ON THE LEADING TERM OF THE EIGENVALUE VARIATION FOR AHARONOV-BOHM OPERATORS WITH A MOVING POLE

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Abstract. We study the behavior of certain eigenvalues for magnetic Aharonov-Bohm operators with half-integer circulation and Dirichlet boundary conditions in a planar domain. We analyze the leading term in the Taylor expansion of the eigenvalue function as the pole moves in the interior of the domain, proving that it is a harmonic homogeneous polynomial and determining its exact coefficients.

1. Introduction

This paper is concerned with the behavior of certain eigenvalues of magnetic Aharonov-Bohm operators with half-integer circulation and Dirichlet boundary conditions in a planar domain.

A remarkable mathematical motivation for the study of Aharonov-Bohm operators with half-integer circulation can be found in the deep relation between nodal domains of eigenfunctions of such operators and spectral minimal partitions of the Dirichlet Laplacian, i.e. partitions of the domain minimizing the largest of the first eigenvalues on the components. From [10] it is known that the boundary of the optimal partition is the union of a finite number of regular arcs; moreover, if the number of half lines meeting at each intersection point is even, then the partition components are nodal domains of an eigenfunction of the Dirichlet Laplacian. On the other hand, partitions with points of odd multiplicity can be obtained as nodal domains by minimizing a certain eigenvalue of an Aharonov-Bohm Hamiltonian with respect to the number and the position of poles, as suggested in [2, 3, 4, 14] and confirmed by the magnetic characterization of minimal partitions given in [8]. The properties of the map associating eigenvalues of magnetic operators to the position of poles and its connection between its critical points and nodal properties of eigenfunctions was further investigated in [5, 12, 13]. The present paper completes the analysis started in [1] and aims at giving the sharp asymptotic expansion for the eigenvalue variation with respect to moving poles.

For every $a = (a_1, a_2) \in \mathbb{R}^2$, the Aharonov-Bohm vector potential with pole a and circulation $1/2$ is defined as

$$A_a(x_1, x_2) = \frac{1}{2}\left(\frac{-(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\right), \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{a\}.$$

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Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain. For every $a \in \Omega$, we consider the eigenvalue problem

$$(E_a) \qquad \begin{cases} (i\nabla + A_a)^2 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

in the weak sense (15).

From classical spectral theory (see e.g. [6, Chapter 6]), the eigenvalue problem (E_a) admits a sequence of real diverging eigenvalues (repeated according to their finite multiplicity) $\lambda_1^a \leq \lambda_2^a \leq \cdots \leq \lambda_j^a \leq \dots$. We are concerned with the behavior of the function $a \mapsto \lambda_j^a$ in a neighborhood of a fixed point $b \in \Omega$; without loss of generality, we can consider $b = 0 \in \Omega$.

Let us assume that there exists $n_0 \geq 1$ such that

$$(1) \qquad \lambda_{n_0}^0 \quad \text{is simple,}$$

and denote

$$\lambda_0 = \lambda_{n_0}^0$$

and, for any $a \in \Omega$,

$$\lambda_a = \lambda_{n_0}^a.$$

In [12, Theorem 1.3] it is proved that, for all $j \geq 1$ such that λ_j^0 is simple, the function $a \mapsto \lambda_j^a$ is analytic in a neighborhood of 0. In particular, under assumption (1), $a \mapsto \lambda_a$ is continuous and, if $a \to 0$, then

$$(2) \qquad \lambda_a \to \lambda_0.$$

Let $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C}) \setminus \{0\}$ (see section 2 for the definition of the functional space $H_0^{1,0}(\Omega, \mathbb{C})$) be a $L^2(\Omega, \mathbb{C})$-normalized eigenfunction of problem (E_0) associated to the eigenvalue $\lambda_0 = \lambda_{n_0}^0$, i.e. satisfying

$$(3) \qquad \begin{cases} (i\nabla + A_0)^2 \varphi_0 = \lambda_0 \varphi_0, & \text{in } \Omega, \\ \varphi_0 = 0, & \text{on } \partial\Omega, \\ \int_\Omega |\varphi_0(x)|^2 \, dx = 1. \end{cases}$$

From [7, Theorem 1.3] (see also [14, Theorem 1.5] and [1, Proposition 2.1]) it is known that

$$(4) \qquad \varphi_0 \text{ has at 0 a zero of order } \frac{k}{2} \text{ for some odd } k \in \mathbb{N},$$

and that there exist $\beta_1, \beta_2 \in \mathbb{C}$ such that $(\beta_1, \beta_2) \neq (0, 0)$ and

$$(5) \qquad r^{-k/2} \varphi_0(r(\cos t, \sin t)) \to e^{i\frac{t}{2}} \left(\beta_1 \cos\left(\frac{k}{2}t\right) + \beta_2 \sin\left(\frac{k}{2}t\right) \right) \quad \text{in } C^{1,\tau}([0, 2\pi], \mathbb{C})$$

as $r \to 0^+$ for any $\tau \in (0, 1)$. We recall that, by [9] (see also [5, Lemma 2.3]), the function $e^{-i\frac{t}{2}} \varphi_0(r(\cos t, \sin t))$ is a (complex) multiple of a real-valued function; therefore (5) implies that the function $\gamma(t) = \beta_1 \cos\left(\frac{k}{2}t\right) + \beta_2 \sin\left(\frac{k}{2}t\right)$ is real-valued up to a complex multiplicative constant and then either $\beta_1 = \gamma(0) = 0$ or $\frac{\beta_2}{\beta_1} = \pm\frac{\gamma(\pi)}{\gamma(0)}$ is real. Then φ_0 has exactly k nodal lines meeting at 0 and dividing the whole angle into k equal parts; these nodal lines are tangent to the k half-lines $\left\{ \left(t, \tan(\alpha_0 + j\frac{2\pi}{k})t\right) : t > 0 \right\}$, $j = 0, 1, \dots, k-1$, where

$$(6) \qquad \alpha_0 = \begin{cases} \frac{2}{k} \operatorname{arccot}\left(-\frac{\beta_2}{\beta_1}\right), & \text{if } \beta_1 \neq 0, \\ 0, & \text{if } \beta_1 = 0. \end{cases}$$

At a deeper study, the rate of convergence of λ_a to λ_0 is strictly related to the number of nodal lines of φ_0 ending at 0. First results in this direction are proved in [5], in which the authors provide some estimates for the rate of convergence (2). A significant improvement of these studies is obtained in [1], where sharp asymptotic behavior of eigenvalues is provided as the pole is approaching an internal zero of an eigenfunction φ_0 of the limiting problem (3) along the half-line tangent to any nodal line of φ_0; more precisely, under assumption (1) and being k as in (4), in [1, Theorem 1.2] it is proved that the limit

$$\lim_{|a|\to 0^+} \frac{\lambda_0-\lambda_a}{|a|^k} \text{ is finite and strictly positive as } a\to 0 \text{ tangentially to a nodal line.} \tag{7}$$

The above positive limit can be expressed in terms of the value $\mathfrak{m}_k$ defined as follows. Let s_0 be the positive half-axis $s_0=[0,+\infty)\times\{0\}$. For every odd natural number k, the function

$$\psi_k(r\cos t, r\sin t) = r^{k/2}\sin\left(\frac{k}{2}\,t\right), \quad r\ge 0, \quad t\in[0,2\pi], \tag{8}$$

is the unique (up to a multiplicative constant) function which is harmonic on $\mathbb{R}^2\setminus s_0$, homogeneous of degree $k/2$ and vanishing on s_0. Let $s:=\{(x_1,x_2)\in\mathbb{R}^2 : x_2=0 \text{ and } x_1\ge 1\}$, $\mathbb{R}^2_+=\{(x_1,x_2)\in\mathbb{R}^2 : x_2>0)\}$, and denote as $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ the completion of $C^\infty_{\rm c}(\overline{\mathbb{R}^2_+}\setminus s)$ under the norm $(\int_{\mathbb{R}^2_+}|\nabla u|^2\,dx)^{1/2}$. By standard minimization methods, the functional

$$J_k : \mathcal{D}^{1,2}_s(\mathbb{R}^2_+)\to\mathbb{R}, \quad J_k(u)=\frac{1}{2}\int_{\mathbb{R}^2_+}|\nabla u(x)|^2\,dx - \int_{\partial\mathbb{R}^2_+\setminus s} u(x_1,0)\frac{\partial\psi_k}{\partial x_2}(x_1,0)\,dx_1,$$

achieves its minimum over the whole space $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ at some function $w_k\in\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$, i.e. there exists $w_k\in\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ such that

$$\mathfrak{m}_k = \min_{u\in\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)} J_k(u) = J_k(w_k). \tag{9}$$

We notice that

$$\mathfrak{m}_k = J_k(w_k) = -\frac{1}{2}\int_{\mathbb{R}^2_+}|\nabla w_k(x)|^2\,dx = -\frac{1}{2}\int_0^1 \frac{\partial_+\psi_k}{\partial x_2}(x_1,0)\,w_k(x_1,0)\,dx_1 < 0, \tag{10}$$

where, for all $x_1>0$, $\frac{\partial_+\psi_k}{\partial x_2}(x_1,0)=\lim_{t\to 0^+}\frac{\psi_k(x_1,t)-\psi_k(x_1,0)}{t}=\frac{k}{2}x_1^{\frac{k}{2}-1}$. In [1] it is proved that the limit in (7) is equal to

$$C_0 = -4(|\beta_1|^2+|\beta_2|^2)\,\mathfrak{m}_k \tag{11}$$

with $(\beta_1,\beta_2)\neq(0,0)$ being as in (5).

From [1, Theorem 1.2] we can easily deduce that, under assumption (1) and being k as in (4), the Taylor polynomials of the function $a\mapsto\lambda_0-\lambda_a$ with center 0 and degree strictly smaller than k vanish.

Lemma 1.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded, open and simply connected domain such that $0\in\Omega$ and let $n_0\ge 1$ be such that the n_0-th eigenvalue $\lambda_0=\lambda^0_{n_0}$ of $(i\nabla+A_0)^2$ on Ω is simple with associated eigenfunctions having in 0 a zero of order $k/2$ with $k\in\mathbb{N}$ odd. For $a\in\Omega$ let $\lambda_a=\lambda^a_{n_0}$ be the n_0-th eigenvalue of $(i\nabla+A_a)^2$ on Ω. Then*

$$\lambda_0-\lambda_a = P(a)+o(|a|^k), \quad \text{as } |a|\to 0^+, \tag{12}$$

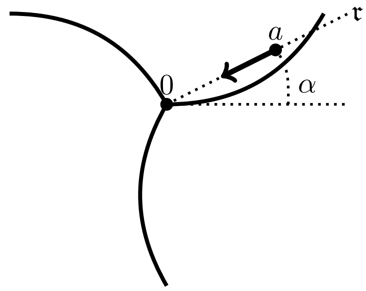

FIGURE 1. $a = |a|(\cos\alpha, \sin\alpha)$ approaches 0 along the direction determined by the angle α.

for some homogeneous polynomial $P \not\equiv 0$ of degree k

$$P(a) = P(a_1, a_2) = \sum_{j=0}^{k} c_j a_1^{k-j} a_2^j. \tag{13}$$

The main result of the present paper is the determination of the exact value of all coefficients of the polynomial P (and hence the sharp asymptotic behavior of $\lambda_a - \lambda_0$ as $a \to 0$ along any direction, see Figure 1).

Theorem 1.2. *Under the same assumptions of Lemma 1.1, let $\alpha \in [0, 2\pi)$. Then*

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to C_0 \cos\big(k(\alpha - \alpha_0)\big) \qquad \text{as } a \to 0 \text{ with } a = |a|(\cos\alpha, \sin\alpha),$$

where α_0 is defined in (6) *and C_0 in* (11).

Remark 1.3. By Theorem 1.2 it follows that the polynomial (13) of Lemma 1.1 is given by

$$P(|a|(\cos\alpha, \sin\alpha)) = C_0 |a|^k \cos(k(\alpha - \alpha_0)).$$

Hence

$$P(a_1, a_2) = C_0\, \mathfrak{Re}\left(e^{-ik\alpha_0}(a_1 + i\, a_2)^k\right),$$

thus yielding $\Delta P = 0$, i.e. the polynomial P in (12)-(13) is *harmonic*.

The proof of Theorem 1.2 is based on a combination of estimates from above and below of the Rayleigh quotient associated to the eigenvalue problem with a fine blow-up analysis for scaled eigenfunctions

$$\frac{\varphi_a(|a|x)}{|a|^{k/2}}, \tag{14}$$

which gives a sharp characterization of upper and lower bounds for eigenvalues. Differently from the blow-up analysis performed in [1] for poles moving tangentially to nodal lines, in the general case of poles moving along any direction we cannot explicitly construct the limit profile of the family of scaled functions (14). Such a difficulty is overcome by studying the dependence of the limit profile on the position of the pole and the symmetry/periodicity properties of its Fourier coefficient with respect to a basis of eigenvectors of an associated angular problem: such symmetry and periodicity turn into some symmetry and periodicity invariances of the polynomial P. A complete classification of homogeneous k-degree polynomials with such periodicity/symmetry invariances then allows us to determine explicitly the polynomial P.

The paper is organized as follows. Section 2 is devoted to recalling some known facts and introducing some notation. In section 3 we prove sharp asymptotics for $\lambda_0 - \lambda_a$ in dependence on the angle α. In section 4 we describe some symmetry properties of the sharp asymptotics, which allow us to prove Theorem 1.2 in section 5.

2. Preliminaries

In this section we present some preliminaries as needed in the forthcoming argument. For every $a \in \Omega$, we introduce the functional space $H^{1,a}(\Omega, \mathbb{C})$ as the completion of

$$\{u \in H^1(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C}) : u \text{ vanishes in a neighborhood of } a\}$$

with respect to the norm

$$\|u\|_{H^{1,a}(\Omega,\mathbb{C})} = \Big(\|\nabla u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})} + \big\| \tfrac{u}{|x-a|} \big\|^2_{L^2(\Omega,\mathbb{C})} \Big)^{1/2},$$

which, in view of the Hardy type inequality proved in [11] (see also [7, Lemma 3.1 and Remark 3.2]), is equivalent to the norm

$$\Big(\|(i\nabla + A_a)u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})} \Big)^{1/2}.$$

We denote as $H^{1,a}_0(\Omega, \mathbb{C})$ the space obtained as the completion of $C^\infty_{\rm c}(\Omega \setminus \{a\}, \mathbb{C})$ with respect to the norm $\| \cdot \|_{H^{1,a}(\Omega,\mathbb{C})}$.

For every $a \in \Omega$, we say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (E_a) in a weak sense if there exists $u \in H^{1,a}_0(\Omega, \mathbb{C}) \setminus \{0\}$ (called an eigenfunction) such that

$$\int_\Omega (i\nabla u + A_a u) \cdot \overline{(i\nabla v + A_a v)}\, dx = \lambda \int_\Omega u\overline{v}\, dx \quad \text{for all } v \in H^{1,a}_0(\Omega, \mathbb{C}). \tag{15}$$

2.1. **Change of coordinates.** Up to a change of coordinates (a rotation), it is not restrictive to assume in (5) that

$$\beta_1 = 0, \tag{16}$$

see [1, Remark 2.2]. Under condition (16), we have that $\alpha_0 = 0$ and one nodal line of φ_0 is tangent the x_1-axis.

2.2. **Polar eigenfunctions.** The limit function in (5) is an eigenfunction of the operator

$$\mathfrak{L}\psi = -\psi'' + i\psi' + \frac{1}{4}\psi$$

acting on 2π-periodic functions. The eigenvalues of $\mathfrak{L}$ are $\big\{\frac{j^2}{4} : j \in \mathbb{N},\ j \text{ is odd}\big\}$; moreover each eigenvalue $\frac{j^2}{4}$ has multiplicity 2 and the functions

$$\psi^j_1(t) = \frac{e^{i\frac{t}{2}}}{\sqrt{\pi}} \cos\Big(\frac{j}{2}t\Big), \quad \psi^j_2(t) = \frac{e^{i\frac{t}{2}}}{\sqrt{\pi}} \sin\Big(\frac{j}{2}t\Big) \tag{17}$$

form an $L^2((0, 2\pi), \mathbb{C})$-orthonormal basis of the eigenspace associated to the eigenvalue $\frac{j^2}{4}$.

2.3. **Angles and approximating eigenfunctions.** As in [5], for every $\alpha \in [0,2\pi)$ and $b = (b_1,b_2) = |b|(\cos\alpha, \sin\alpha) \in \mathbb{R}^2 \setminus \{0\}$, we define

$$\theta_b : \mathbb{R}^2 \setminus \{b\} \to [\alpha, \alpha+2\pi) \quad \text{and} \quad \theta_0^b : \mathbb{R}^2 \setminus \{0\} \to [\alpha, \alpha+2\pi) \tag{18}$$

such that

$$\theta_b(b + r(\cos t, \sin t)) = t \quad \text{and} \quad \theta_0^b(r(\cos t, \sin t)) = t, \quad \text{for all } r > 0 \text{ and } t \in [\alpha, \alpha+2\pi).$$

E.g. if $b_1 > 0$ and $b_2 > 0$ the functions θ_b and θ_0^b are given by

$$\theta_b(x_1,x_2) = \begin{cases} \arctan\frac{x_2-b_2}{x_1-b_1}, & \text{if } x_1 > b_1,\ x_2 \geq \frac{b_2}{b_1}x_1, \\ \frac{\pi}{2}, & \text{if } x_1 = b_1,\ x_2 > b_2, \\ \pi + \arctan\frac{x_2-b_2}{x_1-b_1}, & \text{if } x_1 < b_1, \\ \frac{3}{2}\pi, & \text{if } x_1 = b_1,\ x_2 < b_2, \\ 2\pi + \arctan\frac{x_2-b_2}{x_1-b_1}, & \text{if } x_1 > b_1,\ x_2 < \frac{b_2}{b_1}x_1, \end{cases} \tag{19}$$

$$\theta_0^b(x_1,x_2) = \begin{cases} \arctan\frac{x_2}{x_1}, & \text{if } x_1 > 0,\ x_2 \geq \frac{b_2}{b_1}x_1, \\ \frac{\pi}{2}, & \text{if } x_1 = 0,\ x_2 > 0, \\ \pi + \arctan\frac{x_2}{x_1}, & \text{if } x_1 < 0, \\ \frac{3}{2}\pi, & \text{if } x_1 = 0,\ x_2 < 0, \\ 2\pi + \arctan\frac{x_2}{x_1}, & \text{if } x_1 > 0,\ x_2 < \frac{b_2}{b_1}x_1. \end{cases}$$

We notice that θ_b and θ_0^b are regular except on the half-lines

$$s_b := \{tb : t \geq 1\}, \quad s_0^b := \{t\, b : t \geq 0\}, \tag{20}$$

respectively, whereas the difference $\theta_0^b - \theta_b$ is regular except for the segment $\{tb : t \in [0,1]\}$ from 0 to b.

We also define

$$\theta_0 : \mathbb{R}^2 \setminus \{0\} \to [0, 2\pi)$$

as

$$\theta_0(x_1,x_2) = \begin{cases} \arctan\frac{x_2}{x_1}, & \text{if } x_1 > 0,\ x_2 \geq 0, \\ \frac{\pi}{2}, & \text{if } x_1 = 0,\ x_2 > 0, \\ \pi + \arctan\frac{x_2}{x_1}, & \text{if } x_1 < 0, \\ \frac{3}{2}\pi, & \text{if } x_1 = 0,\ x_2 < 0, \\ 2\pi + \arctan\frac{x_2}{x_1}, & \text{if } x_1 > 0,\ x_2 < 0, \end{cases} \tag{21}$$

so that $\theta_0(\cos t, \sin t) = \theta_0^0(\cos t, \sin t) = t$ for all $t \in [0, 2\pi)$ and θ_0 is regular except for the half-axis $\{(x_1, 0) : x_1 \geq 0\}$. We notice that

$$(\theta_0^b - \theta_0)(r\cos t, r\sin t) = \begin{cases} 0, & \text{if } t \in [\alpha, 2\pi), \\ 2\pi, & \text{if } t \in [0, \alpha). \end{cases} \tag{22}$$

Let us now consider a suitable family of eigenfunctions relative to the approximating eigenvalue λ_a. For all $a \in \Omega$, let $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C}) \setminus \{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_a, i.e. solving

$$\begin{cases} (i\nabla + A_a)^2 \varphi_a = \lambda_a \varphi_a, & \text{in } \Omega, \\ \varphi_a = 0, & \text{on } \partial\Omega, \end{cases} \tag{23}$$

such that

$$\int_\Omega |\varphi_a(x)|^2\,dx = 1 \quad\text{and}\quad \int_\Omega e^{\frac{i}{2}(\theta_0^a-\theta_a)(x)}\varphi_a(x)\overline{\varphi_0(x)}\,dx \text{ is a positive real number,} \tag{24}$$

where φ_0 is as in (3). From (1), (3), (23), (24), and standard elliptic estimates, it follows that $\varphi_a \to \varphi_0$ in $H^1(\Omega,\mathbb{C})$ and in $C^2_{\rm loc}(\Omega\setminus\{0\},\mathbb{C})$ and

$$(i\nabla + A_a)\varphi_a \to (i\nabla + A_0)\varphi_0 \quad \text{in } L^2(\Omega,\mathbb{C}). \tag{25}$$

2.4. **Limit profile in dependence on α.** A key role in the proof of our main result is played by a suitable magnetic-harmonic function in $\mathbb{R}^2$, which will turn out to be the limit of blowed-up sequences of eigenfunctions with poles approaching 0 along the half-line starting from 0 with slope $\tan\alpha$.

For every $p\in\mathbb{R}^2$, we denote by $\mathcal{D}^{1,2}_p(\mathbb{R}^2,\mathbb{C})$ the completion of $C^\infty_{\rm c}(\mathbb{R}^N\setminus\{0\},\mathbb{C})$ with respect to the magnetic Dirichlet norm

$$\|u\|_{\mathcal{D}^{1,2}_p(\mathbb{R}^2,\mathbb{C})} := \bigg(\int_{\mathbb{R}^2}\big|(i\nabla + A_p)u(x)\big|^2\,dx\bigg)^{1/2}.$$

We recall from [11] that functions in $\mathcal{D}^{1,2}_p(\mathbb{R}^2,\mathbb{C})$ satisfy the Hardy type inequality

$$\int_{\mathbb{R}^2}|(i\nabla + A_p)u|^2\,dx \geq \frac14\int_{\mathbb{R}^2}\frac{|u(x)|^2}{|x-p|^2}\,dx;$$

furthermore (see also [7, Lemma 3.1 and Remark 3.2]) the inequality

$$\int_{D_r(p)}|(i\nabla + A_p)u|^2\,dx \geq \frac14\int_{D_r(p)}\frac{|u(x)|^2}{|x-p|^2}\,dx,$$

holds for all $r>0$ and $u\in H^{1,p}(D_r(p),\mathbb{C})$, where $D_r(p)$ denotes the disk of center p and radius r.

If $a=|a|(\cos\alpha,\sin\alpha)\to 0$ with $\alpha\in[0,2\pi)$, i.e. if $a\to 0$ along the line of slope α, all the functions of the blowed-up family (14) are singular at the same point $p=(\cos\alpha,\sin\alpha)$. We will prove in section 3 that the family (14) converges to the limit profile Ψ_p described in the following proposition.

Proposition 2.1. *Let $\alpha\in[0,2\pi)$ and $p=(\cos\alpha,\sin\alpha)$. There exists a unique function $\Psi_p\in H^{1,p}_{\rm loc}(\mathbb{R}^2,\mathbb{C})$ such that*

$$(i\nabla + A_p)^2\Psi_p = 0 \qquad \textit{in } \mathbb{R}^2 \textit{ in a weak } H^{1,p}\textit{-sense}, \tag{26}$$

and

$$\int_{\mathbb{R}^2\setminus D_r}\big|(i\nabla + A_p)(\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}\psi_k)\big|^2\,dx < +\infty, \quad \textit{for any } r>1, \tag{27}$$

where $D_r = D_r(0)$.

Proof. Let η be a smooth cut-off function such that $\eta\equiv 0$ in D_1 and $\eta\equiv 1$ in $\mathbb{R}^2\setminus D_R$ for some $R>1$. We observe that

$$\begin{aligned}\mathcal{F} &= (\Delta\eta)e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}\psi_k - 2i\nabla\eta\cdot(i\nabla + A_p)\Big(e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}\psi_k\Big)\\ &= -(i\nabla + A_p)^2\Big(\eta e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}\psi_k\Big) \in \big(\mathcal{D}^{1,2}_p(\mathbb{R}^2,\mathbb{C})\big)^\star.\end{aligned}$$

Hence, via Lax-Milgram's Theorem there exists a unique solution $g \in \mathcal{D}^{1,2}_p(\mathbb{R}^2,\mathbb{C})$ to the problem

$$(i\nabla + A_p)^2 g = \mathcal{F}, \quad \text{in } \big(\mathcal{D}^{1,2}_p(\mathbb{R}^2,\mathbb{C})\big)^\star.$$

The function $\Psi_p = g + \eta e^{\frac{i}{2}(\theta_p-\theta_0^p)} e^{\frac{i}{2}\theta_0}\psi_k$ satisfies (26) and (27).

To prove uniqueness, it is enough to observe that, if two functions $\Psi_p^1, \Psi_p^2 \in H^{1,p}_{\rm loc}(\mathbb{R}^2,\mathbb{C})$ satisfy (26) and (27), then their difference $\Psi_p^1 - \Psi_p^2$ belongs to the space $\mathcal{D}^{1,2}_p(\mathbb{R}^2,\mathbb{C})$ (see [1, Proposition 4.3]); since $(i\nabla + A_p)^2(\Psi_p^1 - \Psi_p^2) = 0$ in $(\mathcal{D}^{1,2}_p(\mathbb{R}^2,\mathbb{C}))^\star$, we conclude that necessarily $\Psi_p^1 - \Psi_p^2 \equiv 0$. □

Remark 2.2. We observe that from [7, Theorem 1.5] it follows easily that

$$\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)} e^{\frac{i}{2}\theta_0}\psi_k = O(|x|^{-1/2}), \quad \text{as } |x| \to +\infty.$$

3. Sharp asymptotics for $\lambda_0 - \lambda_a$ in dependence on α

In this section we prove some estimates of the eigenvalue variation $\lambda_0 - \lambda_a$, by evaluating the Rayleigh quotient at suitable test functions obtained by manipulation of eigenfunctions. Although this procedure follows the scheme of [1], it presents some additional difficulties requiring nontrivial adaptions, mainly due to the fact that the limit profile of Proposition 2.1 cannot be explicitly constructed as in [1]. We describe below this procedure, referring to [1] for details of arguments already developed there and instead highlighting the differences with [1] and the difficulties in the adaption to the general case of poles moving along a generic direction.

For all $1 \le j \le n_0$ and $a \in \Omega$, let $\varphi_j^a \in H^{1,a}_0(\Omega,\mathbb{C}) \setminus \{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_j^a, i.e. solving

$$\begin{cases} (i\nabla + A_a)^2 \varphi_j^a = \lambda_j^a \varphi_j^a, & \text{in } \Omega,\\ \varphi_j^a = 0, & \text{on } \partial\Omega, \end{cases} \tag{28}$$

such that

$$\int_\Omega |\varphi_j^a(x)|^2\,dx = 1 \quad \text{and} \quad \int_\Omega \varphi_j^a(x)\overline{\varphi_\ell^a(x)}\,dx = 0 \text{ if } j \ne \ell. \tag{29}$$

For $j = n_0$, we choose

$$\varphi^a_{n_0} = \varphi_a, \tag{30}$$

with φ_a as in (23)–(24).

3.1. Estimate from below of $\lambda_0 - \lambda_a$.

Letting $p = (\cos\alpha, \sin\alpha)$, we define the function w_R as the unique solution to the minimization problem

$$\int_{D_R} |(i\nabla + A_p)w_R(x)|^2\,dx = \min\left\{ \int_{D_R} |(i\nabla + A_p)u(x)|^2\,dx : \ u \in H^{1,p}(D_R,\mathbb{C}), \ u = e^{\frac{i}{2}(\theta_p-\theta_0^p)} e^{\frac{i}{2}\theta_0}\psi_k \text{ on } \partial D_R \right\},$$

which then solves

$$\begin{cases} (i\nabla + A_p)^2 w_R = 0, & \text{in } D_R,\\ w_R = e^{\frac{i}{2}(\theta_p-\theta_0^p)} e^{\frac{i}{2}\theta_0}\psi_k, & \text{on } \partial D_R. \end{cases} \tag{31}$$

Arguing as in [1, Subsection 6.2], we can prove the following lemma.

Lemma 3.1. *For $\alpha \in [0,2\pi)$ and $a = |a|(\cos\alpha, \sin\alpha) \in \Omega$, let $\lambda_a \in \mathbb{R}$ and $\varphi_a \in H^{1,a}_0(\Omega,\mathbb{C})$ solve* (23)-(24) *and $\lambda_0 \in \mathbb{R}$ and $\varphi_0 \in H^{1,0}_0(\Omega,\mathbb{C})$ solve* (3)*. If* (1) *and* (4) *hold and* (16) *is satisfied, then, for all $R > \tilde R$ and $a = |a|(\cos\alpha, \sin\alpha) \in \Omega$,*

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \geq g_R(a)$$

where

$$\lim_{|a|\to 0} g_R(a) = i|\beta_2|^2 \tilde\kappa_R,$$

with

$$\tilde\kappa_R = \int_{\partial D_R} \Big(e^{-\frac{i}{2}\theta_p} e^{\frac{i}{2}(\theta_0^p - \theta_0)} (i\nabla + A_p) w_R \cdot \nu - (i\nabla)\psi_k \cdot \nu \Big) \psi_k \, ds \tag{32}$$

being $p = (\cos\alpha, \sin\alpha)$ and ψ_k as in (8)*.*

Proof. The proof follows exactly as in [1, Lemma 6.6], so we omit it. □

For any $R > 1$ let us introduce the following Fourier-type coefficient

$$\upsilon_R(r) := \int_0^{2\pi} e^{-\frac{i}{2}\theta_p(r\cos t, r\sin t)} w_R(r\cos t, r\sin t) e^{\frac{i}{2}\theta_0^p(r\cos t, r\sin t)} \overline{\psi_2^k(t)}\, dt, \quad r \in [1,R], \tag{33}$$

with ψ_2^k defined in (17). Due to jumps of the phases appearing in (33) on s_p (see (20)), the derivation of the equation satisfied by the Fourier-type coefficient υ_R is more delicate than in [1]; hence we give the details in Lemma 3.2 below.

Lemma 3.2. *For any $R > 1$ the function υ_R defined in* (33) *satisfies*

$$\big(r^{-k/2}\upsilon_R(r)\big)' = \frac{c_R}{r^{1+k}}, \quad \textit{in } (1,R), \tag{34}$$

for some $c_R \in \mathbb{C}$.

Proof. To prove (34) it is enough to show that

$$\int_1^R \Big(-\upsilon_R'' - \frac{1}{r}\upsilon_R' + \frac{k^2}{4r^2}\upsilon_R \Big) r\eta(r)\, dr = 0, \quad \text{for all } \eta \in C^\infty_{\rm c}(1,R).$$

By (31), it is easy to see that the function $u(x) := e^{-\frac{i}{2}\theta_p(x)} w_R(x)$ is harmonic in $D_R \setminus s_p$, where s_p is defined in (20). Let us consider an arbitrary function $\eta(r) \in C^\infty_{\rm c}(1,R)$ and the function

$$g(t) := \frac{1}{\sqrt{\pi}} e^{\frac{i}{2}(\theta_0^p - \theta_0)(\cos t, \sin t)} \sin(\tfrac{k}{2}t) = \begin{cases} -\frac{1}{\sqrt{\pi}} \sin(\frac{k}{2}t) & t \in [0,\alpha) \\ \frac{1}{\sqrt{\pi}} \sin(\frac{k}{2}t) & t \in [\alpha, 2\pi). \end{cases}$$

Testing equation $-\Delta u = 0$ with $v(r\cos t, r\sin t) = \eta(r)g(t)$ in $D_R \setminus s_p$, integrating by parts and observing that both v and ∇u jump across s_p, we obtain that

$$
\begin{aligned}
0 &= \int_1^R \bigg(\int_0^{2\pi} \Big(r\partial_r u(r\cos t, r\sin t)\eta'(r)g(t) + \frac{\eta(r)}{r} g'(t)\partial_t u(r\cos t, r\sin t)\Big)\, dt\bigg)\, dr \\
&= -\int_1^R \eta(r) \bigg(\int_0^{2\pi} \Big(\partial_r u(r\cos t, r\sin t) + r\partial^2_{rr} u(r\cos t, r\sin t)\Big) g(t)\, dt\bigg)\, dr \\
&\quad + \int_1^R \frac{\eta(r)}{r} \bigg(\int_0^{2\pi} \partial_t u(r\cos t, r\sin t) g'(t)\, dt\bigg)\, dr \\
&= -\int_1^R \Big(\eta(r)\upsilon_R'(r) + r\eta(r)\upsilon_R''(r)\Big)\, dr + \int_1^R \frac{\eta(r)}{r} \bigg(\int_0^{2\pi} \partial_t u(r\cos t, r\sin t) g'(t)\, dt\bigg)\, dr.
\end{aligned}
$$

A further integration by parts yields

$$
\begin{aligned}
\int_0^{2\pi} \partial_t u(r\cos t, r\sin t) g'(t)\, dt &= -\int_0^{2\pi} u(r\cos t, r\sin t) g''(t)\, dt \\
&\quad + g'_-(2\pi) u(r\cos(2\pi^-), r\sin(2\pi^-)) - g'_+(\alpha) u(r\cos(\alpha^+), r\sin(\alpha^+)) \\
&\quad + g'_-(\alpha) u(r\cos(\alpha^-), r\sin(\alpha^-)) - g'_+(0) u(r\cos(0^+), r\sin(0^+)) \\
&= -\int_0^{2\pi} u(r\cos t, r\sin t) g''(t)\, dt = \frac{k^2}{4}\int_0^{2\pi} u(r\cos t, r\sin t) g(t)\, dt = \frac{k^2}{4}\upsilon_R(r)
\end{aligned}
$$

in view of the fact that $g'_+(0) = g'_-(2\pi)$, $g'_+(\alpha) = -g'_-(\alpha)$, and

$$
\lim_{t\to\alpha^+} u(r\cos(t), r\sin(t)) = -\lim_{t\to\alpha^-} u(r\cos(t), r\sin(t)).
$$

The conclusion then follows. □

For $\alpha \in [0, 2\pi)$ and $p = (\cos\alpha, \sin\alpha)$, let us define the following Fourier-type coefficient of the limit profile Ψ_p

$$
\xi_p(r) := \int_0^{2\pi} e^{-\frac{i}{2}\theta_p(r\cos t, r\sin t)} \Psi_p(r\cos t, r\sin t) e^{\frac{i}{2}\theta_0^p(r\cos t, r\sin t)} \overline{\psi_2^k(t)}\, dt, \quad r \geq 1. \tag{35}
$$

Lemma 3.3. *Let $\tilde\kappa_R$ be as in* (32)*. Then*

$$
\lim_{R\to+\infty} \tilde\kappa_R = ik\sqrt{\pi}(\sqrt{\pi} - \xi_p(1)),
$$

where $\xi_p(r)$ is defined in (35)*.*

Proof. The proof follows from integration of (34) arguing as in the proof of [1, Lemma 6.7]. □

Combining the results of Lemmas 3.1 and 3.3 we have that

$$
\lambda_0 - \lambda_a \geq |a|^k k |\beta_2|^2 \sqrt{\pi}\Big(\xi_p(1) - \sqrt{\pi} + o(1)\Big) \tag{36}
$$

as $a = |a|p \to 0$.

3.2. **Blow-up analysis and Rayleigh quotient for λ_0.** Differently from the lower bound for $\lambda_0-\lambda_a$, the upper bound of the eigenvalue variation presents significant new difficulties with respect to the case of poles moving along nodal lines of φ_0 treated in [1, Subsection 6.1]. Indeed, when the direction along which $a\to 0$ is not a nodal line of φ_0 the value $(\xi_p(1)-\sqrt{\pi})$ can have any sign (and vanish along some directions); this does not allow deriving the exact asymptotic behavior of the normalization term in the blow-up analysis from estimates of the Rayleigh quotient from above and below as done in [1]. On the other hand, from [1] we can derive Lemma 1.1 and hence obtain a control on the size of the eigenvalue variation along any direction.

The proof of Lemma 1.1 is based on the following result (see also [5, Lemma 6.6]).

Lemma 3.4. *Let $Q(x_1,x_2)=\sum_{j=0}^{h}c_jx_1^jx_2^{h-j}$ be a homogeneous polynomial in two variables x_1,x_2 of degree at most $h\in\mathbb{N}$. If there exist $\bar\theta\in[0,2\pi)$ and an odd natural number k such that $k>h$ and*

$$Q\Big(\cos\big(\bar\theta+j\tfrac{2\pi}{k}\big),\sin\big(\bar\theta+j\tfrac{2\pi}{k}\big)\Big)=0,\quad \text{for all } j=0,1,\dots,k-1, \tag{37}$$

then $Q\equiv 0$.

Proof. Up to a rotation, it is not restrictive to assume that $\bar\theta=0$. If $x_1\neq 0$, we can write Q as

$$Q(x_1,x_2)=x_1^h\tilde Q\big(\tfrac{x_2}{x_1}\big),\quad \text{where } \tilde Q(t)=\sum_{j=0}^{h}c_jt^{h-j}.$$

Since k is odd, we have that $\cos\big(j\frac{2\pi}{k}\big)\neq 0$ for all $j=0,1,\dots,k-1$. Then, from assumption (37) it follows that $\tilde Q\big(\tan\big(j\frac{2\pi}{k}\big)\big)=0$ for all $j=0,1,\dots,k-1$. Since k is odd, we also have that $\tan\big(j\frac{2\pi}{k}\big)\neq\tan\big(\ell\frac{2\pi}{k}\big)$ for all $j,l\in\{0,1,\dots,k-1\}$ with $j\neq\ell$. Hence $\tilde Q$ has k distinct zeros. Since $\tilde Q$ is a polynomial of degree at most h and $h<k$, from the Fundamental Theorem of Algebra we conclude $\tilde Q\equiv 0$, i.e. $c_j=0$ for all $j=0,1,\dots,k-1$. Hence $Q\equiv 0$. □

Proof of Lemma 1.1. Since the function $a=(a_1,a_2)\mapsto\lambda_0-\lambda_a$ is C^∞ in a neighborhood of 0 (see [5, Theorem 1.3]), it admits a Taylor expansion up to order k of the form

$$\lambda_0-\lambda_a=\sum_{j=1}^{k}P_j(a_1,a_2)+o(|a|^k),\quad \text{as } |a|\to 0,$$

where, for every $j=1,\dots,k$, $P_j(a_1,a_2)$ is either identically zero or a homogeneous polynomial in the two variables a_1,a_2 of degree j. From [1, Theorem 1.2] (see also (7)) we have that, for every $\ell<k$,

$$P_\ell\Big(\cos\big(\alpha_0+j\tfrac{2\pi}{k}\big),\sin\big(\alpha_0+j\tfrac{2\pi}{k}\big)\Big)=0,\quad \text{for all } j=0,1,\dots,k-1,$$

where α_0 is as in (6) (i.e. $\alpha_0+j\frac{2\pi}{k}$, with $j=0,1,\dots,k-1$, identify the directions of the k half-lines tangent to the nodal lines of the eigenfunctions associated to λ_0). The conclusion follows directly from Lemma 3.4. □

From the expansion (12)–(13) in Lemma 1.1 it follows that

$$|\lambda_a-\lambda_0|=O(|a|^k) \tag{38}$$

as $|a|\to 0$ along any direction. Exploiting (38) we can perform a sharp blow-up analysis prior to the estimate from above of the eigenvalue variation $\lambda_0-\lambda_a$.

Let $\alpha \in [0, 2\pi)$ and $p = (\cos\alpha, \sin\alpha)$. Arguing as in [1] we can prove that, for every $\delta \in (0, 1/4)$, there exist $r_\delta, K_\delta > 0$ such that, for all $R \geq K_\delta$,

$$\text{the family of functions } \left\{\tilde\varphi_a : a = |a|p, |a| < \tfrac{r_\delta}{R}\right\} \text{ is bounded in } H^{1,p}(D_R, \mathbb{C}) \tag{39}$$

where

$$\tilde\varphi_a(x) := \frac{\varphi_a(|a|x)}{\sqrt{H_{a,\delta}}}, \tag{40}$$

and

$$H_{a,\delta} := \frac{1}{K_\delta |a|} \int_{\partial D_{K_\delta |a|}} |\varphi_a|^2 \, ds.$$

Furthermore, from [1, Estimates (113) and (114)] we have that

$$H_{a,\delta} \geq C_\delta |a|^{k+2\delta}, \quad \text{if } |a| < \frac{r_\delta}{K_\delta}, \tag{41}$$

for some $C_\delta > 0$ independent of a, and

$$H_{a,\delta} = O(|a|^{1-2\delta}) \quad \text{as } |a| \to 0. \tag{42}$$

We observe that $\tilde\varphi_a$ weakly solves

$$(i\nabla + A_p)^2 \tilde\varphi_a = |a|^2 \lambda_a \tilde\varphi_a, \quad \text{in } \tfrac{1}{|a|}\Omega = \{x \in \mathbb{R}^2 : |a|x \in \Omega\}, \tag{43}$$

and

$$\frac{1}{K_\delta} \int_{\partial D_{K_\delta}} |\tilde\varphi_a|^2 \, ds = 1. \tag{44}$$

Let $R > 2$. For $|a|$ sufficiently small we define the functions $v_{j,R,a}$ as follows:

$$v_{j,R,a} = \begin{cases} v^{ext}_{j,R,a}, & \text{in } \Omega \setminus D_{R|a|}, \\ v^{int}_{j,R,a}, & \text{in } D_{R|a|}, \end{cases} \qquad j = 1, \dots, n_0, \tag{45}$$

where

$$v^{ext}_{j,R,a} := e^{\frac{i}{2}(\theta_0^a - \theta_a)} \varphi_j^a \quad \text{in } \Omega \setminus D_{R|a|},$$

with φ_j^a as in (28)–(30) and θ_a, θ_0^a as in (18) (notice that $e^{\frac{i}{2}(\theta_0^a - \theta_a)}$ is smooth in $\Omega \setminus D_{R|a|}$), so that it solves

$$\begin{cases} (i\nabla + A_0)^2 v^{ext}_{j,R,a} = \lambda_j^a v^{ext}_{j,R,a}, & \text{in } \Omega \setminus D_{R|a|}, \\ v^{ext}_{j,R,a} = e^{\frac{i}{2}(\theta_0^a - \theta_a)} \varphi_j^a & \text{on } \partial(\Omega \setminus D_{R|a|}), \end{cases}$$

whereas $v^{int}_{j,R,a}$ is the unique solution to the problem

$$\begin{cases} (i\nabla + A_0)^2 v^{int}_{j,R,a} = 0, & \text{in } D_{R|a|}, \\ v^{int}_{j,R,a} = e^{\frac{i}{2}(\theta_0^a - \theta_a)} \varphi_j^a, & \text{on } \partial D_{R|a|}. \end{cases}$$

It is easy to verify that $\dim\big(\operatorname{span}\{v_{1,R,a}, \dots, v_{n_0,R,a}\}\big) = n_0$.

For all $R > 2$ and $a = |a|p \in \Omega$ with $|a|$ small, we define

$$Z_a^R(x) := \frac{v^{int}_{n_0,R,a}(|a|x)}{\sqrt{H_{a,\delta}}}. \tag{46}$$

Arguing as in [1, Lemma 6.2] we can prove that, as a consequence of (39) and the Dirichlet principle,

(47) the family of functions $\left\{Z_a^R : a = |a|p, |a| < \frac{r_\delta}{R}\right\}$ is bounded in $H^{1,0}(D_R, \mathbb{C})$.

We also define

$$W_a(x) := \frac{\varphi_0(|a|x)}{|a|^{k/2}}. \tag{48}$$

As in [1], under assumption (16) and letting k as in (4), from [7, Theorem 1.3 and Lemma 6.1] we have that

$$W_a \to \frac{\beta_2}{\sqrt{\pi}} e^{\frac{i}{2}\theta_0} \psi_k \quad \text{as } |a| \to 0 \tag{49}$$

in $H^{1,0}(D_R, \mathbb{C})$ for every $R > 1$, with β_2 as in (5).

Theorem 3.5. *For every $R > 2$,*

$$\|v_{n_0,R,a} - \varphi_0\|_{H_0^{1,0}(\Omega,\mathbb{C})} = O\Big(\sqrt{H_{a,\delta}}\Big) \quad \textit{as } a = |a|p \to 0.$$

Proof. Let $R > 2$. We first notice that $v_{n_0,R,a} \to \varphi_0$ in $H_0^{1,0}(\Omega, \mathbb{C})$ as $|a| \to 0^+$. Indeed

$$\begin{aligned}\int_\Omega \big|(i\nabla + A_0)(v_{n_0,R,a} - \varphi_0)\big|^2 dx &= \int_\Omega |e^{\frac{i}{2}(\theta_0^a - \theta_a)}(i\nabla + A_a)\varphi_a - (i\nabla + A_0)\varphi_0|^2 dx \\ &+ \int_{D_R} \Big|\sqrt{H_{a,\delta}}(i\nabla + A_0)Z_a^R - |a|^{k/2}(i\nabla + A_0)W_a\Big)\Big|^2 dx \\ &- \int_{D_R} \Big|\sqrt{H_{a,\delta}} e^{\frac{i}{2}(\theta_0^p - \theta_p)}(i\nabla + A_p)\tilde{\varphi}_a - |a|^{k/2}(i\nabla + A_0)W_a\Big)\Big|^2 dx = o(1)\end{aligned}$$

in view of (25), (39), (47), (49) and (42).

From [1, Lemma 7.1] the function

$$\begin{aligned} F : \mathbb{C} \times H_0^{1,0}(\Omega, \mathbb{C}) &\longrightarrow \mathbb{R} \times \mathbb{R} \times (H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C}))^\star \\ (\lambda, \varphi) &\longmapsto \Big(\|\varphi\|^2_{H_0^{1,0}(\Omega,\mathbb{C})} - \lambda_0,\ \mathfrak{Im}\big(\textstyle\int_\Omega \varphi\overline{\varphi_0}\,dx\big),\ (i\nabla + A_0)^2\varphi - \lambda\varphi\Big)\end{aligned}$$

is Fréchet-differentiable at (λ_0, φ_0) and its Fréchet-differential $dF(\lambda_0, \varphi_0)$ is invertible. In the above definition, $(H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C}))^\star$ is the real dual space of $H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C}) = H_0^{1,0}(\Omega, \mathbb{C})$, which is here meant as a vector space over $\mathbb{R}$ endowed with the norm

$$\|u\|_{H_0^{1,0}(\Omega,\mathbb{C})} = \Big(\int_\Omega \big|(i\nabla + A_0)u\big|^2 dx\Big)^{1/2}.$$

Therefore

$$\begin{aligned}&|\lambda_a - \lambda_0| + \|v_{n_0,R,a} - \varphi_0\|_{H_0^{1,0}(\Omega,\mathbb{C})} \\ &\leq \|(dF(\lambda_0, \varphi_0))^{-1}\|_{\mathcal{L}(\mathbb{R}\times\mathbb{R}\times(H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^\star, \mathbb{C}\times H_0^{1,0}(\Omega,\mathbb{C}))} \|F(\lambda_a, v_{n_0,R,a})\|_{\mathbb{R}\times\mathbb{R}\times(H_{0,\mathbb{R}}^{1,0}(\Omega))^\star}(1 + o(1))\end{aligned}$$

as $|a| \to 0^+$. To prove the theorem it is then enough to estimate the norm of

$$\begin{aligned}F(\lambda_a, v_{n_0,R,a}) &= (\alpha_a, \beta_a, w_a) \\ &= \Big(\|v_{n_0,R,a}\|^2_{H_0^{1,0}(\Omega,\mathbb{C})} - \lambda_0, \mathfrak{Im}\left(\textstyle\int_\Omega v_{n_0,R,a}\overline{\varphi_0}\,dx\right), (i\nabla + A_0)^2 v_{n_0,R,a} - \lambda_a v_{n_0,R,a}\Big)\end{aligned}$$

in $\mathbb{R} \times \mathbb{R} \times (H^{1,0}_{0,\mathbb{R}}(\Omega))^\star$. The estimates of β_a and w_a can be performed as in [1, Proof of Theorem 7.2] obtaining that

$$\beta_a = o\big(\sqrt{H_{a,\delta}}\big) \quad \text{and} \quad \|w_a\|_{(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star} = O\big(\sqrt{H_{a,\delta}}\big),$$

as $a = |a|p \to 0$. As far as α_a is concerned, differently from [1], the estimate of [1, Proposition 6.10] which, in the case $\alpha = 0$, implied that $|\lambda_a - \lambda_0| = O(H_{a,\delta})$, is not available after preliminary estimates of the Rayleigh quotient for generic values of α since $(\xi_p(1) - \sqrt{\pi})$ can have any sign. This difficulty can be overcome by observing that (41) and (38) imply that $|\lambda_a - \lambda_0| = O(|a|^{\frac{k}{2}-\delta}\sqrt{H_{a,\delta}})$ and then

$$|\lambda_a - \lambda_0| = o(\sqrt{H_{a,\delta}}),$$

as $a = |a|p \to 0$. Then, from (39) and (47), we obtain that

$$\begin{aligned}
\alpha_a &= \left(\int_{D_{R|a|}} |(i\nabla + A_0)v^{int}_{n_0,R,a}|^2\,dx - \int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\,dx\right) + (\lambda_a - \lambda_0)\\
&= H_{a,\delta}\left(\int_{D_R} |(i\nabla + A_0)Z^R_a|^2\,dx - \int_{D_R} |(i\nabla + A_p)\tilde\varphi_a|^2\,dx\right) + (\lambda_a - \lambda_0)\\
&= o(\sqrt{H_{a,\delta}}),
\end{aligned}$$

as $a = |a|p \to 0$, thus concluding the proof. □

Via Theorem 3.5 and a change of variables, it follows that, letting $\alpha \in [0, 2\pi)$, $p = (\cos\alpha, \sin\alpha)$, and $R > 1$,

$$\int_{\left(\frac{1}{|a|}\Omega\right)\setminus D_R} \left|(i\nabla + A_p)\Big(\tilde\varphi_a(x) - e^{\frac{i}{2}(\theta_p - \theta_0^p)}\frac{|a|^{k/2}}{\sqrt{H_{a,\delta}}}W_a\Big)\right|^2 dx = O(1), \quad \text{as } a = |a|p \to 0. \tag{50}$$

Theorem 3.6. *For $\alpha \in [0, 2\pi)$, $p = (\cos\alpha, \sin\alpha)$ and $a = |a|p \in \Omega$, let $\varphi_a \in H^{1,a}_0(\Omega,\mathbb{C})$ solve (23-24) and $\varphi_0 \in H^{1,0}_0(\Omega,\mathbb{C})$ be a solution to* (3) *satisfying* (1), (4), *and* (16). *Let $\tilde\varphi_a$ and K_δ be as in* (40) *and Ψ_p be as in Proposition 2.1. Then*

$$\lim_{|a|\to 0^+} \frac{|a|^{k/2}}{\sqrt{H_{a,\delta}}} = \frac{1}{|\beta_2|}\sqrt{\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2 ds}} \tag{51}$$

and

$$\tilde\varphi_a \to \frac{\beta_2}{|\beta_2|}\sqrt{\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_p|^2 ds}}\ \Psi_p \quad \text{as } a = |a|p \to 0, \tag{52}$$

in $H^{1,p}(D_R,\mathbb{C})$ for every $R > 1$, almost everywhere and in $C^2_{\rm loc}(\mathbb{R}^2 \setminus \{p\},\mathbb{C})$.

Proof. **Step 1.** We first prove that for every sequence $a_n = |a_n|p$ with $|a_n| \to 0$, there exist $\tilde\Phi \in H^{1,p}_{\rm loc}(\mathbb{R}^2,\mathbb{C})$, $\tilde\Phi \not\equiv 0$, and a subsequence a_{n_ℓ} such that $\tilde\varphi_{a_{n_\ell}} \to \tilde\Phi$ in $H^{1,p}(D_R,\mathbb{C})$ for every $R > 1$, almost everywhere and in $C^2_{\rm loc}(\mathbb{R}^2 \setminus \{p\},\mathbb{C})$ and $\tilde\Phi$ weakly solves

$$(i\nabla + A_p)^2\tilde\Phi = 0, \quad \text{in } \mathbb{R}^2. \tag{53}$$

To prove it, we observe that from (39) it follows that, for every sequence $a_n = |a_n|p$ with $|a_n| \to 0$, by a diagonal process there exists $\tilde\Phi \in H^{1,p}_{\rm loc}(\mathbb{R}^2,\mathbb{C})$, and a subsequence a_{n_ℓ} such that $\tilde\varphi_{a_{n_\ell}} \rightharpoonup \tilde\Phi$ weakly in $H^{1,p}(D_R,\mathbb{C})$ for every $R > 1$ and almost everywhere. $\tilde\Phi \not\equiv 0$

since $\frac{1}{K_\delta}\int_{\partial D_{K_\delta}} |\tilde\Phi|^2\,ds = 1$ thanks to (44) and the compactness of the trace embedding $H^{1,p}(D_{K_\delta},\mathbb{C}) \hookrightarrow L^2(\partial D_{K_\delta},\mathbb{C})$.

Passing to the limit in (43), we have that $\tilde\Phi$ weakly solves (53), whereas, arguing as in the proof of [1, Theorem 8.1], we can prove that the convergence of the subsequence $\tilde\varphi_{a_{n_\ell}}$ to $\tilde\Phi$ is actually strong in $H^{1,p}(D_R,\mathbb{C})$ for every $R>1$. The convergence in $C^2_{\rm loc}(\mathbb{R}^2\setminus\{p\},\mathbb{C})$ follows easily from classical elliptic estimates.

Step 2. We claim that, for every sequence $a_n = |a_n|p$ with $|a_n|\to 0$, there exists a subsequence a_{n_ℓ} such that

$$\lim_{\ell\to+\infty}\frac{|a_{n_\ell}|^{k/2}}{\sqrt{H_{a_{n_\ell},\delta}}}\quad\text{is finite and strictly positive.}$$

To prove the claim, we argue by contradiction, assuming that

(i) either there exists a sequence $a_n = |a_n|p$ with $|a_n|\to 0$ such that $\lim_{n\to+\infty}\frac{|a_n|^{k/2}}{\sqrt{H_{a_n,\delta}}} = 0$

(ii) or there exists a sequence $a_n = |a_n|p$ with $|a_n|\to 0$ such that $\lim_{n\to+\infty}\frac{|a_n|^{k/2}}{\sqrt{H_{a_n,\delta}}} = +\infty$.

If (i) holds, then, by step 1, along a subsequence, $\tilde\varphi_{a_{n_\ell}}\to\tilde\Phi$ in $H^{1,p}(D_R,\mathbb{C})$ for every $R>1$, for some $\tilde\Phi\not\equiv 0$ weakly solving (53). Then from (49), passing to the limit in (50) we would obtain that

$$\int_{\mathbb{R}^2\setminus D_R}|(i\nabla + A_p)\tilde\Phi(x)|^2dx < +\infty,$$

contradicting the fact that $\tilde\Phi\not\equiv 0$ is a non trivial weak solution to (53) (and so cannot have finite energy otherwise by testing the equation we would get that $\tilde\Phi\equiv 0$, see [1, Proof of Proposition 4.3]). Hence case (i) cannot occur.

If (ii) holds, then from (50) we would have, for all $R>2$

$$\frac{|a|^k}{H_{a,\delta}}\int_{D_{2R}\setminus D_R}\left|(i\nabla + A_p)\Big(\tfrac{\sqrt{H_{a,\delta}}}{|a|^{k/2}}\tilde\varphi_a(x) - e^{\frac{i}{2}(\theta_p-\theta_0^p)}W_a\Big)\right|^2dx = O(1),\quad\text{as } a=|a|p\to 0,$$

and hence, in view of (49) and (39), passing to the limit along the sequence we would obtain that

$$\begin{aligned}\frac{|a_n|^k}{H_{a_n,\delta}}&\bigg(\int_{D_{2R}\setminus D_R}\left|(i\nabla + A_p)\Big(e^{\frac{i}{2}(\theta_p-\theta_0^p)}\beta_2 e^{\frac{i}{2}\theta_0}\psi_k\Big)\right|^2dx + o(1)\bigg)\\ &= \frac{|a_n|^k}{H_{a_n,\delta}}\bigg(|\beta_2|^2\int_{D_{2R}\setminus D_R}|\nabla\psi_k|^2dx + o(1)\bigg) = O(1),\quad\text{as } n\to+\infty,\end{aligned}$$

which is not possible if $\lim_{\ell\to+\infty}\frac{|a_{n_\ell}|^{k/2}}{\sqrt{H_{a_{n_\ell},\delta}}} = +\infty$ as in case (ii), since $\int_{D_{2R}\setminus D_R}|\nabla\psi_k|^2dx > 0$. Hence also case (ii) cannot occur and the claim of step 2 is proved.

Step 3. From steps 1 and 2, it follows that, for every sequence $a_n = (|a_n|,0) = |a_n|p$ with $|a_n|\to 0$, there exist $c\in(0,+\infty)$, $\tilde\Phi\in H^{1,p}_{\rm loc}(\mathbb{R}^2,\mathbb{C})$ weakly solving (53), $\tilde\Phi\not\equiv 0$, and a subsequence a_{n_ℓ} such that $\lim_{\ell\to+\infty}\frac{|a_{n_\ell}|^{k/2}}{\sqrt{H_{a_{n_\ell},\delta}}} = c$ and $\tilde\varphi_{a_{n_\ell}}\to\tilde\Phi$ in $H^{1,p}(D_R,\mathbb{C})$ for every

$R>1$ and in $C^2_{\rm loc}(\mathbb{R}^2\setminus\{p\},\mathbb{C})$. Passing to the limit along a_{n_ℓ} in (50) and recalling (49), we obtain that, for every $R>2$,

$$\int_{\mathbb{R}^2\setminus D_R}\left|(i\nabla+A_p)\Big(\tilde\Phi(x)-c\beta_2 e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}\psi_k\Big)\right|^2dx<+\infty.$$

Hence from Proposition 2.1 we conclude that necessarily

$$\tilde\Phi=c\beta_2\Psi_p. \tag{54}$$

Since $\frac{1}{K_\delta}\int_{\partial D_{K_\delta}}|\tilde\Phi|^2\,ds=1$, from (54) and the fact that c is a positive real number, it follows that $c=\frac{1}{|\beta_2|}\big(\frac{K_\delta}{\int_{\partial D_{K_\delta}}|\Psi_p|^2ds}\big)^{1/2}$. Hence we have that

$$\tilde\varphi_{a_{n_\ell}}\to\frac{\beta_2}{|\beta_2|}\sqrt{\frac{K_\delta}{\int_{\partial D_{K_\delta}}|\Psi_p|^2ds}}\Psi_p\quad\text{in } H^{1,p}(D_R,\mathbb{C})\text{ for every } R>1\text{ and in } C^2_{\rm loc}(\mathbb{R}^2\setminus\{p\},\mathbb{C}),$$

and

$$\frac{|a_{n_\ell}|^{k/2}}{\sqrt{H_{a_{n_\ell},\delta}}}\to\frac{1}{|\beta_2|}\sqrt{\frac{K_\delta}{\int_{\partial D_{K_\delta}}|\Psi_p|^2ds}}.$$

Since the above limits depend neither on the sequence $\{a_n\}_n$ nor on the subsequence $\{a_{n_\ell}\}_\ell$, we conclude that the above convergences hold as $|a|\to0^+$, thus proving (51) and (52). □

Remark 3.7. Combining (51) and (52) we deduce that

$$\frac{\varphi_a(|a|x)}{|a|^{k/2}}\to\beta_2\Psi_p\quad\text{as } a=|a|p\to0,$$

in $H^{1,p}(D_R,\mathbb{C})$ for every $R>1$ and in $C^2_{\rm loc}(\mathbb{R}^2\setminus\{p\},\mathbb{C})$. Furthermore, arguing as in [1, Lemma 8.3], from Theorem 3.6 we can deduce that, letting Z_a^R as is (46),

$$Z_a^R\to\frac{\beta_2}{|\beta_2|}\sqrt{\frac{K_\delta}{\int_{\partial D_{K_\delta}}|\Psi_p|^2ds}}\,z_R\quad\text{as } a=|a|p\to0,$$

in $H^{1,0}(D_R,\mathbb{C})$ for every $R>2$, where z_R is the unique solution to

$$\begin{cases}(i\nabla+A_0)^2z_R=0, & \text{in } D_R,\\ z_R=e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p, & \text{on } \partial D_R.\end{cases}$$

Thanks to the convergences of blow-up sequences established in Theorem 3.6 and Remark 3.7, we can now follow closely the arguments of [1, Subsection 6.1, Lemma 9.1] thus obtaining the following upper bound for the difference $\lambda_0-\lambda_a$.

Lemma 3.8. *For $\alpha\in[0,2\pi)$ and $a=|a|(\cos\alpha,\sin\alpha)\in\Omega$, let $\lambda_a\in\mathbb{R}$ and $\varphi_a\in H^{1,a}_0(\Omega,\mathbb{C})$ solve (23-24) and $\lambda_0\in\mathbb{R}$ and $\varphi_0\in H^{1,0}_0(\Omega,\mathbb{C})$ solve* (3). *If* (1) *and* (4) *hold and* (16) *is satisfied, then, for $a=|a|(\cos\alpha,\sin\alpha)$ and $p=(\cos\alpha,\sin\alpha)$,*

$$\limsup_{|a|\to0}\frac{\lambda_0-\lambda_a}{|a|^k}\le|\beta_2|^2\,k\sqrt{\pi}(\xi_p(1)-\sqrt{\pi}),$$

with $\xi_p(r)$ defined in (35).

Collecting (36) and Lemma 3.8 we can state the following result.

Proposition 3.9. *For $\alpha \in [0, 2\pi)$ and $a = |a|(\cos\alpha, \sin\alpha) \in \Omega$, let $\varphi_a \in H^{1,a}_0(\Omega, \mathbb{C})$ and $\lambda_a \in \mathbb{R}$ solve (23-24) and $\lambda_0 \in \mathbb{R}$ and $\varphi_0 \in H^{1,0}_0(\Omega, \mathbb{C})$ solve* (3)*. If* (1) *and* (4) *hold and* (16) *is satisfied, then, for $a = |a|(\cos\alpha, \sin\alpha)$,*

$$\lim_{|a|\to 0} \frac{\lambda_0 - \lambda_a}{|a|^k} = |\beta_2|^2 \; k\sqrt{\pi} f(\alpha),$$

where

$$f : [0, 2\pi) \to \mathbb{R}, \quad f(\alpha) = (\xi_p(1) - \sqrt{\pi}), \quad p = (\cos\alpha, \sin\alpha), \tag{55}$$

with $\xi_p(r)$ defined in (35)*.*

4. Properties of $f(\alpha)$

To prove our main result, we are going to investigate two suitable symmetry properties of the function $f(\alpha)$. Let us define two transformations $\mathcal{R}_1, \mathcal{R}_2$ acting on a general point

$$x = (x_1, x_2) = (r\cos t, r\sin t), \quad r > 0, \; t \in [0, 2\pi),$$

as

$$\mathcal{R}_1(x) = \mathcal{R}_1(x_1, x_2) = M_k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad M_k = \begin{pmatrix} \cos\frac{2\pi}{k} & -\sin\frac{2\pi}{k} \\ \sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} \end{pmatrix} \tag{56}$$

i.e.

$$\mathcal{R}_1(r\cos t, r\sin t) = \Big(r\cos(t + \tfrac{2\pi}{k}), r\sin(t + \tfrac{2\pi}{k})\Big),$$

and

$$\mathcal{R}_2(x) = \mathcal{R}_2(x_1, x_2) = (x_1, -x_2), \tag{57}$$

i.e.

$$\mathcal{R}_2(r\cos t, r\sin t) = (r\cos(2\pi - t), r\sin(2\pi - t)),$$

The transformation $\mathcal{R}_1$ is a rotation of $\frac{2\pi}{k}$ and $\mathcal{R}_2$ is a reflexion through the x_1-axis.

We would like to study how the coefficient $\xi_p(1)$ (see (35)) changes when the above trasformations act on p. In particular, we are going to prove that such a quantity $\xi_p(1)$ is invariant under the transformations $\mathcal{R}_1, \mathcal{R}_2$.

In order to obtain such an invariance, we first study the relation between the limit profiles $\Psi_p(\mathcal{R}_j(x))$ and $\Psi_{\mathcal{R}_j^{-1}(p)}(x)$, $j = 1, 2$.

Lemma 4.1. *For $p = (\cos\alpha, \sin\alpha)$, $\alpha \in [0, 2\pi)$, let Ψ_p be the limit profile introduced in Proposition 2.1 and let $\mathcal{R}_1, \mathcal{R}_2$ be the transformations introduced in* (56) *and* (57)*. Then*

$$\Psi_{\mathcal{R}_1^{-1}(p)} = -e^{-i\frac{\pi}{k}} \big(\Psi_p \circ \mathcal{R}_1\big) \tag{58}$$

and

$$\Psi_{\mathcal{R}_2(p)} = -e^{i\theta_{\mathcal{R}_2(p)}} \big(\Psi_p \circ \mathcal{R}_2\big). \tag{59}$$

Proof. In order to prove (58), we observe that, by direct calculations,

$$\big(A_p \circ \mathcal{R}_1\big)(x) = A_{\mathcal{R}_1^{-1}(p)}(x) M_k^{-1}, \tag{60}$$

$$e^{\frac{i}{2}(\theta_0 \circ \mathcal{R}_1)}(\psi_k \circ \mathcal{R}_1) = -e^{i\frac{\pi}{k}} e^{\frac{i}{2}\theta_0} \psi_k, \tag{61}$$

$$e^{\frac{i}{2}(\theta_0 \circ \mathcal{R}_1)(x)} \nabla\psi_k(\mathcal{R}_1(x)) = -e^{i\frac{\pi}{k}} e^{\frac{i}{2}\theta_0(x)} \nabla\psi_k(x) M_k^{-1}. \tag{62}$$

Furthermore

$$\theta_p(\mathcal{R}_1(x)) = \begin{cases} \theta_{\mathcal{R}_1^{-1}(p)}(x) + \frac{2\pi}{k}, & \text{if } \alpha \in \big[\frac{2\pi}{k}, 2\pi\big), \\ \theta_{\mathcal{R}_1^{-1}(p)}(x) + \frac{2\pi}{k} - 2\pi, & \text{if } \alpha \in \big[0, \frac{2\pi}{k}\big), \end{cases}$$

and

$$\theta_0^p(\mathcal{R}_1(x)) = \begin{cases} \theta_0^{\mathcal{R}_1^{-1}(p)}(x) + \frac{2\pi}{k}, & \text{if } \alpha \in \big[\frac{2\pi}{k}, 2\pi\big), \\ \theta_0^{\mathcal{R}_1^{-1}(p)}(x) + \frac{2\pi}{k} - 2\pi, & \text{if } \alpha \in \big[0, \frac{2\pi}{k}\big), \end{cases}$$

so that

$$\theta_{\mathcal{R}_1^{-1}(p)} - \theta_0^{\mathcal{R}_1^{-1}(p)} = \theta_p \circ \mathcal{R}_1 - \theta_0^p \circ \mathcal{R}_1. \tag{63}$$

Let us denote $\widetilde{\Psi}_p(y) = \Psi_p(\mathcal{R}_1(y))$. By direct calculations we have that, since Ψ_p weakly solves the equation $(i\nabla + A_p)^2\Psi_p = 0$, the function $\widetilde{\Psi}_p$ solves $(i\nabla + (A_p \circ \mathcal{R}_1)M_k)^2\widetilde{\Psi}_p = 0$ and hence, in view of (60),

$$(i\nabla + A_{\mathcal{R}_1^{-1}(p)})^2\widetilde{\Psi}_p = 0, \quad \text{in } \mathbb{R}^2 \text{ in a weak } H^{1,\mathcal{R}_1^{-1}(p)}\text{-sense.} \tag{64}$$

Passing to the limit in (50) and taking into account (49) and Theorem 3.6, we obtain that, for all $R > 1$,

$$\begin{aligned} &\int_{\mathbb{R}^2\setminus D_R} \left|(i\nabla + A_p)\Big(\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p + \theta_0)}\psi_k\Big)\right|^2 dx \\ &\qquad = \int_{\mathbb{R}^2\setminus D_R} \left|(i\nabla + A_p)\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p + \theta_0)} i\nabla\psi_k\right|^2 dx < +\infty. \end{aligned} \tag{65}$$

By the change of variable $x = \mathcal{R}_1(y)$ in the above integral, using (60), (62), and (63) we obtain that

$$\begin{aligned} &\int_{\mathbb{R}^2\setminus D_R} \left|(i\nabla + A_p)\Psi_p(x) - e^{\frac{i}{2}(\theta_p - \theta_0^p + \theta_0)(x)} i\nabla\psi_k(x)\right|^2 dx \\ &= \int_{\mathbb{R}^2\setminus D_R} \left|(i\nabla + A_{\mathcal{R}_1^{-1}(p)})\widetilde{\Psi}_p(y) + e^{\frac{i}{k}\pi} e^{\frac{i}{2}\left(\theta_{\mathcal{R}_1^{-1}(p)} - \theta_0^{\mathcal{R}_1^{-1}(p)} + \theta_0\right)(y)} i\nabla\psi_k(y)\right|^2 dy \\ &= \int_{\mathbb{R}^2\setminus D_R} \left|(i\nabla + A_{\mathcal{R}_1^{-1}(p)})\big(-e^{-\frac{i}{k}\pi}\widetilde{\Psi}_p\big)(y) - e^{\frac{i}{2}\left(\theta_{\mathcal{R}_1^{-1}(p)} - \theta_0^{\mathcal{R}_1^{-1}(p)} + \theta_0\right)(y)} i\nabla\psi_k(y)\right|^2 dy < +\infty. \end{aligned} \tag{66}$$

From (64), (66) and Proposition 2.1 we conclude that

$$-e^{-\frac{i}{k}\pi}\widetilde{\Psi}_p = \Psi_{\mathcal{R}_1^{-1}(p)}$$

thus proving (58).

To prove (59), we first observe that direct calculations yield

$$\big(A_{\mathcal{R}_2(p)} \circ \mathcal{R}_2\big)M^{-1} = -A_p, \quad \text{where } M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{67}$$

$$\psi_k \circ \mathcal{R}_2 = \psi_k, \quad \nabla\psi_k(\mathcal{R}_2(x)) = \nabla\psi_k(x)M^{-1}. \tag{68}$$

Moreover

$$\theta_p(\mathcal{R}_2(x)) = \begin{cases} 4\pi - \theta_{\mathcal{R}_2(p)}(x), & \text{if } \theta_p(\mathcal{R}_2(x)) \in \big(\alpha, \alpha+2\pi\big), \\ 2\pi - \theta_{\mathcal{R}_2(p)}(x), & \text{if } \theta_p(\mathcal{R}_2(x)) = \alpha, \end{cases} \qquad \text{if } \alpha \in (0, 2\pi),$$

$$\theta_p(\mathcal{R}_2(x)) = \begin{cases} 2\pi - \theta_{\mathcal{R}_2(p)}(x) = 2\pi - \theta_p(x), & \text{if } \theta_p(\mathcal{R}_2(x)) \in \big(0, 2\pi\big), \\ -\theta_{\mathcal{R}_2(p)}(x) = 0, & \text{if } \theta_p(\mathcal{R}_2(x)) = 0, \end{cases} \qquad \text{if } \alpha = 0,$$

and

$$\theta_0^p(\mathcal{R}_2(x)) = \begin{cases} 4\pi - \theta_0^{\mathcal{R}_2(p)}(x), & \text{if } \theta_0^p(\mathcal{R}_2(x)) \in \big(\alpha, \alpha+2\pi\big), \\ 2\pi - \theta_0^{\mathcal{R}_2(p)}(x), & \text{if } \theta_0^p(\mathcal{R}_2(x)) = \alpha, \end{cases} \qquad \text{if } \alpha \in (0, 2\pi),$$

$$\theta_0^p(\mathcal{R}_2(x)) = \theta_0(\mathcal{R}_2(x)) = \begin{cases} 2\pi - \theta_0(x), & \text{if } \theta_0(x) \in \big(0, 2\pi\big), \\ -\theta_0(x) = 0, & \text{if } \theta_0(x) = 0, \end{cases} \qquad \text{if } \alpha = 0,$$

so that

$$\theta_0^{\mathcal{R}_2(p)} - \theta_{\mathcal{R}_2(p)} = \theta_p \circ \mathcal{R}_2 - \theta_0^p \circ \mathcal{R}_2, \quad \text{in } \mathbb{R}^2 \setminus \{tp : \, t \in [0,1]\}, \tag{69}$$

and

$$e^{\frac{i}{2}\theta_0(\mathcal{R}_2(y))} = -e^{-\frac{i}{2}\theta_0(y)}, \quad \text{in } \mathbb{R}^2 \setminus \{(x_1, 0) : x_1 \geq 0\}. \tag{70}$$

Let us denote $\widehat{\Psi}_p(y) = -e^{i\theta_{\mathcal{R}_2(p)}}\Psi_p(\mathcal{R}_2(y))$. In view of (67), it is easy to verify that $\widehat{\Psi}_p$ solves

$$(i\nabla + A_{\mathcal{R}_2(p)})^2 \widehat{\Psi}_p = 0, \quad \text{in } \mathbb{R}^2 \text{ in a weak } H^{1,\mathcal{R}_2(p)}\text{-sense.} \tag{71}$$

By the change of variable $x = \mathcal{R}_2(y)$ in the integral (65), using (67), (68), (69), and (70) and observing that, by (22), $e^{-i\left(\theta_0^{\mathcal{R}_2(p)} - \theta_0\right)} \equiv 1$ in $\mathbb{R}^2 \setminus D_R$, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus D_R} \left| (i\nabla + A_p)\Psi_p(x) - e^{\frac{i}{2}(\theta_p - \theta_0^p + \theta_0)(x)} i\nabla\psi_k(x) \right|^2 dx \\
&= \int_{\mathbb{R}^2\setminus D_R} \left| (i\nabla + (A_p \circ \mathcal{R}_2)M)(-\Psi_p \circ \mathcal{R}_2)(y) - e^{-\frac{i}{2}\left(\theta_{\mathcal{R}_2(p)} - \theta_0^{\mathcal{R}_2(p)} + \theta_0\right)(y)} i\nabla\psi_k(y) \right|^2 dy \\
&= \int_{\mathbb{R}^2\setminus D_R} \left| (i\nabla - A_{\mathcal{R}_2(p)})(-\Psi_p \circ \mathcal{R}_2)(y) - e^{-\frac{i}{2}\left(\theta_{\mathcal{R}_2(p)} - \theta_0^{\mathcal{R}_2(p)} + \theta_0\right)(y)} i\nabla\psi_k(y) \right|^2 dy \\
&= \int_{\mathbb{R}^2\setminus D_R} \left| e^{i\left(\theta_{\mathcal{R}_2(p)} - \theta_0^{\mathcal{R}_2(p)} + \theta_0\right)} (i\nabla - A_{\mathcal{R}_2(p)})(-\Psi_p \circ \mathcal{R}_2) - e^{\frac{i}{2}\left(\theta_{\mathcal{R}_2(p)} - \theta_0^{\mathcal{R}_2(p)} + \theta_0\right)} i\nabla\psi_k \right|^2 dy \\
&= \int_{\mathbb{R}^2\setminus D_R} \left| e^{i\theta_{\mathcal{R}_2(p)}} (i\nabla - A_{\mathcal{R}_2(p)})(-\Psi_p \circ \mathcal{R}_2) - e^{\frac{i}{2}\left(\theta_{\mathcal{R}_2(p)} - \theta_0^{\mathcal{R}_2(p)} + \theta_0\right)} i\nabla\psi_k \right|^2 dy \\
&= \int_{\mathbb{R}^2\setminus D_R} \left| (i\nabla + A_{\mathcal{R}_2(p)})\widehat{\Psi}_p - e^{\frac{i}{2}\left(\theta_{\mathcal{R}_2(p)} - \theta_0^{\mathcal{R}_2(p)} + \theta_0\right)} i\nabla\psi_k \right|^2 dy.
\end{aligned} \tag{72}$$

From (71), (72) and Proposition 2.1 we conclude that

$$\widehat{\Psi}_p = \Psi_{\mathcal{R}_2(p)}$$

thus proving (59). □

We are now in position to prove invariance properties of the function $p \mapsto \xi_p(1)$ under the transformations (56) and (57).

Lemma 4.2. *Let $\mathcal{R}_1, \mathcal{R}_2$ be the transformations introduced in* (56)-(57)*, $\alpha \in [0, 2\pi)$, and $p = (\cos\alpha, \sin\alpha)$. Then*

$$\xi_{\mathcal{R}_1^{-1}(p)}(1) = \xi_p(1) \tag{73}$$

and

$$\xi_{\mathcal{R}_2(p)}(1) = \xi_p(1), \tag{74}$$

where ξ_p is defined in (35)*.*

Proof. We first notice that, from (17),

$$\psi_2^k\Big(s + \frac{2\pi}{k}\Big) = -e^{i\frac{\pi}{k}}\psi_2^k(s), \quad \text{for all } s \in \mathbb{R}, \tag{75}$$

and

$$\psi_2^k(2\pi - s) = -e^{-is}\psi_2^k(s), \quad \text{for all } s \in \mathbb{R}. \tag{76}$$

By the change of variable $t = s + \frac{2\pi}{k}$ in the integral defining $\xi_p(1)$, from (75), (63) and (58) we obtain that

$$\begin{aligned}
\xi_p(1) &= \int_0^{2\pi} e^{-\frac{i}{2}(\theta_p - \theta_0^p)(\cos t, \sin t)}\Psi_p(\cos t, \sin t)\overline{\psi_2^k(t)}\, dt \\
&= -e^{-i\frac{\pi}{k}} \int_0^{2\pi} e^{-\frac{i}{2}(\theta_p - \theta_0^p)(\mathcal{R}_1(\cos s, \sin s))}\Psi_p(\mathcal{R}_1(\cos s, \sin s))\overline{\psi_2^k(s)}\, ds \\
&= \int_0^{2\pi} e^{-\frac{i}{2}\left(\theta_{\mathcal{R}_1^{-1}(p)} - \theta_0^{\mathcal{R}_1^{-1}(p)}\right)(\cos s, \sin s)}\Psi_{\mathcal{R}_1^{-1}(p)}(\cos s, \sin s)\overline{\psi_2^k(s)}\, ds \\
&= \xi_{\mathcal{R}_1^{-1}(p)}(1),
\end{aligned}$$

thus proving (73).

By the change of variable $t = 2\pi - s$ in the integral defining $\xi_p(1)$, from (76), (69), (59), and (22) we obtain that

$$\begin{aligned}
\xi_p(1) &= \int_0^{2\pi} e^{-\frac{i}{2}(\theta_p - \theta_0^p)(\cos t, \sin t)}\Psi_p(\cos t, \sin t)\overline{\psi_2^k(t)}\, dt \\
&= -\int_0^{2\pi} e^{-\frac{i}{2}(\theta_p - \theta_0^p)(\mathcal{R}_2(\cos s, \sin s))}\Psi_p(\mathcal{R}_2(\cos s, \sin s))e^{i\theta_0(\cos s, \sin s)}\overline{\psi_2^k(s)}\, ds \\
&= \int_0^{2\pi} e^{-\frac{i}{2}\left(\theta_0^{\mathcal{R}_2(p)} - \theta_{\mathcal{R}_2(p)}\right)(\cos s, \sin s)}e^{-i\theta_{\mathcal{R}_2(p)}(\cos s, \sin s)}\Psi_{\mathcal{R}_2(p)}(\cos s, \sin s)e^{i\theta_0(\cos s, \sin s)}\overline{\psi_2^k(s)}\, ds \\
&= \int_0^{2\pi} e^{-\frac{i}{2}\left(\theta_{\mathcal{R}_2(p)} - \theta_0^{\mathcal{R}_2(p)}\right)(\cos s, \sin s)}e^{-i\left(\theta_0^{\mathcal{R}_2(p)} - \theta_0\right)(\cos s, \sin s)}\Psi_{\mathcal{R}_2(p)}(\cos s, \sin s)\overline{\psi_2^k(s)}\, ds \\
&= \int_0^{2\pi} e^{-\frac{i}{2}\left(\theta_{\mathcal{R}_2(p)} - \theta_0^{\mathcal{R}_2(p)}\right)(\cos s, \sin s)}\Psi_{\mathcal{R}_2(p)}(\cos s, \sin s)\overline{\psi_2^k(s)}\, ds \\
&= \xi_{\mathcal{R}_2(p)}(1),
\end{aligned}$$

thus proving (74). □

Let f be the 2π-periodic extension of the function introduced in (55), i.e.

$$f(\alpha) = \xi_{(\cos\alpha,\sin\alpha)}(1) - \sqrt{\pi}, \quad \text{for all } \alpha \in \mathbb{R}, \tag{77}$$

with ξ_p defined in (35).

Corollary 4.3. *Let f be defined in* (77)*. Then*

$$f(\alpha) = f\big(\alpha + \tfrac{2\pi}{k}\big) \quad \textit{and} \quad f(\alpha) = f(2\pi - \alpha)$$

for all $\alpha \in \mathbb{R}$.

Proof. It is a straightforward consequence of Lemma 4.2. □

5. Proof of the main result

From Lemma 1.1 and Proposition 3.9, it follows that, under assumption (16), the homogeneous polynomial P (13) of degree k appearing in the expansion (12) is given by

$$P(r\cos\alpha, r\sin\alpha) = r^k|\beta_2|^2 k\sqrt{\pi} f(\alpha), \quad r > 0,\ \alpha \in \mathbb{R}, \tag{78}$$

with f as in (77). Furthermore, from Corollary 4.3 and (78), we have that the 2π-periodic function

$$g : \mathbb{R} \to \mathbb{R}, \quad g(\alpha) := P(\cos\alpha, \sin\alpha) = \sum_{j=0}^{k} c_j (\cos\alpha)^{k-j}(\sin\alpha)^j \tag{79}$$

satisfies the periodicity/symmetry conditions

$$g(\alpha) = g\big(\alpha + \tfrac{2\pi}{k}\big) \quad \text{and} \quad g(\alpha) = g(2\pi - \alpha), \quad \text{for all } \alpha \in \mathbb{R}. \tag{80}$$

From [1, Theorem 1.2] we also know that

$$c_0 = g(0) = -4|\beta_2|^2\mathfrak{m}_k > 0, \tag{81}$$

with $\mathfrak{m}_k$ being as in (9)–(10).

Lemma 5.1. *Under the assumptions of Lemma 1.1 and* (16)*, let P be as in* (12)-(13) *and g be defined in* (79)*. Then*

$$g(\alpha) = \frac{c_0}{\prod_{\ell=1}^{k} \sin\big(\frac{\pi}{2k}(2\ell-1)\big)} \prod_{j=1}^{k} \sin\Big(\frac{\pi}{2k}(2j-1) - \alpha\Big), \quad \textit{for all } \alpha \in \mathbb{R}.$$

Proof. From (81) and (80), we have that

$$g\Big(j\frac{2\pi}{k}\Big) > 0 \quad \text{for all } j = 0, 1, \dots, \frac{k-1}{2}. \tag{82}$$

Moreover, from (79) and the oddness of k, we have that

$$g(\alpha + \pi) = -g(\alpha), \quad \text{for all } \alpha \in \mathbb{R}, \tag{83}$$

and hence from (80) we deduce that

$$\begin{aligned} g\Big(\frac{\pi}{k} + j\frac{2\pi}{k}\Big) &= g\Big(\pi + \big(\tfrac{\pi}{k} - \pi + j\tfrac{2\pi}{k}\big)\Big) \\ &= g\Big(\pi + \tfrac{2\pi}{k}\big(j + \tfrac{1-k}{2}\big)\Big) = g(\pi) = -g(0) < 0 \quad \text{for all } j = 0, 1, \dots, \frac{k-1}{2}. \end{aligned} \tag{84}$$

From (82) and (84) we infer that g has at least k distinct zeros $\theta_1, \theta_2, \dots, \theta_k$ in $(0, \pi)$ such that

$$(j-1)\frac{\pi}{k} < \theta_j < j\frac{\pi}{k}, \quad \text{for all } j = 1, \dots, k.$$

In view of this fact, we aim at factorizing the function g. For every $\alpha \in \mathbb{R} \setminus \{\ell\pi : \ell \in \mathbb{Z}\}$ we have that

$$g(\alpha) = (\sin\alpha)^k \widetilde{P}(\cot\alpha) \tag{85}$$

where

$$\widetilde{P}(t) = \sum_{j=0}^{k} c_j t^{k-j}.$$

From (81) the 1-variable polynomial $\widetilde{P}$ has degree k. Furthermore, by (85), $\cot\theta_1, \dots, \cot\theta_k$ are k distinct real zeroes of $\widetilde{P}$. Therefore from the Fundamental Theorem of Algebra it follows that $\widetilde{P}(t) = c_0 \prod_{j=1}^{k}(t - \cot\theta_j)$, and hence, in view of (85),

$$\begin{aligned} g(\alpha) &= c_0(\sin\alpha)^k \prod_{j=1}^{k}(\cot\alpha - \cot\theta_j) = c_0 \prod_{j=1}^{k}(\cos\alpha - \cot\theta_j \sin\alpha) \\ &= c_0 \prod_{j=1}^{k} \frac{1}{\sin\theta_j}(\sin\theta_j\cos\alpha - \cos\theta_j\sin\alpha) = c_0 \prod_{j=1}^{k} \frac{1}{\sin\theta_j}\sin(\theta_j - \alpha), \end{aligned}$$

for all $\alpha \in \mathbb{R} \setminus \{\ell\pi : \ell \in \mathbb{Z}\}$. Then, by continuity, we conclude that

$$g(\alpha) = c_0 \prod_{j=1}^{k} \frac{1}{\sin\theta_j}\sin(\theta_j - \alpha), \quad \text{for all } \alpha \in \mathbb{R}. \tag{86}$$

We notice that (86) implies that the values $\theta_1, \theta_2, \dots, \theta_k$ are the unique zeros of g in the interval $(0, \pi)$. In particular, for every $j \in \{1, \dots, k\}$,

$$\theta_j \text{ is the unique zero of } g \text{ in the interval } \left((j-1)\frac{\pi}{k}, j\frac{\pi}{k}\right). \tag{87}$$

From (80) and (83) we have that

$$\begin{aligned} &g\Big(\theta_1 + (j-1)\frac{\pi}{k}\Big) \\ &= \begin{cases} g(\theta_1), & \text{if } j \text{ is odd,} \\ -g\big(\theta_1 + \pi + (j-1)\frac{\pi}{k}\big) = -g\big(\theta_1 + (j-1+k)\frac{\pi}{k}\big) = -g(\theta_1), & \text{if } j \text{ is even,} \end{cases} \\ &= 0, \end{aligned}$$

and hence, in view of (87) and since $\theta_1 + (j-1)\frac{\pi}{k} \in \big((j-1)\frac{\pi}{k}, j\frac{\pi}{k}\big)$, we have that

$$\theta_j = \theta_1 + (j-1)\frac{\pi}{k}, \quad \text{for all } j = 1, \dots, k. \tag{88}$$

From (80) it follows that $g\big(-\theta_1 + \frac{2\pi}{k}\big) = g(-\theta_1) = g(2\pi - \theta_1) = g(\theta_1) = 0$; therefore, since $-\theta_1 + \frac{2\pi}{k} \in \big(\frac{\pi}{k}, \frac{2\pi}{k}\big)$, from (87) and (88) we can conclude that $-\theta_1 + \frac{2\pi}{k} = \theta_2 = \theta_1 + \frac{\pi}{k}$ and hence $\theta_1 = \frac{\pi}{2k}$. Then from (88) we deduce that

$$\theta_j = \frac{\pi}{2k}(2j-1), \quad \text{for all } j = 1, \dots, k,$$

thus reaching the conclusion in view of (86). □

Lemma 5.2. *Let $k \in \mathbb{N} \setminus \{0\}$. Then*

$$\prod_{j=1}^{k} \sin\Big(\frac{\pi}{2k}(2j-1)-\alpha\Big) = 2^{1-k}\cos(k\alpha)$$

for all $\alpha \in \mathbb{R}$.

Proof. Since the complex numbers $e^{i\frac{2\pi}{k}j}$ with $j = 1, 2, \dots, k$ are k-th distinct roots of unity and k is odd, we have that

$$1 - z^k = (-1)^{k+1}\prod_{j=1}^{k}\Big(e^{i\frac{2\pi}{k}j} - z\Big), \quad \text{for all } z \in \mathbb{C}. \tag{89}$$

Since

$$\begin{aligned}\sin\Big(\frac{\pi}{2k}(2j-1)-\alpha\Big) &= \frac{e^{i\left(\frac{\pi}{2k}(2j-1)-\alpha\right)} - e^{-i\left(\frac{\pi}{2k}(2j-1)-\alpha\right)}}{2i} \\ &= \frac{1}{2i}e^{-i\alpha}e^{-i\frac{\pi}{2k}(2j+1)}\Big(e^{i\frac{2\pi}{k}j} - e^{i(2\alpha+\frac{\pi}{k})}\Big) = \frac{1}{2}e^{-i\alpha}e^{-i\frac{\pi}{2k}(2j+1+k)}\Big(e^{i\frac{2\pi}{k}j} - e^{i(2\alpha+\frac{\pi}{k})}\Big),\end{aligned}$$

from (89) we deduce that

$$\begin{aligned}\prod_{j=1}^{k}\sin\Big(\frac{\pi}{2k}(2j-1)-\alpha\Big) &= \frac{1}{2^k}e^{-ik\alpha}e^{-i\frac{\pi}{k}\sum_{j=1}^{k} j}e^{-i\frac{\pi}{2}(1+k)}\prod_{j=1}^{k}\Big(e^{i\frac{2\pi}{k}j} - e^{i(2\alpha+\frac{\pi}{k})}\Big) \\ &= \frac{1}{2^k}e^{-ik\alpha}e^{-i\pi(1+k)}\Big(1 - e^{i(2k\alpha+\pi)}\Big)(-1)^{k+1} \\ &= \frac{1}{2^k}e^{-ik\alpha}\Big(1 + e^{2ki\alpha}\Big) = \frac{1}{2^k}\Big(e^{-ik\alpha} + e^{ki\alpha}\Big) = 2^{1-k}\cos(k\alpha)\end{aligned}$$

thus proving the lemma. □

Proof of Theorem 1.2. From Lemmas 5.1 and 5.2 it follows that, under the assumptions of Lemma 1.1 and (16), the polynomial P in (12)-(13) is given by

$$P(r\cos\alpha, r\sin\alpha) = -4|\beta_2|^2\mathfrak{m}_k r^k\cos(k\alpha),$$

thus proving the conclusion in the case in which assumption (16) is satisfied. The general case $\beta_1 \neq 0$ can be easily reduced to the case $\beta_1 = 0$ by a change of the Cartesian coordinate system (x_1, x_2) in $\mathbb{R}^2$ which rotates the axes in such a way that the positive x_1-axis is tangent to one of the k nodal lines of φ_0 ending at 0. If $\beta_1 \neq 0$ and α_0 is defined in (6), the nodal lines of φ_0 at 0 have tangent half-lines forming with the x_1-axis angles of $\alpha_0 + \frac{2\pi}{k}j$, $j = 0, 1, \dots, k-1$. If $\tilde\varphi_0(x) = \varphi_0(R(x))$ and $\tilde\varphi_a(x) = \varphi_a(R(x))$ with

$$R(x_1, x_2) = \begin{pmatrix}\cos\alpha_0 & -\sin\alpha_0 \\ \sin\alpha_0 & \cos\alpha_0\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix},$$

it is easy to verify that $\tilde\varphi_0, \tilde\varphi_a$ solve the problems

$$(i\nabla + A_0)^2\tilde\varphi_0 = \lambda_0\tilde\varphi_0, \quad (i\nabla + A_{R^{-1}(a)})^2\tilde\varphi_a = \lambda_a\tilde\varphi_a,$$

in the domain $R^{-1}(\Omega)$. Moreover

$$r^{-k/2}\tilde\varphi_0(r(\cos t, \sin t)) \to \tilde\beta_1 e^{i\frac{t}{2}}\cos\Big(\frac{k}{2}t\Big) + \tilde\beta_2 e^{i\frac{t}{2}}\sin\Big(\frac{k}{2}t\Big) \quad \text{in } C^{1,\tau}([0,2\pi],\mathbb{C})$$

as $r \to 0^+$, where

$$\begin{pmatrix}\tilde\beta_1\\ \tilde\beta_2\end{pmatrix} = e^{i\frac{\alpha}{2}} \begin{pmatrix}\cos(\frac{k}{2}\alpha_0) & -\sin(\frac{k}{2}\alpha_0)\\ \sin(\frac{k}{2}\alpha_0) & \cos(\frac{k}{2}\alpha_0)\end{pmatrix} \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}.$$

From (6) it follows that $\tilde\beta_1 = 0$ and hence $|\tilde\beta_2|^2 = |\beta_1|^2 + |\beta_2|^2$. Since we have already proved the theorem in the case $\beta_1 = 0$, we know that

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to -4|\tilde\beta_2|^2 \mathfrak{m}_k \cos(k\alpha), \quad \text{as } a \to 0 \text{ with } R^{-1}(a) = |a|(\cos\alpha, \sin\alpha),$$

which yields

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to -4\big(|\beta_1|^2 + |\beta_2|^2\big) \mathfrak{m}_k \cos(k(\theta - \alpha_0)), \quad \text{as } a \to 0 \text{ with } a = |a|(\cos\theta, \sin\theta),$$

thus concluding the proof. □

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