

APPROXIMATING CRITICAL PARAMETERS OF BRANCHING RANDOM WALKS

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ABSTRACT. Given a branching random walk on a graph, we consider two kinds of truncations: either by inhibiting the reproduction outside a subset of vertices or by allowing at most m particles per vertex. We investigate the convergence of weak and strong critical parameters of these truncated branching random walks to the analogous parameters of the original branching random walk. As a corollary, we apply our results to the study of the strong critical parameter of a branching random walk restricted to the cluster of a Bernoulli bond percolation.

Keywords: branching random walks, critical parameters, percolation, graphs.

AMS subject classification: 60K35.

1. INTRODUCTION

The BRW is a process which serves as a (rough) model for a population living in a spatially structured environment (the vertices of a – possibly oriented – graph $(X, \mathcal{E}(X))$), where each individual lives in a vertex, breeds and dies at random times and each offspring is placed (randomly) in one of the neighbouring vertices. There is no bound on the number of individuals allowed per site. The vertices may be thought as small ecosystems or squares of soil (with their proximity connections – the edges) and individuals as animals or plants. Depending on the parameters involved and on the nature of $(X, \mathcal{E}(X))$, the population may face almost sure extinction, global survival (i.e. with positive probability at any time there will be at least one individual alive) or local survival (i.e. with positive probability at arbitrarily large times there will be at least one individual alive in a fixed vertex). These matters have been investigated by several authors ([12], [13], [14], [17], [20], [24] only to mention a few, see [16] for more references).

Let us be more precise as to the definition of the process and of the environment. The graph $(X, \mathcal{E}(X))$ is endowed with a weight function $\mu : X \times X \rightarrow [0, +\infty)$ such that $\mu(x, y) > 0$ if and only if $(x, y) \in \mathcal{E}(X)$ (in which case we write $x \rightarrow y$). We call the couple (X, μ) a weighted graph. We require that there exists $K < +\infty$ such that $k(x) := \sum_{y \in X} \mu(x, y) \leq K$ for all $x \in X$ (other conditions will be stated in Section 2).

Given $\lambda > 0$, the branching random walk (BRW(X) or briefly BRW) is the continuous-time Markov process $\{\eta_t\}_{t \geq 0}$, with configuration space \mathbb{N}^X , where each existing particle at x has an exponential lifespan of parameter 1 and, during its life, breeds at the arrival times of a Poisson process of parameter $\lambda k(x)$ and then chooses to send its offspring to y with probability $\mu(x, y)/k(x)$ (note that $(\mu(x, y)/k(x))_{x, y \in X}$ is the transition matrix of a random walk on X). In the literature

one usually finds the particular case $k(x) = 1$ for all $x \in X$ (i.e. the breeding rate is constant among locations – no place is more fertile than others) or, sometimes, the case where $\mu = \mathbb{I}_{\mathcal{E}(X)}$ (i.e. the breeding rate is proportional to the degree and all edges have the same rate).

Two critical parameters are associated to the BRW: the weak (or global) survival critical parameter λ_w and the strong (or local) survival one λ_s . They are defined as

$$\begin{aligned}\lambda_w &:= \inf\{\lambda > 0 : \mathbb{P}^{\delta_{x_0}}(\exists t : \eta_t = \underline{0}) < 1\} \\ \lambda_s &:= \inf\{\lambda > 0 : \mathbb{P}^{\delta_{x_0}}(\exists \bar{t} : \eta_t(x_0) = 0, \forall t \geq \bar{t}) < 1\},\end{aligned}\tag{1.1}$$

where x_0 is a fixed vertex, $\underline{0}$ is the configuration with no particles at all sites and $\mathbb{P}^{\delta_{x_0}}$ is the law of the process which starts with one individual in x_0 . Note that, if the weighted graph is connected then these values do not depend on the initial configuration, provided that this configuration is finite (that is, it has only a finite number of individuals), nor on the choice of x_0 . See Section 2 for a discussion on the values of λ_w and λ_s .

When (X, μ) is infinite (and connected), the BRW is, so to speak, unbounded in two respects: the environment, since individuals may live at arbitrarily large distance from their ancestors (actually n -th generation individuals may live at distance n from the ancestor), and the colonies' size, since an arbitrarily large number of individuals may pile up on any vertex. Hence it is natural to consider “truncated” BRWs where either space or colonies are bounded, and investigate the relationship between these processes and the BRW. Indeed, in the literature one often finds problems tackled first in finite or compact spaces and then reached through a “thermodynamical limit” procedure. One can see easily that it is possible to construct the BRW either from the process on finite sets (spatial truncation) or from the process on infinite space and a bound on the number of particles per site (particles truncation). In both cases the truncated process, for any fixed time t , converges almost surely to the BRW.

First we consider “spatially truncated” BRWs. We choose a family of weighted subgraphs $\{(X_n, \mu_n)\}_{n \in \mathbb{N}}$, such that $X_n \uparrow X$, $\mu_n(x, y) \leq \mu(x, y)$, and $\mu_n(x, y) \xrightarrow{n \rightarrow \infty} \mu(x, y)$ for all x, y . The process $\text{BRW}(X_n)$ can be seen as the $\text{BRW}(X)$ with the constraint that reproductions outside X_n are deleted and the ones from x to y (x, y in X_n) are removed with probability $1 - \mu_n(x, y)/\mu(x, y)$. It is not difficult to see that for any fixed t , as n goes to infinity, the $\text{BRW}(X_n)$ converges to the BRW almost surely. Our first result is that $\lambda_s(X_n) \xrightarrow{n \rightarrow \infty} \lambda_s(X)$ (the latter being the strong survival critical parameter of the BRW on (X, μ)). Indeed we prove a slightly more general result (Theorem 3.2) which allows us to show that if $X = \mathbb{Z}^d$ and X_n is the infinite cluster of the Bernoulli bond percolation of parameter p_n , where $p_n \xrightarrow{n \rightarrow \infty} 1$ sufficiently fast, then $\lambda_s(X_n) \xrightarrow{n \rightarrow \infty} \lambda_s(X)$ almost surely with respect to the percolation probability space (Section 7).

Second we consider BRWs where at most m individuals per site are allowed (thus taking values in $\{0, 1, \dots, m\}^X$). We call this process BRW_m and denote it by $\{\eta_t^m\}_{t \geq 0}$. Note that if $m = 1$ we get the contact process (indeed the BRW_m is sometimes referred to as a “multitype contact

process” – see for instance [19]). It is easily seen that for all fixed t we have $\eta_t^m \xrightarrow{m \rightarrow \infty} \eta_t$ almost surely (see for instance [20] where the authors suggest this limit as a way to construct the BRW). Clearly, for all $m \geq 1$, one may consider the critical parameters λ_w^m and λ_s^m defined as in (1.1) with η_t^m in place of η_t . One of the main questions we investigate in this paper is whether $\lambda_w^m \xrightarrow{m \rightarrow \infty} \lambda_w$ and $\lambda_s^m \xrightarrow{m \rightarrow \infty} \lambda_s$: to our knowledge this was still unknown even for the case where $X = \mathbb{Z}^d$ with μ transition matrix of the simple random walk.

Here is a brief outline of the paper. In Section 2 we state the basic terminology and assumptions needed in the sequel. Section 3 is devoted to the spatial approximation of the strong critical parameter λ_s by finite or infinite sets (see Theorems 3.1 and 3.2 respectively). We note that results on the spatial approximation, in the special case when $X = \mathbb{Z}^d$ and μ is the transition matrix of the simple random walk, were obtained in [18] using a different approach. In Section 4 we introduce the technique we use to prove convergence of the critical parameters of the BRW_m . The technique is essentially a suitable coupling with a supercritical oriented bond percolation: this kind of comparison has been introduced in [5], discussed in [7] and later applied in several papers (see for instance [1], [23], [22] and [8]). Nevertheless the coupling here is quite tricky, therefore we describe it in four steps which can be adapted to different graphs. In Section 5 we prove that λ_s^m converges to λ_s under some assumptions on the graph (Theorem 5.6). As a corollary, we have $\lambda_s^m \xrightarrow{m \rightarrow \infty} \lambda_s$ for \mathbb{Z}^d with the simple random walk. The same approach is used in Section 6 to prove the convergence of the sequence λ_w^m to λ_w when $X = \mathbb{Z}^d$ (see Theorem 6.1 and Corollary 6.2, and Remark 6.3 for a slightly more general class of graphs) or when X is a homogeneous tree (Theorem 6.4). The results of Section 3 are applied in Section 7 in order to study the strong critical parameter of a BRW restricted to a random subgraph generated by a Bernoulli bond percolation process. Section 8 is devoted to final remarks and open questions.

2. TERMINOLOGY AND ASSUMPTIONS

In this section we state our assumptions on the graph (X, μ) ; we also recall the description of the BRW through its generator and the associated semigroup, and discuss the values of λ_w and λ_s .

Given the (weighted) graph (X, μ) , the degree of a vertex x , $\deg(x)$ is the cardinality of the set $\{y \in X : x \rightarrow y\}$; we require that (X, μ) is with bounded geometry, that is $\sup_{x \in X} \deg(x) < +\infty$. Moreover we consider (X, μ) connected, which by our definition of μ (recall that $\mu(x, y) > 0$ if and only if $(x, y) \in \mathcal{E}(X)$) is equivalent to $\mu^{(n)}(x, y) > 0$ for some $n = n(x, y)$, where $\mu^{(n)}$ is the n -th power of the matrix μ . When $(\mu(x, y))_{x, y}$ is stochastic (i.e. $k(x) = 1$ for all $x \in X$), in order to stress this property we use the notation P , $p(x, y)$ and $p^{(n)}(x, y)$ instead of μ , $\mu(x, y)$ and $\mu^{(n)}(x, y)$. Define $d(x, y) = \min\{n : \exists \{x_i\}_{i=0}^n, x_0 = x, x_n = y, x_i \rightarrow x_{i+1}\}$; note that this is a true metric on X if and only if (X, μ) is non oriented.

We need to define the product of two graphs (in our paper these will be space/time products): given two graphs $(X, \mathcal{E}(X))$, $(Y, \mathcal{E}(Y))$ we denote by $(X, \mathcal{E}(X)) \times (Y, \mathcal{E}(Y))$ the weighted graph

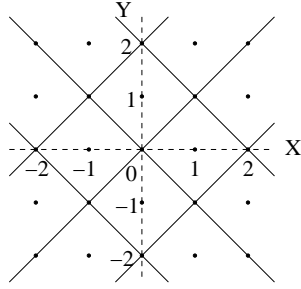


Figure 1: $X \times Y$ ($X = Y = \mathbb{Z}$).

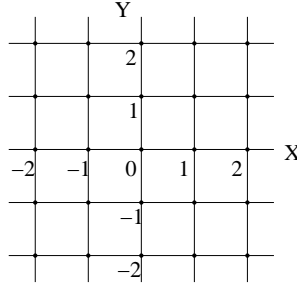


Figure 2: $X \square Y$ ($X = Y = \mathbb{Z}$).

with set of vertices $X \times Y$ and set of edges $\mathcal{E} = \{((x, y), (x_1, y_1)) : (x, x_1) \in \mathcal{E}(X), (y, y_1) \in \mathcal{E}(Y)\}$ (in Figure 1 we draw the connected component of $\mathbb{Z} \times \mathbb{Z}$ containing $(0, 0)$). Besides, by $(X, \mathcal{E}(X)) \square (Y, \mathcal{E}(Y))$ we mean the graph with the same vertex set as before and vertices $\mathcal{E} = \{((x, y), (x_1, y_1)) : (x, x_1) \in \mathcal{E}(X), y = y_1\} \cup \{((x, y), (x_1, y_1)) : x = x_1, (y, y_1) \in \mathcal{E}(Y)\}$ (see Figure 2).

Let $\{\eta_t\}_{t \geq 0}$ be the branching random walk on X with parameter λ , associated to the weight function μ : the configuration space is \mathbb{N}^X and its generator is

$$\mathcal{L}f(\eta) := \sum_{x \in X} \eta(x) \left(\partial_x^- f(\eta) + \lambda \sum_{y \in X} \mu(x, y) \partial_y^+ f(\eta) \right), \quad (2.2)$$

where $\partial_x^\pm f(\eta) := f(\eta \pm \delta_x) - f(\eta)$. Analogously the generator of the BRW $_m$ $\{\eta_t^m\}_{t \geq 0}$ is

$$\mathcal{L}_m f(\eta) := \sum_{x \in X} \eta(x) \left(\partial_x^- f(\eta) + \lambda \sum_{y \in X} \mu(x, y) \mathbb{1}_{[0, m-1]}(\eta(y)) \partial_y^+ f(\eta) \right), \quad (2.3)$$

Note that the configuration space is still \mathbb{N}^X (though one may consider $\{0, 1, \dots, m\}^X$ as well). The semigroup S_t is defined as $S_t f(\eta) := \mathbb{E}^\eta(f(\eta_t))$, where f is any function on \mathbb{N}^X such that the expected value is defined.

The strong and weak survival critical parameters of the BRW clearly depend on the weighted graph (X, μ) ; we denote them by $\lambda_s(X, \mu)$ and $\lambda_w(X, \mu)$ (or simply by $\lambda_s(X)$ and $\lambda_w(X)$ or λ_s and λ_w). Analogously we denote by $\lambda_s^m(X, \mu)$ and $\lambda_w^m(X, \mu)$ (or simply by $\lambda_s^m(X)$ and $\lambda_w^m(X)$ or λ_s^m and λ_w^m) the critical parameters of the BRW $_m$ on (X, μ) . It is known (see for instance [3] and [4]) that $\lambda_s = R_\mu := 1/\limsup_n \sqrt[n]{\mu^{(n)}(x, y)}$ (which is easily seen to be independent of $x, y \in X$ since the graph is connected). On the other hand the explicit value of λ_w is not known in general. Nevertheless in many cases it is possible to prove that $\lambda_w = 1/\limsup_n \sqrt[n]{\sum_{y \in X} \mu^{(n)}(x, y)}$ (see [3] and [4]). In particular if $k(x) = K$ for all $x \in X$ then $\lambda_w = 1/K$; thus if $(\mu(x, y))_{x, y}$ is a stochastic matrix then $\lambda_w = 1$.

The two critical parameters coincide (i.e. there is no pure weak phase) in many cases: if X is finite, or, when $\mu = P$ is stochastic, if $R = 1$. Here are two sufficient conditions for $R = 1$ (when $\mu = P$ is stochastic):

- (1) (X, P) is the simple random walk on a non-oriented graph and the ball of radius n and center x has subexponential growth ($\sqrt[n]{|B_n(x)|} \rightarrow 1$ as $n \rightarrow \infty$).

Indeed for any reversible random walk the following universal lower bound holds

$$p^{(2n)}(x, x) \geq v(x)/v(B_n(x))$$

(see [6, Lemma 6.2]) where v is a reversibility measure. If P is the simple random walk then v is the counting measure and the claim follows.

An explicit example is the simple random walk on \mathbb{Z}^d or on d -dimensional combs (see [25, Section 2.21] for the definition of comb).

- (2) (X, P) is a symmetric, irreducible random walk on an amenable group (see [25]).

3. SPATIAL APPROXIMATION

In this section we consider spatial truncations of the BRW. In Lemma 3.1 and Theorem 3.2 $\{X_n\}_{n \in \mathbb{N}}$ will be a sequence of finite subsets of X such that $X_n \subseteq X_{n+1}$ and $\bigcup_{n=1}^{\infty} X_n = X$; we denote by ${}_n\mu$ the truncation matrix defined by ${}_n\mu := \mu|_{X_n \times X_n}$. We define ${}_nR_\mu := 1/\limsup_{k \rightarrow \infty} \sqrt[k]{{}_n\mu^{(k)}(x, y)}$.

Lemma 3.1. *Let $\{X_n\}_{n \in \mathbb{N}}$ be such that $(X_n, {}_n\mu)$ is connected for all n . Then ${}_nR_\mu \geq {}_{n+1}R_\mu$ for all n and when $X_n \subsetneq X_{n+1}$ we have ${}_nR_\mu > {}_{n+1}R_\mu$. Moreover ${}_nR_\mu \downarrow R_\mu$.*

Proof. This is essentially Theorem 6.8 of [21]. □

The next result is a generalization of this lemma and it goes beyond the pure spatial approximation by finite subsets. We omit the proof which follows easily from Lemma 3.1.

Theorem 3.2. *Let $\{(Y_n, \mu_n)\}_{n \in \mathbb{N}}$ be a sequence of connected weighted graphs and let $\{X_n\}_{n \in \mathbb{N}}$ be such that $Y_n \supseteq X_n$. Let us suppose that $\mu_n(x, y) \leq \mu(x, y)$ for all $n \in \mathbb{N}$, $x, y \in Y_n$ and $\mu_n(x, y) \rightarrow \mu(x, y)$ for all $x, y \in X$. If $(X_n, {}_n\mu)$ is connected for every $n \in \mathbb{N}$ then $\lambda_s(Y_n, \mu_n) \geq \lambda_s(X, \mu)$ and $\lambda_s(Y_n, \mu_n) \xrightarrow{n \rightarrow \infty} \lambda_s(X, \mu)$.*

A simple situation where the previous theorem applies, is the non-oriented case ($\mu(x, y) > 0$ if and only if $\mu(y, x) > 0$) where $X_n = Y_n$ is the ball of radius n with center at a fixed vertex x_0 of X .

Remark 3.3. *If Y_n is finite for all n , then $\lambda_w(Y_n) = \lambda_s(Y_n)$, hence $\lambda_w(Y_n) \rightarrow \lambda_w(X)$ if and only if $\lambda_w(X) = \lambda_s(X)$.*

4. THE COMPARISON WITH AN ORIENTED PERCOLATION

From now on, we suppose that X is countable (otherwise $\lambda_w^n = \lambda_s^n = +\infty$). First of all, we need a coupling between $\{\eta_t\}_{t \geq 0}$ and $\{\eta_t^m\}_{t \geq 0}$: think of $\{\eta_t^m\}_{t \geq 0}$ as obtained from $\{\eta_t\}_{t \geq 0}$ by removing all the births which cause more than m particles to live on the same site. Then we need two other coupled processes. Fix $n_0 \in \mathbb{N}$ and let $\{\bar{\eta}_t\}_{t \geq 0}$ be the process obtained from the BRW $\{\eta_t\}_{t \geq 0}$ by

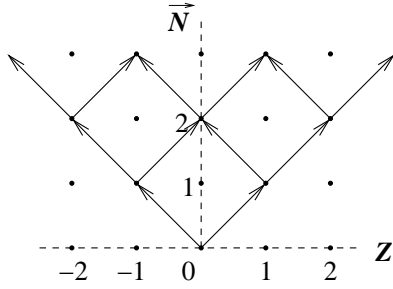


Figure 3: $\mathbb{Z} \times \vec{\mathbb{N}}$.

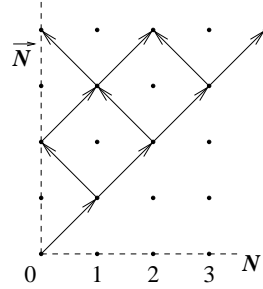


Figure 4: $\mathbb{N} \times \vec{\mathbb{N}}$.

removing all n -th generation particles, with $n > n_0$. Analogously, define $\{\bar{\eta}_t^m\}_{t \geq 0}$ from $\{\eta_t^m\}_{t \geq 0}$. Clearly, $\eta_t \geq \bar{\eta}_t$, $\eta_t \geq \eta_t^m$, $\eta_t^m \geq \bar{\eta}_t^m$ and $\bar{\eta}_t \geq \bar{\eta}_t^m$ for all $t \geq 0$. Note that, by construction, the progenies of a given particle in $\{\bar{\eta}_t\}_{t \geq 0}$ or $\{\bar{\eta}_t^m\}_{t \geq 0}$ lives at a distance from the ancestor not larger than n_0 (and the processes go extinct almost surely).

Our proofs of the convergence of λ_s^m and λ_w^m are essentially divided in the following four steps.

Step 1. Fix a graph $(I, \mathcal{E}(I))$ such that the Bernoulli percolation on $(I, \mathcal{E}(I)) \times \vec{\mathbb{N}}$ has two phases (where we denote by $\vec{\mathbb{N}}$ the oriented graph on \mathbb{N} , that is, (i, j) is an edge if and only if $j = i + 1$).

Note that since the (oriented) Bernoulli bond percolation on $\mathbb{Z} \times \vec{\mathbb{N}}$ and $\mathbb{N} \times \vec{\mathbb{N}}$ has two phases, it is enough to find a copy of the graph \mathbb{Z} or \mathbb{N} as a subgraph of I . This is true for instance for any infinite non-oriented graph (in this paper, we choose either $I = \mathbb{Z}$, or $I = \mathbb{N}$, or $I = X$). Figures 3 and 4 respectively show the components of the products $\mathbb{Z} \times \vec{\mathbb{N}}$ and $\mathbb{N} \times \vec{\mathbb{N}}$ containing all the vertices y such that there exists a path from $(0, 0)$ to y .

Step 2. For all $\lambda > \lambda_w$ (or $\lambda > \lambda_s$) and for every $\varepsilon > 0$ there exists a collection of disjoint sets $\{A_i\}_{i \in I}$ ($A_i \subset X$ for all $i \in I$), $\bar{t} > 0$, and $k \in \mathbb{N} \setminus \{0\}$, such that, for all $i \in I$,

$$\mathbb{P}\left(\forall j : (i, j) \in \mathcal{E}(I), \sum_{x \in A_j} \eta_{\bar{t}}(x) \geq k \mid \eta_0 = \eta\right) > 1 - \varepsilon, \quad (4.4)$$

for all η such that $\sum_{x \in A_i} \eta(x) = k$ and $\eta(x) = 0$ for all $x \notin A_i$. The same holds, for some suitable n_0 , for $\{\bar{\eta}_t\}_{t \geq 0}$ in place of $\{\eta_t\}_{t \geq 0}$.

In the following sections Step 2 will be established under certain conditions (and for suitable choices of $(I, \mathcal{E}(I))$).

Step 3. Let λ , ε , $\{A_i\}_{i \in I}$, \bar{t} and k be chosen as in Step 2. Then for all sufficiently large m , we have that for all $i \in I$,

$$\mathbb{P}\left(\forall j : (i, j) \in \mathcal{E}(I), \sum_{x \in A_j} \eta_{\bar{t}}^m(x) \geq k \mid \eta_0^m = \eta\right) > 1 - 2\varepsilon, \quad (4.5)$$

for all η such that $\sum_{x \in A_i} \eta(x) = k$, $\eta(x) = 0$ for all $x \notin A_i$. The same holds, for some suitable n_0 , for $\{\bar{\eta}_t^m\}_{t \geq 0}$ in place of $\{\eta_t^m\}_{t \geq 0}$.

Step 3 is a direct consequence of Step 2. Indeed let N_t be the total number of particles ever born in the BRW (starting from the configuration η) before time t ; it is clear that N_t is a process bounded above by a branching process with birth rate $K\lambda$, death rate 0 and starting with k particles. If $N_0 < +\infty$ almost surely then for all $t > 0$ we have $N_t < +\infty$ almost surely; hence for all $t > 0$ and $\varepsilon > 0$ there exists $n(t, \varepsilon)$ such that, for all $i \in I$,

$$\mathbb{P}\left(N_t \leq n(t, \varepsilon) \mid \eta_0 = \eta\right) > 1 - \varepsilon,$$

for all η such that $\sum_{x \in A_i} \eta(x) = k$, $\eta(x) = 0$ for all $x \notin A_i$. Define $\bar{n} = n(\bar{t}, \varepsilon)$. We note that for any event A such that $\mathbb{P}\left(A \mid \eta_0 = \eta\right) > 1 - \varepsilon$ we have

$$\mathbb{P}\left(A \mid N_t \leq \bar{n}, \eta_0 = \eta\right) \geq \mathbb{P}\left(A, N_t \leq \bar{n} \mid \eta_0 = \eta\right) \geq 1 - 2\varepsilon. \quad (4.6)$$

Choose $m \geq \bar{n}$: then $\eta_t = \eta_t^m$ for all $t \leq \bar{t}$ on $\{N_{\bar{t}} \leq \bar{n}\} = \bigcap_{t \leq \bar{t}} \{N_t \leq \bar{n}\}$. Thus, (4.4) and (4.6) imply (4.5). The claim for $\bar{\eta}_t^m$ is proven analogously.

Step 4. For all $\lambda > \lambda_w$ (or $\lambda > \lambda_s$) and for every $\varepsilon > 0$, for all sufficiently large m , there exists a one-dependent oriented percolation on $I \times \vec{\mathbb{N}}$ (with probability $1 - 2\varepsilon$ of opening simultaneously all edges from a vertex and 2ε of opening no edges) such that the probability of survival of the BRW_m (starting at time 0 from a configuration η such that $\sum_{x \in A_{i_0}} \eta(x) = k$ and $\eta(x) = 0$ for all $x \notin A_{i_0}$) is larger than the probability that there exists an infinite cluster containing $(i_0, 0)$.

In order to prove Step 4 using Step 3, we need another auxiliary process, namely $\{\hat{\eta}_t\}_{t \geq 0}$ defined from $\bar{\eta}_t$ suppressing all newborns after that the \bar{n} -th particle is born. If $m \geq \bar{n}$, then $\hat{\eta}_t \leq \bar{\eta}_t^m \leq \eta_t^m$ for all $t \geq 0$, and for all $t \leq \bar{t}$, $\hat{\eta}_t = \bar{\eta}_t^m$ on $\{N_{\bar{t}} \leq \bar{n}\}$.

Consider an edge $((i, n), (j, n+1))$ in $(I, \mathcal{E}(I)) \times \vec{\mathbb{N}}$: let it be open if η_t^m has at least k individuals in A_i at time $n\bar{t}$ and in A_j at time $(n+1)\bar{t}$. Thus the probability of weak survival of η_t^m is bounded from below by the probability that there exists an infinite cluster containing $(i_0, 0)$ in this percolation on $I \times \vec{\mathbb{N}}$, and, if A_{i_0} is finite, the probability of strong survival is bounded from below by the probability that the cluster contains infinitely many points in $\{(i_0, l) : l \in \mathbb{N}\}$ (we suppose that we start with k particles in A_{i_0}). Let ν_1 be the associated percolation measure. Unfortunately this percolation is neither independent nor one-dependent. In fact the opening procedure of the edges $((i, n), (j, n+1))$ and $((i_1, n), (j_1, n+1))$ may depend respectively on two different progenies of particles overlapping on a vertex x_0 . This may cause dependence since if in x_0 there are already m particles then newborns are not allowed.

To avoid this difficulty we will choose m sufficiently large and consider another percolation on $I \times \vec{\mathbb{N}}$. Let $\hat{\eta}_{i_0, 0, t}$ (constructed from η_t with the usual removal rules) start with k particles in A_{i_0} : we open all edges $(i_0, 0) \rightarrow (j, 1)$ if $\hat{\eta}_{i_0, 0, \bar{t}}$ has at least k particles in A_j . With a slight abuse of notation we define $\hat{\eta}_t := \hat{\eta}_{i_0, 0, t}$ for all $t \in (0, \bar{t}]$. We construct the process $\{\hat{\eta}_t\}_{t \geq 0}$ by iteration on the time intervals $(n\bar{t}, (n+1)\bar{t}]$ where $n \in \mathbb{N}$. We did it in the case $n = 0$.

Suppose we constructed $\{\widehat{\eta}_t\}_{t \geq 0}$ for $t \in [0, n\bar{t}]$: we construct it now for $t \in (n\bar{t}, (n+1)\bar{t}]$. For all j such that $\widehat{\eta}_{n\bar{t}}$ has at least k particles in A_j we start a process $\{\widehat{\eta}_{j,n,t}\}_{t \geq 0}$ with initial configuration given by k particles in A_j (chosen among those $\widehat{\eta}_{n\bar{t}}$ had there) and zero elsewhere. Note that the $\{\widehat{\eta}_{j,n,t}\}_j$ are independent conditioned to $\widehat{\eta}_{n\bar{t}}$. As before we define $\widehat{\eta}_t := \sum_j \widehat{\eta}_{j,n,t-n\bar{t}}$ for all $t \in (n\bar{t}, (n+1)\bar{t}]$.

Choosing m sufficiently large we have that $\widehat{\eta}_t \leq \bar{\eta}_t^m \leq \eta_t^m$. Indeed it is enough to choose $m \geq 2\bar{n}H$, where $H \in \mathbb{N}$ is the supremum over x of the number of paths of length n_0 which contain a fixed vertex x and \bar{n} is the same as in equation (4.6); H is finite since (X, μ) is with bounded geometry. To the process $\{\widehat{\eta}_t\}_{t \geq 0}$ we associate a percolation ν_2 , such that $\nu_1 \geq \nu_2$, in the following way: we open the edge from (i, n) to $(j, n+1)$ if $\widehat{\eta}_{i,n,\bar{t}}$ has at least k particles in A_j .

By equation (4.5), the probability of opening all edges from (i, n) according to ν_2 (conditioned to the event “ $(i_0, 0)$ is connected to (i, n) ”) is at least $1 - 2\varepsilon$. We observe that the set of open edges from (i, n) depends only on the progenies of the k particles alive in A_i at time $n\bar{t}$ (hence on $\widehat{\eta}_{i,t}$, which are conditionally independent with respect to $\widehat{\eta}_{n\bar{t}}$).

We construct a one-dependent percolation (i.e. a percolation where the openings of edges from different vertices are independent) ν_3 as follows: we open simultaneously all the edges from (i, n) with probability $1 - 2\varepsilon$ and no edge with probability 2ε . Finally, the percolation processes ν_2 and ν_3 can be coupled in such a way that, almost surely, the cluster Ω_3 containing the starting point $(i_0, 0)$ in percolation ν_3 is a subset of the cluster Ω_2 containing $(i_0, 0)$ in percolation ν_2 . Indeed, if $\beta_n := \{(j, m) \in I \times \vec{\mathbb{N}} : m \leq n\}$ then it is not difficult to prove by induction on n that $\mathbb{P}(\Omega_3 \cap \beta_n \subseteq \Omega_2 \cap \beta_n) = 1$ (where \mathbb{P} is the probability associated to the coupled processes). Hence we have that the probability of existence of an infinite cluster containing $(i_0, 0)$ in the percolation ν_2 is larger than or equal to the probability of the same event in the percolation ν_3 . This concludes Step 4.

We note that the trick is to fix a suitable $(I, \mathcal{E}(I))$ and prove Step 2 for all $\lambda > \lambda_w$: then by Steps 4 and 1, for all sufficiently large m , the λ -BRW $_m$ survives with positive probability and we deduce that $\lambda_w^m \xrightarrow{m \rightarrow \infty} \lambda_w$. On the other hand, to show that $\lambda_s^m \xrightarrow{m \rightarrow \infty} \lambda_s$, we need to prove Step 2 with a choice of at least one A_i finite, say A_{i_0} , and I containing a copy of \mathbb{Z} or \mathbb{N} as a subgraph. Indeed the infinite open cluster in a supercritical Bernoulli bond percolation in $\mathbb{Z} \times \vec{\mathbb{N}}$ or $\mathbb{N} \times \vec{\mathbb{N}}$ with probability 1 has an infinite intersection with the set $\{(0, n) : n \in \mathbb{N}\}$. As a consequence, in the supercritical case we have, with positive probability, an infinite open cluster in $\mathbb{Z} \times \vec{\mathbb{N}}$ (resp. $\mathbb{N} \times \vec{\mathbb{N}}$) which contains the origin $(0, 0)$ and infinite vertices of the set $\{(0, n) : n \geq 0\}$. This (again by Steps 3 and 4) implies that, with positive probability, the λ -BRW $_m$ starting with k particles in A_{i_0} has particles alive in A_{i_0} at arbitrarily large times. Being A_{i_0} finite yields the conclusion.

Remark 4.1. *The previous set of steps represents the skeleton of the proofs of Theorems 5.6 and 6.4. In Theorem 6.1 we need a generalization of this approach. We sketch here the main differences. We choose an oriented graph $(W, \mathcal{E}(W))$ and a family of subsets of X , $\{A_{(i,n)}\}_{(i,n) \in W}$ such that*

- *W is a subset of the set $\mathbb{Z} \times \mathbb{N}$ (note that this is an inclusion between sets not between graphs);*
- *for all $n \in \mathbb{N}$ we have that $\{A_{(i,n)}\}_{i:(i,n) \in W}$ is a collection of disjoint subsets of X ;*
- *$(i, n) \rightarrow (j, m)$ implies $m = n + 1$.*

The analog of Step 2 is the following: for all sufficiently large λ (for instance $\lambda > \lambda_s$ or $\lambda > \lambda_w$) and for every $\varepsilon > 0$, there exists $\bar{t} > 0$ and $k \in \mathbb{N}$, such that, for all $n \in \mathbb{N}$, $i \in \mathbb{Z}$, and for all η such that $\sum_{x \in A_{(i,n)}} \eta(x) = k$,

$$\mathbb{P}\left(\forall j : (i, n) \rightarrow (j, n+1), \sum_{x \in A_{(j,n)}} \eta_{(n+1)\bar{t}}(x) \geq k \mid \eta_{m\bar{t}} = \eta\right) > 1 - \varepsilon.$$

Step 3 is the same as before and the percolation described in Step 4 now concerns the graph $(W, \mathcal{E}(W))$ (instead of $(I, \mathcal{E}(I)) \times \vec{\mathbb{N}}$ as it was before).

5. APPROXIMATION OF λ_s BY λ_s^m

We choose the initial configuration as δ_o (where o is a fixed vertex in X) and we first study the expected value of the number of individuals in one site at some time, that is $\mathbb{E}^{\delta_o}(\eta_t(x))$. This is done using the semigroup S_t , indeed if we define the evaluation maps $e_x(\eta) := \eta(x)$ for any $\eta \in \mathbb{N}^X$ and $x \in X$, then $\mathbb{E}^\eta(\eta_t(x)) = S_t e_x(\eta)$.

By standard theorems (see [9], or, since \mathbb{N}^X is not locally compact, [15] and [2]),

$$\frac{d}{dt} S_t e_x \Big|_{t=t_0} = S_{t_0} \mathcal{L} e_x,$$

from which we deduce

$$\frac{d}{dt} \mathbb{E}^\eta(\eta_t(x)) = -\mathbb{E}^\eta(\eta_t(x)) + \lambda \sum_{z \in X} \mu(z, x) \mathbb{E}^\eta(\eta_t(z)). \quad (5.7)$$

It is not difficult to verify that

$$\mathbb{E}^{\delta_{x_0}}(\eta_t(x)) = \sum_{n=0}^{\infty} \mu^{(n)}(x_0, x) \frac{(\lambda t)^n}{n!} e^{-t}. \quad (5.8)$$

Remark 5.1. *For all $x, x_0 \in X$, for all $\lambda > 0$ and $n \in \mathbb{N}$,*

$$\mathbb{E}^{\delta_{x_0}}(\eta_n(x)) \geq \mu^{(n)}(x_0, x) \frac{\lambda^n n^n}{n!} e^{-n} \sim \mu^{(n)}(x_0, x) \frac{\lambda^n}{\sqrt{2\pi n}},$$

and the same inequality holds, if $n_0 \geq n$, with $\bar{\eta}_n$ in place of η_n .

Depending on λ , we may characterize the behaviour of the expected number of descendants at a fixed site.

Lemma 5.2. *Let us fix $x \in X$. If $\lambda < R_\mu$ then $\lim_{t \rightarrow +\infty} \mathbb{E}^{\delta_{x_0}}(\eta_t(x)) = 0$; if $\lambda > R_\mu$ then $\lim_{t \rightarrow +\infty} \mathbb{E}^{\delta_{x_0}}(\eta_t(x)) = +\infty$.*

Proof. Let $\lambda < R_\mu$. For all $\varepsilon > 0$ there exists n_0 such that $\mu^{(n)}(x_0, x) < 1/(R_\mu - \varepsilon)^n$ for all $n \geq n_0$. If $\varepsilon = (R_\mu - \lambda)/2$ then $\lambda^n \mu^{(n)}(x_0, x) \leq \left(\frac{2\lambda}{R_\mu + \lambda}\right)^n$ for all $n \geq n_0$, hence $\mathbb{E}^{\delta_{x_0}}(\eta_t(x)) \leq Q(t)e^{-t} + e^{-t(R_\mu - \lambda)/(R_\mu + \lambda)} \rightarrow 0$ as $t \rightarrow \infty$ (Q is a polynomial of degree at most $n_0 - 1$).

Let $\lambda > R_\mu$. If $k, r \in \mathbb{N}$ are such that $\mu^{(k)}(x_0, x) > 0$ and $\mu^{(ir)}(x, x) > 0$ for all $i \in \mathbb{N}$ then

$$\mathbb{E}^{\delta_{x_0}}(\eta_t(x)) \geq \mu^{(k)}(x_0, x) e^{-t} \sum_{i=0}^{\infty} \mu^{(ir)}(x, x) (\lambda t)^{ir+k} / (ri+k)!$$

Let us define $a_n := \mu^{(nr)}(x, x)$; clearly $a_{n+m} \geq a_n a_m$ and $R_\mu = 1/\lim_{n \rightarrow \infty} \sqrt[n]{a_n}$ (since $\{a_n\}_{n \in \mathbb{N}}$ is supermultiplicative then the limit exists).

We prove now that for any nonnegative, supermultiplicative sequence $\{a_n\}_{n \in \mathbb{N}}$, if $\lambda > R_\mu$ then

$$\lim_{t \rightarrow \infty} e^{-t} \sum_{i=0}^{\infty} a_i \frac{(\lambda t)^{ir+k}}{(ir+k)!} = +\infty.$$

Indeed, let n_0 be such that $a_{n_0} \geq 2^{rn_0} (R + \lambda)^{-rn_0}$ and define $f(t) := e^{-t} \sum_{i=0}^{\infty} a_i (\lambda t)^{ir+k} / (ir+k)!$.

Clearly

$$f(t) \geq e^{-t} \left(\frac{R + \lambda}{2}\right)^k \sum_{i=0}^{\infty} \frac{(\lambda' t)^{in_0r+k}}{(in_0r+k)!}$$

where $\lambda' = 2\lambda/(R + \lambda) > 1$. For all $i \in \mathbb{N}$ we have that

$$\frac{(\lambda' t)^{in_0r+k}}{(in_0r+k)!} + \frac{(\lambda' t)^{in_0r+k+1}}{(in_0r+k+1)!} + \dots + \frac{(\lambda' t)^{in_0r+k+n_0r-1}}{(in_0r+k+n_0r-1)!} \leq \frac{(\lambda' t)^{in_0r+k}}{(in_0r+k)!} \frac{(\lambda' t)^{n_0r} - 1}{\lambda' t - 1}$$

thus

$$f(t) \geq e^{-t} \left(\frac{R + \lambda}{2}\right)^k \frac{\lambda' t - 1}{(\lambda' t)^{n_0r} - 1} \sum_{i=k}^{\infty} \frac{(\lambda' t)^i}{i!} \rightarrow \infty$$

exponentially if $t \rightarrow \infty$ (since $\lambda' > 1$). □

In the following lemma we prove that, when $\lambda > R_\mu$, if at time 0 we have one individual at each of l sites x_1, \dots, x_l , then, given any choice of l sites y_1, \dots, y_l , after some time the expected number of descendants in y_i of the individual in x_i exceeds 1 for all $i = 1, \dots, l$. The proof is easy and we omit it.

Lemma 5.3. *Let us consider a finite set of couples $\{(x_j, y_j)\}_{j=0}^l$; if $\lambda > R_\mu$ then there exists $t = t(\lambda) > 0$ such that $\mathbb{E}^{\delta_{x_j}}(\eta_t(y_j)) > 1, \forall j = 0, 1, \dots, l$. Moreover, $\mathbb{E}^{\delta_{x_j}}(\bar{\eta}_t(y_j)) > 1$ when n_0 is sufficiently large.*

So far we got results on the expected number of individuals, now we show that, when $\lambda > R_\mu$, for all sufficiently large $k \in \mathbb{N}$, given k particles in a site x at time 0, “typically” (i.e. with arbitrarily large probability) after some time we will have at least k individuals in each site of a fixed finite set Y . Analogously, starting with l colonies of size k (in sites x_1, \dots, x_l respectively), each of them

will, after a sufficiently long time, spread at least k descendants in every site of a corresponding (finite) set of sites Y_i .

Lemma 5.4. *Suppose that $\lambda > R_\mu$.*

- (1) *Let us fix $x \in X$, Y a finite subset of X and $\varepsilon > 0$. Then there exists $t = t(\lambda, x) > 0$ (independent of ε), $k(\varepsilon, x, Y, \lambda)$ such that, for all $k \geq k(\varepsilon, x, Y, \lambda)$,*

$$\mathbb{P} \left(\bigcap_{y \in Y} (\eta_t(y) \geq k) \mid \eta_0(x) = k \right) > 1 - \varepsilon.$$

The claim holds also with $\{\bar{\eta}_t\}_{t \geq 0}$ in place of $\{\eta_t\}_{t \geq 0}$ when n_0 is sufficiently large.

- (2) *Let us fix a finite set of vertices $\{x_i\}_{i=1, \dots, m}$, a collection of finite sets $\{Y_i\}_{i=1, \dots, l}$ of vertices of X and $\varepsilon > 0$. Then there exists $t = t(\lambda, \{x_i\}, \{Y_i\})$ (independent of ε), $k(\varepsilon, \{x_i\}, \{Y_i\}, \lambda)$ such that, for all $i = 1, \dots, l$ and $k \geq k(\varepsilon, \{x_i\}, \{Y_i\}, \lambda)$,*

$$\mathbb{P} \left(\bigcap_{y \in Y_i} (\eta_t(y) \geq k) \mid \eta_0(x_i) = k \right) > 1 - \varepsilon.$$

The claim holds also with $\{\bar{\eta}_t\}_{t \geq 0}$ in place of $\{\eta_t\}_{t \geq 0}$ when n_0 is sufficiently large.

Proof.

- (1) If we denote by $\{\xi_t\}_t$ the branching process starting from $\xi_0 = \delta_x$ then, by Lemma 5.3, we can choose t such that $\mathbb{E}^{\delta_x}(\xi_t(y)) > 1$ for all $y \in Y$. We can write $\eta_t(y) = \sum_{j=1}^k \xi_{t,j}(y)$ where $\xi_{t,j}(y)$ denotes the number of descendants in y of the j -th initial particle; note that $\{\xi_{t,j}(y)\}_{j \in \mathbb{N}}$ is an iid family with $\mathbb{E}(\xi_{t,j}(y)) = \mathbb{E}^{\delta_x}(\xi_t(y))$ and $\text{Var}(\xi_{t,j}(y)) =: \sigma_{t,y}^2$. Since $\xi_{t,j}$ is stochastically dominated by a continuous time branching process with birth rate λ , it is clear that $\sigma_{t,y}^2 < +\infty$. Thus by the Central Limit Theorem, given any $\delta > 0$, if k is sufficiently large,

$$\delta \geq \left| \mathbb{P} \left(\sum_{j=1}^k \xi_{t,j}(y) \geq z \right) - 1 + \Phi \left(\frac{z - k\mathbb{E}^{\delta_x}(\xi_t(y))}{\sqrt{k}\sigma_{t,y}} \right) \right|$$

uniformly with respect to $z \in \mathbb{R}$. Whence there exists $k(\delta, x, y)$ such that, for all $k \geq k(\delta, x, y)$,

$$\mathbb{P}(\eta_t(y) \geq k) \geq 1 - \Phi \left(\sqrt{k} \frac{1 - \mathbb{E}^{\delta_x}(\eta_t(y))}{\sigma_{t,y}} \right) - \delta \geq 1 - 2\delta,$$

since $\sqrt{k}(1 - \mathbb{E}^{\delta_x}(\eta_t(y)))/\sigma_{t,y} \rightarrow -\infty$ as $k \rightarrow +\infty$. Take $k(\delta, x, Y) := \max_{y \in Y} k(\delta, x, y) < +\infty$, and let D be the cardinality of Y . Hence, for all $k \geq k_x$,

$$\mathbb{P} \left(\bigcap_{y \in Y} (\eta_t(y) \geq k) \mid \eta_0(x) = k \right) \geq 1 - 2D\delta.$$

The assertion for $\bar{\eta}_t$ follows from Lemma 5.3.

- (2) Let $\{\xi_t\}_{t \geq 0}$ be as before and choose t such that $\mathbb{E}^{\delta_{x_i}}(\xi_t(y)) > 1$ for all $y \in Y_i$ and for all $i = 1, \dots, l$. According to (1) above we fix k_i such that, for all $k \geq k_i$,

$$\mathbb{P} \left(\bigcap_{y \in Y_i} (\eta_t(y) \geq k) \mid \eta_0(x_i) = k \right) \geq 1 - \varepsilon.$$

Take $k \geq \max_{i=1, \dots, l} k_i$ to conclude. The assertion for $\bar{\eta}_t$ follows from Lemma 5.3. □

Remark 5.5. Note that Remark 5.1 and Lemmas 5.3 and 5.4 can be restated for the process $\{\bar{\eta}_t^m\}_{t \geq 0}$ if m is sufficiently large.

We say that (X, μ) is quasi-transitive if there exists a finite partition of X such that for all couples (x, y) in the same class there exists a bijection γ on X satisfying $\gamma(x) = y$ and, for all $a, b \in X$, $\mu(\gamma(a), \gamma(b)) = \mu(a, b)$ (when the last condition holds we say that μ is γ -invariant). In particular if $\mu(x, y) = p(x, y)$ where P is the simple random walk on X then it is γ -invariant for any automorphism γ . Roughly speaking, if μ is γ -invariant then the graph is identical seen from x and $\gamma(x)$.

Theorem 5.6.

If at least one of the following conditions holds

- (1) (X, μ) is quasi-transitive;
- (2) (X, μ) is connected and there exists γ bijection on X such that
 - (a) μ is γ -invariant;
 - (b) for some $x_0 \in X$ we have $x_0 = \gamma^n x_0$ if and only if $n = 0$;

then

$$\lim_{m \rightarrow +\infty} \lambda_s^m = \lambda_s \geq \lim_{m \rightarrow +\infty} \lambda_w^m \geq \lambda_w.$$

Moreover if $\lambda_s = \lambda_w$ then $\lambda_w^m \downarrow_{m \rightarrow +\infty} \lambda_w$.

Proof. Remember that $\lambda_s = R_\mu$.

- (1) Let us collect one vertex from each orbit into the (finite) set $\{x_i\}_{i=1, \dots, l}$ and let $Y_i := \{y \in X : x_i \rightarrow y\}$. Fix $\lambda > R_\mu$, $I = X$, $\mathcal{E}(I) = \{(x, y) : (x, y) \in \mathcal{E}(X) \text{ or } (y, x) \in \mathcal{E}(X)\}$ and $A_x = \{x\}$. Note that $(I, \mathcal{E}(I))$ coincides with $(X, \mathcal{E}(X))$ if the latter is non oriented. By these choices, Lemma 5.4 yields Step 2. To prove that the percolation on $(I, \mathcal{E}(I)) \times \vec{\mathbb{N}}$ has two phases (that is, $(I, \mathcal{E}(I))$ is a suitable choice for Step 1) we note that the existence of the supercritical phase for the Bernoulli percolation on $X \times \vec{\mathbb{N}}$ follows from the fact that the graph \mathbb{N} is a subgraph of X . Moreover in the supercritical Bernoulli percolation on $\mathbb{N} \times \vec{\mathbb{N}}$ with positive probability the infinite open cluster contains $(0, 0)$ and intersects the y -axis

infinitely often. Hence by Steps 3 and 4 we have that, for all sufficiently large m , $\lambda_s^m \leq \lambda$ and this yields the result.

(2) Fix $\lambda > R_\mu$; by Lemma 5.4, for sufficiently large n_0 ,

$$\mathbb{P}\left(\bar{\eta}_t(\gamma x) \geq k \mid \bar{\eta}_0(x) = k\right) > 1 - \varepsilon \quad \text{and} \quad \mathbb{P}\left(\bar{\eta}_t(x) \geq k \mid \bar{\eta}_0(\gamma x) = k\right) > 1 - \varepsilon.$$

This implies

$$\mathbb{P}\left(\bar{\eta}_t(\gamma^n x) \geq k \mid \bar{\eta}_0(\gamma^{n-1}x) = k\right) > 1 - \varepsilon \quad \text{and} \quad \mathbb{P}\left(\bar{\eta}_t(\gamma^{n-1}x) \geq k \mid \bar{\eta}_0(\gamma^n x) = k\right) > 1 - \varepsilon$$

for all $n \in \mathbb{Z}$ since μ is γ -invariant; one more time, Steps 3 and 4 yield $\lambda_s^m \leq \lambda$ (for sufficiently large m) and the claim (here $I = \mathbb{Z}$ and $A_i = \{\gamma^i x_0\}$).

□

6. APPROXIMATION OF λ_w BY λ_w^m

From now on we set $\mu(x, y) = p(x, y)$ where P is a stochastic matrix. We stress that in this case $\lambda_w = 1$. We are concerned with the question whether $\lambda_w^m \downarrow \lambda_w = 1$ or not. Under the hypotheses of Theorem 5.6, this is the case when the BRW has no pure weak phase (i.e. $R = 1$). The interesting case is $R > 1$. Most natural examples are drifting random walks on \mathbb{Z}^d and the simple random walk on homogeneous trees. In both cases we show that $\lambda_w^m \xrightarrow{m \rightarrow \infty} \lambda_w$.

Theorem 6.1. *Let P be a random walk on \mathbb{Z} such that $p(i, i+1) = p$, $p(i, i-1) = q$ and $p(i, i) = 1 - p - q$ for all $i \in \mathbb{Z}$. Then $\lim_{m \rightarrow +\infty} \lambda_w^m = 1 = \lambda_w$.*

Proof. We consider $\alpha, \beta \in (0, 1)$, $\alpha \leq \beta \leq (1 + \alpha)/2$ and write

$$\begin{aligned} p^{(n)}(0, \alpha n) &= \sum_{i=\alpha n}^{(1+\alpha)n/2} \binom{n}{i, i - \alpha n, n - 2i + \alpha n} p^i q^{i - \alpha n} (1 - p - q)^{n - 2i + \alpha n} \\ &\geq \binom{n}{\beta n, (\beta - \alpha)n, (1 - 2\beta + \alpha)n} p^{\beta n} q^{(\beta - \alpha)n} (1 - p - q)^{(1 - 2\beta + \alpha)n} \\ &\stackrel{n \rightarrow \infty}{\sim} \frac{1}{2\pi n \sqrt{\beta(\beta - \alpha)(1 - 2\beta + \alpha)}} \left(\frac{p^\beta q^{\beta - \alpha} (1 - p - q)^{1 - 2\beta + \alpha}}{\beta^\beta (\beta - \alpha)^{\beta - \alpha} (1 - 2\beta + \alpha)^{1 - 2\beta + \alpha}} \right)^n. \end{aligned}$$

Thus if $\lambda > 1$, $\mathbb{E}^{\delta_0}(\eta_n(\alpha n))$ is bounded from below by a quantity which is asymptotic to

$$\frac{1}{(2\pi n)^{3/2} \sqrt{\beta(\beta - \alpha)(1 - 2\beta + \alpha)}} (g_\lambda(\alpha, \beta))^n.$$

where

$$g_\lambda(\alpha, \beta) = \frac{\lambda p^\beta q^{\beta - \alpha} (1 - p - q)^{1 - 2\beta + \alpha}}{\beta^\beta (\beta - \alpha)^{\beta - \alpha} (1 - 2\beta + \alpha)^{1 - 2\beta + \alpha}}.$$

Note that $g_\lambda(p - q, p) = \lambda$, thus we may find $\alpha_1 < \alpha_2 \leq \beta_1 < \beta_2$ (with $\beta_i \leq (1 + \alpha_i)/2$, $i = 1, 2$) such that $g_\lambda(x, y) > 1$, for all $(x, y) \in [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$. By taking $n = \bar{n}$ sufficiently large one can find three distinct integers d_1, d_2 and d_3 such that $\alpha_1 n \leq d_1 < d_2 \leq \alpha_2 n$, $\beta_1 n \leq d_3 \leq \beta_2 n$ and $g_\lambda(d_l/n, d_3/n) > 1$, $l = 1, 2$.

By reasoning as in Lemma 5.4 we have that, for all $\lambda > 1$ and $\varepsilon > 0$, there exists \bar{t} , $k = k(\varepsilon, \lambda)$ such that, for all $i \in \mathbb{Z}$, for all n_0 sufficiently large,

$$\mathbb{P}(\bar{\eta}_{\bar{t}}(i+j) \geq k, j = d_1, d_2 | \bar{\eta}_0 = k\delta_i) > 1 - \varepsilon.$$

Since k and \bar{t} are independent of i we have proven the general version of Step 2 as stated in Remark 4.1 (where $W = \{a(d_1, 1) + b(d_2, 1) : a, b \in \mathbb{N}\}$, $A_{(i,n)} = \{i\}$ and $(i, n) \rightarrow (j, n+1)$ if and only if $j - i = d_1$ or $j - i = d_2$). \square

In view of Corollary 6.2 and Theorem 6.4 it is useful to introduce the concept of local isomorphism which allows to extend some results from \mathbb{Z} to more general graphs. Given two weighted graphs (X, μ) and (I, ν) , we say that a map $f : X \rightarrow I$ is a *local isomorphism* of X on I if for all $x \in X$ and $i \in I$ we have $\sum_{z \in f^{-1}(i)} \mu(x, z) = \nu(f(x), i)$.

In this case it is clear that, if we consider the partition of X given by $\{A_i\}_{i \in I}$ where $A_i := f^{-1}(i)$, we can easily compute the expected number of particles alive at time t in A_i starting from a single particle alive in x at time 0

$$\sum_{z \in A_i} \mathbb{E}_\mu^{\delta_x}(\eta_t(z)) = \mathbb{E}_\nu^{\delta_{f(x)}}(\xi_t(i)) \quad (6.9)$$

(where $\{\xi_t\}_{t \geq 0}$ is a branching random walk on (I, ν)), since $\sum_{z \in f^{-1}(i)} \mu^{(n)}(x, z) = \nu^{(n)}(f(x), i)$, for all $n \in \mathbb{N}$. We note that the latter depends only on $f(x)$ and i . As a consequence $R_\mu \geq R_\nu$.

Corollary 6.2. *If P is a translation invariant random walk on \mathbb{Z}^d then $\lim_{k \rightarrow +\infty} \lambda_w^k = 1 = \lambda_w$.*

Proof. Let $\{Z_n\}_{n \in \mathbb{N}}$ be a realization of the random walk and $A_i = \{x \in \mathbb{Z}^d : x(1) = i\}$. Note that

$$\mathbb{P}(Z_{n+1} \in A_j | Z_n = w) = \tilde{p}(i, j), \quad \forall w \in A_i,$$

where \tilde{P} is a random walk on \mathbb{Z} with $p = p(0, e_1)$, $q = p(0, -e_1)$. Using equation (6.9) and reasoning as in the proof of the previous theorem, we conclude. \square

Remark 6.3. *The argument of the previous corollary may be applied to a more general case: let (Y, Q) be a random walk and (\mathbb{Z}, P) be as in Theorem 6.1. Consider $Y \times \mathbb{Z}$ with transition matrix $\alpha(\mathbb{I}^Y \times P) + (1 - \alpha)(Q \times \mathbb{I}^{\mathbb{Z}})$, where $\alpha \in (0, 1)$ and by \mathbb{I} we denote the identity matrix (on the superscripted space). Using the projection on the second coordinate one proves that $\lim_{m \rightarrow +\infty} \lambda_w^m = 1 = \lambda_w$.*

Theorem 6.4. *If (X, P) is the simple random walk on the homogeneous tree of degree r then $\lim_{m \rightarrow +\infty} \lambda_w^m = 1 = \lambda_w$.*

Proof. Fix an end τ in X and a root $o \in X$ and define the map $h : X \rightarrow \mathbb{Z}$ as the usual height (see [25] page 129). Define $A_k = h^{-1}(k)$, $k \in \mathbb{Z}$ (these sets are usually referred to as horocycles).

The projection of the simple random walk on X onto \mathbb{Z} is a random walk with transition matrix \tilde{P} where $\tilde{p}(a, a+1) = 1 - 1/r$ and $\tilde{p}(a, a-1) = 1/r$. Note that for all $x \in X$

$$\sum_{y \in A_k} p^{(n)}(x, y) = \tilde{p}^{(n)}(h(x), k).$$

By using equation (6.9) and reasoning as in Lemma 5.4 and Theorem 6.1 we have that, for all $i \in \mathbb{N}$, and some integers d_1, d_2 ,

$$\mathbb{P}\left(\sum_{x \in A_{i+j}} \bar{\eta}_n(x) \geq k, j = d_1, d_2 \mid \bar{\eta}_0(x) = \eta\right) > 1 - \varepsilon,$$

for all η such that $\sum_{x \in A_i} \eta(x) = k$ and $\eta(x) = 0$ if $x \notin A_i$, and for all n_0 sufficiently large. The claim follows as in Theorem 6.1. \square

7. BRANCHING RANDOM WALKS IN RANDOM ENVIRONMENT

We use the results of Section 3 to prove some properties of the BRW in random environment.

Let (X, μ) be a non-oriented weighted graph. We consider any subgraph $(Y, \mathcal{E}(Y))$ of $(X, \mathcal{E}(X))$ as a weighted subgraph with weight function $\mathbb{1}_{\mathcal{E}(Y)}\mu$.

Given any $p \in (0, 1]$ we consider the independent Bernoulli bond percolation on $(X, \mathcal{E}(X))$ and we define the random weighted subgraphs $(Y^a, \mathcal{E}(Y^a))$ where $Y^a = X$ and $\mathcal{E}(Y^a)$ is the random set of edges resulting from the percolation process. We define $\lambda_s(Y^a) := \inf_{A \in \mathcal{A}} \lambda_s(A)$ where \mathcal{A} is the random collection of all the connected components of Y^a . This corresponds to the critical (strong) parameter of a BRW where the initial state is one particle alive at time 0 in every connected component of Y^a .

On the other hand, if there exists a nontrivial critical parameter p_c for the Bernoulli percolation on X and if the infinite cluster is unique almost surely then, for all $p > p_c$, we denote by $(Y^c, \mathcal{E}(Y^c))$ the infinite cluster and we consider the critical (strong) parameter $\lambda_s(Y^c)$. Given a sequence $\{p_n\}_{n \in \mathbb{N}}$ such that $p_n \in [0, 1]$ for all $n \in \mathbb{N}$, we consider the sequences $\{Y_n^a\}_{n \in \mathbb{N}}$ and $\{Y_n^c\}_{n \in \mathbb{N}}$ as the results of Bernoulli percolation processes on X with parameters $\{p_n\}_{n \in \mathbb{N}}$. Observe that we do not make any a priori assumption on the joint distributions of these Bernoulli percolation processes.

Here is the main result; we note that, even when $X = \mathbb{Z}^d$, we do not require μ to be the simple random walk.

- Theorem 7.1.**
- (1) If $\sum_n (1 - p_n) < +\infty$ then $\lambda_s(Y_n^a) \rightarrow \lambda_s(X)$ a.s.
 - (2) If (X, μ) is quasi-transitive then $\lambda_s(Y^a) = \lambda_s(X)$ a.s.
 - (3) If $X = \mathbb{Z}^d$ and $\sum_n (1 - p_n) < +\infty$, then $\lambda_s(Y_n^c) \rightarrow \lambda_s(\mathbb{Z}^d)$ a.s.
 - (4) If $X = \mathbb{Z}^d$, μ is translation invariant and $p > p_c$ then $\lambda_s(Y^c) = \lambda_s(\mathbb{Z}^d)$ a.s.

Proof.

- (1) By using the Borel-Cantelli Lemma, we have that any finite connected subgraph of X is eventually contained in a (random) connected component of Y_n^a almost surely (since $\sum_n(1 - p_n^k) < +\infty$ for all $k \in \mathbb{N}$). Theorem 3.2 yields the conclusion.
- (2) In this case if we take an infinite orbit X_0 then for any $m \in \mathbb{N}$, with probability 1, Y^a contains at least one ball $B(z_0, m)$, centered on a vertex $z_0 \in X_0$ and of radius m , with all open edges. Indeed for each m there exists an infinite subset of X_0 , say X_1 , such that the balls $\{B(z, m) : z \in X_1\}$ are pairwise disjoint. Hence the events “all the edges in $B(z, m)$ are open” with $z \in X_1$ are independent and they have the same probability, thus almost surely (with respect to the percolation probability), at least one occurs. Let $B(z_0, m)$ be the corresponding ball (z_0 being random).

Since the critical parameter of a ball $\lambda_s(B(z_0, m))$ does not depend on the choice of $z_0 \in X_0$, then using Theorem 3.2, we have that $\lambda_s(X) \leq \lambda_s(Y^a) \leq \lambda_s(B(z_0, m)) \rightarrow \lambda_s(X)$ as $m \rightarrow +\infty$.

- (3) Note that $p_n > p_c$ eventually, hence $\lambda_s(Y_n^c)$ is well-defined for all sufficiently large n . What we need to prove is that, almost surely, any edge is eventually connected to the infinite cluster. To this aim we apply the FKG inequality obtaining that the probability of the event “the edge (x, y) is open and connected to the infinite cluster Y_n^c ” is bigger than $p_n\theta(p_n)$ (where $\theta(p)$ is the probability that a fixed vertex x is contained in the infinite cluster, when each edge is open with probability p). According to Theorem 8.92 of [10], θ is a differentiable function on $[0, 1]$ hence $1 - p\theta(p) \sim (1 - p)(1 + \theta'(1))$ and this implies $\sum_n(1 - p_n\theta(p_n)) < +\infty$. The Borel-Cantelli Lemma yields the conclusion.
- (4) It is tedious but essentially straightforward to prove that, for any $m \in \mathbb{N}$, with probability 1, Y^c contains an hypercube Q_m of side-length m with all open edges; as before, Theorem 3.2 yields the result.

□

Note that if $p_n \uparrow 1$ and the joint distribution of the family of Bernoulli percolation processes (indexed by n) is the natural monotone coupling, then the conclusions of Theorem 7.1 (1) and (3) hold without requiring $\sum_n(1 - p_n) < +\infty$. This stronger assumption nevertheless allows us to obtain a convergence result for all possible couplings.

8. FINAL REMARKS

At this point the theory of spatial approximation (see Section 3) is quite complete as far as we are concerned with the basic questions on the convergence of the critical parameters. Indeed we proved results in this direction (see Theorem 3.2) for the strong parameter under reasonable assumptions, while the question on the weak critical parameter, in the pure spatial approximation by finite subsets, is uninteresting (see Remark 3.3). It is possible to further investigate the convergence

of the sequence of weak critical parameters under the hypotheses of Theorem 3.2 by using the characterization $\lambda_w = 1/\limsup_n \sqrt[n]{\sum_{y \in X} \mu^{(n)}(x, y)}$ which holds in many cases (see [3] and [4] for details).

As for the approximation of the BRW by BRW_m s, we proved that, on graphs satisfying the hypotheses of Theorem 5.6 (among them, quasi-transitive graphs), $\lambda_s^m \downarrow \lambda_s$ as $m \rightarrow \infty$ and, if there is no weak phase, on \mathbb{Z}^d or on regular trees, $\lambda_w^m \downarrow \lambda_w$ as $m \rightarrow \infty$. Here are some natural questions which, as far as we know, are still open:

- can one weaken the hypothesis of Theorem 5.6 and still prove the convergence of the strong critical parameters?
- when $\lambda_w \neq \lambda_s$, is it still true that $\lambda_w^m \downarrow_{m \rightarrow \infty} \lambda_w$, at least for Cayley graphs or on quasi transitive graphs?

ACKNOWLEDGMENTS

The authors are grateful to Rick Durrett for his invaluable suggestions. They would also like to thank an anonymous referee for her comments which helped to clarify some proofs.

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