based on the fact that the Bonferroni index exhibits greater sensitivity on lower levels of the income distribution than the Gini index. A general class of inequality measures inspired by the Bonferroni index has been explored by Imedio-Olmedo et al. (2011). Giorgi (1998) provides a list of Bonferroni's publications.

Equations (5) and (6) suggest that the Gini and Bonferroni indices are members of the following general class of indices

$$\mathcal{A}_F = \mathbf{E}[v(\mathrm{AV@R}_F(\pi), \mu_F)], \tag{7}$$

where $v(x,y)$ can be any function for which the expectation is well-defined and finite. In the case of the Gini and Bonferroni indices (e.g., Greselin 2014), we have $v(x,y) = 1 - x/y$, which is the relative value of x with respect to y. We call any function $v(x,y)$ used in expressions like (7) a relative-value function throughout this paper. Hence, we can view the index $\mathcal{A}_F$ as the expected utility of being in the society whose income distribution is depicted by the function $\mathrm{AV@R}_F(p)$ and compared with the reference mean income μ_F using an appropriately chosen relative-value function $v(x,y)$. We should note at this point that even though the class of relative-value functions $v(x,y)$ may look large, it is nevertheless prudent to restrict our attention to those that are of the form

$$v(x,y) = \ell(x/y) \tag{8}$$

for some function $\ell(t)$. Indeed, under the natural assumption of positive homogeneity, which means that the equation $v(\lambda x, \lambda y) = v(x,y)$ holds for all $\lambda > 0$, Euler's classical theorem says that we must have Equation (8) for some function $\ell(t)$. The Gini and Bonferroni indices give rise to $\ell(t) = 1 - t$.

Another example of the function $\ell(t)$ arises from the E-Gini index of Chakravarty (1988):

$$\begin{aligned} \mathrm{C}_{F,\alpha} &= 2\bigg(\int_0^1 (t - \mathrm{L}_F(t))^\alpha dt\bigg)^{1/\alpha} \\ &= 2\bigg(\int_0^1 \bigg(1 - \frac{\mathrm{AV@R}_F(\pi)}{\mu_F}\bigg)^\alpha t^\alpha dt\bigg)^{1/\alpha} \\ &= \frac{2}{(\alpha+1)^{1/\alpha}}\Big(\mathbf{E}[v(\mathrm{AV@R}_F(\pi), \mu_F)]\Big)^{1/\alpha}, \end{aligned} \tag{9}$$

where the reference-value function is $v(x,y) = (1 - x/y)^\alpha$, that is, $\ell(t) = (1-t)^\alpha$, and the gamble $\pi \sim \mathrm{Beta}(\alpha+1, 1)$. Zitikis (2002) suggests using $(\alpha+1)^{1/\alpha}$ instead of 2 in the definition of the E-Gini index (see also Zitikis (2003) for additional notes) in which case the right-hand side of Equation (9) turns into the index

$$\widetilde{\mathrm{C}}_{F,\alpha} = \Big(\mathbf{E}[v(\mathrm{AV@R}_F(\pi), \mu_F)]\Big)^{1/\alpha}.$$

In either case, note from the expressions of $\mathrm{C}_{F,\alpha}$ and $\widetilde{\mathrm{C}}_{F,\alpha}$ that it is sometimes useful to transform the index $\mathcal{A}_F$ by some function $w(x)$. We shall elaborate on this point in the next section.

Coming now back to the index $\mathcal{A}_F$, we note that, with the generic relative-value function $v(x,y) = \ell(x/y)$, the index can be rewritten as $\mathbf{E}[\bar{\ell}(B_F(\pi))]$, where $\bar{\ell}(t) = \ell(1-t)$. Hence, we are dealing with the distorted Bonferroni function $\bar{\ell}(B_F(p))$, $0 < p < 1$, which is analogous to the distorted Lorenz function upon which Sordo et al. (2014) have built their research (see Aaberge (2000) for earlier results on the topic). We do not pursue this research venue in the present paper because the Bonferroni function, just like that of Lorenz, incorporates a pre-specified reference measure, which is the mean income μ_F. In what follows, we argue in favour of more flexibility when choosing reference measures, which may even include personal preferences in addition to those of the entire population.

4. From the Mean to Generic Societal References

We now extend the index $\mathcal{A}_F$ to arbitrary references, which we denote by θ_F. Namely, let

$$\mathcal{B}_F = w\Big(\mathbf{E}[v(\mathrm{AV@R}_F(\pi), \theta_F)]\Big),$$

where $w(x)$ is a normalizing function whose main role is to fit the index into the unit interval $[0,1]$, with the value 0 meaning perfect equality (i.e., everybody has the same amount) and 1 meaning extreme inequality (i.e., only one person has something, and thus everything, with the others having nothing). Having the flexibility to manipulate references is important due to a variety of reasons. For example, the use of the mean μ_F can become questionable when population skewness increases, and this has already been noted by, e.g., Gastwirth (2014) who, in his research on the changing income inequality in the U.S. and Sweden, has suggested replacing the mean μ_F by the median $m_F = F^{-1}(0.5)$.

Another example of θ_F that differs from μ_F is provided by the Palma index; we refer to Cobham and Sumner (2013a, 2013b, 2014) for details. Namely, let θ_F be the average of the top 10% of the population incomes, that is, $\theta_F = \frac{1}{0.1}\int_{0.9}^1 F^{-1}(t)dt$. Furthermore, let the normalizing function be $w(x) = x$, the relative-value function $v(x,y) = y/x$, and the (deterministic) gamble $\pi = 0.4$. Under these specifications, the index $\mathcal{B}_F$ becomes the Palma index of economic inequality:

$$\mathrm{P}_F^{40,90} = \frac{\frac{1}{0.1}\int_{0.9}^{1} F^{-1}(t)dt}{\frac{1}{0.4}\int_{0}^{0.4} F^{-1}(t)dt}.$$

Instead of the underlying random variable (e.g., income) X, the researcher might be primarily interested in its transformation (e.g., utility of income) $u(X)$. To tackle this situation, we first incorporate the transformed incomes into our framework by extending the definition of the (financial) average value at risk as follows:

$$\mathrm{AV@R}_{F,u}(p) = \frac{1}{p}\int_0^p u(F^{-1}(t))dt.$$

Note that $\mathrm{AV@R}_{F,u}(1) = \mathbf{E}[u(X)]$, which we can view as the expected utility of X. We have arrived at the extension

$$\mathcal{C}_F = w\Big(\mathbf{E}[v(\mathrm{AV@R}_{F,u}(\pi), \theta_F)]\Big) \tag{10}$$

of the index $\mathcal{B}_F$.

The index $\mathcal{C}_F$ appears to be a minor generalization of the extended intermediate index of Cowell (2003) (see Equation (12) therein), which has been shown to include a large number of well-known indices (in particular, the Generalized Entropy class of indices) and far-reaching new ones. Namely, $\mathcal{C}_F$ reduces to the index of Cowell (2003), which for referencing purposes we denote by $\mathcal{C}_{F,k}$, by choosing $w(x) = A_k(x-1)$ for a certain constant A_k, $u(x) = \phi_k(x)$ for a certain function $\phi_k(x)$, the reference $\theta_F = u(\mu_F)$, the relative-value function $v(x) = x/y$, and the (deterministic) gamble $\pi = 1$; here are the aforementioned quantities that we have not yet specified:

$$A_k = \frac{1+k^2}{\alpha_k^2 - \alpha_k}, \quad \alpha_k = \gamma + \beta k, \quad \phi_k(x) = \frac{1}{\alpha_k}(x+k)^{\alpha_k},$$

where $\gamma \in (-\infty, \infty)$, $\beta \geq 0$, and $k \geq 0$ are parameters. Hence, even though the reason for our use of the letter $\mathcal{C}$ for index (10) is alphabetical, it would only be natural to call $\mathcal{C}_F$ the Cowell general intermediate index, whose special case, called extended intermediate index, appears in Cowell (2003).

The Atkinson (1970) index, which we denote by $\mathrm{A}_{F,\gamma}$, is a special case of $\mathcal{C}_F$. (For many other special cases, we refer to Cowell (2003).) Namely, let the utility function be $u(x) = x^\gamma$ for some $\gamma \in (0,1)$. Furthermore, let the (deterministic) gamble be $\pi = 1$, the reference $\theta_F = u(\mu_F)$, and the relative-value function $v(x,y) = 1 - x/y$. Under these specifications, the index $\mathcal{C}_F$ turns into $1 - \mathbf{E}[X^\gamma]/\mu_F^\gamma$, which after the transformation with the function $w(x) = 1-(1-x)^{1/\gamma}$ becomes the Atkinson index

$$\mathrm{A}_{F,\gamma} = 1 - \frac{(\mathbf{E}[X^\gamma])^{1/\gamma}}{\mu_F}.$$

This index has been highly influential in measuring economic inequality (e.g., Cowell (2011), and references therein) and inspired a variety of extensions and generalization of the Gini index. In addition, Mimoto and Zitikis (2008) have found the Atkinson index useful for developing a statistical inference theory for testing exponentiality, which has been a prominent problem in life-time analysis and, particularly, in reliability engineering.

5. The Donaldson-Weymark-Kakwani Index Revisited and Extended

The Donaldson-Weymark-Kakwani index (Donaldson and Weymark 1980, 1983; Kakwani 1980a, 1980b; Weymark 1981)

$$\mathrm{DWK}_{F,\alpha} = \alpha(\alpha-1)\int_0^1 (1-p)^{\alpha-2}(p - \mathrm{L}_F(p))dp,$$

which is also known as the S-Gini index, has arisen following Atkinson (1970) observation that the Gini index G_F does not take into account social preferences. Via the parameter $\alpha > 1$, the index $DWK_{F,\alpha}$ can reflect different social preferences, with the classical Gini index arising by setting $\alpha = 2$. We note in this regard that a justification for a family of indices to be based on the theory of relative deprivation has been provided by Yitzhaki (1979, 1982).

Just like the Gini index G_F, the index $DWK_{F,\alpha}$ can also be placed within the framework of expected relative value. Indeed, using Equations (3) and (4), we have

$$\begin{aligned} DWK_{F,\alpha} &= \int_0^1 \left(1 - \frac{L_F(p)}{p}\right) f_{\text{Beta}}(p \mid 2, \alpha - 1)dp \\ &= \int_0^1 \left(1 - \frac{AV@R_F(p)}{\mu_F}\right) f_{\text{Beta}}(p \mid 2, \alpha - 1)dp \\ &= \mathbf{E}[v(AV@R_F(\pi_\alpha), \mu_F)] \end{aligned} \tag{11}$$

with the relative-value function $v(x, y) = 1 - x/y$ and the gamble $\pi_\alpha \sim \text{Beta}(2, \alpha - 1)$, whose density is visualized in Figure 2.

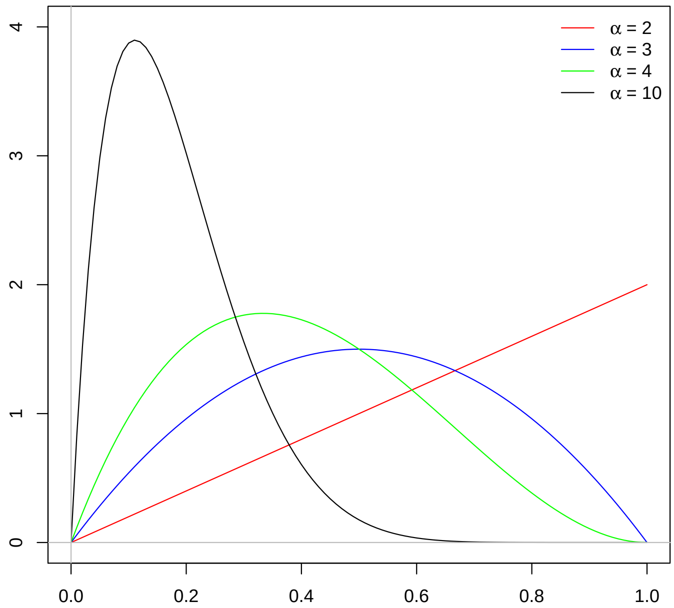

Figure 2. The density of π_α for various values of α.

We next introduce a more flexible index than $DWK_{F,\alpha}$ that allows us to employ more general gambles than π_α. For this, we first introduce a class of generating functions:

(H) Let $h : [0, 1] \to [0, 1]$ be any twice differentiable and convex function (i.e., $h''(p) \geq 0$ for all $p \in (0, 1)$) that satisfies the boundary conditions $h(0) = 0$ and $h(1) = 1$, and such that $h'(0) \neq 1$.

Let π_h denote the gamble whose density $f(p)$ is given by the formula

$$f(p) = \frac{p\, h''(1-p)}{1 - h'(0)} \tag{12}$$

for all $p \in (0,1)$, and $f(p) = 0$ elsewhere. With the relative-value function $v(x,y) = 1 - x/y$, we have (details in Appendix A)

$$\begin{aligned}
\mathrm{DWK}_{F,h} &:= \mathbf{E}[v(\mathrm{AV@R}_F(\pi_h), \mu_F)] \\
&= \frac{1}{1-h'(0)}\left(1 - \frac{1}{\mu_F}\int_0^1 F^{-1}(p)h'(1-p)\,dp\right) \\
&= \frac{1}{1-h'(0)}\left(1 - \frac{1}{\mu_F}\int_0^\infty h(1-F(x))\,dx\right).
\end{aligned} \tag{13}$$

To illustrate, we choose the function $h(p) = p^\alpha$ with any $\alpha > 1$, in which case the gamble π_h follows the density $\alpha(\alpha-1)p(1-p)^{\alpha-2}$; that is, $\pi_h \sim \mathrm{Beta}(2, \alpha-1)$, which means that π_h has the same distribution as the earlier noted gamble π_α. Consequently, $\mathrm{DWK}_{F,h}$ reduces to $\mathrm{DWK}_{F,\alpha}$, and thus Equation (13) reduce to the following expressions of the Donaldson-Weymark-Kakwani index:

$$\begin{aligned}
\mathrm{DWK}_{F,\alpha} &= 1 - \frac{\alpha}{\mu_F}\int_0^1 F^{-1}(p)(1-p)^{\alpha-1}dp \\
&= 1 - \frac{1}{\mu_F}\int_0^\infty (1-F(x))^\alpha dx
\end{aligned} \tag{14}$$

(cf. Donaldson and Weymark (1980, 1983); Yitzhaki (1983); Muliere and Scarsini (1989)).

6. The Wang Risk Measure Revisited and Extended

The index $\mathrm{DWK}_{F,h}$ is based on gambles generated by *convex* functions h. A similar index but based on *concave* generating functions g is called the Wang (or distortion) risk measure, which has been used in actuarial science and financial mathematics for measuring risks. In detail, the risk measure is defined by the formula

$$\mathrm{W}_{F,g} = \int_0^\infty g(1-F(x))\,dx,$$

where $g : [0,1] \to [0,1]$ is a distortion function, meaning that it is non-decreasing and satisfies the boundary conditions $g(0) = 0$ and $g(1) = 1$.

Hence, unlike in the previous section, we now work with concave distortion functions, denoted by g, under which the risk measure $\mathrm{W}_{F,g}$ is coherent (Wang et al. 1997; Wang and Young 1998; Wirch and Hardy 1999; see Artzner et al. (1999) for a general discussion). A classical example of such a distortion function is $g(p) = p^\alpha$ for any $\alpha \in (0,1)$, in which case the Wang risk measure $\mathrm{W}_{F,g}$ reduces to the proportional-hazards-transform risk measure (Wang 1995)

$$\mathrm{PHT}_{F,\alpha} = \int_0^\infty (1-F(x))^\alpha\,dx.$$

For more information on concave versus convex distortion functions in the context of measuring risks, their variability and orderings, we refer to Sordo and Suárez-Llorens (2011), Giovagnoli and Wynn (2012), and references therein.

We next show that the Wang risk measure $\mathrm{W}_{F,g}$ can be placed within the framework of expected relative value. When compared with the index $\mathrm{DWK}_{F,\alpha}$, there are two major changes: First, the function of interest is now the (insurance) average value at risk:

$$\mathrm{AVaR}_F(p) = \frac{1}{1-p}\int_p^1 F^{-1}(t)dt.$$

(Note that when $p = 0$, then $\mathrm{AVaR}_F(p)$ is equal to the mean μ_F.) Second, the function g that generates the distribution of the random position is concave. Specifically, we introduce the following class of generating functions:

(G) Let $g : [0,1] \to [0,1]$ be twice differentiable and concave function (i.e., $g''(p) \le 0$ for all $p \in (0,1)$) that satisfies the boundary conditions $g(0) = 0$ and $g(1) = 1$, and such that $g'(1) \neq 1$.

Any such function g generates the density $f(p)$ of the gamble π_g given by the formula

$$f(p) = \frac{-(1-p)g''(1-p)}{1-g'(1)} \tag{15}$$

for all $p \in (0,1)$, and $f(p) = 0$ elsewhere. With the relative-value function $v(x,y) = y/x - 1$, we have (details in Appendix)

$$\begin{aligned} \mathbf{E}[v(\mu_F, \mathrm{AVaR}_F(\pi_g))] &= \frac{1}{1-g'(1)} \left(\frac{1}{\mu_F} \int_0^1 F^{-1}(p) g'(1-p)\, dp - 1 \right) \\ &= \frac{1}{1-g'(1)} \left(\frac{1}{\mu_F} \int_0^\infty g(1-F(x))\, dx - 1 \right). \end{aligned} \tag{16}$$

Consequently, the Wang risk measure $\mathrm{W}_{F,g}$ can be expressed in terms of the expected relative value $\mathbf{E}[v(\mu_F, \mathrm{AVaR}_F(\pi_g))]$ as follows:

$$\mathrm{W}_{F,g} = \mu_F \Big(\mathbf{E}[v(\mu_F, \mathrm{AVaR}_F(\pi_g))](1-g'(1)) + 1 \Big). \tag{17}$$

When the generating function is $g(t) = t^\alpha$ for any $\alpha \in (0,1)$, then the gamble π_g follows $\mathrm{Beta}(1,\alpha)$ whose density function $\alpha(1-p)^{\alpha-1}$ is depicted in Figure 3.

From Equation (16), we have

$$\mathbf{E}[v(\mu_F, \mathrm{AVaR}_F(\pi_\alpha))] = \frac{1}{1-\alpha} \left(\frac{1}{\mu_F} \int_0^\infty (1-F(x))^\alpha\, dx - 1 \right). \tag{18}$$

Finally, we note the following expression for the proportional-hazards-transform risk measure:

$$\mathrm{PHT}_{F,\alpha} = \mu_F \Big(\mathbf{E}[v(\mu_F, \mathrm{AVaR}_F(\pi_g))](1-\alpha) + 1 \Big).$$

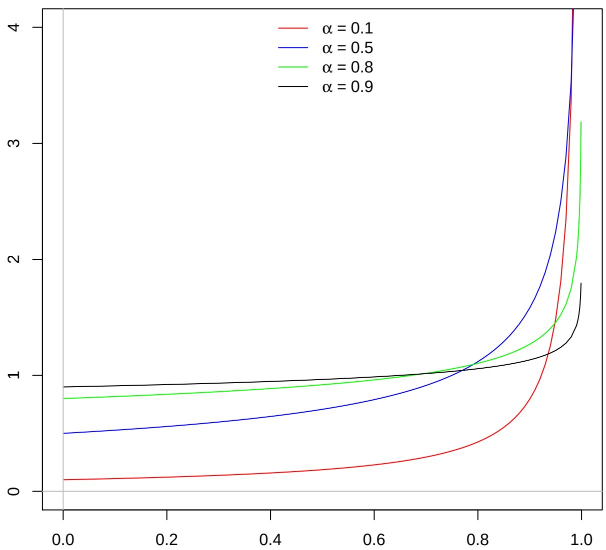

Figure 3. The density of π_g when $g(p) = p^\alpha$ for various values of α.

7. From Collective to Individual References

So far, we have worked with *collective* references. They do not depend on the outcomes of personal gambles and thus apply to all members of the society. Such references may not, however, be always desirable or justifiable. For example, given the outcome 0.4 of the gamble π, meaning that the person is considered to be among the 40% lowest income earners, the person may wish to compare the current position with the hypothetical one of being among the 60% highest income earners. In such situations, we are dealing with *individual* references: their values may depend on outcomes of the personal gamble π.

Hence, for example, the mean μ_F and the median $m_F = F^{-1}(0.5)$ are collective references, but $\theta_F = F^{-1}(\pi)$ is an individual reference because its value depends on the outcome of π. Would the quantile $F^{-1}(\pi)$ be a good reference? There are at least two major reasons against the use of the quantile, which is known in the risk literature as the value-at-risk:

1. The quantile $F^{-1}(p)$ is not robust with respect to realized values p of the random gamble π, in the sense that the quantile may change drastically even for very small changes of p.
2. For a realized value p of π, the quantile $F^{-1}(p)$ is not informative about the values of $F^{-1}(q)$ for $q > p$. Indeed, we may have the same value of $F^{-1}(p)$ irrespective of whether the cdf F is heavy- or light-tailed.

These are serious issues when constructing sound measures of economic inequality and risk. In the risk literature (cf., e.g., McNeil et al. (2005); Meucci (2007); Pflug and Römisch (2007); Cruz (2009); Sandström (2010); Cannata and Quagliariello (2011); and references therein), the problem with quantiles has been overcome by using $\mathrm{AVaR}_F(p)$ whose definition was given in the previous section. For example, adopting $\mathrm{AVaR}_F(p)$ as our (individual) reference θ_F and using the normalizing function $w(x) = x$, the earlier introduced index $\mathcal{B}_F$ turns into the Zenga (2007) index

$$\begin{aligned} \mathrm{Z}_F &= \int_0^1 \left(1 - \frac{\mathrm{AV@R}_F(p)}{\mathrm{AVaR}_F(p)}\right) dp \\ &= \mathbf{E}[v(\mathrm{AV@R}_F(\pi), \mathrm{AVaR}_F(\pi))] \end{aligned} \tag{19}$$

with the relative-value function $v(x, y) = 1 - x/y$ and the gamble $\pi \sim \mathrm{Beta}(1, 1)$. Hence, the Zenga index Z_F is the average with respect to all percentiles $p \in (0, 1)$ of the relative deviations of the mean income of the poor (i.e., those whose incomes are below the poverty line $F^{-1}(p)$) from the corresponding mean income of the rich, that is, of those whose incomes are above the poverty line $F^{-1}(p)$. We refer to Greselin et al. (2013) for a more detailed discussion of the relative nature of the Gini and Zenga indices, and their comparison.

8. Relative Measure of Risk

Many risk measures that we find in the literature are designed to measure *absolute* heaviness of the right-hand tail of the underlying loss distribution. Suppose now that we wish to measure the severity of large (e.g., insurance) losses *relative* to small ones. Note that this problem is very similar to that tackled by Zenga (2007) in the context of economic inequality. Hence, following the same path but now using the relative-value function $v(x, y) = y/x - 1$ and generic gamble π, we arrive at the relative measure of risk

$$\mathrm{R}_F = \mathbf{E}[v(\mathrm{AV@R}_F(\pi), \mathrm{AVaR}_F(\pi))], \tag{20}$$

which, in the spirit of expected utility, can be rewritten as

$$\mathrm{R}_F = \mathbf{E}[\mathrm{R}_F(\pi)], \tag{21}$$

where the role of utility function is played by the risk function

$$\mathrm{R}_F(p) = \frac{\mathrm{AVaR}_F(p)}{\mathrm{AV@R}_F(p)} - 1.$$

In what follows, we explore properties of this risk measure, using the notation R_X instead of R_F to simplify the presentation.

Proposition 1. *We have the following statements:*

1. *If the risk X is constant, that is, $X = d$ for some constant $d > 0$, then* $\mathrm{R}_X = 0$.
2. *Multiplying X by any constant $d > 0$ does not change the relative measure of risk, that is,* $\mathrm{R}_{dX} = \mathrm{R}_X$.
3. *Adding any constant $d > 0$ to the risk X decreases the relative measure of risk, that is,* $\mathrm{R}_{X+d} \leq \mathrm{R}_X$.

We have relegated the proof of Proposition 1 to Appendix. We next comment on the meaning of the three properties spelled out in the proposition. First, given that we are dealing with a *relative* measure of risk, properties 1 and 2 are self-explanatory. As to property 3, it says that lifting up the risk by any positive constant decreases its riskiness. This is natural because lifting up diminishes the relative variability of the risk. This, in turn, suggests that ordering of the relative risk measures should be done, for example, in terms of the Lorenz ordering, which is one of the most used tools for comparing the variability of economic-size distributions. This leads to the following property:

Proposition 2. *If risks X and Y follow the Lorenz ordering $X \leq_{\mathrm{L}} Y$, then* $\mathrm{R}_X \leq \mathrm{R}_Y$.

The proof of Property 2 is provided in Appendix, where the basic definition of Lorenz ordering can also be found. It is related to the notion of ordering based on the generalized, also called absolute, Lorenz curve (e.g., Ramos et al. 2000; Sriboonchita et al. 2010; and references therein). This leads us directly to a closely related property called the Pigou-Dalton principle of transfers. In the context of economic inequality, the principle says that progressive (i.e., from rich to poor) rank-order and mean-preserving transfers should decrease the value of inequality measures. Hence, in the context of risk, the transfers should be risk decreasing. Formally (cf., Vergnaud 1997), X is less risk-unequal than Y in the Pigou-Dalton sense, denoted by $X \leq_{\mathrm{PD}} Y$, if and only if $\mu_X = \mu_Y$ and $X \leq_{\mathrm{L}} Y$. Hence, $X \leq_{\mathrm{PD}} Y$ is sometimes denoted by $X \leq_{\mathrm{L},=} Y$ (cf. Denuit et al. 2005). The following property is now obvious.

Proposition 3. *If a Pigou-Dalton risk-increasing transfer turns risk X into Y so that $X \leq_{\mathrm{PD}} Y$, then* $\mathrm{R}_X \leq \mathrm{R}_Y$.

To have an idea of how the Pigou-Dalton transfers act, we recall (e.g., Shaked and Shanthikumar 2007; Sriboonchita et al. 2010) that given X and Y with densities f_X and f_Y, respectively, and assuming that their means are equal, if the sign of the difference $f_X - f_Y$ changes twice according to the pattern $(+, -, +)$, then $X \leq_{\mathrm{L}} Y$. Examples of parametric distributions with such pdf's can be found in, e.g., Kleiber and Kotz (2003); see also references therein.

In what follows, we discuss an example based on the Zenga (2010) distribution that has shown remarkably good performance in terms of goodness-of-fit on a number of real income data sets. It is a very flexible three-parameter distribution with Pareto-type right-hand tail and whose density is

$$f_{\mathrm{Zenga}}(x \mid \mu, \alpha, \theta) = \begin{cases} \dfrac{1}{2\mu\,\mathrm{Beta}(\alpha,\theta)} \left(\dfrac{x}{\mu}\right)^{-1.5} \displaystyle\int_0^{x/\mu} t^{\alpha+0.5-1}(1-t)^{\theta-2}dt, & x < \mu, \\[2ex] \dfrac{1}{2\mu\,\mathrm{Beta}(\alpha,\theta)} \left(\dfrac{\mu}{x}\right)^{1.5} \displaystyle\int_0^{\mu/x} t^{\alpha+0.5-1}(1-t)^{\theta-2}dt, & x \geq \mu, \end{cases}$$

where μ is the scale parameter, which also happens to be the mean of the distribution, and θ and α are two shape parameters that affect, respectively, the center and the tails of the distribution. We have depicted the Zenga density in Figure 4. For further details on this distribution and its uses, we refer to Zenga (2010), Zenga et al. (2011), Zenga et al. (2012), and Arcagni and Zenga (2013).

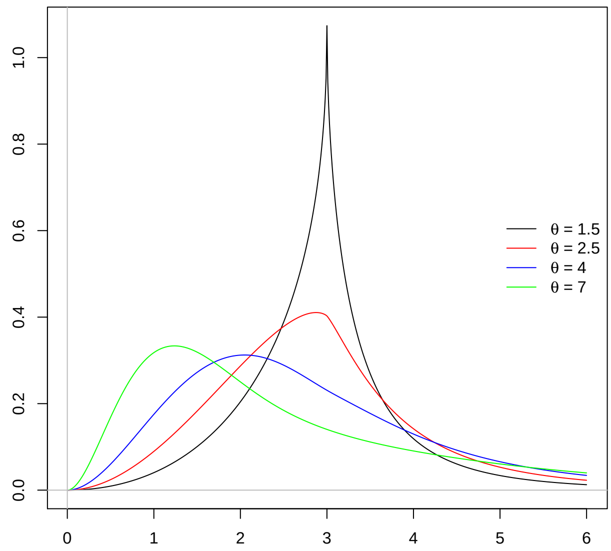

Figure 4. $\text{Zenga}(2, 2, \theta)$ density for various values of θ.

To see the effects of the Pigou-Dalton transfers in the case of the Zenga distribution, the following theorem is particularly useful.

Theorem 1 (Arcagni and Porro 2013)**.** *Assume* $X \sim \text{Zenga}(\mu_X, \alpha_X, \theta_X)$ *and* $Y \sim \text{Zenga}(\mu_Y, \alpha_Y, \theta_Y)$, *where all the parameters are positive. When* $\alpha_X \geq \alpha_Y$ *and* $\theta_X \leq \theta_Y$, *then* $X \leq_{\text{PD}} Y$.

9. Conclusions: A General Index of Inequality and Risk

The right-hand sides of Equations (19) and (20), which are identical, barring their different relative-value functions $v(x, y)$, give rise to a very general measure of inequality:

$$\mathbf{E}[v(\text{AV@R}_F(\pi), \text{AVaR}_F(\pi^*))],$$

where π and π^* are two gambles, which could be dependent or independent, degenerate or not. Obviously, when $\pi = \pi^*$, then we have either the Zenga index of economic inequality or the relative measure of risk, depending on the choice of the relative-value function. Furthermore, if $\pi^* = 0$, then we have $\text{AVaR}_F(\pi^*) = \mu_F$ and thus $\mathbf{E}[v(\text{AV@R}_F(\pi), \mu_F)]$, which is the Bonferroni index B_F. By appropriately choosing relative-value functions and personal gambles, we can reproduce a number of other measures of economic inequality and risk, but the Chakravarty and Atkinson indices require some little extension:

$$\mathcal{E}_F = w\Big(\mathbf{E}\big[v\big(\text{AV@R}_{F,u}(\pi), \text{AVaR}_{F,u^*}(\pi^*)\big)\big]\Big), \tag{22}$$

where u and u^* are two utility functions, and

$$\text{AVaR}_{F,u^*}(p) = \frac{1}{1-p}\int_p^1 u^*(F^{-1}(t))dt.$$

Note that $\text{AVaR}_{F,u^*}(0) = \mathbf{E}[u^*(X)]$. All the examples that we have mentioned in this paper, and also many other ones that appear in the literature, are special cases of the just introduced index $\mathcal{E}_F$. Table 1 provides a summary.

Table 1. Special cases of index (22) with $u^*(x) = x$ in all the rows.

	π	π^*	$w(x)$	$v(x,y)$	$u(x)$
Atkinson $A_{F,\alpha}$	1	0	$1-(1-x)^{1/\gamma}$	$1-x/y$	x^γ
Bonferroni B_F	Beta(1,1)	0	x	$1-x/y$	x
Chakravarty $C_{F,\alpha}$	Beta$(\alpha+1,1)$	0	$2(\alpha+1)^{-1/\alpha}x^{1/\alpha}$	$(1-x/y)^\alpha$	x
Inequality index $\widetilde{C}_{F,\alpha}$	Beta$(\alpha+1,1)$	0	$x^{1/\alpha}$	$(1-x/y)^\alpha$	x
Cowell $\mathcal{C}_{F,k}$	1	0	$A_k(x-1)$	x/y	$\phi_k(x)$
Cowell's Generalized Entropy class	1	0	linear	x/y	$\phi(x)$
Donaldson-Weymark-Kakwani $DWK_{F,\alpha}$	Beta$(2,\alpha-1)$	0	x	$1-x/y$	x
Inequality index $DWK_{F,h}$	π_h	0	x	$1-x/y$	x
Gini G_F	Beta(2,1)	0	x	$1-x/y$	x
Palma $P_F^{40,90}$	0.4	0.9	x	y/x	x
Risk measure R_F	Any	$\pi^*=\pi$	x	$y/x-1$	x
Wang $W_{F,g}$	1	π_g	$\mu_F(x(1-g'(1))+1)$	$y/x-1$	x
Proportional hazards transform $PHT_{F,\alpha}$	1	Beta$(1,\alpha)$	$\mu_F(x(1-\alpha)+1)$	$y/x-1$	x
Zenga Z_F	Beta(1,1)	$\pi^*=\pi$	x	$1-x/y$	x

We conclude with the note that, in the examples throughout this paper, the gambles π and π^* have been such that either they are identical (i.e., $\pi = \pi^*$) or one of them is degenerate (e.g., $\pi = 1$ or $\pi^* = 0$). There is no reason why this should always be the case: the two gambles can be dependent but not necessarily identical or degenerate. This suggests that, in general, modeling probability distributions of the pair (π, π^*) can be conveniently achieved by, for example, specifying marginal distributions of the gambles π and π^*, as well as dependence structures between them using, e.g., appropriately chosen copulas. For methodological and applications-driven developments related to copulas, we refer to the monographs of Nelsen (2006), Jaworski et al. (2010), Jaworski et al. (2013), and references therein.

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Appendix A. Technicalities

Proof of Equation (13). Since the relative-value function is $v(x,y) = 1 - x/y$, we have

$$\mathrm{DWK}_{F,h} = 1 - \frac{1}{\mu_F}\int_0^1 \mathrm{AV@R}_F(p) f(p)\,dp, \tag{A1}$$

where $f(p)$ is the density function of the gamble π_h defined by Equation (12). The following are straightforward calculations:

$$\begin{aligned}
\int_0^1 \mathrm{AV@R}_F(p) f(p)\,dp &= \int_0^1 \frac{1}{p}\left(\int_0^p F^{-1}(t)\,dt\right) f(p)\,dp \\
&= \int_0^1 F^{-1}(t)\left(\int_t^1 \frac{1}{p} f(p)\,dp\right)dt \\
&= \frac{1}{1-h'(0)}\int_0^1 F^{-1}(t)\big(h'(1-t) - h'(0)\big)\,dt \\
&= \frac{1}{1-h'(0)}\left(\int_0^1 F^{-1}(t) h'(1-t)\,dt - h'(0)\mu_F\right).
\end{aligned}$$

Combining this result with Equation (A1), we obtain the first equation of (13). Since

$$\begin{aligned}\int_0^1 F^{-1}(t)h'(1-t)\,dt &= \int_0^\infty \left(\int_0^1 \mathbf{1}\{F^{-1}(t) > x\} h'(1-t)\,dt \right) dx \\ &= \int_0^\infty \left(\int_0^1 \mathbf{1}\{t > F(x)\} h'(1-t)\,dt \right) dx \\ &= \int_0^\infty \left(\int_{F(x)}^1 h'(1-t)\,dt \right) dx \\ &= \int_0^\infty h(1-F(x))\,dx,\end{aligned} \tag{A2}$$

we have the second equation of (13). □

Proof of Equation (16). Since the relative-value function is $v(x,y) = y/x - 1$, we have

$$\mathbf{E}[v(\mu_F, \mathrm{AVaR}_F(\pi_g))] = \frac{1}{\mu_F} \int_0^1 \mathrm{AVaR}_F(p) f(p)\,dp - 1, \tag{A3}$$

where $f(p)$ is the density function of the gamble π_g defined by Equation (15). The following are straightforward calculations:

$$\begin{aligned}\int_0^1 \mathrm{AVaR}_F(p) f(p)\,dp &= \int_0^1 \frac{1}{1-p} \left(\int_p^1 F^{-1}(t)\,dt \right) f(p)\,dp \\ &= \int_0^1 F^{-1}(t) \left(\int_0^t \frac{f(p)}{1-p}\,dp \right) dt.\end{aligned}$$

Applying definition (15) of the density function $f(p)$, we obtain

$$\begin{aligned}\int_0^1 \mathrm{AVaR}(p) f(p)\,dp &= \frac{1}{1-g'(1)} \int_0^1 F^{-1}(t)\big(g'(1-t) - g'(1)\big)\,dt \\ &= \frac{1}{1-g'(1)} \left(\int_0^1 F^{-1}(t) g'(1-t)\,dt - g'(1)\mu_F \right).\end{aligned} \tag{A4}$$

Combining Equations (A3) and (A4), we obtain the first equation of (16). Using Equation (A2) with g instead of h, we arrive at the second equation of (16). □

Remark A1. From the mathematical point of view, Equation (A4) is elementary, but it was a pivotal observation that allowed Jones and Zitikis (2003) to initiate the development of statistical inference for the Wang (or distortion) risk measure. Since then, numerous statistical results have appeared on risk measures: parametric and non-parametric, light- and heavy-tailed cases have been explored in great detail by many authors. To illustrate the challenges that arise in the heavy-tailed context, we refer to Necir and Meraghni (2009), and Necir et al. (2007) for the proportional hazards transform; Necir et al. (2010), and Rassoul (2013) for the tail conditional expectation; and Brahimi et al. (2012) for general distortion risk measures.

Proof of Proposition 1. Property 1 follows from the fact that, if $X = d$ for any constant $d > 0$, then $F_X^{-1}(p) = d$ and so $\mathrm{AVaR}_X(p) = \mathrm{AV@R}_X(p)$ for every $p \in (0,1)$. Property 2 follows from the fact that if $d > 0$, then $F_{dX}^{-1}(p) = dF_X^{-1}(p)$ and so $\mathrm{AVaR}_{dX}(p)/\mathrm{AV@R}_{dX}(p) = \mathrm{AVaR}_X(p)/\mathrm{AV@R}_X(p)$ for every $p \in (0,1)$. Property 3 follows from the fact that $F_{X+d}^{-1}(p) = F_X^{-1}(p) + d$ for every d, and so the bound $\mathrm{AV@R}_X(p) \le \mathrm{AVaR}_X(p)$ together with the assumed positivity of d imply

$$\frac{\mathrm{AVaR}_{X+d}(p)}{\mathrm{AV@R}_{X+d}(p)} = \frac{\mathrm{AVaR}_X(p) + d}{\mathrm{AV@R}_X(p) + d} \le \frac{\mathrm{AVaR}_X(p)}{\mathrm{AV@R}_X(p)}.$$

The latter bound is equivalent to $\mathrm{R}_{X+d}(p) \leq \mathrm{R}_X(p)$ for every $p \in (0,1)$, which establishes the bound $\mathrm{R}_{X+d} \leq \mathrm{R}_X$. □

Proof of Proposition 2. We first recall (Arnold 1987; Aaberge 2000) that the Lorenz ordering $X \leq_{\mathrm{L}} Y$ means the bound $\mathrm{L}_X(p) \geq \mathrm{L}_Y(p)$ for all $p \in [0,1]$. Since

$$\begin{aligned}\mathrm{R}_X(p) &= \frac{1-\mathrm{L}_X(p)}{\mathrm{L}_X(p)} \frac{p}{1-p} - 1 \\ &= \frac{p}{(1-p)\mathrm{L}_X(p)} - \frac{p}{1-p} - 1,\end{aligned}$$

the Lorenz ordering $X \leq_{\mathrm{L}} Y$ is equivalent to the R-ordering $X \leq_{\mathrm{R}} Y$, which means $\mathrm{R}_X(p) \leq \mathrm{R}_Y(p)$ for all $p \in (0,1)$. The latter bound and Equation (21) conclude the verification of Proposition 2. □

Remark A2. With the above introduced notion of R-ordering, we can rephrase Proposition 2 as follows: if $X \leq_{\mathrm{R}} Y$, then $\mathrm{R}_X \leq \mathrm{R}_Y$. For detailed treatments of various notions of stochastic orders, we refer to Shaked and Shanthikumar (2007); Li and Li (2013); and Sriboonchita et al. (2010).

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