A nonexistence result for a nonlinear elliptic equation with singular and decaying potential

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Abstract

Several existence and nonexistence results are known for positive solutions $u \in D^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^{-\alpha}\,dx) \cap L^p(\mathbb{R}^N)$ to the equation

$$-\triangle u + \frac{A}{|x|^\alpha} u = u^{p-1} \quad \text{in } \mathbb{R}^N \setminus \{0\}, \quad N \geq 3,\ A, \alpha > 0,\ p > 2,$$

resting upon compatibility conditions between α and p. Letting $2_\alpha := 2N/(N-\alpha)$ and $2^*_\alpha := 2(2N-2+\alpha)/(2N-2-\alpha)$, the problem is still open for $0 < \alpha < 2$ and $2_\alpha < p \leq 2^*_\alpha$, for $2 < \alpha < N$ and $2^*_\alpha \leq p < 2_\alpha$, and for $N \leq \alpha < 2N-2$ and $p \geq 2^*_\alpha$. Here we give a negative answer to the problem of the existence of radial solutions in the first open case.

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1 Introduction

In this paper we consider the following nonlinear problem:

$$\begin{cases} -\triangle u + \dfrac{A}{|x|^\alpha} u = u^{p-1} & \text{in } \mathbb{R}^N \setminus \{0\},\ N \geq 3 \\ u > 0 & \text{in } \mathbb{R}^N \setminus \{0\} \\ u \in H^1_\alpha \cap L^p(\mathbb{R}^N) \end{cases} \tag{1}$$

where $A, \alpha > 0$, $p > 2$ and $H^1_\alpha := D^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^{-\alpha}\,dx)$ is the natural energy space related to the equation. We deal with problem (1) in the classical sense, that is, speaking about *solutions* to (1) we will always mean *classical solutions* (cf. Remark 1 below).

Problems like (1) arise for instance in the search of solitary waves for nonlinear Schrödinger and Klein-Gordon equations with potential (see e.g. [17, Chapter 7], [6], [8], the overviews in [1], [9] and the

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monographs [18], [24]) and (1) itself is a radial model problem for the so-called *zero mass* case (see [4], [7] and the references therein). In this respect, the requirement $u \in H^1_\alpha \cap L^p(\mathbb{R}^N)$ plays a preeminent role, since it is necessary for the energy of the particle represented by the solution to be finite.

Though it can be considered of quite recent investigation, problem (1) has already some history and several existence and nonexistence results are known, resting upon compatibility conditions between α and p (see [2] for a related cylindrical problem). At our knowledge, the first results are due to Terracini [22], who both proved that (1) has no solution if

$$\begin{cases} \alpha = 2 \\ p \neq 2^* \end{cases} \quad \text{or} \quad \begin{cases} \alpha \neq 2 \\ p = 2^* \end{cases}, \qquad 2^* := \frac{2N}{N-2},$$

and explicitly found all the radial solutions of (1) for $(\alpha, p) = (2, 2^*)$. As usual, 2^* denotes the critical exponent for the Sobolev embedding in dimension $N \geq 3$. The problem was subsequently addressed in [11], where it was proved that (1) has no solution if

$$\begin{cases} 0 < \alpha < 2 \\ p > 2^* \end{cases} \quad \text{or} \quad \begin{cases} \alpha > 2 \\ 2 < p < 2^* \end{cases}.$$

On the other hand, the authors obtained the existence of a radial solution to (1) provided that

$$\begin{cases} 0 < \alpha < 2 \\ 2^* + \frac{\alpha-2}{N-2} < p < 2^* \end{cases} \quad \text{or} \quad \begin{cases} \alpha > 2 \\ 2^* < p < 2^* + \frac{\alpha-2}{N-2} \end{cases}.$$

The existence and nonexistence results of [11] were then extended in [5], by showing that (1) has no solution also if

$$\begin{cases} 0 < \alpha < 2 \\ 2 < p \leq 2_\alpha \end{cases} \quad \text{or} \quad \begin{cases} 2 < \alpha < N \\ p \geq 2_\alpha \end{cases}, \qquad 2_\alpha := \frac{2N}{N-\alpha},$$

and obtaining a radial solution for every pair (α, p) such that

$$\begin{cases} 0 < \alpha < 2 \\ 2^* + 2\frac{\alpha-2}{N-2} < p < 2^* \end{cases} \quad \text{or} \quad \begin{cases} \alpha > 2 \\ 2^* < p < 2^* + 2\frac{\alpha-2}{N-2} \end{cases}.$$

A further extension of this existence condition were found in [19], [20], where the authors proved that (1) has a radial solution for all the pairs (α, p) satisfying

$$\begin{cases} 0 < \alpha < 2 \\ 2^*_\alpha < p < 2^* \end{cases} \quad \text{or} \quad \begin{cases} 2 < \alpha < 2N-2 \\ 2^* < p < 2^*_\alpha \end{cases} \quad \text{or} \quad \begin{cases} \alpha \geq 2N-2 \\ p > 2^* \end{cases}, \quad 2^*_\alpha := 2\frac{2N-2+\alpha}{2N-2-\alpha}.$$

All these known results are portrayed in the below picture of the αp-plane, where the nonexistence regions are shaded in light gray (and include both the lines $p = 2^*$ and $p = 2_\alpha$, except for the pair $(\alpha, p) = (2, 2^*)$), while dark gray means existence. The problem is still open for the pairs (α, p) in the white regions of the picture, namely, for

$$\begin{cases} 0 < \alpha < 2 \\ 2_\alpha < p \leq 2^*_\alpha \end{cases}, \qquad \begin{cases} 2 < \alpha < N \\ 2^*_\alpha \leq p < 2_\alpha \end{cases} \quad \text{and} \quad \begin{cases} N \leq \alpha < 2N-2 \\ p \geq 2^*_\alpha \end{cases}.$$

In this paper we give a negative answer to the problem of radial solutions to (1) in the first of such cases.

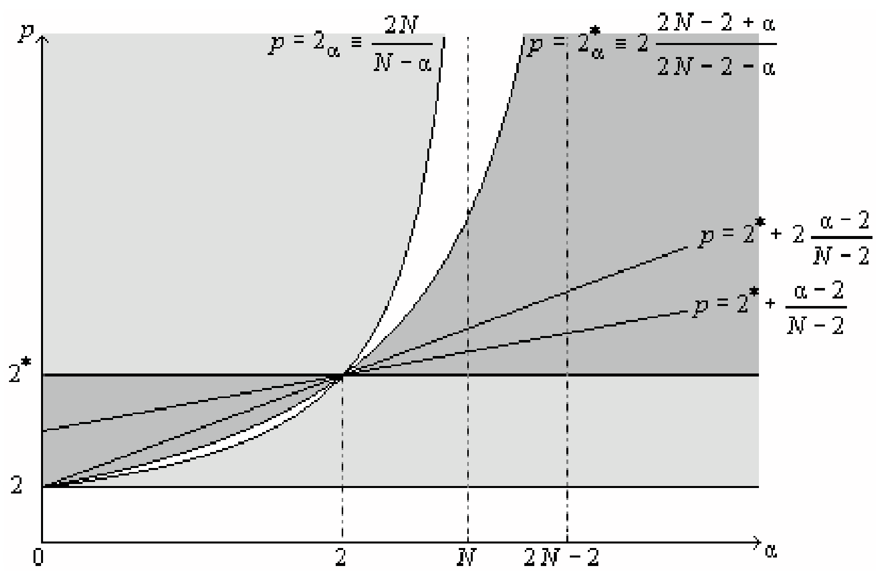

Figure 1: Existence (dark gray) and nonexistence (light gray) regions.

Remark 1 *In some of the above mentioned existence results, the authors only concern themselves with nonnegative radial weak solutions to (1) in the sense of* H_α^{-1}*, the dual space of* H_α^1 *(in [19], [20], only weak solutions in the sense of the dual space of the radial subspace of* H_α^1 *are considered, but the symmetric criticality type results of [3] apply, yielding solutions in the sense of* H_α^{-1}*). However, Schauder regularity theory and the strong maximum principle (c.f. also Remark 8 below) assure that all such solutions are actually positive classical solutions to (1).*

Our nonexistence result is the following.

Theorem 2 *Let* $0 < \alpha < 2$ *and* $2_\alpha < p \le 2_\alpha^*$*. Then (1) has no radial solution.*

Observe that, although we are concerned with classical solutions, Theorem 2 also prevents the existence of nonnegative radial weak solutions in the H_α^{-1} sense, by the same reasons used in Remark 1.

Theorem 2 will be proved in Section 3 and needs a more refined argument than the one used in [5], [11], [22], where the nonexistence results were all obtained by Pohozaev type identities. In fact, we will combine a Pucci-Serrin type identity (see [16]), which we deduce by an argument of [12], with a suitable asymptotic estimate (Lemma 10), which derives from our next result.

Theorem 3 *Let* $0 < \alpha < 2$ *and* $2_\alpha < p < 2^*$*. Assume that* u *is a radial solution of*

$$\begin{cases} -\triangle u + \dfrac{A}{|x|^\alpha} u = u^{p-1} & \text{in } \mathbb{R}^N \setminus \{0\}, \ N \ge 3 \\ u > 0 & \text{in } \mathbb{R}^N \setminus \{0\} \\ u \in H_\alpha^1 \end{cases} . \tag{2}$$

Then, as $x \to 0$, one has

$$u(x) = \begin{cases} O(1) & \text{if } p < 2^* - 1 \\ O(\ln|x|) & \text{if } p = 2^* - 1 \\ O\left(|x|^{-\frac{N-2}{2}(p-2^*+1)}\right) & \text{if } p > 2^* - 1 \end{cases}. \tag{3}$$

Theorem 3 will be proved in Section 2 and, besides yielding Theorem 2, it is interesting on its own, since it also covers the existence case $2^*_\alpha < p < 2^*$ (some results on the asymptotic behaviour of solutions at infinity can be found in [17], [13]). Observe that all the cases of (3) improve the estimate of a well known Radial Lemma for $D^{1,2}(\mathbb{R}^N)$ (see [10, Lemma A.III], where the proof also works for $0 < |x| < 1$). Moreover, they are all possible for $2_\alpha < p < 2^*$ (even for $2_\alpha < p \leq 2^*_\alpha$) if $N < 6$, whereas only the third case occurs if $N \geq 7$.

Our proof of Theorem 3 will proceed as follows. First, we will consider the ODE problem associated to the radial solutions of (2) and, after rescaling, we will recover its solutions as fixed points of a suitable integral operator, which is expressed in terms of the modified Bessel functions of the first and second kind (Lemma 6). Then we will show that such fixed points need to satisfy suitable estimates (Theorem 9), by exploiting a version of the already mentioned Radial Lemma (Lemma 4), the monotonicity of the integral operator and the well known behaviour of the Bessel functions at the origin. Such estimates yield Theorem 3 by rescaling back.

Some useful properties of the modified Bessel functions are collected in the Appendix. For a complete treatment, we refer the reader to [15], [21] and the monumental monograph [23].

Notations. We end this introductory section by summarizing the notations of most frequent use throughout the paper.

- We denote by $2^* := 2N/(N-2)$ the critical exponent for the Sobolev embedding in dimension $N \geq 3$. Moreover we denote $2_\alpha := 2N/(N-\alpha)$ and $2^*_\alpha := 2(2N-2+\alpha)/(2N-2-\alpha)$.
- We set $\mathbb{R}_+ := (0, +\infty)$.
- If $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$, is a measurable set, $\rho : \Omega \to \mathbb{R}_+$ is a measurable function and $1 \leq q \leq \infty$, then $L^q(\Omega, \rho(z)\,dz)$ is the usual real Lebesgue space with respect to the measure $\rho(z)\,dz$ (dz stands for the Lebesgue measure on $\mathbb{R}^d$).
- $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ is the usual Sobolev space, which identifies with the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the norm of the gradient.
- I_ν and K_ν are the modified Bessel functions of order ν, of the first and second kind respectively.
- o and O are the usual Landau symbols. Moreover, by $f(t) \sim g(t)$ and $f(t) \asymp g(t)$ as $t \to t_0$ we respectively mean $\lim_{t \to t_0} f(t)/g(t) = 1$ and $\lim_{t \to t_0} f(t)/g(t) = \ell \in \mathbb{R} \setminus \{0\}$.

2 Asymptotic estimates for radial solutions at the origin

In this section we assume $0 < \alpha < 2$ and $2_\alpha < p < 2^*$. As one can easily check, the problem of radial solutions $u(x) = \phi(|x|)$ to (2) is equivalent to the following ODE problem:

$$\begin{cases} -\phi'' - \dfrac{N-1}{r}\phi' + \dfrac{A}{r^\alpha}\phi = \phi^{p-1} & \text{in } \mathbb{R}_+ = (0, +\infty) \\ \phi > 0 & \text{in } \mathbb{R}_+ \\ r^{-\frac{\alpha}{2}}\phi,\ \phi' \in L^2(\mathbb{R}_+, r^{N-1}dr) \end{cases} \tag{4}$$

(cf. the proof of Lemma 4 below). Making the change of variable

$$r = r(t) = \left(\frac{2-\alpha}{2\sqrt{A}} t \right)^{\frac{2}{2-\alpha}} \tag{5}$$

and defining

$$v(t) = \phi(r(t)) \quad \text{for all } t > 0, \tag{6}$$

one has

$$t = t(r) = \frac{2\sqrt{A}}{2-\alpha} r^{\frac{2-\alpha}{2}}, \qquad \phi(r) = v(t(r)) = v\left(\frac{2\sqrt{A}}{2-\alpha} r^{\frac{2-\alpha}{2}} \right), \tag{7}$$

so that

$$\phi'(r) = v'(t) \frac{dt}{dr} = \sqrt{A} v'(t) r^{-\frac{\alpha}{2}}$$

and

$$\begin{aligned} \phi''(r) &= \sqrt{A} \left(v''(t) \frac{dt}{dr} r^{-\frac{\alpha}{2}} - \frac{\alpha}{2} v'(t) r^{-\frac{\alpha+2}{2}} \right) = \sqrt{A} \left(v''(t) \sqrt{A} r^{-\alpha} - \frac{\alpha}{2} v'(t) r^{-\frac{\alpha+2}{2}} \right) \\ &= A v''(t) r^{-\alpha} - \sqrt{A} \frac{\alpha}{2} v'(t) r^{-\frac{\alpha+2}{2}}. \end{aligned}$$

Plugging into the equation of (4) we get

$$-A v''(t) r^{-\alpha} + \left(\frac{\alpha}{2} - N + 1 \right) \sqrt{A} v'(t) r^{-\frac{\alpha+2}{2}} + \frac{A}{r^\alpha} v(t) = v(t)^{p-1}$$

and multiplying both sides by r^α / A we obtain

$$-v''(t) + \frac{\alpha - 2N + 2}{2\sqrt{A}} v'(t) r^{\frac{\alpha-2}{2}} + v(t) = \frac{r^\alpha}{A} v(t)^{p-1}.$$

Since $r^\alpha = \left(\frac{2-\alpha}{2\sqrt{A}} \right)^{\frac{2\alpha}{2-\alpha}} t^{\frac{2\alpha}{2-\alpha}}$ and $r^{\frac{\alpha-2}{2}} = \frac{2\sqrt{A}}{2-\alpha} \frac{1}{t}$, the equation of (4) turns thus out to be equivalent to

$$-v'' - \frac{2N-2-\alpha}{2-\alpha} \frac{1}{t} v' + v = \left(\frac{2-\alpha}{2A^{1/\alpha}} \right)^{\frac{2\alpha}{2-\alpha}} t^{\frac{2\alpha}{2-\alpha}} v^{p-1}.$$

Observing that

$$r^{N-1-\alpha} dr = \left(\frac{2-\alpha}{2\sqrt{A}} t \right)^{\frac{2(N-1-\alpha)}{2-\alpha}} \frac{1}{\sqrt{A}} \left(\frac{2-\alpha}{2\sqrt{A}} t \right)^{\frac{\alpha}{2-\alpha}} dt = (\text{const.})\, t^{\frac{2N-2-\alpha}{2-\alpha}} dt$$

and $\phi'(r)^2 = A v'(t)^2 r^{-\alpha}$, one has

$$\int_0^{+\infty} \phi(r)^2 r^{N-1-\alpha} dr = (\text{const.}) \int_0^{+\infty} v(t)^2 t^{\frac{2N-2-\alpha}{2-\alpha}} dt$$

and

$$\int_0^{+\infty} \phi'(r)^2 r^{N-1} dr = A \int_0^{+\infty} v'(t(r))^2 r^{N-1-\alpha} dr = (\text{const.}) \int_0^{+\infty} v'(t)^2 t^{\frac{2N-2-\alpha}{2-\alpha}} dt. \tag{8}$$

As a conclusion, setting

$$\nu := \frac{N-2}{2-\alpha}, \qquad B := \left(\frac{2-\alpha}{2A^{1/\alpha}}\right)^{\frac{2\alpha}{2-\alpha}},$$

problem (4) is equivalent to

$$\begin{cases} -v'' - \dfrac{2\nu+1}{t} v' + v = B t^{\frac{2\alpha}{2-\alpha}} v^{p-1} & \text{in } \mathbb{R}_+ \\ v > 0 & \text{in } \mathbb{R}_+ \\ v \in H \end{cases} \tag{9}$$

where

$$H := H^1(\mathbb{R}_+, t^{\frac{2N-2-\alpha}{2-\alpha}} dt) := \left\{ v \in L^2(\mathbb{R}_+, t^{\frac{2N-2-\alpha}{2-\alpha}} dt) : v' \in L^2(\mathbb{R}_+, t^{\frac{2N-2-\alpha}{2-\alpha}} dt) \right\}.$$

Note that $\nu, B > 0$.

The next lemma is a version of a well known Radial Lemma [10] and states some properties of the functions in H.

Lemma 4 *Every $v \in H$ is continuous on $\mathbb{R}_+$ (up to the choice of a representative) and satisfies*

$$|v(t)| \leq C_{N,A,\alpha} \|v'\|_{2,\alpha} \frac{1}{t^{\nu}} \quad \text{for all } t > 0, \tag{10}$$

where $\|v'\|_{2,\alpha}$ is the norm of v' in $L^2(\mathbb{R}_+, t^{\frac{2N-2-\alpha}{2-\alpha}} dt)$ and the constant $C_{N,A,\alpha}$ only depends on N, A and α.

Proof. Let $v \in H$ and let ϕ be defined by (5)-(7). Then $\phi(|x|)$ belongs to $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$. Indeed $r^{-\frac{\alpha}{2}} \phi \in L^2(\mathbb{R}_+, r^{N-1} dr)$ implies that $\phi(|x|) \in L^2(\mathbb{R}^N, |x|^{-\alpha} dx) \subset L^1_{\text{loc}}(\mathbb{R}^N)$ and $\phi \in L^2((1,+\infty))$, while $\phi' \in L^2(\mathbb{R}_+, r^{N-1} dr)$ implies $\phi' \in L^2((1,+\infty))$, as well as that the gradient of $\phi(|x|)$ is in $L^2(\mathbb{R}^N)$; hence $\phi \in H^1((1,+\infty))$ and thus $\lim_{|x|\to\infty} \phi(|x|) = 0$, which yields $u \in L^{2^*}(\mathbb{R}^N)$ by Sobolev inequality (see the version given in [14, Theorem 8.3]).

So, by [10, Lemma A.III] (where the proof actually works for every $x \neq 0$), ϕ is continuous on $\mathbb{R}_+$ (up to the choice of a representative) and satisfies

$$|\phi(r)| \leq C_N \left(\int_0^{+\infty} \phi'(r)^2 r^{N-1} dr \right)^{1/2} \frac{1}{r^{\frac{N-2}{2}}} \quad \text{for all } r > 0,$$

where the constant C_N only depends on N. This gives (10) by (8) and (5). ■

We now consider the linear equation associated to the equation of (9), whose general solution can be expressed in terms of the modified Bessel functions of the first and second kind (see Appendix).

Lemma 5 *For any $g \in C(\mathbb{R}_+)$, the general solution of the equation*

$$-v'' - \frac{2\nu+1}{t} v' + v = g(t) \quad \text{in } \mathbb{R}_+ \tag{11}$$

is

$$v(t;c_1,c_2) = \tag{12}$$
$$= \frac{1}{t^\nu}\left\{\left(c_1-\int_1^t s^{1+\nu}K_\nu(s)\,g(s)\,ds\right)I_\nu(t)+\left(c_2+\int_1^t s^{1+\nu}I_\nu(s)\,g(s)\,ds\right)K_\nu(t)\right\},$$

where $c_1, c_2 \in \mathbb{R}$ *are arbitrary constants and* I_ν *and* K_ν *are the modified Bessel functions of order* ν*, of the first and second kind respectively.*

Proof. Taking into account that I_ν and K_ν are linearly independent solutions of the modified Bessel equation

$$-v''-\frac{1}{t}v'+\left(1+\frac{\nu^2}{t^2}\right)v=0 \quad \text{in } \mathbb{R}_+,$$

one easily checks that $t^{-\nu}I_\nu$ and $t^{-\nu}K_\nu$ are linearly independent solutions of the homogeneous equation associated to (11). On the other hand, the function

$$\widetilde{v}(t)=\frac{1}{t^\nu}\left(K_\nu(t)\int_1^t s^{1+\nu}I_\nu(s)\,g(s)\,ds-I_\nu(t)\int_1^t s^{1+\nu}K_\nu(s)\,g(s)\,ds\right)$$

is a particular solution of equation (11), since one has

$$\begin{aligned}
&-\widetilde{v}''(t)-\frac{2\nu+1}{t}\widetilde{v}'(t)+\widetilde{v}(t)\\
=&-\frac{1}{t^\nu}\left(K_\nu''(t)+\frac{1}{t}K_\nu'(t)-\left(1+\frac{\nu^2}{t^2}\right)K_\nu(t)\right)\int_1^t s^{1+\nu}I_\nu(s)\,g(s)\,ds+\\
&+\frac{1}{t^\nu}\left(I_\nu''(t)+\frac{1}{t}I_\nu'(t)-\left(1+\frac{\nu^2}{t^2}\right)I_\nu(t)\right)\int_1^t s^{1+\nu}K_\nu(s)\,g(s)\,ds+\\
&+t\left(I_\nu(t)\,K_{\nu+1}(t)+K_\nu(t)\,I_{\nu+1}(t)\right)g(t)
\end{aligned}$$

and the following identity holds: $I_\nu(t)\,K_{\nu+1}(t)+K_\nu(t)\,I_{\nu+1}(t)=\frac{1}{t}$ for all $t>0$. ■

In the following, for the sake of brevity, we will denote

$$H_+ := \{v\in H : v>0\}$$

and

$$I(t):=t^{\frac{N+\alpha}{2-\alpha}}I_\nu(t) \quad \text{and} \quad K(t):=t^{\frac{N+\alpha}{2-\alpha}}K_\nu(t) \quad \text{for every } t>0.$$

Furthermore, we will make an extensive use of the following estimates (see the Appendix for more accurate asymptotic equivalences):

- as $t\to 0^+$ one has

$$\frac{I_{\nu+1}(t)}{t}\asymp I_\nu(t)\asymp t^\nu,\quad tK_{\nu+1}(t)\asymp K_\nu(t)\asymp t^{-\nu},\quad I(t)\asymp t^{\frac{N+\alpha}{2-\alpha}+\nu},\quad K(t)\asymp t^{\frac{N+\alpha}{2-\alpha}-\nu}; \tag{13}$$

- as $t\to+\infty$ one has

$$I_\nu(t)\asymp\frac{e^t}{\sqrt{t}},\quad K_\nu(t)\asymp\frac{e^{-t}}{\sqrt{t}},\quad I(t)\asymp t^{\frac{N+\alpha}{2-\alpha}-\frac{1}{2}}e^t,\quad K(t)\asymp t^{\frac{N+\alpha}{2-\alpha}-\frac{1}{2}}e^{-t}. \tag{14}$$

Note that $(N+\alpha)/(2-\alpha)=\nu+1+2\alpha/(2-\alpha)$.

Lemma 6 *Let $v \in H_+$. Then v is a solution to problem (9) if and only if*

$$v(t) = \frac{B}{t^\nu}\left\{ I_\nu(t) \int_t^{+\infty} K(s)\, v(s)^{p-1}\, ds + K_\nu(t) \int_0^t I(s)\, v(s)^{p-1}\, ds \right\} \quad \text{for all } t > 0. \tag{15}$$

Remark 7 *The integrals $\int_t^{+\infty} K(s)\, v(s)^{p-1}\, ds$ and $\int_0^t I(s)\, v(s)^{p-1}\, ds$ are finite for every $v \in H_+$ and $t > 0$, since:*

- *$K(s) \asymp s^{\frac{N+\alpha}{2-\alpha} - \frac{1}{2}} e^{-s}$ and $v(s) = O(s^{-\nu})$ as $s \to +\infty$ (see (14) and Lemma 4);*
- *from (13) and Lemma 4, it follows that*

$$I(s)\, v(s)^{p-1} \asymp s^{\frac{N+\alpha}{2-\alpha}+\nu} v(s)^{p-1} = s^{\frac{N+\alpha}{2-\alpha}+\nu} O\left(s^{-\nu(p-1)}\right) = O\left(s^{\frac{N+\alpha}{2-\alpha}+2\nu-\nu p}\right)$$

as $s \to 0^+$, where

$$\frac{N+\alpha}{2-\alpha} + 2\nu - \nu p + 1 = \nu\left(2^* + 1 - p\right) > 0.$$

Proof. Clearly, v solves (9) if (15) holds, since for all $t > 0$ one has

$$\begin{aligned} v(t) \;=\; & \frac{I_\nu(t)}{t^\nu}\left(B \int_1^{+\infty} K(s)\, v(s)^{p-1}\, ds - B \int_1^t K(s)\, v(s)^{p-1}\, ds \right) + \\ & + \frac{K_\nu(t)}{t^\nu}\left(B \int_0^1 I(s)\, v(s)^{p-1}\, ds + B \int_1^t I(s)\, v(s)^{p-1}\, ds \right) \end{aligned}$$

and thus v is of the form (12) with $g(t) = Bt^{2\alpha/(2-\alpha)} v(t)^{p-1}$ continuous on $\mathbb{R}_+$. In order to prove the "only if" part of the lemma, assume that v is a solution of problem (9). Then, using Lemma 5 with $g(t) = Bt^{2\alpha/(2-\alpha)} v(t)^{p-1}$, there exist two unique constants $c_1 = c_1(v), c_2 = c_2(v) \in \mathbb{R}$ such that

$$v(t) = \frac{1}{t^\nu}\left\{ \left(c_1 - B \int_1^t K(s)\, v^{p-1}(s)\, ds \right) I_\nu(t) + \left(c_2 + B \int_1^t I(s)\, v^{p-1}(s)\, ds \right) K_\nu(t) \right\} \tag{16}$$

for all $t > 0$. Set

$$\begin{aligned} \Phi_1 \;=\; \Phi_1(t) &:= \left(c_1 - B \int_1^t K(s)\, v^{p-1}(s)\, ds \right) \frac{I_\nu(t)}{t^\nu}, \\ \Phi_2 \;=\; \Phi_2(t) &:= \left(c_2 + B \int_1^t I(s)\, v^{p-1}(s)\, ds \right) \frac{K_\nu(t)}{t^\nu} \end{aligned}$$

in such a way that $v = \Phi_1 + \Phi_2$, and assume by contradiction that

$$c_1 \neq B_1 := B \int_1^{+\infty} K(s)\, v(s)^{p-1}\, ds,$$

where $B_1 < +\infty$ by Remark 7. This implies

$$c_1 - B \int_1^t K(s)\, v(s)^{p-1}\, ds \asymp 1 \quad \text{as } t \to +\infty$$

and hence, as $t \to +\infty$, one gets

$$\begin{aligned}\Phi_1\Phi_2 &= t^{-2\nu} I_\nu(t) K_\nu(t) \left(c_1 - B\int_1^t K(s)\, v(s)^{p-1}\, ds\right)\left(c_2 + B\int_1^t I(s)\, v(s)^{p-1}\, ds\right)\\ &\asymp \left(c_2 + B\int_1^t I(s)\, v(s)^{p-1}\, ds\right) t^{-2\nu-1}\end{aligned}$$

and

$$\Phi_1^2 = t^{-2\nu} I_\nu(t)^2 \left(c_1 - B\int_1^t K(s)\, v(s)^{p-1}\, ds\right)^2 \asymp t^{-2\nu-1}e^{2t}.$$

Now we distinguish two cases, according to the value of the limit

$$\lim_{t\to+\infty}\left(c_2 + B\int_1^t I(s)\, v(s)^{p-1}\, ds\right),$$

which exists since $I(s)\, v(s)^{p-1} > 0$. If the limit is finite, we readily get

$$\Phi_1\Phi_2 \asymp t^{-2\nu-1} \asymp e^{-2t}\Phi_1^2 = o\left(\Phi_1^2\right) \quad \text{as } t\to+\infty.$$

If the limit is infinite, then, by De L'Hôpital's rule, we obtain

$$\begin{aligned}\lim_{t\to+\infty}\frac{\Phi_1\Phi_2}{\Phi_1^2} &= (\text{const.})\lim_{t\to+\infty}\frac{c_2 + B\int_1^t I(s)\, v(s)^{p-1}\, ds}{e^{2t}} \overset{H}{=} (\text{const.})\lim_{t\to+\infty}\frac{I(t)\, v(t)^{p-1}}{e^{2t}}\\ &= (\text{const.})\lim_{t\to+\infty}\frac{t^{\frac{N+\alpha}{2-\alpha}-\frac{1}{2}}e^t O\left(t^{-\nu(p-1)}\right)}{e^{2t}} = 0.\end{aligned}$$

So, in any case, we have $\Phi_1\Phi_2 = o\left(\Phi_1^2\right)$ as $t\to+\infty$ and hence

$$v^2 = \Phi_1^2 + 2\Phi_1\Phi_2 + \Phi_2^2 \geq \Phi_1^2 + 2\Phi_1\Phi_2 \sim \Phi_1^2 \asymp t^{-2\nu-1}e^{2t} \quad \text{as } t\to+\infty.$$

This implies $v \notin L^2(\mathbb{R}_+, t^{\frac{2N-2-\alpha}{2-\alpha}}dt)$, which is false by hypothesis, and thus it must be $c_1 = B_1$. Substituting into (16), we obtain

$$v(t) = \frac{1}{t^\nu}\left\{BI_\nu(t)\int_t^{+\infty} K(s)\, v(s)^{p-1}\, ds + \left(c_2 + B\int_1^t I(s)\, v(s)^{p-1}\, ds\right) K_\nu(t)\right\} \tag{17}$$

for all $t > 0$. We now prove that

$$c_2 = B_2 := B\int_0^1 I(s)\, v(s)^{p-1}\, ds,$$

where $B_2 < +\infty$ by Remark 7. Taking the derivative of (17) and using the identities

$$K_\nu'(t) - \frac{\nu}{t}K_\nu(t) = -K_{\nu+1}(t), \qquad I_\nu'(t) - \frac{\nu}{t}I_\nu(t) = I_{\nu+1}(t)$$

and $I(t)K_\nu(t) - K(t)I_\nu(t) = t^{\frac{N+\alpha}{2-\alpha}} I_\nu(t)K_\nu(t) - t^{\frac{N+\alpha}{2-\alpha}} K_\nu(t)I_\nu(t) = 0$ on $\mathbb{R}_+$, we get

$$\begin{aligned}
&v'(t) = \\
&= -\frac{\nu}{t^{\nu+1}}\left\{BI_\nu(t)\int_t^{+\infty} K(s)v(s)^{p-1}\,ds + \left(c_2 + B\int_1^t I(s)v(s)^{p-1}\,ds\right)K_\nu(t)\right\} + \\
&\quad + \frac{1}{t^\nu}\left\{BI'_\nu(t)\int_t^{+\infty} K(s)v(s)^{p-1}\,ds - BI_\nu(t)K(t)v(t)^{p-1}\right\} + \\
&\quad + \frac{1}{t^\nu}\left\{\left(c_2 + B\int_1^t I(s)v(s)^{p-1}\,ds\right)K'_\nu(t) + BK_\nu(t)I(t)v(t)^{p-1}\right\} \\
&= \frac{B}{t^\nu}v(t)^{p-1}\left(K_\nu(t)I(t) - I_\nu(t)K(t)\right) + \\
&\quad + \frac{B}{t^\nu}\left(I'_\nu(t) - \frac{\nu}{t}I_\nu(t)\right)\int_t^{+\infty} K(s)v(s)^{p-1}\,ds + \\
&\quad + \frac{1}{t^\nu}\left(K'_\nu(t) - \frac{\nu}{t}K_\nu(t)\right)\left(c_2 + B\int_1^t I(s)v(s)^{p-1}\,ds\right) \\
&= \frac{B}{t^\nu}I_{\nu+1}(t)\int_t^{+\infty} K(s)v(s)^{p-1}\,ds - \frac{1}{t^\nu}K_{\nu+1}(t)\left(c_2 + B\int_1^t I(s)v(s)^{p-1}\,ds\right).
\end{aligned}$$

Setting

$$\begin{aligned}
\Psi_1 &= \Psi_1(t) := \frac{I_{\nu+1}(t)}{t^\nu}\int_t^{+\infty} K(s)v(s)^{p-1}\,ds, \\
\Psi_2 &= \Psi_2(t) := \frac{K_{\nu+1}(t)}{t^\nu}\left(c_2 + B\int_1^t I(s)v(s)^{p-1}\,ds\right),
\end{aligned} \tag{18}$$

in such a way that $v' = \Psi_1 + \Psi_2$, we show that

$$\Psi_1 \in L^2\left((0,1), t^{\frac{2N-2-\alpha}{2-\alpha}}dt\right). \tag{19}$$

If $\int_0^{+\infty} K(s)v(s)^{p-1}\,ds < +\infty$, then one has

$$\Psi_1 \asymp \frac{I_{\nu+1}(t)}{t^\nu} \asymp t \quad \text{as } t \to 0^+ \tag{20}$$

and hence

$$t^{\frac{2N-2-\alpha}{2-\alpha}}\Psi_1^2 \asymp t^{\frac{2N-2-\alpha}{2-\alpha}+2} = t^{\frac{2N+2-3\alpha}{2-\alpha}} \quad \text{as } t \to 0^+$$

with

$$\frac{2N+2-3\alpha}{2-\alpha} + 1 = 2\frac{N+2-2\alpha}{2-\alpha} > 2\frac{N-2}{2-\alpha} > 0,$$

which implies (19). Otherwise, if $\int_0^{+\infty} K(s)v(s)^{p-1}\,ds = +\infty$, we observe that

$$\frac{\nu}{2}\left(\frac{2\alpha}{N-2} - p\right) < \frac{\nu}{2}\left(\frac{2\alpha}{N-2} - \frac{2N}{N-\alpha}\right) < 0$$

and apply De L'Hôpital's rule: we obtain

$$\lim_{t\to0^+}\frac{\int_t^{+\infty}K(s)\,v(s)^{p-1}\,ds}{t^{\frac{\nu}{2}\left(\frac{2\alpha}{N-2}-p\right)}}\overset{H}{=}(\text{const.})\lim_{t\to0^+}\frac{K(t)\,v(t)^{p-1}}{t^{\frac{\nu}{2}\left(\frac{2\alpha}{N-2}-p\right)-1}}$$
$$=(\text{const.})\lim_{t\to0^+}\frac{t^{\frac{N+\alpha}{2-\alpha}-\nu}O\left(t^{-\nu(p-1)}\right)}{t^{\frac{\nu}{2}\left(\frac{2\alpha}{N-2}-p\right)-1}}$$
$$=(\text{const.})\lim_{t\to0^+}O\left(t^{\frac{N+2-\alpha}{2-\alpha}-\frac{\nu}{2}p}\right)=0,$$

since

$$\frac{N+2-\alpha}{2-\alpha}-\frac{\nu}{2}p=\frac{\nu}{2}\left(2^*-p+2\frac{2-\alpha}{N-2}\right)>0.$$

So, recalling (18) and (20), one has

$$\Psi_1=\frac{I_{\nu+1}(t)}{t^\nu}o\left(t^{\frac{\nu}{2}\left(\frac{2\alpha}{N-2}-p\right)}\right)=o\left(t^{\frac{\nu}{2}\left(\frac{2\alpha}{N-2}-p\right)+1}\right)\quad\text{as } t\to0^+$$

and hence

$$t^{\frac{2N-2-\alpha}{2-\alpha}}\Psi_1^2=o\left(t^{\frac{2N-2-\alpha}{2-\alpha}+\nu\left(\frac{2\alpha}{N-2}-p\right)+2}\right)\quad\text{as } t\to0^+$$

with

$$\frac{2N-2-\alpha}{2-\alpha}+\nu\left(\frac{2\alpha}{N-2}-p\right)+3=\nu\left(2^*-p+2\frac{2-\alpha}{N-2}\right)>0,$$

which gives (19) again. Therefore $v'\in L^2((0,1)\,,t^{\frac{2N-2-\alpha}{2-\alpha}}dt)$ implies

$$\Psi_2\in L^2\left((0,1)\,,t^{\frac{2N-2-\alpha}{2-\alpha}}dt\right).$$

But this is impossible if $c_2\neq B_2$, since $c_2\neq B_2$ implies

$$\Psi_2\sim(c_2-B_2)\frac{K_{\nu+1}(t)}{t^\nu}\asymp\frac{1}{t^{2\nu+1}}\quad\text{as } t\to0^+,$$

whence

$$t^{\frac{2N-2-\alpha}{2-\alpha}}\Psi_2^2\asymp t^{\frac{2N-2-\alpha}{2-\alpha}-4\nu-2}\quad\text{as } t\to0^+$$

with

$$\frac{2N-2-\alpha}{2-\alpha}-4\nu-1=-2\nu<0.$$

So it must be $c_2=B_2$ and (15) then follows from (17). ■

Remark 8 *Checking the proof of Lemma 6, one readily sees that (15) also holds for every nonnegative $v\in H$ satisfying equation (9). This directly yields, without the use of the maximum principle, that every nontrivial nonnegative solution $v\in H$ of equation (9) is strictly positive on $\mathbb{R}_+$. Indeed, since $I_\nu(t)\,,K_\nu(t)\,,I(t)\,,K(t)>0$ for all $t>0$, if there exists $t_0>0$ such that $v(t_0)=0$ then (15) implies*

$$\int_{t_0}^{+\infty}K(s)\,v(s)^{p-1}\,ds=\int_0^{t_0}I(s)\,v(s)^{p-1}\,ds=0,$$

which means $v=0$ on $\mathbb{R}_+$.

Theorem 9 *Assume that v is a solution of problem (9). Then, as $t \to 0^+$, one has*

$$v(t) = \begin{cases} O(1) & \text{if } p < 2^* - 1 \\ O(\ln t) & \text{if } p = 2^* - 1 \\ O\left(t^{\nu(2^*-1-p)}\right) & \text{if } p > 2^* - 1 \end{cases} . \tag{21}$$

Observe that all the cases of (21) are possible for $2_\alpha < p < 2^*$ if $N < 6$, while only the third case occurs if $N \geq 7$.

Proof. By Lemmas 6 and 4, for every $t > 0$ we have

$$v(t) = \frac{B}{t^\nu}\left\{ I_\nu(t) \int_t^{+\infty} K(s)\, v(s)^{p-1}\, ds + K_\nu(t) \int_0^t I(s)\, v(s)^{p-1}\, ds \right\}$$

and

$$v(t) \leq C_{N,A,\alpha} \|v'\|_{2,\alpha} \frac{1}{t^\nu}.$$

Then, for every $t > 0$, one has

$$\begin{aligned} v(t) \ &\leq \ C_{N,A,\alpha}^{p-1} \|v'\|_{2,\alpha}^{p-1} \frac{B}{t^\nu} \left\{ I_\nu(t) \int_t^{+\infty} \frac{K(s)}{s^{\nu(p-1)}} ds + K_\nu(t) \int_0^t \frac{I(s)}{s^{\nu(p-1)}} ds \right\} \\ &=: B\, C_{N,A,\alpha}^{p-1} \|v'\|_{2,\alpha}^{p-1}\, w(t) \end{aligned} \tag{22}$$

with obvious definition of $w(t)$. Note that $w(t) \in \mathbb{R}$, by the same reasons used in Remark 7. We now study the behaviour of $w(t)$ as $t \to 0^+$.

By estimates (13) and De L'Hôpital's rule, one obtains

$$\int_0^t \frac{I(s)}{s^{\nu(p-1)}} ds \asymp t^{\nu(2^*+1-p)}$$

and hence, since $t^{-\nu} K_\nu(t) \asymp t^{-2\nu}$, we have

$$\frac{K_\nu(t)}{t^\nu} \int_0^t \frac{I(s)}{s^{\nu(p-1)}} ds \asymp t^{\nu(2^*-1-p)}.$$

In particular, since both sides are positive, there exists $C_1 > 0$ such that

$$\frac{K_\nu(t)}{t^\nu} \int_0^t \frac{I(s)}{s^{\nu(p-1)}} ds = C_1 t^{\nu(2^*-1-p)} + o\left(t^{\nu(2^*-1-p)}\right) \tag{23}$$

(one can also check that $C_1 = 1/\left(2\nu^2(2^*+1-p)\right)$).

Assume $2^* - 1 - p > 0$. Then

$$w(t) = \frac{I_\nu(t)}{t^\nu} \int_t^{+\infty} \frac{K(s)}{s^{\nu(p-1)}} ds + \frac{K_\nu(t)}{t^\nu} \int_0^t I(s) \frac{I(s)}{s^{\nu(p-1)}} ds = \frac{I_\nu(t)}{t^\nu} \int_t^{+\infty} \frac{K(s)}{s^{\nu(p-1)}} ds + o(1).$$

Since $K(s)\, s^{-\nu(p-1)} \asymp s^{\frac{N+\alpha}{2-\alpha} - \nu p}$ (see (13)) and

$$\frac{N+\alpha}{2-\alpha} - \nu p > \frac{N+\alpha}{2-\alpha} - \nu(2^*-1) = -1,$$

we have $\int_t^{+\infty} K(s)\, s^{-\nu(p-1)} ds \asymp 1$, whence

$$\frac{I_\nu(t)}{t^\nu} \int_t^{+\infty} \frac{K(s)}{s^{\nu(p-1)}} ds \asymp 1$$

because $t^{-\nu} I_\nu(t) \asymp 1$. Therefore $w(t) \asymp 1$ and the first estimate of (21) then follows from (22).

Now we assume $2^* - 1 - p < 0$. Then we have $K(s)\, s^{-\nu(p-1)} \asymp s^{\frac{N+\alpha}{2-\alpha} - \nu p}$ (see (13)) and

$$\frac{N+\alpha}{2-\alpha} - \nu p < \frac{N+\alpha}{2-\alpha} - \nu(2^* - 1) = -1, \tag{24}$$

so that $\int_0^{+\infty} K(s)\, s^{-\nu(p-1)} ds = +\infty$. By estimates (13) and De L'Hôpital's rule, we get

$$\int_t^{+\infty} \frac{K(s)}{s^{\nu(p-1)}} ds \asymp t^{\nu(2^*-1-p)},$$

which gives

$$\frac{I_\nu(t)}{t^\nu} \int_t^{+\infty} \frac{K(s)}{s^{\nu(p-1)}} ds \asymp t^{\nu(2^*-1-p)},$$

since $t^{-\nu} I_\nu(t) \asymp 1$. In particular, since both sides are positive, there exists $C_2 > 0$ such that

$$\frac{I_\nu(t)}{t^\nu} \int_t^{+\infty} \frac{K(s)}{s^{\nu(p-1)}} ds = C_2 t^{\nu(2^*-1-p)} + o\left(t^{\nu(2^*-1-p)}\right) \tag{25}$$

(one can also check that $C_2 = -1/\left(2\nu^2(2^* - 1 - p)\right)$). Therefore, by (25) and (23), we obtain

$$\begin{aligned} w(t) &= \frac{I_\nu(t)}{t^\nu} \int_t^{+\infty} \frac{K(s)}{s^{\nu(p-1)}} ds + \frac{K_\nu(t)}{t^\nu} \int_0^t I(s) \frac{I(s)}{s^{\nu(p-1)}} ds \\ &= (C_1 + C_2)\, t^{\nu(2^*-1-p)} + o\left(t^{\nu(2^*-1-p)}\right) \end{aligned}$$

and thus the third estimate of (21) follows from (22).

Finally, we assume $2^* - 1 - p = 0$. We get $\int_0^{+\infty} K(s)\, s^{-\nu(p-1)} ds = +\infty$ again (the inequality (24) becomes an equality), but the estimates (13) and De L'Hôpital's rule now give

$$\int_t^{+\infty} \frac{K(s)}{s^{\nu(p-1)}} ds \asymp \ln t$$

and hence

$$\frac{I_\nu(t)}{t^\nu} \int_t^{+\infty} \frac{K(s)}{s^{\nu(p-1)}} ds \asymp \ln t.$$

In particular, since the left hand side is positive, there exists $C_3 > 0$ such that

$$\frac{I_\nu(t)}{t^\nu} \int_t^{+\infty} \frac{K(s)}{s^{\nu(p-1)}} ds = -C_3 \ln t + o(\ln t)$$

(one can also check that $C_3 = 1/(2\nu)$). So, by (23), we conclude that

$$w(t) = -C_3 \ln t + o(\ln t) + C_1 + o(1) = -C_3 \ln t + o(\ln t)$$

and therefore the second estimate of (21) follows from (22). ■

Proof of Theorem 3. It readily follows from Theorem 9, by the change of variables (5)-(7). ■

3 Nonexistence result

In this section we assume

$$0<\alpha<2 \quad \text{and} \quad \frac{2N}{N-\alpha}<p\leq 2\frac{2N-2+\alpha}{2N-2-\alpha} \tag{26}$$

and consider the problem (4) of the radial solutions $u(x)=\phi(|x|)$ of (1) which belongs to $L^p(\mathbb{R}^N)$, that is,

$$\begin{cases} -\phi''-\dfrac{N-1}{r}\phi'+\dfrac{A}{r^\alpha}\phi=\phi^{p-1} & \text{in } \mathbb{R}_+ \\ \phi>0 & \text{in } \mathbb{R}_+ \\ r^{-\frac{\alpha}{2}}\phi,\phi'\in L^2(\mathbb{R}_+,r^{N-1}dr) \\ \phi\in L^p(\mathbb{R}_+,r^{N-1}dr) \end{cases}. \tag{27}$$

We set

$$\beta:=\frac{\alpha p}{p-2}. \tag{28}$$

Notice that, since $\alpha>0$, the second condition of (26) is equivalent to

$$\frac{2N-2+\alpha}{2}\leq\beta<N.$$

Moreover, one has

$$\beta-2\geq\frac{2N-2+\alpha}{2}-2=\frac{2N-6+\alpha}{2}>0.$$

Lemma 10 *Assume that ϕ is a solution of problem (27) (with conditions (26)). Then*

$$\lim_{r\to0^+}r^{\beta-2}\phi(r)^2=0.$$

Proof. It follows from Theorem 3. Indeed, if $p>2^*-1$, then one has

$$r^{\beta-2}\phi(r)^2=r^{\beta-2}O\left(r^{(2^*-1-p)(N-2)}\right)=O\left(r^{(2^*-1-p)(N-2)+\beta-2}\right) \quad \text{as } r\to0^+,$$

where

$$\begin{aligned}(2^*-1-p)(N-2)+\beta-2 &\geq \left(2^*-1-2\frac{2N-2+\alpha}{2N-2-\alpha}\right)(N-2)+\frac{2N-2+\alpha}{2}-2 \\ &= \frac{(2-\alpha)(6N-6+\alpha)}{2(2N-2-\alpha)}>0.\end{aligned}$$

The other cases are obvious, since $\beta>2$. ■

Proof of Theorem 2. For the sake of contradiction, we assume the ϕ is a solution of problem (27) (with conditions (26)). Rewriting the equation of (27) in the following form

$$r^{1-N}\left(r^{N-1}\phi'\right)'-\frac{A}{r^\alpha}\phi+\phi^{p-1}=0 \quad \text{in } \mathbb{R}_+$$

and testing it with $r^{\beta-1}\phi(r)$ on an arbitrary interval $[a,b]\subset\mathbb{R}_+$, we get

$$\int_a^b \left(\left(r^{N-1}\phi'\right)' r^{\beta-N}\phi - Ar^{\beta-1-\alpha}\phi^2 + r^{\beta-1}\phi^p\right) dr = 0. \tag{29}$$

Integrating by parts twice, one finds that

$$\begin{aligned}\int_a^b \left(r^{N-1}\phi'\right)' r^{\beta-N}\phi\, dr &= \left[r^{\beta-1}\phi'\phi\right]_a^b - \int_a^b \left(r^{\beta-1}\left(\phi'\right)^2 + (\beta-N)\, r^{\beta-2}\phi\phi'\right) dr \\ &= \left[r^{\beta-1}\phi'\phi\right]_a^b - \int_a^b r^{\beta-1}\left(\phi'\right)^2 dr + \\ &\quad -\frac{\beta-N}{2}\left[r^{\beta-2}\phi^2\right]_a^b + \frac{(\beta-N)(\beta-2)}{2}\int_a^b r^{\beta-3}\phi^2 dr,\end{aligned}$$

so that, plugging into (29), we obtain

$$\begin{aligned}&\left[r^{\beta-1}\phi'\phi - \frac{\beta-N}{2} r^{\beta-2}\phi^2\right]_a^b - \int_a^b r^{\beta-1}\left(\phi'\right)^2 dr + \frac{(\beta-N)(\beta-2)}{2}\int_a^b r^{\beta-3}\phi^2 dr + \\ &\qquad - A\int_a^b r^{\beta-1-\alpha}\phi^2 dr + \int_a^b r^{\beta-1}\phi^p dr = 0.\end{aligned} \tag{30}$$

We now define

$$E(r) := \frac{1}{2}\phi'(r)^2 - \frac{1}{2}\frac{A}{r^\alpha}\phi(r)^2 + \frac{1}{p}\phi(r)^p \quad \text{and} \quad E_\beta(r) := r^\beta E(r) \quad \text{for all } r>0. \tag{31}$$

Taking the derivative of E and using the equation, we get

$$\begin{aligned}E'(r) &= \phi''\phi' + \frac{\alpha}{2}\frac{A}{r^{\alpha+1}}\phi^2 - \frac{A}{r^\alpha}\phi\phi' + \phi^{p-1}\phi' \\ &= \left(-\frac{N-1}{r}\phi' + \frac{A}{r^\alpha}\phi - \phi^{p-1}\right)\phi' + \frac{\alpha}{2}\frac{A}{r^{\alpha+1}}\phi^2 - \frac{A}{r^\alpha}\phi\phi' + \phi^{p-1}\phi' \\ &= -\frac{N-1}{r}\left(\phi'\right)^2 + \frac{\alpha}{2}\frac{A}{r^{\alpha+1}}\phi^2\end{aligned}$$

and hence

$$\begin{aligned}E_\beta(b) - E_\beta(a) &= \int_a^b \left(\beta r^{\beta-1}E(r) + r^\beta E'(r)\right) dr \\ &= \left(\frac{\beta}{2} - N + 1\right)\int_a^b r^{\beta-1}\left(\phi'\right)^2 dr + \\ &\quad + \frac{A(\alpha-\beta)}{2}\int_a^b r^{\beta-\alpha-1}\phi^2 dr + \frac{\beta}{p}\int_a^b r^{\beta-1}\phi^p dr.\end{aligned} \tag{32}$$

Multiplying (30) by β/p and adding side by side to (32), we finally obtain

$$\left(\frac{\beta}{p} + \frac{\beta}{2} - N + 1\right)\int_a^b r^{\beta-1}\left(\phi'\right)^2 dr + A\left(\frac{\alpha-\beta}{2} + \frac{\beta}{p}\right)\int_a^b r^{\beta-\alpha-1}\phi^2 dr$$

$$+\frac{\beta\left(N-\beta\right)\left(\beta-2\right)}{2p}\int_a^b r^{\beta-3}\phi^2dr=\frac{\beta}{p}\left[r^{\beta-1}\phi'\phi-\frac{\beta-N}{2}r^{\beta-2}\phi^2\right]_a^b+E_\beta\left(b\right)-E_\beta\left(a\right), \tag{33}$$

where the second term of the left hand side actually vanishes, since $(\alpha-\beta)/2+\beta/p=0$ thanks to the definition (28) of β.

We now use the integrability properties (27) of ϕ and ϕ'. Since $\beta<N$, we have

$$\beta-1-N+\frac{\alpha}{2}<\frac{\alpha}{2}-1<0\quad\text{and}\quad\beta-3<N-3<N-1-\alpha,$$

so that

$$\begin{aligned}\int_1^{+\infty}r^{\beta-2}\left|\phi'\right|\phi\,dr&=\int_1^{+\infty}r^{\frac{N-1}{2}}\left|\phi'\right|r^{\frac{N-1-\alpha}{2}}\phi\,r^{\beta-1-N+\frac{\alpha}{2}}dr\\&\leq\int_1^{+\infty}r^{\frac{N-1}{2}}\left|\phi'\right|r^{\frac{N-1-\alpha}{2}}\phi\,dr\\&\leq\left(\int_1^{+\infty}r^{N-1}\left(\phi'\right)^2dr\right)^{1/2}\left(\int_1^{+\infty}r^{N-1-\alpha}\phi^2dr\right)^{1/2}<\infty\end{aligned}$$

and

$$\begin{aligned}&\int_1^{+\infty}\left(r^{\beta-3}\phi^2+r^{\beta-1}\left(\phi'\right)^2+r^{\beta-1-\alpha}\phi^2+r^{\beta-1}\phi^p\right)dr\\\leq&\int_1^{+\infty}\left(r^{\beta-1-\alpha}\phi^2+r^{N-1}\left(\phi'\right)^2+r^{N-1-\alpha}\phi^2+r^{N-1}\phi^p\right)dr<\infty.\end{aligned}$$

This implies

$$\liminf_{r\to+\infty}\left(r^{\beta-1}\left|\phi'\right|\phi+r^{\beta-2}\phi^2+r^\beta\left(\phi'\right)^2+r^{\beta-\alpha}\phi^2+r^\beta\phi^p\right)=0$$

and thus there exists a sequence $b_n\to+\infty$ such that

$$\lim_{n\to\infty}b_n^{\beta-1}\phi'\left(b_n\right)\phi\left(b_n\right)=\lim_{n\to\infty}b_n^{\beta-2}\phi\left(b_n\right)^2=\lim_{n\to\infty}E_\beta\left(b_n\right)=0.$$

Evaluating (33) with $b=b_n$ and passing to the limit, we find

$$\begin{aligned}\gamma_1\int_a^{+\infty}r^{\beta-1}\left(\phi'\right)^2dr+\gamma_2\int_a^{+\infty}r^{\beta-3}\phi^2dr&=-\frac{\beta}{p}a^{\beta-1}\phi'\left(a\right)\phi\left(a\right)+\\&\quad-\frac{\beta\left(N-\beta\right)}{2p}a^{\beta-2}\phi\left(a\right)^2-E_\beta\left(a\right),\end{aligned} \tag{34}$$

where

$$\gamma_1:=\frac{\beta}{p}+\frac{\beta}{2}-N+1,\quad\gamma_2:=\frac{\beta\left(N-\beta\right)\left(\beta-2\right)}{2p}.$$

We study the two sides of identity (34) separately. We have

$$\gamma_1=\frac{2N-2-\alpha}{2\left(p-2\right)}\left(2\frac{2N-2+\alpha}{2N-2-\alpha}-p\right)\geq0$$

and $\gamma_2 > 0$ (recall that $\beta < N$ and $\beta > 2$). As a consequence, since $\phi > 0$, there exist two constants $a_0, \gamma_0 > 0$ such that

$$\forall a \leq a_0, \quad \gamma_1 \int_a^{+\infty} r^{\beta-1} \left(\phi'\right)^2 dr + \gamma_2 \int_a^{+\infty} r^{\beta-3} \phi^2 dr \geq \gamma_0 > 0. \tag{35}$$

On the other hand, Lemma 10 assures that

$$\lim_{a \to 0^+} a^{\beta-2} \phi(a)^2 = 0,$$

which also gives

$$\lim_{a \to 0^+} a^{\beta-\alpha} \phi(a)^2 = \lim_{a \to 0^+} a^{2-\alpha} a^{\beta-2} \phi(a)^2 = 0.$$

Therefore, briefly denoting the right hand side of (34) by $F(a)$ and substituting the definitions (31), we infer that

$$\begin{aligned} F(a) &= -\frac{\beta}{p} a^{\beta-1} \phi'(a) \phi(a) - \frac{\beta(N-\beta)}{2p} a^{\beta-2} \phi(a)^2 - \frac{1}{2} a^\beta \phi'(a)^2 + \frac{A}{2} a^{\beta-\alpha} \phi(a)^2 + \\ &\quad -\frac{1}{p} a^\beta \phi(a)^p - \frac{1}{2} \frac{\beta^2}{p^2} a^{\beta-2} \phi(a)^2 + \frac{1}{2} \frac{\beta^2}{p^2} a^{\beta-2} \phi(a)^2 \\ &= -\frac{1}{2} a^{\beta-2} \left(a\phi'(a) + \frac{\beta}{p} \phi(a) \right)^2 - \frac{1}{p} a^\beta \phi(a)^p + o(1)_{a \to 0^+} \leq o(1)_{a \to 0^+}. \end{aligned} \tag{36}$$

So, from (34), (35) and (36) it follows that $\forall a \leq a_0$ one has $0 < \gamma_0 \leq F(a) \leq o(1)_{a \to 0^+}$, which is a contradiction. ■

4 Appendix

This Appendix is devoted to a summary of the most useful properties of the Bessel functions used in the paper. For a complete treatment, we refer the reader to [15], [21] and [23].

For every $\nu \in \mathbb{R}$ and $t \in \mathbb{R}_+$, the *modified Bessel function of the first kind of order* ν is defined as

$$I_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^\infty \frac{1}{k!} \frac{1}{\Gamma(\nu+k+1)} \left(\frac{t}{2}\right)^{2k},$$

where Γ is the usual Gamma function and $1/\Gamma(-n) = 0$ for $n \in \mathbb{N}$. The *modified Bessel function of the second kind of order* ν (also known as *Macdonald's function*) is defined as

$$K_\nu(t) = \frac{\pi}{2} \frac{I_{-\nu}(t) - I_\nu(t)}{\sin(\pi\nu)} \quad \text{if } \nu \notin \mathbb{Z}$$

and $K_n(t) = \lim_{\nu \to n} K_\nu(t)$ if $n \in \mathbb{Z}$. These functions are linearly independent real solutions of the *modified Bessel equation in* $\mathbb{R}_+$, namely,

$$-u'' - \frac{1}{t} u' + \left(1 + \frac{\nu^2}{t^2}\right) u = 0 \quad \text{in } \mathbb{R}_+,$$

and satisfy the following identities on $\mathbb{R}_+$:

- $K_{\nu+1}(t) I_\nu(t) + I_{\nu+1}(t) K_\nu(t) = \frac{1}{t}$;
- $I'_\nu(t) - \frac{\nu}{t} I_\nu(t) = I_{\nu+1}(t)$;
- $K'_\nu(t) - \frac{\nu}{t} K_\nu(t) = -K_{\nu+1}(t)$.

For every $\nu > 0$, both I_ν and K_ν are strictly positive on $\mathbb{R}_+$ and the following asymptotic estimates hold:

$$I_\nu(t) \sim \frac{1}{2^\nu \Gamma(\nu+1)} t^\nu \quad \text{and} \quad K_\nu(t) \sim \frac{\Gamma(\nu)}{2^{1-\nu}} t^{-\nu} \quad \text{as } t \to 0^+,$$

$$I_\nu(t) \sim \sqrt{\frac{1}{2\pi}} \frac{e^t}{\sqrt{t}} \quad \text{and} \quad K_\nu(t) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-t}}{\sqrt{t}} \quad \text{as } t \to +\infty.$$

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