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ON MARGINAL DEFORMATIONS OF $\mathcal{N} = 4$ SUPER YANG–MILLS THEORY

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To my family

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Chapter 1

Introduction

In the last decades, string theory has become the most important and studied model for a microscopic description of the fundamental physics including gravity. According to its historical ideas, at distances of the order of the Planck scale, the pointlike particles description of nature breaks up and a new one–dimensional object (the string) must be introduced. The oscillations of these fundamental constituents give rise to a spectrum of particles with different energies and spin which look like localized pointlike objects to a low energy observer. Moreover, all string theories contain a massless spin–two particle which can consistently interact with others particles only by gravitational means. Therefore string theory turns out to be a fundamental model which has the appeal to contain gravity in a natural way and which can be reduced to classical quantum field theory in the low energy limit. Even though it has not been possible to test any of its predictions so far, string theory keeps attracting growing attention for its beautiful mathematical formulation and for the high non–triviality of the challenges it poses.

Besides the approach to string theory as a microscopic model for nature, in the last ten years a new strong tendency showed up, recovering with some surprise the original motivation string theory was introduced for. In fact, string theory was originally formulated in the late 1960s as an attempt to explain the large number of mesons and hadrons that were being produced in the accelerator experiments of those days. Hadronic resonances seemed to exist with rather high spin so that it was not plausible that they were all fundamental. The idea was to consider all these particles as different oscillation modes of a single element: the string. Some of the characteristics of the hadron spectrum turned out to be well described by the string model. For instance, the mass m of the lightest hadron with fixed spin obeys the relation $m^2 \sim T\,J^2 + const$; the relation between mass and angular momentum can be understood if we consider the hadron as a rotating relativistic string of mass m and tension T. On the other hand, some phenomenological as well as theoretical aspects were not in accord with the string model and led to its progressive decline. After the discovery of non-abelian gauge theories and their successful application to strong interactions, the idea to give a stringy description of hadrons was definitively abandoned.

After few years, however, there was a strong revival of the idea of a string–gauge connection, since 't Hooft [1] observed that a $SU(N)$ gauge theory could be considered under a particular limit, namely sending $N \to \infty$, in such a way that its diagrammatic perturbative expansion resembled very much a genus expansion of a string theory with coupling constant proportional to $1/N$. This argument suggested that could exist proper string descriptions for the planar limit of general gauge theories. That is to say, a single physical system could have two dual descriptions, one in terms of a canonical quantum field theory and one other in terms of a string theory; these formulations could be appropriate and best suited for different energetic regimes. This nice proposal remained a rather abstract possibility for more than twenty years, until Maldacena suggested a specific realization of it known as the AdS/CFT conjecture [2, 3, 4].

As we can deduce by its name, the correspondence is realized by a particular conformal invariant field theory. Conformal invariance is a very non–trivial property to ask for a four–dimensional quantum field theory but supersymmetry, along with its consequent non–renormalization theorems, comes in some help. In fact the maximally (rigid) supersymmetric field theory in four dimensional space–time, the so called $\mathcal{N} = 4$ Super Yang–Mills, turns out to be a full quantum conformal field theory and therefore is invariant under the four dimensional conformal group $SO(4, 2)$. Beside this global symmetry [1], the theory is invariant under the $SU(4)$ R–symmetry group. A string theory dual formulation of $\mathcal{N} = 4$ SYM should live in a background which respects the symmetries of the field theory. Locally there is only one space with $SO(4, 2)$ isometry: five–dimensional Anti–de–Sitter space. Moreover, we expect strings associated to $\mathcal{N} = 4$ SYM to be supersymmetric too and thus live in a ten dimensional space. Since the gauge theory has an $SU(4) \simeq SO(6)$ global symmetry it is natural to think that the five extra dimensions needed should be given by an S^5 space. Therefore it would be plausible if the superstring theory associated to $\mathcal{N} = 4$ SYM lived in the ten dimensional space $AdS_5 \times S^5$.

This heuristic argument just provides a symmetry based indication about the dual theories which enter the correspondence. More precisely, the conjecture is deduced by taking two different low energy limits of a single system and comparing the results. The starting system consists in Type IIB strings propagating in ten–dimensional Minkowsky background where N coincident D3–brane have been placed. Then this system is analyzed at low energies considering two different points of view. On the one hand it is studied the low energy dynamics on the world–volume of the D–branes, which decouples from the bulk gravity dynamics and gives rise to $\mathcal{N} = 4$ SYM. On the other hand, the system is studied in its supergravity approximation, searching for a solution which carries the right fluxes and charges. Then, in the low energy limit, the near horizon physics of the solution decouples from bulk free supergravity and leads to $AdS_5 \times S^5$ geometry. Hence it is natural to identify the two different systems found in the low energy limiting process. As a final result, the AdS/CFT conjecture states that $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ is in duality correspondence with Type IIB strings moving on a $AdS_5 \times S^5$ background with N flux unit of five–form Ramond–Ramond field strength F_5.

[1] We will give a much more detailed and precise description of the global symmetries of $\mathcal{N} = 4$ Super Yang–Mills in the main text.

Having recognized these dual descriptions, an entire dictionary between the two theories is drawn in order to compare them. For instance, the spectrum of scaling dimensions of gauge invariant operators in the CFT is put in correspondence with the spectrum of energies of string states.

However a direct comparison between quantities in the two theories is not easy to perform. In fact it is important to notice that the two descriptions are perturbatively valid in different regimes of the coupling constants. The precise parameter connection is given by:

$$g^2 N = \lambda = \frac{R^4}{\alpha'^2}\ , \qquad \frac{1}{N} = \frac{4\pi g_s}{\lambda} \tag{1.0.1}$$

where g is the Yang-Mills coupling constant, α' the inverse string tension, g_s is the topological expansion parameter in string theory and R the common radius of $AdS_5 \times S^5$. As we can see from (1.0.1), after taking the planar limit, the perturbative regime in the field theory is obtained for small values of the 't Hooft coupling λ. From the string side, sending $N \to \infty$ corresponds to neglect g_s corrections. Then the perturbative α' expansion is valid for big values of λ. It becomes clear that the duality is of the strong/weak type and, if we want to check its validity, we cannot use pure perturbative results.

It is fundamental then to find exact quantities, objects that do not depend on the energy scale. For instance, from the field theory side, one could study non-renormalization properties of some special operators, or search for those particular sectors for which the anomalous dimension can be computed exactly with some non–perturbative argument. If one is able to find such a kind of objects, then a direct comparison with the correspondent string energy spectrum can be done. The chiral primary operators (CPO) of $\mathcal{N} = 4$ SYM are an example of protected objects which are suitable to check the correspondence. Lot of tests have been done computing anomalous dimensions, energies and correlation functions and up to now a successful matching has been obtained. As a consequence of these and other checks, nowadays we consider the AdS/CFT correspondence well established and not anymore a simple conjecture.

The natural sequel was to extend the original conjecture to some more general and less constrained theories, with the aim to get as near as possible to models of phenomenological interest. A first obvious attempt was to try and break the maximal supersymmetry of $\mathcal{N} = 4$ to some lower degree adding proper terms to the Lagrangian and arguing the correspondent deformation in the supergravity side. This issue has been pursued along different directions and considering various limits. One the most intriguing possibility is to try to break supersymmetry while preserving superconformal invariance in the deforming process. We will call this process *marginal deformation* of AdS/CFT conjecture.

If we want to analyze the field theory side of this procedure, we should consider adding marginal operators to the $\mathcal{N} = 4$ action in order not to break scale invariance. The exact marginality of an operator at the quantum level is not an easy property to determine. However, in [5] a systematic analysis of this problem has been done and the form of the most general marginal deformation of $\mathcal{N} = 4$ SYM has been found. Leigh-Strassler (LS) deformed theories were object of studies ([63]-[71]) even before their role in AdS/CFT context was completely clarified. Then, when Lunin and Maldacena [6] (see also [7]) found the supergravity solution dual to one particular subclass of these deformations (the so called β–deformations), we had a renewed interest in their analysis. In fact, with Lunin-Maldacena conjecture a more general correspondence involving an $\mathcal{N}=1$ superconformal theory and a brand new playground to check the string–gauge duality ideas emerged.

The aim of this thesis is to give a throughout analysis of the field theory side of this new correspondence and, in more generality, to give a detailed description of the whole class of marginal deformations of $\mathcal{N}=4$ SYM. We will start with an introductory part (Chapter 2), where, after a description of $\mathcal{N}=4$ SYM, the whole issue of its marginal deformations will be treated in details. We then conclude Chapter 2 introducing the Lunin–Maldacena supergravity solution dual to the β–deformed theory and briefly explaining how this was found.

With Chapter 3 and 4 we start reviewing and extending the results obtained in ([8]-[12]). Chapter 3 is devoted to analyze the problem of finding under what conditions the deformed actions obtained following the Leigh–Strassler arguments are actually superconformal. In fact, as will become clear after reading Chapter 2, LS analysis of marginal deformations of $\mathcal{N}=4$ SYM gives the final result for the superconformal actions in a somewhat implicit form. According to LS procedure, we should consider the following $\mathcal{N}=1$ action:

$$\begin{aligned} S &= \int d^8z \ \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \ \mathrm{Tr}(W^\alpha W_\alpha) + \\ &+ih\int d^6z \ \mathrm{Tr}\left(q\,\Phi_1\Phi_2\Phi_3 - \frac{1}{q}\,\Phi_1\Phi_3\Phi_2\right) + \frac{i\,h'}{3}\int d^6z \ \mathrm{Tr}(\Phi_1^3+\Phi_2^3+\Phi_3^3) + \\ &+i\bar{h}\int d^6\bar{z} \ \mathrm{Tr}\left(\frac{1}{\bar{q}}\,\overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3 - \bar{q}\,\overline{\Phi}_1\overline{\Phi}_3\overline{\Phi}_2\right) - \frac{i\,\bar{h}'}{3}\int d^6\bar{z} \ \mathrm{Tr}(\overline{\Phi}_1^3+\overline{\Phi}_2^3+\overline{\Phi}_3^3) \end{aligned} \tag{1.0.2}$$

which we write here in superfield formalism (with notations introduced in the main text) and that can be easily reduced to $\mathcal{N}=4$ SYM by choosing $h'=0$, $q=1$ and $h=g$. However, this action does not describe a superconformal theory by itself because its couplings must satisfy a specific constraint in order to select a fixed point of the renormalization group. As we will see in details, the superconformal condition on the couplings turns out to coincide with the requirement of vanishing anomalous dimension for the elementary fields:

$$\gamma_\Phi(h,h',q,g)=0 \tag{1.0.3}$$

Therefore, in order to obtain the actual superconformal action, the above constraint must be explicitly solved. The solution of (1.0.3) is not known a priori and, in general, it is not at all trivial. One of our aims will be therefore to give an explicit realization of the fixed point condition in order to finally deal with a true superconformal action.

In Chapter 3 we will study extensively this problem considering the subclass of the theories in (1.0.2) obtained by choosing $h' = 0$. These theories are known as β–deformations of $\mathcal{N} = 4$ SYM because it is often used the reparametrization $q \equiv e^{i\pi\beta}$. We will see that, depending on the choice made on the values for the couplings, qualitatively different scenarios will show up, revealing very subtle differences between similarly deformed theories. We will start analysing this problem in the planar case, which is the relevant one in the AdS/CFT context, and then we will extend the analysis to finite N values. We will see that for the simple case of planar real β-deformation (the most significant in the AdS/CFT context) an exact solution for the constraint (1.0.3) can be given. This is done in a simple and elegant way just exploiting a formal analogy that can be found between the real β–deformed theory theory and non commutative quantum field theories. As a result the exact form of the planar superconformal action will be given, providing a very non trivial example of four dimensional $\mathcal{N} = 1$ supersymmetric quantum conformal theory.

In all the other cases, the problem must be approached in a perturbative fashion and its solution will become more involved. We will see that a physical (not scheme dependent) perturbative definition of the surface of fixed points can be given for complex values of the β parameter only for the full finite N theory. In fact the physical request of scheme independence in the definition of the theory is achieved thanks to subleading contributions. Therefore, taking the planar limit we will be forced to restrict to the already mentioned real–β case.

As an aside of this analysis, we will use the marginally deformations of $\mathcal{N} = 4$ SYM to study and extend the validity of finiteness theorems for the gauge beta–functions.

In Chapter 4 a very different problem will be addressed. As we already stressed, since it has been found a new and more general example of AdS/CFT correspondence, it is then obvious to try to reproduce the checks done for the original case and find whether they work or not for the deformed one. In this context, we will concentrate on the relevant problem of finding operator protected under renormalization. We specialize on the structure of the chiral ring of the theory and propose an alternative procedure, with respect to the resolution of the mixing problem, to find CPO's in a general superconformal theory. The method we propose is based on the perturbative computation of the effective superpotential and we will see that in some cases it drastically simplify the calculations with respect to the classical method. Using this procedure we give a detailed description of the chiral ring of the marginally deformed theories.

We conclude this introduction, raising the attention to the huge amount of literature that has been produced with regard of marginal deformation of AdS/CFT. In [72] the effective action of these theories have been analyzed while instanton calculations were performed in [73]. For considerations about string configurations relevant for the correspondence see also [95]-[102]. Integrability issues have been treated for instance in [74]-[85]. D-brane configurations [103]-[114] or relation with dipole theories [115]-[118] were also studied. For further interesting papers see [86]-[94].

Chapter 2

$\mathcal{N} = 4$ SYM and its Marginal Deformations

Our starting point is $\mathcal{N} = 4$ super Yang–Mills theory. In this Chapter we describe its most important features, with particular attention to its finiteness properties. Then we show how we can deform the theory breaking supersymmetry down to $\mathcal{N} = 1$ while preserving superconformal invariance. As a result, we obtain the most general marginal deformation of $\mathcal{N} = 4$ SYM which will be the object of interest in the following Chapters. We conclude this introductory part by giving a brief description of the AdS/CFT interpretation of these marginally deformed quantum field theories and describing their dual supergravity geometries.

2.1 $\mathcal{N} = 4$ SYM and finiteness

Four dimensional $\mathcal{N} = 4$ SYM is a very special quantum field theory because, on the one hand it is simple to treat and analyze because of its rich symmetry structure; on the other, its constrained nature does not prevent it to exhibit some very non trivial properties. In this Section we will review some of these characteristics, focusing on what will turn out to be useful in the description of the marginal deformations of the theory itself. Meanwhile we will introduce our main conventions, referring the reader to [13] for further details.

$\mathcal{N} = 4$ SYM is the maximally supersymmetric quantum field theory in four dimensional space-time because supersymmetry transformations are generated by sixteen real supercharges; a larger number would require the inclusion of gravity in the theory. The request of maximal supersymmetry totally constrain both the matter content and the form of the action of the theory. To be specific, the $\mathcal{N} = 4$ super Yang–Mills multiplet consists in six real scalars ϕ_i, four complex Weyl spinors λ_i and a gauge field A_μ. All of these fields are massless and transform in the adjoint representation of the gauge group which will be chosen to be $SU(N)$ (of course the gauge group is not *a priori* determined by supersymmetry).

As for the action, there exist several equivalent descriptions encoding the above matter content and the correct supersymmetry invariance. For instance, one could choose different component formulations or use $\mathcal{N}=1$ or $\mathcal{N}=2$ superfields. We will find convenient to use $\mathcal{N}=1$ superfield formalism in order to easily take advantage of D-algebra techniques. In this framework, the field content just described is reorganized in the following way: three chiral superfields $\Phi_{1,2,3}$ and their complex conjugates contain the six real scalars and three of the four complex Weyl spinors; a real vector superfield V includes the gauge field A_μ and the remaining Weyl spinor as a gaugino. In terms of these $\mathcal{N}=1$ superfields the action can be written as:

$$\begin{aligned} S &= \int d^8z \, \mathrm{Tr}\left(e^{-gV}\,\overline{\Phi}_i\, e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \, \mathrm{Tr}\,(W^\alpha W_\alpha) + \\ &+ i\,g \int d^6z \, \mathrm{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) + i\,g\int d^6\bar{z}\, \mathrm{Tr}(\overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3 - \overline{\Phi}_1\overline{\Phi}_3\overline{\Phi}_2) \qquad (2.1.1) \end{aligned}$$

where we have defined $d^8z \equiv d^4x\, d^2\theta\, d^2\bar{\theta}$ and $d^6z \equiv d^4x\, d^2\theta$. We also introduced $W_\alpha = i\,\overline{D}^2(e^{-gV}D_\alpha\, e^{gV})$, the superfield strength associated to V. In what follows we will be interested only in studying perturbative properties of $\mathcal{N}=4$ SYM and its deformations and the gauge coupling g will be chosen to be real. In this way the quadratic term in W_α is hermitian up to surface contributions. It is worth noticing that in order to have maximal supersymmetry it is crucial that the superpotential coupling coincides with the gauge coupling g, so that the theory depends only on one parameter.

Let us now analyse the classical symmetries of the theory. Of course, by construction the theory is invariant under $\mathcal{N}=4$ extended super–Poincare' transformations, whose generators form an algebra which has an $SU(4)_R$ automorphism group, the R–symmetry group of the theory. Moreover only dimensionless couplings appear in the action, so that we have classical scaling invariance in addition to the super–Poincare' one. It can be shown that all of these transformations sum up to build the $\mathcal{N}=4$ superconformal algebra (see [13] for details). This algebra generates all the classical global symmetry transformations of the theory. Notice that in the formalism we have chosen (i.e. using $\mathcal{N}=1$ superfield representations), only an $SU(3)\times U(1)$ subgroup of the R-symmetry group remains manifest: the Φ's transform in the $\mathbf{3}$ of $SU(3)$, the antichiral superfields in the $\overline{\mathbf{3}}$ and V is a singlet under $SU(3)$. The $U(1)$ factor behaves like a sort of residual R-symmetry for the $\mathcal{N}=1$ superfield notation. The remaining transformations that complete this subgroup to $SU(4)_R$ cannot be directly read from the action as they are too complicated but they can be of course written out [13].

Having introduced the action and the classical symmetries of the theory we now turn on describing what happens at the quantum level. We will consider the so called superconformal phase of $\mathcal{N}=4$ SYM, choosing the vacuum with null expectation values for all the scalar moduli. In this way $\mathcal{N}=4$ SYM can be shown to be a finite (conformal invariant) full quantum theory.

A conformal quantum theory is defined by requiring that the beta functions of the coupling constants exactly vanish. Therefore, to analyze this feature in the case of $\mathcal{N}=4$ SYM, we just need to consider the running of the coupling g, looking for instance at its renormalization as a superpotential parameter. In superfield formalism, from dimensional analysis, it is clear that a vertex coupling three chiral superfields is finite. Then, if we are interested in computing the beta function of g, we should only concentrate on the renormalization of the Φ wave function. This is to say that we just need to compute the propagator of the chiral superfield and the corresponding multiplicative renormalization factor Z_g. Hence, using dimensional regularization, the beta function will read

$$\beta(g) \sim g^2 \frac{\partial Z_g^{(1)}}{\partial g} \tag{2.1.2}$$

where $Z_g^{(1)}$ is the coefficient of the simple pole term in the Fourier transform of the chiral superfield propagator. Remembering the definition of anomalous dimension of the elementary field γ_Φ, we immediately deduce that

$$\beta(g) \sim \gamma_\Phi \tag{2.1.3}$$

We then observe that non–renormalization properties of the superpotential relate beta functions to anomalous dimensions of elementary fields. In the $\mathcal{N} = 4$ SYM case the relation is trivial because we just have a single coupling g and a very symmetric superpotential. However, as we will see in the next Sections, this fact is still valid for more general supersymmetric theories.

Coming back to $\mathcal{N} = 4$ SYM, from (2.1.3) we see that to check out finiteness one has to study perturbatively the pole structure of the Φ two–point function, showing that there are no divergences loop by loop. At one and two–loop level, component formulations were sufficient to demonstrate finiteness [14, 15]. Taking advantage of the power of superspace techniques it has also been possible to prove that β_g is vanishing up to three loops [16][17][18].

Besides these perturbative checks, there exist proofs of all order finiteness of the theory. In [20] this result is achieved using a light cone superspace formulation while in [21] an anomaly based approach has been used. In [19] the use of extended superspace jointly with non–renormalization properties related to the $\mathcal{N} = 2$ formalism has been sufficient to prove the finiteness of the theory. In view of all these results, from now on we will *assume* that $\mathcal{N} = 4$ SYM is *finite* at all orders of perturbation theory. This assumption is crucial, as will be the starting point for the whole analysis of marginal deformations of $\mathcal{N} = 4$ SYM that we will pursue in the next Chapters.

2.2 Leigh-Strassler analysis

We now turn to consider the most general marginal deformation of $\mathcal{N} = 4$ SYM. For the reasons discussed in the introduction, we would like to modify the action in (2.1.1) adding terms which (partially) break supersymmetry while preserve superconformal invariance. In order to do this we have to carefully choose the operators we want to add to the Lagrangian. We remind that, in a n dimensional field theory, the qualitative behaviour of operators under renormalization group flow can be classified in terms of their scaling dimension Δ. If we call μ the energy scale at which we evaluate the operators, we can distinguish three different cases:

- Relevant operators ($\Delta < n$) for which the normalization grows with increasing μ
- Irrelevant operators ($\Delta > n$), which dies away with renormalization flow
- Marginal operators ($\Delta = n$) whose normalization does not depend on μ

If we want to deform the theory consistently with conformal invariance, it is clear that we should only concentrate on marginal operators. Otherwise we would introduce a dimensionful parameter in the theory therefore breaking scale invariance. As soon as we add to the action a term $\delta\mathcal{L} = h\,\mathcal{O}$, where $\mathcal{O}$ is marginal even at the quantum level, new fixed points are generated starting from the $\mathcal{N}=4$ ones and are parametrized varying h. As a consequence we obtain a line of conformal field theories but break supersymmetry down to $\mathcal{N}=1$. In general proving quantum marginality of an operator turns out to be a difficult task. The situation simplify slightly in the case of supersymmetric theories. In [6] a systematic analysis of this problem has been performed and we are going to review it hereafter.

The analysis is based on the structure of beta functions for the gauge and superpotential couplings in a generic $\mathcal{N}=1$ theory. Given a superpotential term of the form:

$$W = h\,\Phi_1....\Phi_n \tag{2.2.1}$$

the beta function for the h coupling can be written as:

$$\beta_h \sim \left(-d_W + \sum_k \left[d(\Phi_k) + \frac{1}{2}\,\gamma_{\Phi_k}\right]\right) \tag{2.2.2}$$

where d_W is the canonical dimension of the superpotential term W, $d(\Phi_k)$ is the canonical dimension of the field Φ_k and γ_{Φ_k} is its anomalous dimension. For what concern the gauge beta function of a generic $\mathcal{N}=1$ theory, it is known [22] that there exists a specific scheme [23, 24] in which it can be written as:

$$\beta_g \sim \left(\left[3C_2(G) - \sum_k T(R_k)\right] + \sum_k T(R_k)\,\gamma_{\Phi_k}\right) \tag{2.2.3}$$

where $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge group of the theory and $T(R_k)$ is the quadratic Casimir of the representation in which Φ_k appears. The relations (2.2.2) and (2.2.3) are valid for a generic $\mathcal{N}=1$ theory. Starting from these expressions for the beta functions one can examine the conditions for a fixed point, thus requiring that they vanish simultaneously. In a general quantum field theory, associated to k couplings we have k independent beta functions. Therefore fixed points (if exist) turn out to be isolated points in the space of the couplings. However it may happen that some of the beta functions are linearly dependent. In this case we only have p equations on k parameters, where p is the number of independent beta functions. So, if a solution to the conformal conditions exists, it generically selects a $(k-p)$–dimensional surface of fixed points in the space of couplings. Translation on this surface corresponds to varying a coupling constant associated to a marginal operator which remains marginal even at the quantum level. As a consequence, the idea is to find out whether it is possible that conformal conditions coming from (2.2.2) and (2.2.3) become linearly dependent. In our case, as we want to study marginal deformations of the action in (2.1.1), we would just have to concentrate on a superpotential of the form of the $\mathcal{N}=4$ SYM one with the addition of classically marginal terms. Then we would have to study the problem of proportionality of the beta functions with the help of (2.2.2) and (2.2.3).

In order to easily visualize this procedure let us consider, as an introductory exercise, the following $\mathcal{N}=1$ action:

$$\begin{aligned} S \; &= \int d^8z \; \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \; \mathrm{Tr}\left(W^\alpha W_\alpha\right) + \\ &+ i\,h \int d^6z \; \mathrm{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) + i\,\bar{h}\int d^6\bar{z} \; \mathrm{Tr}(\overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3 - \overline{\Phi}_1\overline{\Phi}_3\overline{\Phi}_2) \end{aligned} \tag{2.2.4}$$

which differs from the one of $\mathcal{N}=4$ SYM just for the superpotential coupling constant that is now chosen to be generical. Once again, the gauge group is $SU(N)$ and all the fields are in the adjoint representation. This theory can be thought to be obtained by adding the cubic Φ interactions to the free theory without superpotential. We can now use equation (2.2.2) substituting the values $d_W=3$ and $d(\Phi_k)=1$ and obtaining:

$$\beta_h \sim \gamma_\Phi \tag{2.2.5}$$

Note that in the present case, as for $\mathcal{N}=4$ SYM, the anomalous dimensions γ_{Φ_k} are the same for all k and we are denoting their common value with γ_Φ. Now we consider the gauge beta function in (2.2.3) and impose $T(R_k)=C_2(G)$, as all the fields appear in the adjoint representation. Then it's easy to see that:

$$\beta_g \sim \gamma_\Phi \tag{2.2.6}$$

So we conclude that both the gauge and chiral beta functions are proportional to the anomalous dimension of the elementary field Φ. In order to have fixed points we just have to solve the constraint $\gamma_\Phi(h,g)=0$, which defines a curve in the space of the couplings h and g. In principle we could try to satisfy this constraint order by order in perturbation theory. Up to the order we were able to push the perturbative calculation, we should obtain that $\gamma_\Phi = 0$ *iff* $|h|^2 = g^2$. As already stated, this result could be demonstrated to all orders in perturbation theory using the full $\mathcal{N}=4$ supersymmetry but it is worth noticing that the approach to select finite theories just described makes only use of $\mathcal{N}=1$ supersymmetry and thus can be generalized to more interesting cases. As a final result we obtain the curve of fixed points in the space of couplings (Fig. 2.1) that, in the example we are treating, coincides with $\mathcal{N}=4$ theory at different values of the coupling g.

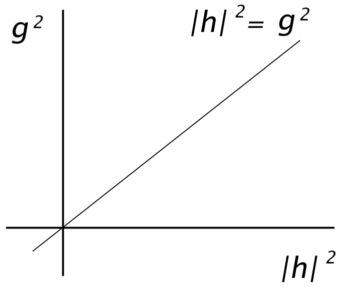

Figure 2.1: Fixed point line in the space of the parameters

2.3 Marginal deformations of $\mathcal{N}=4$ SYM

The procedure described in the previous paragraph can be applied to the case of marginal deformation of the action in (2.1.1). The superpotential of $\mathcal{N}=4$ SYM can be written as:

$$\frac{1}{3}\sum_{ijk} f^{ijk}\,\mathrm{Tr}(\Phi_i\Phi_j\Phi_k) \tag{2.3.1}$$

where the coefficients f^{ijk} are completely antisymmetric in i,j,k. The general classically marginal deformation consists in adding terms like

$$\sum_{ijk} d^{ijk}\,\mathrm{Tr}(\Phi_i\Phi_j\Phi_k) \tag{2.3.2}$$

where now d^{ijk} is completely symmetric in i,j,k and therefore we are adding 10 independent superpotential terms. Now we want to study which of these operators keeps being marginal even at the quantum level for the deformed action. Once these combinations have been selected one finds the form of the most general marginal deformation of $\mathcal{N}=4$ SYM.
One can easily see [6] that among the contributions in (2.3.2) one should consider only:

$$\delta W = d_1\mathrm{Tr}(\Phi_1\Phi_2\Phi_3+\Phi_1\Phi_3\Phi_2)+d_2\mathrm{Tr}(\Phi_1^3+\Phi_2^3+\Phi_3^3) \tag{2.3.3}$$

Thus we add these terms to the $\mathcal{N}=4$ action and conveniently rename the coupling constants obtaining:

$$\begin{aligned} S \;=\; & \int d^8z\ \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right)+\frac{1}{2g^2}\int d^6z\ \mathrm{Tr}(W^\alpha W_\alpha)\ + \\ +ih\int d^6z\ \mathrm{Tr}&\left(q\,\Phi_1\Phi_2\Phi_3-\frac{1}{q}\,\Phi_1\Phi_3\Phi_2\right)+\frac{i\,h'}{3}\int d^6z\ \mathrm{Tr}(\Phi_1^3+\Phi_2^3+\Phi_3^3)\ + \\ +i\bar{h}\int d^6\bar{z}\ \mathrm{Tr}&\left(\frac{1}{\bar{q}}\,\overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3-\bar{q}\,\overline{\Phi}_1\overline{\Phi}_3\overline{\Phi}_2\right)-\frac{i\,\bar{h}'}{3}\int d^6\bar{z}\ \mathrm{Tr}(\overline{\Phi}_1^3+\overline{\Phi}_2^3+\overline{\Phi}_3^3) \end{aligned} \tag{2.3.4}$$

We can then check using (2.2.2) and (2.2.3) that:

$$\beta_{h'} \sim \beta_{hq} \sim \beta_{\frac{h}{q}} \sim \beta_g \sim \gamma_\Phi \tag{2.3.5}$$

Thus the action in (2.3.4), supplied with the condition

$$\gamma_\Phi(h, h', q, g) = 0 \tag{2.3.6}$$

defines the most general marginal deformation of $\mathcal{N} = 4$ SYM. We can immediately reduce to the undeformed theory putting $h' = 0$, $q = 1$, $h = g$.

We stress again that the action (2.3.4) alone does not describe a conformal invariant quantum field theory because we additionally have to ask that $\gamma_\Phi(h, h', q, g) = 0$. In general, this constraint selects a 3–complex manifold of fixed points in the space of couplings. Of course it is of vital importance to try to find an explicit realization of the conformal condition on the parameters. This issue has been studied extensively choosing different deformed actions and will be the subject Chapter 3. As we will see, we can distinguish qualitatively different behaviours associated to different choices for the value of the couplings and a very rich and intriguing scenario will show up.

Let us finally describe the symmetries of the action in (2.3.4). Of course, supersymmetry is broken from $\mathcal{N} = 4$ down to $\mathcal{N} = 1$ and the R–symmetry group from $SU(4)_R$ to $U(1)_R$. However, some discrete symmetries survive as a left-over of this breaking. In fact we have two Z_3 symmetries: the first consists in cyclic permutations of (Φ_1, Φ_2, Φ_3) and a second corresponds to

$$(\Phi_1, \Phi_2, \Phi_3) \rightarrow (\Phi_1, z\Phi_2, z^2\Phi_3) \tag{2.3.7}$$

where z is a cubic root of unity. Moreover, the action is invariant under

$$\Phi_i \leftrightarrow \Phi_j \ , \quad i \neq j \qquad \text{and} \qquad q \rightarrow -\frac{1}{q} \tag{2.3.8}$$

In [25] it has been advocated that the full discrete symmetry group of (2.3.4) is given by the trihedral group $\Delta(27)$.

A far more rich symmetry pattern shows up if we restrict to the case $h' = 0$. In this case the theory has action

$$\begin{aligned} S \;=\; & \int d^8z \; \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \; \mathrm{Tr}(W^\alpha W_\alpha) \\ & + ih \int d^6z \; \mathrm{Tr}(\; q\; \Phi_1\Phi_2\Phi_3 - \frac{1}{q}\; \Phi_1\Phi_3\Phi_2 \;) \\ & + i\bar{h} \int d^6\bar{z} \; \mathrm{Tr}(\; \frac{1}{\bar{q}}\; \overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3 - \bar{q}\; \overline{\Phi}_1\overline{\Phi}_3\overline{\Phi}_2 \;) \qquad , \qquad q \equiv e^{i\pi\beta} \end{aligned} \tag{2.3.9}$$

and it is better known as β–deformed $\mathcal{N} = 4$ SYM for the particular reparametrization of q in terms of the complex β coupling.

One can easily see that the action in (2.3.9) is invariant under an extra global $U(1)_1 \times U(1)_2$ non–R symmetry group, which acts on the fields as

$$\begin{array}{llcl} U(1)_1 : & (\Phi_1, \Phi_2, \Phi_3) & \rightarrow & \left(\Phi_1, e^{i\,\alpha_1}\,\Phi_2, e^{-i\,\alpha_1}\,\Phi_3\right) \\ U(1)_2 : & (\Phi_1, \Phi_2, \Phi_3) & \rightarrow & \left(e^{-i\,\alpha_2}\,\Phi_1, e^{i\,\alpha_2}\,\Phi_2, \Phi_3\right) \end{array} \qquad (2.3.10)$$

For simplicity, we will choose the charge values to be $\alpha_1 = \alpha_2 = 1$. The presence of this extra global symmetry turned out to be fundamental in the discovery of the gravity dual of the β–deformed theory, as we will explain in the next Section. The gravity dual of the much less symmetric action in (2.3.4) still has to be found.

2.4 The Lunin-Maldacena solution

The theory in (2.3.9) is the most interesting case of marginal deformation of $\mathcal{N} = 4$ SYM because it has a gravity dual description in the context of AdS/CFT correspondence [6]. In this Section, for completeness, we briefly review the essential properties of the supergravity solution dual to β–deformed $\mathcal{N} = 4$ SYM.

The general idea of Lunin and Maldacena is the following: if we start with a gravity background with two $U(1)$ symmetries that are realized geometrically (i.e. the geometry contains a two torus), one can generate a new, non singular solution by performing the $SL(2, R)$ transformation:

$$\tau \equiv B + i\sqrt{g} \longrightarrow \tau_\beta = \frac{\tau}{1 + \beta\tau} \qquad (2.4.1)$$

where B is the two–form field, $\sqrt{g}$ is the volume of the two–torus and β is, for the moment, a real parameter. The substitution in (2.4.1) should be seen as a solution generating transformation. Namely, one has to reduce the ten dimensional theory to eight dimensions on the two torus. The eight dimensional gravity theory is invariant under $SL(2, R)$ transformations acting on τ. The deformation (2.4.1) is one particular element of $SL(2, R)$. This particular element has the interesting property that it produces a non-singular metric if the original metric was non-singular. The $SL(2, R)$ transformation could only produce singularities when $\tau \to 0$. But we see from (2.4.1) that $\tau_\beta = \tau + o(\tau^2)$ for small τ. Therefore, near the possible singularities the ten dimensional metric is actually same as the original metric, which was non-singular by assumption.

Then it is conjectured that applying the transformation (2.4.1) to the $AdS_5 \times S_5$ geometry one obtains a supergravity solution that is dual to the β–deformed theory of action (2.3.9). Let us begin by finding the deformed solution and then we will motivate the above conjecture. In the notations of [6], one rewrites the S_5 metric in the following form:

$$
\begin{aligned}
\frac{ds^2}{R^2} &= \sum_{i=1}^{3} d\mu_i^2 + \mu_i^2 d\phi_i^2 \ , \qquad \text{with} \quad \sum_i \mu_i^2 = 1 \\
\frac{ds^2}{R^2} &= d\alpha^2 + s_\alpha^2 d\theta^2 + c_\alpha^2 (d\psi - d\varphi_2)^2 + s_\alpha^2 c_\theta^2 (d\psi + d\varphi_1 + d\varphi_2)^2 + s_\alpha^2 s_\theta^2 (d\psi - d\varphi_1)^2 \\
&= d\alpha^2 + s_\alpha^2 d\theta^2 + \frac{9 c_\alpha^2 s_\alpha^2 s_{2\theta}^2}{4 c_\alpha^2 + s_\alpha^2 s_{2\theta}^2} d\psi^2 + \qquad (2.4.2) \\
&\quad + s_\alpha^2 \left[d\varphi_1 + c_\theta^2 d\varphi_2 + c_{2\theta} d\psi \right]^2 + (c_\alpha^2 + s_\alpha^2 s_\theta^2 c_\theta^2) \left[d\varphi_2 + \frac{(-c_\alpha^2 + 2 s_\alpha^2 s_\theta^2 c_\theta^2)}{c_\alpha^2 + s_\alpha^2 s_\theta^2 c_\theta^2} d\psi \right]^2
\end{aligned}
$$

where $s_\alpha = \sin\alpha$, $c_\alpha = \cos\alpha$, etc. Now the two $U(1)$ symmetries act by shifting φ_1 and φ_2 so that the two torus has a metric given by the last line in (2.4.2). One can compute the τ parameter of this two torus

$$
\tau = i\sqrt{g_0} = i[R^2 s_\alpha^2 (c_\alpha^2 + s_\alpha^2 s_\theta^2 c_\theta^2)]^{1/2} = iR(\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2)^{1/2} \qquad (2.4.3)
$$

where $R = (4\pi g_s N)^{1/4}$. Then one applies the transformation (2.4.1) and finds the solution corresponding to the gravity dual of the deformed theory

$$
\begin{aligned}
ds^2 &= R^2 \left[ds^2_{AdS_5} + \sum_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \hat\beta^2 G \mu_1^2\mu_2^2\mu_3^2 (\sum_i d\phi_i)^2 \right] \\
G^{-1} &= 1 + \hat\beta^2 (\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2) \ , \qquad \hat\beta = R^2\beta \ , \qquad R^4 \equiv 4\pi e^{\phi_0} N \\
e^{2\phi} &= e^{2\phi_0} G \\
B^{NS} &= \hat\beta R^2 G (\mu_1^2\mu_2^2 d\phi_1 d\phi_2 + \mu_2^2\mu_3^2 d\phi_2 d\phi_3 + \mu_3^2\mu_1^2 d\phi_3 d\phi_1) \qquad (2.4.4) \\
C_2 &= -3\beta(16\pi N) w_1 d\psi \ , \qquad \text{with} \quad dw_1 = c_\alpha s_\alpha^3 s_\theta c_\theta d\alpha d\theta \\
C_4 &= (16\pi N)(w_4 + G w_1 d\phi_1 d\phi_2 d\phi_3) \\
F_5 &= (16\pi N)(\omega_{AdS_5} + G\omega_{S^5}) \ , \qquad \omega_{S^5} = dw_1 d\phi_1 d\phi_2 d\phi_3 \ , \qquad \omega_{AdS_5} = dw_4
\end{aligned}
$$

where ω_{S^5} is the volume element of a unit radius S^5 and the metric is written in string frame.

To motivate the conjectured duality we note the following facts. From the field theory side the superpotential in (2.3.9) can be written as:

$$\mathrm{Tr}\left(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2\right) = \mathrm{Tr}\left(\Phi_1 \star \Phi_2 \star \Phi_3 - \Phi_1 \star \Phi_3 \star \Phi_2\right) \tag{2.4.5}$$

where we have introduced the star–product:

$$f \star g = e^{i\pi\beta\left(Q_f^{(1)}Q_g^{(2)} - Q_f^{(2)}Q_g^{(1)}\right)} f \cdot g \tag{2.4.6}$$

and $(Q^{(1)}, Q^{(2)})$ are the charges under the $U(1)_1 \times U(1)_2$ symmetry of (2.3.10). Deforming the theory is then in complete formal analogy with the insertion of a Moyal star–product in the presence of a non-commutative background of matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The difference stays in the fact that the phase introduced by (2.4.6) depends on charges instead of momenta.

From the string theory side, we know that on a stack of D-branes in a background with $B \neq 0$ we have a non commutative field theory with non–commutative parameter θ:

$$\theta \sim \frac{1}{\tau} \tag{2.4.7}$$

Then it follows that the transformation

$$\frac{1}{\tau} \to \frac{1}{\tau} + \beta \tag{2.4.8}$$

can be interpreted as switching on a non-commutative parameter $\theta = \beta$. This correspondence motivated the conjectured duality between the supergravity background of (2.4.4) and the β–deformed SYM theory in (2.3.9).

All we have said so far is valid for the case of a real β parameter. The case of complex β is more complicated and could present some subtleties. In general, it can be recovered by performing $SL(2,R)_s$ transformations of the solutions of the real β case . Here we are referring to the $SL(2,R)_s$ symmetry group of the ten dimensional theory, which should not be confused with the $SL(2,R)$ group that we used in (2.4.1). In other words, one starts with $AdS_5 \times S^5$ and performs a more general $SL(3,R)$ transformation in the eight dimensional theory.

The resulting solution is the following

$$
\begin{aligned}
ds^2 &= R_E^2 G^{-1/4}\left[ds^2_{AdS_5} + \sum_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \frac{|\gamma - \tau_s\sigma|^2}{\tau_{2s}} R_E^4 \mu_1^2\mu_2^2\mu_3^2 (\sum_i d\phi_i)^2\right] \\
e^{-\phi} &= \tau_{2s} G^{-1/2} H^{-1} \ , \qquad \chi = \tau_{2s}\sigma(\gamma - \tau_{1s}\sigma) H^{-1} g_{0,E} + \tau_{1s} \\
G^{-1} &\equiv 1 + \frac{|\gamma - \tau_s\sigma|^2}{\tau_{2s}} g_{0,E} \ , \qquad H \equiv 1 + \tau_{2s}\sigma^2 g_{0,E} \ , \quad \tau_s = \tau_{1s} + i\tau_{2s} \\
g_{0,E} &= R_E^4(\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2) \ , \qquad R_E^4 = 4\pi N \qquad (2.4.9) \\
B^{NS} &= \frac{\gamma - \tau_{1s}\sigma}{\tau_{2s}} R_E^4 G w_2 - \sigma 12 R_E^4 w_1 d\psi \\
C_2 &= [-\tau_{2s}\sigma + \frac{\tau_{1s}}{\tau_{2s}}(\gamma - \tau_{1s}\sigma)] R_E^4 G w_2 - \gamma 12 R_E^4 w_1 d\psi \\
& \quad dw_1 = c_\alpha s_\alpha^3 s_\theta c_\theta d\alpha d\theta \ , \qquad w_2 = (\mu_1^2\mu_2^2 d\phi_1 d\phi_2 + \mu_2^2\mu_3^2 d\phi_2 d\phi_3 + \mu_3^2\mu_1^2 d\phi_3 d\phi_1) \\
F_5 &= 4R_E^4(\omega_{AdS_5} + G\omega_{S^5}) \ , \qquad \omega_{S^5} = dw_1 d\phi_1 d\phi_2 d\phi_3
\end{aligned}
$$

where we wrote $\beta = \gamma - \tau_s\,\sigma$ in terms of the complex structure τ_s of the two–torus and two real parameters γ and σ; the metric is in the Einstein frame.

Chapter 3

Conformality and Finiteness

This Chapter is devoted to the study of an explicit realization of the condition (2.3.6) we must impose on the coupling constants of the action (2.3.4) in order to obtain a superconformal theory. This issue has been widely analyzed in literature ([26]-[34]). Here we present a "state of the art" of our contribution on the subject, describing in detail the features associated to different deformation choices. We start from the simplest case, the planar real β–deformed theory, where an exact solution to the problem has been found [8]. Then we add an imaginary part to the β parameter and describe how the scenario changes [10, 11]. Finally, although we loose the direct connection with AdS/CFT, we relax the planar condition considering the theory at finite N [12].

As an aside, we will use [11, 12] the deformed models to check and generalize the validity of finiteness theorems for the gauge beta functions first proposed in [35, 36].

3.1 Planar real β–deformation

Let us start by analysing the simplest marginal deformation of $\mathcal{N} = 4$ SYM, namely the $SU(N)$ β–deformed theory, where β is chosen to be real. To get contact with the supergravity dual we will consider the theory in the planar limit. We write again the action of the theory:

$$\begin{aligned} S \;=\; & \int d^8z \ \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \ \mathrm{Tr}(W^\alpha W_\alpha) \\ & + ih\int d^6z \ \mathrm{Tr}(\ q\ \Phi_1\Phi_2\Phi_3 - \frac{1}{q}\ \Phi_1\Phi_3\Phi_2\ ) \\ & + i\bar{h}\int d^6\bar{z} \ \mathrm{Tr}(\ \frac{1}{\bar{q}}\ \overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3 - \bar{q}\ \overline{\Phi}_1\overline{\Phi}_3\overline{\Phi}_2\ ) \qquad , \qquad q \equiv e^{i\pi\beta} \end{aligned} \tag{3.1.1}$$

where now β and g are real parameters, while h is in general left complex. We remind that the superconformal invariance condition (i.e. vanishing of beta functions) can be expressed as the vanishing of the anomalous dimensions of the elementary superfields.

Therefore, in order to study superconformal invariance, it is sufficient to focus on the divergent corrections to the propagators of the elementary fields. As we already stated, the condition $\gamma_\Phi(h, q, g) = 0$ can be in general satisfied only perturbatively. However, in the particular case of real β parameter, one is able to find the form of the surface of fixed points at all order of perturbation theory [8]. Therefore, finding the *exact* relation between the coupling constants which ensures conformal invariance, we provide a very non–trivial example of planar $\mathcal{N} = 1$ superconformal action in four space–time dimensions.

Let us see in detail how we can derive this result. In the β–deformed theory we consider a generic L–loop diagram contributing to the propagator of the Φ_i superfield. The crucial observation is the following: If we prove that at the planar level, as long as $q\bar{q} = 1$, this diagram does not depend on q, then we are sure that $|h|^2 = g^2$ is the exact solution of the superconformal invariance equations. In fact, if it is independent of q, the corresponding perturbative contribution is the same for any deformed theory, independently of the choice of the q–deformation. In particular, it is the same for any deformed theory ($q \neq 1$) and for the underformed one ($q = 1$). Focusing on the undeformed case we can conclude that $|h|^2 = g^2$ is the exact condition for the planar superconformal invariance, since $q = 1$ and $|h|^2 = g^2$ bring us back to the $\mathcal{N} = 4$ case which is known to be exactly superconformal. The independence of the perturbative corrections on q allows to extend this statement to any deformed theory.

To conclude the proof we need to show that the contribution from a generic self–energy planar diagram never depends on q. We can focus on diagrams containing only matter vertices because adding vector propagators cannot introduce any q-dependence. We exploit the formal analogy between the deformed theory and noncommutative (nc) field theory. As observed in Section 2.4, the deformed potential can be written as

$$ih \int d^6z \ \mathrm{Tr}(\Phi_1 \star \Phi_2 \star \Phi_3 - \Phi_1 \star \Phi_3 \star \Phi_2) \ + \ \mathrm{h.c.} \tag{3.1.2}$$

where

$$f \star g = e^{i\pi\beta Q_f^{(i)} M_{ij} Q_g^{(j)}} f \cdot g \ , \tag{3.1.3}$$

$Q^{(i)}$, $i = 1, 2$ being the non–R–symmetry $U(1)_1 \times U(1)_2$ charges and M the antisymmetric matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. When drawing a Feynman diagram we can consider the flow of the charges inside the diagram. Observing that the charges are conserved at any vertex and propagate through the straight lines we can formally identify them with the ordinary momenta in noncommutative diagrams. A known property of planar diagrams in nc field theory is that the star product phase factors dependent on the loop momenta cancel out (for a proof see [37, 38, 39]) and only an overall phase depending on the external momenta survives. In our case, exploiting the formal identification of charges with momenta, we can use the same arguments to conclude that any planar diagram will have a phase factor from (3.1.3) depending only on the configuration of the external charges.

In the particular case of self–energy diagrams the overall phase factor is given by:

$$V(Q_{\Phi_1}, Q_{\bar{\Phi}_1}) = e^{i\pi\beta\, Q^{(i)}_{\Phi_1}\, M_{ij}\, Q^{(j)}_{\bar{\Phi}_1}} = e^{i\pi\beta\left(Q^{(1)}_{\Phi_1}Q^{(2)}_{\bar{\Phi}_1} - Q^{(2)}_{\Phi_1}Q^{(1)}_{\bar{\Phi}_1}\right)} = 1$$

In other words, any self–energy planar diagram always contains an equal number of $q = \frac{1}{\bar{q}}$ and $\bar{q} = \frac{1}{q}$ vertices. This concludes the proof of the q independence of perturbative self–energy corrections.

As a conclusion, we state [8] that in the $N \to \infty$ limit the exact condition for superconformal invariance is simply $|h|^2 = g^2$. Therefore, in the planar limit the theory described by the action

$$\begin{aligned} S = &\int d^8z \ \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \ \mathrm{Tr}W^\alpha W_\alpha \\ &+ig\int d^6z \ \mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2) + h.c. \end{aligned} \tag{3.1.4}$$

represents a $\mathcal{N} = 1$ superconformal invariant theory for any value of β real without additional conditions on the couplings. In the context of the AdS/CFT correspondence [2, 3, 4] this is exactly the theory whose strong coupling phase is described by the supergravity dual found in [6]. We conclude our analysis by noticing that the proof we have presented makes repeated use of the requirement $q\bar{q} = 1$ (β real) and cannot be immediately extended to more general cases.

3.2 Planar complex β–deformation

The aim of this Section is to study how the conformal invariance condition can be implemented for the case of complex β–deformation of $\mathcal{N}=4$ SYM. The simple addition of an imaginary part to the β parameter turns out to change dramatically the whole scenario. In fact an exact treatment of the problem in analogy with the case of real β is not possible. The solution must be found from a perturbative point of view as we are going to describe.

We rewrite here the action of the theory:

$$\begin{aligned} S &= \int d^8z \ \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \ \mathrm{Tr}(W^\alpha W_\alpha) \\ &\quad +ih\int d^6z \ \mathrm{Tr}(\ q\ \Phi_1\Phi_2\Phi_3 - \frac{1}{q}\ \Phi_1\Phi_3\Phi_2\ ) \\ &\quad +i\bar{h}\int d^6\bar{z} \ \mathrm{Tr}(\ \frac{1}{\bar{q}}\ \overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3 - \bar{q}\ \overline{\Phi}_1\overline{\Phi}_3\overline{\Phi}_2\ ) \qquad , \qquad q\equiv e^{i\pi\beta} \end{aligned} \tag{3.2.1}$$

where now we are considering β and h complex and a real g. We immediately notice that the phase of h can always be reabsorbed by a field redefinition, so that the effective number of independent real parameters in the superpotential is actually three.

Since we are interested in studying the theory in the planar limit, from now on we will be considering 't Hooft rescaled quantities

$$h\ \rightarrow\ \frac{h}{\sqrt{N}} \qquad g\ \rightarrow\ \frac{g}{\sqrt{N}} \tag{3.2.2}$$

in our perturbative analysis. In order to perform higher order perturbative calculations it is very efficient to rely on $\mathcal{N}=1$ superspace techniques. Supergraphs are evaluated performing standard D-algebra in the loops and the corresponding divergent integrals are computed using dimensional regularization in $D=4-2\epsilon$. We have collected superspace Feynmann rules togheter with some relevant color identities in Appendices A and B. All the bosonic integrals necessary to the computations can be found in Appendix C. In Subsection 3.2.1, we will try to impose loop by loop the condition $\gamma_\Phi(h,q,g)=0$ and we will see that, differently from the real β case, the one–loop condition is not sufficient to ensure finiteness at higher loops. To deal with this problem, we will be led to expand the superpotential couplings in terms of the gauge one, following the coupling constant reduction (CCR) program [40]. Then a study of the pole structure arising from this expansion is performed [10].

In Subsection 3.2.2, the vanishing conditions for the gauge and chiral beta functions are discussed within the CCR context. We will see that scheme dependence problems in the definition of the theory arise, suggesting that the fixed point surface is well defined only for real values of the β parameter [11]. Moreover we will discuss the validity of finiteness theorems [35, 36] for the gauge beta function, extending them to the CCR framework [11, 12].

3.2.1 Pole analysis and CCR program

Let us begin by trying to require direct finiteness for the two-point chiral correlator [10]. When $q\bar{q} \neq 1$, in order to isolate the relevant terms and drastically simplify the analysis, it is convenient [28] to study the condition of conformal invariance considering the difference between contributions computed in the β-deformed theory and the corresponding ones in the $\mathcal{N} = 4$ SYM theory (which is finite and with vanishing beta function). The simplification is due essentially to the following facts: when computing the difference between graphs in the β-deformed and in the $\mathcal{N} = 4$ theory we do not need to consider diagrams that contain only gauge-type vertices since their contributions is the same in the two theories. Instead we concentrate on divergent graphs that contain either only chiral vertices or mixed chiral and gauge vertices. In fact the relevant terms come from the chiral vertices that are actually different in the two theories. Addition of vectors simply modifies the color due to the chiral vertices by the multiplication of g^2 factors which are the same for both theories.

We can now begin our perturbative analysis. At one loop the analysis is very simple and mimics exactly what happens in the β real case. The divergent supergraphs are shown in Fig. 3.1

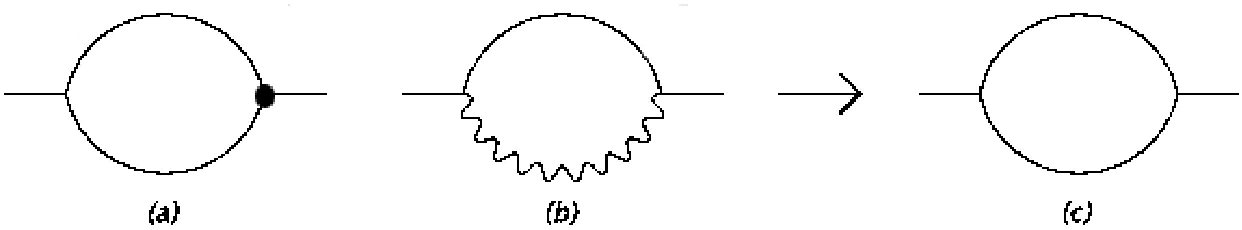

Figure 3.1: Supergraphs contributing at one loop. The wavy line represents a gauge superfield propagator while the full line a chiral superfield one. The dot indicates antichiral vertices.

The D-algebra is the same for the two configurations and its completion gives rise to a logarithmically divergent momentum integral of Fig. 3.1(c). The diagram in Fig. 3.1(b) containing a vector line is the same in the $\mathcal{N} = 4$ and in the β-deformed SYM theory, since it only depends on the gauge coupling g. The diagram in Fig. 3.1(a) contains the chiral couplings: in the deformed theory it gives a contribution

$$\frac{|h|^2}{(4\pi)^2}\left(q\bar{q} + \frac{1}{q\bar{q}}\right)\frac{1}{\epsilon} \tag{3.2.3}$$

while in the $\mathcal{N} = 4$ theory it is proportional to g^2

$$\frac{2g^2}{(4\pi)^2}\,\frac{1}{\epsilon} \tag{3.2.4}$$

where we remind that we are using rescaled quantities in (3.2.2) and refer the reader to Appendix C for the integral values. In order to achieve finiteness one has to impose that the difference between the two results be finite. This implies that to this order the β-deformed theory is conformal invariant if

$$|h|^2 \left(q\bar{q} + \frac{1}{q\bar{q}} \right) = 2g^2 \tag{3.2.5}$$

Now we consider higher-loop contributions. Since we look at the difference between the two-point correlators computed in the β-deformed theory and in the $\mathcal{N} = 4$ SYM, we do not need to consider diagrams that contain only gauge-type vertices their contributions being the same in the two theories. Therefore we concentrate on divergent graphs that contain either only chiral vertices or mixed chiral and gauge vertices. Moreover we observe that a chiral loop can close only if it has the same number of chiral and antichiral vertices, i.e. no polygonal configurations with an odd number of vertices are possible. With these rules in mind it is straightforward to analyze the two- and three-loop contributions. At two loops we have the diagrams shown in Fig. 3.2.

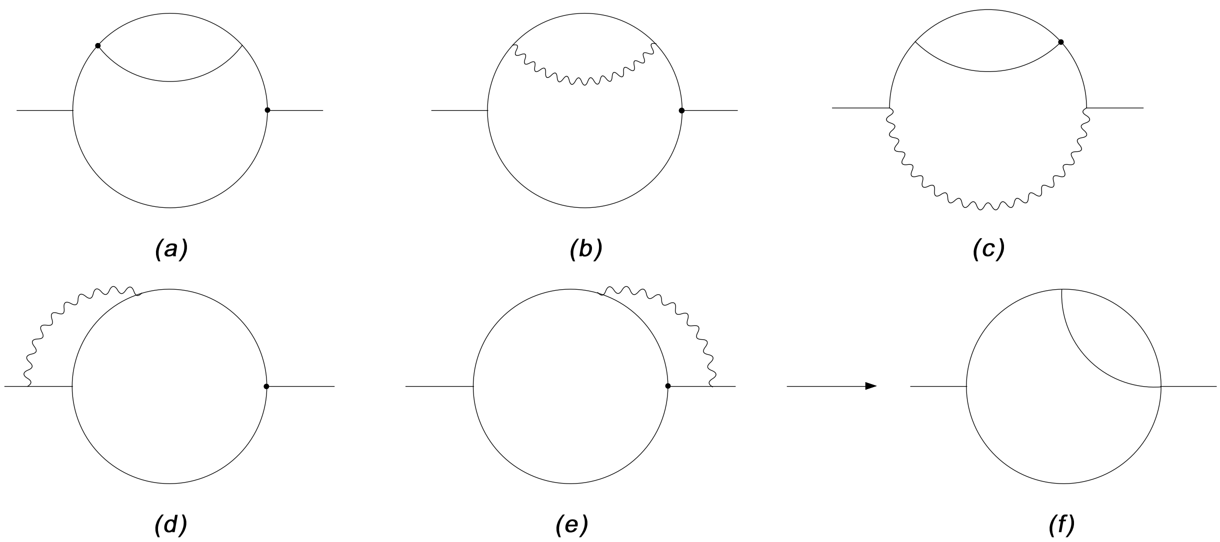

Figure 3.2: Supergraphs contributing at two loops

For all the different configurations the D-algebra leads to the same bosonic integral in Fig. 3.2(f) (see Appendix C). It is very simple to compute the various color factors: we have for the β-deformed theory

$$\begin{aligned} Fig.\, 3.2(a) \quad &\longrightarrow \quad -2 \left[|h|^2 \left(q\bar{q} + \frac{1}{q\bar{q}} \right) \right]^2 \\ Fig.\, 3.2(b) + 3.2(c) + 3.2(d) + 3.2(e) \quad &\longrightarrow \quad 2 \left[|h|^2 \left(q\bar{q} + \frac{1}{q\bar{q}} \right) \right] g^2 \end{aligned} \tag{3.2.6}$$

while correspondingly for $\mathcal{N} = 4$ SYM we find

$$\begin{aligned} Fig.\,3.2(a) &\longrightarrow -8g^4 \\ Fig.\,3.2(b) + 3.2(c) + 3.2(d) + 3.2(e) &\longrightarrow 4g^4 \end{aligned} \tag{3.2.7}$$

If we compute the difference of the results in (3.2.6) and in (3.2.7) and use the condition in (3.2.5), we obtain a zero result. This means that the relation we found at one loop ensures finiteness also at two loops. In fact repeating a similar analysis at three loops one can easily show that (3.2.5) makes the divergent diagrams computed in the deformed theory equal to the corresponding ones in the $\mathcal{N} = 4$ SYM. In the planar limit under the condition in (3.2.5) the two-point correlators do coincide up to three loops [28]. Up to this order the situation is completely parallel to the case of the real β-deformation: there $q\bar{q} = 1$ and the condition in (3.2.5) was simply given by $|h|^2 = g^2$. This condition was actually sufficient to implement finiteness of the two-point correlator in the planar limit to *all orders* in perturbation theory . Moreover the two-point correlator of the real β-deformed theory becomes *exactly* equal to the one computed in the $\mathcal{N} = 4$ theory.

Now we proceed in the study of the β-complex case and examine the situation at four loops. We will find that at this order we are forced to modify the condition in (3.2.5). This should not come as a surprise because of the following reason: as explained above the divergence at one loop is linked to the color factor of the chiral bubble in Fig. 3.1(a) and this leads to the condition in (3.2.5). At two and three loops divergent graphs are constructed either by inserting vector lines on chiral bubbles or by assembling chiral bubbles together. Since the addition of vectors simply modifies the color due to the chiral vertices by the multiplication of g^2 factors, in both cases the condition in (3.2.5) suffices to achieve finiteness. In fact this same reasoning applies also to all the four-loop diagrams that either contain vector lines on chiral bubbles or consist of various arrangements of chiral bubbles: for all these cases the condition in (3.2.5) makes these graphs equal to the corresponding ones in the $\mathcal{N} = 4$ theory. The novelty is that at four loops a new type of chiral divergent structure does arise. We will be able to implement the cancelation of divergences at order g^8 but, differently from the real β case, when β is complex finite parts survive in the β-deformed two-point function which are absent in the corresponding $\mathcal{N} = 4$ two-point function.

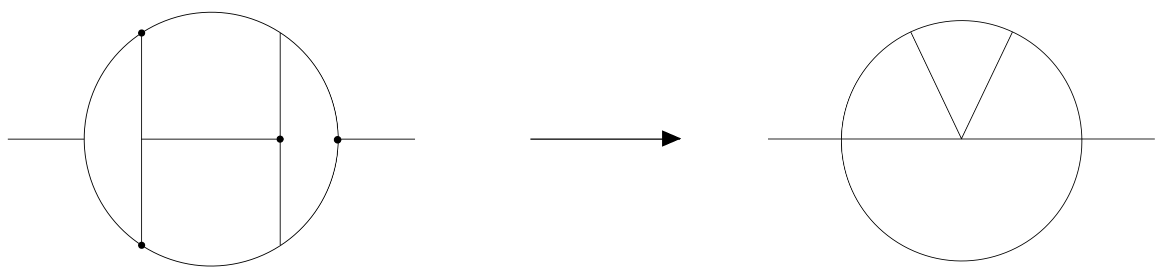

Figure 3.3: Four-loop supergraph and its associated relevant bosonic integral

The new type of chiral supergraph, i.e. not containing chiral bubble insertions, is the one drawn in Fig. 3.3. Completing the D-algebra in the loops one obtains the bosonic graph depicted in the same Figure. The corresponding integral (see appendix C) is divergent

$$\begin{aligned} I_4 &= -\int \frac{d^nk\ d^nq\ d^nr\ d^nt}{(2\pi)^{4n}} \frac{1}{k^2(k+t)^2(q+k)^2(q+r)^2(q+p)^2t^2r^2(t+r)^2} \\ &= -5\ \zeta(5)\ \frac{1}{(4\pi)^8}\frac{1}{\epsilon}\ \frac{1}{(p^2)^{4\epsilon}} \end{aligned} \tag{3.2.8}$$

The color factor is also easily computed: one has to sum over all the various possibilities at the chiral vertices and in so doing one finds

$$C_4 = |h|^8 \left[(q\bar{q})^4 + \frac{1}{(q\bar{q})^4} + 6\right] \tag{3.2.9}$$

The factor in (3.2.9) can be rewritten as

$$C_4 = \frac{|h|^8}{2} \left[\left(q\bar{q} + \frac{1}{q\bar{q}}\right)^4 + \left(q\bar{q} - \frac{1}{q\bar{q}}\right)^4\right] \tag{3.2.10}$$

In this way it is easy to compare the result with the one we would have obtained in $\mathcal{N}=4$ SYM. In fact using the condition in (3.2.5) we find that the β-deformed two-point function at four loops differs from the corresponding $\mathcal{N}=4$ two-point function by the contribution

$$J_4 = -\frac{5}{2}\ \zeta(5)\ \frac{1}{(4\pi)^8}\ \frac{1}{\epsilon}\ \frac{1}{(p^2)^{4\epsilon}}|h|^8 \left(q\bar{q} - \frac{1}{q\bar{q}}\right)^4 \tag{3.2.11}$$

If we want the propagator to be finite this term has to be cancelled. The only way out is to modify the relation in (3.2.5), so that a contribution from a lower-loop order might cancel the unwanted four-loop divergence.

In the spirit of [5], in the space of the coupling constants we are looking for a surface of renormalization group fixed points. To this end we set:

$$h_1 \equiv hq \qquad\qquad h_2 \equiv \frac{h}{q} \tag{3.2.12}$$

and reparametrize these couplings in terms of the gauge coupling g. In fact since in the planar limit for each diagram the color factors from chiral vertices is always in terms of the products $|h_1|^2$ and $|h_2|^2$ we express directly $|h_1|^2$ and $|h_2|^2$ as power series in the coupling g^2 as follows

$$\begin{aligned} |h_1|^2 &= a_1g^2 + a_2g^4 + a_3g^6 + \dots \\ |h_2|^2 &= b_1g^2 + b_2g^4 + b_3g^6 + \dots \end{aligned} \tag{3.2.13}$$

Therefore, in the planar case, the number of independent real superpotential couplings which enter the color structure and thus we need to reparametrize, is just two.

In Section 3.3 we will see that keeping N finite all of the three independent real couplings will enter the color structures. This difference will turn out to be crucial for a physically meaningful definition of the superconformal theory. In the planar case, the two coefficients a_i and b_i will be determined by imposing that divergences from various loop orders, subtracted by the corresponding $\mathcal{N}=4$ results, vanish order by order in the g^2 expansion.

In order to make the comparison with the $\mathcal{N}=4$ calculation simpler we find convenient to determine the general structure of the color factors of the relevant diagrams. At L–loop order the color factor is a homogeneous polynomial in $|h_1|^2, |h_2|^2$ and g^2 of degree L. Moreover, as a consequence of the invariance of the theory under the global symmetry $h_1 \leftrightarrow -h_2$ and $\Phi_i \leftrightarrow \Phi_j$, $i \neq j$, it has to be symmetric under $|h_1|^2 \leftrightarrow |h_2|^2$. These properties, together with the requirement of having a smooth limit to $(2g^2)^L$ in the $\mathcal{N}=4$ limit ($|h_1|^2, |h_2|^2 \to g^2$), constrain the L–loop color factor to have the following form [1]

$$F^{(L)}(|h_1|^2 + |h_2|^2) + (|h_1|^2 - |h_2|^2)^2 \; G^{(L-2)}(|h_1|^2, |h_2|^2) \tag{3.2.14}$$

with $F^{(L)}(2g^2) = (2g^2)^L$. The functions $F^{(L)}$ and $G^{(L-2)}$ depend also on the coupling g^2, but for notational simplicity we have chosen not to write it explicitly. They are homogeneous polynomials of degrees L and $(L-2)$ respectively, symmetric in $|h_1|^2, |h_2|^2$. Their general form is

$$\begin{aligned} F^{(L)}(|h_1|^2 + |h_2|^2) &= \sum_{k=0}^{L} (|h_1|^2 + |h_2|^2)^k \; (2g^2)^{L-k} \; f_k \\ G^{(L-2)}(|h_1|^2, |h_2|^2) &= \sum_{k=0}^{[(L-2)/2]} (|h_1|^2 - |h_2|^2)^{2k} \; \mathcal{P}^{(L-2-2k)}(|h_1|^2, |h_2|^2) \end{aligned} \tag{3.2.15}$$

with constant coefficients f_k satisfying $\sum_{k=0}^{L} f_k = 1$ and $\mathcal{P}^{(L-2-2k)}$ homogeneous polynomials not vanishing for $|h_1|^2 = |h_2|^2$.

We note that for pure chiral diagrams, the ones we will be mainly interested in, there is no g^2–dependence in $F^{(L)}$ and $G^{(L-2)}$ and, in particular, $F^{(L)}(|h_1|^2+|h_2|^2) = (|h_1|^2+|h_2|^2)^L$.

At L–loop order, after we take the difference with the $\mathcal{N}=4$ result what is left over is given by

$$\Gamma^{(L)} = \left[F^{(L)}(|h_1|^2 + |h_2|^2) - (2g^2)^L + (|h_1|^2 - |h_2|^2)^2 \; G^{(L-2)}(|h_1|^2, |h_2|^2)\right] \; I_{div}^{(L)} \tag{3.2.16}$$

where $I_{div}^{(L)}$ denotes the divergent factor from the L-loop integral. Finally summing over all loops and using the expansions in (3.2.13) we end up with

$$\begin{aligned} \sum_L \Gamma^{(L)} &= \sum_L \left[F^{(L)}(h_1^2 + h_2^2) - (2g^2)^L + (h_1^2 - h_2^2)^2 \; G^{(L-2)}(h_1^2, h_2^2)\right] \; I_{div}^{(L)} \\ &= \sum_k A_k \; (g^2)^k \end{aligned} \tag{3.2.17}$$

[1] We do not worry about an overall normalization factor since it is irrelevant for our general argument

Finiteness is achieved imposing

$$A_k = 0 \tag{3.2.18}$$

order by order in g^2.

Thus we go back to the one-loop calculation and apply concretely the general procedure described above. From the results quoted in (3.2.3) and (3.2.4) we see that $G^{(-1)} = 0$ and find

$$\Gamma^{(1)} = \left[F^{(1)}(|h_1|^2 + |h_2|^2) - (2g^2)\right] \; I^{(1)}_{div} = \frac{1}{(4\pi)^2} \; \left[|h_1|^2 + |h_2|^2 - 2g^2\right] \; \frac{1}{\epsilon} \tag{3.2.19}$$

Therefore using the expansions in (3.2.13) at order g^2 we have to impose the condition

$$\mathcal{O}(g^2): \qquad A_1 = 0 \qquad \longrightarrow \qquad a_1 + b_1 - 2 = 0 \tag{3.2.20}$$

In fact since we have shown that the condition in (3.2.5) ensures finiteness up to three loops, up to order g^6, we find the following additional requirements

$$\begin{aligned} \mathcal{O}(g^4): \qquad & A_2 = 0 \qquad \longrightarrow \qquad a_2 + b_2 = 0 \\ \mathcal{O}(g^6): \qquad & A_3 = 0 \qquad \longrightarrow \qquad a_3 + b_3 = 0 \end{aligned} \tag{3.2.21}$$

At this point it should be clear that, according to the procedure we have illustrated above, we do not need consider anymore diagrams containing insertions of chiral bubbles like the one in Fig. 3.1(a): once the condition (3.2.20) is satisfied these diagrams do not lead to new divergent contributions. Therefore at every loop order we have to isolate diagrams corresponding to new chiral structures with eventually vector propagators inserted on them.

Now we reexamine the results we have obtained up to four loops, i.e. up to order g^8. From the four-loop calculation (see eqs. (3.2.8) and (3.2.10)) we have

$$\begin{aligned} & -\frac{5}{2}\,\zeta(5)\,\frac{1}{(4\pi)^8}\,\frac{1}{\epsilon}\,|h|^8 \left\{ \left(q\bar{q} + \frac{1}{q\bar{q}}\right)^4 + \left(q\bar{q} - \frac{1}{q\bar{q}}\right)^4 \right\} \\ = \; & -\frac{5}{2}\,\zeta(5)\,\frac{1}{(4\pi)^8}\,\frac{1}{\epsilon}\,\left[(|h_1|^2 + |h_2|^2)^4 + (|h_1|^2 - |h_2|^2)^4\right] \end{aligned} \tag{3.2.22}$$

Therefore we find

$$\Gamma^{(4)} = -\frac{5}{2}\,\zeta(5)\,\frac{1}{(4\pi)^8}\,\frac{1}{\epsilon}\,\left[(|h_1|^2 + |h_2|^2)^4 - (2g^2)^4 + (|h_1|^2 - |h_2|^2)^4\right] \tag{3.2.23}$$

Now we insert into (3.2.17) the results we have found so far, i.e. (3.2.19) and (3.2.23) and use the expansions in (3.2.13) with the conditions in (3.2.20) and (3.2.21). In this way we find that the finiteness condition at order g^8 is satisfied if

$$\mathcal{O}(g^8): \qquad A_4 = 0 \quad \longrightarrow \quad a_4 + b_4 - \frac{5}{2}\,\zeta(5)\,\frac{1}{(4\pi)^6}(a_1 - b_1)^4 = 0 \tag{3.2.24}$$

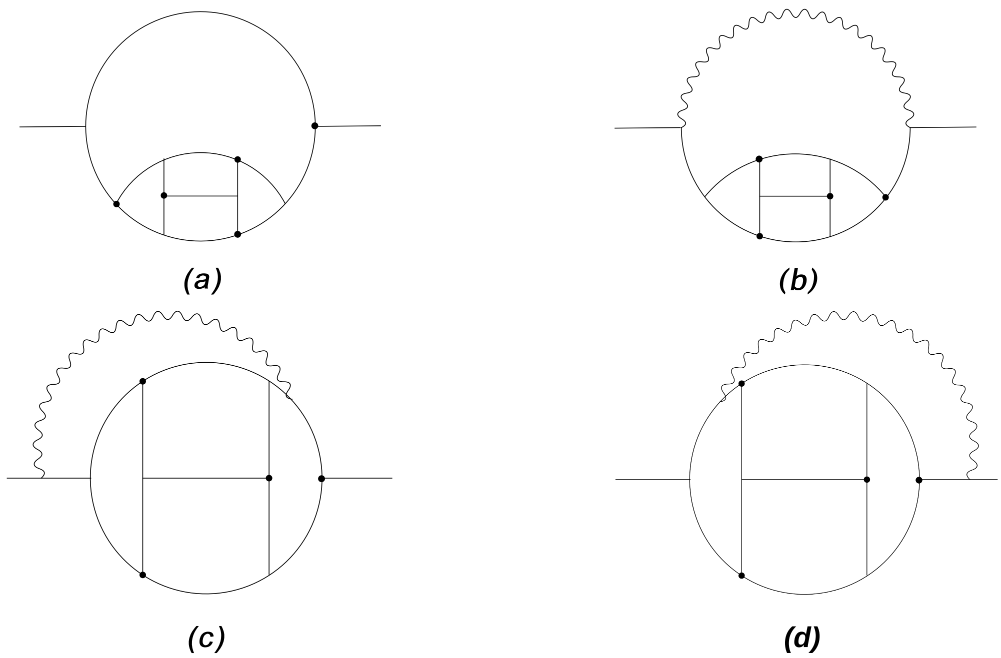

Figure 3.4: Planar supergraphs with $1/\epsilon^2$ divergences at five loops

Up to this point we have ensured that the two-point function is finite up to the order g^8. The finite contributions explicitly depend on q and vanish in the corresponding terms of the $\mathcal{N} = 4$ theory. The next step leads us to order g^{10}: we have to consider the new five-loop diagrams and the two-loop diagrams that will talk to the five-loop graphs once the condition (3.2.24) is imposed. Following the procedure described so far, i.e. implementing the finiteness condition order by order in the couplings, at the order g^8 we ended up adding contributions coming from one-loop integrals and from four-loop integrals. Now these structures show up at order g^{10} as subdivergences in two-loop and five-loop integrals respectively and they are responsible for the insurgence of $1/\epsilon^2$-pole terms. In Fig. 3.2 and in Fig. 3.4 we have drawn the two- and five-loop diagrams which give rise to $1/\epsilon^2$-pole terms. Having cancelled divergences at lower orders one might be tempted to believe that these $1/\epsilon^2$ terms would automatically add up to zero. Indeed this would be the case if we were cancelling divergences order by order in loops. As emphasized above we are proceeding order by order in the coupling g^2. At the order g^8 imposing the relation (3.2.24) we have cancelled the $1/\epsilon$ pole from the one-loop diagram in Fig. 3.1(c) with the $1/\epsilon$ pole appearing from the graph at four loops in Fig. 3.3. Essentially if we write schematically the one-loop result as

$$A \, \frac{1}{\epsilon} \, \frac{1}{(p^2)^\epsilon} \tag{3.2.25}$$

and the four-loop result as

$$B\,\frac{1}{\epsilon}\,\frac{1}{(p^2)^{4\epsilon}} \tag{3.2.26}$$

imposing the relation in (3.2.24) we have set $A + B = 0$. When we go one loop higher we have to deal with the bosonic integrals shown in Fig. 3.5

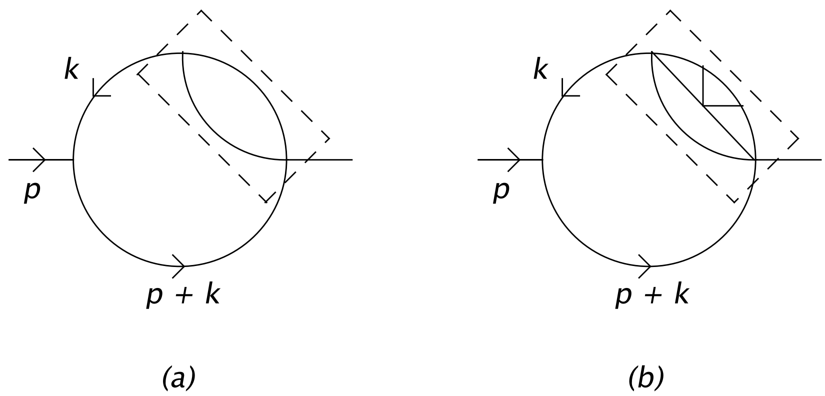

Figure 3.5: Subtraction of subdivergences at order g^{10}

The $1/\epsilon^2$ term in Fig.3.5(a) arises from

$$A\,\frac{1}{\epsilon}\int d^nk\frac{1}{(p+k)^2(k^2)^{1+\epsilon}} \qquad \longrightarrow \qquad A\,\frac{1}{\epsilon}\,\Gamma(2\epsilon) \tag{3.2.27}$$

The $1/\epsilon^2$ term in Fig.3.5(b) arises from

$$B\,\frac{1}{\epsilon}\int d^nk\frac{1}{(p+k)^2(k^2)^{1+4\epsilon}} \qquad \longrightarrow \qquad B\,\frac{1}{\epsilon}\,\Gamma(5\epsilon) \tag{3.2.28}$$

It is clear that setting $A + B = 0$ is not enough to cancel the $1/\epsilon^2$ poles.

In order to check this general argument we have computed the $1/\epsilon^2$ divergent terms explicitly. At order g^{10} from the two-loop graphs shown in Fig. 3.2, denoting with I_2 the divergent integral in Fig. 3.5(a) we have

$$Fig.3a \;=\; -2\left(q\bar{q}+\frac{1}{q\bar{q}}\right)^2|h|^4I_2 \qquad \longrightarrow \qquad -8(a_4+b_4)I_2 \tag{3.2.29}$$

$$Fig.3b \;=\; +4\left(q\bar{q}+\frac{1}{q\bar{q}}\right)|h|^2g^2I_2 \qquad \longrightarrow \qquad +4(a_4+b_4)I_2 \tag{3.2.30}$$

$$Fig.3c \;=\; +2\left(q\bar{q}+\frac{1}{q\bar{q}}\right)|h|^2g^2I_2 \qquad \longrightarrow \qquad +2(a_4+b_4)I_2 \tag{3.2.31}$$

$$Fig.3d \;=\; -2\left(q\bar{q}+\frac{1}{q\bar{q}}\right)|h|^2g^2I_2 \qquad \longrightarrow \qquad -4(a_4+b_4)I_2 \tag{3.2.32}$$

$$Fig.3e \;=\; -2\left(q\bar{q}+\frac{1}{q\bar{q}}\right)|h|^2g^2I_2 \qquad \longrightarrow \qquad -4(a_4+b_4)I_2 \qquad (3.2.33)$$

Summing up all of the contributions we get

$$-6(a_4+b_4)I_2 \qquad \longrightarrow \qquad -15\;\zeta(5)\;\frac{1}{(4\pi)^6}(a_1-b_1)^4\;\frac{1}{(4\pi)^4}\frac{1}{2\epsilon^2} \qquad (3.2.34)$$

where we have used the relation in (3.2.24). In the same way from the five-loop graphs shown in Fig. 3.4, denoting with I_5 the divergent integral in Fig. 3.5(b), we obtain (see Appendix C)

$$Fig.5a \;=\; +2\left((q\bar{q})^4+\frac{1}{(q\bar{q})^4}+6\right)\left(q\bar{q}+\frac{1}{q\bar{q}}\right)|h|^{10}I_5 \qquad \longrightarrow \qquad +2(a_1-b_1)^4I_5 \qquad (3.2.35)$$

$$Fig.5b \;=\; -2\left((q\bar{q})^4+\frac{1}{(q\bar{q})^4}+6\right)|h|^8g^2I_5 \qquad \longrightarrow \qquad -(a_1-b_1)^4I_5 \qquad (3.2.36)$$

$$Fig.5c \;=\; +2\left((q\bar{q})^4+\frac{1}{(q\bar{q})^4}+6\right)|h|^8g^2I_5 \qquad \longrightarrow \qquad +(a_1-b_1)^4I_5 \qquad (3.2.37)$$

$$Fig.5d \;=\; +2\left((q\bar{q})^4+\frac{1}{(q\bar{q})^4}+6\right)|h|^8g^2I_5 \qquad \longrightarrow \qquad +(a_1-b_1)^4I_5 \qquad (3.2.38)$$

Summing up all of the contributions we get

$$3(a_1-b_1)^4I_5 \qquad \longrightarrow \qquad 3(a_1-b_1)^4\;\frac{1}{(4\pi)^{10}}\frac{\zeta(5)}{\epsilon^2} \qquad (3.2.39)$$

Clearly the terms in (3.2.34) and (3.2.39) do not add up to zero and in fact they reproduce the mismatch anticipated in (3.2.27) and (3.2.28) when $A+B=0$. Therefore at order g^{10} the cancelation of the $1/\epsilon^2$ poles requires that (see also (3.2.20) and (3.2.24))

$$a_1=b_1=1 \qquad\qquad a_4+b_4=0 \qquad (3.2.40)$$

Once the conditions in (3.2.40) have been imposed, at the order g^{10} all the $1/\epsilon$ divergences from diagrams at five and two loops are automatically cancelled. Thus at this order the only divergence comes from the one-loop bubble and we are forced to impose

$$a_5+b_5=0 \qquad (3.2.41)$$

At this stage one may ask why there are no further contributions from 5–loop diagrams to $1/\epsilon^2$ pole at order g^{10}. In fact we will show now that all the other 5–loop graphs

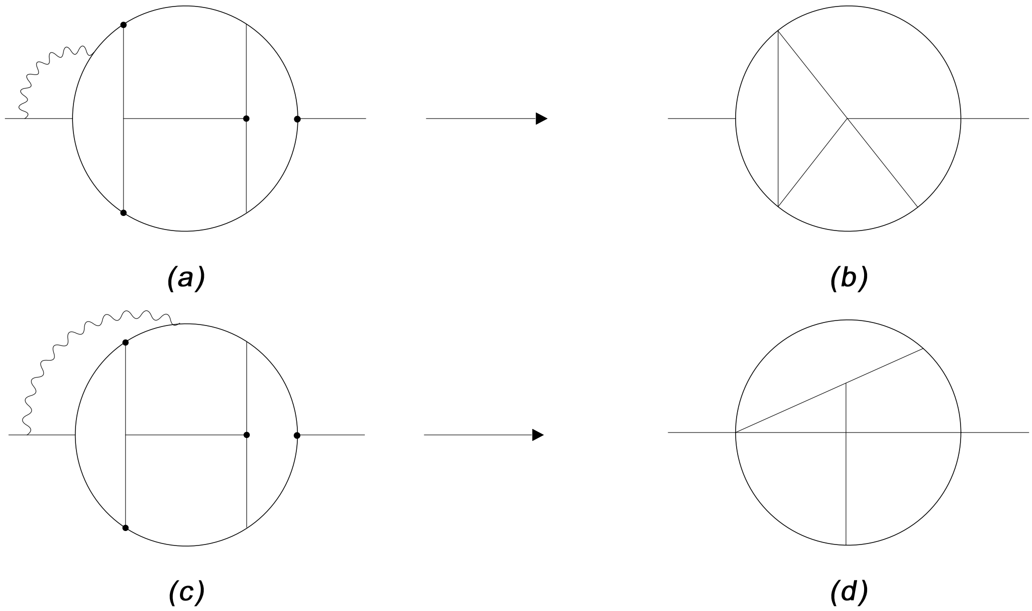

Figure 3.6: Diagrams with a gauge line connected to the external leg

potentially $1/\epsilon^2$–divergent actually do not contribute. To this end let us consider all the possible diagrams obtained by adding a gauge propagator to the 4–loop new chiral graph of Fig. 3.3. Let us start looking at diagrams with a gauge line connected to the external leg. Beside the ones in Fig. 3.4, we can draw also the diagrams in Fig. 3.6 where it has been shown the bosonic integrals appearing after the completion of the D-algebra. They are clearly $1/\epsilon$–divergent since they do not contain subdivergences, so we will neglect them. Let us see what happens if we insert a gauge propagator in order to correct an antichiral vertex. The diagrams one can draw are depicted in Fig. 3.7.

They still diverge as $1/\epsilon$, since they do contain just triangles. Analogously, if we look at a gauge correction to a chiral vertex, we get the diagrams in Fig. 3.8, which are still just $1/\epsilon$–divergent.

The situation does not change if we consider the two diagrams in Fig. 3.9 while things change drastically if we consider the diagrams depicted in Fig. 3.10. After completing the D-algebra one is left with a large number of diagrams, most of which diverge as $1/\epsilon$, as they are made up of triangles. Still there appear some non–trivial contributions presenting subdivergences. In particular both of the graphs in Fig. 3.10 produce three potentially $1/\epsilon^2$–divergent diagrams, depicted in Fig. 3.11.

Quite surprisingly, the graph (c) with two derivatives exactly cancels the contribution coming from the graphs (a) and (b). We have thus demonstrated that only the diagrams in Fig. 3.4 actually contribute to the $1/\epsilon^2$ pole at order g^{10}.

Before proceeding to the next order g^{12}, let us note that this pattern of cancelling divergences between the one-loop bubble in Fig. 3.1(a) and the four-loop diagram in Fig.

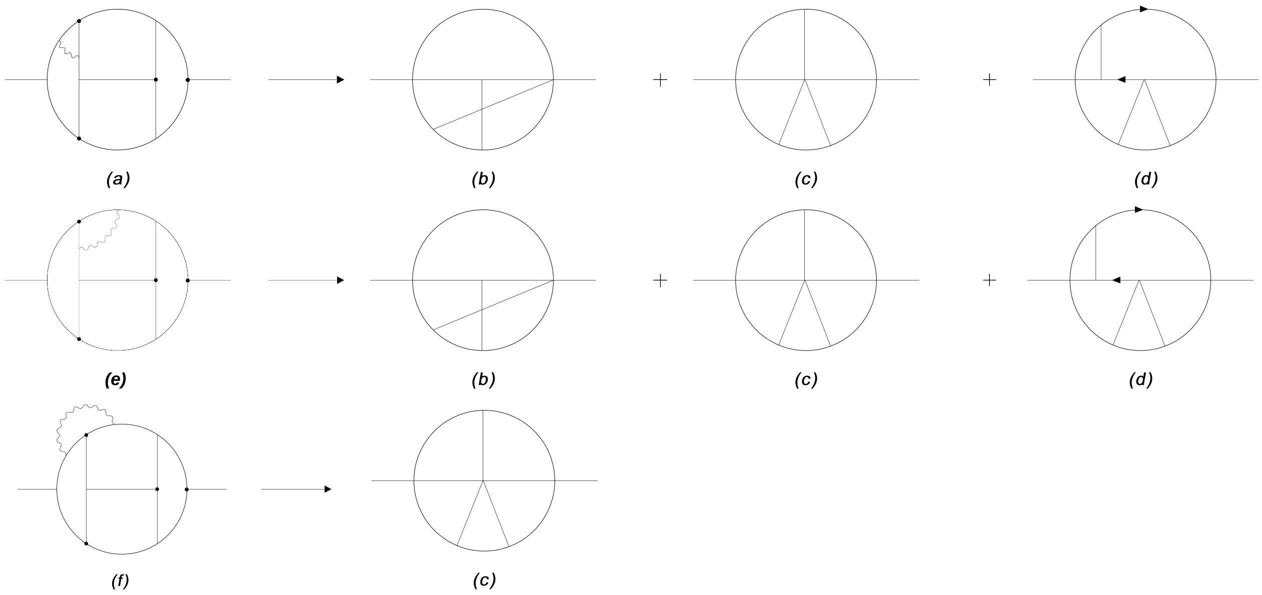

Figure 3.7: Diagrams with a gauge correction to an antichiral vertex

3.3 will repeat itself at order g^{16}, while the cancelation of the $1/\epsilon^2$ poles will show up at the order g^{18} and will involve again the diagrams at two and five loops that we have just considered. Indeed at this stage from the divergent contribution of the four-loop diagrams, using the conditions imposed so far on the coefficients of the expansions in (3.2.13), the first divergence will be proportional to

$$[(a_2 - b_2)\ g^4]^4 = (2a_2)^4\ g^{16} \tag{3.2.42}$$

So for the time being, having ensured finiteness of the theory up to the order g^{10}, we proceed and examine the situation at six loops. The new divergent chiral diagrams are shown in Fig. 3.12: they are all logarithmically divergent.
Their color factor is easily evaluated: it can be written in the following form

$$(|h_1|^2 + |h_2|^2)^6\ +\ (|h_1|^2 - |h_2|^2)^4\ (\frac{5}{3}|h_1|^4 + \frac{2}{3}|h_1|^2|h_2|^2 + \frac{5}{3}|h_2|^4) \tag{3.2.43}$$

Thus we find that in the g^2 expansion the first nonvanishing term from the six-loop divergence will be proportional to

$$[(a_2 - b_2)\ g^4]^4\ g^4 = (2a_2)^4\ g^{20} \tag{3.2.44}$$

Thus once again to the order g^{12} the only divergence arises from the one-loop bubble and its cancelation requires

$$a_6 + b_6 = 0 \tag{3.2.45}$$

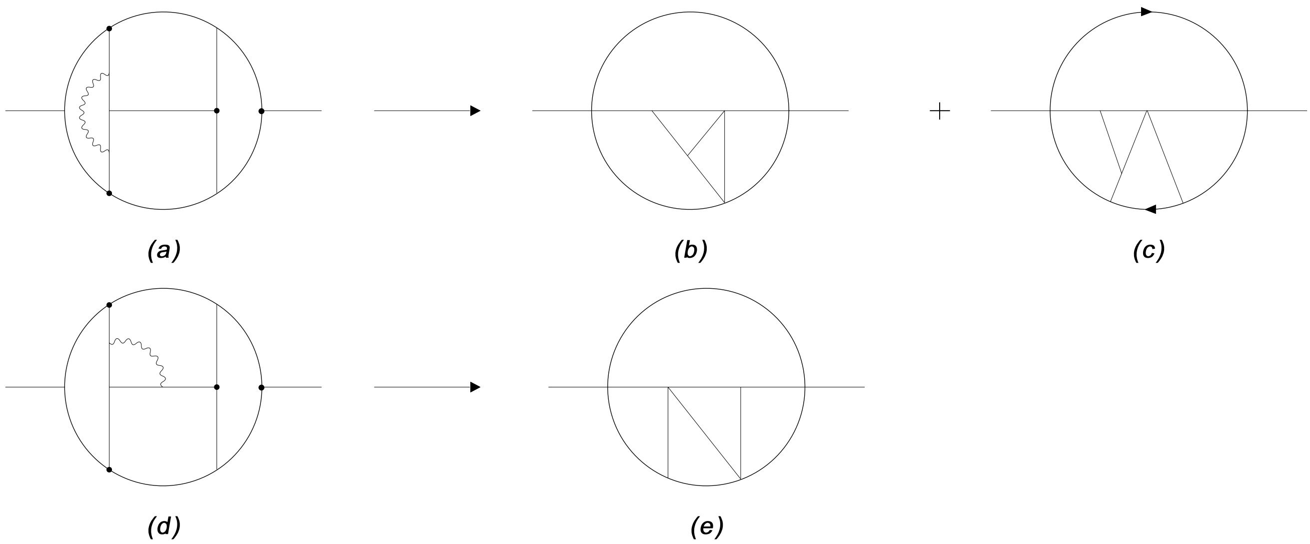

Figure 3.8: Diagrams with a gauge correction to a chiral vertex

We keep on going and look for divergent terms at the order g^{14}. The diagrams at seven loops have a color factor proportional to

$$(|h_1|^2 + |h_2|^2)^7 \ + \ (|h_1|^2 - |h_2|^2)^4 \ (3|h_1|^6 + 5|h_1|^4|h_2|^2 + 5|h_1|^2|h_2|^4 + 3|h_2|^6) \qquad (3.2.46)$$

which, using the expansion in (3.2.13), gives as first relevant term

$$[(a_2 - b_2)\ g^4]^4 \ g^6 = (2a_2)^4 \ g^{22} \qquad (3.2.47)$$

Therefore once again the only divergence at the order g^{14} comes from one loop and leads to the condition

$$a_7 + b_7 = 0 \qquad (3.2.48)$$

In accordance with the general discussion around equations (3.2.14, 3.2.15) and what we have found by the explicit calculations we have reported up to seven loops, we write the L–loop color structure of the pure chiral divergent diagrams in the following form

$$(|h_1|^2+|h_2|^2)^L \ + \ (|h_1|^2-|h_2|^2)^4 \ [\alpha(|h_1|^2)^{L-4}+\beta(|h_1|^2)^{L-5}|h_2|^2+\cdots+\gamma(|h_2|^2)^{L-4}] \quad (3.2.49)$$

We note that the only arbitrary assumption with respect to the general form that one can infer from (3.2.14, 3.2.15) is the absence of a term proportional to $(|h_1|^2 - |h_2|^2)^2$. Even if we do not have a general argument for the absence of such a term we are very well supported by the results up to seven loops illustrated so far.

If we take into account the conditions found so far for the coefficients in (3.2.13), then (3.2.49) immediately implies that the various diagrams at L loops will give contributions in the g^2 expansion whose first relevant term is proportional to

$$[(a_2 - b_2)\ g^4]^4 \ g^{2L-8} = (2a_2)^4 \ g^{2L+8} \qquad (3.2.50)$$

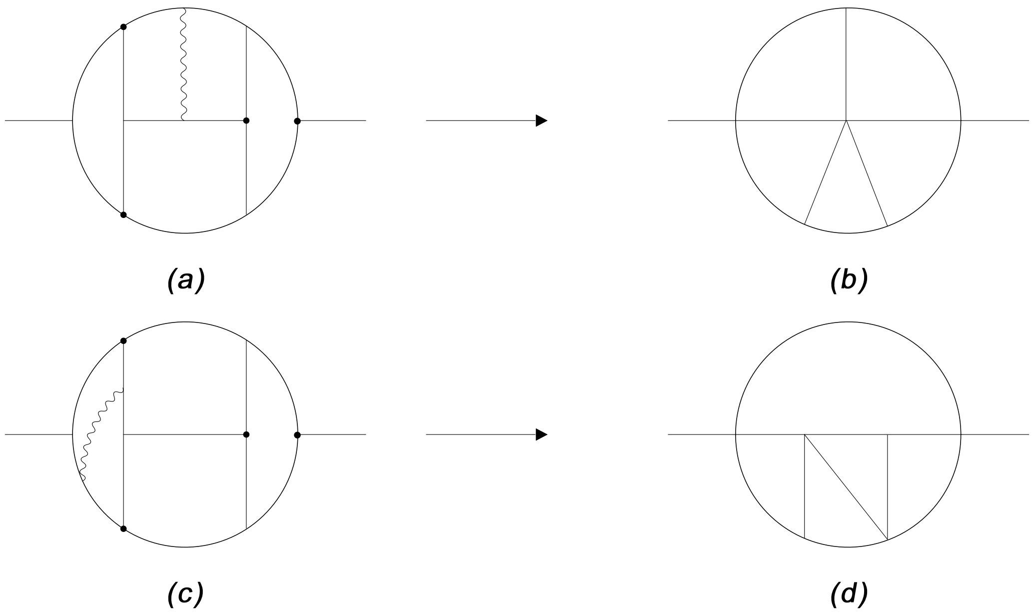

Figure 3.9: Two diagrams with an internal gauge correction

The conclusion is that diagrams at six loops or higher will start contributing at the earliest when we reach order g^{20}, as we have explicitly seen in (3.2.44) and (3.2.47). Therefore if we now turn to the order g^{16}, as previously anticipated, the only divergent contributions come from the one-loop bubble proportional to $a_8 + b_8$ and from the four-loop diagram proportional to a_2^4 (see eq. (3.2.42)). In order for the divergences to cancel at this order we have to require

$$\mathcal{O}(g^{16}) : \qquad A_8 = 0 \quad \longrightarrow \quad a_8 + b_8 - \frac{5}{2}\,\zeta(5)\,\frac{1}{(4\pi)^6}(a_2 - b_2)^4 = 0 \qquad (3.2.51)$$

Going up to the order g^{18} we have to cancel the $1/\epsilon^2$ poles from the two and five-loop diagrams: following the same steps as before we are forced to impose

$$a_8 + b_8 = 0 \qquad\qquad a_2 = b_2 = 0 \qquad (3.2.52)$$

With these conditions on the coefficients in the expansion (3.2.13), at order g^{18} the $1/\epsilon$ poles come only from the one-loop bubble and they cancel out once

$$a_9 + b_9 = 0 \qquad (3.2.53)$$

Since in (3.2.52) we have imposed $a_2 = 0$, automatically we find that the various divergences from six, seven, $\ldots$, L-loop diagrams are pushed up

$$\begin{array}{lcl} 6 \text{ loops} & \longrightarrow & [(a_3 - b_3)\; g^6]^4\; g^4 = (2a_3)^4\; g^{28} \\ 7 \text{ loops} & \longrightarrow & [(a_3 - b_3)\; g^6]^4\; g^6 = (2a_3)^4\; g^{30} \\ \ldots & \ldots & \ldots \\ \text{L loops} & \longrightarrow & [(a_3 - b_3)\; g^6]^4\; g^{2L-8} = (2a_3)^4\; g^{2L+16} \end{array} \qquad (3.2.54)$$

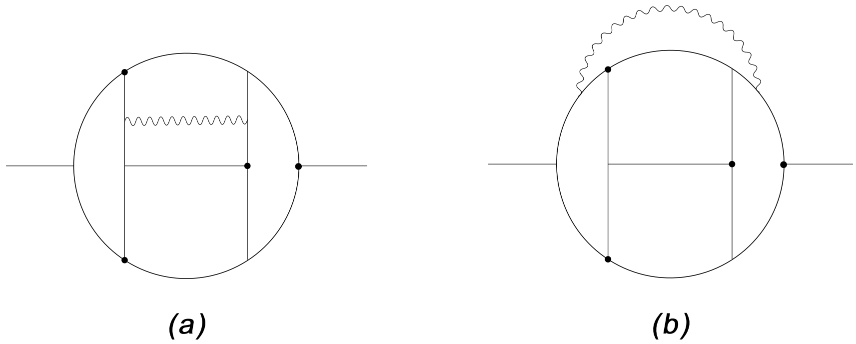

Figure 3.10: Diagrams potentially $1/\epsilon^2$–divergent

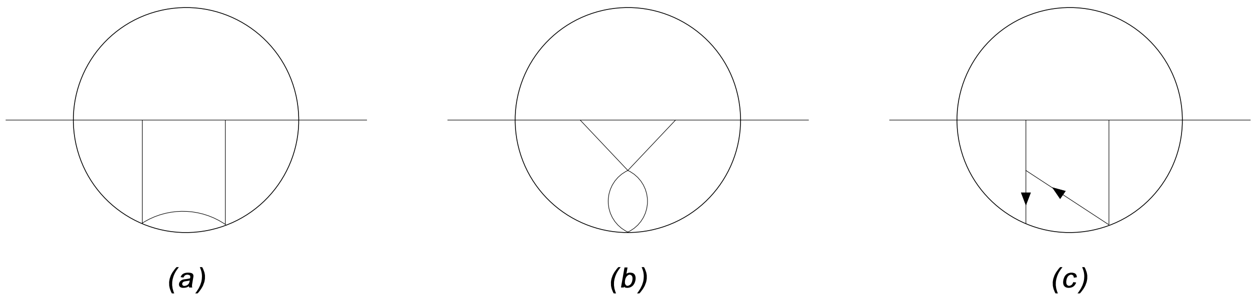

Figure 3.11: $1/\epsilon^2$–divergent bosonic diagrams coming from graphs in Fig. 3.10

It becomes clear that everything is ruled by the cancelation of divergences at one and four loop and by the subsequent cancelation of the $1/\epsilon^2$ poles at two and five loops. This happens at the order $(g^2)^{4k}$ and at the order $(g^2)^{4k+1}$ respectively. The new chiral graphs at six loops and higher never enter the game due to the specific form of their color structure as in (3.2.49). The mechanism works as follows: up to the order $(g^2)^{4k-1}$ we find that the coefficients have to satisfy

$$a_1 = b_1 = 1 \qquad a_{j-1} = 0 \qquad a_{4j-1} + b_{4j-1} = 0 \qquad j = 2, \ldots, k \tag{3.2.55}$$

At $\mathcal{O}((g^2)^{4k})$ in order to cancel the divergent contributions from one and four loops we have to impose

$$a_{4k} + b_{4k} - \frac{5}{2}\,\zeta(5)\,\frac{1}{(4\pi)^6}(a_k - b_k)^4 = 0 \tag{3.2.56}$$

Then at $\mathcal{O}((g^2)^{4k+1})$ the divergences from two and five loops need to be cancelled and we are forced to require

$$a_{4k} + b_{4k} = 0 \qquad a_k = b_k = 0 \tag{3.2.57}$$

Finally this leads to

$$a_1 = b_1 = 1 \qquad a_n = b_n = 0 \qquad n = 2, 3, \ldots \tag{3.2.58}$$

Figure 3.12: New planar chiral diagrams at six loops

These conclusions have been drawn based on the general expression given in (3.2.49) for the color structure of pure chiral diagrams where we have assumed the absence of a term quadratic in $(h_1^2 - h_2^2)$. Now which control do we have on this assumption in the higher-loop divergent chiral diagrams? We have computed explicitly all the color structures up to ten loops; with the help of Mathematica we have evaluated the color factors of arbitrarily chosen higher-loop graphs; in addition we have explicit formulas for several classes of chiral diagrams. We have found consistently that all of them can be cast in the form given in (3.2.49). So we conclude that the only way to reach finiteness for the chiral propagator with an expansion such as in (3.2.13), is to restrict to the case β real. We emphasize that this result is independent of the renormalization scheme: had we done the calculation using a different scheme the condition of finiteness would have led us to the solution β real.

We stress that our investigation has been carried on perturbatively, ignoring completely possible nonperturbative effects. In particular, we have assumed the gauge coupling constant to be real. It would be interesting to extend our analysis to g complex and to understand if the embedding of all the couplings in a complex manifold leads to nontrivial superconformal conditions.

3.2.2 Conformal condition and scheme dependence problem

In this Subsection we will relax the direct finiteness requirement [11]. We want to find the condition that the couplings have to satisfy in the large N limit in order to guarantee the vanishing of the chiral and gauge beta functions. We will find that in this case complex values of β are allowed but the resulting conformal invariant theory depends on the renormalization scheme.

Let us consider once again the action in (3.2.2) and compute perturbatively in the large N limit the chiral and gauge beta functions. The request of vanishing beta functions will identify a conformal field theory.

First we consider the chiral beta function β_h. It is well-known that in minimal subtraction scheme β_h is proportional to the anomalous dimension γ of the elementary fields and the condition $\beta_h = 0$ can be conveniently traded with $\gamma = 0$. In our case, even working in a generic scheme, one can easily convince oneself that at a given order in g^2 the proportionality relation between β_h and γ gets affected only by terms proportional to lower order contributions to γ. Therefore, if we set $\gamma = 0$ order by order in the coupling, we are guaranteed that β_h vanishes as well.

Thus we impose the vanishing of γ which we obtain from the computation of the two-point chiral correlator. Up to three loops nothing new happens: the condition in (3.2.5) insures the vanishing of γ up to the order g^6 and correspondingly also β_h is zero. Moreover up to this order we can use the results in [36] and we are guaranteed that also the gauge beta function β_g is zero up to the order g^9. This is easily understood since in spite of the redefinition in (3.2.13) the requirement of vanishing anomalous dimensions up to order g^6 works order by order in the loop expansion so that general finiteness theorems [35, 36] hold. At this stage the coefficients in (3.2.13) have to satisfy

$$\begin{aligned} \mathcal{O}(g^2): &\qquad a_1 + b_1 - 2 = 0 \\ \mathcal{O}(g^4): &\qquad a_2 + b_2 = 0 \\ \mathcal{O}(g^6): &\qquad a_3 + b_3 = 0 \end{aligned} \tag{3.2.59}$$

Things become more subtle at $\mathcal{O}(g^8)$: here the only way to achieve the vanishing of γ is to mix contributions from one loop with contributions from four loops. Repeating the calculation of the divergent integrals, the result is proportional to

$$\frac{1}{\epsilon}\left[A\left(\frac{\mu^2}{p^2}\right)^{\epsilon} + B\left(\frac{\mu^2}{p^2}\right)^{4\epsilon}\right] = \frac{1}{\epsilon}\left(A+B\right) + \left(A+4B\right)\ln\left(\frac{\mu^2}{p^2}\right) + \mathcal{O}(\epsilon) \tag{3.2.60}$$

where we have introduced

$$A \equiv \frac{1}{(4\pi)^2}(a_4 + b_4) \qquad\qquad B \equiv -\frac{5}{2}\zeta(5)\frac{1}{(4\pi)^8}(a_1 - b_1)^4 \tag{3.2.61}$$

and we have explicitly indicated the factors coming from dimensionally regulated integrals at one and four loops (here p is the external momentum and μ is the standard renormalization mass). The anomalous dimension is given directly by the finite *log* term in (3.2.60) and then we see that at order g^8 the vanishing of the anomalous dimension γ requires

$$\mathcal{O}(g^8): \qquad A + 4B = 0 \tag{3.2.62}$$

We emphasize that at this order this condition ensures the vanishing of γ and β_h , but as it appears in (3.2.60) the theory is *not finite*. We will come back to this point and discuss its implications below. First we want to show that the condition in (3.2.62) is sufficient to insure that β_g is zero up to the order g^{11}.

Contributions to the gauge beta function at $\mathcal{O}(g^{11})$ come from two- and five-loop diagrams. Using standard superspace methods the two-loop calculation is straightforward, but at five loops the number of diagrams involved is large and the calculation looks rather daunting.[2]

[2] We recall that in [23] a calculation of similar difficulty was attempted: the four-loop gauge beta function including nonplanar graphs. In that case the relevant coefficient was obtained by an indirect assumption because a direct calculation was too involved.

In fact using the background field method and covariant supergraph techniques we are able to perform this high loop calculation exactly. We take advantage of the results obtained in [36] where the structure of higher-loop ultraviolet divergences in SYM theories was analyzed using the superspace background field method and supergraph covariant D-algebra [59]. Using this approach contributions to the effective action beyond one loop can be written in terms of the vector connection Γ_a and the field strengths W_α, $\bar{W}_{\dot{\alpha}}$, but not of the spinor connection Γ_α. This result allows to draw strong conclusions on the structure of UV divergences in SYM theories. It was shown [36] that in regularization by dimensional reduction UV divergent terms can be obtained by computing a special subset of all possible supergraphs. The reasoning can be summarized as follows: at any loop order (with the exception of one loop), after subtraction of UV and IR divergences, the infinite part of contributions to the effective action is local and gauge invariant. By superspace power counting and gauge invariance it must have the form

$$\Gamma_\infty = z(\epsilon)\ \mathrm{Tr} \int d^4x\ d^4\theta\ \Gamma^a \Gamma_b (\delta_a{}^b - \hat{\delta}_a{}^b) \tag{3.2.63}$$

where Γ^a is the vector connection from the expansion of the covariant derivatives, i.e. $\nabla_a = \partial_a - i\Gamma_a$, produced in the course of the D-algebra. $z(\epsilon)$ is a singular factor from momentum integration of divergent supergraphs and the n-dimensional $\hat{\delta}_a{}^b$ is produced from symmetric integration. Using the rules of dimensional reduction and the Bianchi identities in terms of covariant derivatives one can show that

$$\mathrm{Tr} \int d^4x\ d^4\theta\ \Gamma^a \Gamma_b (\delta_a{}^b - \hat{\delta}_a{}^b) = -\epsilon\ \mathrm{Tr} \int d^4x\ d^2\theta\ W^\alpha W_\alpha \tag{3.2.64}$$

From the above relation it is clear that in order to obtain a divergence the coefficient $z(\epsilon)$ in (3.2.63) must contain at least a $1/\epsilon^2$ pole. Moreover the complete result can be obtained by calculating tadpole-type contributions with a $\Gamma^a \Gamma_b \delta_a{}^b$ vertex and then covariantizing them by the substitution $\delta_a{}^b \to \delta_a{}^b - \hat{\delta}_a{}^b$. Thanks to all of this even the five loop computation becomes manageable.

We describe here the main steps that apply both to the two-loop and to the five-loop calculation. As emphasized above we need to consider graphs with only internal chiral lines. Thus, according to the rules in [59], at a given order in loop one draws vacuum diagrams with chiral covariant propagators and ∇^2, $\bar{\nabla}^2$ factors at the chiral vertices. Now, in order to reduce as much as possible the number of terms produced in the course of the ∇-algebra, we do not perform the covariant ∇-integration at this stage but modify the procedure as follows. We want to single out tadpole-type contributions proportional to $\Gamma^a \Gamma_a$, therefore we have to figure out which are the potential sources of such terms. The explicit representation of the chiral covariant propagators is given by

$$\Box_+ = \frac{1}{2}\nabla^a \nabla_a - iW^\alpha \nabla_\alpha - \frac{i}{2}(\nabla^\alpha W_\alpha) \qquad \Box_- = \frac{1}{2}\nabla^a \nabla_a - i\bar{W}^{\dot{\alpha}} \bar{\nabla}_{\dot{\alpha}} - \frac{i}{2}(\bar{\nabla}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}) \tag{3.2.65}$$

Therefore in the expressions above we can disregard the terms involving the field strengths since they do not enter the structure in (3.2.63). The $\Gamma^a \Gamma_a$ terms can arise only from the expansion of the covariant operator $\nabla^a \nabla_a$ or from contracted covariant derivatives $\nabla^a \ldots \nabla_a$

Figure 3.13: Vacuum diagrams: (a) two-loops and (b) five-loops contributions

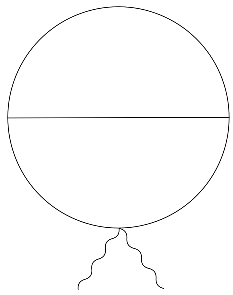

Figure 3.14: Bosonic two-loop integral

produced while performing the ∇-algebra. The net result is that we can immediately expand the covariant propagators as follows

$$\frac{1}{\Box_\pm} \quad \rightarrow \quad \frac{1}{\frac{1}{2}\nabla^a\nabla_a} \quad \rightarrow \quad \frac{1}{\Box} + \frac{1}{2}\,\frac{1}{\Box}\Gamma^a\Gamma_a\frac{1}{\Box} \tag{3.2.66}$$

where $\Box = \frac{1}{2}\,\partial^a\partial_a$ is the flat propagator. We can drop all the rest since it will not contribute to the structure we are looking for. In this way we obtain two types of diagrams:
I. the ones with flat D^2 and $\bar{D}^2$ factors at the vertices, flat propagators and one $\Gamma^a\Gamma_a$ insertion, for which now standard D-algebra can be performed
and
II. the vacuum diagrams with flat propagators but ∇^2, $\bar{\nabla}^2$ factors at the chiral vertices in which the $\Gamma^a\Gamma_a$ vertex will have to appear after completion of the ∇-algebra.
The relevant terms will be the ones that after subtraction of ultraviolet and infrared subdivergences give rise to $1/\epsilon^2$ contributions.

At the two-loop level the analysis is very simple: the vacuum diagram to be considered is shown in Fig. 3.13(a). Following the procedure described above, it is straightforward to realize that only I-type diagrams can give rise to $1/\epsilon^2$ poles and so the calculation reduces to the one presented in [59].

We briefly summarize it here. Expanding the covariant propagators as in (3.2.66) one obtains three times the diagram in Fig. 3.14 which corresponds to the term

$$\frac{1}{2}\ \mathrm{Tr}\ (\Gamma^a\Gamma_a) \int \frac{d^nk\ d^nq}{(2\pi)^{2n}} \frac{1}{q^2(q+k)^2k^4} \tag{3.2.67}$$

This integral contains a one-loop ultraviolet subdivergence and it is infrared divergent. It is convenient to remove the IR divergence using the R^* subtraction procedure of [41]. After UV and IR subtraction one isolates the $1/\epsilon^2$ term and rewrites the result in a covariant form. Using (3.2.64) it can be recast in the standard divergent part of the two-loop effective action giving a total contribution (see Appendix C)

$$\frac{1}{(4\pi)^2}\ \frac{3}{4}A\ \frac{1}{\epsilon}\ \mathrm{Tr} \int d^4x\ d^2\theta\ W^\alpha W_\alpha \tag{3.2.68}$$

where we have reinserted the A factor defined in (3.2.61).

Now we are ready to tackle the five-loop calculation which amounts to start with the vacuum diagram in Fig. 3.13(b). First we consider the I-type diagrams. In this case expanding the covariant propagators as in (3.2.66) we produce twelve times the diagram in Fig. 3.15. We perform standard D-algebra in the loops and look for a contribution that after subtraction of IR and UV subdivergences gives rise to a $1/\epsilon^2$ divergent term. One easily obtains a single contribution corresponding to the bosonic integral shown in Fig. 3.15

$$\frac{1}{2}\ \mathrm{Tr}\ (\Gamma^a\Gamma_a) \int \frac{d^nk\ d^nq\ d^nr\ d^ns\ d^nt}{(2\pi)^{5n}} \frac{1}{r^2(r+q)^2s^2(s+q)^2t^2(t+r)^2(t+s)^2(q+k)^2k^4} \tag{3.2.69}$$

The IR divergence is treated as before via R^* subtraction [41] so that, inserting all the factors, the final result is given by (see Appendix C)

$$\frac{1}{(4\pi)^2}\ \frac{6}{5}B\ \frac{1}{\epsilon}\ \mathrm{Tr} \int d^4x\ d^2\theta\ W^\alpha W_\alpha \tag{3.2.70}$$

with B defined in (3.2.61).

In the class of II-type diagrams we have to analyze the vacuum diagram in Fig. 3.16. We operate directly with the covariant spinor derivatives, pushing them through the propagators. Unlike in ordinary D-algebra, covariant spinor derivatives and space-time derivatives contained in the propagators do not commute but it is easy to realize that they generate field strength factors which are not interesting for our calculation. Thus we can commute the ∇_α's through the $\Box^{-1}$. The relevant contributions arise when we produce terms like

$$\nabla^2\bar{\nabla}^2\nabla^2 = \Box_-\nabla^2 \quad \rightarrow \quad -\frac{1}{2}\ \Gamma^a\Gamma_a\nabla^2 \qquad\qquad \bar{\nabla}^2\nabla^2\bar{\nabla}^2 = \Box_+\bar{\nabla}^2 \quad \rightarrow \quad -\frac{1}{2}\ \Gamma^a\Gamma_a\bar{\nabla}^2$$

$$\nabla_\alpha\bar{\nabla}_{\dot{\alpha}}\nabla^2 = i\nabla_a\nabla^2 \quad \rightarrow \quad \Gamma_a\nabla^2 \qquad\qquad \bar{\nabla}_{\dot{\alpha}}\nabla_\alpha\bar{\nabla}^2 = i\nabla_a\bar{\nabla}^2 \quad \rightarrow \quad \Gamma_a\bar{\nabla}^2 \tag{3.2.71}$$

After all these preliminary observations, now one has to perform the covariant ∇-algebra explicitly and isolate the diagrams that could produce $1/\epsilon^2$ ultraviolet divergences.

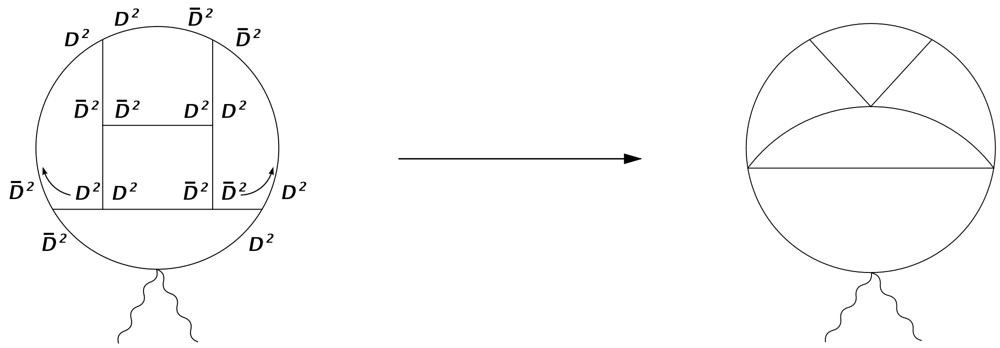

Figure 3.15: Five-loop supergraph and its associated relevant bosonic integral

It turns out that some cleverness must be used in order to reduce the number of the resulting contributions. We show in Fig. 3.16 the successive manipulations that we used to obtain the final answer. As indicated in the Figure the first integration by parts of the $\bar{\nabla}^2$ factor produces three terms: we have denoted by

$$// \equiv \frac{\frac{1}{2}\nabla^a\nabla_a}{\Box} \rightarrow 1 - \frac{1}{2}\,\frac{\Gamma^a\Gamma_a}{\Box} \qquad \blacktriangleright \equiv \nabla_a = \partial_a - i\Gamma_a \tag{3.2.72}$$

At this stage we have to work separately on the three graphs and complete the ∇-algebra by disregarding contributions which do not contain $1/\epsilon^2$ divergent terms. (An example of diagram which is not interesting is the one shown in Fig. 3.17. It arises from the second diagram in Fig. 3.16 and would produce only $1/\epsilon$ divergent terms.) In fact if we judiciously move the ∇'s only very few relevant terms are generated, the ones schematically shown in Fig. 3.18. Now it is straightforward to show that by integration by parts these potentially relevant graphs do cancel out completely.

In conclusion, the only relevant contributions to the gauge beta function at order g^{11} come from (3.2.68) and (3.2.70). Using the ordinary prescription to compute beta functions, we find

$$\mathcal{O}(g^{11}) : \qquad \beta_g = 0 \qquad \Leftrightarrow \qquad A + 4B = 0 \tag{3.2.73}$$

Therefore a single condition on the A and B coefficients is sufficient to define the theory at its conformal point up to the order g^8 and to insure that, despite of the non-finiteness of the theory, the gauge beta function vanishes at the next order.

Now we want to come back to the fact that at order g^8 we have found that the theory subject to the condition (3.2.62) for its renormalized couplings is not finite. In order to understand the implications of the lack of finiteness on the conformal condition, we need to consider the counterterm which renormalize the propagator at this order. As it follows from (3.2.60) this will be proportional to the divergence in the form

$$g^8\ (A+B)\ (\frac{1}{\epsilon} + \rho) \tag{3.2.74}$$

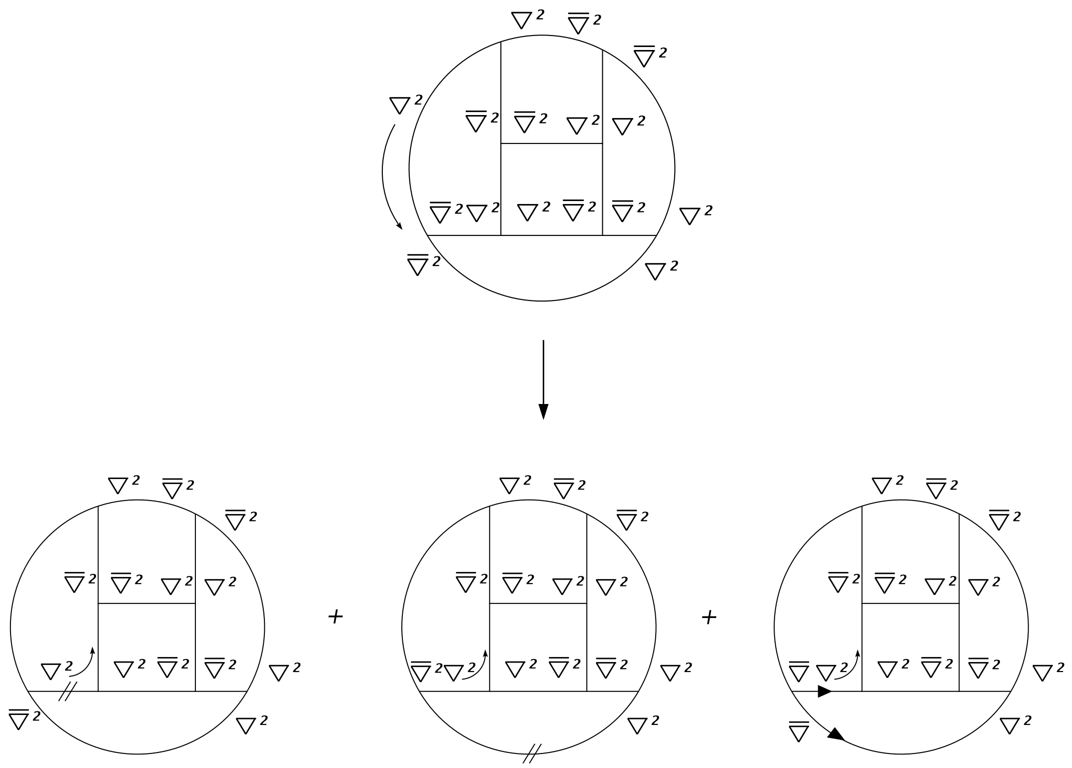

Figure 3.16: Five-loop vacuum diagram and ∇-algebra operations

where ρ is an arbitrary constant linked to the choice of a finite renormalization. It is worth noticing that the results obtained so far are completely independent of the subtraction scheme we have adopted. In fact even for the calculation of β_g at $\mathcal{O}(g^{11})$ the arbitrary parameter ρ does not enter in the evaluation of the coefficient of the $1/\epsilon^2$ poles from which we read β_g. The issue that now we want to address is what happens to the next order.

If we were to push the conformal invariance condition one order higher we should compute the chiral beta function at order g^{10}. We have several sources of nontrivial contributions to γ at this order: one coming from the one-loop bubble proportional to $(a_5 + b_5)$, one from two-loop diagrams and one from five-loop diagrams. In addition we need take into account the contribution from the counterterm in (3.2.74) which gives

$$g^{10}\ (A+B)\ (\frac{1}{\epsilon}+\rho)\ \frac{1}{\epsilon}\ \left(\frac{\mu^2}{p^2}\right)^{\epsilon} \tag{3.2.75}$$

This last contribution is necessary to appropriately subtract diagrams that contain subdivergences at two and five loops, i.e. the ones that contain $1/\epsilon^2$ poles considered in Subsection 3.2.1. The condition for vanishing γ, obtained as usual from the finite *log* terms, gives an algebraic equation involving A, B and $(a_5 + b_5)$ which, together with (3.2.62) allows to determine A and B parametrically and not necessarily vanishing. However the result

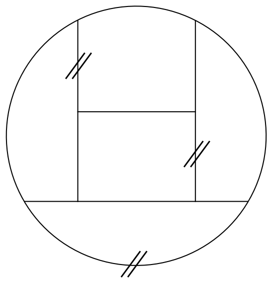

Figure 3.17: Example of diagram not contributing to the $\frac{1}{\epsilon^2}$ divergence

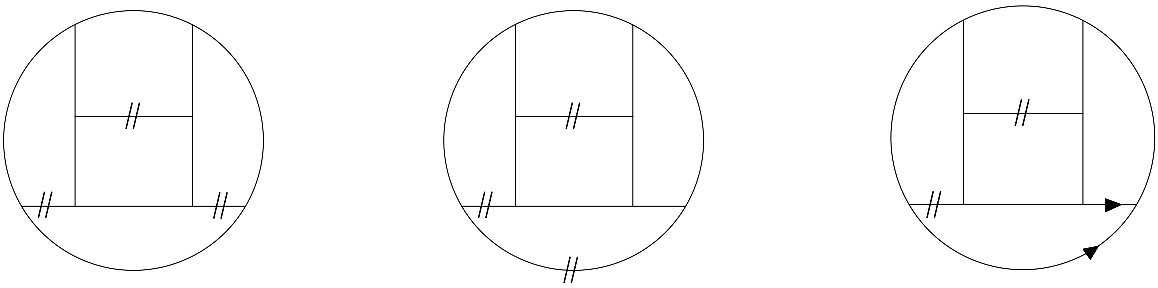

Figure 3.18: Relevant bosonic integrals associated to the five-loop graph

depends unavoidably on the arbitrary constant ρ which appears in the form

$$(A + B)\ \rho \qquad (3.2.76)$$

If we wanted to kill the scheme dependence of the result we would need to impose $A+B=0$ which together with (3.2.62) would lead immediately to $A = B = 0$, i.e. the theory is finite and $\mathrm{Im}\beta = 0$.

The comparison of these results with the ones of the previous Subsection leads to the conclusion that the requirement of conformal invariance via the vanishing of the beta functions is less restrictive than requiring finiteness but the result is scheme dependent.

Pushing the calculations higher we expect to draw the same conclusion: conformal invariance via vanishing beta functions allows for $\mathrm{Im}\beta \neq 0$ but this value and ultimately the conformal theory depend on the choice of the particular renormalization scheme we use.

Differential renormalization approach

In order to show that our findings do not depend on the particular regularization used, we reconsider the calculation of the chiral propagator up to the order g^8 in the scheme of differential renormalization.

Differential renormalization works strictly in four dimensions. In its original formulation

[42] it is a renormalization without regularization, i.e. it allows for a direct computation of renormalized quantities without the intermediate step of regularizing divergent integrals. In coordinate space the procedure consists in replacing locally singular functions (functions which do not admit a Fourier transform) with suitable distributions defined by differential operators acting on regular functions, where the derivatives have to be understood in distributional sense. The simplest example is the function $1/(x^2)^2$ from the one–loop contribution to $\Gamma^{(2)}$. This function has a non-integrable singularity in $x = 0$. The prescription required by differential renormalization in order to subtract such a singularity is

- We substitute
$$\frac{1}{x^4} \;\rightarrow\; -\frac{1}{4}\Box\frac{\log M^2x^2}{x^2} \tag{3.2.77}$$
where M is identified with the mass scale of the theory.

- We understand derivatives in the distributional sense, i.e.
$$\int d^4x f(x)\Box\frac{\log M^2x^2}{x^2} \equiv \int d^4x \Box f(x)\frac{\log M^2x^2}{x^2} \tag{3.2.78}$$
for any regular function f.

The two expressions in (3.2.77) are identical as long as $x \neq 0$, whereas they differ by a singular term for $x \rightarrow 0$. The substitution (3.2.77) can then be understood as adding a suitable counterterm [43]-[45]:

$$\frac{1}{x^4} = -\frac{1}{4}\Box\frac{\log M^2x^2}{x^2} \;+\; c(\alpha)\delta^{(4)}(x) \tag{3.2.79}$$

where $c(\alpha)$ can be computed in some regularization scheme and becomes singular when the regularization parameter α is removed.

Having in mind to study conformal invariance and/or finiteness for the deformed theory we need compute both the renormalized chiral propagator and its divergent contributions. As long as we are interested in $\Gamma^{(2)}_R$ we apply the standard differential renormalization prescription (3.2.77) order by order in g^2, whereas in order to identify the divergent counterterms which in (3.2.77) are automatically subtracted we need introduce a regularization prescription. We compute divergences using the analytic regularization [46].

As noticed above we are interested in computing the difference $(\Gamma^{(2)}_{\text{deformed}} - \Gamma^{(2)}_{\mathcal{N}=4})$. Thus at one-loop in coordinate space the contribution from the self-energy diagram is

$$\begin{aligned}\Gamma^{(2)} &= \frac{1}{x^4}\,(|h_1|^2 + |h_2|^2 - 2g^2)\frac{1}{(4\pi^2)^2} \qquad (3.2.80)\\ &= \frac{1}{x^4}\,\left[(a_1+b_1-2)g^2 + (a_2+b_2)g^4 + (a_3+b_3)g^6 + (a_4+b_4)g^8 + \cdots\right]\frac{1}{(4\pi^2)^2}\end{aligned}$$

We renormalize this amplitude by the prescription (3.2.77). At order g^2 we find the condition (3.2.20) which guarantees finiteness and vanishing of the beta functions.

As already discussed, once the condition (3.2.20) is satisfied we can neglect all higher loop diagrams which contain bubbles. In particular, at two and three loops we do not find relevant diagrams. Therefore, at orders g^4 and g^6 only the one-loop expression (3.2.80) contributes and the conditions (3.2.59) are sufficient to cancel the renormalized and the divergent parts of $1/x^4$.

At order g^8 the pattern changes since besides the contribution from (3.2.80) we have the new diagram in Fig. 3.3. After D-algebra, in configuration space it corresponds to

$$-\frac{1}{2}(a_1-b_1)^4 g^8 \frac{1}{(4\pi^2)^8}\,\frac{1}{x^2}\int \frac{d^4y\,d^4z\,d^4w}{y^2z^2(y-z)^2(y-w)^2(z-w)^2(x-y)^2(x-w)^2} \tag{3.2.81}$$

This expression has a singularity for $x \sim y \sim z \sim w \sim 0$. To compute its finite part, away from $x=0$ it is convenient to rescale the integration variables as $y \to |x|y$, $z \to |x|z$ and $w \to |x|w$. We are then left with

$$-\frac{1}{2}(a_1-b_1)^4 g^8 \frac{1}{(4\pi^2)^8}\,\frac{1}{x^4}\int \frac{d^4y\,d^4z\,d^4w}{y^2z^2(y-z)^2(y-w)^2(z-w)^2(1-y)^2(1-w)^2} \tag{3.2.82}$$

The integral is finite and uniformly convergent for $x \to 0$. It has been computed e.g. in [47] and it gives $20\pi^6\zeta(5)$. At order g^8, summing this contribution to the one-loop result and renormalizing $1/x^4$ as in (3.2.77) we obtain

$$\Gamma^{(2)}_R|_{g^8} = (A+4B)\left(-\frac{1}{4\pi^2}\Box\frac{\log M^2x^2}{x^2}\right) \tag{3.2.83}$$

where A and B are given in (3.2.61).

Therefore, the condition of vanishing γ from $\Gamma^{(2)}_R$ requires $A+4B=0$. This is exactly the condition we have found working in dimensional regularization and momentum space. This is consistent with the fact that the Fourier transform of $\Box\frac{\log M^2x^2}{x^2}$ is $4\pi^2\log p^2/M^2$.

Now we concentrate on the evaluation of the divergent contributions from the one-loop self-energy diagram and from the four-loop diagram in Fig. 3.3. Using analytic regularization in four dimensions, at one loop and order g^8 we have (for simplicity we neglect (2π) factors)

$$A\frac{1}{(x^2)^{2+2\lambda}} \tag{3.2.84}$$

whereas at four loops we need to evaluate the integral

$$-\frac{1}{2}(a_1-b_1)^4 g^8\ \frac{1}{(x^2)^{1+\lambda}}\times \tag{3.2.85}$$
$$\int \frac{d^4y\,d^4z\,d^4w}{(y^2)^{1+\lambda}(z^2)^{1+\lambda}[(y-z)^2]^{1+\lambda}[(y-w)^2]^{1+\lambda}[(z-w)^2]^{1+\lambda}[(x-y)^2]^{1+\lambda}[(x-w)^2]^{1+\lambda}}$$

Dimensional analysis allows to compute this integral and obtain $(20\zeta(5)+O(\lambda))\frac{1}{(x^2)^{1+7\lambda}}$. This gives the final answer $4B/(x^2)^{2+8\lambda}$ for the diagram in Fig. 3.3.

Now using the general identity

$$\frac{1}{(x^2)^{2+\alpha\lambda}} \sim \frac{1}{\alpha\lambda}\, \delta^{(4)}(x) \;+\; O(\lambda^0) \tag{3.2.86}$$

and summing the one and four-loop results we find that the divergent contribution is

$$A\frac{1}{(x^2)^{2+2\lambda}} + 4B\frac{1}{(x^2)^{2+8\lambda}} \;\rightarrow\; (A+B)\,\frac{1}{2\lambda}\,\delta^{(4)}(x) \tag{3.2.87}$$

Therefore the cancellation of divergences at order g^8 requires $A + B = 0$. If we were to compute the divergences arising at order g^{10} we would find results in total agreement with the results found using dimensional regularization. Going higher in loops we would meet the same pattern an infinite number of times and we would be led to the final result for the coefficients as in (3.2.58).

Collecting the results of this Section, the general situation can be then summarized as follows. If we impose the cancellation of UV divergences at a given order we obtain conditions on the coefficients in the expansion (3.2.13) which do not set automatically to zero the contribution to the chiral beta function at the same order. In particular, in the planar limit the first nontrivial order where this happens is g^8. However, if we move one order higher and still require the cancellation of divergences we obtain more restrictive conditions on the coefficients and as a by–product all the beta functions at that order vanish. This pattern repeats itself at any order in perturbation theory and leads to the following result: The finiteness condition selects a unique expansion (3.2.13) for $h_i(g)$ which corresponds to $\sinh(2\pi \mathrm{Im}\beta) \sim (h_1^2 - h_2^2) = 0$, i.e. to a *real* deformation parameter β.

On the other hand, if we implement superconformal invariance by requiring directly vanishing beta functions regardless of finiteness we obtain less restrictive conditions on the coefficients in (3.2.13) and more general solutions $h_i(g)$ to the renormalization group equation $F(g, h_i) = 0$ which defines the surface of fixed points. These solutions correspond in general to theories which are not finite and allow for a *complex* deformation parameter.

However, while finiteness is a well-defined, scheme independent and then physically meaningful statement, the conditions $\beta_h, \beta_g = 0$ turn out to be scheme dependent, and thus the physical meaning and the predictive power of a quantum theory defined by these conditions is not clear.

In the presence of coupling constant reduction we are not guaranteed that finiteness theorems [35, 36] for the gauge beta function are true in their standard version. However, pushing the perturbative calculation up to five loops, we have found that given the vanishing of the chiral beta function at order g^9, then the gauge beta function is automatically zero at order g^{11}. Our result suggests that the finiteness theorems might be generalized as follows: If the matter chiral beta function vanishes up to the order (g^n) then the gauge beta function vanishes as well up to the order (g^{n+2}).

3.3 Finite N β–deformation

In this Section we consider the $SU(N)$ β-deformed $\mathcal{N}=4$ super Yang–Mills theory working perturbatively with a complex deformation parameter β at finite N [12]. As in the previous Section we address the issue of finding a surface of renormalization fixed points by requiring the theory to have vanishing beta functions and using the coupling constant reduction (CCR) procedure.

First, we concentrate on the chiral beta function β_h up to $\mathcal{O}(g^7)$. If we want to work with a well–defined and a physically meaningful quantum field theory, the condition $\beta_h=0$ should not be affected by scheme dependence. Scheme independence of the conformal definition of the theory introduces a further constraint on the couplings. Here comes the novelty with respect to the planar case studied in Section 3.2. The planar limit involves only two of the three independent superpotential constants in (3.2.2) and scheme independence of the theory forces β to be *real*. On the other hand, keeping N finite, all of the three parameters $|h_1|^2, |h_2|^2, |h_3|^2$ enter the superconformal condition allowing for a *complex* deformed theory which is scheme–independent at least at $\mathcal{O}(g^{10})$. We expect this pattern should hold even for higher orders.

Then we consider the gauge beta function β_g. In Section 3.2.2 a generalization of the standard finiteness theorems has been proposed: if $\beta_h=0$ up to $\mathcal{O}(g^{2n+1})$ then $\beta_g=0$ up to $\mathcal{O}(g^{2n+3})$. This statement has been checked in the planar limit for $n=4$ using an alternative procedure for covariant ∇–algebra. Here we provide another highly non–trivial confirmation of this proposal in the non–planar theory for $n=3$. Moreover, we explicitly check that the simplified ∇–algebra technique used in Section 3.2.2 is equivalent to the standard one.

3.3.1 Chiral beta function and conformal condition

Let us write down one more time the $\mathcal{N}=1$ β–deformed action:

$$
\begin{aligned}
S \;=\; & \int d^8z\ \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z\ \mathrm{Tr}(W^\alpha W_\alpha) \\
& + ih\int d^6z\ \mathrm{Tr}(\ q\ \Phi_1\Phi_2\Phi_3 - \frac{1}{q}\ \Phi_1\Phi_3\Phi_2\ ) \\
& + i\bar{h}\int d^6\bar{z}\ \mathrm{Tr}(\ \frac{1}{\bar{q}}\ \overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3 - \bar{q}\ \overline{\Phi}_1\overline{\Phi}_3\overline{\Phi}_2\ ) \qquad q\equiv e^{i\pi\beta}
\end{aligned}
\tag{3.3.1}
$$

Here h and β are complex couplings and g is the real gauge coupling constant. Although we work at finite N, we will be considering once again 't Hooft rescaled quantities

$$
h \to \frac{h}{\sqrt{N}} \qquad g \to \frac{g}{\sqrt{N}} \tag{3.3.2}
$$

in order to easily make contact with the planar limit. For what we said in Section 3.2.1, the number of real independent couplings in the superpotential is three. For later convenience we choose them to be $|h_1|^2$, $|h_2|^2$ and $|h_3|^2$, where

$$h_1 \equiv h\,q \qquad\qquad h_2 \equiv \frac{h}{q} \qquad\qquad h_3 \equiv h\,q - \frac{h}{q} \tag{3.3.3}$$

In complete analogy with the analysis of the previous Sections we apply the coupling constant reduction program and express the renormalized Yukawa couplings in terms of the gauge one:

$$\begin{aligned} |h_1|^2 &= a_1 g^2 + a_2 g^4 + a_3 g^6 + \ldots \\ |h_2|^2 &= b_1 g^2 + b_2 g^4 + b_3 g^6 + \ldots \\ |h_3|^2 &= c_1 g^2 + c_2 g^4 + c_3 g^6 + \ldots \end{aligned} \tag{3.3.4}$$

To single out a conformal theory we will ask for the chiral and gauge beta functions to vanish. Let us start analyzing β_h, computing as usual the two–point chiral correlator.

In Section 3.2 this issue has been analyzed by considering the planar limit where only two independent real constants enter the color factors, namely $|h_1|^2$ and $|h_2|^2$. As a result the definition of the conformal theory was found to be scheme dependent as long as β was complex. In the non–planar case all of the three parameters enter the calculation of the two–point chiral correlator. We will see that this difference will be important in the definition of the fixed point surface [12].

Let us start at order g^2, considering the difference between divergent diagrams in the β–deformed and in the $\mathcal{N} = 4$ theory. This amounts to the evaluation of the chiral bubbles in Fig. 3.19 which give the following divergent contribution to the chiral propagator

$$\frac{1}{(4\pi)^2}\left[|h_1|^2 + |h_2|^2 - \frac{2}{N^2}|h_3|^2 - 2g^2\right]\,\frac{1}{\epsilon}\,\left(\frac{\mu^2}{p^2}\right)^{\epsilon} \tag{3.3.5}$$

where we have explicitly indicated the factors coming from dimensionally regulated integral (here p is the external momentum and μ is the standard renormalization mass).
At this stage, in order to obtain a vanishing chiral beta function, the following condition has to be imposed

$$\mathcal{O}(g^2): \qquad\qquad a_1 + b_1 - \frac{2}{N^2}c_1 = 2 \tag{3.3.6}$$

Moreover, by now it should be clear that

$$|h_1|^2 + |h_2|^2 - \frac{2}{N^2}|h_3|^2 = 2g^2 \tag{3.3.7}$$

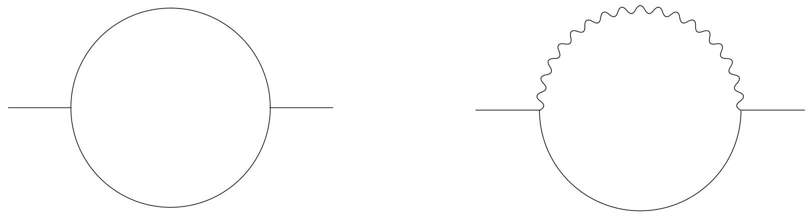

Figure 3.19: One loop diagrams

ensures $\gamma = 0$ up to two loops. So, looking at the chiral two–point contribution (3.3.5) at order g^4, we have the following additional requirement

$$\mathcal{O}(g^4): \qquad a_2 + b_2 - \frac{2}{N^2} c_2 = 0 \tag{3.3.8}$$

It is easy to see that equations (3.3.6) and (3.3.8) reduce to the (3.2.59) in the large N limit. When we move up to the next order the situation becomes more involved with respect to the planar case. In fact, working with finite N we need to consider the non–planar graph in Fig. 3.20, whose contribution is (see Appendix C):

$$\frac{1}{(4\pi)^6}\, 2\zeta(3)\, \mathcal{F}\, \frac{1}{\epsilon} \left(\frac{\mu^2}{p^2}\right)^{3\epsilon} \tag{3.3.9}$$

where $\mathcal{F} \equiv \mathcal{F}\left(|h_1|^2, |h_2|^2, |h_3|^2, N^2\right)$ reads [28, 9]

$$\mathcal{F} = \frac{N^2-4}{N^4}\, |h_3|^2 \left[\frac{N^2+5}{N^2}\, (|h_3|^2)^2 - 3|h_3|^2(|h_1|^2 + |h_2|^2) + 3(|h_1|^2 - |h_2|^2)^2\right] \tag{3.3.10}$$

Notice that the color factor in (3.3.10) is suppressed as $1/N^2$ for large N. Due to the expansion in (3.3.4) both the one loop (3.3.5) and three loops (3.3.9) structures contribute to the evaluation of γ at $\mathcal{O}(g^6)$. The final result can be recast as

$$\frac{1}{\epsilon}\left[A\left(\frac{\mu^2}{p^2}\right)^{\epsilon} + \frac{B}{N^2}\left(\frac{\mu^2}{p^2}\right)^{3\epsilon}\right] \tag{3.3.11}$$

where we have defined for simplicity

$$A \equiv \frac{1}{(4\pi)^2}\, (a_3 + b_3 - \frac{2}{N^2} c_3) \tag{3.3.12}$$

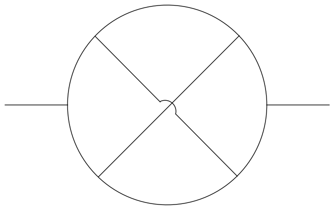

Figure 3.20: Three loop non–planar diagram

$$B \equiv \frac{2\zeta(3)}{(4\pi)^6}\,\frac{N^2-4}{N^2}\;c_1\;\left[\frac{N^2+5}{N^2}\;c_1^2-3c_1(a_1+b_1)+3(a_1-b_1)^2\right] \tag{3.3.13}$$

The vanishing condition of the anomalous dimension at order g^6 can be read directly from the finite *log* term in (3.3.11):

$$\mathcal{O}(g^6):\qquad\qquad A+\frac{3B}{N^2}=0 \tag{3.3.14}$$

We emphasize that at this order the condition for the vanishing of γ and β_h is completely scheme independent. However, from now on we will have to care about the scheme dependence in the definition of the fixed points. To see this, let us consider the counterterm needed at this stage to properly renormalize the propagator in an arbitrary scheme:

$$g^6\;(A+\frac{B}{N^2})\;(\frac{1}{\epsilon}+\rho) \tag{3.3.15}$$

where ρ is a constant related to the choice of finite renormalization. In fact, if we were to push the conformal invariance condition one order higher we should compute the chiral beta function at order g^9. We expect to have several sources of nontrivial contributions to γ at this order: one coming from the one–loop bubble proportional to $(a_4+b_4-\frac{2}{N^2}c_4)$, then from two–loop, three–loop and four–loop diagrams. All of the diagrams containing subdivergences, namely the two and four loop contributions, will be subtracted making use of the appropriate counterterms. To be specific, a term like

$$g^8\;(A+\frac{B}{N^2})\;(\frac{1}{\epsilon}+\rho)\;\frac{1}{\epsilon}\;\left(\frac{\mu^2}{p^2}\right)^{\epsilon} \tag{3.3.16}$$

will appear in the calculation of γ. Therefore the requirement of vanishing anomalous dimension depends unavoidably on the arbitrary constant ρ which appears in the form

$$(A+\frac{B}{N^2})\;\rho \tag{3.3.17}$$

If we wanted to kill the scheme dependence of the result we would also need to impose the vanishing of the combination $A + B/N^2$ which together with (3.3.14) would lead immediately to $A = B = 0$. The crucial observation is that in the non–planar case we deal with three parameters and at this stage we have enough freedom to eliminate the scheme dependence from the conformal condition without reducing to the real β case. In fact, the constraint $A = 0$ gives

$$a_3 + b_3 - \frac{2}{N^2} c_3 = 0 \tag{3.3.18}$$

while the condition $B = 0$ combined with equation (3.3.6) yields

$$\begin{cases} a_1 + b_1 = 2 \\ c_1 = 0 \end{cases} \tag{3.3.19}$$

or, if $c_1 \neq 0$

$$\begin{cases} a_1 + b_1 = 2\left(1 + \frac{c_1}{N^2}\right) \\ \\ a_1 - b_1 = \pm\sqrt{2\, c_1 \, \left(1 - \frac{N^2-1}{6\, N^2}\, c_1\right)} \end{cases} \tag{3.3.20}$$

These solutions allow for a non vanishing imaginary part of β (which is proportional to the combination $|h_1|^2 - |h_2|^2$). At the same time, they define the surface of renormalization fixed points without any ambiguity related to the choice of regularization scheme. It is clear that in the planar limit only the condition coming from $A = 0$ survives as the $B = 0$ condition is subleading. So we are left with $a_3 + b_3 = 0$, in complete agreement with the result found in (3.2.59).

If we move to the next order, a new scenario will show up. Having imposed (3.3.19) or (3.3.20) only three graphs will contribute to the anomalous dimension at order g^8 (Fig. 3.21).

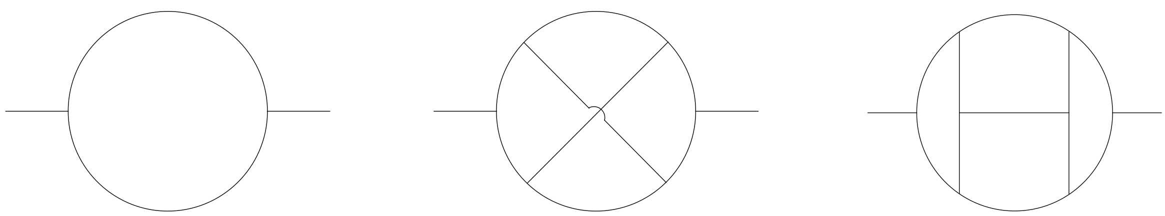

Figure 3.21: Diagrams contributing to γ at order g^8

Since these diagrams are primitively divergent (no subdivergences are present) the condition for $\gamma = 0$ at this order turns out to be completely scheme independent. In fact we have to consider the following expression:

$$\frac{1}{\epsilon}\left[A'\left(\frac{\mu^2}{p^2}\right)^{\epsilon} + \frac{B'}{N^2}\left(\frac{\mu^2}{p^2}\right)^{3\epsilon} + H\left(\frac{\mu^2}{p^2}\right)^{4\epsilon}\right] \tag{3.3.21}$$

where we have denoted

$$A' \equiv \frac{1}{(4\pi)^2}\left(a_4 + b_4 - \frac{2}{N^2}\,c_4\right) \tag{3.3.22}$$

$$B' \equiv \frac{6\,\zeta(3)}{(4\pi)^6}\frac{N^2-4}{N^2}\left[(a_1-b_1)^2 c_2 + c_1\Big(\frac{N^2-1}{N^2}\,c_1 c_2 + 4\,(a_1-b_1)(a_2-\frac{c_2}{N^2}) - 4c_2\Big)\right] \tag{3.3.23}$$

$$H \equiv -\frac{5\,\zeta(5)}{2(4\pi)^8}\left[(a_1-b_1)^4 + (a_1+b_1)^4 + \frac{1}{N^2}\,f\Big(a_1, b_1, c_1, \frac{1}{N^2}\Big) - \frac{16(N^2+12)}{N^2}\right] \tag{3.3.24}$$

where f can be read from Appendix D and we have used the relations (3.3.6) and (3.3.8). The vanishing of γ reads

$$\mathcal{O}(g^8): \qquad\qquad A' + \frac{3B'}{N^2} + 4H = 0 \tag{3.3.25}$$

Again, in order to remove scheme dependence from the $\mathcal{O}(g^{10})$ conformal condition we have to impose:

$$A' + \frac{B'}{N^2} + H = 0 \tag{3.3.26}$$

At this stage, independently of the choice (3.3.19) or (3.3.20), we have enough parameters to solve both equations without restricting to the real β case as in the planar theory. On the other hand, if one sends $N \to \infty$, equations (3.3.25) and (3.3.26) reduce to the ones found in Section 3.2. This large N limit turns out to be smooth and does not present any sort of singularity, so there is no contradiction between our results and those found in [11]. We observe that a scheme–independent definition of the complex β conformal theory can be achieved only thanks to subleading coefficients which are projected out by the planar limit.

3.3.2 Gauge beta function and finiteness theorems

Now we turn to consider the gauge beta function. Standard finiteness theorems [35, 36] ensure the vanishing of β_g at L+1–loops once β_h has been set to zero at L–loops. Here, as

Figure 3.22: Two and four loop vacuum diagrams

a consequence of coupling constant reduction, we are forced to work order by order in g^2 instead of loop by loop and it is not obvious that such theorems still hold. Nevertheless in Section 3.2.2 it was shown that in the planar β–deformed theory the vanishing condition for β_h at $\mathcal{O}(g^9)$ was sufficient to have vanishing β_g at $\mathcal{O}(g^{11})$. This result was a strong indication that finiteness theorems could be generalized as follows: if the matter chiral beta function vanishes up to order g^{2n+1} then the gauge beta function vanishes as well up to order g^{2n+3}. Here we are going to check this result at finite N and for $n = 3$. In order to do this, we take advantage of covariant supergraph techniques combined with background field method [59]. The standard procedure consists in looking at vacuum diagrams at a given perturbative order and performing covariant ∇–algebra. Then by expanding propagators one extracts tadpole type contributions with vector connections as external legs. Moreover one only selects diagrams containing at least a $1/\epsilon^2$ pole (see [36] for details). In the present case, contributions to the gauge beta function at $\mathcal{O}(g^9)$ come from two and four loop vacuum diagrams (Fig. 3.22).

The analysis of the two loop diagram is straightforward and completely analogous to the one in [36]. Expanding the covariant propagators one obtains three times the diagram in Fig. 3.23 which corresponds to the term

$$\frac{1}{2}\,\mathrm{Tr}\,(\Gamma^a\Gamma_a)\int\frac{d^nk\;d^nq}{(2\pi)^{2n}}\frac{1}{q^2(q+k)^2k^4} \tag{3.3.27}$$

where Γ_a is the vector connection.

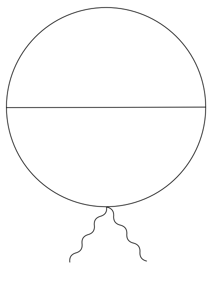

Figure 3.23: Two loops tadpole diagram

This integral contains a one–loop ultraviolet subdivergence and it is infrared divergent. It is convenient to remove the IR divergence using the R^* subtraction procedure of [41]. After UV and IR subtractions one isolates the $1/\epsilon^2$ term and rewrites the result in a covariant form (see Appendix C), obtaining the following contribution to the two loop effective action:

$$\frac{1}{(4\pi)^2} \frac{3\,(N^2-1)}{4\,N} \, A \, \frac{1}{\epsilon} \, \mathrm{Tr} \int d^4x \, d^2\theta \, W^\alpha W_\alpha \tag{3.3.28}$$

where we have inserted the A factor defined in (3.3.12).

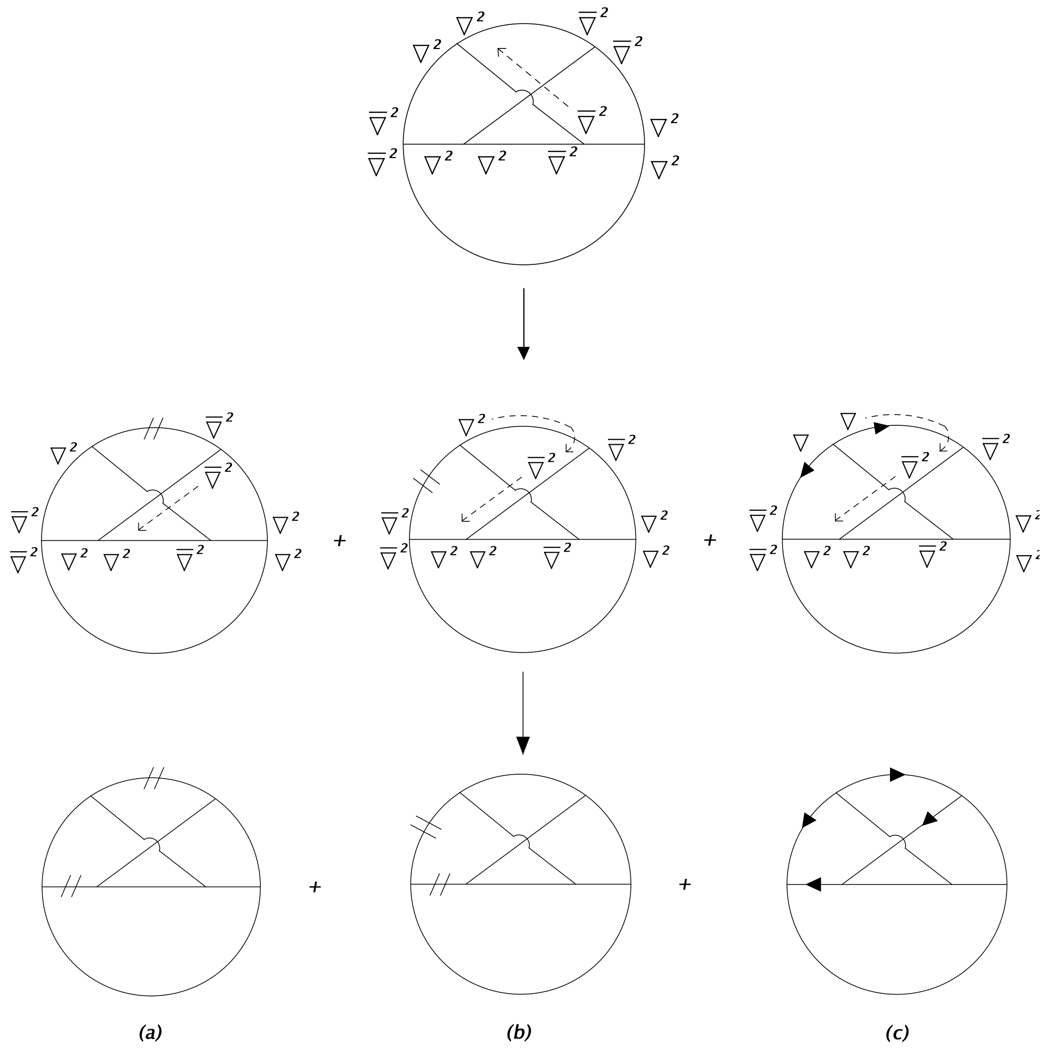

Figure 3.24: ∇–algebra operations on four–loop vacuum diagram

Now we turn to consider the four loop contributions. In this case the computation is much more involved because we need to perform very non trivial ∇–algebra operations. In Section 3.2.2 an analogous problem was solved by using an alternative procedure, though different from the one just described which turned out to be too hard to deal with. Here we want to consider both methods and show that they indeed give the same result. Let us start with the standard procedure. A detailed explanation of ∇–algebra operations can be found in Fig. 3.24. Starting from the top vacuum diagram and performing integration by parts we end up with three different graphs. Each of them gives rise to a single bosonic diagram: Fig. 3.24 (a), (b), (c), where we have denoted

$$// \equiv \frac{\frac{1}{2}\nabla^a\nabla_a}{\Box} \rightarrow 1 - \frac{1}{2}\,\frac{\Gamma^a\Gamma_a}{\Box} \qquad \Box \equiv \frac{1}{2}\,\partial^a\,\partial_a \qquad \blacktriangleright \equiv \nabla_a = \partial_a - i\Gamma_a \quad (3.3.29)$$

Now we are ready to expand the covariant propagators to extract tadpole–type contributions. It is easy to see that Fig. 3.24(a) and (b) diagrams are equivalent and give rise to the tadpole graphs shown in Fig. 3.25.

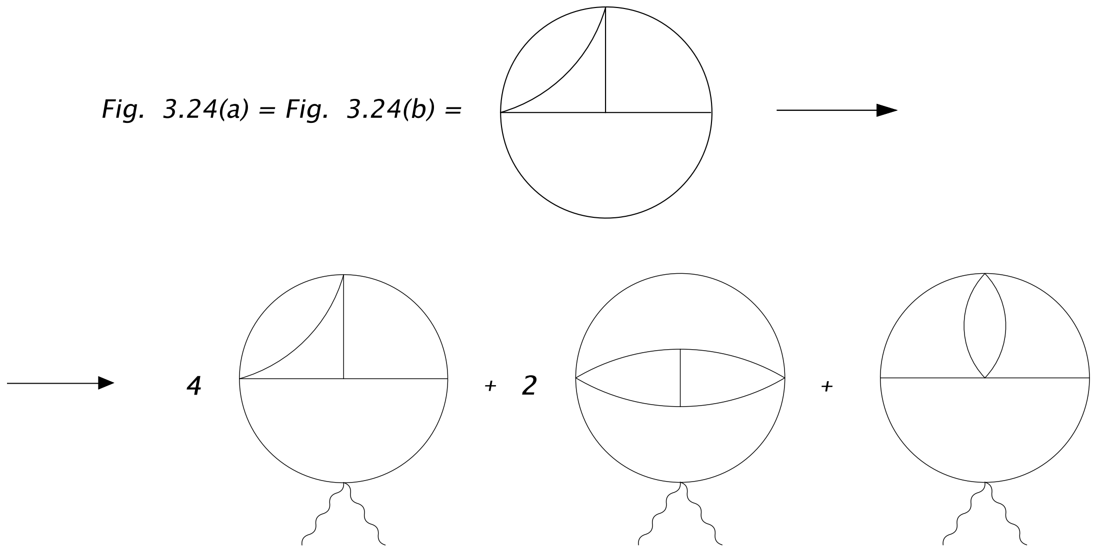

Figure 3.25: Tadpole contributions from propagator expansions of diagrams in Fig. 3.24(a) and (b)

Analogously the Fig. 3.24(c) diagram can be expanded to give the relevant tadpole contributions as indicated in Fig. 3.26. The latter integrals are much harder to compute because of the presence of four derivatives, indicated by the black arrows. However, after some appropriate integrations by parts, they can be reduced to simpler scalar integrals, as depicted in Fig. 3.27. Notice that in the whole procedure we have neglected all tadpole graphs with $1/\epsilon$ divergences, which do not contribute to the four–loop effective action.

Now we just need to sum up the various contributions generated by Fig. 3.24(a), (b) and (c) diagrams. Actually there is no need to compute all these integrals explicitly thanks to

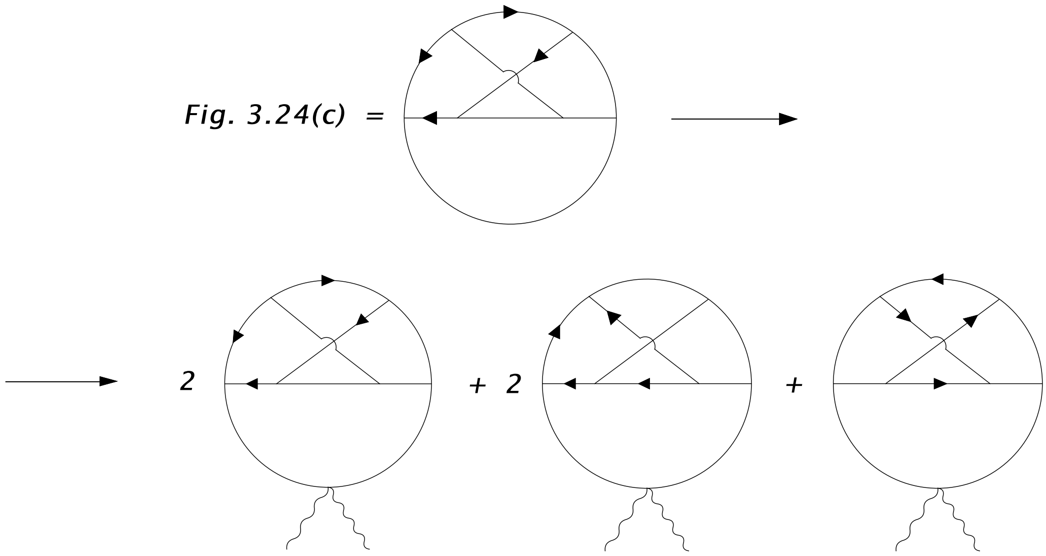

Figure 3.26: Tadpole contributions from relevant propagator expansions of diagram (c)

a beautiful diagrammatic cancellation. In fact, the only surviving terms sum up to give nine times the same diagram, shown in Fig. 3.28. The corresponding bosonic integral is:

$$\frac{1}{2}\,\mathrm{Tr}(\Gamma^a\Gamma_a)\int\frac{d^nk\,d^nq\,d^nr\,d^nt}{(2\pi)^{4n}}\,\frac{1}{k^4\,q^2\,t^2\,(r-q)^2\,(r+t)^2\,(t+q)^2\,(r+k)^2} \qquad (3.3.30)$$

So the total four–loop contribution to the effective action, after inserting color and combinatorial factors and subtracting IR and UV subdivergences is given by (see Appendix C):

$$\frac{1}{(4\pi)^2}\,\frac{9\,(N^2-1)}{8\,N^3}\,B\,\frac{1}{\epsilon}\,\mathrm{Tr}\int d^4x\,d^2\theta\,W^\alpha W_\alpha \qquad (3.3.31)$$

with B defined as in (3.3.13). This completes the computation of the four loops contribution with the standard method.

Had we followed the alternative procedure developed in Section 3.2.2 we would have first expanded each of the nine propagators of the four–loop vacuum diagram in Fig. 3.13 and then performed ∇–algebra. In this case, the only possible contributions would come from two types of diagrams:
I. the ones with flat D^2 and $\bar{D}^2$ factors at the vertices, flat propagators and one tadpole insertion, for which now standard D–algebra can be performed
and

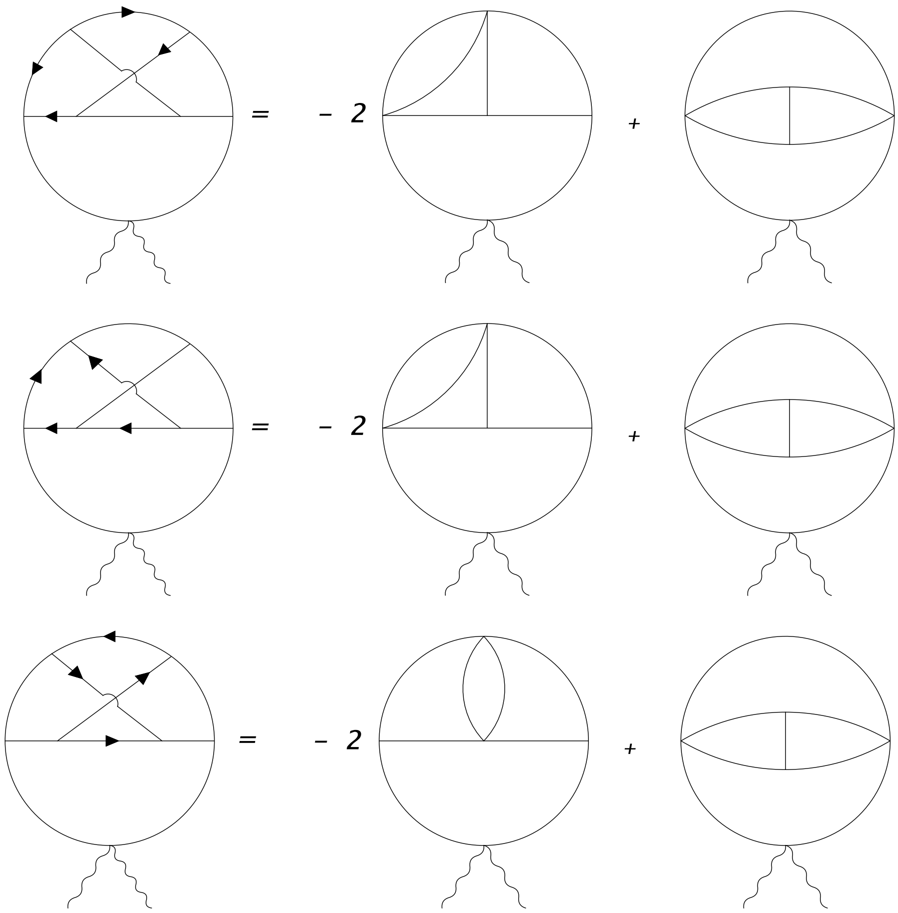

Figure 3.27: Scalar reduction of integrals with derivatives

II. the vacuum diagrams with flat propagators but ∇^2 and $\bar{\nabla}^2$ at the chiral vertices in which the tadpole insertion will have to appear after completion of the ∇–algebra.
Analogously to Section 3.2.2, it is easy to see that only type I diagrams contribute. The computation is now straightforward. As the vacuum diagram is completely symmetric we have nine equivalent choices for the propagator to expand. Once a choice has been made the standard D–algebra gives rise to a unique contribution, producing precisely the result depicted in Fig. 3.28. We have therefore checked that as expected the two methods actually give the same answer.
Now we come back to the computation of the gauge beta function and combine (3.3.28) and (3.3.31). We can easily read the vanishing condition at order g^9:

$$A + \frac{3\,B}{N^2} = 0 \tag{3.3.32}$$

which is exactly the one obtained by requiring the vanishing of β_h at order g^7. Thus we provide one more confirmation that finiteness theorems for the gauge beta functions hold even in the CCR context.

Fig. 3.24(a) + (b) + (c) = 9

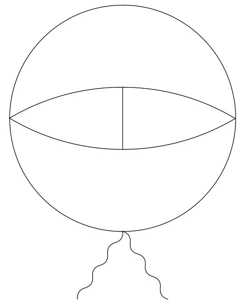

Figure 3.28: Four loops total contribution to the gauge beta function

Chapter 4

Chiral Ring and Protected Operators

A considerable effort has been devoted to provide tests of the AdS/CFT correspondence in its marginal deformed version. As for the $AdS_5 \times S^5$ original correspondence, perturbative properties of the field theory have been investigated. One of the most effective perturbative test consists in verifying non–renormalization properties of composite operators in the chiral ring; alternatively one can check anomalous dimension expressions perturbatively for those sectors where an exact result can be given and then it is possible to get contact with the gravity side.

In this Chapter we would like to review some of these aspects, starting from an example of computation of the anomalous dimension γ for spin–2 operators [1] of the form $\mathrm{Tr}(\Phi_1^J \Phi_2)$ [8]. For this class of operators, an exact expression for γ can be given. This expression will be checked to be valid up to two loops to provide an example of how this kind of calculations are performed in our notations.

Then we turn to analyze the chiral ring of the marginally deformed theories in a more systematic way [9]. We will work at finite N and then take into account mixing among sectors with different trace structures. Exploiting the definition of quantum chiral ring we reduce the determination of protected operators up to order n in perturbation theory to the evaluation of the effective superpotential up to order $(n-1)$. Precisely, from the knowledge of the effective superpotential we determine perturbatively all the quantum descendant operators of naive scale dimension Δ_0, and find the CPO's as the operators which are orthogonal order by order to the descendants.

For the β–deformed theory we investigate the spin–2 sector and applying our procedure to simple cases ($\Delta_0 = 4, 5$) we determine the protected operators up to three loops. In the sectors we have studied we can always define descendant operators which do not receive quantum corrections. This seems to be a general property of the spin–2 operators: Despite the nontrivial appearance of finite perturbative corrections to the effective action, the quantum descendant operators defined in terms of the effective superpotential coincide with their expressions given in terms of the classical superpotential (up to possible mixing among them).

[1] We use the notation of [78] and call "spin–n" the sector containing operators made by products of n different flavors.

We then investigate the spin–3 sector where, due to the appearance of Konishi–like anomalies, we need restrict our analysis at two loops in order to avoid dealing with mixed gauge/scalar operators. Up to this order the descendant operators we consider are the classical ones. However, in this sector we expect higher order corrections to the descendants to appear together with a nontrivial dependence on the anomaly term. Therefore, the non–renormalization properties of the descendant operators that we find for the spin–2 sector are not a general feature of the theory.

We generalize our procedure to the study of protected operators for the $\mathcal{N} = 1$ superconformal theory associated to the full Leigh–Strassler deformation (2.3.4). Even if the gravity dual of this theory is not known yet, it is anyway interesting to figure out the general structure of its chiral ring. Still at finite N, we study explicitly the weight–2 and weight–3 sectors up to two loops and perform a preliminary analysis of the general sectors at least at lowest order in the couplings. An interesting result we find is that, because of the discrete Z_3 symmetries of the theory, the sectors corresponding to conformal weights which are multiple of 3 have a different operator structure from the other ones.

4.1 A first simple example

To introduce our procedure we begin with a simple example of perturbative computation of γ for a class of operators whose anomalous dimension is known exactly [8]. We consider the case of planar real β–deformed theory at his superconformal point $|h|^2 = g^2$. Then we choose to analyze the class of non–protected operators [2]

$$\mathcal{O}_J = \mathrm{Tr}(\Phi_1^J \Phi_2) \tag{4.1.1}$$

These operators are charged under the $U(1)_1 \times U(1)_2$ global symmetry group with charges $(1, 1-J)$.

Using the equations of motion from the action (2.3.9) (from now on we neglect factors of $e^{\pm gV}$ since they are not relevant to our purposes)

$$\bar{D}^2\, \overline{\Phi}_3^a = -ih \Phi_1^b \Phi_2^c\, [q(abc) - \frac{1}{q}(acb)] \tag{4.1.2}$$

it is easy to see that

$$\mathcal{O}_J = \frac{i}{h[q - \frac{1}{q}]} \bar{D}^2 \mathrm{Tr}(\Phi_1^{J-1} \overline{\Phi}_3) + \frac{1}{N} \mathrm{Tr}(\Phi_1^{J-1}) \mathrm{Tr}(\Phi_1 \Phi_2) \tag{4.1.3}$$

As long as $J > 1$, in the large N limit the operator $\mathcal{O}_J$ becomes descendant of the primary $\mathrm{Tr}(\Phi_1^{J-1}\overline{\Phi}_3)$, whereas for finite N the combination $\mathcal{O}_J - \frac{1}{N}\mathrm{Tr}(\Phi_1^{J-1})\mathrm{Tr}(\Phi_1\Phi_2)$ is descendant. The exceptional case $J = 1$ corresponds to the chiral primary operator whose protection has been proven perturbatively in [26, 27].

[2] The choice of Φ_1 and Φ_2 superfields is totally arbitrary and we expect the operators $\mathrm{Tr}(\Phi_i^J \Phi_k)$, for any i, k with $i \neq k$ to have similar quantum properties. We will comment on this point later on.

4.1.1 Perturbative evaluation of the anomalous dimensions

We compute the anomalous dimension of $\mathcal{O}_J$ in (4.1.1) perturbatively, up to two loops. For generic values of J we perform the calculation in the large N limit in order to avoid dealing with mixing with multitrace operators.

In order to compute anomalous dimensions we evaluate one–point correlators $\langle \mathcal{O}_J e^{S_{int}} \rangle$ where S_{int} is the sum of the interaction terms. Divergent contributions proportional to the operator itself are removed by a multiplicative renormalization which in dimensional regularization reads

$$\mathcal{O}_J^{(bare)} \equiv \mathcal{O}_J \left(1 + \sum_{k=0}^{\infty} \frac{a_k(\lambda, q, N)}{\epsilon^k} \right) \equiv Z \mathcal{O}_J \tag{4.1.4}$$

where we have introduced the 't Hooft coupling $\lambda = \frac{g^2 N}{4\pi^2}$. We have not indicated the explicit dependence on the h coupling since we are at the superconformal point $|h|^2 = g^2$.

The anomalous dimension is then given by

$$\gamma \equiv 2\lambda \frac{d a_1(\lambda, q, N)}{d\lambda} \tag{4.1.5}$$

Therefore, at any order it is easily read from the simple pole divergence.

At the lowest order the only contribution to the one–point function for the operator $\mathcal{O}_J$ is the one given in Fig. 4.1 where, using the notation introduced in [48], the horizontal bold line indicates the spacetime point where the operator is inserted.

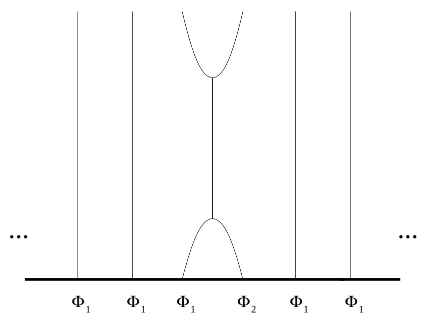

Figure 4.1: One–loop contribution to the $\mathcal{O}_J$ anomalous dimension

The corresponding contribution is proportional to the self–energy integral

$$I_1 \equiv \int d^n k \frac{1}{k^2 (p-k)^2} \sim \frac{1}{(4\pi)^2} \frac{1}{\epsilon} \tag{4.1.6}$$

Evaluating the color factor, the combinatorics and taking into account a minus sign from D–algebra we obtain

$$\text{Diagram 1} \quad \rightarrow \quad -\frac{1}{\epsilon} \, |q - \frac{1}{q}|^2 \frac{|h|^2 N}{(4\pi)^2} \tag{4.1.7}$$

Using the one–loop superconformal condition in the planar limit ($g^2 = |h|^2$) and the definition (4.1.5) we immediately find the one–loop anomalous dimension

$$\gamma^{(1)} = \frac{1}{2}\Big|q - \frac{1}{q}\Big|^2 \lambda \tag{4.1.8}$$

At two loops (order λ^2) the diagrammatic contributions are drawn in Fig. 4.2.

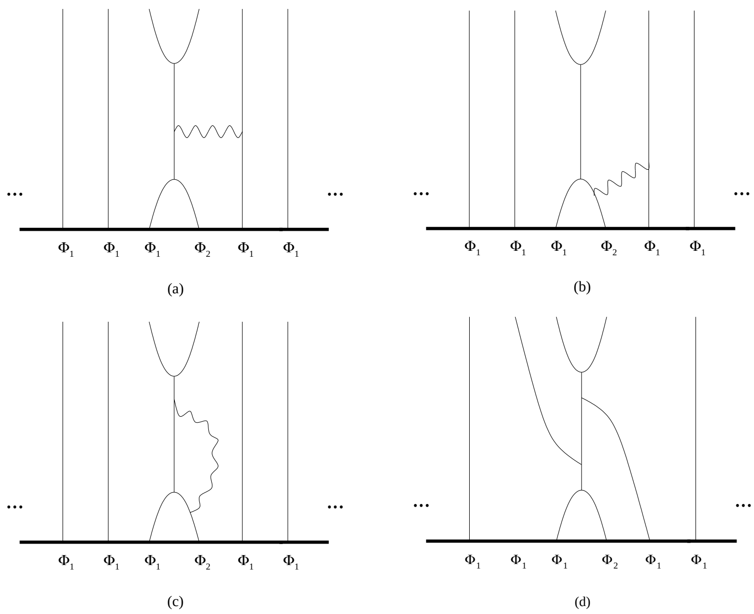

Figure 4.2: Two–loop contributions to the $\mathcal{O}_J$ anomalous dimension

Performing the D–algebra we reduce all the diagrams to ordinary Feynman diagrams containing the loop structure as in Fig. 4.3.

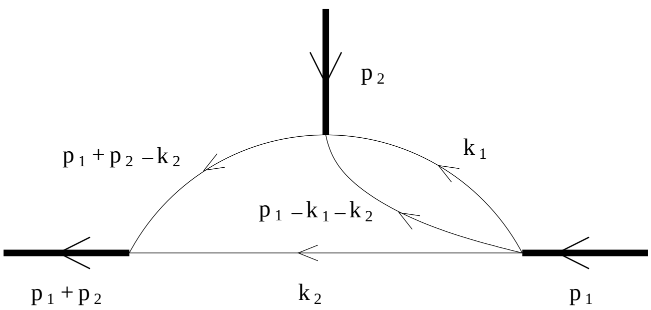

Figure 4.3: The two–loop bosonic integral

The associated momentum integral is

$$I_2 \equiv \int d^n k_1 d^n k_2 \frac{1}{k_1^2(p_1-k_1-k_2)^2 k_2^2(p_1+p_2-k_2)^2} \tag{4.1.9}$$

As long as we are only concerned with UV divergences we can safely set one of the external momenta to zero. Thus the graph is easily evaluated being proportional to two nested self–energies. We obtain (in the G-scheme [60])

$$I_2 \sim \frac{1}{(4\pi)^4}\ \frac{1}{2\epsilon^2}(1+5\epsilon)\frac{1}{(p^2)^{2\epsilon}} \tag{4.1.10}$$

where we have kept only divergent terms. Performing the subtraction of the subdivergence we finally have

$$[I_2]_{sub} \sim \frac{1}{(4\pi)^4}\ \left[-\frac{1}{2\epsilon^2}+\frac{1}{2\epsilon}\right] \tag{4.1.11}$$

Computing the combinatorics, the color factors and taking into account minus signs from the vector propagator we find that the factors in front of (4.1.11) for the various diagrams are

$$\begin{aligned}
Fig.\,4.2(a) \quad &\rightarrow \quad -2(q-\frac{1}{q})(\bar{q}-\frac{1}{\bar{q}})g^2|h|^2N^2 \\
Fig.\,4.2(b) \quad &\rightarrow \quad 2(q-\frac{1}{q})(\bar{q}-\frac{1}{\bar{q}})g^2|h|^2N^2 \\
Fig.\,4.2(c) \quad &\rightarrow \quad 2(q-\frac{1}{q})(\bar{q}-\frac{1}{\bar{q}})g^2|h|^2N^2 \\
Fig.\,4.2(d) \quad &\rightarrow \quad -(q-\frac{1}{q})(\bar{q}-\frac{1}{\bar{q}})\left(\frac{q}{\bar{q}}+\frac{\bar{q}}{q}\right)|h|^4N^2
\end{aligned} \tag{4.1.12}$$

Summing all the contributions, using the planar superconformal condition $|h|^2 = g^2$ and the definition (4.1.5), we find

$$\gamma^{(2)} = -\frac{1}{8}\left|q-\frac{1}{q}\right|^4\lambda^2 \tag{4.1.13}$$

We observe that the diagrams contributing to the anomalous dimensions for our operators are exactly the same as the ones for BMN operators in $\mathcal{N}=4$ SYM in the planar limit [49, 48]. In fact, up to this order the calculation is exactly the same under the formal identification $|q-\frac{1}{q}|^2 \leftrightarrow -(e^{i\phi}+e^{-i\phi}-2)$, where ϕ is the phase of BMN operators [49, 48, 50]. We expect that the same pattern will persist at any order in perturbation theory. In particular, as in the BMN case, the graphs relevant for the calculation are only the ones where the interactions are close to the "impurity" Φ_2: at L–loop order the interactions may involve at most the Φ_1 lines which are L–steps far away from the impurity. As an important consequence, in the large J limit the anomalous dimensions do not grow with J.

In conclusion we note that the result we have found for the anomalous dimensions of the operators $\mathrm{Tr}(\Phi_1^J\Phi_2)$ at large N is actually valid for any operator of the form $\mathrm{Tr}(\Phi_i^J\Phi_k)$

with $i \neq k$. In fact the superpotential is invariant under cyclic permutation of (Φ_1, Φ_2, Φ_3), and in addition it becomes invariant if non–cyclic exchanges of fields are accompanied by

$$q \to -\frac{1}{q} \tag{4.1.14}$$

Since the anomalous dimensions are proportional to powers of the effective coupling $\alpha \equiv \lambda \Big| q - \frac{1}{q} \Big|^2$ which is invariant under (4.1.14) we conclude that the result is valid for any operator of the form $\mathrm{Tr}(\Phi_i^J \Phi_k)$, $i \neq k$.

4.1.2 The exact anomalous dimensions

Motivated by the formal correspondence of the previous calculation with the BMN case, we are going to compute the *exact* anomalous dimensions in the large N, large J limit by using the procedure introduced in [50] for BMN operators. In the context of β–deformed theories this procedure has first been applied to the more general BMN operator class [71]. The result we present here can be considered as a particular case of the one found in [71].

We concentrate on the operator $\mathcal{O}_{J+1}$ which, as follows from eq. (4.1.3), in the planar limit satisfies

$$\bar{D}^2 \mathcal{U}_J = -ih[q - \frac{1}{q}] \mathcal{O}_{J+1} \tag{4.1.15}$$

where we have defined

$$\mathcal{U}_J \equiv \mathrm{Tr}(\Phi_1^J \overline{\Phi}_3) \tag{4.1.16}$$

As already noticed, this shows that the $\mathcal{O}_{J+1}$ operators are descendants of the $\mathcal{U}_J$ ones. Being part of the same superconformal multiplet they share the renormalization properties, i.e. they will have the same scaling dimension and the same perturbative corrections to their overall normalization. Moreover since $\mathcal{U}_J$ is not a Konishi-like operator it is not affected by the Konishi anomaly.

As discussed in details in [50], in any $N = 1$ superconformal field theory the two–point function for a *primary* operator $\mathcal{A}_{s,\bar{s}}$ is fixed and given by $(z \equiv (x, \theta, \bar{\theta}))$

$$< \mathcal{A}_{(s,\bar{s})}(z) \bar{\mathcal{A}}_{(s,\bar{s})}(z') >= f_{\mathcal{A}}(g^2, N, h, \bar{h}) \left\{ \frac{1}{2} D^\alpha \bar{D}^2 D_\alpha + \frac{w}{4(\Delta_0 + \gamma)} [D^\alpha, \bar{D}^{\dot{\alpha}}] i\partial_{\alpha\dot{\alpha}} \right.$$
$$\left. + \frac{(\Delta_0 + \gamma)^2 + w^2 - 2(\Delta_0 + \gamma)}{4(\Delta_0 + \gamma)(\Delta_0 + \gamma - 1)} \Box \right\} \frac{\delta^4(\theta - \theta')}{|x - x'|^{2(\Delta_0 + \gamma)}} \tag{4.1.17}$$

where $\Delta_0 = s + \bar{s}$ is the tree–level dimension of the operator, $\omega = s - \bar{s}$ is its R–symmetry charge [3] and γ is the exact anomalous dimension.

The relation (4.1.17) can be straightforwardly applied to our primary operators $\mathcal{U}_J$. The analysis of the two-point correlator for the $\mathcal{O}_J$'s is somewhat subtler since, as we see

[3] We assume ω not to renormalize. In fact, once the R–symmetry of the elementary fields is fixed by requiring the exact R–symmetry of the superpotential, any composite operator has a fixed charge given by the sum of the charges of its elementary constituents.

from eq. (4.1.15), these chiral operators are not primaries and in principle the relation (4.1.17) cannot be applied to their correlators. However, as we are going to show, in the large J limit these operators turn out to behave as CPO's and (4.1.17) can be safely used.

To this end we remind that in general given a *chiral* operator $\mathcal{A}$, the condition for the operator to be non–protected (anomalous dimension acquired) is equivalent to the condition that its chiral nature is not maintained under superconformal transformations, i.e. $\bar{D}(\delta_{\bar{S}}\mathcal{A}) \sim \{\bar{S},\bar{D}\}\mathcal{A} \neq 0$ (see for instance [51]). In fact, writing schematically the superconformal algebra relation for a scalar operator as $\{\bar{S},\bar{D}\} = \Delta - \omega$, we have

$$\{\bar{S},\bar{D}\}\mathcal{A} = (\Delta-\omega)\mathcal{A} = [(\Delta_0+\gamma)-\omega]\,\mathcal{A} = \gamma\mathcal{A} \tag{4.1.18}$$

where we have used $\Delta = \Delta_0 + \gamma$ and for a chiral operator $\omega = \Delta_0$. Therefore if $\gamma \neq 0$, $\bar{S}\mathcal{A}$ is not chiral anymore. Viceversa, if $\{\bar{S},\bar{D}\}\mathcal{A} = 0$, then $\bar{S}\mathcal{A}$ is still chiral and the dimension is protected by the well–known condition $\Delta = \omega$.

An alternative proof goes through the simple observation that the conditions

$$s + \bar{s} = \Delta_0 + \gamma \qquad\qquad s - \bar{s} = \Delta_0 \tag{4.1.19}$$

imply

$$s = \Delta_0 + \frac{\gamma}{2} \qquad\qquad \bar{s} = \frac{\gamma}{2} \tag{4.1.20}$$

The appearance of $\bar{s} \neq 0$ signals the lack of chirality of the quantum operator.

We now apply the previous argument to our operators O_J to prove that in the large N, large J limit the violation of chirality is suppressed and they behave as CPO's. In the limit of large R–symmetry $\omega = J$ it is more natural to consider

$$\frac{1}{J}\{\bar{S},\bar{D}\}\mathcal{O}_J = \left(\frac{\Delta_0+\gamma}{J} - 1\right)\mathcal{O}_J = \frac{\gamma}{J}\mathcal{O}_J \tag{4.1.21}$$

As discussed in the previous Section, at any fixed order in perturbation theory the anomalous dimension γ does not grow with J. It follows that in the large J limit the r.h.s. of eq. (4.1.21) is suppressed and the operator behaves as a chiral primary. In particular, in this limit it is consistent to apply eq. (4.1.17) for the evaluation of its two–point function.

Supported by these considerations we can now proceed exactly as in [50] and find

$$\begin{aligned}
&< \bar{D}^2\mathcal{U}_J(z) D^2\bar{\mathcal{U}}_J(z') >= \\
&= \frac{N^{J+1}}{(4\pi^2)^{J+1}} f\ \bar{D}^2\left\{\frac{1}{2}D^\alpha\bar{D}^2D_\alpha + \frac{J-1}{4(J+1+\gamma)}[D^\alpha,\bar{D}^{\dot{\alpha}}]i\partial_{\alpha\dot{\alpha}}\right. \\
&\quad \left.+\frac{(J+1+\gamma)^2+(J-1)^2-2(J+1+\gamma)}{4(J+1+\gamma)(J+\gamma)}\Box\right\} D^2\frac{\delta^4(\theta-\theta')}{|x-x'|^{2(J+1+\gamma)}} \\
&= \frac{N^{J+1}}{(4\pi^2)^{J+1}} f\ (\gamma^2+2\gamma)\bar{D}^2D^2\frac{\delta^4(\theta-\theta')}{|x-x'|^{2(J+2+\gamma)}}
\end{aligned} \tag{4.1.22}$$

and

$$< \mathcal{O}_{J+1}(z)\bar{\mathcal{O}}_{J+1}(z') >= \frac{N^{J+2}}{(4\pi^2)^{J+2}} f \bar{D}^2 D^2 \frac{\delta^4(\theta-\theta')}{|x-x'|^{2(J+2+\gamma)}} \tag{4.1.23}$$

where f is the common normalization function not fixed by superconformal invariance.

From the relation (4.1.15) the two correlators are related by

$$\langle \bar{D}^2 \mathcal{U}_J(z) D^2 \bar{\mathcal{U}}_J(z') \rangle \;=\; |h|^2 \Big| q - \frac{1}{q} \Big|^2 \langle \mathcal{O}_{J+1}(z)\bar{\mathcal{O}}_{J+1}(z') \rangle \tag{4.1.24}$$

Therefore, inserting in (4.1.24) the explicit expressions (4.1.22) and (4.1.23) we end up with an algebraic equation

$$\gamma^2 + 2\gamma = |h|^2 \Big| q - \frac{1}{q} \Big|^2 \frac{N}{4\pi^2} \tag{4.1.25}$$

which allows to find the exact expression for the anomalous dimensions

$$\begin{aligned} \gamma &= -1 + \sqrt{1 + |h|^2 \Big| q - \frac{1}{q} \Big|^2 \frac{N}{4\pi^2}} \\ &= \frac{1}{2} |h|^2 \Big| q - \frac{1}{q} \Big|^2 \frac{N}{4\pi^2} \;-\; \frac{1}{8} |h|^4 \Big| q - \frac{1}{q} \Big|^4 \frac{N^2}{(4\pi^2)^2} \;+\; \cdots \end{aligned} \tag{4.1.26}$$

Up to the second order this expression coincides with the perturbative results obtained in the previous Section.

We note that our operators $\mathcal{O}_J$ can be thought as dual to the 0–modes of the BMN sector considered in [6, 71, 79, 78]. Formula (4.1.26) is in agreement with the results presented in those papers for the spectrum of the 0–modes.

4.2 The chiral ring of the β-deformed theory

After the simple example of the previous Section, we start a systematic analysis of protected operators focusing on the chiral scalar sector of the theory [9]. Once again we first consider the real β–deformed theory (2.3.9) but keeping N finite to provide a more general analysis. We remind [52] that, for a generic $\mathcal{N} = 1$ SYM theory, scalar operators in the chiral ring can be constructed as products of scalar chiral superfields Φ_i and/or times $(W^\alpha W_\alpha)$, where W_α is the chiral field strength. In what follows we will focus only on the Φ–sector, neglecting operators with a dependence on W_α.

In [67, 68, 6] the single–trace sector of the chiral ring has been identified as given by chiral operators of the form $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3})$ with weight $\Delta_0 = J_1 + J_2 + J_3$ and $(J_1, J_2, J_3) = (J, 0, 0), (0, J, 0), (0, 0, J), (J, J, J)$. In [26, 27] it has been shown perturbatively that also the assignments $(J_1, J_2, J_3) = (1, 1, 0), (1, 0, 1), (0, 1, 1)$ give protected operators.

This classification identifies the CPO's according to their dimension and their charges with respect to the two $U(1)$ global invariances of the theory. However, it does not give any information on the precise form of the protected operator corresponding to a given set

(J_1, J_2, J_3), which turns out to be in general a linear combination of single–trace operators with different order of the fields inside the trace. Moreover, if we work at finite N, mixing with multi–trace operators is also allowed.

A first example has been studied in [26] for the weight–3 sector. There, it has been shown that the correct expression for the protected operator corresponding to $(J_1, J_2, J_3) = (1, 1, 1)$ is a linear combination

$$\mathrm{Tr}(\Phi_1\Phi_2\Phi_3) + \alpha\mathrm{Tr}(\Phi_1\Phi_3\Phi_2) \tag{4.2.1}$$

where at one–loop

$$\alpha = \frac{(N^2-2)\bar{q}^2+2}{N^2-2+2\bar{q}^2} \tag{4.2.2}$$

showing an explicit dependence on the coupling β.

We are interested in the generalization of this result to higher loops in order to investigate whether and how the linear combination gets modified order by order. Moreover, we extend this analysis to other sectors of the chiral ring in order to discuss mixing at finite N.

In general, given a set of primary operators $\mathcal{O}_i$ with the same dimension Δ_0 and the same global charges, we can read their anomalous dimensions perturbatively from the matrix of the two–point correlation functions. Precisely, this matrix has the form

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \frac{1}{x^{2\Delta_0}}\left(A_{ij} - \rho_{ij}\log\mu^2x^2 \ + \ \cdots \ \right) \tag{4.2.3}$$

where dots stay for higher powers in $\log\mu^2x^2$. Here A is the mixing matrix, whereas ρ signals the appearance of anomalous dimensions. Both matrices are given as power series in the couplings.

In order to determine the anomalous dimensions we need to diagonalize the two matrices by performing the linear transformation $\mathcal{O}' = L\mathcal{O}$ which maps the operators into an orthogonal basis of quasi–primaries. In a perturbative approach it is easy to see [53, 54] that the diagonalization of the ρ matrix at order n fixes the correct orthogonalization (resolution of the mixing) at order $(n-1)$ uniquely, up to a residual rotation among operators with the same anomalous dimension. This means that in general an order n calculation is required to determine the anomalous dimensions at this order and the correct linear combinations of operators $\mathcal{O}_i$ at order $(n-1)$ which correspond to quasi–primaries with well–defined anomalous dimensions up to order n.

In our case, since we are interested into **chiral primary operators**, the procedure to determine perturbatively the correct linear combination which corresponds to a protected operator is made simpler if we also use the definition of chiral ring.

In our conventions the chiral ring is the set of chiral operators which cannot be written, by using the equations of motion, as $\bar{D}^2X$, being X any scalar operator.

In general, given a set of linearly independent chiral operators $\mathcal{C}_i$, $i = 1, \cdots, s$ characterized by the same classical scale dimension Δ_0 and the same charges under the two $U(1)$ flavor groups they will mix and we need solve the mixing in order to compute their anomalous dimensions. Since we are working with chiral operators, we know a priori that once we have orthogonalized as $\mathcal{C}'_i = L_{ij}\mathcal{C}_j$ in order to have well–defined quasi–primary operators, some of them will turn out to be descendant, i.e. they can be written as $\bar{D}^2 X$ for some proper operator X. The remaining operators will be necessarily primary chirals with vanishing anomalous dimensions.

Exploiting this simple observation, in order to find the correct expression for the protected operators, we then proceed as follows: In a given (J_1, J_2, J_3) sector, we first select all the descendants, that is all the linear combinations

$$\mathcal{D}_i = \sum_j d_j^{(i)} \mathcal{C}_j \tag{4.2.4}$$

which satisfy the condition

$$\mathcal{D}_i = \bar{D}^2 X_i \tag{4.2.5}$$

Let us suppose that there are $i = 1, \cdots, r \leq s$ independent linear combinations of this type. Then, for a generic operator $\mathcal{P} = \sum_j c_j \mathcal{C}_j$ we impose the orthogonality condition

$$\langle \mathcal{P} \bar{\mathcal{D}}_i \rangle = 0 \qquad i = 1, \cdots, r \tag{4.2.6}$$

where $\bar{\mathcal{D}}$ indicates the hermitian conjugate of $\mathcal{D}$. These constraints provide r equations for the s unknowns c_j. In this way we select a $(s - r)$-dimensional subspace of operators orthogonal to the descendant ones. We can choose an appropriate (orthogonal) basis in this subset, obtaining $(s - r)$ independent operators which are protected. This procedure has been already applied in the undeformed $\mathcal{N} = 4$ case [55].

The problem of determining the CPO's of the theory is then translated into the problem of finding *all* the linear combinations of operators which satisfy the condition (4.2.5). In particular, since we are interested into a perturbative determination of the chiral ring we need find descendants which solve eq. (4.2.5) order by order in perturbation theory. This can be done by introducing a perturbative definition of quantum chiral ring, as we are now going to explain in detail.

4.2.1 The perturbative quantum chiral ring

As previously discussed, the chiral ring is defined as the set of chiral operators orthogonal to null operators, i.e. linear combinations of chirals which can be written in the form $\bar{D}^2 X$, X primary. At the classical level a linear combination (4.2.4) gives rise to a null operator every time the coefficients $d_j^{(i)}$ are such that the operator $\mathcal{D}_i$ can be rewritten as a product of chiral superfields times $\frac{\delta W}{\delta \Phi_k}$, where W is the classical superpotential[4]

$$W = ih \left[q \,\mathrm{Tr}(\Phi_1 \Phi_2 \Phi_3) - \bar{q} \,\mathrm{Tr}(\Phi_1 \Phi_3 \Phi_2) \right] \tag{4.2.7}$$

[4]This is true only for operators which are not affected by Konishi–like anomalies or as long as these anomalies do not enter the actual calculation.

Indeed, if this is the case, we can use the classical equations of motion $\bar{D}^2\bar{\Phi}_k = -\frac{\delta W}{\delta \Phi_k}$ to express the operator as in (4.2.5). It follows that we can alternatively define the chiral ring as

$$\mathcal{C} = \{\text{chiral op.'s } \mathcal{P} \mid \langle \mathcal{P}\bar{\mathcal{D}} \rangle = 0, \text{ for any } \mathcal{D} \sim (...\Phi..\Phi..\frac{\delta W}{\delta \Phi})\} \tag{4.2.8}$$

where in $\mathcal{D}$ we do not indicate trace structures and flavor charges explicitly. In the undeformed $\mathcal{N} = 4$ theory, an immediate consequence of the definition (4.2.8) is that all the CPO's correspond to completly symmetric representations of the $SU(3) \subset SU(4)$ R–symmetry group [4].

This definition for the chiral ring allows for a straightforward generalization at the quantum level. Since the quantum dynamics of the elementary superfields is driven by the effective superpotential rather than the classical W, it appears natural to define the quantum chiral ring as

$$\mathcal{C}_Q = \{\text{chiral op.'s } \mathcal{P} \mid \langle \mathcal{P}\bar{\mathcal{D}}_Q \rangle = 0, \text{ for any } \mathcal{D}_Q \sim (..\Phi...\Phi...\frac{\delta W_{eff}}{\delta \Phi})\} \tag{4.2.9}$$

where now $\mathcal{D}_Q$ is a *quantum* null operator. Using the quantum equations of motion $\bar{D}^2 \frac{\delta K}{\delta \Phi_i} = -\frac{\delta W_{eff}}{\delta \Phi_i}$ where K is the effective Kähler potential which takes into account possible perturbative D–term corrections, it is easy to see that $\mathcal{D}_Q$ is a null operator at the quantum level. In the undeformed $\mathcal{N} = 4$ case the symmetries of the theory constrain $\mathcal{D}_Q$ to be proportional to $\mathcal{D}$ and the quantum chiral ring coincides with the classical one (4.2.8).

When W_{eff} is determined perturbatively, eq. (4.2.9) gives a perturbative definition of chiral ring. Precisely, given W_{eff} at a fixed perturbative order[5]

$$W_{eff} = W + \lambda W_{eff}^{(1)} + \lambda^2 W_{eff}^{(2)} + \cdots + \lambda^L W_{eff}^{(L)} \tag{4.2.10}$$

we can construct independent descendants [6] at that order as

$$\mathcal{D} = \mathcal{D}_0 + \lambda\mathcal{D}_1 + \lambda^2\mathcal{D}_2 + \cdots + \lambda^L\mathcal{D}_L \qquad , \qquad \mathcal{D}_i = \Phi \ldots \frac{\delta W_{eff}^{(i)}}{\delta \Phi} \tag{4.2.11}$$

and determine the protected operators $\mathcal{P}$ by imposing the orthogonality condition $\langle \mathcal{P}\bar{\mathcal{D}} \rangle = 0$ order by order. Since $\mathcal{P}$ will be in general a linear combination of single/multitrace operators, these conditions allow to determine the coefficients of the linear combination order by order in the couplings. If we set

$$\mathcal{P} = \mathcal{P}_0 + \lambda\mathcal{P}_1 + \lambda^2\mathcal{P}_2 + \cdots + \lambda^L\mathcal{P}_L \tag{4.2.12}$$

[5] In principle, perturbative corrections to W_{eff} would depend on both g and h couplings. Here we mean to use the superconformal invariance condition to express $|h|^2$ as a function of g^2 and write the perturbative expansion in powers of the 't Hooft coupling $\lambda = \frac{g^2 N}{4\pi^2}$.

[6] As long as we are interested in orthogonalizing with respect to the whole space generated by the descendants, we do not need the precise form of pure descendants, but just a suitable set of linear independent states. From now on we will refer to this definition of quantum descendants.

the perturbative corrections $\mathcal{P}_j$ will be determined by

$$\begin{aligned}
O(\lambda^0) \quad &: \qquad \langle \mathcal{P}_0 \bar{\mathcal{D}}_0 \rangle_0 = 0 \\
O(\lambda^1) \quad &: \qquad \langle \mathcal{P}_0 \bar{\mathcal{D}}_1 \rangle_0 + \langle \mathcal{P}_0 \bar{\mathcal{D}}_0 \rangle_1 + \langle \mathcal{P}_1 \bar{\mathcal{D}}_0 \rangle_0 = 0 \\
\vdots \qquad & \qquad\qquad \vdots \\
O(\lambda^L) \quad &: \qquad \langle \mathcal{P}_0 \bar{\mathcal{D}}_L \rangle_0 + \langle \mathcal{P}_0 \bar{\mathcal{D}}_{L-1} \rangle_1 + \cdots + \langle \mathcal{P}_0 \bar{\mathcal{D}}_0 \rangle_L + \langle \mathcal{P}_1 \bar{\mathcal{D}}_{L-1} \rangle_0 + \cdots + \langle \mathcal{P}_L \bar{\mathcal{D}}_0 \rangle_0 = 0
\end{aligned} \tag{4.2.13}$$

where $\langle\ \rangle_j$ stands for the two–point function at order λ^j.

Conditions (4.2.13) together with the general statement that orthogonalization at order $(n-1)$ is sufficient for having well–defined quasi–primary operators at order n, brings us to formulate the following prescription: In order to determine perturbatively the correct form of chiral operators with vanishing anomalous dimension at order n it is sufficient to determine the effective superpotential at order $(n-1)$, select all the descendant operators at that order by (4.2.11) and impose the conditions (4.2.13) up to order $(n-1)$. In so doing, we gain a perturbative order at each step. Moreover, in order to have all the descendants at a given order it is sufficient to compute the effective superpotential once for all.

As follows from its definition, the structure of the chiral ring is directly related to the structure of the effective superpotential. Therefore, the perturbative corrections to the CPO's depend on the perturbative corrections to the effective superpotential. In particular, this explains universality properties of the protected operators we will discuss later, as for example the fact that in any case the orthogonalization at tree level is sufficient for the protection up to two loops.

4.2.2 The effective superpotential at two–loops

Since we are dealing with a superconformal (finite) theory any correction to the effective action must be finite. By definition, the effective superpotential corresponds to perturbative, finite F–terms evaluated at zero momenta. It is given by *local* contributions which are constrained by dimensions, $U(1) \times U(1)$ flavor symmetry charges, reality and symmetry (2.3.8) to have necessarily the form

$$W_{eff} = ih \left[b \ \mathrm{Tr}(\Phi_1 \Phi_2 \Phi_3) - \bar{b} \ \mathrm{Tr}(\Phi_1 \Phi_3 \Phi_2) \right] \ + \ \mathrm{h.c.} \tag{4.2.14}$$

The constant b is given as an expansion in the couplings, $b = q(1 + b_1 \lambda + b_2 \lambda^2 + \cdots)$, with coefficients b_j which are functions of q and N, whereas $\bar{b}$ is the hermitian conjugate. We note that in principle the symmetries of the theory would only constrain the form of the superpotential to $W_{eff} = \{ih\,[b(q)\ \mathrm{Tr}(\Phi_1\Phi_2\Phi_3) + b(-\bar{q})\ \mathrm{Tr}(\Phi_1\Phi_3\Phi_2)]\ +\ \mathrm{h.c.}\}$. However, it is easy to show that $b(-\bar{q}) = -\overline{b(q)}$ since the b_j coefficients are rational functions of q^2 with real coefficients (loop diagrams always give real contributions and they always contain an even number of extra chiral vertices compared to the tree–level vertex).

At a given order L we can have two kinds of corrections to W_{eff}: Corrections which do not mix the two terms in the superpotential and are then of the form

$$W^{(L)}_{eff} \sim \lambda^L \ W \tag{4.2.15}$$

where W is the classical superpotential. These contributions do not affect the structure of the descendant operators at order L since $\frac{\delta W_{eff}^{(L)}}{\delta\Phi} \sim \frac{\delta W}{\delta\Phi}$ and $\mathcal{D}_L \sim \mathcal{D}_0$. As a consequence at order L the correlation function $\langle \mathcal{P}_0 \bar{\mathcal{D}}_L \rangle_0$ in (4.2.13) vanishes and the protected operator is determined only by loop corrections to its two–point function with descendants of lower orders.

The second kind of corrections to W_{eff} mixes the two terms in W and gives rise to a linear combination $W_{eff}^{(L)}$ of the form (4.2.14) which is not proportional to the classical superpotential anymore. For these corrections the request for the protected operator to be orthogonal to a descendant proportional to $\frac{\delta W_{eff}^{(L)}}{\delta\Phi}$ modifies in general its structure by contributions of order λ^L proportional to $\langle \mathcal{P}_0 \bar{\mathcal{D}}_L \rangle_0$.

In this Section we evaluate explicitly the effective superpotential up to two loops. Our result is useful for determining the correct CPO's up to three loops.

The diagrams contributing to the effective superpotential up to this order are given in Fig. 4.4 where the grey bullets indicate the one–loop corrections to the chiral and gauge–chiral vertices, respectively.

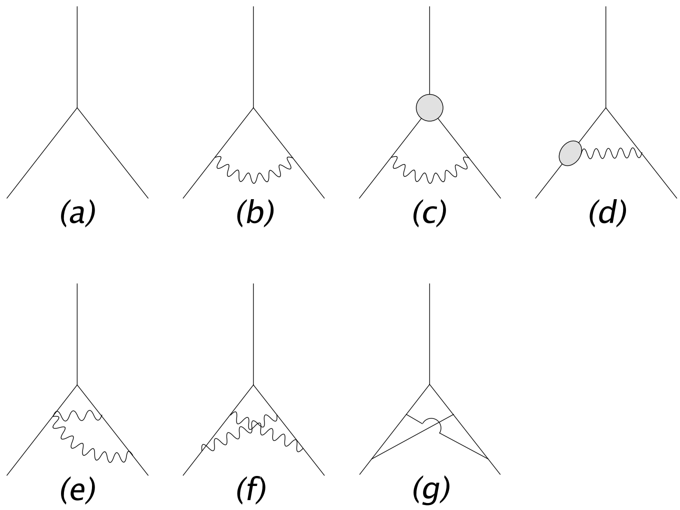

Figure 4.4: Diagrams contributing to the effective superpotential up to two loops.

These corrections are exactly the ones of the undeformed $\mathcal{N} = 4$ theory once we use the one–loop superconformal invariance condition:

$$|h|^2 \left[1 - \frac{1}{N^2} \left| q - \frac{1}{q} \right|^2 \right] = g^2 \tag{4.2.16}$$

The one–loop diagram 4.4b), compared with the tree level diagram 4.4a), does not contain any extra q–deformed vertex. Moreover, using standard color identities it is easy to see that its contribution is proportional to λW, where W is the classical superpotential.

The same happens at two loops for the diagrams 4.4c), 4.4d) and 4.4e) which do not contain any extra q–deformed vertex and have a color structure which does not mix the two traces, so reproducing W.

Diagram 4.4f) vanishes for color reasons.

Diagram 4.4g) contains four extra q–deformed vertices. Moreover, by direct inspection one can easily see that the nonplanar chiral structure which corrects the tree level diagram mixes nontrivially the two terms of W. As a result at two loops the superpotential undergoes a nontrivial modification of the form

$$W^{(2)}_{eff} \sim ih \left[q\, P \; \mathrm{Tr}(\Phi_1\Phi_2\Phi_3) - \bar{q}\, \bar{P} \; \mathrm{Tr}(\Phi_1\Phi_3\Phi_2) \right] \; + \; \mathrm{h.c.} \tag{4.2.17}$$

with

$$P = \frac{(q^2-1)^3[N^2+3+q^2(3N^2-10+7q^2)]}{q^2[q^4+1+(N^2-2)q^2]^2} \tag{4.2.18}$$

Here we have used $\bar{q} = 1/q$. We note that the nontrivial q–dependence of this diagram is a direct consequence of its nonplanarity. In fact, as discussed in Section 3.1, planar diagrams depend on the particular combination $q\bar{q} = 1$, while the nonplanar ones have generically nontrivial phases. Moreover, a q–dependence has also been introduced by using the superconformal condition (4.2.16) to express the coefficient $|h|^4$ from the four chiral vertices in terms of λ^2.

To evaluate the various contributions from Fig. 4.4 we first perform D–algebra to reduce superdiagrams to ordinary loop diagrams and compute the corresponding integrals in momentum space. As reported in Appendix C the one and two–loop integrals are all finite and they give a well–defined, local value for external momenta set to zero. Therefore, collecting all the contributions, at two loops the superpotential has the structure (4.2.14) with

$$b = q \left[(1 + \lambda c_1 + \lambda^2 c_2) + \lambda^2 \frac{3}{8} \zeta(3) P \right] \tag{4.2.19}$$

where the coefficients c_1, c_2 are numbers, independent of q and N, determined by the loop integrals 4.4b) and 4.4c)–4.4e), respectively (we do not need their explicit values).

It follows that in general a descendant at this order will have the form

$$\mathcal{D}_Q = (1 + \lambda c_1 + \lambda^2 c_2)\mathcal{D}_0 + \lambda^2 \mathcal{D}_2 \tag{4.2.20}$$

with $\mathcal{D}_2 \neq \mathcal{D}_0$.

4.2.3 Chiral Primary Operators in the spin–2 sector

The $(J, 1, 0)$ flavor:

We start considering operators of the form $\mathrm{Tr}(\Phi_1^J \Phi_2)$. In this case, due to the ciclicity of the trace, there is no ambiguity in the ordering of the operators inside the trace. In the large N limit these operators do not belong to the chiral ring, they are descendants and their anomalous dimensions have been computed exactly in Section 4.1 for J large. However, for finite N they can mix with multitraces and give rise to linear combinations of single and multi–trace operators which are protected. We are going to construct them perturbatively up to three loops. For simplicity we consider first the particular cases of $J = 3, 4$ and postpone the discussion for generic J at the end of this Section.

The $(3, 1, 0)$ case:

The first nontrivial example where mixing conspires to give rise to protected operators is for $J = 3$. This sector contains the two operators

$$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^3\Phi_2) \qquad , \qquad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2) \tag{4.2.21}$$

Using the classical equations of motion (4.1.2), it is easy to see that

$$\bar{D}^2\mathrm{Tr}(\Phi_1^2 e^{-gV}\bar{\Phi}_3 e^{gV}) = \mathrm{Tr}\left(\Phi_1^2\frac{\delta W}{\delta\Phi_3}\right) = -ih\,(q-\bar{q})\,[\mathrm{Tr}(\Phi_1^3\Phi_2) - \frac{1}{N}\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2)] \tag{4.2.22}$$

and a descendant can be constructed as (we always forget about the normalization of the operators)

$$\mathcal{D}_0 = \mathcal{O}_1 - \frac{1}{N}\mathcal{O}_2 \tag{4.2.23}$$

The knowledge of $\mathcal{D}_0$ allows us to determine the one–loop protected operator. We consider the linear combination

$$\mathcal{P}_0 = \mathcal{O}_1 + \alpha_0\,\mathcal{O}_2 \tag{4.2.24}$$

which, for any $\alpha_0 \neq -\frac{1}{N}$, gives an operator in the chiral ring. We then impose the orthogonality condition $\langle\mathcal{P}_0\,\bar{\mathcal{D}}_0\rangle_0 = 0$ and find

$$\alpha_0 = -\frac{N^2-6}{2N} \tag{4.2.25}$$

This result coincides with the one found in [28] where the one–loop CPO has been determined by diagonalizing directly the one–loop anomalous dimension matrix.

In order to extend our analysis to higher loops we need establish the correct form of the descendant operator order by order, as described in Section 4.2.1. If we look at its perturbative definition (4.2.11) and the way the equations of motion work in this case, we easily realize that as long as the effective superpotential has the structure (4.2.14) we obtain

$$\mathrm{Tr}\left(\Phi_1^2\frac{\delta W_{eff}}{\delta\Phi_3}\right) = -ih\left(b-\bar{b}\right)[\mathrm{Tr}(\Phi_1^3\Phi_2) - \frac{1}{N}\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1\Phi_2)] \tag{4.2.26}$$

whatever b might be (determined perturbatively at a given order). It follows that the linear combination on the r.h.s. of this equation, which is nothing but the operator (4.2.23), is always a descendant operator independently of the order we have computed the coefficient

b. Therefore we conclude that (4.2.23) is the *exact* quantum descendant up to an overall coupling–dependent normalization factor, that is $\mathcal{D}_Q \sim \mathcal{D}_0$.

An alternative way [55] to establish the relation $\mathcal{D}_Q \sim \mathcal{D}_0$ is to consider the combination

$$\bar{D}^2 \mathrm{Tr}(\Phi_1^2 e^{-gV} \bar{\Phi}_3 e^{gV}) + ih\,(q - \bar{q})\,[\mathrm{Tr}(\Phi_1^3 \Phi_2) - \frac{1}{N} \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1 \Phi_2)] \tag{4.2.27}$$

which is zero at tree level and check that it is order by order orthogonal to the three monomials $\bar{D}^2 \mathrm{Tr}(\Phi_1^2 e^{-gV} \bar{\Phi}_3 e^{gV})$, $\mathrm{Tr}(\Phi_1^3 \Phi_2)$ and $\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1 \Phi_2)$, separately. In fact, if this is the case, there is no extra mixing of the linear combination (4.2.27) with the three operators at the quantum level and (4.2.27) must be necessarily zero at any order in perturbation theory. We have checked the absence of mixing perturbatively up to two loops confirming our conclusion.

In order to determine the protected operator we consider the linear combination

$$\mathcal{P} = \mathcal{O}_1 + \alpha\, \mathcal{O}_2 \tag{4.2.28}$$

with α given as an expansion in λ

$$\alpha = \alpha_0 + \alpha_1\, \lambda + \alpha_2\, \lambda^2 \, + O(\lambda^3) \tag{4.2.29}$$

In the notation previously introduced, we have $\mathcal{P}_0 = \mathcal{O}_1 + \alpha_0 \mathcal{O}_2$ with α_0 already determined in (4.2.25) and $\mathcal{P}_j = \alpha_j \mathcal{O}_2$.

As a consequence of the relation $\mathcal{D}_Q \sim \mathcal{D}_0$ the orthogonality conditions (4.2.13) become

$$O(\lambda) \;: \qquad \langle \mathcal{P}_0 \bar{\mathcal{D}}_0 \rangle_1 + \langle \mathcal{P}_1 \bar{\mathcal{D}}_0 \rangle_0 = 0 \tag{4.2.30}$$

$$O(\lambda^2) : \qquad \langle \mathcal{P}_0 \bar{\mathcal{D}}_0 \rangle_2 + \langle \mathcal{P}_1 \bar{\mathcal{D}}_0 \rangle_1 + \langle \mathcal{P}_2 \bar{\mathcal{D}}_0 \rangle_0 = 0 \tag{4.2.31}$$

The first condition (4.2.30) gives

$$\alpha_1 = - \frac{\langle (\mathcal{O}_1 + \alpha_0 \mathcal{O}_2) \bar{\mathcal{D}}_0 \rangle_1}{\langle \mathcal{O}_2 \bar{\mathcal{D}}_0 \rangle_0} \tag{4.2.32}$$

In order to select the diagrams which contribute to the two–point function at the numerator we note that the tree level correlation function at the denominator, when computed in momentum space and in dimensional regularization ($n = 4 - 2\epsilon$), is $1/\epsilon$ divergent. This divergence signals the well–known short distance singularity of any two–point function of a conformal field theory.

If the denominator of (4.2.32) goes as $1/\epsilon$, in the numerator we can consider only divergent diagrams (finite diagrams would not contribute in the $\epsilon \to 0$ limit). It is easy to show that at this order the only diagram which we need take into account is the one in Fig. 4.5 where on the left hand side we have an insertion of the operator $(\mathcal{O}_1 + \alpha_0 \mathcal{O}_2)$ while on the right hand side we have $\bar{\mathcal{D}}_0$.

By a direct calculation one realizes that if α_0 is chosen as in (4.2.25) this diagram vanishes. The reason is very simple to understand: If we cut the diagram vertically at

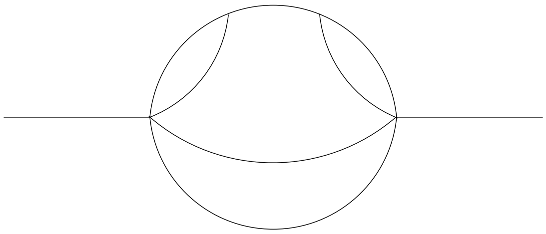

Figure 4.5: One–loop diagram contributing to the evaluation of α_1.

the very right end, close to the $\bar{\mathcal{D}}_0$ vertex, from the calculation it comes out that the left part would be nothing but a one–loop divergent contribution to the operator $(\mathcal{O}_1 + \alpha_0 \mathcal{O}_2)$ which vanishes since α_0 has been determined just to give a protected (not renormalized) operator at one–loop.

From the one–loop constraint we then read $\alpha_1 = 0$ and the expression (4.2.24) with α_0 as in (4.2.25) corresponds to the protected chiral operator up to two loops.

Next we analyze the constraint (4.2.31). Setting $\mathcal{P}_1 = 0$ there, we obtain

$$\alpha_2 = -\frac{\langle (\mathcal{O}_1 + \alpha_0 \mathcal{O}_2)\, \bar{\mathcal{D}}_0 \rangle_2}{\langle \mathcal{O}_2\, \bar{\mathcal{D}}_0 \rangle_0} \tag{4.2.33}$$

and consequently the exact expression for the CPO up to three loops.

Again we select only divergent diagrams contributing to the numerator. They are given in Fig. 4.6. We have not drawn diagrams associated to the two–loop anomalous dimension of the operator $(\mathcal{O}_1 + \alpha_0 \mathcal{O}_2)$ which vanish when α_0 is chosen as in (4.2.25).

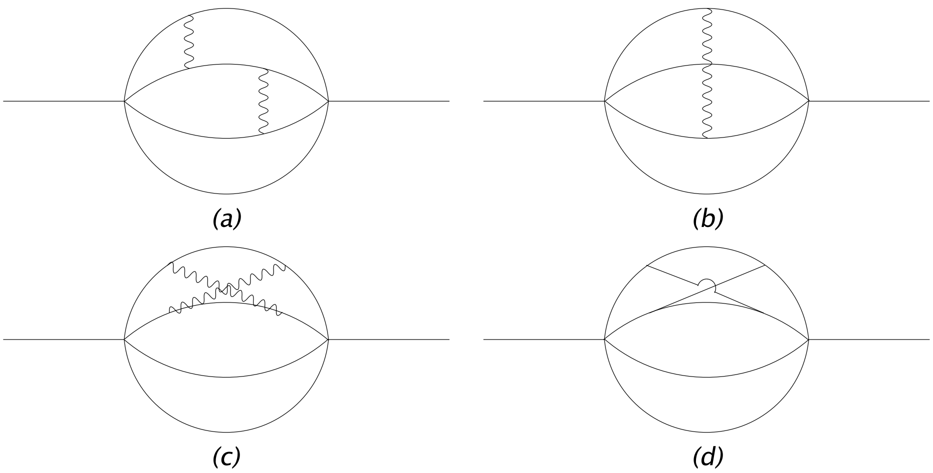

Figure 4.6: Two-loop diagrams contributing to the evaluation of α_2.

These diagrams contribute nontrivially to α_2 since, cutting the graphs at the very right hand side, their left parts cannot be recognized as corrections to the tree–level operator (nontrivial mixing between $\mathcal{O}_1$ and $\mathcal{O}_2$ occurs). Evaluating the diagrams by using the results in Appendix C we obtain

$$\alpha_2 = \frac{9(N^2-9)(q^2-1)^2[(N^4-8N^2-8)(q^4+1)+2(N^4+8)q^2]}{80N[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \qquad (4.2.34)$$

where we have used the one–loop superconformal condition (4.2.16) to express all the contributions of Fig. 4.6 in terms of λ^2 and set $\bar{q} = 1/q$.

Therefore the protected operator $\mathcal{P}$ up to three–loops can be written as

$$\mathcal{P} = \mathcal{O}_1 - \frac{N^2-6}{2N}(1 + r\,\lambda^2)\,\mathcal{O}_2 \qquad (4.2.35)$$

with

$$r = \frac{\alpha_2}{\alpha_0} = -\frac{9(N^2-9)(q^2-1)^2[(N^4-8N^2-8)(q^4+1)+2(N^4+8)q^2]}{40(N^2-6)[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \qquad (4.2.36)$$

We note that in the 't Hooft limit, $N \to \infty$ and λ fixed, $\mathcal{O}_2$ dominates and gives the protected operator up to three loops. This is consistent with the fact that, in the absence of mixing, the only primary operators in a given Δ_0 sector are necessarily products of single–trace primaries $\mathrm{Tr}(\Phi_1^m)$ and $\mathrm{Tr}(\Phi_1\Phi_2)$.

The $(4,1,0)$ case:

It is interesting to analyze this case in detail since it is the first case where more than one descendant appears.

This sector contains three independent operators

$$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^4\Phi_2) \quad , \quad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1^3)\mathrm{Tr}(\Phi_1\Phi_2) \quad , \quad \mathcal{O}_3 = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1^2\Phi_2) \qquad (4.2.37)$$

Using the classical equations of motion (4.1.2), we can write

$$\bar{D}^2\mathrm{Tr}(\Phi_1^3 e^{-gV}\bar{\Phi}_3 e^{gV}) = \mathrm{Tr}\left(\Phi_1^3\frac{\delta W}{\delta\Phi_3}\right) = -ih\,(q-\bar{q})\,[\mathrm{Tr}(\Phi_1^4\Phi_2) - \frac{1}{N}\mathrm{Tr}(\Phi_1^3)\mathrm{Tr}(\Phi_1\Phi_2)] \qquad (4.2.38)$$

$$\bar{D}^2\left[\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1 e^{-gV}\bar{\Phi}_3 e^{gV})\right] = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}\left(\Phi_1\frac{\delta W}{\delta\Phi_3}\right) = -ih\,(q-\bar{q})\,\mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_1^2\Phi_2) \qquad (4.2.39)$$

Therefore, in this case we can consider the two descendants

$$\mathcal{D}_0^{(1)} = \mathcal{O}_1 - \frac{1}{N}\mathcal{O}_2 \qquad , \qquad \mathcal{D}_0^{(2)} = \mathcal{O}_3 \qquad (4.2.40)$$

or any linear combination which realizes an orthogonal basis in the subspace of weight–5 descendants.

As in the previous example it is easy to prove that, given the particular structure (4.2.14) of the effective superpotential and the way the equations of motion enter the calculation, the linear combinations $\mathcal{D}_0^{(1)}$ and $\mathcal{D}_0^{(2)}$ provide two independent descendants even at the quantum level.

Proceeding as before we consider the linear combination

$$\mathcal{P} = \mathcal{O}_1 + \alpha\,\mathcal{O}_2 + \beta\,\mathcal{O}_3 \tag{4.2.41}$$

and choose the constants α and β (expanded in powers of λ) by requiring $\mathcal{P}$ to be orthogonal to the two descendants up to two loops.

Solving the constraints $\langle \mathcal{P}_0 \bar{\mathcal{D}}_0^{(i)} \rangle_0$ at tree level we determine the correct expression for the operator characterized by a vanishing one–loop anomalous dimension

$$\mathcal{P}_0 = \mathcal{O}_1 - \frac{N^2-12}{3N}\,\mathcal{O}_2 - \frac{2}{N}\,\mathcal{O}_3 \tag{4.2.42}$$

As in the previous case, this operator is automatically orthogonal to $\mathcal{D}_0^{(1)}$ and $\mathcal{D}_0^{(2)}$ also at one loop and so we expect it to be protected up to two loops.

The orthogonality at two loops can be imposed exactly as in the previous case and allows to determine the corrections α_2 and β_2. The diagrams contributing are still the ones in Fig. 4.6 with one extra free chiral line running between the two vertices. Performing the calculation we find the final expression for the operator protected up to three loops

$$\mathcal{P} = \mathcal{O}_1 - \frac{N^2-12}{3N}\,(1+s_1\,\lambda^2)\,\mathcal{O}_2 - \frac{2}{N}\,(1+s_2\,\lambda^2)\,\mathcal{O}_3 \tag{4.2.43}$$

where

$$\begin{aligned} s_1 = \frac{\alpha_2}{\alpha_0} &= \frac{(N^2-16)(q^2-1)^2\,[(11N^2+21)(q^4+1)+2(N^2-21)q^2]}{4(N^2-12)[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \\ s_2 = \frac{\beta_2}{\beta_0} &= -\frac{(N^2-16)(q^2-1)^2[(N^2+5)(q^4+1)+2(N^2-5)q^2]}{8[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \end{aligned} \tag{4.2.44}$$

Again, the coefficients depend on N in such a way that in the large N limit only the $\mathcal{O}_2$ operator in (4.2.37) survives in agreement with the chiral ring content of the theory in the planar limit.

We note that these coefficients, as well as r in (4.2.36) are real. This is a consequence of the fact that in the sectors studied so far the descendant operators are q–independent and the two–point correlation functions are real.

The previous analysis can be applied to the generic operators of the form $(\Phi_1^J \Phi_2)$. The peculiar pattern $\mathcal{D}_Q \sim \mathcal{D}_0$ for the descendants occurs in any $(J,1,0)$ sector since it only depends on the particular structure of the superpotential and the particular way the equations of motion work for this class of operators. Therefore, the determination of CPO's proceeds as before. In particular, we expect the tree level orthogonality condition to be still sufficient for protection up to two loops since the only one–loop diagram relevant for the calculation would be the vanishing one–loop anomalous dimension diagram in Fig. 4.5. At two loops diagrams of the kind drawn in Fig. 4.6 should be still the only relevant ones.

Without entering the details of the calculations which would be quite involved and not very illuminating, we can determine the dimension of the corresponding chiral ring subspace, i.e. the number of independent protected operators corresponding to $U(1)$ flavors $(J, 1, 0)$.

To be definite we consider J even ($J = 2p$). In this case the list of chirals we can construct is

$$
\begin{array}{lll}
\text{single}-\text{trace} & \mathrm{Tr}(\Phi_1^{2p}\Phi_2) & \\
\text{double}-\text{trace} & \mathrm{Tr}(\Phi_1^{m_1})\ \mathrm{Tr}(\Phi_1^{2p-m_1}\Phi_2) & m_1 = 2, \cdots, 2p-1 \\
\text{triple}-\text{trace} & \mathrm{Tr}(\Phi_1^{m_1})\ \mathrm{Tr}(\Phi_1^{m_2})\ \mathrm{Tr}(\Phi_1^{2p-m_1-m_2}\Phi_2) & \\
 & \qquad m_1 = 2, \cdots, p-1, \quad m_2 = m_1, \cdots, 2p-1-m_1 & \\
\vdots & & \\
p-\text{trace} & \mathrm{Tr}(\Phi_1^2)\cdots \mathrm{Tr}(\Phi_1^2)\ \mathrm{Tr}(\Phi_1^2\Phi_2)\ ,\ \mathrm{Tr}(\Phi_1^3)\ \mathrm{Tr}(\Phi_1^2)\cdots \mathrm{Tr}(\Phi_1^2)\ \mathrm{Tr}(\Phi_1\Phi_2) &
\end{array}
\tag{4.2.45}
$$

In order to find how many independent primaries we can construct out of (4.2.45) we need first count how many descendants of the form (4.2.5) we have. As explained in the previous simple examples, given the generic n–trace, $\Delta_0 = J$ sector, null conditions come from considering the operators

$$
\mathrm{Tr}(\Phi_1^{m_1})\cdots \mathrm{Tr}(\Phi_1^{m_{n-1}})\bar{D}^2 \mathrm{Tr}(\Phi_1^{2p-1-m_1-\ldots-m_{n-1}} e^{-gV}\bar{\Phi}_3 e^{gV}) \tag{4.2.46}
$$

as long as $2p - 1 - m_1 - \ldots - m_{n-1} \geq 1$. In fact, once we act with $\bar{D}^2$ on $\bar{\Phi}_3$ and use the equations of motion (4.1.2) we generate the linear combination

$$
\begin{aligned}
&\mathrm{Tr}(\Phi_1^{m_1})\cdots \mathrm{Tr}(\Phi_1^{m_{n-1}})\mathrm{Tr}(\Phi_1^{2p-m_1-\ldots-m_{n-1}}\Phi_2) \\
&\qquad - \frac{1}{N}\mathrm{Tr}(\Phi_1^{m_1})\cdots \mathrm{Tr}(\Phi_1^{m_{n-1}})\mathrm{Tr}(\Phi_1^{2p-1-m_1-\ldots-m_{n-1}})\mathrm{Tr}(\Phi_1\Phi_2)
\end{aligned}
\tag{4.2.47}
$$

which is then a descendant. Therefore, the complete list of descendants is

$$
\begin{array}{lll}
\text{single}-\text{trace} & \bar{D}^2\ \mathrm{Tr}(\Phi_1^{2p-1} e^{-gV}\bar{\Phi}_3 e^{gV}) & \\
\text{double}-\text{trace} & \bar{D}^2\left[\mathrm{Tr}(\Phi_1^{m_1})\ \mathrm{Tr}(\Phi_1^{2p-1-m_1} e^{-gV}\bar{\Phi}_3 e^{gV})\right] & m_1 = 2, \cdots, 2p-2 \\
\text{triple}-\text{trace} & \bar{D}^2\left[\mathrm{Tr}(\Phi_1^{m_1})\ \mathrm{Tr}(\Phi_1^{m_2})\ \mathrm{Tr}(\Phi_1^{2p-1-m_1-m_2} e^{-gV}\bar{\Phi}_3 e^{gV})\right] & \\
 & \qquad m_1 = 2, \cdots, p-1, \quad m_2 = m_1, \cdots, 2p-2-m_1 & \\
\vdots & & \\
p-\text{trace} & \bar{D}^2\left[\mathrm{Tr}(\Phi_1^2)\cdots\ \mathrm{Tr}(\Phi_1^2)\ \mathrm{Tr}(\Phi_1 e^{-gV}\bar{\Phi}_3 e^{gV})\right] &
\end{array}
\tag{4.2.48}
$$

Counting how many operators we have in (4.2.45) and subtracting the number of descendants in (4.2.48) we find that the number of protected chiral operators is $\sum_{n=2}^{p} X_n$ where

X_n is the number of partitions of $(2p-1)$ objects into $(n-1)$ boxes with at least 2 objects per box. Analogously, the number of chiral primary operators for J odd is $\sum_{n=2}^{p+1} X_n$.

This result is consistent with the number of primary operators which survive in the large N limit where mixing effects disappear and the chiral ring reduces to products of single–trace operators $\mathrm{Tr}(\Phi_1^k)$, $\mathrm{Tr}(\Phi_1\Phi_2)$.

The $(2,2,0)$ flavor:

In the class of more general operators with weights $(J_1, J_2, 0)$ we consider the particular case $J_1 = J_2 = 2$. This sector contains four operators, two single– and two double–traces

$$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^2\Phi_2^2) \quad , \quad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1\Phi_2\Phi_1\Phi_2)$$
$$\mathcal{O}_3 = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_2^2) \quad , \quad \mathcal{O}_4 = \mathrm{Tr}(\Phi_1\Phi_2)\mathrm{Tr}(\Phi_1\Phi_2) \tag{4.2.49}$$

Using the classical equations of motion (4.1.2), we can write

$$\bar{D}^2\left[\mathrm{Tr}(\Phi_1\Phi_2 e^{-gV}\bar{\Phi}_3 e^{gV}) - \mathrm{Tr}(\Phi_2\Phi_1 e^{-gV}\bar{\Phi}_3 e^{gV})\right] = -ih(q+\bar{q})[\mathcal{O}_2 - \mathcal{O}_1]$$
$$\bar{D}^2\left[\mathrm{Tr}(\Phi_1\Phi_2 e^{-gV}\bar{\Phi}_3 e^{gV}) + \mathrm{Tr}(\Phi_2\Phi_1 e^{-gV}\bar{\Phi}_3 e^{gV})\right] = -ih(q-\bar{q})[\mathcal{O}_1 + \mathcal{O}_2 - \frac{2}{N}\mathcal{O}_4] \tag{4.2.50}$$

We note that on the right hand side of these equations the q–dependence is still factored out as it happened in the previous cases (see eqs. (4.2.22, 4.2.39)). Therefore, tree level descendants can be defined as linear combinations

$$\mathcal{D}_0^{(1)} = \mathcal{O}_2 - \mathcal{O}_1$$
$$\mathcal{D}_0^{(2)} = \mathcal{O}_1 + \mathcal{O}_2 - \frac{2}{N}\mathcal{O}_4 \tag{4.2.51}$$

Because of their q–independence these operators correspond indeed to a suitable choice of quantum descendants.

The general structure of a chiral primary operator in this sector is

$$\mathcal{P} = \alpha\mathcal{O}_1 + \beta\,\mathcal{O}_2 + \gamma\,\mathcal{O}_3 + \delta\,\mathcal{O}_4 \tag{4.2.52}$$

where the coefficients are determined order by order by the orthogonality conditions $\langle\mathcal{P}\bar{\mathcal{D}}_0^{(1)}\rangle$ and $\langle\mathcal{P}\bar{\mathcal{D}}_0^{(2)}\rangle$. Having two conditions for four unknowns we expect to single out two protected operators.

At tree level, for the particular choice $\alpha_0 = 2, \beta_0 = 1$ and $\alpha_0 = 1, \beta_0 = -1$, we find

$$\mathcal{P}^{(1)} = 2\mathcal{O}_1 + \mathcal{O}_2 - \frac{N^2-6}{2N}(\mathcal{O}_3 + 2\mathcal{O}_4)$$
$$\mathcal{P}^{(2)} = \mathcal{O}_1 - \mathcal{O}_2 - \frac{N}{4}\mathcal{O}_3 + N\mathcal{O}_4 \tag{4.2.53}$$

These are one–loop protected operators and coincide with the ones found in [28]. They are not orthogonal but a basis can be easily constructed by considering linear combinations.

According to the general pattern already discussed for the previous cases we expect the operators (4.2.53) to be protected up to two loops. The condition for these operators to be protected up to three loops requires instead nontrivial λ^2–corrections to (4.2.53) which can be determined by solving the orthogonality constraints at this order. The diagrams contributing nontrivially to the 2–point functions are still the ones in Fig. 4.6. Since the final expressions are quite unreadable, we find convenient to fix $\alpha_2 = \beta_2 = 0$ for both the CPO's and we obtain

$$\begin{aligned} \mathcal{P}^{(1)} &= 2\mathcal{O}_1 + \mathcal{O}_2 - \frac{N^2-6}{2N}(1+t_1\,\lambda^2)\mathcal{O}_3 - \frac{N^2-6}{N}(1+t_2\,\lambda^2)\mathcal{O}_4 \\ \mathcal{P}^{(2)} &= \mathcal{O}_1 - \mathcal{O}_2 - \frac{N}{4}(1+u_1\,\lambda^2)\mathcal{O}_3 + N(1+u_2\,\lambda^2)\mathcal{O}_4 \end{aligned} \tag{4.2.54}$$

where

$$\begin{aligned} t_1 &= -\frac{9(N^2-9)(q^2-1)^2[(N^4-6N^2-4)(q^4+1)+2(N^4-2N^2+4)q^2]}{20(N^2-6)[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \\ t_2 &= \frac{9(N^2-9)(N^2+2)(q^2-1)^4}{10(N^2-6)[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \end{aligned} \tag{4.2.55}$$

and

$$\begin{aligned} u_1 &= -\frac{9(q^2-1)^2[(N^6-9N^4-16N^2+18)(q^4+1)+2(N^6-14N^4+34N^2-18)q^2]}{20N^2[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \\ u_2 &= \frac{9(q^2-1)^2[(N^4-31N^2-18)(q^4+1)-2(7N^4-13N^2-18)q^2]}{40N^2[q^4+1+(N^2-2)q^2]^2}\,\zeta(3) \end{aligned} \tag{4.2.56}$$

4.2.4 Chiral Primary Operators in the spin–3 sector

This sector contains operators of the form $(\Phi_1^k\Phi_2^l\Phi_3^m)$ with all possible trace structures.

The simplest case is for $k = l = m = 1$ and involves the two weight–3 operators

$$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1\Phi_2\Phi_3) \qquad , \qquad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1\Phi_3\Phi_2) \tag{4.2.57}$$

As already mentioned, the correct one–loop expression for the protected operator has been determined in [26] by computing directly the anomalous dimension at that order. It turns out that the protected operator is a linear combination of the two operators (4.2.57) with coefficient α as in (4.2.2). The result has been confirmed in [28] by using a simplified approach based on the evaluation of the difference between the one–loop two–point function of the deformed theory and the one for the $\mathcal{N} = 4$ case. This approach is very convenient since it avoids computing many graphs containing gauge vertices but, as recognized by the authors, in this case it cannot be pushed beyond one loop.

Using our procedure, we can easily re-derive the Freedman–Gursoy result by working at tree level and extend it to two–loops by performing a one–loop calculation. The correct application of our procedure beyond this order would require a substantial modification in the definition of quantum chiral ring (4.2.9) since in this sector descendants of Konishi–like operators are present and the equations of motion need be supplemented by the Konishi anomaly term. As a consequence the corresponding chiral ring sector necessarily contains operators depending on $W^\alpha W_\alpha$.

In fact, from the anomalous conservation equation for the Konishi current we can write

$$\bar{D}^2 \mathrm{Tr}(e^{-gV}\bar{\Phi}_i e^{gV}\Phi_i) = -3ih[q\,\mathrm{Tr}(\Phi_1\Phi_2\Phi_3) - \bar{q}\mathrm{Tr}(\Phi_1\Phi_3\Phi_2)] + \frac{1}{32\pi^2}\mathrm{Tr}(W^\alpha W_\alpha) \quad (4.2.58)$$

We remind that in our conventions $W_\alpha = i\bar{D}^2(e^{-gV}D_\alpha e^{gV})$ and it is at least of order g. From the previous identity it follows that a descendant operator has to be constructed out of the two operators (4.2.57) plus the anomaly term

$$\mathcal{D}_0 = q\mathcal{O}_1 - \bar{q}\mathcal{O}_2 + \frac{i}{96\pi^2\,h}\mathrm{Tr}(W^\alpha W_\alpha) \quad (4.2.59)$$

However, since the operator $\mathrm{Tr}(W^\alpha W_\alpha)$ is of order g^2 and has vanishing tree level two–point function with $\mathcal{O}_1$ and $\mathcal{O}_2$ it does not enter the orthogonality conditions at tree level and one–loop. Therefore we can safely use our procedure to find CPO's up to two loops forgetting about the anomaly.

Thus we consider the linear combination

$$\mathcal{P}_0 = \mathcal{O}_1 + \alpha_0\,\mathcal{O}_2 \quad (4.2.60)$$

for any value of $\alpha_0 \neq -\bar{q}^2$. In order to determine the exact expression for the CPO at one–loop we need impose the operator to be orthogonal to the descendant (4.2.59) at tree level. A simple calculation proves that $\langle\mathcal{P}_0\bar{\mathcal{D}}_0\rangle_0 = 0$ iff α_0 is given in (4.2.2), in agreement with the result of [26].

At one loop first we need determine the correct expression for the descendant at this order. As it follows from the calculations of Section 4.2.2 at one loop the effective superpotential is proportional to the tree level W and the corresponding descendant operator is still proportional to $\mathcal{D}_0$ in eq. (4.2.59). Given the generic linear combination $\mathcal{P} = \mathcal{O}_1 + (\alpha_0 + \alpha_1\lambda)\,\mathcal{O}_2$ we then impose the orthogonality condition up to order λ to uniquely determine α_1 as in (4.2.32). As in the previous examples, if α_0 is given in (4.2.2) the α_1 coefficient is identically zero being this a consequence of the one–loop protection of $\mathcal{P}_0$. Therefore the expression (4.2.60) with α_0 given in (4.2.2) corresponds to the protected chiral operator up to two loops.

The next case we investigate is for $k = 2$, $l = m = 1$. There are five operators

$$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^2\Phi_2\Phi_3) \quad , \quad \mathcal{O}_2 = \mathrm{Tr}(\Phi_1^2\Phi_3\Phi_2) \quad , \quad \mathcal{O}_3 = \mathrm{Tr}(\Phi_1\Phi_2\Phi_1\Phi_3)$$
$$\mathcal{O}_4 = \mathrm{Tr}(\Phi_1^2)\mathrm{Tr}(\Phi_2\Phi_3) \quad , \quad \mathcal{O}_5 = \mathrm{Tr}(\Phi_1\Phi_2)\mathrm{Tr}(\Phi_1\Phi_3) \quad (4.2.61)$$

Using the classical equations of motion (4.1.2) we can write three descendants

$$\begin{aligned}\mathcal{D}_0^{(1)} &= q\mathcal{O}_3 - \bar{q}\mathcal{O}_2 - \frac{1}{N}(q-\bar{q})\mathcal{O}_5 \\ \mathcal{D}_0^{(2)} &= q\mathcal{O}_1 - \bar{q}\mathcal{O}_3 - \frac{1}{N}(q-\bar{q})\mathcal{O}_5 \\ \mathcal{D}_0^{(3)} &= q\mathcal{O}_1 - \bar{q}\mathcal{O}_2 - \frac{1}{N}(q-\bar{q})\mathcal{O}_4\end{aligned} \tag{4.2.62}$$

We expect to find out two protected operators of the form

$$\mathcal{P} = \alpha\,\mathcal{O}_1 + \beta\,\mathcal{O}_2 + \gamma\,\mathcal{O}_3 + \delta\,\mathcal{O}_4 + \epsilon\,\mathcal{O}_5 \tag{4.2.63}$$

By imposing the tree-level orthogonality condition with respect to the three $\mathcal{D}_0^{(i)}$ we can fix for instance γ, δ and ϵ in terms of α and β. The calculation proceeds exactly as in the previous case and we find

$$\begin{aligned}\gamma &= \frac{\alpha[q^4-2q^2+1-N^2]-\beta[(1-N^2)q^4-2q^2+1]}{N^2(q^4-1)} \\ \delta &= \frac{\alpha[(N^2+2)q^4+2(N^2-2)q^2+N^4-5N^2+2]}{2N^3(q^4-1)} \\ &\quad -\frac{\beta[(N^4-5N^2+2)q^4+2(N^2-2)q^2+N^2+2]}{2N^3(q^4-1)} \\ \epsilon &= \frac{\alpha[2(N^2+1)q^4+(N^4-4)q^2+N^4-4N^2+2]}{N^3(q^4-1)} \\ &\quad -\frac{\beta[(N^4-4N^2+2)q^4+(N^4-4)q^2+2(N^2+1)]}{N^3(q^4-1)}\end{aligned} \tag{4.2.64}$$

We expect these operators to have a vanishing anomalous dimension at one loop. If we set $\alpha=\beta=1$ and $\alpha=-\beta=1$, we recover the two protected operators found in [28].

As in the previous cases, the operators $\mathcal{D}_0^{(1)}$, $\mathcal{D}_0^{(2)}$ and $\mathcal{D}_0^{(3)}$ keep being good descendants at one loop. Moreover, the one–loop orthogonality conditions do not modify the CPO's (4.2.63, 4.2.64) and we expect these operators to have a vanishing two–loop anomalous dimension.

If we were to push our calculation beyond this order we should first determine the descendant operators at two loops. It is easy to realize that in this case the relation $\mathcal{D}_Q \sim \mathcal{D}_0$ does not hold anymore, for two simple reasons:

1) At higher orders the Konishi anomaly cannot be ignored anymore. In particular, the correct expression for the descendant operators from two loops on will have a nontrivial dependence on $(W^\alpha W_\alpha)$.

2) Differently from the spin–2 case, the nontrivial corrections to the effective superpotential which appear at two loops determine nontrivial corrections to the descendants since in this case they depend on q not only through an overall coefficient (see eq. (4.2.62)).

4.3 The full Leigh–Strassler deformation

From a field theory point of view it is interesting to investigate the quantum properties of the full Leigh-Strassler $\mathcal{N} = 1$ deformation of the $\mathcal{N} = 4$ SYM theory given by the action we found in Section 2.3:

$$\begin{aligned} S \;=\; & \int d^8z \; \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \; \mathrm{Tr}(W^\alpha W_\alpha) + \\ +ih & \int d^6z \; \mathrm{Tr}\left(q\,\Phi_1\Phi_2\Phi_3 - \frac{1}{q}\,\Phi_1\Phi_3\Phi_2\right) + \frac{i\,h'}{3}\int d^6z \; \mathrm{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3) + \\ +i\bar{h} & \int d^6\bar{z} \; \mathrm{Tr}\left(\frac{1}{\bar{q}}\,\overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3 - \bar{q}\,\overline{\Phi}_1\overline{\Phi}_3\overline{\Phi}_2\right) - \frac{i\,\bar{h}'}{3}\int d^6\bar{z} \; \mathrm{Tr}(\overline{\Phi}_1^3 + \overline{\Phi}_2^3 + \overline{\Phi}_3^3) \end{aligned} \tag{4.3.1}$$

The equations of motion derived from (4.3.1) are

$$\begin{aligned} \bar{D}^2(e^{-gV}\bar{\Phi}_1^a e^{gV}) &= -ih\Phi_2^b\Phi_3^c\,[q(abc) - \bar{q}(acb)] - ih'\Phi_1^b\Phi_1^c(abc) \\ \bar{D}^2(e^{-gV}\bar{\Phi}_2^b e^{gV}) &= -ih\Phi_1^a\Phi_3^c\,[q(abc) - \bar{q}(acb)] - ih'\Phi_2^a\Phi_2^c(abc) \\ \bar{D}^2(e^{-gV}\bar{\Phi}_3^c e^{gV}) &= -ih\Phi_1^a\Phi_2^b\,[q(abc) - \bar{q}(acb)] - ih'\Phi_3^a\Phi_3^b(abc) \end{aligned} \tag{4.3.2}$$

where once again keeping β real we can interchange $\bar{q} \leftrightarrow \frac{1}{q}$. As widely discussed, the requirement of vanishing anomalous dimensions of the elementary chiral superfields guarantees the theory to be superconformal invariant. Since the three chirals have the same anomalous dimension due to the cyclic Z_3 symmetry, superconformal invariance requires a single condition $\gamma(g, h, h', \beta) = 0$ and we find a three–dimensional complex manifold of fixed points. We evaluate the anomalous dimension of the chiral superfield Φ_i up to two loops. The calculation can be carried on exactly as in the case of $h' = 0$ by taking into account that compared to the previous case the present action contains three extra chiral vertices of the form $\frac{ih'}{6} d_{abc}\Phi_i^a\Phi_i^b\Phi_i^c$, $i = 1, 2, 3$.

As long as we deal with diagrams which do not contain the new h' vertices we have exactly the same contributions as in the $h' = 0$ theory. We only need evaluate all the diagrams which contain these extra vertices.

At one loop, besides the h–chiral and the mixed gauge–chiral self–energy diagrams [27] we have a h'–chiral self–energy graph whose contribution is proportional to $|h'|^2$. This new diagram modifies the one–loop superconformal condition (4.2.16) as

$$\left[|h|^2\left(1 - \frac{1}{N^2}\,|q - \bar{q}|^2\right) + |h'|^2\frac{N^2 - 4}{2N^2}\right] = g^2 \tag{4.3.3}$$

in agreement with [56, 64, 71]. As for the $h' = 0$ case it is easy to verify that the one–loop condition is sufficient to guarantee the vanishing of the beta functions (i.e. superconformal invariance) up to two loops.

Once the theory is made finite we are interested in the perturbative evaluation of *finite* corrections to the superpotential. In this case the symmetries of the theory force the effective superpotential to have the form

$$W_{eff} = ih \int d^6z \mathrm{Tr}[b(q)\, \Phi_1\Phi_2\Phi_3 + b(-\bar{q})\, \Phi_1\Phi_3\Phi_2] + \frac{ih'}{3} d \int d^6z \mathrm{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3) \;+\; \text{h.c.} \tag{4.3.4}$$

where the coefficients b and d are determined as double power expansions in the couplings h and h' [7]. In particular, the invariance under cyclic permutations of the superfields requires the d correction to be the same for the three Φ_i^3 terms, whereas the other global symmetries force the particular q dependence of the corrections to $(\Phi_1\Phi_2\Phi_3)$ and $(\Phi_1\Phi_3\Phi_2)$. We note that in this case we cannot apply the previous arguments (see the discussion after eq. (4.2.14)) to state that $b(-\bar{q}) = -\overline{b(q)}$ since the perturbative corrections to $(\Phi_1\Phi_2\Phi_3)$ and $(\Phi_1\Phi_3\Phi_2)$ are not always proportional to q times functions of q^2. In fact, it is still true that diagrams contributing to the effective potential contain an even number of extra chiral vertices compared to the tree level diagrams, but part of these vertices could be h'–vertices not carrying any q–dependence.

The topologies of diagrams contributing to the superpotential up to two loops are still the ones in Fig. 4.4 where now chiral vertices may be either h or h' vertices. Performing the explicit calculation as in Section 4.2.2 we discover that at one loop the various terms in the superpotential do not mix and receive separate corrections still proportional to the classical terms. Precisely, we find that $W_{eff}^{(1)}$ coincides with W, up to an overall constant coefficient. This is also true at two loops for the diagrams 4.4c), 4.4d) and 4.4e), whereas the diagram 4.4g) with all possible configurations of h and h' vertices mixes nontrivially the various terms of the superpotential. Similarly to what happens for the β–deformed theory, this leads to a nontrivial correction $W_{eff}^{(2)}$ which has the form (4.3.4) but with the b and d coefficients nontrivially corrected by functions of q and N. We then expect descendant operators to get modified at this order as in the previous case (see discussion around eq. (4.2.20)).

The exact supergravity dual of the theory (4.3.1) is still unknown even if few steps towards it have been undertaken in [63]. However, it is interesting to investigate the nature of composite operators of the superconformal field theory waiting for the discovery of the exact correspondence of these operators to superstring states.

The chiral ring for the h'-deformed theory is not known in general (however, see [68, 25]). Compared to the chiral ring of the β–deformed theory ($h' = 0$) which contains operators of the form $\mathrm{Tr}(\Phi_i^J)$, $\mathrm{Tr}(\Phi_1^J\Phi_2^J\Phi_3^J)$ plus the particular operators $\mathrm{Tr}(\Phi_i\Phi_j)$, $i \neq j$, we expect the chiral ring of the present theory to be more complicated because of the lower number of global symmetries present.

Here we exploit the general procedure described in Section 4.2 to move the first steps towards the determination of chiral primary operators. In particular, we concentrate on the first simple cases of matter chiral operators with dimensions $\Delta_0 = 2, 3$ and study how turning on the h'-interaction may affect their quantum properties. We then take advantage of these results to make a preliminary discussion of the CPO content for generic

[7] Here we use the superconformal condition (4.3.3) to express g^2 as a function of h and h'. Any other choice would be equally acceptable.

scale dimensions.

4.3.1 Chiral ring: The $\Delta_0 = 2$ sector

Weight–2 chiral operators are $\text{Tr}(\Phi_i^2)$ and $\text{Tr}(\Phi_i\Phi_j)$, $i \neq j$. These operators can be classified as in Table 4.1 according to their charge $\mathcal{Q}$ with respect to the Z_3 symmetry (2.3.7).

$\mathcal{Q} = 0$	$\mathcal{Q} = 1$	$\mathcal{Q} = 2$
$\mathcal{O}_{11} = \text{Tr}(\Phi_1^2)$	$\mathcal{O}_{33} = \text{Tr}(\Phi_3^2)$	$\mathcal{O}_{22} = \text{Tr}(\Phi_2^2)$
$\mathcal{O}_{23} = \text{Tr}(\Phi_2\Phi_3)$	$\mathcal{O}_{12} = \text{Tr}(\Phi_1\Phi_2)$	$\mathcal{O}_{13} = \text{Tr}(\Phi_1\Phi_3)$

Table 4.1: Operators with $\Delta_0 = 2$.

The charged sectors can be obtained from the $\mathcal{Q} = 0$ one by successive applications of cyclic Z_3–permutations $\Phi_i \to \Phi_{i+1}$. This is the reason why the three sectors contain the same number of operators. In the $h' = 0$ theory their anomalous dimensions have been computed up to two loops and found to be vanishing [26, 27]. According to our discussion, this was an expected result since for these operators there is no way to use the equations of motion (4.1.2) to write them as $\bar{D}^2 X$. Therefore they must be necessarily primaries and belong to the classical chiral ring. Since this sector does not contain descendants this property is maintained at the quantum level. In the $h' = 0$ case these operators have different $U(1)$ flavor charges and do not mix. The matrix of their two–point functions is then diagonal and receives finite corrections at two loops [27].

The same analysis can be applied in the present case. Again, there is no way to write these operators as descendants by using the classical equations of motion (4.3.2). Therefore, we expect them to belong to the chiral ring.

In order to check that these operators do not get renormalized but their correlators might receive finite corrections we compute directly their two–point functions.

The smaller number of global symmetries surviving the h'–deformation do not prevent the operators to mix. For instance the operator $\text{Tr}(\Phi_1^2)$ can mix with $\text{Tr}(\Phi_2\Phi_3)$ since they have the same charge under the Z_3 symmetry (2.3.7). Therefore, we need to compute the non–diagonal matrix of their two–point functions.

To this purpose we concentrate on the operators $\mathcal{O}_{11}$ and $\mathcal{O}_{23}$ and evaluate all the correlators up to two loops. The calculation goes exactly as in the $h' = 0$ theory with the understanding of adding contributions from diagrams containing the new h'–vertices.

At one–loop, as in the undeformed and the β–deformed cases we do not find any divergent nor finite contributions to the two–point functions as long as the superconformal condition (4.3.3) holds.

At two loops the topologies of diagrams which contribute to $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11} \rangle$ and $\langle \mathcal{O}_{23}\bar{\mathcal{O}}_{23} \rangle$ are the ones in Fig. 4.7.

Here the grey bullets indicate two–loop corrections to the chiral propagator and one–loop corrections to the mixed gauge–chiral vertex. Using the superconformal condition

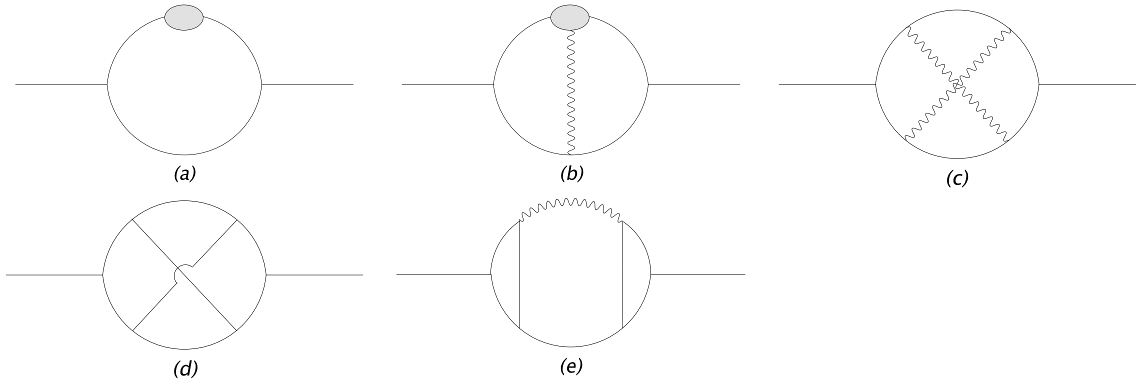

Figure 4.7: Two–loop diagrams for $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$ and $\langle \mathcal{O}_{23}\bar{\mathcal{O}}_{23}\rangle$.

(4.3.3) their q, h, h' dependence disappears and these corrections coincide with the ones of the $\mathcal{N}=4$ theory [16, 61, 62]. Therefore the first three diagrams give the same kind of contribution to both correlators.

The last two diagrams contain chiral vertices and they instead differ in the two cases for the number of h vs. h' insertions: Diagram 4.7d) gives contributions proportional to $|h|^4$ and $|h'|^4$ to $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$, and contributions proportional to $|h|^4$ and $|h|^2|h'|^2$ to $\langle \mathcal{O}_{23}\bar{\mathcal{O}}_{23}\rangle$. Analogously, diagram 4.7e) contributes to $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$ with a term proportional to $g^2|h'|^2$ and to $\langle \mathcal{O}_{23}\bar{\mathcal{O}}_{23}\rangle$ with $g^2|h|^2$.

Diagrams contributing to the mixed two–point function $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{23}\rangle$ at two loops are of the type 4.7d) with two h and two h' vertices (contributions proportional to $\bar{h}^2h'^2$), with three h and one h' (contributions proportional to $|h|^2\bar{h}'h$) and 4.7e) with one h and one h' vertices (contributions proportional to $g^2h\bar{h}'$).

Performing the D–algebra and computing the corresponding loop integrals in momentum space and dimensional regularization, it is easy to verify that the diagrams 4.7a)–d) have at most $1/\epsilon$ poles which correspond to finite corrections to the two–point functions when transformed back to the configuration space.

The only potential source of anomalous dimension terms would be the graph 4.7e) since, after D–algebra, the corresponding integral has a $1/\epsilon^2$ pole, that is a $\log(\mu^2x^2)$ divergence in configuration space. However, when computing the correlators $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{11}\rangle$ and $\langle \mathcal{O}_{11}\bar{\mathcal{O}}_{23}\rangle$ this diagram gives a vanishing color factor, whereas for the third correlator there is a complete cancellation between the contribution corresponding to a particular configuration of the $\bar{\Phi}_2$, $\bar{\Phi}_3$ lines coming out from the $\bar{\mathcal{O}}_{23}$ vertex and the one with the two lines interchanged (the same happens in the $h'=0$ theory [27]).

Therefore, all the correlators in configuration space are two–loop finite, the anomalous dimension matrix vanishes and the two operators are protected up to this order.

It is interesting to give the explicit result for the two–loop corrections to the correlators.

We find

$$\begin{aligned}
\langle \mathrm{Tr}(\Phi_1^2)(z_1)\mathrm{Tr}(\bar{\Phi}_1^2)(z_2)\rangle_{2-\mathrm{loops}} &\sim \frac{\delta^{(4)}(\theta_1-\theta_2)}{[(x_1-x_2)^2]^2}\,\mathcal{F}_1 \\
\langle \mathrm{Tr}(\Phi_2\Phi_3)(z_1)\mathrm{Tr}(\bar{\Phi}_2\bar{\Phi}_3)(z_2)\rangle_{2-\mathrm{loops}} &\sim \frac{\delta^{(4)}(\theta_1-\theta_2)}{[(x_1-x_2)^2]^2}\,\mathcal{F}_2 \qquad (4.3.5)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_1 &= \Bigg[|h|^4\frac{N^2-4}{N^2}\,|q-\bar{q}|^2\left(\frac{N^2-1}{4N^2}\,|q-\bar{q}|^2-1\right) \\
&+ |h'|^4\frac{(N^2-20)(N^2-4)}{8N^4} - |h|^2|h'|^2\frac{N^2-4}{2N^2}\left(1-\frac{1}{N^2}\,|q-\bar{q}|^2\right)\Bigg] \qquad (4.3.6)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_2 &= \Bigg[|h|^4\frac{N^2-4}{4N^4}\,|q-\bar{q}|^4 + |h'|^4\frac{(N^2-4)^2}{8N^4} \\
&\qquad + |h|^2|h'|^2\frac{N^2-4}{2N^2}\left(3-\frac{N^2-5}{N^2}\,|q-\bar{q}|^2\right)\Bigg] \qquad (4.3.7)
\end{aligned}$$

We note that all the g^4 contributions cancel and we are left with expressions which vanish in the $\mathcal{N}=4$ limit ($\beta = h' = 0$, $|h|^2 = g^2$). Moreover, both the contributions survive in the large N limit in contradistinction to the $h'=0$ case where $\mathcal{F}_2$ is subleading [27].

4.3.2 Chiral ring: The $\Delta_0 = 3$ sector

The next sector we investigate contains operators with naive scale dimension $\Delta_0 = 3$. We classify them according to their Z_3–charge as in Table 4.2.

$\mathcal{Q}=0$	$\mathcal{Q}=1$	$\mathcal{Q}=2$
$\mathcal{O}_1 = \mathrm{Tr}(\Phi_1^3)$	$\mathcal{O}_6 = \mathrm{Tr}(\Phi_1^2\Phi_2)$	$\mathcal{O}_9 = \mathrm{Tr}(\Phi_1^2\Phi_3)$
$\mathcal{O}_2 = \mathrm{Tr}(\Phi_2^3)$	$\mathcal{O}_7 = \mathrm{Tr}(\Phi_2^2\Phi_3)$	$\mathcal{O}_{10} = \mathrm{Tr}(\Phi_3^2\Phi_2)$
$\mathcal{O}_3 = \mathrm{Tr}(\Phi_3^3)$	$\mathcal{O}_8 = \mathrm{Tr}(\Phi_3^2\Phi_1)$	$\mathcal{O}_{11} = \mathrm{Tr}(\Phi_2^2\Phi_1)$
$\mathcal{O}_4 = \mathrm{Tr}(\Phi_1\Phi_2\Phi_3)$		
$\mathcal{O}_5 = \mathrm{Tr}(\Phi_1\Phi_3\Phi_2)$		

Table 4.2: Operators with $\Delta_0 = 3$.

We note that the neutral sector does not contain the same number of operators as the charged ones. This is due to the fact that, in contradistinction to the previous case, the $\mathcal{Q}=0$ sector is closed under the application of cyclic permutations $\Phi_i \to \Phi_{i+1}$ and transformations (2.3.8). Therefore, we cannot generate the charged sectors from the neutral one by using these mappings.

The charged sectors are also closed under permutations but they get exchanged under transformations (2.3.8). This is the reason why they still have the same number of operators.

We first focus on the set of operators with $\mathcal{Q} = 0$. As for the $h' = 0$ theory, in this sector the Konishi anomaly enters the game when we try to use the equations of motion to write descendants which involve $\mathcal{O}_4$ and $\mathcal{O}_5$. However the Konishi anomaly can be neglected as long as we are interested in the construction of CPO's up to two loops. We will then restrict our analysis at this order.

Using the equations of motion (4.3.2) we can write three descendant operators

$$\begin{aligned} \mathcal{D}^{(1)} &= h\,(q\,\mathcal{O}_4 - \bar{q}\,\mathcal{O}_5) + h'\,\mathcal{O}_1 \\ \mathcal{D}^{(2)} &= h\,(q\,\mathcal{O}_4 - \bar{q}\,\mathcal{O}_5) + h'\,\mathcal{O}_2 \\ \mathcal{D}^{(3)} &= h\,(q\,\mathcal{O}_4 - \bar{q}\,\mathcal{O}_5) + h'\,\mathcal{O}_3 \end{aligned} \tag{4.3.8}$$

According to the discussion of Section 4.2 we expect to single out two protected operators. We consider the most general linear combination

$$\mathcal{P} = \alpha\,\mathcal{O}_1 + \beta\,\mathcal{O}_2 + \gamma\,\mathcal{O}_3 + \delta\,\mathcal{O}_4 + \epsilon\,\mathcal{O}_5 \tag{4.3.9}$$

and require tree-level orthogonality to the three descendants. These constraints provide the condition $\alpha = \beta = \gamma \equiv a$ (as expected because of the Z_3 symmetries of this sector) and the extra relation

$$3a\bar{h}'(N^2-4)q + \bar{h}\left[\delta\,\left(N^2-2+2q^2\right) - \epsilon\,\left((N^2-2)q^2+2\right)\right] = 0 \tag{4.3.10}$$

which can be used to express a in terms of two arbitrary constants.

Any CPO in this sector has then the following form

$$\mathcal{P} = a\,(\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3) + \delta\,\mathcal{O}_4 + \epsilon\,\mathcal{O}_5 \tag{4.3.11}$$

An explicit check on its two–point function at one loop leads to $\langle \mathcal{P}\,\bar{\mathcal{P}} \rangle_1$ finite, independently of the choice of δ and ϵ. One can choose the two constants in order to select two mutually orthogonal operators.

As it happened in the previous cases, these operators are guaranteed to be protected up to two loops as a consequence of their one–loop protection plus the result $W^{(1)}_{eff} \sim W$ which insures that the classical descendants (4.3.8) keep being good descendants also at one loop.

The sectors characterized by Z_3 charges $\mathcal{Q} = 1, 2$ do not contain protected operators. In fact, one can see that any charged operator in Table 4.2 can be written as $\mathcal{O}_i = \bar{D}^2 X_i$ by using the classical equations of motion. We expect this result to be valid at any order of perturbation theory since the structure of the effective superpotential for what concerns its superfield dependence cannot change.

To summarize, in the $\Delta_0 = 3$ sector we have found two protected operators which are linear combinations of $\mathrm{Tr}(\Phi_i^3)$, $i = 1,2,3$, $\mathrm{Tr}(\Phi_1\Phi_2\Phi_3)$ and $\mathrm{Tr}(\Phi_1\Phi_3\Phi_2)$. We note that among all possible weight–3 operators these are the only ones which belong to the chiral ring of the β–deformed theory. The rest of weight–3 operators which were descendants for $h' = 0$ keep being descendants.

The protected operators we have found are neutral under the Z_3 symmetry (2.3.7). As discussed in [68], the neutral sector of the chiral ring (the untwisted sector) coincides with the center of the quantum algebra generated by the F–terms constraints. In particular, for the h'–deformation one element of the center has been constructed explicitly (eq. (4.83) in [68]). This element coincides with one of the CPO's (4.3.11) we have found, once we set $\mathcal{D}^{(i)} = 0$ in the chiral ring (see eq. (4.3.8)), use these identities to express the operator $\mathcal{O}_5$ in terms of the other ones and make a suitable choice for the coefficients δ and ϵ.

4.3.3 Comments on the general structure of the chiral ring

The $\Delta_0 = 2, 3$ sectors studied in the previous Section are very peculiar and do not provide enough information to guess the structure of the sectors for generic scale dimension. In fact, for $\Delta_0 = 2$ no descendants are present and we cannot even apply the orthogonality procedure to construct CPO's. The $\Delta_0 = 3$ sector contains only protected operators which are Z_3 neutral and are linear combinations of "old" CPO's, that is operators which were protected for $h' = 0$.

A naive generalization of our results to higher dimensional sectors would lead to the conjecture that the chiral ring for the h'–deformed theory, at least for what concerns its neutral sector with $\Delta_0 = 3J$, would be given by linear combinations of $\mathrm{Tr}(\Phi_i^{3J})$ and $\mathrm{Tr}(\Phi_1^J\Phi_2^J\Phi_3^J)$. However, we expect more general operators of the form $\mathrm{Tr}(\Phi_1^{3J-m-n}\Phi_2^m\Phi_3^n)$, $m + 2n = \mathrm{mod}(3)$ to appear. Moreover, nontrivial Z_3–charged sectors should appear for $\Delta_0 = 3J$ even if they are absent in the particular case $\Delta_0 = 3$.

To investigate these issues we should extend our analysis to higher dimensional sectors and this would require quite a bit of technical effort. However, without entering any calculative detail, but simply performing dimensional and Z_3–charge balances we can figure out few general properties of the $\mathcal{Q}$–sectors of the chiral ring.

We consider the generic chiral operator $\mathcal{O}_1 = (\Phi_1^a\Phi_2^b\Phi_3^c)$ for any trace structure with scale dimension $\Delta_0 = a + b + c$ and Z_3–charge $\mathcal{Q}_1 \equiv b + 2c$ with respect to the symmetry (2.3.7).

We now perform $\Phi_i \leftrightarrow \Phi_j$ exchanges according to the symmetry (2.3.8) and Z_3 permutations. In this way of doing we generate all the operators with the same trace structure in a given Δ_0 sector. Let us consider for example the operators $\mathcal{O}_2 = (\Phi_2^a\Phi_1^b\Phi_3^c)$ and $\mathcal{O}_3 = (\Phi_3^a\Phi_1^b\Phi_2^c)$ obtained by a $\Phi_1 \leftrightarrow \Phi_2$ exchange and a cyclic permutation, respectively. They have charges $\mathcal{Q}_2 = a + 2c$ and $\mathcal{Q}_3 = 2a + c$. It is easy to see that if $\Delta_0 = 3J$ then $\mathcal{Q}_2 = \mathcal{Q}_3 = 0\ (\mathrm{mod}(3))$ *iff* $\mathcal{Q}_1 = 0\ (\mathrm{mod}(3))$. This property holds for any operator that we can construct from $\mathcal{O}_1$ by the application of the two discrete symmetries. On the other hand, if $\mathcal{Q}_1 = 1, 2(\mathrm{mod}(3))$ operators obtained from it by cyclic permutations still maintain the same charge, but the application of field exchanges (2.3.8) map charge–1 operators into charge–2 operators and viceversa.

Therefore, for $\Delta_0 = 3J$ the $\mathcal{Q} = 0$ class is closed under the action of Z_3–permutations and (2.3.8) symmetry, and being independent, may contain a different number of operators compared to the charged sectors which instead are related by (2.3.8) mappings. In particular, as it happens for $\Delta_0 = 3$ charged classes of the chiral ring might be empty while the corresponding neutral one is not.

If $\Delta_0 \neq 3J$ a simple calculation leads to the conclusion that starting from operators with zero Z_3–charge we generate operators with $\mathcal{Q} = 1$ by applying $\Phi_1 \leftrightarrow \Phi_2$ if $\Delta_0 = 3J+1$ and a cyclic permutation if $\Delta_0 = 3J+2$. Correspondingly, we obtain operators with $\mathcal{Q} = 2$ by applying a cyclic permutation in the first case and a $\Phi_1 \leftrightarrow \Phi_2$ exchange in the second case. Therefore, in any sector with $\Delta_0 \neq 3J$ the number of operators with $\mathcal{Q} = 1$ is the same as the ones with $\mathcal{Q} = 2$ and coincides with the number of neutral operators.

If we apply the same reasoning to the descendant operators of each sector (to simplify the analysis we work at large N to avoid mixing among different trace structures) we discover that every time $\Delta_0 \neq 3J$ the descendants of the charged classes can be obtained from the neutral ones by field exchanges. As a consequence, the three classes contain the same number of descendants and then the *same* number of protected operators.

To summarize, the sectors of the chiral ring behave differently according to their scale dimension: If $\Delta_0 \neq 3J$ the corresponding operators are equally split into the three $\mathcal{Q}$ classes. On the contrary, if $\Delta_0 = 3J$ the neutral class is independent and may contain a different number of CPO's.

As a further example we have studied the $\Delta_0 = 4$ operators. In the large N limit and at the lowest order in perturbation theory we have found that the neutral single–trace sector contains one independent CPO (we have eight single–trace chirals and seven descendants). Therefore, we conclude that also the charged sectors contain one single protected operator and we know how to construct it once we have found the $\mathcal{Q} = 0$ operator explicitly. In the single–trace sector the protected operator turns out to be a linear combination of

$$\begin{array}{l} \mathrm{Tr}(\Phi_1^4) \\ \mathrm{Tr}(\Phi_1\Phi_2^3) \quad , \quad \mathrm{Tr}(\Phi_1\Phi_3^3) \quad , \quad \mathrm{Tr}(\Phi_2^2\Phi_3^2) \quad , \quad \mathrm{Tr}(\Phi_2\Phi_3\Phi_2\Phi_3) \\ \mathrm{Tr}(\Phi_1^2\Phi_2\Phi_3) \quad , \quad \mathrm{Tr}(\Phi_1^2\Phi_3\Phi_2) \quad , \quad \mathrm{Tr}(\Phi_1\Phi_2\Phi_1\Phi_3) \end{array} \tag{4.3.12}$$

It remains the open question whether for $\Delta_0 = 3J$, $J > 1$, the charged sectors are trivial as in the weight–3 case. A systematic analysis of the charged protected operators is a difficult task in general. However, working at large N it is easy to realize that for J *even* and $J > 1$, there are nontrivial protected operators for $\mathcal{Q} = 1$ and $\mathcal{Q} = 2$. These are operators with the $3J$ chiral superfields split into the maximal number of traces allowed by $SU(N)$, i.e. $3J/2$. In fact, for these operators it is impossible to exploit the equations of motion and write them as descendants. For J *odd* the same arguments do not lead to any definite conclusion. However, we expect to generate nontrivial charged protected operators by multiplying the neutral CPO's of weight 3 previously constructed by $3(J-1)/2$ traces containing two operators each and carrying the right Z_3 charge.

4.4 Summary and results

In this Chapter we have focused on the perturbative structure of the matter (not gauge) quantum chiral ring defined as in (4.2.9) in terms of the effective superpotential. According to our general prescription, CPO's can be determined by imposing order by order the orthogonality condition (4.2.6) to all the descendants of a given sector. This requires constructing first the descendants as a power expansion in the couplings. According to the definition (4.2.9), this can be easily accomplished once the effective superpotential is known at a given order.

For the real β–deformed theory (2.3.9) we have studied quite extensively the spin–2 sector of the theory. For the particular examples of weights $(J,1,0)$ and $(2,2,0)$ we have considered, a special pattern arises which allows for a drastic simplification in the study of the orthogonality condition: In any of these sectors descendants can be always constructed at tree level which turn out to be good independent descendants even at the quantum level. This is due to the particular form (4.2.14) of the superpotential and the peculiar way the equations of motion work which allow for constructing q–independent descendants, insensible to the quantum corrections of the theory. This property persists even for other examples of the form $(J_1,J_2,0)$. Therefore, we conjecture that it might be a property of the entire spin–2 sector: For any weight $(J_1,J_2,0)$ quantum descendant operators can be constructed which coincide with the descendants determined classically.

We have then studied the spin–3 sector. In this case the determination of quantum descendants of weights (J_1,J_2,J_3) cannot ignore the Konishi anomaly term. Being its effect of order λ it only enters nontrivially the orthogonality condition from two loops on, that is it will affect the form of the protected operators at least at three loops. For weights $(1,1,1)$ and $(2,1,1)$ we have determined the CPO's up to two loops. In particular, for the first case we have proved that up to this order the correct CPO is the one found in [26]. Higher order calculations would require computing two–point correlation functions between matter chiral operators and $\mathrm{Tr}(W^\alpha W_\alpha)$. It would be interesting to pursue this direction since it represents the first case where the descendant operators, apart from acquiring an explicit dependence on the Konishi anomaly term, get modified nontrivially at the quantum level due to the nontrivial corrections to the superpotential which start appearing at order λ^2.

We have extended our procedure to the study of protected operators for the full Leigh–Strassler deformation. We can think of this theory as a marginal perturbation of the β–deformed theory induced by the h'–terms in (4.3.1). In this case the determination of the complete chiral ring is a difficult task and only few insights have been discussed in [68, 25]. We have moved few steps in this direction by studying perturbatively the simple $\Delta_0 = 2,3$ sectors. For operators of scale dimension two we have found that the h'–deformed theory has still the same CPO's as the $h'=0$ one, i.e. $\mathrm{Tr}(\Phi_i^2)$ and $\mathrm{Tr}(\Phi_i\Phi_j)$, $i\neq j$.

For the $\Delta_0 = 3$ sector we have found a two–dimensional plane of CPO's given as linear combinations of the CPO's of the corresponding $h'=0$ theory, i.e. $\mathrm{Tr}(\Phi_i^3)$ and $\mathrm{Tr}(\Phi_1\Phi_2\Phi_3)$. In fact, in this case the lower number of global symmetries surviving the deformation allows for mixing among the operators who were protected in the previous case and belonged to different $U(1) \times U(1)$ sectors. The class of protected operators we have found contains the central element of the quantum algebra proposed in [68].

What turns out is that in the $\Delta_0 = 2$ sector the chiral ring is made by operators which are both charged and neutral with respect to the Z_3–symmetry (2.3.7) that the theory inherits from the parent $h' = 0$ theory. On the other hand, in the $\Delta_0 = 3$ sector *all* CPO's we can construct are neutral under (2.3.7). The generalization of our results to higher dimensional sectors leads to the result that the chiral ring for the h'–deformed theory can be divided into two subsets: Sectors with scale dimension $\Delta_0 = 3J$ have an independent $\mathcal{Q} = 0$ class which may contain in general a different number of CPO's. Instead, whenever $\Delta_0 \neq 3J$ we can generate the chiral primary operators of the charged classes from neutral CPO's by the use of the other discrete symmetries, i.e. cyclic permutations of the three superfields and the symmetry (2.3.8). It then follows that the three classes contain the same number of protected operators. In particular, for any non–empty neutral sector (for instance $\Delta_0 = 2, 4$) the corresponding charged ones are nontrivial. Neutral CPO's will be in general linear combinations of operators of the form $\mathrm{Tr}(\Phi_1^{J-m-n}\Phi_2^m\Phi_3^n)$ with $m+2n = 3p$.

The Z_3 periodicity we have found in the chiral ring structure should have a counterpart in the spectrum of BPS states of the dual supergravity theory. Therefore, it might be of some help in the construction of the dual spectrum.

For *all* the cases we have investigated the CPO's do not get corrected at one–loop, whereas they start being modified at order λ^2. This one–loop non–renormalization found for a large class of chiral operators is probably universal for all the CPO's and might be traced back to the one–loop non–renormalization properties of the theories. Precisely, the conditions (4.2.16, 4.3.3) which insure superconformal invariance at one–loop are maintained at two loops, i.e. the superconformal theories at one and two loops are the same. It is then natural to speculate that the corresponding chiral rings should be the same. The theory instead changes at three loops where the superconformal condition gets modified by terms of order λ^2 [28]. Therefore we expect that at this order the chiral ring will be modified by effects of the same order.

Appendix A

Color Conventions and Identities

We give a brief description of color conventions and a series of useful identities involving the group generators. For a general simple Lie algebra we have:

$$[T_a, T_b] = i f_{abc} T^c \tag{A.0.1}$$

where T_a are the generators and f_{abc} the structure constants. The matrices T_a are normalized as

$$Tr\,(T_a T_b) = \delta_{ab} \tag{A.0.2}$$

We specialize to the case of $SU(N)$ Lie algebra whose generators T_a, $a = 1, \ldots, N^2 - 1$ are taken in the fundamental representation, i.e. they are $N \times N$ traceless matrices. The basic relation which allows to deal with products of the T_a is the following

$$T^a_{ij}\, T^a_{kl} = \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right) \tag{A.0.3}$$

From this identities, we can easily obtain all the identities used to compute the color structures associated to the Feynmann diagrams relevant for the two–point correlation functions.

In particular fusion and splitting rules between traces with contracted inidices are repeatedly used in the computation of colours. We introduce the notation $\mathrm{Tr}(T^a T^b T^c \ldots) \equiv (abc \ldots)$. If we need to contract two indices (say c) appearing in different traces the *fusion rule* states that

$$\begin{aligned}(a_1 \ldots a_{n-1}\, c\, a_{n+1} \ldots a_N)\,(b_1 \ldots b_{m-1}\, c\, b_{m+1} \ldots b_M) &= \\ = (a_{n+1} \ldots a_N a_1 \ldots a_{n-1} b_{m+1} \ldots b_M b_1 \ldots b_{m-1}) &+ \\ -\frac{1}{N}\,(a_1 \ldots a_{n-1} a_{n+1} \ldots a_N)\,(b_1 \ldots b_{m-1} b_{m+1} \ldots b_M)&\end{aligned} \tag{A.0.4}$$

If we need to contract two indices c appearing inside the same trace the *splitting rule* states that

$$(a_1 \dots a_{n-1}\, c\, a_{n+1} \dots a_{m-1}\, c\, a_{m+1} \dots a_N) = \tag{A.0.5}$$

$$= (a_{m+1} \dots a_N a_1 \dots a_{n-1})(a_{n+1} \dots a_{m-1}) - \frac{1}{N}(a_1 \dots a_{n-1} a_{n+1} \dots a_{m-1} a_{m+1} \dots a_N)$$

These rules can be used to derive some useful identities

$$f_{acd} f_{bcd} = 2N \delta_{ab} \tag{A.0.6}$$

$$f_{abm} f_{cdm} + f_{cbm} f_{dam} + f_{dbm} f_{acm} = 0. \tag{A.0.7}$$

$$(abcd)(abcd) = \frac{1}{N^2}(N^2-1)(N^2+3) \tag{A.0.8}$$

$$(abcd)(abdc) = -\frac{1}{N^2}(N^2-1)(N^2-3) \tag{A.0.9}$$

$$(abcd)(dcba) = \frac{1}{N^2}(N^2-1)(N^4-3N^2+3) \tag{A.0.10}$$

$$(cadb) f_{cme} f_{dmf} = -(\delta_{ea}\delta_{fb} + \delta_{fa}\delta_{eb}) \tag{A.0.11}$$

$$(cdab) f_{cme} f_{dmf} = \delta_{ef}\delta_{ab} + N\,(efab) \tag{A.0.12}$$

Appendix B

Superspace Feynmann Rules and Notations

We resume here the main ingredients necessary to perform perturbative computations in superfield formalism. From the action:

$$
\begin{aligned}
S \;=\; & \int d^8z \; \mathrm{Tr}\left(e^{-gV}\overline{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \; \mathrm{Tr}W^\alpha W_\alpha \; + \qquad \text{(B.0.1)} \\
& + ih\int d^6z \; \mathrm{Tr}\left(q\,\Phi_1\Phi_2\Phi_3 - \frac{1}{q}\,\Phi_1\Phi_3\Phi_2\right) + \frac{i\,h'}{3}\int d^6z \; \mathrm{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3) + \; h.c.
\end{aligned}
$$

quantized in the Feynman gauge we can read the superfield propagators and the vertices. We write $V = V^a T_a$, $\Phi_i = \Phi_i^a T_a$ where T_a are $SU(N)$ matrices in the fundamental representation (see Appendix B and notations therein). Then the propagators are:

$$
\begin{aligned}
\langle V^a\, V^b\rangle = & \; -\delta^{ab}\frac{1}{p^2}\,\delta^{(4)}(\theta_1-\theta_2) \\
\langle \Phi_i^a\, \overline{\Phi}_j^b\rangle = & \; \delta_{ij}\delta^{ab}\frac{1}{p^2}\,\delta^{(4)}(\theta_1-\theta_2) \qquad \text{(B.0.2)}
\end{aligned}
$$

and the three–point vertices

$$
\begin{aligned}
(\Phi_1\Phi_2\Phi_3)_{\,\mathrm{vertex}} \;\; & \rightarrow \;\; ih\,\Phi_1^a\Phi_2^b\Phi_3^c\left[q(abc) - \frac{1}{q}(acb)\right] \qquad \text{(B.0.3)} \\
(\overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3)_{\,\mathrm{vertex}} \;\; & \rightarrow \;\; -i\bar{h}\,\overline{\Phi}_1^a\overline{\Phi}_2^b\overline{\Phi}_3^c\left[\bar{q}(acb) - \frac{1}{\bar{q}}(abc)\right] \\
(\overline{\Phi}_i V\Phi_i)_{\,\mathrm{vertex}} \;\; & \rightarrow \;\; g\,\overline{\Phi}_i^a V^b\Phi_i^c\;\;[(abc)-(acb)] \\
(\Phi_i^3)_{\,\mathrm{vertex}} \;\; & \rightarrow \;\; \frac{h'}{3}\,\Phi_i^a\Phi_i^b\Phi_i^c\,(abc) \qquad\qquad i=1,\ldots,3
\end{aligned}
$$

After performing D-algebra inside the loops the computation is reduced to the resolution of standard bosonic integrals, calculated in dimensional regularization in $n = 4 - 2\epsilon$ dimensions. A list of relevant integrals is given in Appendix C.

Appendix C

Relevant Integrals

In this Appendix we list the results for loop integrals that we have used along the calculations. All the integrals are computed in the framework of dimensional regularization and choosing the G–scheme [60]. We work in n dimensions, with $n = 4 - 2\epsilon$ and in momentum space. Having this in mind we now list the relevant integrals and their $1/\epsilon$ expansion. The momentum integrals can be computed by making repeated use of the one loop results

$$\begin{aligned} \int \frac{d^n k}{(k^2)^a \left[(p-k)^2\right]^b} &= \frac{\Gamma\left(a+b-\frac{n}{2}\right)\,\Gamma\left(\frac{n}{2}-a\right)\,\Gamma\left(\frac{n}{2}-b\right)}{\Gamma(a)\,\Gamma(b)\,\Gamma(n-a-b)}\,\frac{1}{(p^2)^{a+b-\frac{n}{2}}} \\ \int d^n k\,\frac{k_{\alpha\dot\alpha}}{(k^2)^a \left[(p-k)^2\right]^b} &= \frac{1}{2}\,\frac{n-2a}{n-a-b}\,p_{\alpha\dot\alpha}\int \frac{d^n k}{(k^2)^a \left[(p-k)^2\right]^b} \end{aligned} \tag{C.0.1}$$

and, for the more complicated ones, by using uniqueness methods (see for instance [47]).

We begin with the integrals relevant for Chapter 3 computations. At one loop we are interestd in the bubble integral of Fig. 3.1(f)

$$I_1 = \int \frac{d^n k}{(2\pi)^n}\,\frac{1}{k^2\,(p+k)^2} = \frac{1}{(4\pi)^2}\,\frac{1}{\epsilon}\,\frac{1}{(p^2)^\epsilon} + \mathcal{O}(\epsilon^0) \tag{C.0.2}$$

At two loops we have to consider the bosonic integral in Fig. 3.2 or Fig. 3.5(a)

$$I_2 = \int \frac{d^n k\, d^n q}{(2\pi)^{2n}}\,\frac{1}{k^2\,q^2\,(q+k)^2\,(p+k)^2} = \frac{1}{(4\pi)^4}\,\frac{1}{\epsilon^2}\,\frac{1}{(p^2)^{2\epsilon}}\left[\frac{1}{2} + \mathcal{O}(\epsilon)\right] \tag{C.0.3}$$

At three loops we need the value for the bosonic integral in Fig. 3.20

$$I_3 = \int \frac{d^n k\, d^n q\, d^n s}{(2\pi)^{3n}}\,\frac{1}{k^2\,q^2\,(p+s)^2\,(k+s)^2\,(k+q)^2\,(q-s)^2} = \frac{1}{(4\pi)^6}\,\frac{1}{\epsilon}\,\frac{1}{(p^2)^{3\epsilon}}\left[2\,\zeta(3) + \mathcal{O}(\epsilon)\right] \tag{C.0.4}$$

At four loops we have the integral associated to Fig. 3.3

$$\begin{aligned} I_4 &= \int \frac{d^n k\, d^n q\, d^n r\, d^n t}{(2\pi)^{4n}} \frac{1}{k^2\,(k+t)^2\,(q+k)^2\,(q+r)^2\,(q+p)^2\,t^2\,r^2\,(t+r)^2} = \\ &= \frac{1}{(4\pi)^8}\,\frac{1}{\epsilon}\,\frac{1}{(p^2)^{4\epsilon}}\,[5\,\zeta(5) + \mathcal{O}(\epsilon)] \end{aligned} \tag{C.0.5}$$

and at five loops the one in Fig. 3.5(b)

$$\begin{aligned} I_5 &= \int \frac{d^n s\, d^n k\, d^n q\, d^n r\, d^n t}{(2\pi)^{5n}} \frac{1}{s^2\,t^2\,r^2\,k^2\,(s+t)^2\,(q+s)^2\,(q+r)^2\,(k+q)^2\,(t+r)^2\,(p+k)^2} = \\ &= \frac{1}{(4\pi)^{10}}\,\frac{1}{\epsilon^2}\,\frac{1}{(p^2)^{5\epsilon}}\,[\zeta(5) + \mathcal{O}(\epsilon)] \end{aligned} \tag{C.0.6}$$

Finally we give the expressions for the tadpole type integrals used for the gauge beta–function computation, after IR and UV subtraction. At two loop we consider the integral in Fig. 3.14 of in Fig. 3.23

$$\int \frac{d^n k\ d^n q}{(2\pi)^{2n}} \frac{1}{q^2(q+k)^2 k^4} \longrightarrow \frac{1}{(4\pi)^4}\left(-\frac{1}{2\epsilon^2} + \mathcal{O}(1/\epsilon)\right) \tag{C.0.7}$$

and its five loop companion of Fig. 3.15

$$\begin{aligned} &\int \frac{d^n k\ d^n q\ d^n r\ d^n s\ d^n t}{(2\pi)^{5n}} \frac{1}{r^2(r+q)^2 s^2(s+q)^2 t^2(t+r)^2(t+s)^2(q+k)^2 k^4} \longrightarrow \\ &\longrightarrow \frac{1}{(4\pi)^{10}}\left(-\frac{4\,\zeta(5)}{\epsilon^2} + \mathcal{O}(1/\epsilon)\right) \end{aligned} \tag{C.0.8}$$

The four loop integral of Fig. 3.28

$$\int \frac{d^n k\, d^n q\, d^n r\, d^n t}{(2\pi)^{4n}}\, \frac{1}{k^4\,q^2\,t^2\,(r-q)^2\,(r+t)^2\,(t+q)^2\,(r+k)^2} \longrightarrow \frac{1}{(4\pi)^8}\left(-\frac{3}{2}\,\frac{\zeta(3)}{\epsilon^2} + \mathcal{O}(1/\epsilon)\right)$$

We stress that these last three integrals must be multiplied by the color and combinatoric factors of the correspondent graphs and, after using eq. (3.2.64) one can obtain the formulae in (3.2.68), (3.2.70) and (3.3.31).

We turn to the integrals used in Chapter 4. We begin by considering the momentum integrals associated to the one–loop and two–loop diagrams in Fig. 4.4 for the perturbative corrections to the superpotential. At one loop, after performing D–algebra, the diagram

4.4(b) gives the standard triangle contribution [57]. Assigning external momenta p_i ($p_1 + p_2 + p_3 = 0$) we have

$$p_3^2 \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2 (q-p_2)^2 (q+p_1)^2} = \frac{1}{(4\pi)^2} \, \Phi^{(1)}(x,y) + \mathcal{O}(\epsilon) \tag{C.0.9}$$

where

$$x \equiv \frac{p_1^2}{p_3^2} \quad \text{and} \quad y \equiv \frac{p_2^2}{p_3^2} \tag{C.0.10}$$

The p_3^2 in front of the integral is produced by D–algebra. The function $\Phi^{(1)}(x,y)$ can be represented as a parametric integral

$$\Phi^{(1)}(x,y) = -\int_0^1 \frac{d\xi}{y\,\xi^2 + (1-x-y)\xi + x} \left(\log \frac{y}{x} + 2 \log \xi \right) \tag{C.0.11}$$

Since we look for a local contribution to the superpotential we are interested in the result of the integral for external momenta set to zero. A consistent way [58] to take the limit of vanishing external momenta is to set $p_i^2 = m^2$ for any i so having $x, y = 1$ and let the IR cut–off m^2 going to zero at the end of the calculation. In the limit we obtain a finite local result [58]

$$-\int_0^1 d\xi \frac{\log \xi(1-\xi)}{1-\xi(1-\xi)} \tag{C.0.12}$$

At two loops two types of integrals appear. From diagrams 4.4(c) and 4.4(d) we have integrals of the form

$$\begin{aligned} (p_3^2)^2 \int \frac{d^n q \, d^n r}{(2\pi)^{2n}} \frac{1}{(r+p_1)^2 (q+p_1)^2 (r-p_2)^2 (q-p_2)^2 r^2 (q-r)^2} &= \\ = \frac{1}{(4\pi)^4} \, \Phi^{(2)}(x,y) + \mathcal{O}(\epsilon) \end{aligned} \tag{C.0.13}$$

with x and y as in (C.0.10). The function $\Phi^{(2)}(x,y)$ is defined by [57]

$$\Phi^{(2)}(x,y) = -\frac{1}{2} \int_0^1 \frac{d\xi}{y\,\xi^2 + (1-x-y)\xi + x} \log \xi \left(\log \frac{y}{x} + \log \xi \right) \left(\log \frac{y}{x} + 2 \log \xi \right) \tag{C.0.14}$$

As in the one–loop case, the limit $x, y \to 1$ gives a finite local contribution to the effective superpotential.

From diagrams 4.4(c)–(g) this kind of integral also appears

$$p_3^2 \int \frac{d^n q \, d^n r}{(2\pi)^{2n}} \frac{1}{q^2 \, r^2 (q-r)^2 (q-p_3)^2 (r-p_3)^2} = \frac{1}{(4\pi)^4} \, 6\zeta(3) + \mathcal{O}(\epsilon) \tag{C.0.15}$$

where one of the external momenta has been already set to zero (in this case we can safely set one of the external momenta to zero from the very beginning since we do not introduce fake IR divergences). This is already the local finite contribution we obtain by setting also $p_3^2 = 0$.

When we deal with two-point correlation functions, at tree-level we have ($k = \Delta_0$ is the free scale dimension of the operators involved and p is the external momentum)

$$\int \frac{d^n q_1 \dots d^n q_{k-1}}{(2\pi)^{n(k-1)}} \frac{1}{q_1^2 (q_2 - q_1)^2 (q_3 - q_2)^2 \dots (p - q_{k-1})^2}$$
$$= \frac{1}{\epsilon}\left[\frac{1}{(4\pi)^2}\right]^{k-1} \frac{(-1)^k}{[(k-1)!]^2} (p^2)^{k-2-(k-1)\epsilon} + \mathcal{O}(1) \qquad \text{(C.0.16)}$$

At two loops we are interested in the four diagrams listed in Fig. 4.6. From the graph 4.6(a) we obtain

$$\int \frac{d^n q_3 \dots d^n q_{k-1}}{(2\pi)^{n(k+1)}} \frac{1}{(q_4 - q_3)^2 \dots (p - q_{k-1})^2} \times$$
$$\int \frac{d^n k\, d^n l\, d^n r\, d^n s \quad r^2 (q_3 - l)^2}{k^2 l^2 (k-l)^2 (r-k)^2 (r-l)^2 (s-l)^2 (r-s)^2 (q_3 - r)^2 (q_3 - s)^2} \qquad \text{(C.0.17)}$$
$$= \frac{1}{\epsilon}\left[\frac{1}{(4\pi)^2}\right]^{k+1} \frac{(-1)^k (k-1)}{[(k-1)!]^2 (k+1)} [6\zeta(3) - 20\zeta(5)] (p^2)^{k-2-(k+1)\epsilon} + \mathcal{O}(1)$$

The momentum integral for the graph 4.6(b) gives

$$\int \frac{d^n q_3 \dots d^n q_{k-1}}{(2\pi)^{n(k+1)}} \frac{-q_3^2}{(q_4 - q_3)^2 \dots (p - q_{k-1})^2} \times$$
$$\int \frac{d^n k\, d^n l\, d^n r\, d^n s}{k^2 l^2 (k-l)^2 (r-k)^2 (s-l)^2 (r-s)^2 (q_3 - r)^2 (q_3 - s)^2} \qquad \text{(C.0.18)}$$
$$= \frac{1}{\epsilon}\left[\frac{1}{(4\pi)^2}\right]^{k+1} \frac{(-1)^k (k-1)}{[(k-1)!]^2 (k+1)} 40\zeta(5) (p^2)^{k-2-(k+1)\epsilon} + \mathcal{O}(1)$$

Finally, the graphs 4.6(c) and 4.6(d) lead to the same contribution

$$\int \frac{d^n r\, d^n q_2 \dots d^n q_{k-1}}{(2\pi)^{n(k+1)}} \frac{1}{(q_2 - r)^2 (q_3 - q_2)^2 \dots (p - q_{k-1})^2} \times$$
$$\int \frac{d^n k\, d^n l}{k^2 l^2 (k-l)^2 (r-k)^2 (r-l)^2} \qquad \text{(C.0.19)}$$
$$= \frac{1}{\epsilon}\left[\frac{1}{(4\pi)^2}\right]^{k+1} \frac{(-1)^k (k-1)}{[(k-1)!]^2 (k+1)} 6\zeta(3) (p^2)^{k-2-(k+1)\epsilon} + \mathcal{O}(1)$$

Appendix D

Color of H-Type Diagram

In this Appendix we report the full non–planar expression for the color of the four loop diagram depicted in Fig. 3.21:

$$
\begin{aligned}
K_4 &= \frac{1}{2}\Big[(|h_1|^2+|h_2|^2)^4+(|h_1|^2-|h_2|^2)^4\Big]+\\
&+ \frac{4}{N^2}\Big[|h_3|^8-4|h_3|^6(|h_1|^2+|h_2|^2)+2|h_3|^4(3|h_1|^4+4|h_1|^2|h_2|^2+3|h_2|^4)+\\
&- 2|h_3|^2(3|h_1|^6+5|h_1|^4|h_2|^2+5|h_1|^2|h_2|^4+3|h_2|^6)+\\
&+ (|h_1|^8+8|h_1|^6|h_2|^2+6|h_1|^4|h_2|^4+8|h_1|^2|h_2|^6+|h_2|^8)\Big]+\\
&- \frac{4}{N^4}\Big[5|h_3|^8-20|h_3|^6(|h_1|^2+|h_2|^2)+12|h_3|^4(|h_1|^4+|h_1|^2|h_2|^2+|h_2|^4)+\\
&- 8|h_3|^2(|h_1|^6-|h_1|^4|h_2|^2-|h_1|^2|h_2|^4+|h_2|^6)\Big]+\\
&- \frac{4}{N^6}\Big[10|h_3|^8+32|h_3|^6(|h_1|^2+|h_2|^2)-8|h_3|^4|h_1|^2|h_2|^2\Big]+\frac{256}{N^8}|h_3|^8
\end{aligned}
$$

From this formula one can easily obtain the explicit value of the f function in (3.3.24):

$$
\begin{aligned}
f &= 8\Big[a_1^4+8a_1^3b_1+6a_1^2b_1^2+8a_1b_1^3+b_1^4-2(a_1+b_1)(3a_1^2+2a_1b_1+3b_1^2)c_1+\\
&+ 2(3a_1^2+4a_1b_1+3b_1^2)c_1^2-4(a_1+b_1)c_1^3+c_1^4\Big]+\frac{8}{N^2}\Big[8(a_1-b_1)^2(a_1+b_1)c_1+\\
&- 12(a_1^2+a_1b_1+b_1^2)c_1^2+20(a_1+b_1)c_1^3-5c_1^4\Big]+\frac{8}{N^4}\Big[8a_1b_1c_1^2-32(a_1+b_1)c_1^3-10c_1^4\Big]+\\
&+ \frac{512}{N^6}c_1^4
\end{aligned}
$$

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