1 Introduction

The analysis of the inequality is a crucial issue in our society. All the governments have the important task to take under control and eventually to reduce the inequality among their citizens. Since the beginning of the last century, the scientific literature started to think about the inequality, by formalizing the issue, and by proposing methods for measuring it. Basically there two kinds of mathematical-statistical tools for evaluating the inequality: the curves and the indexes. An inequality curve can be seen as a pointwise measure of inequality, while an inequality index is a synthetic measure of it.

Nowadays in the literature, the most used curve in inequality analysis is the Lorenz curve. It is usually denoted by $L(p)$, and it can be defined in several equivalent ways. One of them is the following one (see Pietra [10], and Gastwirth [5]):

$$L_X(p) = \frac{1}{\mathbb{E}(X)} \int_0^p F_X^{-1}(t)\, dt \qquad p \in [0, 1], \tag{1}$$

where F_X and $\mathbb{E}(X)$ denote the distribution function and the finite positive expected value of the non-negative continuous random variable X, respectively. Another classical inequality curve strictly related with the Lorenz one, is the Bonferroni curve introduced by Bonferroni in [3]. Using the same notation than before, it can be defined as follows:

$$B_X(p) = \frac{1}{p\mathbb{E}(X)} \int_0^p F_X^{-1}(t)\, dt \qquad p \in (0, 1]. \tag{2}$$

This paper deals with the inequality curves, and it is organized as follows: in the next Section the definition of the inequality $I(p)$ curve is presented for the discrete case and for the continuous one; Section 3 describes the main features of the $I(p)$ curve, with a particular focus on the partial order based on it. The same section provides the analytical form of the $I(p)$ curve for some classical distribution models used for modeling the income distribution. Section 4 is devoted to an empirical analysis based on real data, while the final section cointains some brief conclusions.

2 A good alternative

Beyond the well-known Lorenz and Bonferroni curves, in the last years, two relatively new tools suitable for evaluating the inequality, have been proposed and they continue to obtain more and more attention. They are the $I(p)$ curve and the index I, introduced by Zenga in [20]. The definition for the discrete case is provided in the next subsection; the extension to the continuous one is described later.

2.1 The discrete case

Let the couples $\{(x_j, n_j) : j = 1, 2, \ldots, s;\ 0 \leq x_1 < \cdots < x_s;\ \sum_{j=1}^{s} n_j = N\}$ be the frequency distribution of a non-negative variable X. For $j = 1, 2, \ldots, s$, consider:

- $N_j = \sum_{i=1}^{j} n_i$
- $p_j = N_j / N$
- $Q_j = \sum_{i=1}^{j} x_i n_i$.

The basic idea is, for each $j = 1, 2, \ldots s$, to split the observations into two collectively exhaustive and mutually exclusive groups: the first one contains the smallest values of X (named *lower group*); the second one contains all the remaining values (named *the upper group*). For each group, it is possible to calculate the mean, obtaining the lower and the upper mean, respectively. The lower mean can be therefore defined as

$$\overset{-}{M}_{(p_j)} = \frac{Q_j}{N_j} = \frac{1}{N_j} \sum_{i=1}^{j} x_i n_i \qquad j = 1, \ldots, s$$

and the upper mean as

$$\overset{+}{M}_{(p_j)} = \begin{cases} \dfrac{T - Q_j}{N - N_j} & j = 1, \ldots, s-1 \\ x_s & j = s, \end{cases}$$

where T denotes the total sum, that is:

$$T = Q_s = \sum_{i=1}^{s} x_i n_i.$$

Then for $j = 1, 2, \ldots, s$, the pointwise measure $I(p_j)$ can be calculated as:

$$I(p_j) = \frac{\overset{+}{M}_{(p_j)} - \overset{-}{M}_{(p_j)}}{\overset{+}{M}_{(p_j)}} = 1 - \frac{\overset{-}{M}_{(p_j)}}{\overset{+}{M}_{(p_j)}}. \tag{3}$$

Following the procedure described in Zenga [20], by using the s couples $(p_j, I(p_j))$, a very useful graphical representation of the diagram of inequality $I(p_j)$ can be easily obtained. An example of such diagram is shown in Figure 1. It is worth remarking that the $j-$th rectangle has length given by the interval $(p_{j-1}, p_j]$ and width equal to the value $I(p_j)$. The synthetic inequality index I

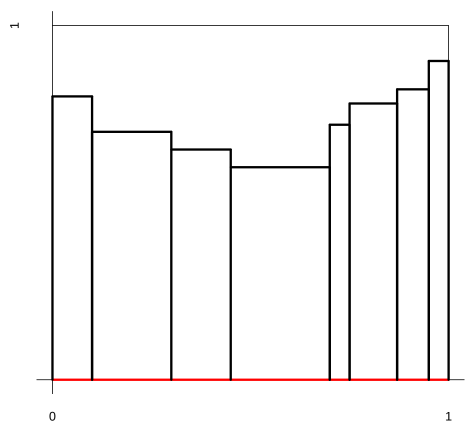

Figure 1: An example of the diagram of inequality $I(p_j)$

can be calculated as the weighted arithmetic mean of the values taken on by the pointwise measure $I(p_j)$ with weights n_j/N. It coincides with the sum of the areas of the s rectangles, it can be evaluated as

$$I = \sum_{j=1}^{s} I(p_j) \cdot \frac{n_j}{N},$$

and it can be also seen as the area below the $I(p_j)$ diagram. Figure 2 shows the $I(p_j)$ diagram for the two extreme situations: the case of minimun inequality is shown in the left panel, whereas the case of maximum inequality in the right one. In the former case

$$I_{min} = 0,$$

while in the latter one

$$I_{max} = 1 - \frac{1}{N^2}.$$

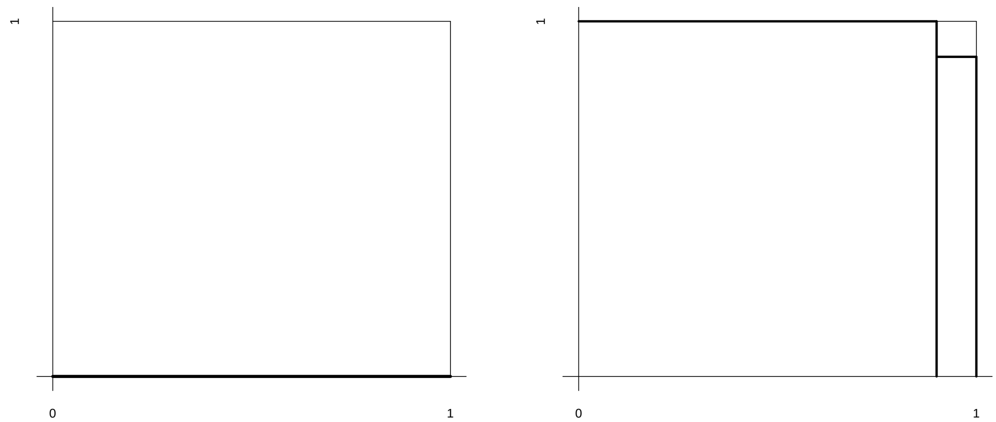

Figure 2: The pointwise measure $I(p_j)$ for the two extreme situations: minimun inequality (on the left), maximun inequality (on the right)

2.2 The continuous case

The pointwise measure defined in (3) can also be generalized to the continuous case. In such case, let X be a non-negative continuous random variable, with support (a, b), with probability density function f, with distribution function F and with positive expectation μ. The lower and the upper means can be defined as

$$\bar{M}_{(p)} = \frac{1}{p}\int_0^p F^{-1}(y)\,dy, \qquad p \in (0,1),$$

and

$$\overset{+}{M}_{(p)} = \frac{1}{1-p}\int_p^1 F^{-1}(y)\,dy \qquad p \in (0,1),$$

where F^{-1} denotes the inverse of the distribution function F, or if needed, the generalized inverse function given by:

$$F^{-1}(p) = \begin{cases} \inf\{y : F(y) \geq p\} & \text{if } p \in (0,1] \\ \inf\{y : F(y) > p\} & \text{if } p = 0. \end{cases}$$

In analogy with the discrete case, Zenga in [20] defines the inequality curve $I(p)$ as:

$$I(p) = \frac{\overset{+}{M}_{(p)} - \overset{-}{M}_{(p)}}{\overset{+}{M}_{(p)}} = 1 - \frac{\overset{-}{M}_{(p)}}{\overset{+}{M}_{(p)}} \qquad p \in (0, 1).$$

As for the $I(p_j)$ diagram, the graph of the inequality $I(p)$ curve lies in the unitary square $[0, 1]^2$, and the area below it is the value of the inequality index I, that is:

$$I = \int_0^1 I(p)\, dp.$$

Figure 3 shows a typical behaviour (U-shaped) of the $I(p)$ curve for a real income distribution.

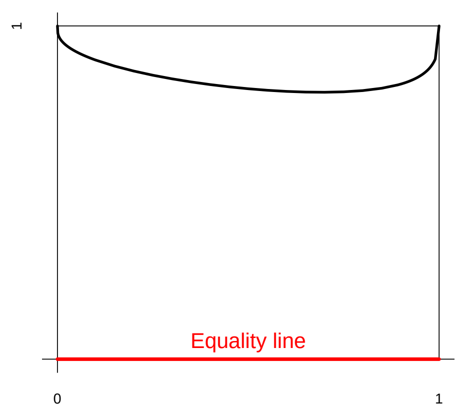

Figure 3: A tipical behaviour of the $I(p)$ curve

3 Features of the $I(p)$ curve

In Zenga [20] it is proved that there exists an analytic relationship between the $I(p)$ curve and the most used inequality curves. More in detail, the link between the $I(p)$ curve and the Lorenz curve $L(p)$ defined in (1) is given by:

$$I(p) = \frac{p - L(p)}{p[1 - L(p)]} \qquad p \in (0, 1), \tag{4}$$

while between the $I(p)$ curve and the Bonferroni curve $B(p)$ defined in (2), the following relationship holds true:

$$I(p) = \frac{1 - B(p)}{1 - pB(p)} \qquad p \in (0, 1). \tag{5}$$

The two aforementioned formulas (formula (4) and (5)) prove that there is a one-to-one relationship between the $I(p)$ curve and the other most used inequality curves.

An important property of the $I(p)$ curve is that it has not a pre-established behavior. This feature is not so common, since it is well-known that both the Lorenz and the Bonferroni curves are necessarily increasing functions of p (see for example Sarabia et al. [16]). For the $I(p)$ curve this restriction does not apply. For this reason the $I(p)$ curve can be considered more flexible than the other ones, in fact some different real situations bring to very different $I(p)$ curves, but not so different $L(p)$ or $B(p)$ curves (see for further investigation Maffenini and Polisicchio [8]). All this gives to the $I(p)$ curve a major capability to capture information of real situations than the other inequality curves.

The more flexibility of the $I(p)$ curve is also revealed by the fact that the Lorenz curve of the continuous random variable X evaluated at $p \in [0, 1]$ is the ratio of $\int_0^p F_X^{-1}(t)\, dt$ over the expected value of X. So, it is trivial that $L(0) = 0$ and $L(1) = 1$, no matter how the variable X is. For this reason the explaining power of the Lorenz curve vanishes for values of p close to 0 and close to 1. Such restriction does not apply to the $I(p)$ curve, in fact, if (a, b) is the support of the continuous random variable X (with $0 \leq a < b \leq +\infty$), Polisicchio in [11] showed that

$$\lim_{p \to 0^+} I(p) = 1 - \frac{a}{\mathbb{E}(X)}$$

and

$$\lim_{p \to 1^-} I(p) = 1 - \frac{\mathbb{E}(X)}{b}.$$

This means that near the boundary of the domain, the $I(p)$ curve is related to the support of X, and therefore it is more explanatory than Lorenz one and so much more suitable for inequality analysis around lower or upper values of X. A similar discussion holds for the Bonferroni curve.

Another valuable issue concerns the interpretation of the values assumed by the $I(p)$ curve. It is well-known that, if the random variable X models the incomes, $L_X(p^*) = L^*$ means that the "poorest" proportion p^* of the considered population owns the proportion L^* of the total income. On the other hand, $B_X(p^*) = B^*$ means that the mean of the income of the "poorest" proportion p^* is B^* times the average income of the whole population. The interpretation of the $I(p)$ curve is quite different, since it provides a clearer information. It follows by the definition that if the $I(p)$ curve is equal to I^* at $p = p^*$, it means that the income mean of the "poorest" proportion p^* of the considered population is $(1 - I^*)$-times the income mean of the remaining population. In other words, the $I(p)$ curve compares the means of two groups partitioning the population (the lower and the upper group): this approach seems to be more informative than the comparison between a group and the total of the population.

A conventional application of the inequality curves regards the partial orders. Starting from an inequality curve, it is usually possible to define a partial ordering. The following two definitions characterize the very well-known ordering based on Lorenz curve and the ordering based on the Bonferroni curve, respectively.

Definition 1 *Let X and Y be two continuous non-negative random variables, both with finite and positive expected value. X is said to be larger (or more unequal) than Y in the Lorenz ordering (and it is denoted by $X \geq_L Y$), iff*

$$L_X(p) \leq L_Y(p) \qquad \forall p \in (0, 1)$$

where $L_X(p)$ and $L_Y(p)$ are the value of the Lorenz curve of X and that of Y at p, respectively.

Definition 2 *Let X and Y be two continuous non-negative random variables, both with finite and positive expected value. X is said to be larger (or more unequal) than Y in the Bonferroni ordering (and it is denoted by $X \geq_B Y$), iff*

$$B_X(p) \geq B_Y(p) \qquad \forall p \in (0, 1)$$

where $B_X(p)$ and $B_Y(p)$ are the value of the Bonferroni curve of X and that of Y at p, respectively.

Many papers in the literature deal with the partial order based on the Lorenz curve; whereas some examples of papers dealing with the order based on the Bonferroni curve are: Tarsitano [19], Giorgi and Crescenzi [6], Pundir et al. [14]. In analogy to the two aforementioned orderings, Polisicchio and Porro in [12] introduced the order based on the $I(p)$ curve, by the following definition.

Definition 3 *Let X and Y be two continuous non-negative random variables, both with finite and positive expected value. X is said to be larger (or more unequal) than Y in the ordering based on the $I(p)$ curve (and it is denoted by $X \geq_I Y$), iff*

$$I_X(p) \geq I_Y(p) \qquad \forall p \in (0,1)$$

where $I_X(p)$ and $I_Y(p)$ are the value of the $I(p)$ curve of X and that of Y at p, respectively.

The links among different orders and their relationships with inequality have been deeply studied (see for example Atkinson, [2] and Muliere and Scarsini, [9]). Keeping this in mind, Polisicchio and Porro in [12] stated and proved the following equivalence lemma.

Lemma 1 (Equivalence Lemma) *Let X and Y be two continuous non-negative random variables, both with finite and positive expected value. Then:*

$$X \geq_L Y \Leftrightarrow X \geq_I Y.$$

This lemma makes evident the coherence of the $I(p)$ and the Lorenz curves, in fact two distributions are ordered for Lorenz ordering if and only if they are ordered for the ordering based on $I(p)$ curve, too. It is important to remark that these two orders are only partial orders, meaning that there are some distributions with crossing $L(p)$ curves and therefore with crossing $I(p)$ curves, that are not ordinable for all $p \in (0,1)$. In order to collocate the equivalence lemma in a broader framework, the following result is reported. A detailed proof and more insights can be found in Porro [13].

Theorem 1 *Let X and Y be two continuous non-negative random variables, with the same finite and positive expected value. Then all the following statements are equivalent:*

i) $X \geq_L Y$
ii) $X \geq_I Y$
iii) $X \geq_B Y$
iv) $X \geq_{CV} Y$
v) $X \geq_2 Y$

where $\geq_{CV}$ and $\geq_2$ denote the convex order and the second-order stochastics dominance, respectively.

Refer to Shaked and Shanthikumar [17] for the definitions and features of the convex order and the second-order stochastics dominance.

3.1 Some examples

This subsection provides the analytical form of the $I(p)$ curve for some classical distribution models used in income-distribution analysis. The considered models are: the log-normal, the Pareto, the Dagum, and the Singh-Maddala distributions.

Let X be a random variable with log-normal distribution, depending on the parameters $\gamma \in \mathbb{R}$ and $\delta > 0$. Then it can be proved that the $I(p)$ curve of X is given by:

$$I(p) = \frac{p - \phi[\phi^{-1}(p) - \delta]}{p\{1 - \phi[\phi^{-1}(p) - \delta]\}} \qquad p \in (0, 1)$$

where ϕ denotes the normal standard distribution function.

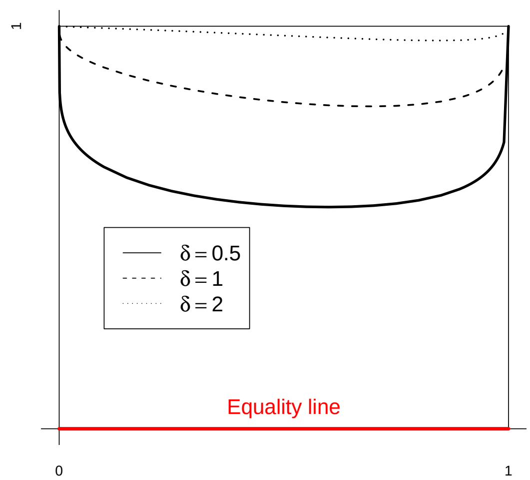

Figure 4: The $I(p)$ curve for the log-normal distribution

Figure 4 shows the $I(p)$ curves for the log-normal distribution for different values of the parameter δ. It is worth noting that the $I(p)$ curve for the log-normal distribution does not depend on γ, since such parameter is a scale parameter. It can be proved that for the log-normal model, the parameter δ is a direct inequality indicator: the bigger value it assumes, the more inequality (see more details in Porro [13]).

The Pareto model, depending on the parameters $x_0 > 0$ and $\theta > 1$ (in order to have finite expected value), is characterized by the distribution function

$$F(x) = 1 - (x^{-1}x_0)^{\theta}\mathbf{I}_{\{x>x_0\}}(x),$$

and it has the $I(p)$ curve given by:

$$I(p) = \frac{1-(1-p)^{\frac{1}{\theta}}}{p} \qquad p \in (0,1).$$

Such curve does not depend on the parameter x_0, but only on θ. It can be proved that, for any fixed $p \in (0,1)$ as θ increases, the $I(p)$ curve decreases, and so inequality does. This means that the distribution parameter θ is an inverse inequality indicator for the $I(p)$ curve. In the literature, it is well-known that the parameter θ of the Pareto distribution is an inverse inequality indicator also for the Lorenz and the Bonferroni curves. In Figure 5 the $I(p)$

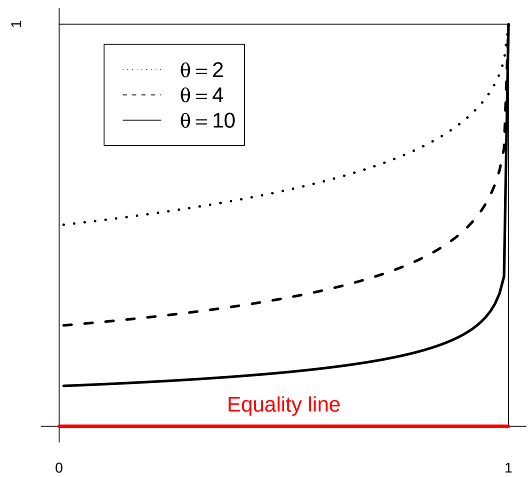

Figure 5: The $I(p)$ curve for the Pareto distibution

curves of the Pareto distribution for different values of the parameter θ are shown.

A more sophisticated model used for representing the income distribution is the Dagum distribution described for the first time in [4]. The distribution function, depending on the three positive parameters λ, θ and β, is given by

$$F(x) = (1+\lambda x^{-\theta})^{-\beta}\mathbf{I}_{\{x>0\}}(x).$$

In order to have finite expected value, θ must be also greater than 1. The $I(p)$ curve of the Dagum distribution is:

$$I(p) = \frac{p - B\left(p^{\frac{1}{\beta}}; \beta + \frac{1}{\theta}; 1 - \frac{1}{\theta}\right)}{p\left[1 - B\left(p^{\frac{1}{\beta}}; \beta + \frac{1}{\theta}; 1 - \frac{1}{\theta}\right)\right]} \qquad p \in (0,1). \tag{6}$$

where $B(x;a;b)$ denotes the incomplete Beta function defined as

$$B(x;a;b) = \int_0^x \frac{1}{B(a;b)} u^{a-1}(1-u)^{b-1}\,du \qquad x \in (0,1),\ a > 0,\ b > 0,$$

and $B(a;b)$ is the function Beta with parameters a e b, that is:

$$B(a;b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx \qquad a > 0,\ b > 0.$$

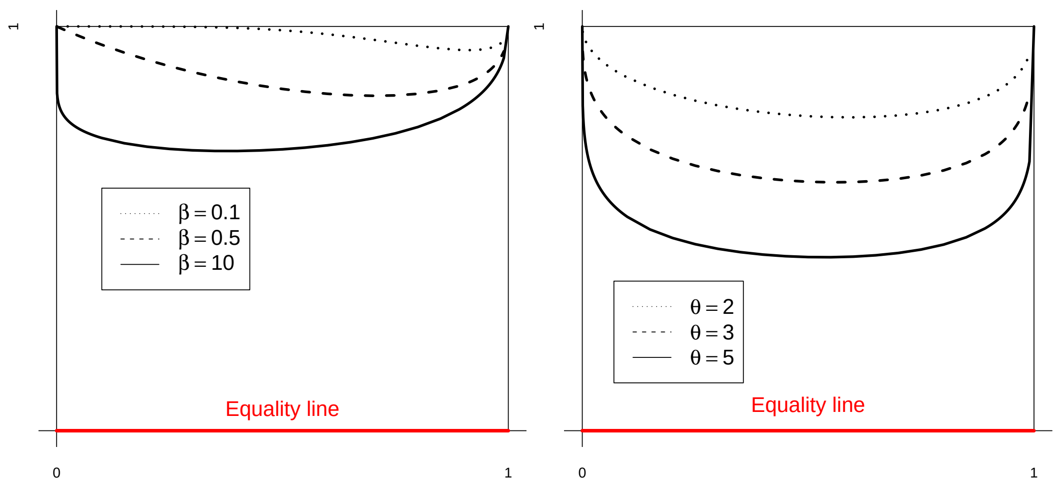

Figure 6: Some $I(p)$ curves for the Dagum distribution: $\theta = 2$, and β varies (left panel); $\beta = 1$ and θ varies (right panel)

In Figure 6 some $I(p)$ curves for the Dagum distribution are shown with different values of the parameters: in the left panel, θ is fixed and equal to 2, while β takes on different values; in the right panel the value of the parameter β is fixed and equal to 1, while θ changes. Two important remarks need to be mentioned: the first one is that λ for the Dagum model is a scale parameter, therefore as aspected, the $I(p)$ curve does not depend on it. The second remark regards the remaining two parameters: as one parameter is fixed, then the other one is an inverse inequality indicator. Such dynamic can be observed also in the Figure 6.

The last model considered is the Singh-Maddala distribution, also known in the literature as Burr Type XII distribution. The distribution function of this model, proposed for the first time in [18] is given by

$$F(x) = 1 - (1 + \lambda^{-\beta} x^{\beta})^{-\delta} \mathbf{I}_{\{x>0\}}(x),$$

where the three parameters λ, β and δ must be all positive. The further condition $\delta > 1/\beta$ must be satisfied in order to have finite expected value. In that case, the inequality $I(p)$ curve is:

$$I(p) = \frac{p - B\left(1 - (1-p)^{\frac{1}{\delta}}; 1 + \frac{1}{\beta}; \delta - \frac{1}{\beta}\right)}{p\left[1 - B\left(1 - (1-p)^{\frac{1}{\delta}}; 1 + \frac{1}{\beta}; \delta - \frac{1}{\beta}\right)\right]} \qquad p \in (0,1).$$

In Figure 7 some $I(p)$ curves for the Singh-Maddala distribution are shown

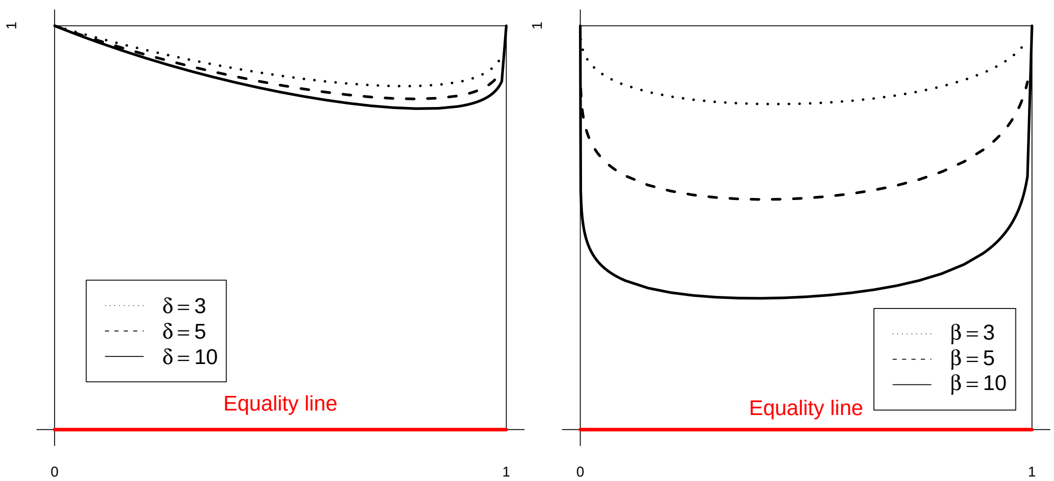

Figure 7: Some $I(p)$ curves for the Singh-Maddala distribution: $\beta = 0.5$, and δ varies (left panel); $\delta = 1$ and β varies (right panel)

with different values of the parameters: in the left panel, β is fixed and equal to 0.5, while δ takes on different values; in the right panel the value of the parameter δ is fixed and equal to 1, while β changes. Also for the Singh-Maddala model, λ is a scale parameter, therefore any inequality curve must not depend on it. The other two remaining parameters play the same role seen for the Dagum model: as one parameter is fixed, the other one is an inverse inequality indicator.

It is now important to highlight the following remark.

Remark 1 *All the inverse (direct) inequality indicators for the $I(p)$ curve presented in this section are also inverse (direct) inequality indicator for the Lorenz curve and for the Bonferroni curve. This characteristic reveals an important and valuable coherence of all these three inequality curves.*

The following table summarizes the expressions of the $I(p)$ curve for the analyzed models used to represent the income distributions.

Model	$I(p)$ **curve**
Pareto	$I(p) = \dfrac{1-(1-p)^{1/\theta}}{p}$
Log-normal	$I(p) = \dfrac{p - \Phi[\Phi^{-1}(p) - \delta]}{p[1 - \Phi(\Phi^{-1}(p) - \delta)]}$
Dagum	$I(p) = \dfrac{p - B\left(p^{1/\beta}; \beta + 1/\theta; 1 - 1/\theta\right)}{p[1 - B\left(p^{1/\beta}; \beta + 1/\theta; 1 - 1/\theta\right)]}$
Singh Maddala	$I(p) = \dfrac{p - B\left(1-(1-p)^{\frac{1}{\delta}}; 1 + 1/\beta; \delta - 1/\beta\right)}{p\left[1 - B\left(1-(1-p)^{\frac{1}{\delta}}; 1 + 1/\beta; \delta - 1/\beta\right)\right]}$

4 Application to real data

In this section some Lorenz curves and some $I(p)$ curves for the income distribution of the metropolitan area of Flint are presented. Data came from the IPUM-USA database (see Ruggles et al., [15]). The income distributions considered regard the years 1980, 1990, 2000 and 2010. The income analyzed is the *total individual income* of the income earners. All the negative and the zero incomes have been removed from the dataset. In order to compare the incomes in different years, the data have been adjusted with the Consumer Price Index factors (CPI 1999 = 1). The data came from different surveys, therefore the sample sizes over years are not equal. The following table summarizes the sample sizes used in this empirical application.

Year	1980	1990	2000	2010
Sample size	13645	10524	6166	1016

Another relevant difference is that the distribution of 1980 is topcoded: all the values greater than a threshold are censored and put equal to the value of the threshold, which therefore represents the maximum admissible value of the distribution. In order to overcome this issue, since data of other years are not censored, the topcoded observations in 1980 have been replaced using the procedure proposed in Jenkins et al. [7] and in Armour et al. [1]. In such way the distribution is prolonged, by performing a multi-imputation procedure. Figure 8 shows the Lorenz curves for the considered income distributions. The

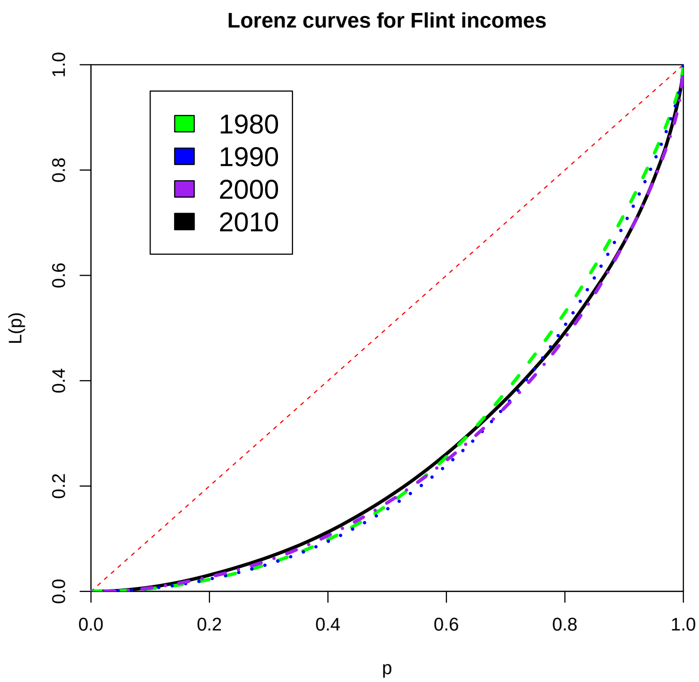

Figure 8: The Lorenz curve of Flint income in different years

curves point out a medium-high level of inequality, but actually no relevant difference over years is highlighted: it is difficult to obtain more details about the dynamic evolution of the inequality. In Figure 9, the corresponding $I(p)$ curves are drawn. By an analysis of these curves instead, some dissimilarities among the years can be found. The behaviour of the $I(p)$ curves over years shows much clearly that the inequality for lower incomes (related to the low values of p) decreased from 1980 to 2010, while for higher incomes (related to the values of p approaching to 1) the situation is very different: for such kind of incomes, the inequality increased in the considered time range. A possible interpretation of such dynamic is that from 1980 to 2010, the lower incomes became flat, causing a decreasing of inequality, while the the upper incomes grew up, causing an increase of inequality. Remaks of such kind are difficult to be observed in the Lorenz curve, but they can really help the researchers to better understand the social phenomena, that otherwise can be remain hidden.

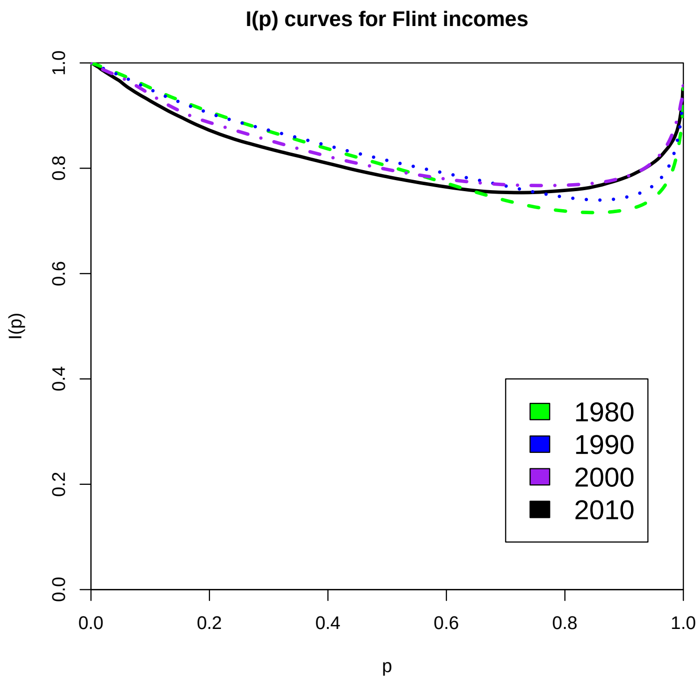

Figure 9: The $I(p)$ curve of Flint income in different years

5 Final remarks

In this paper some features of the inequality $I(p)$ curve have been presented. This curve can be considered more explanatory and flexibile than other inequality curves. The interpretation of its values is very intuitive and easily understandable. All these characteristics play a fundamental role, especially in the applicative field. For this reason the $I(p)$ curve seem to be a valid alternative to the classical inequality curves, like the Lorenz and the Bonferroni ones.

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