WALL REPULSION AND MUTUAL INTERFACE REPULSION: AN HARMONIC CRYSTAL MODEL IN HIGH DIMENSIONS

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ABSTRACT. We consider two independent lattice harmonic crystals in dimension $d \geq 3$ constrained to live in the upper half plane and to lie one above the other in a large region. We identify the leading order asymptotics of this model, both from the point of view of probability estimates and of pathwise behavior: this gives a rather complete picture of the phenomenon via a detailed analysis of the underlying entropy–energy competition. From the technical viewpoint, with respect to earlier work on sharp constants for harmonic entropic repulsion, this model is lacking certain monotonicity properties and the main tool that allows to overcome this difficulty is the comparison with suitable *rough substrate* models.

1. INTRODUCTION

1.1. **The model.** An *harmonic crystal* or *lattice free field* on $\mathbb{Z}^d$ is the centered Gaussian field $\varphi = \{\varphi_x\}_{x \in \mathbb{Z}^d}$, $d \geq 3$, with covariance $\mathrm{cov}(\varphi_x, \varphi_y) = G(x, y)$, where $G(\cdot, \cdot) = \chi G^\star(\cdot, \cdot)$; χ is a positive number and $G^\star(\cdot, \cdot)$ is the Green function of a random walk $\{X_j\}_{j=0,1,\dots}$ on $\mathbb{Z}^d$ with 1–step transition probability $Q : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, 1]$. The following properties are satisfied:

- Q is symmetric and shift invariant: for every $x, y \in \mathbb{Z}^d$

$$Q(x, y) = Q(y, x) \quad \text{and} \quad Q(x, y) = Q(x - y, 0); \tag{1.1}$$

- The walk has finite range: there exists a positive integer R such that $Q(0, x) = 0$ if $|x| \geq R$;
- The walk is *irreducible* and $Q(0, 0) = 0$.

It is well know that under these conditions the walk is transient and the Green function is finite. Let us recall (notationally) different expressions of the Green function:

$$G^\star(x, y) = \sum_{j=0}^{\infty} \mathbf{P}_x (X_j = y) = \sum_{j=0}^{\infty} Q^j(x, y) = (-\Delta_Q)^{-1} (x, y), \tag{1.2}$$

where we introduce the notation $\mathbf{P}_x (\mathbf{E}_x)$ for the law (the expectation) of X when $X_0 = x$. Moreover $\Delta_Q = Q - \mathbb{1}$. Given Q, we introduce an alternative to the Euclidean norm on $\mathbb{R}^d$: for $r \in \mathbb{R}^d$

$$|r|_Q = \sqrt{\sum_{x \in \mathbb{Z}^d} (x \cdot r)^2 Q(0, x)}. \tag{1.3}$$

We associate to Q the symmetric $d \times d$ matrix M_Q defined by requiring $|r|_Q^2 = (r, M_Q r)$ for every r. Of course, M_Q is invertible.

We work with two independent harmonic crystals: $\varphi = \{\varphi_x\}_x$ and $\psi = \{\psi_x\}_x$. The covariance of φ will be denoted by $G_1(\cdot, \cdot)$ and it is proportional to the Green function of a random walk

with transition probability Q_1. The quantities referring to ψ will have instead the subscript 2. We may choose to represent (φ, ψ) on $\Omega = \mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d}$: $\mathbb{R}$ is equipped with the standard Euclidean topology, Ω is equipped with the product topology and the σ–algebra that we choose for $\mathbb{R}$ and Ω, unless otherwise stated, is the Borel one, denoted as $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(\Omega)$. On $(\Omega, \mathcal{B}(\Omega))$ the probability measure will be simply denoted by $\mathbb{P}$ ($\mathbb{E}$ for the expectation). In particular $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$.

We define the event

$$\Omega_A^+ = \{(\varphi, \psi) \,:\, \psi_x \geq \varphi_x \geq 0 \text{ for every } x \in A\}. \tag{1.4}$$

Given D, an open bounded subset of $\mathbb{R}^d$ which contains the origin and whose boundary is piecewise smooth, we use the notation $\Omega_N^+ = \Omega_{D_N}^+$, where $D_N = ND \cap \mathbb{Z}^d$.

We are interested in the asymptotic properties, as $N \to \infty$, of the trajectories of the coupled field (φ, ψ) under the measure $\mathbb{P}(\cdot|\Omega_N^+)$. Needless to say, this is a model for two interfaces that, in the region D_N, are forced not to intersect, with ψ on top of φ, and both cannot enter the negative half–space, which acts as a hard wall. It is well known (see [9] for a review of the literature on the subject) that a hard–wall of linear size N, pushes harmonic crystals, or more general interface models, to infinity as $N \to \infty$. For finite N, the interface is repelled to a typical height $O(\log N)$ above the hard wall. This effect is purely of entropic nature and in fact it goes under the name of *entropic repulsion*. A sharp analysis of this phenomenon in the Gaussian setting (see [2] and [9] for the most updated results) reveals a subtle *energy–entropy competition* that can be unraveled in great detail. Such completeness is of course due to the possibility of performing exact (Gaussian) computations, but an attentive analysis of the arguments reveals another crucial ingredient: the systematic exploitation of the monotonicity properties of the harmonic crystal.

The repulsion effects that we study in our model are more complex, due to the mutual action of the two fields that leads to loosing, partially, the desired monotonicity properties. Multi–interface phenomena is a topic of great interest for which mathematical results are rather limited, even in the $d = 1$ case: one of the ultimate aims in the field is to study *a gas of non intersecting interfaces* confined by external forces or by hard walls in a large domain. We refer in particular to [7] in which the authors consider the case of one interface constrained between two walls as a caricature of this challenging problem. We refer to [7] also for further references on multi–interface phenomena: we stress however that, with respect to the work we just mentioned, our interest is in determining the precise leading asymptotics and not just rough bounds.

1.2. **Main results.** For $i = 1, 2$ define the i–capacity of D as

$$\text{Cap}_i(D) = \inf \left\{ \chi_i^{-1} \, \||\partial f|_{Q_i}\|_2^2 \,:\, f \in C_0^\infty(\mathbb{R}^d; [0, \infty)), \ f(r) = 1 \text{ for every } r \in D \right\}$$

$$= \sup_{f \in L^\infty(D; [0, \infty))} \frac{\left(\int_D f(r)\mathrm{d}r\right)^2}{\int_D \int_D f(r)f(r')\chi_i g_Q(r - r') \,\mathrm{d}r\mathrm{d}r'}, \tag{1.5}$$

in which $\|\cdot\|_2$ is the $\mathbb{L}_2$ norm of $\cdot$, ∂ denotes the gradient in $\mathbb{R}^d$ and

$$g_{Q_i}(r) = \frac{\Gamma(d/2)}{(d - 2)\pi^{d/2}(\det(M_{Q_i}))^{1/2} \left(r, M_{Q_i}^{-1} r\right)^{(d-2)/2}}. \tag{1.6}$$

We stress also that

$$\lim_{x \to \infty} \frac{G_i^\star(0, x)}{g_{Q_i}(x)} = 1. \tag{1.7}$$

The equivalence between the two formulas for the capacity in (1.5) can be found for example in [3] and for the existence of the limit in (1.7) we refer to [12]. Set moreover $G_i = G_i(0, 0)$.

Normally it would be of course more natural to define the i–capacity of D without the χ_i^{-1} factor. We made the choice of introducing this factor in order to keep several formulas, starting already with (1.8) just below, uniform with analogous formulas appearing in related works and because the i–capacity, as we define it, appears naturally in evaluating the *lowest energy cost* for translating φ or ψ in D_N.

Theorem 1.1. *The following two results hold:*

(1) *The Laplace asymptotics of the probability of $\mathbb{P}(\Omega_N^+)$ is identified:*

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N}\log \mathbb{P}\left(\Omega_N^+\right) = -2\left[\left(\sqrt{G_1+G_2}+\sqrt{G_1}\right)^2 \mathrm{Cap}_2(D) + G_1\mathrm{Cap}_1(D)\right]. \qquad (1.8)$$

(2) *For every $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\beta}$ and $\overline{\beta}$ such that $\underline{\alpha} < \sqrt{4G_1} < \overline{\alpha}$ and $\underline{\beta} < \sqrt{4(G_1+G_2)} < \overline{\beta}$ and for every $\varepsilon > 0$ we have*

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\left\{x \in D_N : \frac{\varphi_x}{\sqrt{\log N}} \in (\underline{\alpha},\overline{\alpha}),\ \frac{\psi_x}{\sqrt{\log N}} \in (\underline{\beta},\overline{\beta})\right\}\right| \leq (1-\varepsilon)\left|D_N\right| \Big| \Omega_N^+\right) = 0. \qquad (1.9)$$

1.3. **About the results and the strategy of the proof.** We have already remarked the importance of *monotonicity* properties of the field and the fact that results are for now mostly restricted to the harmonic crystal. The first of these properties is due to the fact that the covariance of the harmonic field is pointwise positive, so the field is *positively correlated* or, in statistical mechanics language, it satisfies the Fortuin–Kasteleyn–Ginibre (FKG) property. Moreover both mean and covariance have a random walk representation that allows more comparison arguments (and, of course, sharp estimates). For the moment there exists no general technique to get sharp constants for more general models and, as a matter of fact, very interesting cases, like the case of the *membrane* fields [9, A.12], are out of reach for the moment (see [11] for some estimates in a broad Gaussian class).

The model we present enjoys some monotonicities, but not all the ones that would allow a direct application of the ideas developed up to now. If we concentrate on probability estimates, cf. part 1 of Theorem 1.1, the most serious obstacles appear in proving the lower bound. This is a specificity of our model, for which random walk representation tools are still applicable, and one should not be lead to think that it is a typical situation.

In order to explain the strategy of the proof let us recall that a measure μ on $\mathbb{R}^A$, $A \subseteq \mathbb{Z}^d$, *stochastically dominates* another measure $\tilde{\mu}$, defined on the same space, if for every non decreasing bounded measurable function f on the partially ordered set $\mathbb{R}^A$ we have $\int f\mathrm{d}\mu \geq \int f\mathrm{d}\tilde{\mu}$ (notation: $\mu \succ \tilde{\mu}$). We say moreover that μ satisfies the FKG property if $\int fg\,\mathrm{d}\mu \geq \int f\mathrm{d}\mu \cdot \int g\,\mathrm{d}\mu$, g a non decreasing function too. Of course the notion of monotonicity of an event E is simply the monotonicity of its indicator function $\mathbf{1}_E$.

Harmonic crystals enjoy the FKG properties, but Ω_N^+, as well as other events associated to the lower bounds arguments that yield sharp results in the cases solved up to now, is not a monotonic event. We overcome this difficulty by writing $\mathbb{P}(\Omega_N^+) = \mathbb{P}(\Omega_N^+|\Omega_{1,N}^+)\mathbb{P}(\Omega_{1,N}^+)$, $\Omega_{1,N}^+ = \{(\varphi,\psi) : \varphi_x \geq 0$ for every $x \in D_N\}$, and then we estimate $\mathbb{P}(\Omega_N^+|\Omega_{1,N}^+)$ by conditioning on φ (sharp estimates on $\mathbb{P}(\Omega_{1,N}^+)$ are already known [5]). Conditioning on φ leads to a single interface repulsion problem, but this time the hard wall is the fixed configuration φ, typical with respect to $\mathbb{P}_1(\cdot|\varphi_x \geq 0$ for $x \in D_N)$ acting on the random field ψ. This is what we call a *rough substrate* repulsion problem: the model enjoys all the monotonicity properties we desire, but it lacks translation invariance and it adds the problem of understanding the effect of rare large excursions of the substrate φ, a strongly

correlated field in entropic repulsion, on the field ψ. It turns out that combining on one side the precise estimates available on $\mathbb{P}_1(\mathrm{d}\varphi|\varphi_x \geq 0$ for $x \in D_N)$ ([5], [8]), Brascamp–Lieb inequalities [6] and the rough substrate estimates of [1], one obtains the optimal lower bound. This a priori is not obvious: notice in particular that [1] deals with the case of a substrate modeled by independent random variables.

One way of understanding why such a procedure yields optimal results is hidden in the pathwise behavior of $\mathbb{P}(\cdot|\Omega_N^+)$, that is part 2 of Theorem 1.1. This pathwise behavior is at first somewhat surprising and it can be informally read as follows:

- There is no *push down* effect of ψ on φ: φ is repelled to the *same* (in the sense of leading asymptotic) height as when ψ is absent;
- We could replace the field φ with Gaussian independent random variables of variance G_1 centered around $\sqrt{4G_1 \log N}$ and the pathwise behavior, in the sense of the result in part 2 of Theorem 1.1, would be unchanged.

Notice that the role of the two underlying walks is somewhat downscaled: while in the probability estimates the asymptotics of the walks are still relevant, the pathwise behavior turns out to be rather universal, since it depends only on the variances.

Still about pathwise estimates, we stress that in Section 4 we prove estimates on the conditioned field which go beyond what we report in part 2 of Theorem 1.1. In general however we have tried to present concise arguments and we did not try to get results that are uniform in x, except for the upper bound on φ, see Proposition 4.2, where the statement is a direct consequence of known results. We stress that getting a uniform estimate on ψ analogous to the one for φ would yield automatically the extension of the results to the case of three interfaces and, by iteration, any finite number. We believe that such a result can be extracted by combining the ideas of [5], of [8] and of what we present here, but it does not appear to be straightforward.

1.4. **More preliminaries, notations and organization of the paper.** We will repeatedly use the following entropy inequality: if P and $\widetilde{P}$ are two probability measures, $\widetilde{P} \ll P$, and E is a positive $\widetilde{P}$–probability event then if we set $H(\widetilde{P}|P) = \widetilde{E}[\log(\mathrm{d}\widetilde{P}/\mathrm{d}P)]$ by Jensen's inequality we have (see e.g. [5] or [9, A.3])

$$\log\left(\frac{P(E)}{\widetilde{P}(E)}\right) \geq -\frac{1}{\widetilde{P}(E)}\left[H(\widetilde{P}|P) + e^{-1}\right]. \tag{1.10}$$

By $\mathcal{F}^\varphi$, respectively $\mathcal{F}^\psi$, we denote the σ–algebra generated by the field φ, respectively ψ. When an event E is in $\mathcal{F}^\varphi$ or in $\mathcal{F}^\psi$, then we will commit frequent abuse of notation by considering it at the same time as a subset of $\mathbb{R}^{\mathbb{Z}^d}$. For example $\Omega_{1,N}^+$ may mean, according to the context, $\{\varphi : \varphi_x \geq 0$ for every $x \in D_N\}$ or $\{(\varphi, \psi) : \varphi_x \geq 0$ for every $x \in D_N\}$. Abuse of notation will also be committed in systematically not distinguishing between random and numerical variables.

The plan of the paper is straightforward: in Section 2 we prove the lower bound corresponding to part 1 of Theorem 1.1, while the upper bound may be found in Section 3. The proof of part 2 is instead in Section 4, split in four propositions.

2. Probability lower bounds

In this section we prove the lower bound for the limit in part 1 of Theorem 1.1. The proof consists of two parts:

(1) First (Lemma 2.1) we exploit the sharp results available for $\mathbb{P}_{1,N}^{+}(\cdot) := \mathbb{P}_1(\cdot|\Omega_{1,N}^{+})$ and a version of the Brascamp–Lieb inequalities to find an upper bound on the upward excursions. The average height of the trajectories of $\mathbb{P}_{1,N}^{+}(\cdot)$ in D_N is $\approx \sqrt{4G_1 \log N}$: the Brascamp–Lieb inequality provides a sharp concentration of the measure and gives a good upper bound on the number of points in which the field is above $\sqrt{a \log N}$, $a > 4G_1$. In short, this step identifies a set $E_N \subset \Omega$ whose probability is close to 1 for N large and for which we have suitable upper bounds on high level ($O(\sqrt{\log N})$) excursions of φ.

(2) Then, in Proposition 2.2, we obtain the desired lower bound on $\mathbb{P}(\Omega_N^{+})$ by separating the problem into estimating from below $\mathbb{P}(\Omega_{1,N}^{+})$, a problem already solved in [5], and $\mathbb{P}_2(\psi_x \geq \varphi_x$ for every $x \in D_N)$ for $\varphi \in E_N$. This last term is clearly an entropic repulsion problem in presence of an inhomogenous wall or, in other words, in presence of a random quenched substrate: we dealt with this kind of estimate in [1, Proposition 2.1] and there are only minor modifications in this case: since a priori it may not be clear to everybody that the problem is the same and since the notations are necessarily rather different we choose to detail these steps.

2.1. Upper bound on the high excursions of $\mathbb{P}_{1,N}^{+}$. We will prove the following result:

Lemma 2.1. *For every $\alpha > 0$ and $\eta > 0$*

$$\lim_{N \to \infty} \mathbb{P}_{1,N}^{+}\left(\left|\left\{x \in D_N : \varphi_x \geq (\sqrt{\alpha} + \sqrt{4G_1 + \eta})\sqrt{\log N}\right\}\right| \geq |D_N| \, N^{-\frac{\alpha}{2G_1}}\right) = 0. \qquad (2.1)$$

Proof. Let us recall the following two results:

- A uniform asymptotic control on $\mathbb{E}_{1,N}^{+}[\varphi_x]$ is known:

$$\lim_{N \to \infty} \sup_{x \in D_N} \left| \frac{\mathbb{E}_{1,N}^{+}[\varphi_x] - \sqrt{4G_1 \log N}}{\sqrt{\log N}} \right| = 0. \qquad (2.2)$$

  This result is proven in [8, Lemma 3.3] for the basic case of $Q_1(x,y) = 1/2d$ if $|y - x| = 1$. The generalization to the finite range case considered here is a lengthy book–keeping exercise that we leave to the interested reader. Here we will make use only of the upper bound on $\mathbb{E}_{1,N}^{+}[\varphi_x]$ corresponding to the full estimate in (2.2).

- Set $M_N^{+}(x) = \min\{t : \mathbb{P}_{1,N}^{+}(\varphi_x \leq t) \geq 1/2\}$, that is $M_N^{+}(x)$ is the (unique) median of φ_x when the latter is distributed according to $\mathbb{P}_{1,N}^{+}$. By [10, Theorem 1.1] we have that for every positive β

$$\mathbb{P}_{1,N}^{+}\left(\varphi_x - M_N^{+}(x) \geq \beta\right) \leq \mathbb{P}\left(\varphi_x \geq \beta\right), \qquad (2.3)$$

  and by [10, Remark 2.3] we know that mean and median of log–concave perturbations of Gaussian measures cannot be too far from each other:

$$\sup_{N} \left| \mathbb{E}_{1,N}^{+}[\varphi_x] - M_N^{+}(x) \right| \leq \sqrt{2G_1/\pi}. \qquad (2.4)$$

By combining the results we just stated and an elementary upper bound on the tail of a Gaussian random variable we directly obtain that for every $\eta > 0$ we can find N_0 such that if $N \geq N_0$

$$\sup_{x \in D_N} \mathbb{P}_{1,N}^{+}\left(\varphi_x \geq (\sqrt{\alpha} + \sqrt{4G_1 + \eta})\sqrt{\log N}\right) \leq \mathbb{P}(\varphi_0 \geq \sqrt{\alpha \log N}) \leq \left(\frac{G_1}{2\pi\alpha \log N}\right)^{1/2} N^{-\alpha/(2G_1)}, \qquad (2.5)$$

for every $\alpha > 0$. It is therefore clear that, with obvious definition of c, we have

$$\mathbb{E}_{1,N}^{+}\left[\left|\left\{x \in D_N : \varphi_x \geq (\sqrt{\alpha} + \sqrt{4G_1 + \eta})\sqrt{\log N}\right\}\right|\right] \leq \frac{c}{\sqrt{\alpha \log N}}|D_N| N^{-\frac{\alpha}{2G_1}}, \qquad (2.6)$$

for N sufficiently large. We now apply the Markov inequality and the proof is complete. $\qquad\square$

2.2. **The lower bound via quenched estimates.** We are now ready to prove the lower bound for the limit in part 1 of Theorem 1.1.

Proposition 2.2.

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}\log N}\log \mathbb{P}\left(\Omega_N^{+}\right) \geq -2\left[\left(\sqrt{G_1 + G_2} + \sqrt{G_1}\right)^2 \mathrm{Cap}_2(D) + G_1\mathrm{Cap}_1(D)\right]. \qquad (2.7)$$

Proof. We define an auxiliary field $\widetilde{\varphi}$ as a function of φ. Fix a large integer $\overline{k}$ and define $\theta = \sqrt{4G_1}(1 + (1/2\overline{k}))/\overline{k}$, $\widetilde{k} = \lfloor(\sqrt{3dG_1})/\theta\rfloor$, $\gamma = \sqrt{4G_1 + \eta}$ ($\eta > 0$) and

$$\widetilde{\varphi}_x = \begin{cases} (\gamma + \theta)\sqrt{\log N}, & \text{if } \varphi_x \leq (\gamma + \theta)\sqrt{\log N}, \\ (\gamma + k\theta)\sqrt{\log N}, & \text{if } \varphi_x \in ((\gamma + (k-1)\theta)\sqrt{\log N}, (\gamma + k\theta)\sqrt{\log N}] \text{ for } k = 2, 3, \ldots, \overline{k}, \\ (\gamma + \widetilde{k}\theta)\sqrt{\log N}, & \text{if } \varphi_x \in ((\gamma + \overline{k}\theta)\sqrt{\log N}, (\gamma + \widetilde{k}\theta)\sqrt{\log N}], \\ \infty & \text{if } \varphi_x > (\gamma + \widetilde{k}\theta)\sqrt{\log N}, \end{cases}$$
$$(2.8)$$

and set $L_N(k) = \{x \in D_N : \widetilde{\varphi}_x = (\gamma + k\theta)\sqrt{\log N}\}$, $N_k = |L_N(k)|$, for $k \in \{1, 2, \ldots, \overline{k}, \widetilde{k}, \infty\}$. Observe that if $\varphi \in \{N_\infty = 0\}$, then $\widetilde{\varphi} \geq \varphi$.

Let us introduce the set $E_N \in \mathcal{F}^{\varphi}$ specified by the following three conditions:

$$\begin{aligned} N_k &\leq cN^{d - \frac{(k-1)^2\theta^2}{2G_1}} \qquad \text{for all } k = 1, \ldots, \overline{k}, \\ N_{\widetilde{k}} &\leq cN^{d - \frac{\overline{k}^2\theta^2}{2G_1}}, \\ N_\infty &= 0. \end{aligned} \qquad (2.9)$$

By Lemma 2.1 we can choose $c = c(D)$ so that $\mathbb{P}_{1,N}^{+}(E_N)$ tends to 1 as N tends to infinity. From now on we choose N such that $\mathbb{P}_{1,N}^{+}(E_N) \geq 1/2$.

Now observe that

$$\begin{aligned} \mathbb{P}\left(\psi_x \geq \varphi_x \geq 0 \text{ for } x \in D_N\right) &\geq \mathbb{P}\left(\{\psi_x \geq \varphi_x \geq 0 \text{ for } x \in D_N\} \cap E_N\right) \\ &\geq \mathbb{P}\left(\{\psi_x \geq \widetilde{\varphi}_x, \varphi_x \geq 0 \text{ for } x \in D_N\} \cap E_N\right) \\ &= \mathbb{E}\left(\mathbb{P}_2\left(\psi_x \geq \widetilde{\varphi}_x \text{ for } x \in D_N\right)(\varphi); E_N \big| \Omega_{1,N}^{+}\right) \mathbb{P}\left(\Omega_{1,N}^{+}\right) \qquad (2.10) \\ &\geq \frac{1}{2} \inf_{\varphi \in E_N} \mathbb{P}_2\left(\psi_x \geq \widetilde{\varphi}_x \text{ for } x \in D_N\right) \mathbb{P}(\Omega_{1,N}^{+}). \end{aligned}$$

Let us simplify a bit the notation by setting $\widetilde{E}_N = \{\widetilde{\varphi}(\varphi) : \varphi \in E_N\}$. Since, by [5, Theorem 1.1], $\log \mathbb{P}(\Omega_{1,N}^{+})$ is asymptotic to $-2G_1\mathrm{Cap}_1(D)N^{d-2}\log N$, it is enough to show that for every $\widetilde{\varphi} \in \widetilde{E}_N$ we have

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}\log N}\log \mathbb{P}_2(\psi_x \geq \widetilde{\varphi}_x, \text{ for } x \in D_N) \geq -2\left(\sqrt{G_1} + \sqrt{G_1 + G_2}\right)^2 \mathrm{Cap}_2(D). \qquad (2.11)$$

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Note that, by the FKG inequality, we have that

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}\log N}\log\mathbb{P}_2\left(\psi_x \geq \widetilde{\varphi}_x, \text{ for } x \in D_N\right) \geq$$

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}\log N}\log\mathbb{P}_2\left(F\right) + \liminf_{N\to\infty} \frac{1}{N^{d-2}\log N}\log\mathbb{P}_2\left(\psi_x \geq \widetilde{\varphi}_x \text{ for } x \in D_N^-\right)$$

$$=: \mathbf{T}_1 + \mathbf{T}_2, \quad (2.12)$$

where $F := \{\psi : \psi_x \geq (\gamma + \widetilde{k}\theta)\sqrt{\log N}, x \in L_N(\widetilde{k})\}$, $D_N^- = D_N \setminus L_N(\widetilde{k})$.

We first show that $\mathbf{T}_1 = 0$. Let T_σ be defined by $(T_\sigma\varphi)_x = \varphi_x + \sigma_x$, for $\sigma \in \Omega_N$ and $x \in \mathbb{Z}^d$, where $\sigma_x = \sqrt{2G_2(d+2)\log N} + (\gamma + \widetilde{k}\theta)\sqrt{\log N}$ for $x \in L_N(\widetilde{k})$ and $\sigma_x = 0$ otherwise. Then by direct computation one shows that

$$H\left(\mathbb{P}_2 T_\sigma^{-1}|\mathbb{P}_2\right) \leq \chi_2^{-1}\left(\sqrt{2G_2(d+2)} + \gamma + \widetilde{k}\theta\right)^2 N_{\widetilde{k}}\log N. \quad (2.13)$$

By the definition of θ and (2.9) we have that for N and $\overline{k}$ sufficiently large

$$H\left(\mathbb{P}_2 T_\sigma^{-1}|\mathbb{P}_2\right) \leq N^{d-2(1+(1/3\overline{k}))^2}. \quad (2.14)$$

Moreover by using the FKG inequality we obtain that for N sufficiently large

$$\mathbb{P}_2 T_\sigma^{-1}\left(F\right) = \mathbb{P}_2\left(\psi_x \geq -\sqrt{2G_2(d+2)\log N} \text{ for every } x \in L_N(\widetilde{k})\right)$$

$$\geq \prod_{x \in L_N(\widetilde{k})} \mathbb{P}_2\left(\psi_x \geq -\sqrt{2G_2(d+2)\log N}\right) \geq (1 - (1/N^{d+1}))^{N^d} \geq 1/2, \quad (2.15)$$

and therefore by applying the entropy inequality (1.10) we obtain

$$\mathbb{P}_2\left(F\right) \geq \exp\left(-N^{d-2(1+(1/4\overline{k}))^2}\right), \quad (2.16)$$

for sufficiently large N, which shows that $\mathbf{T}_1 = 0$.

We are left with evaluating $\mathbf{T}_2$. Set $\alpha_N = \alpha\sqrt{\log N}$, $\alpha > 0$, $(\sigma_N)_x = \alpha_N f(x/N)$, $f \in C_0^\infty(\mathbb{R}^d; [0, \infty))$ and $f(r) = 1$ if $r \in D$. Let $\mathbb{P}_{2,N} = \mathbb{P}_2 T_{\sigma_N}^{-1}$ and $\widetilde{\mathbb{P}}_{2,N}(\cdot) = \mathbb{P}_{2,N}(\cdot|\widetilde{F})$, where $\widetilde{F} = \{\psi_x \geq \widetilde{\varphi}_x, x \in D_N^-\}$. Therefore $\mathrm{d}\widetilde{\mathbb{P}}_{2,N}/\mathrm{d}\mathbb{P}_2 = (\mathrm{d}\widetilde{\mathbb{P}}_{2,N}/\mathrm{d}\mathbb{P}_{2,N})(\mathrm{d}\mathbb{P}_{2,N}/\mathrm{d}\mathbb{P}_2)$ and by the entropy inequality (1.10) we have

$$\log\mathbb{P}_2\left(\widetilde{F}\right) \geq -H\left(\widetilde{\mathbb{P}}_{2,N}|\mathbb{P}_2\right) - e^{-1}$$

$$= -H\left(\widetilde{\mathbb{P}}_{2,N}|\mathbb{P}_{2,N}\right) - \widetilde{\mathbb{E}}_{2,N}\left(\log\left(\frac{\mathrm{d}\mathbb{P}_{2,N}}{\mathrm{d}\mathbb{P}_2}\right)\right) - e^{-1} =: -H_1 - H_2 - e^{-1}. \quad (2.17)$$

First of all by direct evaluation and FKG we have

$$H_1 = -\log\mathbb{P}_2 T_{\sigma_N}^{-1}\left(\widetilde{F}\right) \leq -\sum_{k=1}^{\overline{k}} N_k \log\left(1 - \mathbb{P}_{2,N}\left(\psi_0 < (\gamma + k\theta - \alpha)\sqrt{\log N}\right)\right). \quad (2.18)$$

One checks directly that if

$$\alpha > \gamma + \overline{k}\theta = \sqrt{4G_1 + \eta} + \sqrt{4G_1}\left(1 + \frac{1}{2\overline{k}}\right), \quad (2.19)$$

and

$$\frac{(k-1)^2\theta^2}{2G_1} + \frac{(\sqrt{4G_1 + \eta} + k\theta - \alpha)^2}{2G_2} > 2, \quad (2.20)$$

7

for all $k \leq \overline{k}$, then each of the $\overline{k}$ summands in (2.18) is $o(N^{d-2})$, and therefore negligible:

$$\lim_{N \to \infty} \frac{1}{N^{d-2}} H_1 = 0. \tag{2.21}$$

Observe that, by (2.19) and (2.20), a more explicit assumption that implies (2.21) is

$$\alpha > \sqrt{4G_1 + \eta} + k\theta + \sqrt{4G_2 - (k-1)^2\theta^2 \left(\frac{G_2}{G_1}\right)}, \quad \text{for every } k \leq \overline{k}. \tag{2.22}$$

It is then easy to see that (2.19) and (2.22) are satisfied if

$$\alpha > \sqrt{4G_1 + \eta} + \sqrt{4(G_1 + G_2)} + \theta. \tag{2.23}$$

Therefore under this hypothesis on α the estimate (2.21) holds.

Let us consider H_2: observe that

$$\log\left(\frac{d\mathbb{P}_{2,N}}{d\mathbb{P}_2}(T_{\sigma_N}\psi)\right) = \frac{\alpha_N^2}{2}\sum_x f(x/N)\sum_y \left(f(x/N) - f(y/N)\right)\chi_2^{-1}Q_2(x,y)$$
$$+ \alpha_N \sum_x f(x/N)\sum_y (\psi_x - \psi_y)\chi_2^{-1}Q_2(x,y), \tag{2.24}$$

and therefore

$$\frac{H_2}{N^{d-2}\log N} = \frac{\alpha_N^2}{N^d \log N}\sum_x Nf(x/N)\sum_y N\left(f(x/N) - f(y/N)\right)\chi_2^{-1}Q_2(x,y)$$
$$+ \frac{\alpha_N}{N^{d-2}\log N}\mathbb{E}_2\left[\sum_x f(x/N)\sum_y (\psi_x - \psi_y)\chi_2^{-1}Q_2(x,y)\Big| T_{\sigma_N}^{-1}\widetilde{F}\right] \tag{2.25}$$
$$=: C_N + R_N.$$

It is easy to see that C_N converges for $N \to \infty$ to $\alpha^2\chi_2^{-1}\|\|\partial f|_{Q_2}\|^2/2$. We show now that $\lim_{N \to \infty} R_N = 0$ if (2.23) holds. Observe in fact that

$$\sum_x f(x/N)\sum_y (\psi_x - \psi_y)\chi_2^{-1}Q_2(x,y) = -\chi_2^{-1}\sum_x (\Delta_2 f(\cdot/N))(x)\psi_x \sim \mathcal{N}\left(0, s_N^2\right), \tag{2.26}$$

where $s_N^2 = \chi_2^{-2}(\|\partial f|_{Q_2}\|_2^2/2)N^{d-2}(1 + o(1))$. We use now the following consequence of Jensen inequality (Y a random variable, E a positive probability event, $t > 0$)

$$\mathbb{E}\left[Y|E\right] \leq \frac{1}{t}\log\mathbb{E}\left[\exp(tY)\right] - \frac{1}{t}\log\mathbb{P}\left(E\right), \tag{2.27}$$

to obtain with $t = N^{d-2}$ that

$$|R_N| \leq \frac{1}{t}\log\mathbb{E}_2\left[\exp\left(\frac{t\alpha_N}{N^{d-2}\log N}\sum_x\sum_y f(x/N)(\psi_x - \psi_y)\chi_2^{-1}Q_2(x,y)\right)\right] - \frac{1}{t}\log\mathbb{P}_2\left(T_{\sigma_N}^{-1}\widetilde{F}\right)$$
$$\leq \frac{\alpha^2 s_N^2}{2N^{d-2}\log N} + \frac{H_1}{N^{d-2}} = o(1). \tag{2.28}$$

This shows that under the hypothesis (2.23) on α

$$\lim_{N \to \infty}\frac{1}{N^{d-2}\log N}H_2 = \frac{\alpha^2}{2}\chi_2^{-1}\|\|\partial f|_{Q_2}\|_2^2. \tag{2.29}$$

The thesis is obtained by optimizing the choices of f and α, by the definition of the capacity (1.5) and using the fact that θ and η can be taken arbitrarily small. $\qquad\square$

3. Probability upper bounds

Proposition 3.1.

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}\log N}\log\mathbb{P}\left(\Omega_N^+\right) \leq -2\left[\left(\sqrt{G_1+G_2}+\sqrt{G_1}\right)^2\mathrm{Cap}_2(D) + G_1\mathrm{Cap}_1(D)\right]. \quad (3.1)$$

Proof. Let us choose an even natural number L larger than $2R$, recall that R is larger than the ranges of the random walks, and for $y \in 2L\mathbb{Z}^d$ let us set

$$B(y) = B_L(y) = \left\{x : \max_{i=1,\ldots,d}|x_i - y_i| \in [L/2, L/2+R]\right\}, \quad (3.2)$$

and Λ_c is the set of $y \in 2L\mathbb{Z}^d$ such that $B(y) \subset D_N$. Set also $\Lambda = \cup_{y\in\Lambda_c}B(y)$. We denote by $\mathcal{F}_A^{\varphi,\psi}$ the σ–algebra generated by φ_x and ψ_x, $x \in A$. We have that

$$\begin{aligned}
\mathbb{P}\left(\Omega_N^+\right) &\leq \mathbb{P}\left(\Omega_{\Lambda\cup\Lambda_c}^+\right) \\
&= \mathbb{E}\left[\prod_{y\in\Lambda_c}\mathbb{P}\left(\psi_y \geq \varphi_y \geq 0\big|\mathcal{F}_{B(y)}^{\varphi,\psi}\right); \Omega_\Lambda^+\right],
\end{aligned} \quad (3.3)$$

in which we used the Markov property of the fields φ and ψ.

Observe that, if $y \in \Lambda_c$ is fixed, $q^{(i)}(z) = q_L^{(i)}(z)$ is the probability that the Q_i–random walk leaving at y hits $B(y)$ at z and $M_y^{(i)}(\sigma) = \sum_{z\in B(y)}q^{(i)}(z)\sigma_z$, $\sigma \in \mathbb{R}^{\mathbb{Z}^d}$, the law of φ_y and ψ_y, conditioned to $\mathcal{F}_{B(y)}^{\varphi,\psi}$, is the law of two independent Gaussian random fields of mean $M_y^{(1)}(\varphi)$ and $M_y^{(2)}(\psi)$ respectively, and of variance $G_{1,L}$ and $G_{2,L}$ respectively (positive numbers with the property that $G_{i,L} \nearrow G_i$ as $L \nearrow \infty$).

We now choose a small positive number κ and consider the *inner κ–discretization* of D: that is for $r \in \kappa\mathbb{Z}^d$, set $A_r = r + [0,\kappa)^d$ and define $I = \{r \in \kappa\mathbb{Z}^d : A_r \subset D\}$ (assume $I \neq \emptyset$). We are interested in this decomposition at the lattice level or, more precisely, on the $2L$–rarefied lattice level (the sublattice Λ_c of centers): so define $C_r = NA_r\cap\Lambda_c$ and remark that $c_{\kappa,N} = |C_r|/(N\kappa/2L)^d$ tends to a finite non zero limit as $N \to \infty$ and then $\kappa \to 0$.

For $\eta, \alpha, \beta > 0$ let us now consider the events

$$\begin{aligned}
E_{\eta,\alpha} &= \left\{\text{there exists } r \in I \text{ such that } |\{y \in C_r : M_y^{(1)}(\varphi) \leq \sqrt{\alpha\log N}\}| \geq \eta|C_r|\right\}, \\
F_{\eta,\beta} &= \left\{\text{there exists } r \in I \text{ such that } |\{y \in C_r : M_y^{(2)}(\psi) - M_y^{(1)}(\varphi) \leq \sqrt{\beta\log N}\}| \geq \eta|C_r|\right\}.
\end{aligned} \quad (3.4)$$

For what follows η will be chosen smaller than $1/4$.

Observe that on $E_{\eta,\alpha}$

$$
\begin{aligned}
\prod_{y \in \Lambda_c} \mathbb{P}\left(\psi_y \geq \varphi_y \geq 0 \big| \mathcal{F}_{B(y)}^{\varphi,\psi}\right) &\leq \prod_{y \in \Lambda_c} \mathbb{P}\left(\varphi_y \geq 0 \big| \mathcal{F}_{B(y)}^{\varphi,\psi}\right) \\
&= \prod_{y \in \Lambda_c}\left(1 - \Phi\left(-\frac{M_y^{(1)}(\varphi)}{\sqrt{G_{1,L}}}\right)\right) \\
&\leq \left(1 - \Phi\left(-\sqrt{\frac{\alpha \log N}{G_{1,L}}}\right)\right)^{\eta|C_r|},
\end{aligned}
\tag{3.5}
$$

where r is an arbitrary element in I. Then for N sufficiently large and suitable choices of positive constants c' and c'' we have that

$$
\begin{aligned}
\mathbb{E}\left[\prod_{y \in \Lambda_c} \mathbb{P}\left(\varphi_y \geq 0 \big| \mathcal{F}_{B(y)}\right); E_{\eta,\alpha}\right] &\leq \left(1 - \frac{c'}{\sqrt{(\alpha \log N / G_{1,L})}} N^{-\alpha/2G_{1,L}}\right)^{c_{\kappa,N}\eta(N\kappa/2L)^d} \\
&\leq \exp\left(-c'' N^{d-(\alpha/2G_{1,L})}\right),
\end{aligned}
\tag{3.6}
$$

which is negligible (recall that we want to prove (3.1)) if $\alpha < 4G_{1,L}$. In an analogous way, one proves that one may choose $c > 0$ such that

$$
\begin{aligned}
\mathbb{E}\left[\prod_{y \in \Lambda_c} \mathbb{P}\left(\psi_y \geq \varphi_y \geq 0 \big| \mathcal{F}_{B(y)}\right); F_{\eta,\beta}\right] &\leq \left(1 - \Phi\left(-\sqrt{\frac{\beta \log N}{G_{1,L} + G_{2,L}}}\right)\right)^{\eta|C_r|} \\
&\leq \exp\left(-c N^{d-(\beta/(G_{1,L}+G_{2,L}))}\right),
\end{aligned}
\tag{3.7}
$$

which is negligible if $\beta < 4(G_{1,L} + G_{2,L})$.

Let us then assume that $\alpha < 4G_{1,L}$ and $\beta < 4(G_{1,L} + G_{2,L})$. We may therefore replace the event Ω_Λ^+ with $\Omega_\Lambda^+ \cap E_{\eta,\alpha}^{\complement} \cap F_{\eta,\beta}^{\complement}$ in the rightmost expression of (3.3).

If $(\varphi, \psi) \in \Omega_\Lambda^+ \cap E_{\eta,\alpha}^{\complement} \cap F_{\eta,\beta}^{\complement}$ then for every $r \in I$ there are at least $(1 - 2\eta)|C_r|$ sites $y \in C_r$ such that $M_y^{(1)}(\varphi) > \sqrt{\alpha \log N}$ and $M_y^{(2)}(\psi) - M_y^{(1)}(\varphi) > \sqrt{\beta \log N}$ and in the remaining sites $M_y^{(1)}(\varphi) \geq 0$ and $M_y^{(2)}(\psi) - M_y^{(1)}(\varphi) \geq 0$. This in turn implies that

$$
M_y^{(1)}(\varphi) > \sqrt{\alpha \log N} \quad \text{and} \quad M_y^{(2)}(\psi) > \left(\sqrt{\beta} + \sqrt{\alpha}\right)\sqrt{\log N},
\tag{3.8}
$$

for at least $(1 - 4\eta)|C_r|$ sites $y \in C_r$, and $M_y^{(1)}(\varphi) \geq 0$ and $M_y^{(2)}(\psi) \geq 0$ elsewhere. Therefore for every choice of $f_r \geq 0$, $\widetilde{f}_r \geq 0$, $r \in I$, (on $\Omega_\Lambda^+ \cap E_{\eta,\alpha}^{\complement} \cap F_{\eta,\beta}^{\complement}$) we have that

$$
\sum_{r \in I} f_r \frac{1}{|C_r|} \sum_{y \in C_r} M_y^{(1)}(\varphi) \geq (1 - 4\eta)\sqrt{\alpha \log N} \sum_{r \in I} f_r
\tag{3.9}
$$

$$
\sum_{r \in I} \widetilde{f}_r \frac{1}{|C_r|} \sum_{y \in C_r} M_y^{(2)}(\psi) \geq (1 - 4\eta)\left(\sqrt{\beta} + \sqrt{\alpha}\right)\sqrt{\log N} \sum_{r \in I} \widetilde{f}_r.
\tag{3.10}
$$

Therefore it suffices to find an upper bound on the probability that (3.9) and (3.10) happen together. Two observations are in order: first it suffices to treat the probability of (3.9) independently of (3.10) (φ and ψ are independent!) and, secondly, this two problems are effectively only one problem

(a Gaussian computation), that has been already treated in detail in [9, §3.6], see also [1] and [5]. We sum up the net result: if we set for $f \in L^\infty(D; [0, \infty))$

$$\mathbf{C}_i(f) = \frac{\left(\int_D f(r)\mathrm{d}r\right)^2}{\int_D \int_D f(r)f(r')\chi_i g_Q(r - r')\,\mathrm{d}r\mathrm{d}r'}, \tag{3.11}$$

then for every $\alpha < 4G_{1,L}$, $\beta < 4(G_{1,L} + G_{2,L})$, we have that

$$\limsup_{N \to \infty} \frac{1}{N^{d-2} \log N} \log \mathbb{P}(\Omega_N^+) \leq -(1 - 4\eta)^2 \left(\frac{(\sqrt{\beta} + \sqrt{\alpha})^2}{2} \mathbf{C}_2(\widetilde{f}) + \frac{\alpha}{2} \mathbf{C}_1(f)\right), \tag{3.12}$$

for every $f, \widetilde{f} \in L^\infty(D; [0, \infty))$. Let then $\alpha \nearrow 4G_{1,L}$, $\beta \nearrow 4(G_{1,L} + G_{2,L})$, $L \nearrow \infty$, $\eta \searrow 0$ $\kappa \searrow 0$, optimize over the choice of f and $\tilde{f}$ to recover the i–capacities of D (recall (1.5)) and the proof of Proposition 3.1 is complete. $\qquad\square$

4. Repulsion phenomena

In this section we prove part 2 of Theorem 1.1, along with some further (and sharper) estimates. We will use the compact notation $\mathbb{P}_N^+(\cdot)$ for $\mathbb{P}(\cdot|\Omega_N^+)$.

4.1. **Lower and upper bounds for φ.** Both the lower and the upper bound for φ claimed in part 2 of Theorem 1.1 are obtained by reducing the problem to known pathwise estimates on the repulsion action of a wall on a single interface.

Proposition 4.1. *For all $\delta > 0$ and $a < \sqrt{4G_1}$,*

$$\lim_{N \to \infty} \mathbb{P}_N^+ \left(|\{x \in D_N : \varphi_x < \sqrt{a \log N}\}| \geq \delta|D_N|\right) = 0. \tag{4.1}$$

Proof. Set

$$A_N = \left\{(\varphi, \psi) : |\{x \in D_N : \varphi_x < \sqrt{a \log N}\}| \geq \delta|D_N|\right\}, \tag{4.2}$$

and observe that

$$\mathbb{P}_N^+ (A_N) \leq \frac{\mathbb{P}\left(A_N \cap \Omega_{1,N}^+\right)}{\mathbb{P}(\Omega_N^+)}. \tag{4.3}$$

From part 1 of Theorem 1.1 we know that

$$\mathbb{P}(\Omega_N^+) \geq \exp(-cN^{d-2} \log N), \tag{4.4}$$

for $c > 2\left[\left(\sqrt{G_1 + G_2} + \sqrt{G_1}\right)^2 \mathrm{Cap}_2(D) + G_1 \mathrm{Cap}_1(D)\right]$ and N sufficiently large. On the other hand, since $a < \sqrt{4G_1}$, there exists $\epsilon > 0$ such that

$$\mathbb{P}(A_N \cap \Omega_{1,N}^+) \leq \exp(-N^{d-2+\epsilon}), \tag{4.5}$$

for N sufficiently large. This result is a statement involving φ alone and, while not explicitly stated, it has been already established in [5, Section 4], see [9, Prop. 3.2] for a more explicit and concise treatment. We remark also that, strictly speaking, the needed argument is also included in this work, but in the slightly more involved context of proving a lower bound for ψ (see footnote in the proof of Proposition 4.3).

By inserting (4.4) and (4.5) into (4.3), we see $\mathbb{P}_N^+(A_N)$ vanishes as N tends to infinity and the proof is complete. $\qquad\square$

Proposition 4.2. *We have that*

$$\limsup_{N \to \infty} \sup_{x \in D_N} \frac{\mathbb{E}_N^+ [\varphi_x]}{\sqrt{\log N}} \leq \sqrt{4G_1}, \tag{4.6}$$

and for all $\delta > 0$ and $b > \sqrt{4G_1}$,

$$\lim_{N \to \infty} \mathbb{P}_N^+ \left(|\{x \in D_N : \varphi_x > \sqrt{b \log N}\}| \geq \delta |D_N| \right) = 0. \tag{4.7}$$

Proof. Here we simply observe that for every ψ such that $\psi_x \geq 0$ and every increasing event $B \in \mathcal{F}^\psi$

$$\mathbb{P}_N^+ \left(B \big| \mathcal{F}^\psi \right) (\psi) \leq \mathbb{P}_1 \left(B \big| \Omega_{1,N}^+ \right). \tag{4.8}$$

The term on the left is in fact equal to $\mathbb{P}_1(\cdot | 0 \leq \varphi_x \leq \psi_x$ for $x \in D_N)$, with ψ a fixed configuration and φ random: what (4.8) is saying is therefore that the field φ constrained between two walls is dominated by φ constrained just from below, see [9, Appendix B] for a proof. Therefore (4.6) follows directly from [5, Lemma 4.7] and (4.7) from [5, Proposition 4.1]. As a side remark, (4.7) follows from (4.6) by using Brascamp–Lieb inequalities, cf. [8], [9] and [10]. $\square$

4.2. Lower bounds for ψ.

Proposition 4.3. *For all $\delta > 0$ and $a < \sqrt{4G_1} + \sqrt{4(G_1 + G_2)}$,*

$$\lim_{N \to \infty} \mathbb{P}_N^+ \left(|\{x \in D_N : \psi_x < a\sqrt{\log N}\}| \geq \delta |D_N| \right) = 0. \tag{4.9}$$

Proof. The thesis is proven once we show that

$$\lim_{N \to \infty} \mathbb{P}_N^+ \left(|\{x \in \Lambda_c : \psi_x < a\sqrt{\log N}\}| \geq \delta |\Lambda_c| \right) = 0, \tag{4.10}$$

since it suffices to repeat a finite number of times, proportional to L^d, the same estimate by shifting the L–subgrid and the corresponding set of centers.

Call B_N the event whose $\mathbb{P}_N^+$–probability is evaluated in (4.10): in view of the lower bound on the probability of Ω_N^+ (cf. part 1 of Theorem 1.1), it suffices to show that for every $a < \sqrt{4G_1} + \sqrt{4(G_1 + G_2)}$

$$\limsup_{N \to \infty} \frac{1}{N^{d-2} \log N} \log \mathbb{P} \left(B_N \cap \Omega_N^+ \right) \leq -C, \tag{4.11}$$

for a sufficiently large C. We are going to prove (4.11) with $C = +\infty$[1].

The next step is to remark that

$$\mathbb{P} \left(B_N \cap \Omega_N^+ \right) \leq \mathbb{P} \left(B_N \cap \Omega_{\Lambda \cup \Lambda_c}^+ \right), \tag{4.12}$$

and proceed with an estimate that is just a rough version of the first part of the proof of Proposition 3.1: for any fixed positive η, α and β, define

$$\begin{aligned}
\widetilde{E}_{\eta,\alpha} &= \left\{ |\{y \in \Lambda_c : M_y^{(1)}(\varphi) \leq \sqrt{\alpha \log N}\}| \geq \eta |\Lambda_c| \right\}, \\
\widetilde{F}_{\eta,\beta} &= \left\{ |\{y \in \Lambda_c : M_y^{(2)}(\psi) - M_y^{(1)}(\varphi) \leq \sqrt{\beta \log N}\}| \geq \eta |\Lambda_c| \right\}.
\end{aligned} \tag{4.13}$$

---

[1]With reference to formula (4.5), the explicit estimates that we exhibit show that the probability in (4.10), as well as the one in (4.11), are bounded above by $\exp(-N^{d-2+\epsilon})$ for some $\epsilon > 0$ and N sufficiently large.

Let us observe that

$$\mathbb{P}(\Omega^+_{\Lambda \cup \Lambda_c} \cap \widetilde{E}_{\eta,\alpha}) \leq \mathbb{E}\left[\prod_{y \in \Lambda_c} \mathbb{P}\left(\varphi_y \geq 0 | \mathcal{F}^{\varphi,\psi}_{B(y)}\right); \Omega^+_N \cap \widetilde{E}_{\eta,\alpha}\right] \tag{4.14}$$

and, on $\widetilde{E}_{\eta,\alpha}$, if $\alpha < 4G_{1,L}$ for some positive constants c and ϵ

$$\prod_{y \in \Lambda_c} \mathbb{P}\left(\varphi_y \geq 0 | \mathcal{F}^{\varphi,\psi}_{B(y)}\right) \leq \left(1 - \frac{1}{\sqrt{c \log N}} N^{-\alpha/(2G_{1,L})}\right)^{cN^d L^{-d}} \tag{4.15}$$

$$\leq \exp(-N^{d-2+\epsilon}).$$

Analogously one shows that $\mathbb{P}(\Omega_{\Lambda \cup \Lambda_c} \cap \widetilde{F}_{\eta,\beta})$ is bounded by $\leq \exp(-N^{d-2+\epsilon})$ if $\beta < 4(G_{1,L} + G_{2,L})$. In view of (4.11), we are left with estimating $\mathbb{P}(B_N \cap \Omega^+_{\Lambda \cup \Lambda_c} \cap \widetilde{E}^{\complement}_{\eta,\alpha} \cap \widetilde{F}^{\complement}_{\eta,\beta})$, for α and β in the range that we have just chosen. We observe then that, by definition of $\widetilde{E}_{\eta,\alpha}$ and of $\widetilde{F}_{\eta,\beta}$,

$$\widetilde{E}^{\complement}_{\eta,\alpha} \cap \widetilde{F}^{\complement}_{\eta,\beta} \subset B'_N := \left\{ |\{y \in \Lambda_c : M_y^{(2)}(\psi) > (\sqrt{\alpha} + \sqrt{\beta})\sqrt{\log N}\}| > (1 - 2\eta)|\Lambda_c| \right\}, \tag{4.16}$$

thus $\mathbb{P}(B_N \cap \Omega^+_{\Lambda \cup \Lambda_c} \cap \widetilde{E}^{\complement}_{\eta,\alpha} \cap \widetilde{F}^{\complement}_{\eta,\beta}) \leq \mathbb{P}(B_N \cap B'_N)$. Now we choose $\eta < \delta/2$ (recall that δ is fixed) and α and β sufficiently close respectively to $4G_1$ and $4(G_1 + G_2)$ so that $\varepsilon := \sqrt{\alpha} + \sqrt{\beta} - a > 0$ (of course, to do this we have to choose L sufficiently large). With these choices we obtain

$$B_N \cap B'_N \subset \left\{ |\{y \in \Lambda_c : \psi_y < \sqrt{a \log N}, M_y^{(2)}(\psi) > (\sqrt{\alpha} + \sqrt{\beta})\sqrt{\log N}\}| > (\delta - 2\eta)|\Lambda_c| \right\}$$
$$\subset B''_N := \left\{ |\{y \in \Lambda_c : |\psi_y - M_y^{(2)}(\psi)| > \epsilon\sqrt{\log N}\}| > (\delta - 2\eta)|\Lambda_c| \right\}. \tag{4.17}$$

Since $\{\psi_y - M_y^{(2)}(\psi)\}_{y \in \Lambda_c}$ are Gaussian IID variables we have that for some $c > 0$

$$\mathbb{P}(B''_N) \leq \mathbb{P}\left(\frac{1}{|\Lambda_c|} \sum_{y \in \Lambda_c} |\psi_y - M_y^{(2)}(\psi)| > \epsilon(\delta - 2\eta)\sqrt{\log N}\right) \leq \exp(-cN^d \log N). \tag{4.18}$$

Therefore $\mathbb{P}(B_N \cap \Omega^+_{\Lambda \cup \Lambda_c} \cap \widetilde{E}^{\complement}_{\eta,\alpha} \cap \widetilde{F}^{\complement}_{\eta,\beta}) \leq \exp(-cN^d \log N)$ and the proof is complete. $\square$

4.3. Upper bounds for ψ. We first prove a result for $\mathbb{E}^+_N(M_N^\Lambda(\psi))$, where $M_N^\Lambda(\psi) = \sum_{x \in \Lambda_N} \psi_x/|\Lambda_N|$, Λ subset of $\mathbb{R}^d$ with piecewise smooth boundary and $\Lambda_N = N\Lambda$.

Proposition 4.4. *For every Λ we have that*

$$\limsup_{N \to \infty} \frac{\mathbb{E}^+_N[M_N^\Lambda(\psi)]}{\sqrt{\log N}} \leq \sqrt{4G_1} + \sqrt{4(G_1 + G_2)} \tag{4.19}$$

Proof. As for the arguments in Section 2, this proof relies on estimates for the *rough substrate* model that arises when one conditions with respect to $\mathcal{F}^\varphi$. More precisely we will make use of

$$\mathbb{E}^+_N\left(u(\psi)|\mathcal{F}^\varphi\right)(\varphi) = \mathbb{E}_2\left(u(\psi)|\psi_x \geq \varphi_x \text{ for } x \in D_N\right), \tag{4.20}$$

that holds for every measurable $u : \mathbb{R}^{\mathbb{Z}^d} \longrightarrow \mathbb{R}$ such that $\mathbb{E}[u(\psi)] < \infty$. To avoid possible misunderstanding due to abuse of notation, we stress that in (4.20) φ on both sides is a numerical variable, while ψ is a random variable. Therefore the ψ–marginal of $\mathbb{P}^+_N(\cdot|\mathcal{F}^\varphi)(\varphi)$ is a repulsion model for the field ψ constrained above the fixed rough substrate φ.

In evaluating $\mathbb{E}^+_N[M_N^\Lambda(\psi)]$ via conditioning on $\mathcal{F}^\varphi$, the random substrate φ is distributed according to $\mathbb{P}^+_{1,N}(\mathrm{d}\varphi)$. We may therefore use again the fact that in the proof of Proposition 2.2 we have

identified a set $E_N \in \mathcal{F}^\varphi$ such that $\mathbb{P}_{1,N}^+(E_N) \overset{N\to\infty}{\longrightarrow} 1$. Since we are evaluating the expectation of the unbounded random variable $M_N^\Lambda(\psi)$, we have to take care of what happens also for *atypical* substrates, that is what happens on $E_N^{\complement}$.

Both in considering $\varphi \in E_N$ and $\varphi \in E_N^{\complement}$, the following procedure, already employed in [1, Prop. 4.6] and in [9, Prop. 3.2], turns out to be helpful: for every $\alpha \geq 0$

$$\mathbb{P}_2 \left(\cdot \, \big| \psi_x \geq \varphi_x \text{ for } x \in D_N \right) \prec \mathbb{P}_2 T_\alpha^{-1} \left(\cdot \, \big| \psi_x \geq \varphi_x \text{ for } x \in D_N \right), \tag{4.21}$$

where T_α is a short–cut notation for T_σ, $\sigma \in \mathbb{R}^{\mathbb{Z}^d}$ and $\sigma_x = \alpha$ for every x, so that

$$\mathbb{E}_2 \left[M_N^\Lambda(\psi) \big| \psi_x \geq \varphi_x \text{ for } x \in D_N \right] \leq \alpha + \mathbb{E}_2 \left[M_N^\Lambda(\psi) \big| \psi_x \geq \varphi_x - \alpha \text{ for } x \in D_N \right]. \tag{4.22}$$

The proof of (4.21) may be found for example in [9, Appendix B]. We observe moreover that, by using equation (2.27) with $Y = M_N^\Lambda(\psi)$, $t = \rho N^{d-2}$ ($\rho > 0$) and $E = \{\psi_x \geq \varphi_x - \alpha \text{ for } x \in D_N\}$, one obtains that there exists a constant $c(\Lambda)$ such that

$$\mathbb{E}_2 \left[M_N^\Lambda(\psi) \big| \psi_x \geq \varphi_x - \alpha \text{ for } x \in D_N \right] \leq c(\Lambda) - \frac{1}{\rho N^{d-2}} \log \mathbb{P}_2 \left(\psi_x \geq \varphi_x - \alpha \text{ for } x \in D_N \right). \tag{4.23}$$

Let us now consider the case $\varphi \in E_N$ and choose $\alpha = (\sqrt{4G_1} + \sqrt{4(G_1 + G_2)} + \delta)\sqrt{\log N}$, $\delta > 0$. We claim that in this case

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}_2 \left(\psi_x \geq \varphi_x - \alpha \text{ for } x \in D_N \right) = 0, \tag{4.24}$$

so that, by (4.22) and (4.23), we conclude that

$$\limsup_{N\to\infty} \frac{\mathbb{E}_N^+ \left[M_N^\Lambda(\psi); E_N \right]}{\sqrt{\log N}} \leq \sqrt{4G_1} + \sqrt{4(G_1 + G_2)}. \tag{4.25}$$

We prove claim (4.24) by exploiting the tools developed in the Proof of Proposition 2.2. We have

$$\frac{1}{N^{d-2}} \log \mathbb{P}_2 \left(\psi_x \geq \varphi_x - \alpha \text{ for } x \in D_N \right) \geq \frac{1}{N^{d-2}} \log \mathbb{P}_2 \left(\psi_x \geq \widetilde{\varphi}_x - \alpha \text{ for } x \in D_N \right)$$

$$\geq \frac{1}{N^{d-2}} \log \mathbb{P}_2 \left(\psi_x \geq \widetilde{\varphi}_x - \alpha \text{ for } x \in D_N^- \right) +$$

$$\frac{1}{N^{d-2}} \log \mathbb{P}_2 \left(\psi_x \geq (\gamma + \widetilde{k}\theta)\sqrt{\log N} - \alpha \text{ for } x \in L_N\left(\widetilde{k}\right) \right)$$

$$=: \mathbf{T}_2(N) + \mathbf{T}_1(N), \quad (4.26)$$

where $\gamma = \sqrt{4G_1} + \eta$ and $\widetilde{\varphi}$, D_N^- and $L_N(\widetilde{k})$ are defined in Section (2). Of course the fact that $\mathbf{T}_2$, defined in (2.12), is equal to zero implies also that $\lim_{N\to\infty} \mathbf{T}_2(N) = 0$. To get to the same conclusion for $\mathbf{T}_1(N)$ we apply the FKG inequality:

$$\mathbf{T}_2(N) \geq \sum_{k=1}^{\overline{k}} \frac{N_k}{N^{d-2}} \log \left(1 - \mathbb{P}_2 \left(\psi_0 < \left(k\theta - \sqrt{4(G_1 + G_2)} + \eta - \delta \right) \sqrt{\log N} \right) \right), \tag{4.27}$$

for which, since $N_k \leq cN^{d - \frac{(k-1)^2\theta^2}{2G_1} + \epsilon}$, we get the following lower bound:

$$-\frac{c}{\sqrt{\log N}} \sum_{k=1}^{\overline{k}} N^{2 - \frac{(k-1)^2\theta^2}{2G_1} + \epsilon - \frac{(k\theta - 2\sqrt{G_1+G_2} + \eta - \delta)^2}{2G_2}}. \tag{4.28}$$

This term is negligible if

$$\frac{(k-1)^2\theta^2}{2G_1} + \frac{(k\theta - 2\sqrt{G_1 + G_2} + \eta - \delta)^2}{2G_2} > 2 \tag{4.29}$$

for all $k \leq \overline{k}$ and this is true whenever $|\eta - \delta| > \theta$, that is for θ sufficiently small, which is achieved by choosing $\overline{k}$ sufficiently large. Therefore (4.24) is proven.

In view of (4.25) we are therefore left with showing

$$\limsup_{N\to\infty} \frac{\mathbb{E}_N^+\left[M_N^\Lambda(\psi); E_N^\complement\right]}{\sqrt{\log N}} \leq 0. \tag{4.30}$$

Configurations in $E_N^\complement$ will be split according to the decomposition $\mathbb{R}^{\mathbb{Z}^d} = \cup_{j=0}^\infty \Omega_j$, where

$$\begin{aligned}
\Omega_0 &= \left\{\varphi : \max_{x\in D_N} \varphi_x \leq K\sqrt{\log N}\right\}, \\
\Omega_j &= \left\{\varphi : \max_{x\in D_N} \varphi_x \in (K + j - 1, K + j]\sqrt{\log N}\right\}, \quad \text{for } j = 1, 2, \dots
\end{aligned} \tag{4.31}$$

where K is a (sufficiently large) constant that will be chosen below. If $\varphi \in \Omega_0$ then $\varphi_x - (K + \sqrt{4G_2} + \delta)\sqrt{\log N} \leq -(\sqrt{4G_2} + \delta)\sqrt{\log N}$ for every $x \in D_N$ and by applying the FKG inequality one obtains that

$$\lim_{N\to 0} \frac{1}{N^{d-2}} \log \mathbb{P}_2\left(\psi_x > -(\sqrt{4G_2} + \delta)\sqrt{\log N} \text{ for } x \in D_N\right) = 0, \tag{4.32}$$

for every $\delta > 0$. Once we apply these considerations to (4.22) and (4.23) with $\alpha = (K + \sqrt{4G_2} + \delta)\sqrt{\log N}$ we find

$$\frac{\mathbb{E}_2\left[M_N^\Lambda(\psi)\big|\psi_x \geq \varphi_x \text{ for } x \in D_N\right]}{\sqrt{\log N}} \leq 2K, \tag{4.33}$$

where we have chosen $K > \sqrt{4G_2} + \delta$ (and N sufficiently large). Therefore

$$\frac{\mathbb{E}_N^+\left[M_N^\Lambda(\psi); E_N^\complement \cap \Omega_0\right]}{\sqrt{\log N}} \leq 2K\mathbb{P}\left(E_N^\complement\right) \xrightarrow{N\to\infty} 0. \tag{4.34}$$

We are left with focusing on $\varphi \in \Omega_j$, $j = 1, 2, \dots$: the estimate is technically almost identical to the case of Ω_0. It is a matter of using again (4.22) and (4.23): this time we choose $\alpha = (K + j + \sqrt{4G_2} + \delta)\sqrt{\log N}$ and we obtain

$$\mathbb{E}_2\left[M_N^\Lambda(\psi)\big|\psi_x \geq \varphi_x \text{ for } x \in D_N\right] \leq$$
$$K + j + \sqrt{4G_2} + \delta + \frac{c(\Lambda)}{\sqrt{\log N}} - \frac{1}{N^{d-2}\sqrt{\log N}} \log \mathbb{P}_2\left(\psi_x \geq -\left(\sqrt{4G_2} + \delta\right)\sqrt{\log N} \text{ for } x \in D_N\right)$$
$$\leq 3K + j. \quad (4.35)$$

We have spelled out these steps to stress that the estimate is not asymptotic in N: it holds for $N \geq N_0$ and N_0 is independent of j. We finally have

$$\mathbb{E}_N^+\left[M_N^\Lambda(\psi); E_N^\complement \cap \cup_{j=1}^\infty \Omega_j\right] \leq \mathbb{E}_N^+\left[M_N^\Lambda(\psi); \cup_{j=1}^\infty \Omega_j\right] \leq \sum_{j=1}^\infty (3K + j)\mathbb{P}_{1,N}^+\left(\Omega_j\right). \tag{4.36}$$

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By (2.6) (applying the Markov inequality) we obtain that $\mathbb{P}_{1,N}^+(\Omega_j) \leq cN^{d-(K+j-1-\sqrt{4G_1}-\delta)^2/2G_1}$, $\delta > 0$. Therefore the series in the right–hand side of (4.36) is summable and, if $K > \sqrt{2dG_1} + \sqrt{4G_2}$, it vanishes as $N \to \infty$. Therefore

$$\lim_{N\to\infty} \mathbb{E}_N^+ \left[M_N^\Lambda(\psi); E_N^{\complement} \cap \cup_{j=1}^\infty \Omega_j \right] = 0. \tag{4.37}$$

Putting together (4.34) and (4.37) we obtain (4.30) and the proof is complete. $\square$

Remark 4.5. It is immediate to see that Proposition 4.3 and the fact that $\psi_x \geq 0$ for every $x \in D_N$ if $(\varphi, \psi) \in \Omega_N^+$ imply that the lower bound corresponding to (4.19), so that we have that for every $\Lambda \subseteq D$

$$\lim_{N\to\infty} \frac{\mathbb{E}_N^+[M_N^\Lambda(\psi)]}{\sqrt{\log N}} = \sqrt{4G_1} + \sqrt{4(G_1+G_2)}. \tag{4.38}$$

Finally it is not too difficult to see that Proposition 4.4 is compatible with Proposition 4.3 only if the density of sites in which ψ is above $\sqrt{4G_1} + \sqrt{4(G_1+G_2}+\epsilon$, any $\epsilon > 0$, is negligible. Namely:

Proposition 4.6. *For all $\delta > 0$ and $b > \sqrt{4G_1} + \sqrt{4(G_1+G_2)}$,*

$$\lim_{N\to\infty} \mathbb{P}_N^+ \left(|\{x \in D_N : \psi_x > b\sqrt{\log N}\}| \geq \delta |D_N| \right) = 0. \tag{4.39}$$

References

[1] D. Bertacchi and G. Giacomin, *Enhanced interface repulsion from quenched hard–wall randomness*, Probab. Theory Rel. Fields, published online (link.springer.de), September 2002.

[2] E. Bolthausen, *Random walk representations and entropic repulsion for gradient models*, Infinite Dimensional Stochastic Analysis, Koninklijke Nederlandse Akademie van Wetenschappen, Ph. Clément et al. eds., (2000), 55–84.

[3] E. Bolthausen and J.D. Deuschel, *Critical large deviations for Gaussian fields in the phase transition regime*, Ann. Prob. 21 (1994), 1876–1920.

[4] E. Bolthausen, J.-D. Deuschel and G. Giacomin, *Entropic repulsion and the maximum of the two dimensional free field*, Ann. Probab. **29** (2001).

[5] E. Bolthausen, J.-D. Deuschel and O. Zeitouni, *Entropic repulsion of the lattice free field*, Commun. Math. Phys. **170** (1995), 417–443.

[6] H.J. Brascamp and E. Lieb, *On extensions of the Brun–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Funct. Anal. **22** (1976), 366-389.

[7] J. Bricmont, A. el Mellouki and J. Fröhlich, *Random surfaces in statistical mechanics: roughening, rounding, wetting*, J. Stat. Phys. **42** (1986), 743–798.

[8] J.-D. Deuschel and G. Giacomin, *Entropic repulsion for the free field: pathwise characterization in $d \geq 3$*, Commun. Math. Phys. **206**, 447–462 (1999).

[9] G. Giacomin, *Aspects of statistical mechanics of random surfaces*, notes of the lectures given at IHP in the fall 2001, Preprint (available at the web page of the author).

[10] G. Giacomin, *On stochastic domination in the Brascamp–Lieb framework*, preprint (2001), accepted for publication on Math. Proc. Cambridge Phil. Soc..

[11] H. Sakagawa, *Entropic repulsion for a Gaussian lattice field with certain finite range interactions*, preprint (2002).

[12] F. Spitzer, *Principles of random walks*, Springer–Verlag, second edition (1976).

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