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4-valent graphs of order $6p^2$ admitting a group of automorphisms acting regularly on arcs

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Abstract

In this paper we classify the 4-valent graphs having $6p^2$ vertices, with p a prime, admitting a group of automorphisms acting regularly on arcs. As a corollary, we obtain the 4-valent one-regular graphs having $6p^2$ vertices.

Keywords: One-regular graphs, 4-valent, Cayley graphs.

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1 Introduction

For a finite, simple and undirected graph X, we use $V(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, arc set and automorphism group, respectively. The graph X is said to be B-vertex-transitive, respectively B-arc-transitive, if B is a subgroup of $\mathrm{Aut}(X)$ acting transitively on $V(X)$, respectively $A(X)$. When $B = \mathrm{Aut}(X)$, the prefix B in the above notation is omitted. Moreover, X is said to be *one-regular* if $\mathrm{Aut}(X)$ acts regularly on $A(X)$, that is, X is arc-transitive and $|\mathrm{Aut}(X)| = |A(X)|$.

Obviously, one-regular graphs are connected, and a graph of valency 2 is one-regular if and only if it is a cycle. The first example of a 3-valent one-regular graph was constructed by Frucht [10], with 432 vertices. Later on, a considerable amount of work has been done on 3-valent one-regular graphs as part of the more general problem dealing with the classification of the 3-valent arc-transitive graphs (see [5, 6, 7, 8, 19, 30]). Marušič and Pisanski [17] have fully classified the 3-valent one-regular Cayley graphs on a dihedral group, and Kwak et al. [14] have similarly classified those of valency 5. Moreover, more

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recently, Feng and Li [9] have classified one-regular graphs of square-free order and of prime valency. Tetravalent one-regular graphs have also received considerable attention (see [1, 2, 15, 16, 24, 25, 28, 29]).

In this paper we are concerned with the classification of the 4-valent one-regular graphs. We recall that such graphs are already classified when their orders are a prime or the product of two (not necessarily distinct) primes [3, 21, 22, 26, 29, 28]. Moreover, for p and q primes, the classification of the 4-valent one-regular graphs of order $4p^2$ or $2pq$ is given in [4, 31]. In this context we prove the following.

Theorem 1.1. *Let p be a prime and let X be a 4-valent graph of order $6p^2$ admitting a group of automorphisms acting regularly on $A(X)$. Then one of the following holds:*

(i) *X is isomorphic to $C(2;3p^2,1)$, $C^{\pm 1}(p;6,2)$, $Y_{p,\pm 1}$, $Y_{p,\pm\sqrt{3}}$, $Z_{p,\pm\sqrt{-1}}$ or $Z_{p,\pm\sqrt{-3}}$ (see Section 3 for the definition of these graphs);*

(ii) *X is a Cayley graph over G with connection set S where*

- **(a)** $G = \langle x, y \mid x^p = y^{6p} = [x,y] = 1\rangle$ *and* $S = \{y, y^{-1}, xy, (xy)^{-1}\}$*, or*
- **(b)** $G = \langle x, y, z \mid x^p = y^{3p} = z^2 = [x,y] = [x,z] = 1, y^z = y^{-1}\rangle$ *and* $S = \{xz, x^{-1}z, x^{\varepsilon}yz, x^{-\varepsilon}yz\}$ *(here $\varepsilon^2 \equiv -1 \pmod{p}$ and $p \equiv 1 \pmod{4}$), or*
- **(c)** $G = \langle x, y, z \mid x^{p^2} = y^3 = z^2 = [x,y] = [x,z] = 1, y^z = y^{-1}\rangle$ *and* $S = \{xz, x^{-1}z, xyz, x^{-1}yz\}$*, or*
- **(d)** $G = \langle x, y, z \mid x^{p^2} = y^3 = z^2 = [x,y] = [x,z] = 1, y^z = y^{-1}\rangle$ *and* $S = \{xz, x^{-1}z, x^{\varepsilon}yz, x^{-\varepsilon}yz\}$ *(here $\varepsilon^2 \equiv -1 \pmod{p^2}$), or*
- **(e)** $G = \langle x, y, z, t \mid x^p = y^p = z^3 = t^2 = [x,y] = [x,z] = [x,t] = [y,z] = [y,t] = 1, z^t = z^{-1}\rangle$ *and* $S = \{xt, x^{-1}t, yzt, y^{-1}zt\}$*;*

(iii) *$p \in \{2,3,5\}$ and X is described in Section 6.*

The definition of the graphs in part (i) requires a fair amount of notation and terminology, so we do not include their description in this introductory section. Observe that if $p \leq 7$, then $|V(X)| = 24, 54, 150$ or 294. Since a complete census of the 4-valent arc-transitive graphs of order at most 640 has been recently obtained by Potočnik, Spiga and Verret [18, 19], for $p \in \{2,3,5,7\}$, the 4-valent graphs of order $6p^2$ admitting a group of automorphisms acting regularly on $A(X)$ can be downloaded (in `magma` format) from [18].

The proof of Theorem 1.1 is based on "normal quotient" techniques. This method is very powerful and allows to obtain results like Theorem 1.1 when the order of the graph has a prime factorization that is not too complicated (the multiplicity and the number of prime factors are both small). However there are two natural limits to this technique. First, as the order of the graph X becomes more complicated, the local properties of the quotient graph X_N might not be strong enough to be lifted to the graph X we start with (to see a concrete example of this situation see "Case $X_P = O$" in the proof of Theorem 1.1). Second, we believe that results like Theorem 1.1 are useful only when the list of graphs is not too long or too cumbersome to use. Therefore, although some of our arguments apply to graphs having order more complicated than $6p^2$ we do not pursue this classification here because of the natural complications describing each possible family.

A direct application of Theorem 1.1 gives the following.

Corollary 1.2. *Let p be a prime and let X be a 4-valent one-regular graph X of order $6p^2$. Then one of the following holds:*

(i) *X is isomorphic to $Y_{p,\pm 1}$, $Y_{p,\pm\sqrt{3}}$, $Z_{p,\pm\sqrt{-1}}$ or $Z_{p,\pm\sqrt{-3}}$ (see Section 3 for the definition of these graphs);*

(ii) *X is a Cayley graph over G with connection set S where*

(a) $G = \langle x, y \mid x^p = y^{6p} = [x, y] = 1\rangle$ *and* $S = \{y, y^{-1}, xy, (xy)^{-1}\}$, *or*

(b) $G = \langle x, y, z \mid x^p = y^{3p} = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle$ *and* $S = \{xz, x^{-1}z, x^{\varepsilon}yz, x^{-\varepsilon}yz\}$ *(here* $\varepsilon^2 \equiv -1 \pmod p$ *and* $p \equiv 1 \pmod 4$*), or*

(c) $G = \langle x, y, z \mid x^{p^2} = y^3 = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle$ *and* $\{xz, x^{-1}z, xyz, x^{-1}yz\}$, *or*

(d) $G = \langle x, y, z \mid x^{p^2} = y^3 = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle$ *and* $\{xz, x^{-1}z, x^{\varepsilon}yz, x^{-\varepsilon}yz\}$ *(here* $\varepsilon^2 \equiv -1 \pmod{p^2}$*);*

(iii) $p \in \{2, 3, 5\}$ *and X is described in Section 6.*

Observe that we are not claiming that every graph in Corollary 1.2 is one-regular.

The structure of the paper is elementary: in Section 2 we introduce the notation and some basic results that we will need for our proof of Theorem 1.1. Then in Sections 3 and 4 we present some graphs revelant to our investigation. In Section 5 we prove Theorem 1.1 and Corollary 1.2. In Section 6 we give the graphs in part (iii) of Theorem 1.1, and in part (iii) of Corollary 1.2.

Acknowledgements. A toast to our friend Primož Potočnik for being constructively critical.

2 Preliminaries

In this section, we introduce some notations and definitions as well as some preliminary results which will be used later in the paper.

Let X be a connected vertex-transitive graph, and let $B \leq \mathrm{Aut}(X)$ be vertex-transitive on X. Suppose that there is some group N such that $1 \neq N \lhd B$, and N is intransitive in its action on $V(X)$. The *normal quotient* X_N is the graph whose vertices are the orbits of N on $V(X)$, with an edge between two distinct vertices v^N and w^N in $V(X_N)$, if and only if there is an edge of X between v_0 and w_0, for some $v_0 \in v^N$ and some $w_0 \in w^N$. Normal quotients were introduced by Praeger in [20] and they turned out to be an invaluable tool for the classification of certain families of vertex-transitive graphs. In fact, our proof of Theorem 1.1 heavily relies on normal quotient techniques.

For a positive integer n, we denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, by $\mathbb{Z}_n^*$ the invertible elements of $\mathbb{Z}_n$, by D_{2n} the dihedral group of order $2n$, and by C_n and K_n the cycle and the complete graph of order n, respectively. For a group G and a subset S of G with $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with connection set S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$.

For the benefit of the reader, we report here [12, Theorems 1.1 and 1.2] and [11, Theorem 1.1], respectively.

Theorem 2.1 ([12, Theorem 1.1])**.** *Let X be a connected B-arc-transitive 4-valent graph, and let N be a minimal normal p-subgroup of B with orbits on $V(X)$ of size p^s. Let K denote the kernel of the action of B on N orbits and let α be a vertex of X. If the quotient X_N is a cycle of length $r \geq 3$, then one of the following holds:*

(a) $p = 2$ *and* $X = C(2; r, s)$*;*

(b) p *is odd and, if* $|K_\alpha| = 2^s$*, then* $X = C^{\pm 1}(p; st, s)$ *or* $X = C^{\pm\varepsilon}(p; 2st, s)$ *for some* $t \geq 1$.

Theorem 2.2 ([12, Theorem 1.2])**.** *Let* X *be a connected* B*-arc-transitive* 4*-valent graph, and let* $N \cong \mathbb{Z}_p^2$ *(*p *an odd prime) be a minimal normal subgroup of* B *with orbits on* $V(X)$ *of size* p^s*. Let* K *denote the kernel of the action of* B *on* N*-orbits and let* α *be a vertex of* X*. If the quotient* X_N *is a cycle of length* $r \geq 3$*, then one of the following holds:*

(a) $s = 1$ *or* 2*,* $K_\alpha \cong \mathbb{Z}_2^s$ *and* $X = C^{\pm 1}(p; st, s)$ *or* $X = C^{\pm\varepsilon}(p; 2st, s)$ *for some* $t \geq 1$*;*

(b) $s = 2$*,* $K_\alpha \cong \mathbb{Z}_2$ *and* $X = C^{\pm 1}(p; 2t, 2)$*, or* X *belongs to one of two families described in [12, Lemmas* 8.4 *and* 8.7*].*

(The graphs $C(2; r, s)$, $C^{\pm 1}(p; st, s)$, $C^{\pm\varepsilon}(p; 2st, s)$ and the graphs in [12, Lemma 8.4 and 8.7] are define in Section 3.)

Theorem 2.3 ([11, Theorem 1.1])**.** *Let* X *be a connected* 4*-valent* B*-arc-transitive graph. For each normal subgroup* N *of* B*, one of the following holds:*

(a) N *is transitive on* $V(X)$*;*

(b) X *is bipartite and* N *acts transitively on each part of the bipartition;*

(c) N *has* $r \geq 3$ *orbits on* $V(X)$*, the quotient graph* X_N *is a cycle of length* r*, and* B *induces the full automorphism group* D_{2r} *on* X_N*;*

(d) N *has* $r \geq 5$ *orbits on* $V(X)$*,* N *acts semiregularly on* $V(X)$*, the quotient graph* X_N *is a connected* 4*-valent* B/N*-symmetric graph, and* X *is a* B*-normal cover of* X_N*.*

3 Some families of graphs

In this section we present some of the graphs relevant to Theorem 1.1 and Corollary 1.2. All of these graphs were introduced in the pivotal paper [12] of Gardiner and Praeger. (Here we do not include the description of all the graphs in [12], but only those that are closely related to our investigation.)

3.1 The graphs $C(2; r, 1)$

The graph $C(2; r, 1)$ is the lexicographic product of a cycle of length r and an edgeless graph on 2 vertices. In other words, $V(C(2; r, 1)) = \mathbb{Z}_2 \times \mathbb{Z}_r$ with (u, i) being adjacent to (v, j) if and only if $i - j \in \{-1, 1\}$.

From [12, Definition 2.1], it follows that $C(2; r, 1)$ is not one-regular.

3.2 The graphs $C^{\pm 1}(p; st, s)$

(We will be only interested to the case $s \in \{1, 2\}$.) The graph $C^{\pm 1}(p; st, s)$ has vertex set $\mathbb{Z}_p^s \times \mathbb{Z}_{st}$. To describe the adjacencies write every vertex of $C^{\pm 1}(p; st, s)$ as

$$v = ((x_0, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{s-1}), ms + i),$$

with $x_0, \dots, x_{s-1} \in \mathbb{Z}_s$, $0 \leq i < s$ and $0 \leq m < t$. Then v is adjacent to

$$((x_0, \dots, x_{i-1}, x_i \pm 1, x_{i+1}, \dots, x_{s-1}), ms + i + 1)$$

and

$$((x_0, \dots, x_{i-1} \pm 1, x_i, x_{i+1}, \dots, x_{s-1}), ms + i - 1).$$

The group $\langle a_0\rangle \times \langle a_1\rangle \times \cdots \times \langle a_{s-1}\rangle \cong \mathbb{Z}_p^s$ acts on $V(C^{\pm 1}(p; st, s))$ in the natural way, inducing translations on the first s coordinates and leaving the coordinate in $\mathbb{Z}_{st}$ unchanged. Moreover, $C^{\pm 1}(p; st, s)$ admits the automorphism σ defined by

$$((x_0, x_1, \dots, x_{s-1}), q)^\sigma = ((x_{s-1}, x_0, \dots, x_{s-2}), q + 1).$$

A computation shows that $C^{\pm 1}(p; st, s)$ is a Cayley graph over $G = \langle a_0, \dots, a_{s-1}, \sigma\rangle$. Observe that when $s = 1$ we have $G \cong \mathbb{Z}_p \times \mathbb{Z}_t$.

From [12, Definition 2.2], it follows that $C^{\pm}(p; t, 1)$ is a normal Cayley graph over G with connection set $S = \{a_0\sigma, (a_0\sigma)^{-1}, a_0^{-1}\sigma, (a_0^{-1}\sigma)^{-1}\}$, that is, $C^{\pm 1}(p; t, 1) = \mathrm{Cay}(G, S)$ and $G \trianglelefteq \mathrm{Aut}(C^{\pm 1}(p; t, 1))$. Moreover, if $C^{\pm 1}(p; st, s)$ is one-regular, then $s = 1$.

3.3 The graphs $C^{\pm\varepsilon}(p; 2st, s)$

We introduce this family only for the case that concerns us, that is, $s = 1$. Let p be a prime with $p \equiv 1 \pmod 4$ and let ε be a square root of $-1 \pmod p$. The graph $C^{\pm\varepsilon}(p; 2t, 1)$ has vertex set $\mathbb{Z}_p \times \mathbb{Z}_{2t}$ and the vertex $v = (x, m)$ is adjacent to

$(x \pm \varepsilon, m - 1)$ and $(x \pm 1, m + 1)$	if m is even, and
$(x \pm 1, m - 1)$ and $(x \pm \varepsilon, m + 1)$	if m is odd.

It is easy to check that the mappings a, τ, σ defined by

$$\begin{aligned}(x, m)^a &= (x + 1, m)\\ (x, m)^\tau &= (x, 1 - m)\\ (x, m)^\sigma &= (\varepsilon x, q + 1)\end{aligned}$$

are automorphisms of $C^{\pm\varepsilon}(p; 2t, 1)$. Furthermore, a computation shows that, when t is odd, $C^{\pm\varepsilon}(p; 2t, 1)$ is a Cayley graph over $G = \langle a, \sigma^4, \tau\rangle = \langle a\rangle \times \langle\sigma^4, \tau\rangle \cong \mathbb{Z}_p \times D_{2t}$.

From [12, Definition 2.3], it follows that, for t odd, $C^{\pm\varepsilon}(p; 2t, 1)$ is a normal Cayley graph over G with connection set $S = \{a\tau, (a\tau)^{-1}, a^\varepsilon\sigma^{2t+2}\tau, (a^\varepsilon\sigma^{2t+2}\tau)^{-1}\}$, that is, $C^{\pm\varepsilon}(p; t, 1) = \mathrm{Cay}(G, S)$ and $G \trianglelefteq \mathrm{Aut}(C^{\pm\varepsilon}(p; 2t, 1))$.

For defining the rest of the graphs we recall the concept of *coset graph*. For a group B, a subgroup H and an element $b \in B$, the coset graph $\mathrm{Cos}(B, H, b)$ is the graph with vertex set the set of right cosets $B/H = \{Hg \mid g \in B\}$ and edge set $\{\{Hg, Hbg\} \mid g \in B\}$. The following proposition is due to Sabidussi [23].

Proposition 3.1. *Let H be a core-free subgroup of B and let $b \in B$ with $B = \langle H, b\rangle$ and $b^{-1} \in HbH$. Then $\Gamma = Cos(B, H, b)$ is a connected B-arc-transitive graph of valency $|H : H \cap H^b|$.*

3.4 The graphs arising from [12, Lemma 8.4]: $Y_{p,\pm 1}$ and $Y_{p,\pm\sqrt{3}}$

The graphs described in [12, Section 8 and Lemma 8.4] are very general, here we describe only the graphs relevant to the scope of this article.

Let $N = \langle a_0, a_1 \rangle$ be an elementary abelian p-group of order p^2 and let K be the group with presentation

$$K = \langle \sigma, \tau, w \mid \sigma^6 = \tau^2 = w^2 = [\sigma, w] = [\tau, w] = 1, \sigma^\tau = \sigma^{-1} \rangle.$$

Clearly, $K = \langle \sigma, \tau \rangle \times \langle w \rangle \cong D_{12} \times C_2$. Fix $u \in \{-1, 1\}$. We let K act on N via

$$\begin{array}{rcl} a_0^\sigma &=& a_1 \\ a_1^\sigma &=& a_0^{-1} a_1^u \end{array}, \quad \begin{array}{rcl} a_0^\tau &=& a_1 \\ a_1^\tau &=& a_0 \end{array}, \quad \begin{array}{rcl} a_0^w &=& a_0^{-1} \\ a_1^w &=& a_1^{-1} \end{array}.$$

Set $B_u = N \rtimes K$ and $H_u = \langle \tau, w \rangle$. Now the graph $Y_{p,u}$ is defined as $\mathrm{Cos}(B_u, H_u, \sigma^{-1} a_0)$. From Proposition 3.1, it follows that $Y_{p,u}$ is a B_u-arc-transitive graph of valency 4 with $6p^2$ vertices and B_u acts regularly on $A(Y_{p,u})$. Moreover, from [12, Section 8], we have $Y_{p,1} \cong Y_{p,-1}$, and for $p \geq 5$, we have $\mathrm{Aut}(Y_{p,u}) = B_u$.

Next we define $Y_{p,\pm\sqrt{3}}$. Suppose $p \geq 5$ and let $u \in \mathbb{Z}_p$ be a square root of $3 \pmod p$. (Observe that, from the law of quadratic reciprocity, for this example to exist we need $p \equiv \pm 1 \mod 12$.) Let $N = \langle a_0, a_1 \rangle$ be an elementary abelian p-group of order p^2 and let K be the group with presentation

$$K = \langle \sigma, \tau, w \mid \sigma^{12} = \tau^2 = w^2 = [\sigma, w] = [\tau, w] = 1, \sigma^6 = w, \sigma^\tau = \sigma^{-1} \rangle.$$

Clearly, $K = \langle \sigma, \tau \rangle \cong D_{24}$. We let K act on N via

$$\begin{array}{rcl} a_0^\sigma &=& a_1 \\ a_1^\sigma &=& a_0^{-1} a_1^u \end{array}, \quad \begin{array}{rcl} a_0^\tau &=& a_1 \\ a_1^\tau &=& a_0 \end{array}, \quad \begin{array}{rcl} a_0^w &=& a_0^{-1} \\ a_1^w &=& a_1^{-1} \end{array}.$$

Set $B_u = N \rtimes K$ and $H_u = \langle w, \tau \rangle$. As for $Y_{p,\pm 1}$, the graph $Y_{p,u}$ is defined by $\mathrm{Cos}(B_u, H_u, \sigma^{-1} a_0)$. From Proposition 3.1, it follows that $Y_{p,u}$ is a B_u-arc-transitive graph of valency 4 with $6p^2$ vertices and B_u acts regularly on $A(Y_{p,u})$. Moreover, from [12, Section 8], we have $Y_{p,\sqrt{3}} \cong Y_{p,-\sqrt{3}}$ and $\mathrm{Aut}(Y_{p,u}) = B_u$.

3.5 The graphs arising from [12, Lemma 8.7]: $Z_{p,\pm\sqrt{-3}}$ and $Z_{p,\pm\sqrt{-1}}$

As for Section 3.4, the graphs described in [12, Section 8 and Lemma 8.7] are very general, here we present only the graphs relevant to the scope of this article.

We start by defining $Z_{p,\pm\sqrt{-3}}$. Suppose $p \geq 5$ and let $u \in \mathbb{Z}_p$ be a square root of $-3 \pmod p$. (Observe that, from the law of quadratic reciprocity, for this example to exist we need $p \equiv 1 \mod 6$.) Let $N = \langle a_0, a_1 \rangle$ be an elementary abelian p-group of order p^2 and let K be the group with presentation

$$K = \langle \sigma, \tau, w \mid \sigma^6 = \tau^4 = w^2 = [\sigma, w] = [\tau, w] = 1, \tau^2 = w, \sigma^\tau = \tau^2 \sigma^{-1} \rangle.$$

We let K act on N via

$$\begin{array}{rcl} a_0^\sigma &=& a_1 \\ a_1^\sigma &=& a_0 a_1^u \end{array}, \quad \begin{array}{rcl} a_0^\tau &=& a_1 \\ a_1^\tau &=& a_0^{-1} \end{array}, \quad \begin{array}{rcl} a_0^w &=& a_0^{-1} \\ a_1^w &=& a_1^{-1} \end{array}.$$

Set $B_u = N \rtimes K$ and $H_u = \langle \tau \rangle$. Now the graph $Z_{p,u}$ is defined by $\mathrm{Cos}(B_u, H_u, \sigma^{-1} a_0)$. From Proposition 3.1, it follows that $Z_{p,u}$ is a B_u-arc-transitive graph of valency 4 with $6p^2$ vertices and B_u acts regularly on $A(Z_{p,u})$. Moreover, from [12, Section 8], we have $Z_{p,\sqrt{-3}} \cong Z_{p,-\sqrt{-3}}$ and $\mathrm{Aut}(Z_{p,u}) = B_u$.

Finally, we define $Z_{p,\pm\sqrt{-1}}$. Suppose $p \geq 5$ and let $u \in \mathbb{Z}_p$ be a square root of $-1 \pmod p$. (Observe that, from the law of quadratic reciprocity, for this example to exist we need $p \equiv 1 \mod 4$.) Let $N = \langle a_0, a_1 \rangle$ be an elementary abelian p-group of order p^2 and let K be the group with presentation

$$K = \langle \sigma, \tau, w \mid \sigma^{12} = \tau^4 = w^2 = [\sigma, w] = [\tau, w] = 1, \sigma^6 = \tau^2 = w, \sigma^\tau = \sigma^{-1} \rangle.$$

We let K act on N via

$$\begin{array}{lcl} a_0^\sigma & = & a_1 \\ a_1^\sigma & = & a_0 a_1^u \end{array}, \quad \begin{array}{lcl} a_0^\tau & = & a_1 \\ a_1^\tau & = & a_0^{-1} \end{array}, \quad \begin{array}{lcl} a_0^w & = & a_0^{-1} \\ a_1^w & = & a_1^{-1} \end{array}.$$

Set $B_u = N \rtimes K$ and $H_u = \langle \tau \rangle$. As for $Z_{p,\pm\sqrt{-3}}$, the graph $Z_{p,u}$ is defined by $\mathrm{Cos}(B_u, H_u, \sigma^{-1} a_0)$. From Proposition 3.1, it follows that $Z_{p,u}$ is a B_u-arc-transitive graph of valency 4 with $6p^2$ vertices and B_u acts regularly on $A(Z_{p,u})$. Moreover, from [12, Section 8], we have $Z_{p,\sqrt{-1}} \cong Z_{p,-\sqrt{-1}}$ and $\mathrm{Aut}(Z_{p,u}) = B_u$.

4 Graphs for Theorem 1.1 (ii)

In this section we describe the examples introduced in Theorem 1.1 (ii) (and their relation to the graphs in Section 3). For simplicity here we assume that $p \geq 5$. The graphs described in Sections 4.1 and 4.6 were already introduced with much more information and details in [28], here we include yet again their construction for making our note self-contained.

4.1 4-Valent normal Cayley graphs over $\mathbb{Z}_p \times \mathbb{Z}_{6p}$ with $p \geq 5$ prime

We describe the connected normal 4-valent arc-transitive Cayley graphs over the group $\mathbb{Z}_p \times \mathbb{Z}_{6p}$.

Let p be a prime and let G be the group given by generators and relations

$$G = \langle x, y \mid x^p = y^{6p} = [x, y] = 1 \rangle.$$

Let S be a subset of G and assume that $X = \mathrm{Cay}(G, S)$ is connected, normal, 4-valent and arc-transitive. Since X is normal and arc-transitive and since G is abelian of exponent $6p$, we see that S consists of elements of order $6p$. Denote by $\mathcal{S}_{6p}$ the elements of G of order $6p$. Therefore

$$S \subseteq \mathcal{S}_{6p} = \{x^a y^b \mid a \in \mathbb{Z}_p, b \in \mathbb{Z}_{6p}^*\}.$$

It is clear that $\mathrm{Aut}(G)$ acts transitively on $\mathcal{S}_{6p}$ by conjugation. In particular, replacing S by a suitable $\mathrm{Aut}(G)$-conjugate, we may assume that $y \in S$. Therefore

$$S = \{y, y^{-1}, x^u y^v, x^{-u} y^{-v}\},$$

for some $u \in \mathbb{Z}_p^*$ and for some $v \in \mathbb{Z}_{6p}^*$.

Let $B = \{\varphi \in \mathrm{Aut}(G) \mid y^\varphi = y\}$. Given $\varphi \in B$, we have

$$\varphi \quad : \quad \left\{ \begin{array}{lcl} x & \mapsto & x^a y^{6b} \\ y & \mapsto & y \end{array} \right.$$

with $a, b \in \mathbb{Z}_p$ and $a \neq 0$. Note that every invertible element of $\mathbb{Z}_{6p}$ is of the form $1 + 6b$ or $-1 + 6b$, for some $b \in \mathbb{Z}_p$. Therefore, we may choose $a, b \in \mathbb{Z}_p$ with $(xy)^\varphi = x^u y^v$ or

$(xy^{-1})^\varphi = x^u y^v$. Thus, replacing S by a suitable B-conjugate, we may assume that either $xy \in S$ or $xy^{-1} \in S$, that is,

$$\begin{array}{rcl} S &=& \{y, y^{-1}, xy, x^{-1}y^{-1}\}, \text{ or} \\ S &=& \{y, y^{-1}, xy^{-1}, x^{-1}y\}. \end{array}$$

Let φ be the automorphism of G with $x^\varphi = x$ and $y^\varphi = y^{-1}$. Clearly, φ maps the first possibility for S onto the second. Therefore, we may assume that

$$S = \{y, y^{-1}, xy, x^{-1}y^{-1}\}.$$

The graph X is in Theorem 1.1 (ii) (a). Also, using [12, Definition 2.2], we see that X is isomorphic to $C^{\pm 1}(p; 6p, 1)$.

4.2 4-Valent normal Cayley graphs over $\mathbb{Z}_p \times D_{6p}$ with $p \geq 5$ prime

We describe the connected normal 4-valent arc-transitive Cayley graphs over the group $\mathbb{Z}_p \times D_{6p}$.

Let p be a prime and let G be the group given by generators and relations

$$G = \langle x, y, z \mid x^p = y^{3p} = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle.$$

Let S be a subset of G and assume that $X = \mathrm{Cay}(G, S)$ is connected, normal, 4-valent and arc-transitive. Since X is normal and arc-transitive, every element of S has the same order. The elements of G of odd order lie in $\langle x, y\rangle$ and the involutions of G lie in $\langle y, z\rangle$. Since X is connected, $G = \langle S\rangle$ and hence S consists of elements of order $2p$. Denote by $\mathcal{S}_{2p}$ the elements of G of order $2p$. Therefore

$$S \subseteq \mathcal{S}_{2p} = \{x^a y^b z \mid b \in \mathbb{Z}_{3p}, a \in \mathbb{Z}_p^*\}.$$

We now consider the action of the automorphism group $\mathrm{Aut}(G)$ of G on $\mathcal{S}_{2p}$. Let $\varphi \in \mathrm{Aut}(G)$. Clearly, φ is uniquely determined by the images of x, y and z. By considering the element orders of G, we need to have

$$\varphi \ : \ \begin{cases} x & \mapsto \ x^a y^{3b} \\ y & \mapsto \ x^c y^d \\ z & \mapsto \ y^e z. \end{cases}$$

Since $[x, z] = 1$, the element x^φ needs to commute with z^φ and so, with a direct computation, we see that $3b = 0$. Also, as $y^z = y^{-1}$, we have $(y^\varphi)^{z^\varphi} = (y^\varphi)^{-1}$, but this happens only for $c = 0$. Summing up,

$$\varphi : \begin{cases} x & \mapsto \ x^a \quad \text{with } a \in \mathbb{Z}_p^* \\ y & \mapsto \ y^d \quad \text{with } d \in \mathbb{Z}_{3p}^* \\ z & \mapsto \ y^e z. \end{cases} \tag{4.1}$$

This shows that all elements of $\mathcal{S}_{2p}$ are $\mathrm{Aut}(G)$-conjugate to xz. Therefore, replacing S by a suitable conjugate under $\mathrm{Aut}(G)$, we may assume that $xz \in S$. In particular, $S = \{xz, x^{-1}z, x^u y^v z, x^{-u} y^v z\}$, for some u, v. As $G = \langle S\rangle \leq \langle x, z, y^v\rangle$, we have that $v \in \mathbb{Z}_{3p}^*$.

Let $B = \{\varphi \in \mathrm{Aut}(G) \mid (xz)^\varphi = xz\}$. From (4.1), we see that $\psi \in B$ only if $x^\psi = x$ and $z^\psi = z$. In particular, replacing S by a suitable B-conjugate, we may assume that $x^u yz \in S$, that is, $v = 1$. Thus $S = \{xz, x^{-1}z, x^u yz, x^{-u}yz\}$.

Let $\varphi \in \mathrm{Aut}(G)$ with $S^\varphi = S$ and $(xz)^\varphi = x^u yz$ (recall that such an automorphism exists because X is a normal arc-transitive Cayley graph over G). From (4.1), we have $x^\varphi = x^u$, $z^\varphi = yz$ and $y^\varphi = y^d$ (for some d coprime to $3p$). Now, $(x^u yz)^\varphi = x^{u^2} y^{d+1} z \in S$. If $x^{u^2} y^{d+1} z \in \{x^u yz, x^{-u} yz\}$, then $d = 0$, contradicting the fact that d is coprime to $3p$. Therefore $x^{u^2} y^{d+1} z \in \{xz, x^{-1}z\}$ and $u^2 = \pm 1$. Thus we have one of the following two possibilities for S:

$$\begin{aligned} S &= \{xz, x^{-1}z, xyz, x^{-1}yz\}, \text{ or} \\ S &= \{xz, x^{-1}z, x^\varepsilon yz, x^{-\varepsilon} yz\}, \text{ where } \varepsilon^2 = -1 \end{aligned}$$

(note that in the second case $p \equiv 1 \pmod 4$). Now we focus our attention to the first possibility for S. Take

$$\psi : \begin{cases} x & \mapsto & x \\ y & \mapsto & y^{-1} \\ z & \mapsto & yz. \end{cases}$$

Since ψ fixes set-wise S, we have $G \rtimes \langle \psi \rangle \le \mathrm{Aut}(X)$. We see that

$$(z\psi)^2 = z\psi z\psi = z(\psi z \psi) = zz^\psi = zyz = y^z = y^{-1}$$

and so $z\psi$ has order $6p$. Moreover $x^{z\psi} = (x^z)^\psi = x^\psi = x$ and x commutes with $z\psi$. Therefore $\langle x, z\psi \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{6p}$. It is immediate to see that $\langle x, z\psi \rangle$ acts regularly on the vertices of X and so X is a Cayley graph over $\mathbb{Z}_p \times \mathbb{Z}_{6p}$. In particular, from the discussion in Section 4.1 (or with a direct computation) we obtain that X is isomorphic to the graph in Theorem 1.1 (ii) (a).

Therefore we may assume that $p \equiv 1 \pmod 4$ and that $S = \{xz, x^{-1}z, x^\varepsilon yz, x^{-\varepsilon} yz\}$ where $\varepsilon^2 = -1$. The graph X is in Theorem 1.1 (ii) (b). Also, using [12, Definition 2.3] (or Section 3.3), we see that X is isomorphic to $C^{\pm\varepsilon}(p; 6p, 1)$.

4.3 4-Valent normal Cayley graphs over $\mathbb{Z}_p \times \mathbb{Z}_p \times D_6$ with $p \ge 5$ prime

We proceed as in the previous two examples. Let p be a prime and let G be the group given by generators and relations

$$G = \langle x, y, z, t \mid x^p = y^p = z^3 = t^2 = [x, y] = [x, z] = [x, t] = [y, z] = [y, t] = 1, \\ z^t = z^{-1} \rangle.$$

Let S be a subset of G and assume that $X = \mathrm{Cay}(G, S)$ is connected, normal, 4-valent and arc-transitive. As the elements of G of odd order lie in $\langle x, y, z \rangle$ and the involutions of G lie in $\langle z, t \rangle$, we must have that S consists of elements of order $2p$. Since $\langle x, y \rangle$ and $\langle z, t \rangle$ are characteristic subgroups of G, we have $\mathrm{Aut}(G) \cong \mathrm{Aut}(\langle x, y \rangle) \times \mathrm{Aut}(\langle z, t \rangle) \cong \mathrm{GL}_2(p) \times D_6$. It follows easily from this description of $\mathrm{Aut}(G)$ and the connectivity of X that we may assume that

$$S = \{xt, x^{-1}t, yzt, y^{-1}zt\}.$$

Thus we obtain the graphs in Theorem 1.1 (ii) (e).

We show that X is not one-regular and hence it is not relevant for Corollary 1.2. Take

$$\psi : \begin{cases} x & \mapsto & x^{-1} \\ y & \mapsto & y \\ z & \mapsto & z \\ t & \mapsto & t. \end{cases}$$

Clearly, ψ defines an automorphism of G that fixes set-wise S. Since ψ fixes the neighbour yzt of 1 and maps xt to $x^{-1}t$, we see that $\mathrm{Aut}(X)$ is not regular on $A(X)$.

4.4 4-Valent normal Cayley graphs over $\mathbb{Z}_{3p} \times D_{2p}$ with $p \geq 5$ prime

Let G be the group given by generators and relations

$$G = \langle x, y, z \mid x^{3p} = y^p = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1} \rangle.$$

Let S be a subset of G and assume that $X = \mathrm{Cay}(G, S)$ is connected, normal, 4-valent and arc-transitive. As the elements of G of odd order lie in $\langle x, y \rangle$, the involutions of G lie in $\langle y, z \rangle$ and the elements of order $2p$ lie in $\langle x^3, y, z \rangle$, we must have that S consists of elements of order $6p$. Denote by $\mathcal{S}_{6p}$ the elements of G of order $6p$. Therefore

$$S \subseteq \mathcal{S}_{6p} = \{x^a y^b z \mid a \in \mathbb{Z}_{3p}^*, b \in \mathbb{Z}_p\}.$$

Arguing as in the previous examples, we see that $\mathrm{Aut}(G)$ acts transitively on $\mathcal{S}_{6p}$ and hence we may assume that $xz \in S$. In particular,

$$S = \{xz, x^{-1}z, x^u y^v z, x^{-u} y^v z\},$$

for some $u \in \mathbb{Z}_{3p}^*$ and some $v \in \mathbb{Z}_p^*$.

Let $B = \{\varphi \in \mathrm{Aut}(G) \mid (xz)^\varphi = xz\}$. Clearly, if $(xz)^\varphi = xz$, then $x^\varphi = x$ and $z^\varphi = z$ because $z = (xz)^{3p}$ and $x^2 = (xz)^2$. Using this observation, it is easy to see that the elements $\varphi \in B$ are of the form

$$\varphi : \begin{cases} x & \mapsto & x \\ y & \mapsto & x^{3a} y^b \\ z & \mapsto & z, \end{cases}$$

for some $a, b \in \mathbb{Z}_p$ with $b \neq 0$. Therefore, we may choose a and b with $(x^u y^v z)^\varphi = xyz$ or $(x^u y^v z)^\varphi = x^{-1}yz$. Therefore (as usual), replacing S by a suitable B-conjugate, we may assume that

$$S = \{xz, x^{-1}z, xyz, x^{-1}yz\}.$$

Take

$$\psi : \begin{cases} x & \mapsto & x \\ y & \mapsto & y^{-1} \\ z & \mapsto & yz. \end{cases}$$

Clearly, ψ defines an automorphism of G. Since ψ fixes set-wise S, we have $G \rtimes \langle \psi \rangle \leq \mathrm{Aut}(X)$. We see that

$$(z\psi)^2 = z\psi z\psi = z(\psi z\psi) = zz^\psi = zyz = y^z = y^{-1}$$

and so $z\psi$ has order $2p$. Moreover $x^{z\psi} = (x^z)^\psi = x^\psi = x$ and x commutes with $z\psi$. Therefore $\langle x, z\psi\rangle \cong \mathbb{Z}_{3p} \times \mathbb{Z}_{2p} \cong \mathbb{Z}_p \times \mathbb{Z}_{6p}$. It is immediate to see that $\langle x, z\psi\rangle$ acts regularly on the vertices of X and so X is a Cayley graph over $\mathbb{Z}_p \times \mathbb{Z}_{6p}$. In particular, from the discussion in Section 4.1 (or with a direct computation) we obtain that X is isomorphic to the graph in Theorem 1.1 (ii) (a).

4.5 4-Valent normal Cayley graphs over $\mathbb{Z}_{p^2} \times D_6$ with $p \geq 5$ prime

Let G be the group given by generators and relations

$$G = \langle x, y, z \mid x^{p^2} = y^3 = z^2 = [x, y] = [x, z] = 1, y^z = y^{-1}\rangle.$$

Let S be a subset of G and assume that $X = \mathrm{Cay}(G, S)$ is connected, normal, 4-valent and arc-transitive. A moment's thought gives that S consists of elements of order $2p^2$. Denote by $\mathcal{S}_{2p^2}$ the elements of G of order $2p^2$. Therefore

$$S \subseteq \mathcal{S}_{2p^2} = \{x^a y^b z \mid a \in \mathbb{Z}_{p^2}^*, b \in \mathbb{Z}_3\}.$$

Since $\langle x\rangle$ and $\langle y, z\rangle$ are characteristic subgroups of G, we have $\mathrm{Aut}(G) = \mathrm{Aut}(\langle x\rangle) \times \mathrm{Aut}(\langle y, z\rangle) \cong \mathbb{Z}_{p^2}^* \times D_6$ and $\mathrm{Aut}(G)$ acts transitively on $\mathcal{S}_{2p^2}$. Hence we may assume that $xz \in S$. In particular, $S = \{xz, x^{-1}z, x^u y^v z, x^{-u} y^v z\}$, for some $u \in \mathbb{Z}_{p^2}^*$ and some $v \in \mathbb{Z}_3^*$. Replacing S by a suitable $\mathrm{Aut}(G)$-conjugate, we may assume that $v = 1$, that is,

$$S = \{xz, x^{-1}z, x^u yz, x^{-u} yz\}.$$

Let $\varphi \in \mathrm{Aut}(G)$ with $S^\varphi = S$ and $(xz)^\varphi = x^u yz$ (recall that such an automorphism exists because X is a normal arc-transitive Cayley graph over G). From the description of $\mathrm{Aut}(G)$, we have $x^\varphi = x^u$ and $z^\varphi = yz$ and $y^\varphi = y^d$ (for some $d \in \mathbb{Z}_3^*$). Now, $(x^u yz)^\varphi = x^{u^2} y^{d+1} z \in S$. As $d \neq 0$, we obtain $d = -1$ and $u^2 = \pm 1$. Thus we have one of the following two possibilities for S:

$$\begin{aligned} S &= \{xz, x^{-1}z, xyz, x^{-1}yz\}, \text{ or} \\ S &= \{xz, x^{-1}z, x^\varepsilon yz, x^{-\varepsilon} yz\}, \text{ where } \varepsilon^2 = -1 \end{aligned}$$

(note that in the second case $p \equiv 1 \pmod 4$). In particular, we obtain that X is isomorphic to the graph in Theorem 1.1 (ii) (c) or (d).

4.6 4-Valent normal Cayley graphs over $\mathbb{Z}_{p^2} \times \mathbb{Z}_6$ with $p \geq 5$

This case is by far the easiest to deal with and we leave it to the conscious reader. There is only one graph arising, namely the graph in Theorem 1.1 (ii) (c).

5 Proof of Theorem 1.1 and Corollary 1.2

We start with some technical preliminary lemmas that will be useful in the proof of Theorem 1.1.

Lemma 5.1. *Let p be a prime with $p \geq 5$ and let N be a normal subgroup of G with $|N| = p$ and $G/N \cong \mathbb{Z}_p \times \mathbb{Z}_6$ or $G/N \cong \mathbb{Z}_p \times D_6$. Then G is isomorphic to one of the following groups*

(i) $\mathbb{Z}_{p^2} \times \mathbb{Z}_6$ *or* $\mathbb{Z}_{p^2} \times D_6$*; or*

(ii) $\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes \mathbb{Z}_6)$ *or* $\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes D_6)$.

Proof. Let P be a Sylow p-subgroup of G. Since G is soluble, G contains a subgroup Q with $|Q| = 6$. As $G/N \cong \mathbb{Z}_p \times \mathbb{Z}_6$ or $G/N \cong \mathbb{Z}_p \times D_6$, we have that $QN/N \cong Q$ (that is, $Q \cong \mathbb{Z}_6$ or $Q \cong D_6$) and that Q centralizes P/N. If P is cyclic, then Q centralizes P by [13, Theorem 1.4]. So $G = P \times Q$ and part (i) follows.

Suppose that P is an elementary abelian p-group. The action of Q by conjugation on P endows P of a structure of an $\mathbb{Z}_pQ$-module. As Q has order coprime to p, the $\mathbb{Z}_pQ$-module N is completely reducible, that is, $P = N \times N'$, for some normal subgroup N' of G of size p. As Q centralizes P/N, we see that Q centralizes N'. Thus $G = N' \times (N \rtimes Q)$ and part (ii) follows. □

Lemma 5.2. *Let p be a prime with $p \geq 7$ and let X be a connected normal 4-valent arc-transitive Cayley graph over one of the groups G in Lemma 5.1 (ii). Let P be a Sylow p-subgroup of G. Assume that $P \lhd G$ and that $X_P = C_6$. Then either $G \cong \mathbb{Z}_p \times \mathbb{Z}_{6p}$, or $G \cong \mathbb{Z}_{3p} \times D_{2p}$, or $G \cong \mathbb{Z}_p \times D_{6p}$, or $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times D_6$.*

Proof. From Lemma 5.1 (ii), we have $G = \langle x \rangle \times (\langle y \rangle \rtimes \langle z, t \rangle)$ with $\langle x, y \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $|z| = 3$ and $|t| = 2$. Moreover, $[z, t] = 1$ if $G \cong \mathbb{Z}_p \times (\mathbb{Z}_p \rtimes \mathbb{Z}_6)$ and $z^t = z^{-1}$ if $G \cong \mathbb{Z}_p \times (\mathbb{Z}_p \rtimes D_6)$. If z centralizes y, then $G = \langle x \rangle \times (\langle yz \rangle \rtimes \langle t \rangle)$. In particular, $G \cong \mathbb{Z}_p \times \mathbb{Z}_{6p}$ if t centralizes both y and z, $G \cong \mathbb{Z}_p \times D_{6p}$ if t inverts both y and z, $G \cong \mathbb{Z}_{3p} \times D_{2p}$ if t centralizes z and inverts y, and $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times D_6$ if t inverts z and centralizes y. In remains to consider the case that z does not centralize y. We show that this case actually does not arise (here we use the fact that G admits a normal Cayley graph).

Note that the automorphism group of $\mathbb{Z}_p$ is cyclic and hence D_6 cannot act faithfully as a group of automorphisms on $\mathbb{Z}_p$. Since we are assuming that z does not centralize y, we must have that $G \cong \mathbb{Z}_p \times (\mathbb{Z}_p \rtimes \mathbb{Z}_6)$, that is, $[z, t] = 1$. Let S be a subset of G with $X = \mathrm{Cay}(G, S)$. Write $A = \mathrm{Aut}(X)$. Since $\langle x \rangle$ is a characteristic subgroup of G and since $G \lhd A$, we obtain that $\langle x \rangle \lhd A$ and $Y = X_{\langle x \rangle}$ is a normal quotient having $6p$ vertices. In particular, Y has valency 2 or 4. Let K be the kernel of the action of A on $V(Y)$ and assume that Y is a cycle. In particular, $A/K \cong D_{12p}$. Now D_{12p} contains exactly two regular subgroups, one isomorphic to $\mathbb{Z}_{6p}$ and the other to D_{6p}. Therefore

$$\frac{GK}{K} \cong \frac{G}{G \cap K} = \frac{G}{\langle x \rangle} \cong \langle y \rangle \rtimes \langle z, t \rangle$$

is isomorphic either to $\mathbb{Z}_{6p}$ or to D_{6p}, contradicting the fact that z does not centralize y. Thus Y is a 4-valent Cayley graph over $GK/K \cong \langle y \rangle \rtimes \langle z, t \rangle$ and $K = \langle x \rangle$.

Now, P/K has order p and is therefore a minimal normal subgroup of A/K. Also, $Y_{P/K} \cong X_P \cong C_6$ and so we are in the position to apply Theorem 2.1 (b) to A/K, P/K and Y. Hence $Y = C^{\pm 1}(p; 6, 1)$ or $Y = C^{\pm \varepsilon}(p; 6, 1)$. From Sections 3.2 and 3.3 we see that Y is a Cayley graph over $\mathbb{Z}_p \times \mathbb{Z}_6$ or over $\mathbb{Z}_p \times D_6$. However, $\langle y \rangle \rtimes \langle z, t \rangle$ is isomorphic to neither of these groups, a contradiction. □

In what follows we denote by O the graph $K_6 - 6K_2$, that is, K_6 with a perfect matching removed. Clearly, O is connected, 4-valent and $|V(O)| = 6$. We refer to O as the *Octahedral* graph.

Proof of Theorem 1.1. We first consider the case that $p \geq 11$. Let B be a subgroup of $\mathrm{Aut}(X)$ acting regularly on $A(X)$, let B_v be the stabilizer in B of the vertex $v \in V(X)$ and let P be a Sylow p-subgroup of B. We show that P is normal in B. Since $|B| = |A(X)| = 24p^2$, Sylow's theorems show that the number of Sylow p-subgroups of B is equal to $|B : \mathbf{N}_B(P)| = 1 + kp$, for some $k \geq 0$. If $k = 0$, then P is normal in B and thus we may assume that $k \geq 1$. Now, $1 + kp$ divides 24 and this is possible if and only if $k = 1$ and $p = 23$, or $k = 1$ and $p = 11$. Suppose that $k = 1$ and $p = 23$. Now $|B : \mathbf{N}_B(P)| = 24$. So $\mathbf{N}_B(P) = P$ and $\mathbf{C}_B(P) = \mathbf{N}_B(P)$. Therefore, by the Burnside's p-complement theorem [27, page 76], we see that B has a normal subgroup N of order 24. In particular, P acts by conjugation as a group of automorphisms on N. As a group of order 24 does not admit non-trivial automorphisms of order 23, we see that P centralizes B. Thus $B \cong N \times P$ and P is normal in B. Finally, suppose that $k = 1$ and $p = 11$. Now, $|B : \mathbf{N}_B(P)| = 12$. Consider the permutation group $\bar{B}$ induced by the action of B on the cosets of $\mathbf{N}_B(P)$. Now, $\bar{B}$ has degree 12 and has order divisible by 11 because (by hypothesis) P is not normal in B. Therefore $\bar{B}$ is a 2-transitive group whose order divides $24 \cdot 11^2$. A quick inspection on the list of 2-transitive groups of degree 12 in `magma` shows that this is impossible. This final contradiction gives that P is normal in B.

As usual, we denote by X_P the normal quotient of X via P. The rest of the proof is a case-by-case analysis depending upon the structure of the normal quotient X_P. At the end of the proof of each case we use the symbol $\blacksquare$ to mean that the theorem is proved in the case under consideration.

As $|B_v| = 4$, we have $B_v \cap P = 1$ and P acts semiregularly on $V(X)$. Thus the orbits of P on $V(X)$ have size p^2 and $|V(X_P)| = 6$. In particular, X_P is either the cycle C_6 or the octahedral graph O (depending on whether X_P has valency 2 or 4).

CASE A: $X_P = O$.

Fix v a vertex of X and let K be the kernel of the action of B on $V(X_P)$. As X_P has valency 4, we obtain $|B/K| = 24$ and $K = P$. Now, the automorphism group W of O is isomorphic to the wreath product $\mathbb{Z}_2 \wr \mathrm{Sym}(3)$, which has order 48. Therefore B/P is isomorphic to a subgroup of index 2 in W. Labelling the vertices of O as $\{1,2,3,4,5,6\}$ (so that $\{\{1,2\},\{3,4\},\{5,6\}\}$ is the system of imprimitivity for W), we may assume that

$$W = \langle (1,2), (3,4), (5,6), (1,3)(2,4), (1,5)(2,6) \rangle.$$

The group W has exactly three subgroups of index 2. Namely,

$$\begin{aligned} W_1 &= \langle (1,2)(3,4), (1,2)(5,6), (1,3)(2,4), (1,5)(2,6) \rangle, \\ W_2 &= \langle (1,2), (3,4), (5,6), (1,3,5)(2,4,6) \rangle, \\ W_3 &= \langle (1,2)(3,4), (1,2)(5,6), (1,3,5)(2,4,6), (1,2)(3,6)(4,5) \rangle. \end{aligned}$$

The group B/P acts by conjugation as a group of automorphisms on P. So, if $B/P \cong W_i$, then W_i admits an action as a group of automorphisms on P. Assume that B/P acts faithfully on P, that is, $\mathbf{C}_B(P) = P$. In particular, W_i admits a faithful irreducible action on P. Suppose that P is cyclic. Then $\mathrm{Aut}(P)$ is cyclic and so B/P is cyclic. However, W_1, W_2 and W_3 are not cyclic, a contradiction. Thus P is an elementary abelian p-group and $\mathrm{Aut}(P) \cong \mathrm{GL}_2(p)$ (the group of 2×2 invertible matrices). It is clear that the group $\mathrm{SL}_2(p)$ has index 2 in $\mathrm{GL}_2(p)$ and that $\mathrm{SL}_2(p)$ contains a unique element of order 2. This shows that W_i has a normal subgroup T with W_i/T cyclic and with T containing at most one involution. A direct inspection on W_1, W_2 and W_3 shows that such a normal subgroup T does not exist. Therefore B/P does not act faithfully on P and $\mathbf{C}_B(P) > P$.

Another direct inspection on W_1, W_2 and W_3 shows that W_1 and W_3 contain a unique minimal normal subgroup (namely, $\langle(1,2)(3,4),(1,2)(5,6)\rangle$, which has order 4) and W_2 contains exactly two minimal normal subgroups ($\langle(1,2)(3,4)(5,6)\rangle$ having size 2 and $\langle(1,2)(3,4),(1,2)(5,6)\rangle$ having size 4). Therefore, $\mathbf{C}_B(P)$ contains a minimal normal subgroup Q with $|Q| = 2$ or $|Q| = 4$. Suppose that $|Q| = 2$ (and so $B/P \cong W_2$). Now, $W_2/\langle(1,2)(3,4)(5,6)\rangle \cong \mathrm{Alt}(4)$ the alternating group on 4 letters. Arguing as in the previous paragraph, we see that $\mathrm{Alt}(4)$ cannot act faithfully on P. Therefore $|\mathbf{C}_B(P) : P| \geq 4$ and $\mathbf{C}_B(P)$ contains a minimal normal subgroup R with $|R| = 4$. So, replacing Q by R if necessary, we may assume that $|Q| = 4$.

Since Q is characteristic in $\mathbf{C}_B(P)$, we get that Q is normal in B. As 4 does not divide $|V(X)|$, we get that $|Q_v| = 2$ and the Q-orbits have size 2. So, the quotient graph X_Q is a cycle of length $3p^2$. Now from Theorem 2.1 (a), we obtain $X = C(2; 3p^2, 1)$ and X is as in Theorem 1.1 (i). ■

For the remainder of the proof we may assume that $X_P = C_6$.

CASE B: P is a minimal normal subgroup of B.

Fix v a vertex of X and let K be the kernel of the action of B on $V(X_P)$. As $|B| = 24p^2$ and as X_P has valency 2, we have $B/K \cong D_{12}$ and $|K_v| = 2$. So, we see that Theorem 2.2 (b) applies (with $s = 2$), and X is isomorphic to $C^{\pm 1}(p; 6, 2)$ or to one of the graphs defined in [12, Lemmas 8.4 and 8.7]. From Sections 3.2, 3.4 and 3.5, Theorem 1.1 (i) holds. ■

For the remainder of the proof we may assume that B has a minimal normal subgroup N with $N \leq P$ and $|N| = p$. Now by Theorem 2.3 the normal quotient X_N is either C_{6p} or a 4-valent graph.

CASE C: $X_N = C_{6p}$.

Let K be the kernel of the action of B on $V(X_N)$. As $|B| = 24p^2$ and X_N is 2-valent, we have $B/K \cong D_{12p}$ and $|K_v| = 2$. So, we are in the position to apply Theorem 2.1 (b) (with $s = 1$) and thus X is isomorphic to either $C^{\pm 1}(p; 6p, 1)$ or to $C^{\pm \varepsilon}(p; 6p, 1)$. From Sections 4.1 and 4.2, we get that in the first case Theorem 1.1 part (ii) (a) holds and in the second case Theorem 1.1 part (ii) (b) holds. ■

For the remainder of the proof we may assume that X_N is a 4-valent graph. So, X is a regular cover of X_N. Denote by K the kernel of the action of B on $V(X_P)$ and recall that $|K| = 2p^2$ because $X_P = C_6$. Also, note that the normal quotient $(X_N)_{P/N}$ is isomorphic to X_P. Now P/N is a minimal normal subgroup of B/N with orbits of size p. The kernel of the action of B/N on X_P is K/N and $|K_v P/P| = 2$. Therefore Theorem 2.1 (b) applies (with $s = 1$ and with B replaced by B/N), and so $X_N = C^{\pm 1}(p; 6, 1)$ or $X_N = C^{\pm \varepsilon}(p; 6, 1)$. From Sections 3.2 and 3.3, we see that $C^{\pm 1}(p; 6, 1)$ is a normal Cayley graph over $\mathbb{Z}_p \times \mathbb{Z}_6$ and that $C^{\pm \varepsilon}(p; 6, 1)$ is a normal Cayley graph over $\mathbb{Z}_p \times D_6$.

Let G/N be the normal subgroup of B/N acting regularly on $V(X_N)$, with $G/N \cong \mathbb{Z}_p \times \mathbb{Z}_6$ or with $G/N \cong \mathbb{Z}_p \times D_6$. Clearly, G acts regularly on $V(X)$ and thus X is a normal Cayley graph over G. From Lemma 5.1, we see that G is isomorphic either to $\mathbb{Z}_{p^2} \times \mathbb{Z}_6$, or $\mathbb{Z}_{p^2} \times D_6$, or $\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes \mathbb{Z}_6)$, or $\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes D_6)$.

If G is isomorphic to $\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes \mathbb{Z}_6)$, or $\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes D_6)$, then from Lemma 5.2 G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{6p}$, or $\mathbb{Z}_p \times D_{6p}$, or $\mathbb{Z}_p \times \mathbb{Z}_p \times D_6$, or $\mathbb{Z}_{3p} \times D_{2p}$. Now the proof follows from Sections 4.1, 4.2, 4.3 and 4.4.

If G is isomorphic either to $\mathbb{Z}_{p^2} \times \mathbb{Z}_6$ or $\mathbb{Z}_{p^2} \times D_6$, then the proof follows from Sections 4.5 and 4.6.

It remains to consider the case that $p \leq 7$. When $p = 7$, we see from [18, 19] that there are seven 4-valent arc-transitive graphs on $6p^2$ vertices. A computer computation shows that six of these graphs are isomorphic to one of the graphs defined in Theorem 1.1 (i) and (ii), and the seventh does not admit a group of automorphisms acting regularly on arcs. When $p = 5$, we see from [18, 19] that there are ten 4-valent arc-transitive graphs on $6p^2$ vertices and they all admit a group of automorphisms acting regularly on arcs. It is a computer computation checking that eight of these graphs are isomorphic to one of the graphs defined in Theorem 1.1 (i) and (ii), and the remaining two are given in Section 6. When $p = 3$, we see from [18, 19] that there are five 4-valent arc-transitive graphs on $6p^2$ vertices and they all admit a group of automorphisms acting regularly on arcs. It is a computer computation checking that three of these graphs are isomorphic to one of the graphs defined in Theorem 1.1 (i) and (ii), and the remaining two are given in Section 6. When $p = 2$, the result follows again with a straightforward computation. □

Proof of Corollary 1.2. Observe that from Sections 3.1, 3.2 and 4.3 the graphs $C(2; 3p^2, 1)$, $C^{\pm 1}(p; 6, 2)$ and the graphs in Theorem 1.1 (ii) (e) are not one-regular. Now the result follows immediately from Theorem 1.1. □

6 Description of the graphs in Theorem 1.1 part (iii) and Corollary 1.2 (iii)

New now describe the exceptional graphs in Theorem 1.1 (iii) and Corollary 1.2 (iii).
CASE $p = 2$.

(i) $X = \mathrm{Cay}(\langle x \rangle, \{x, x^{-1}, x^5, x^{-5}\})$ where $\langle x \rangle$ is a cyclic group of order 24. Now, X is a normal one-regular Cayley graph.

(ii) $X = \mathrm{Cay}(\langle x \rangle, \{x, x^{-1}, x^7, x^{-7}\})$ where $\langle x \rangle$ is a cyclic group of order 24. Now, X is a normal one-regular Cayley graph.

(iii) $X = \mathrm{Cay}(\mathrm{SL}(2,3), S)$ and with connection set

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right\}.$$

Now, X is a normal one-regular Cayley graph.

(iv) $X = \mathrm{Cay}(\mathrm{SL}(2,3), S)$ and with connection set

$$S = \left\{ \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \right\}.$$

Now, X is a normal one-regular Cayley graph.

(v) $X = \mathrm{Cay}(G, S)$ with $G = \langle (1,2,3), (1,2), (4,5,6,7) \rangle \cong D_6 \times \mathbb{Z}_4$ and $S = \{(1,3)(4,6)(5,7), (1,2,3)(4,5,6,7), (2,3)(4,6)(5,7), (1,3,2)(4,7,6,5)\}$. Now, X is neither a normal Cayley graph nor one-regular.

CASE $p = 3$.

(i) X is the Cayley graph $\mathrm{Cay}(G, S)$, where G is the subgroup of $\mathrm{GL}(3, 3)$ generated by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and with connection set

$$\left\{ \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right.$$
$$\left. \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Now, it is a computation to verify that X is not one-regular.

(ii) X is the Cayley graph $\mathrm{Cay}(G, S)$, where G is the subgroup of $\mathrm{GL}(3, 3)$ generated by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and with connection set

$$\left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Now, it is a computation to verify that X is one-regular.

CASE $p = 5$.

(i) X is the coset graph $\mathrm{Cos}(G, H, g)$ where

$$G = \langle (1, 2, 3, 4, 5), (2, 5)(3, 4), (6, 7, 8, 9, 10), (8, 9, 10) \rangle \cong D_5 \times \mathrm{Alt}(5),$$

$H = \langle (1, 5)(2, 4)(6, 9)(8, 10), (6, 10)(8, 9) \rangle$ and $g = (1, 3)(4, 5)(6, 10)(7, 8)$. Now, $|\,\mathrm{Aut}(X)| = 1200$ and hence X is not one-regular.

(ii) X is the coset graph $\mathrm{Cos}(G, H, g)$ where

$$G = \langle (1, 2, 3, 4, 5), (2, 5)(3, 4), (6, 7, 8, 9, 10), (8, 9, 10) \rangle \cong D_5 \times \mathrm{Alt}(5),$$

$H = \langle (1, 5)(2, 4)(7, 10)(8, 9), (7, 9)(8, 10) \rangle$ and $g = (1, 2)(3, 5)(6, 7)(8, 9)$. Now, $|\,\mathrm{Aut}(X)| = 600$ and hence X is one-regular.

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