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Dottorato di Ricerca in Matematica Pura e Applicata

Localization for Riesz Means on compact Riemannian manifolds

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A chi sa
di non sapere.

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Introduction

This thesis deals with the problem of localization for the Riesz means for eigenfunction expansions of the Laplace-Beltrami operator. The classical Riemann localization principle states that if an integrable function of one variable vanishes in an open set, then its trigonometric Fourier expansion converges to zero in this set. This localization principle fails in $\mathbb{R}^d$ with $d \geq 2$. In order to recover localization one has to use suitable summability methods, such as the Bochner-Riesz means.

In Chapter 1 we focus on the compact rank one symmetric spaces case. While in Chapter 2 we show how some of the results obtained in the first chapter can be generalize to smooth compact and connected Riemannian manifolds.

Bochner-Riesz means and compact rank one symmetric spaces

Let $\mathcal{M}$ be a d-dimensional compact rank one symmetric space and denote by $0 = \lambda_0^2 < \lambda_1^2 < \lambda_2^2 < \dots$ the eigenvalues and by $\mathcal{H}_n$ the corresponding eigenspaces of the Laplace-Beltrami operator Δ, the spherical harmonics of degree n. To every square integrable function, and more generally tempered distribution, one can associate a Fourier series:

$$f(x) \;=\; \sum_{n=0}^{+\infty} Y_n f(x)\,,$$

where $Y_n f(x)$ is the orthogonal projection of $f(x)$ on $\mathcal{H}_n$. These Fourier series converge in the metric of $L^2(\mathcal{M})$ and in the topology of distributions, but in general one cannot ensure the pointwise convergence. For this reason we introduce the summability method of Bochner-Riesz means:

$$S_R^\alpha f(x) \;=\; \sum_{\lambda_n < R} \left(1 - \frac{\lambda_n^2}{R^2}\right)^\alpha \, Y_n f(x)\,.$$

In the definition of Bochner-Riesz means, the index α gives the degree of smoothness of the multiplier and this is related to the decay of the associate kernel. In particular, when $\alpha = 0$ one obtains the spherical partial sums, which are a natural analogue of the partial sums of one-dimensional Fourier series in the Euclidean space $\mathbb{R}$. There are examples of the failure of localization in Hölder, Lebesgue and Sobolev spaces. See [3], [4], [7], and [8], [29], [35] for the role of antipodal points for spherical harmonic expansions. See also the examples in Chapter 1. Despite the negative results, it has been proved in [5] and [36] that there is an almost everywhere localization principle for square integrable functions on $\mathcal{M}$; that is, if a function in $L^2(\mathcal{M})$ vanishes almost everywhere in an open set, then $\sum_{n=0}^{+\infty} Y_n f(x) = 0$ for almost every x in this open set. See also [11], [12], [19] and [46] for the corresponding result on the Euclidean spaces. The works of Bastis and Meaney deal with almost everywhere localization of Bochner-Riesz means of order $\alpha = 0$ for square integrable functions. On the other hand, it is known that for square integrable functions localization holds everywhere above the so-called critical index $\alpha = (d-1)/2$, while for integrable functions the critical index is $\alpha = d - 1$. See [8] and Theorem 1.6 below. In [2] Ahmedov studies the almost everywhere convergence of Bochner-Riesz means at critical line $\alpha = (d-1)(\frac{1}{p} - \frac{1}{2})$ for functions in $L^p(\mathbb{S}^d)$. Finally, in [13], [15] and [21] the dimension of sets for which localization fails is studied.

In this work we continue this line of research in the area of exceptional sets in harmonic analysis. In particular we prove that for Bochner-Riesz means of order α of p integrable functions on compact rank one symmetric spaces localization holds, with a possible exception in a set of point of suitable Hausdorff dimension. More generally we consider localization for distributions in Sobolev spaces. The Bessel potential $G^\gamma f(x)$, $-\infty < \gamma < +\infty$, of a tempered distribution $f(x) = \sum_{n=0}^{+\infty} Y_n f(x)$ is the tempered distribution defined by

$$G^\gamma f(x) = \sum_{n=0}^{+\infty} \left(1 + \lambda_n^2\right)^{-\gamma/2} Y_n f(x) .$$

Our Sobolev spaces are the spaces of potentials of functions in $L^p(\mathcal{M})$.

Main localization results

Our first result is an analogue for Bochner-Riesz means of the pointwise localization result of Meaney in [36] for spherical sums.

Assume that $f(x)$ is a tempered distribution on $\mathcal{M}$, with spherical harmonic *expansion* $\sum_{n=0}^{+\infty} Y_n f(x)$*. Also assume that* $f(x) = 0$ *for all* x *in a ball* $\{|x - \mathbf{o}| < \varepsilon\}$*, with radius* $\varepsilon > 0$ *and centre* $\mathbf{o}$*. Then*

$$\lim_{R \to +\infty} \{S_R^\alpha f(\mathbf{o})\} = 0 \quad \Longleftrightarrow \quad \lim_{n \to +\infty} \left\{n^{-\alpha} Y_n f(\mathbf{o})\right\} = 0 .$$

This means that, in order to understand the localization properties of the Bochner-Riesz means when $f(x)$ is in a particular space of functions on $\mathcal{M}$, it is enough to study the pointwise behaviour of the terms $n^{-\alpha} Y_n f(x)$ as $n \to +\infty$ and $f(x)$ ranges over the space of functions.

As shown by Bochner in [7], see also [46], the critical index for pointwise localization of Bochner-Riesz means on Euclidean spaces of dimension d is $(d-1)/2$. On the other hand, as shown by Kogbetlianz in [29] and Bonami and Clerc in [8], spheres and projective spaces are different, since antipodal points come into play. See also the paper of Hörmander [27] for the study of asymptotic properties of the spectral functions and summability of eigenfunction expansions for elliptic differential operators. In our second result we revisit this problem of pointwise localization. In particular, we determine the critical index for each compact rank one symmetric space for the pointwise localization when $f(x)$ is in $L^p(\mathcal{M})$, $1 \leq p \leq +\infty$.

Let $\alpha \geq 0$*,* $-\infty < \gamma < +\infty$ *and* $1 \leq p \leq +\infty$*. Assume also that:*

(1) $\mathcal{M}$ *is the sphere* $\mathbb{S}^d$ *and* $\alpha_p = d/p - 1$*;*

(2) $\mathcal{M}$ *is the real projective space* $P^d(\mathbb{R})$ *and* $\alpha_p = (d-1)/2$*;*

(3) $\mathcal{M}$ *is the complex projective space* $P^d(\mathbb{C})$ *and* $\alpha_p = (d-4)/2 + 2/p$*;*

(4) $\mathcal{M}$ *is the quaternionic projective space* $P^d(\mathbb{H})$ *and* $\alpha_p = (d-6)/2 + 4/p$*;*

(5) $\mathcal{M}$ *is the Cayley projective space* $P^{16}(Cay)$ *and* $\alpha_p = 3 + 8/p$*.*

Then $\max\{(d-1)/2, \alpha_p\}$ *is the critical index for the pointwise localization of the Bochner-Riesz means of* $G^\gamma f(x)$ *when* $f(x)$ *is in* $L^p(\mathcal{M})$*.*

This result, in the case of the sphere $\mathbb{S}^d$, is slightly better than the corresponding in [8], where only the case $\gamma = 0$ is considered and the condition for localization is $\alpha \geq (d-1)/2$ and $\alpha > d/p - 1$. We use a different technique: in [8] Bonami and Clerc use estimates on the kernel, while we estimate its Fourier transform. Observe also that when $\mathcal{M} = \mathbb{S}^d$ the antipodal manifold is composed of only one point and the singularities concentrate more than in the projective case. Finally, when $p = 1$, the critical index for $\alpha + \gamma$ has a geometric interpretation:

$$\{dimension\ of\ the\ space\} \ - \ \frac{1}{2}\{dimension\ of\ the\ antipodal\ manifold\} \ - \ 1\ .$$

Our third result revisits and extends the almost everywhere localization result of Bastis [5] and Meaney [36].

Assume that $\varepsilon > 0$, $-\infty < \gamma < +\infty$, $\alpha \geq 0$ and $1 \leq p \leq 2$. If $f(x) \in L^p(\mathcal{M})$ and $G^\gamma f(x) = 0$ in an open set Ω, then the following hold:

(1) $\alpha + \gamma = (d-1)\left(\frac{1}{p} - \frac{1}{2}\right)$, then for almost every point in Ω,

$$\lim_{R\to+\infty} \{S_R^\alpha G^\gamma f(x)\} \ = \ 0\ ;$$

(2) If $\alpha + \gamma > (d-1)\left(\frac{1}{p} - \frac{1}{2}\right)$, then

$$\lim_{R\to+\infty} \{S_R^\alpha G^\gamma f(x)\} \ = \ 0$$

at all points in this open set Ω, with possible exceptions in a set with Hausdorff dimension at most $\delta = d - p\left(\alpha + \gamma - (d-1)\left(\frac{1}{p} - \frac{1}{2}\right)\right)$.

The case $\alpha = \gamma = 0$ and $p = 2$ of this theorem is the above quoted result on the almost everywhere localization for spherical partial sums of square integrable functions. Indeed when $p = 2$ a more precise result holds.

Assume that $\alpha \geq 0$ and $0 \leq \alpha + \gamma < (d-1)/4$. If $f(x) \in L^2(\mathcal{M})$ and $G^\gamma f(x) = 0$ in an open set Ω, then

$$\lim_{R\to+\infty} \{S_R^\alpha G^\gamma f(x)\} \ = \ 0$$

at all points in the open set Ω, with possible exceptions in a set with Hausdorff dimension at most $\delta = d - 2(\alpha + \gamma)$.

Extension to compact Riemannian manifolds

In Chapter 2 we shall generalize some of the above results. In particular, with the notation $\{\lambda^2\}$ for the eigenvalues of the Laplace-Beltrami operator on the compact and connected Riemannian manifold $\mathcal{M}$ and $\{\varphi_\lambda(x)\}$ for the related eigenfunctions, we prove that pointwise one obtains the following result.

Let $f(x)$ be a tempered distribution on $\mathcal{M}$. If $\alpha \geq 0$ and $f(x) = 0$ for all x in a ball $\{|x-y| < \varepsilon\}$, with radius $\varepsilon > 0$ and centre y, then

$$\lim_{R\to+\infty}\left\{R^{-\alpha}\sup_{0\leq h\leq 1}\left|\sum_{R\leq\lambda\leq R+h}\widehat{f}(\lambda)\varphi_\lambda(y)\right|\right\} = 0 \quad\Longrightarrow\quad \lim_{R\to+\infty}\{S_R^\alpha f(y)\} = 0.$$

This result extends a result of Meaney in [33], in particular it gives an alternative proof when $\alpha = 0$. When $f(x)$ is a square integrable function on $\mathcal{M}$, a partial result for almost everywhere localization holds.

If $f(x) = 0$ in an open set Ω of $\mathcal{M}$ and $R_j - R_{j-1} > \delta > 0$ for every $j = 1, 2, \ldots,$ then for almost every x in Ω,

$$\lim_{j\to+\infty}\{|S_{R_j}^0 f(x)|\} = 0\,.$$

The full result of Bastis [5] and Meaney [36] occurs when one considers a manifold $\mathcal{M}$ with eigenvalues that group together. Finally, we state the partial analogous of the last result seen for compact rank one symmetric spaces for the almost everywhere localization for the Bochner-Riesz means of $G^\gamma f(x)$ when $f(x)$ is in $L^2(\mathcal{M})$.

If $f(x) \in L^2(\mathcal{M})$, $\alpha \geq 0$, $0 \leq \alpha+\gamma < d/2$, $\delta = d - 2(\alpha+\gamma)$ and $G^\gamma f(x) = 0$ in an open set Ω, then

$$\lim_{R\to+\infty}\{S_R^\alpha G^\gamma f(x)\} = 0$$

at all points in the open set Ω, with possible exceptions in a set with Hausdorff dimension at most δ.

Chapter 1

Localization for Riesz Means on compact rank one symmetric spaces

This chapter deals with spherical harmonic expansions on spheres and projective spaces. In particular, we determine sufficient conditions for the pointwise and almost everywhere localization for Riesz means for eigenfunction expansions of the Laplace-Beltrami operator on compact rank one symmetric spaces. Furthermore, we also estimate the Hausdorff dimension of the divergent set.

1.1 Harmonic analysis on compact rank one symmetric spaces

In what follows we shall denote by $|x-y|$ the geodesic distance between the two points x and y in $\mathcal{M}$. A two-point homogeneous space is a Riemannian manifold with the property that for every two pairs of points x_1, x_2 and y_1, y_2 with $|x_1 - x_2| = |y_1 - y_2|$, there is an isometry g of $\mathcal{M}$ such that $x_1 = gy_1$ and $x_2 = gy_2$. Birkhoff has called this property two-point homogeneity. Wang in [50] has shown that any compact two-point homogeneous space is isometric to a compact rank one symmetric space, that is:

i) the sphere $\mathbb{S}^d = SO(d+1)/SO(d)$
   $d = 1, 2, 3, \ldots;$

ii) the real projective space $P^d(\mathbb{R}) = SO(d+1)/O(d)$
$d = 2, 3, 4, \ldots;$

iii) the complex projective space $P^d(\mathbb{C}) = SU(l+1)/S(U(l) \times U(1))$
$d = 4, 6, 8, \ldots$ and $l = d/2$;

iv) the quaternionic projective space $P^d(\mathbb{H}) = Sp(l+1)/Sp(l) \times Sp(1)$
$d = 8, 12, 16, \ldots$ and $l = d/4$;

v) the Cayley projective plane $P^{16}(\mathrm{Cay})$.

Here d denotes the real dimension of any one of these spaces, $O(d)$, $U(d)$, $Sp(d)$ denote the orthogonal, unitary and symplectic groups of order d, and $S(\cdot)$ denotes the formation of a subgroup of matrices of unit determinant. Without loss of generality, one can renormalise the metric and the measure so that the total measure of $\mathcal{M}$ is 1 and the diameter of $\mathcal{M}$ is π. If $\mathbf{o}$ is a fixed point in $\mathcal{M}$, then $\mathcal{M}$ can be identified with the homogeneous space G/K, where G is the maximal connected group of isometries of $\mathcal{M}$ and K is the stabilizer of $\mathbf{o}$ in G. The measure dx is induced by the normalised left Haar measure dg on G: for any fixed point $\mathbf{o}$ in $\mathcal{M}$ and any function $f(x)$ integrable on $\mathcal{M}$,

$$\int_{\mathcal{M}} f(x)\, dx \;=\; \int_G f(g\mathbf{o})\, dg \,.$$

In particular, the convolution on the group G induces a convolution on the manifold $\mathcal{M}$.

For what follows we need the concept of radial functions and antipodal points. We say that a function $f(g)$ on the isometry group G is *right K-invariant* if, for every g in G and k in K,

$$f(gk) \;=\; f(g)\,.$$

A function $f(g)$ on G is *bi-K-invariant* if, for every g in G and k in K,

$$f(kg) \;=\; f(gk) \;=\; f(g)\,.$$

Functions and distributions on $\mathcal{M} = G/K$ can be identify with right K-invariant functions and distributions on G. It suffices to put $f(g) = f(x)$ whenever $g\mathbf{o} = x$. A function is radial around $\mathbf{o}$ if $f(x)$ only depends on $|x - \mathbf{o}|$. Radial functions on $\mathcal{M}$ correspond to bi-K-invariant functions on G. Indeed, by the two-point homogeneity, K fixes $\mathbf{o}$ and acts transitively on the set of points at a given distance from $\mathbf{o}$. The points with distance from $\mathbf{o}$ equal to the diameter of $\mathcal{M}$ are the antipodal points of $\mathbf{o}$ and they form the so called antipodal manifold of $\mathbf{o}$ on $\mathcal{M}$.

If one denotes by $\mathcal{A}(t)$, $0 \le t \le \pi$, the surface measure of a sphere of radius t in $\mathcal{M}$, then

$$\mathcal{A}(t) \;=\; C\left(\sin\frac{t}{2}\right)^{M} (\sin t)^{N} \;,$$

where the constant $C > 0$ is chosen so that $\int_0^\pi \mathcal{A}(t)dt = 1$. In particular, M is the dimension of the antipodal manifold, $M + N + 1$ is the dimension d of the manifold $\mathcal{M}$, and these parameters are as follows (see [26], p.168):

$\mathcal{M}$	M	N	$\mathcal{M}$	M	N
$\mathbb{S}^d$	0	$d-1$	$P^d(\mathbb{H})$	$d-4$	3
$P^d(\mathbb{R})$	$d-1$	0	$P^{16}(\mathrm{Cay})$	8	7
$P^d(\mathbb{C})$	$d-2$	1			

If $f(t)$ is integrable on $[0,\pi]$ with respect to the measure $\mathcal{A}(t)dt$ then, for any $\mathbf{o} \in \mathcal{M}$,

$$\int_{\mathcal{M}} f(|x-\mathbf{o}|)\,dx \;=\; \int_0^\pi f(t)\,\mathcal{A}(t)\,dt\,.$$

1.1.1 The zonal spherical functions

The compact rank one symmetric spaces admit an isometry invariant second order differential operator, the Laplace-Beltrami operator Δ. The spectrum of this operator is discrete, real and non-negative. One can arrange the eigenvalues in increasing order: $0 = \lambda_0^2 < \lambda_1^2 < \lambda_2^2 < \ldots$. More precisely:

$$\lambda_n^2 \;=\; sn\,(sn + a + b + 1)\;.$$

The index n ranges over all non-negative integers and $s = 1$ if $\mathcal{M} = \mathbb{S}^d$, $P^d(\mathbb{C})$, $P^d(\mathbb{H})$ or $P^{16}(\mathrm{Cay})$, while $s = 2$ if $\mathcal{M} = P^d(\mathbb{R})$. The parameters a and b and the eigenvalues λ_n^2 are given by the following table:

$\mathcal{M}$	a	b	λ_n^2
$\mathbb{S}^d$	$(d-2)/2$	$(d-2)/2$	$n(n+d-1)$
$P^d(\mathbb{R})$	$(d-2)/2$	$(d-2)/2$	$2n(2n+d-1)$
$P^d(\mathbb{C})$	$(d-2)/2$	0	$n(n+\frac{d}{2})$
$P^d(\mathbb{H})$	$(d-2)/2$	1	$n(n+1+\frac{d}{2})$
$P^{16}(\text{Cay})$	7	3	$n(n+11)$

The eigenspaces $\mathcal{H}_n$ corresponding to the eigenvalues λ_n^2 are finite-dimensional, invariant and irreducible under the group action, and they are mutually orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_{\mathcal{M}} f(x)\,\overline{g(x)}\,dx\,.$$

Moreover, if $L^2(\mathcal{M})$ denotes the space of square integrable functions on $\mathcal{M}$, $L^2(\mathcal{M}) = \bigoplus_{n=1}^{+\infty} \mathcal{H}_n$.

Now, for each integer $n \geq 0$ let $d_n = \mathit{dimension}\{\mathcal{H}_n\}$ and $\{Y_{n,j}(x)\}_{j=1}^{d_n}$ be an orthonormal basis of $\mathcal{H}_n$. The dimensions of the eigenspaces $\mathcal{H}_n$ can be computed explicitly, but here it suffices to say that there exist two positive constants c and C such that, for every n,

$$c\,(1+n)^{d-1} \leq d_n \leq C\,(1+n)^{d-1}\,.$$

The Fourier expansion of a square integrable function, and more generally of a tempered distribution, is given by

$$f(x) = \sum_{n=0}^{+\infty} Y_n f(x) = \sum_{n=0}^{+\infty}\left\{\sum_{j=1}^{d_n} \widehat{f}(n,j)\,Y_{n,j}(x)\right\}\,,$$

with

$$\widehat{f}(n,j) = \int_{\mathcal{M}} f(x)\,\overline{Y_{n,j}(x)}\,dx\,.$$

It is convenient to rewrite the orthogonal projection $Y_n f(x)$ of $f(x)$ onto $\mathcal{H}_n$ as a convolution:

$$\begin{aligned} Y_n f(x) &= \sum_{j=1}^{d_n} \widehat{f}(n,j)\, Y_{n,j}(x) \\ &= \int_{\mathcal{M}} f(y) \left\{ \sum_{j=1}^{d_n} Y_{n,j}(x)\, \overline{Y_{n,j}(y)} \right\} dy \\ &= \int_{\mathcal{M}} f(y)\, Z_n(x,y)\, dy \,. \end{aligned}$$

Definition 1.1. The functions $Z_n(x,y)$ defined above, i.e.

$$Z_n(x,y) = \sum_{j=1}^{d_n} Y_{n,j}(x)\, \overline{Y_{n,j}(y)}\,,$$

are the *zonal spherical functions* of degree n and pole x.

It is an important feature of rank one symmetric spaces that the zonal spherical functions are given in terms of Jacobi polynomials: If $t = |x-y|$, then

$$Z_n(x,y) = \sum_{j=1}^{d_n} Y_{n,j}(x)\, \overline{Y_{n,j}(y)} = d_n \frac{P_n^{(a,b)}(\cos(t/s))}{P_n^{(a,b)}(1)}\,. \tag{1.1}$$

For all these properties of symmetric spaces and Jacobi polynomials see, for example, [26], [30] and [46].

The following lemma gives an estimate for the size of the zonal spherical functions, and plays a crucial role in the problem of localization.

Lemma 1.1. *With the notation $Z_n(x,y) = Z_n(\cos t)$, if $|x-y| = t$, the following estimates hold:*

(1) For every x and y,

$$|Z_n(x,y)| \le Z_n(x,x) = d_n \le C\,(1+n)^{d-1}\,;$$

(2) If $0 \le t \le \pi/2$, then

$$|Z_n(x,y)| \le C\,(1+n)^{d-1}\,(1+nt)^{-(d-1)/2}\,;$$

(3) If $\pi/2 \le t \le \pi$, then

$$|Z_n(x,y)| \le \begin{cases} C\,(1+n)^{d-1}\,(1+n(\pi-t))^{-(d-1)/2} & \textit{if } \mathcal{M} = \mathbb{S}^d\,, \\ C\,(1+n)^{(d-1)/2} & \textit{if } \mathcal{M} = P^d(\mathbb{R})\,, \\ C\,(1+n)^{d/2}\,(1+n(\pi-t))^{-1/2} & \textit{if } \mathcal{M} = P^d(\mathbb{C})\,, \\ C\,(1+n)^{(d+2)/2}\,(1+n(\pi-t))^{-3/2} & \textit{if } \mathcal{M} = P^d(\mathbb{H})\,, \\ C\,(1+n)^{11}\,(1+n(\pi-t))^{-7/2} & \textit{if } \mathcal{M} = P^{16}(Cay)\,. \end{cases}$$

Proof. The proof of (1) follows from (1.1).

The Jacobi polynomials $P_n^{(a,b)}(\cos t)$ have an asymptotic expansion, as $n \to \infty$, in terms of the Bessel functions:

$$\left(\sin \frac{t}{2}\right)^a \left(\cos \frac{t}{2}\right)^b \frac{P_n^{(a,b)}(\cos t)}{P_n^{(a,b)}(1)}$$

$$= \frac{\Gamma(a+1)}{(n+(a+b+1)/2)^a} \left(\frac{t}{\sin t}\right)^{\frac{1}{2}} J_a\left((n+(a+b+1)/2)\,t\right)$$

$$+ \begin{cases} t^{a+2}\, O\left(1\right) & \text{if} \quad 0 < t \le cn^{-1}\,, \\ t^{1/2}\, O\left(n^{-a-3/2}\right) & \text{if} \quad cn^{-1} \le t \le \pi - \epsilon\,. \end{cases}$$

Here $c, \epsilon > 0$ are fixed and $J_a(x)$ is the Bessel function of order a. See [47], p.197. On the other hand, the Bessel functions satisfy the estimate (see [31]):

$$|J_a(x)| \quad \le \quad C \, \min\left\{|x|^a, \; |x|^{-1/2}\right\} \, .$$

In particular, if $0 \le t \le \pi/2$,

$$\left| \frac{P_n^{(a,b)}(\cos t)}{P_n^{(a,b)}(1)} \right| \le C \, \left(1+nt\right)^{-a-1/2} \, .$$

This gives the estimate for $0 \le t \le \pi/2$ in (2). Observe that this estimate depends only on the dimension of the symmetric space.

The estimates when $\pi/2 \le t \le \pi$ in (3) are similar and they follow from the symmetry relation

$$P_n^{(a,b)}(-x) \;=\; (-1)^n \, P_n^{(b,a)}(x)$$

and the estimate

$$\left| \frac{P_n^{(a,b)}(\cos t)}{P_n^{(a,b)}(1)} \right| \le C \left(1+n\right)^{b-a} \; \left(1+n\left(\pi-t\right)\right)^{-b-1/2} \, .$$

□

A couple of observations: In the estimate of $Z_n(x, y)$, the exponent of n when $x = y$ is equal to

$$\{dimension\; of\; the\; space\} \; - \; 1\, .$$

While, when y is in the antipodal manifold of x, the exponent of n becomes

$$\{dimension\; of\; the\; space\} \; - \; \frac{1}{2} \{dimension\; of\; the\; antipodal\; manifold\} \; - \; 1\, .$$

1.1.2 Bessel potentials and Sobolev spaces

We conclude this section by recalling the definition and some properties of the Bessel potential and the associated Bessel kernel.

Definition 1.2. Let $f(x)$ be a tempered distribution. The *Bessel potential* of $f(x)$ of order γ, $-\infty < \gamma < +\infty$, is defined as

$$G^\gamma f(x) \ = \ \int_{\mathcal{M}} G^\gamma(x,y)\, f(y)\, dy\,,$$

where

$$G^\gamma(x,y) \ = \ \sum_{n=0}^{+\infty} \left(1+\lambda_n^2\right)^{-\gamma/2} \ Z_n(x,y)$$

is the associated *Bessel kernel.*

Of course, when $\gamma \leq 0$ or when $f(x)$ is not a function, write $G^\gamma f(x)$ as an integral is an abuse of notation. The Bessel potentials on Euclidean spaces are presented in [45]. The properties on a manifold are essentially the same.

Lemma 1.2. *If $\gamma > 0$, then the Bessel kernel $G^\gamma(x,y)$ is positive and integrable, and it is smooth in $\{|x-y| \neq 0\}$. Moreover, if $0 < \gamma < d$, then $G^\gamma(x,y) \approx |x-y|^{\gamma-d}$ when $|x-y| \to 0$.*

Proof. It follows from the definition of the Gamma function that, for $\gamma > 0$,

$$\left(1+\lambda^2\right)^{-\gamma/2} \ = \ \frac{1}{\Gamma(\gamma/2)} \int_0^{+\infty} t^{\,\frac{\gamma}{2}-1}\, e^{-t(1+\lambda^2)}\, dt\,.$$

Therefore $G^\gamma(x,y)$ can be subordinated to the heat kernel:

$$\begin{aligned} G^\gamma(x,y) \ &= \ \sum_{n=0}^{+\infty} \left(1+\lambda_n^2\right)^{-\gamma/2} \ Z_n(x,y) \\ &= \ \frac{1}{\Gamma(\gamma/2)} \int_0^{+\infty} t^{\,\frac{\gamma}{2}-1}\, e^{-t} \left(\sum_{n=0}^{+\infty} e^{-\lambda_n^2 t}\, Z_n(x,y) \right) dt\,. \end{aligned}$$

The heat kernel is smooth and positive and it satisfies some Gaussian estimates. More precisely, there exists smooth functions $u_k(x,y)$ such that, if t is small,

$$0 \ < \ \sum_{n=0}^{+\infty} e^{-\lambda_n^2 t}\, Z_n(x,y) \ = \ (4\pi t)^{-d/2}\, e^{-|x-y|^2/(4t)} \left(\sum_{k=0}^{N} t^k\, u_k(x,y) + O(t^{n+1}) \right)\,.$$

See [16]. The estimates for the Bessel kernel follows by integrating these estimates. $\square$

Our treatment will concern localization for distributions in Sobolev spaces. One way of defining the Sobolev spaces $W^{\gamma,p}(\mathcal{M})$ is as the image of $L^p(\mathcal{M})$ under the action of the Bessel potentials $(I-\Delta)^{-\gamma/2}$. Then, our Sobolev spaces are the spaces of potentials of functions in $L^p(\mathcal{M})$.

1.2 Decomposition of the Bochner-Riesz kernel

In what follows we consider operators of the form

$$Tf(x) \;=\; \sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} m(\lambda_n)\, \widehat{f}(n,j)\, Y_{n,j}(x)\,.$$

We shall always assume that $m(\lambda)$ is an even function on $-\infty < \lambda < +\infty$ with tempered growth, i.e. $|m(\lambda)| \le C(1+\lambda)^k$ for some k. The multiplier $m(\lambda)$ is the Fourier transform of a tempered distribution and, formally,

$$Tf(x) \;=\; \int_{\mathcal{M}} \left(\sum_{n=0}^{+\infty} m(\lambda_n)\, Z_n(x,y) \right) f(y)\, dy\,.$$

Hence, the convolution of a zonal kernel $T(x,y) = \sum_{n=0}^{+\infty} m(\lambda_n)\, Z_n(x,y)$ with a tempered distribution $f(x)$ is that tempered distribution whose Fourier transform is the pointwise product between the Fourier transform of $T(x,y)$ and $f(x)$. The Bochner-Riesz means are an example of such operators.

Definition 1.3. Let $f(x)$ be a tempered distribution. The *Bochner-Riesz means* of $f(x)$ of complex order α are defined as

$$\begin{aligned} S_R^\alpha f(x) \;&=\; \sum_{n=0}^{+\infty} \left(1-\frac{\lambda_n^2}{R^2}\right)_+^\alpha \left(\sum_{j=1}^{d_n} \widehat{f}(n,j)\, Y_{n,j}(x) \right) \\ &=\; \int_{\mathcal{M}} S_R^\alpha(x,y)\, f(y)\, dy\,, \end{aligned}$$

where

$$S_R^\alpha(x,y) \;=\; \sum_{n=0}^{+\infty} \left(1-\frac{\lambda_n^2}{R^2}\right)_+^\alpha Z_n(x,y)$$

is the associated *Bochner-Riesz kernel.*

Then, with the above notations,

$$m_R(\lambda_n) \;=\; \left(1-\frac{\lambda_n^2}{R^2}\right)_+^\alpha .$$

The main tool in our localization result is a decomposition of $S_R^\alpha(x,y)$ into a kernel with small support plus a remainder. A natural decomposition is

$$S_R^\alpha(x,y) \;=\; S_R^\alpha(x,y)\chi_{\{|x-y|<\varepsilon\}}(x,y) \;+\; S_R^\alpha(x,y)\left(1-\chi_{\{|x-y|<\varepsilon\}}(x,y)\right) \;.$$

This decomposition has been exploited for example in [23]. Here we exploit a sort of smoothed version of the one above, and we decompose $S_R^\alpha(x,y)$ into a kernel with small support $\{|x-y|\leq\varepsilon\}$ and a kernel with small Fourier transform.

Let $\varepsilon>0$ and let $\psi(\lambda)$ be an even test function with cosine Fourier transform

$$\begin{cases} \widehat{\psi}(\tau) \;=\; 1 & \text{if} \quad |\tau|\leq\varepsilon/2\,, \\ \widehat{\psi}(\tau) \;=\; 0 & \text{if} \quad |\tau|\geq\varepsilon\,. \end{cases}$$

This implies that

$$\int_{\mathbb{R}} \psi(\lambda)\,d\lambda \;=\; 1$$

and

$$\int_{\mathbb{R}} \psi(\lambda)\,\lambda^n\,d\lambda \;=\; 0 \quad n=1,2,\ldots\;,$$

namely $\psi(\lambda)$ has mean one and all other moments are zero. Then, if we denote by $m_R*\psi(\lambda)$ the convolution on $\mathbb{R}$, i.e.

$$m_R*\psi(\lambda) \;=\; \int_{\mathbb{R}} m_R(\lambda-\tau)\psi(\tau)\,d\tau\;,$$

one has that $m_R*\psi(\lambda)$ is a good approximation of $m_R(\lambda)$ when $|\lambda-R|>1/\varepsilon$ and we can write

$$\begin{aligned} S_R^\alpha f(x) \;&=\; \sum_{n=0}^{+\infty}\sum_{j=1}^{d_n} m_R(\lambda_n)\,\widehat{f}(n,j)\,Y_{n,j}(x) \\ &=\; \sum_{n=0}^{+\infty} m_R*\psi(\lambda_n)\,\left(\sum_{j=1}^{d_n}\widehat{f}(n,j)\,Y_{n,j}(x)\right) \\ &+\; \sum_{n=0}^{+\infty}\left(m_R(\lambda_n)-m_R*\psi(\lambda_n)\right)\,\left(\sum_{j=1}^{d_n}\widehat{f}(n,j)\,Y_{n,j}(x)\right) \\ &=\; A_Rf(x)\;+\;B_Rf(x)\,. \end{aligned}$$

The operators A_R and B_R are associated to the kernels

$$A_R(x,y) \;=\; \sum_{n=0}^{+\infty} m_R*\psi(\lambda_n)\,Z_n(x,y)\;,$$

$$B_R(x,y) \;=\; \sum_{n=0}^{+\infty}\left(m_R(\lambda_n)-m_R*\psi(\lambda_n)\right)\,Z_n(x,y)\,.$$

The kernel $A_R(x,y)$ has small support.

Lemma 1.3. *The kernel $A_R(x,y)$ has support in $\{|x-y|\le\varepsilon\}$. In particular, if a tempered distribution $f(x)$ vanishes in an open set Ω then, for all $x\in\Omega$ with distance $\{x,\partial\Omega\}>\varepsilon$,*

$$A_R f(x) \;=\; 0\,.$$

Proof. Let $\cos(\tau\sqrt{\Delta})f(x)$ be the solution of the Cauchy problem for the wave equation in $\mathbb{R}\times\mathcal{M}$,

$$\begin{cases} \dfrac{\partial^2}{\partial\tau^2}\,u(\tau,x) \;+\; \Delta u(\tau,x) \;=\; 0\,, \\ u(0,x) \;=\; f(x)\,, \quad \dfrac{\partial}{\partial\tau}\,u(0,x) \;=\; 0\,. \end{cases}$$

Solving the wave equation by separation of variables, one obtains

$$\cos\left(\tau\sqrt{\Delta}\right) f(x) \;=\; \sum_{n=0}^{+\infty}\sum_{j=1}^{d_n} \cos(\tau\lambda_n)\,\widehat{f}(n,j)\,Y_{n,j}(x)\,.$$

Hence, in the distribution sense,

$$\begin{aligned} A_R f(x) \;&=\; \sum_{n=0}^{+\infty}\sum_{j=1}^{d_n} m_R*\psi(\lambda_n)\,\widehat{f}(n,j)\,Y_{n,j}(x) \\ =\; &\sum_{n=0}^{+\infty}\sum_{j=1}^{d_n}\left(\int_0^{+\infty}\widehat{m_R*\psi}(\tau)\;\cos(\tau\lambda_n)\,d\tau\right)\widehat{f}(n,j)\,Y_{n,j}(x) \\ =\; &\int_0^{+\infty}\widehat{m_R*\psi}(\tau)\,\left(\sum_{n=0}^{+\infty}\sum_{j=1}^{d_n}\cos(\tau\lambda_n)\,\widehat{f}(n,j)\,Y_{n,j}(x)\right)d\tau \\ =\; &\int_0^{+\infty}\widehat{m_R}(\tau)\,\widehat{\psi}(\tau)\;\cos\left(\tau\sqrt{\Delta}\right)\;f(x)\,d\tau\,. \end{aligned}$$

By assumption $\widehat{\psi}(\tau)=0$ if $|\tau|\ge\varepsilon$. Moreover, by the finite propagation of waves (see [17]), if $f(x)=0$ in Ω, then also $\cos(\tau\sqrt{\Delta})f(x)=0$ for every $x\in\Omega$ and $\tau<$ *distance* $\{x,\partial\Omega\}$. Then the lemma follows. □

The Fourier transform of the kernel $B_R(x,y)$ is small.

Lemma 1.4. *If $m_R(\lambda)=(1-\lambda^2/R^2)_+^\alpha$, then for every $k>0$ and $A>0$ there exist $C>0$ and $h>0$ such that for every complex α with $0\le\mathrm{Re}(\alpha)\le A$ and every $R>1$,*

$$|m_R(\lambda)-m_R*\psi(\lambda)| \;\le\; C\,(1+|\alpha|)^h\,R^{-\mathrm{Re}(\alpha)}\;(1+|R-\lambda|)^{-k}\,.$$

Proof. First observe that for every τ,

$$R - |\lambda-\tau| \le |R-\lambda| + |\tau| .$$

So we can write

$$\begin{aligned}|m_R(\lambda-\tau)| &= R^{-2\,\mathrm{Re}(\alpha)} \left(R+|\lambda-\tau|\right)^{\mathrm{Re}(\alpha)} \left(R-|\lambda-\tau|\right)_+^{\mathrm{Re}(\alpha)} \\ &\le 2^{\mathrm{Re}(\alpha)} R^{-\mathrm{Re}(\alpha)} \left(|R-\lambda|+|\tau|\right)^{\mathrm{Re}(\alpha)} .\end{aligned}$$

Since $\psi(\lambda)$ is a test function, if $|R-\lambda| \le 1$ then

$$\begin{aligned}&|m_R(\lambda) - m_R * \psi(\lambda)| \le |m_R(\lambda)| + |m_R * \psi(\lambda)| \\ &\le 2^{\mathrm{Re}(\alpha)} R^{-\mathrm{Re}(\alpha)} \left\{ |R-\lambda|^{\mathrm{Re}(\alpha)} + \int_{\mathbb{R}} \left(|R-\lambda| + |\tau|\right)^{\mathrm{Re}(\alpha)} |\psi(\tau)|\, d\tau \right\} \\ &\le 2^{\mathrm{Re}(\alpha)} R^{-\mathrm{Re}(\alpha)} \left\{ 1 + \int_{\mathbb{R}} \left(1+|\tau|\right)^{\mathrm{Re}(\alpha)} |\psi(\tau)|\, d\tau \right\} = C\, R^{-\mathrm{Re}(\alpha)} .\end{aligned}$$

Now consider the case $|R-\lambda| \ge 1$. Observe that for every $l \in \mathbb{N}$ there exists a polynomial $P_l(\lambda)$ of degree l such that

$$\frac{\partial^l}{\partial\lambda^l}\left(1-\frac{\lambda^2}{R^2}\right)_+^{\alpha} = R^{-l}\, P_l\left(\frac{\lambda}{R}\right) \left(1-\frac{\lambda^2}{R^2}\right)_+^{\alpha-l} .$$

The coefficients of $P_l(\lambda)$ are dominated by $(1+|\alpha|)^l$, therefore

$$\left| \frac{\partial^l}{\partial\lambda^l} m_R(\lambda) \right| \le C\, (1+|\alpha|)^l\, R^{-\mathrm{Re}(\alpha)}\, |R-\lambda|^{\mathrm{Re}(\alpha)-l} .$$

By assumption $\psi(\lambda)$ has mean one, so

$$|m_R(\lambda) - m_R * \psi(\lambda)| = \left| \int_{\mathbb{R}} \left(m_R(\lambda-\tau) - m_R(\lambda)\right) \psi(\tau)\, d\tau \right| .$$

Since the positive moments of $\psi(\lambda)$ are zero, for this last integral we can write

$$\int_{\mathbb{R}} \left(m_R(\lambda-\tau) - m_R(\lambda)\right) \psi(\tau) d\tau = \int_{\mathbb{R}} \left(m_R(\lambda-\tau) - \sum_{l=0}^{L-1} \frac{(-\tau)^l}{l!} \frac{\partial^l}{\partial\lambda^l} m_R(\lambda) \right) \psi(\tau) d\tau,$$

where L is independent of R and will be specified later. Splitting the integration on $\mathbb{R}$ into $\{|\tau| \le |R-\lambda|/2\}$ and $\{|\tau| \ge |R-\lambda|/2\}$, one gets

$$\begin{aligned}&|m_R(\lambda) - m_R * \psi(\lambda)| \le \int_{\{|\tau|\ge|R-\lambda|/2\}} |m_R(\lambda-\tau)|\, |\psi(\tau)|\, d\tau \\ &+ \sum_{l=0}^{L-1} \frac{1}{l!} \left| \frac{\partial^l}{\partial\lambda^l} m_R(\lambda) \right| \int_{\{|\tau|\ge|R-\lambda|/2\}} \left|\tau^l\right| |\psi(\tau)|\, d\tau \\ &+ \frac{1}{L!} \sup_{\{|\tau|\le|R-\lambda|/2\}} \left\{ \left| \frac{\partial^L}{\partial\lambda^L} m_R(\lambda-\tau) \right| \right\} \int_{\{|\tau|\le|R-\lambda|/2\}} \left|\tau^L\right| |\psi(\tau)|\, d\tau .\end{aligned}$$

We have the following estimates:

$$\begin{aligned}
&\int_{\{|\tau|\geq|R-\lambda|/2\}} |m_R(\lambda-\tau)|\ |\psi(\tau)|\ d\tau \\
&\leq\ 2^{\operatorname{Re}(\alpha)}\ R^{-\operatorname{Re}(\alpha)} \int_{\{|\tau|\geq|R-\lambda|/2\}} (\ |R-\lambda|+|\tau|\ )^{\operatorname{Re}(\alpha)}\ |\psi(\tau)|\ d\tau \\
&\leq\ 6^{\operatorname{Re}(\alpha)}\ R^{-\operatorname{Re}(\alpha)} \int_{\{|\tau|\geq|R-\lambda|/2\}} |\tau|^{\operatorname{Re}(\alpha)}\ |\psi(\tau)|\ d\tau \\
&\leq\ C\, R^{-\operatorname{Re}(\alpha)}\ |R-\lambda|^{-k}\ ,
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{\partial^l}{\partial\lambda^l}\, m_R(\lambda)\right| \int_{\{|\tau|\geq|R-\lambda|/2\}} \left|\tau^l\right|\ |\psi(\tau)|\ d\tau \\
&\leq\ C\ (1+|\alpha|)^l\ R^{-\operatorname{Re}(\alpha)}\ |R-\lambda|^{\operatorname{Re}(\alpha)-l} \int_{\{|\tau|\geq|R-\lambda|/2\}} |\tau^l|\ |\psi(\tau)|\ d\tau \\
&\leq\ C\ (1+|\alpha|)^l\ R^{-\operatorname{Re}(\alpha)}\ |R-\lambda|^{-k}\ ,
\end{aligned}$$

$$\begin{aligned}
&\sup_{\{|\tau|\leq|R-\lambda|/2\}} \left\{ \left|\frac{\partial^L}{\partial\lambda^L}\, m_R(\lambda-\tau)\right| \right\} \int_{\{|\tau|\leq|R-\lambda|/2\}} \left|\tau^L\right|\ |\psi(\tau)|\ d\tau \\
&\leq\ C\ (1+|\alpha|)^L\ R^{-\operatorname{Re}(\alpha)}\ |R-\lambda|^{\operatorname{Re}(\alpha)-L} \int_{\mathbb{R}} \left|\tau^L\right|\ |\psi(\tau)|\ d\tau \\
&\leq\ C\ (1+|\alpha|)^L\ R^{-\operatorname{Re}(\alpha)}\ |R-\lambda|^{\operatorname{Re}(\alpha)-L}\ .
\end{aligned}$$

The thesis follows by taking $L\geq\operatorname{Re}(\alpha)+k$ and $h=L+1$. □

Our first result is an exact analogue for Bochner-Riesz means of the result of Meaney in [36] for spherical sums.

Theorem 1.5. *Assume that $f(x)$ is a tempered distribution on $\mathcal{M}$, with spherical harmonic expansion $\sum_{n=0}^{+\infty} Y_n f(x)$. Also assume that $f(x)=0$ for all x in a ball $\{|x-\mathbf{o}|<\varepsilon\}$, with radius $\varepsilon>0$ and centre $\mathbf{o}$. Then the following are equivalent:*

(1) $\lim_{R\to+\infty}\{S_R^\alpha f(\mathbf{o})\}=0$,

(2) $\lim_{n\to+\infty}\{n^{-\alpha}\, Y_n f(\mathbf{o})\}=0$.

Proof. A necessary condition for the pointwise Bochner-Riesz summability of

$$S_R^\alpha f(\mathbf{o})\ =\ \sum_{\lambda_n<R}\left(1-\frac{\lambda_n^2}{R^2}\right)^\alpha\ Y_n f(\mathbf{o})$$

is that $\{\lambda_n^{-\alpha} Y_n f(\mathbf{o})\} \to 0$ when $n \to \infty$. See [53], Theorem 1.22 of Chapter III, for the corresponding result for Cesàro means. Hence (1) implies (2).

Conversely, assume that $f(x) = 0$ in $\{|x - \mathbf{o}| < \varepsilon\}$. By Lemma 1.3, $A_R f(\mathbf{o}) = 0$. By Lemma 1.4,

$$|B_R f(\mathbf{o})| \le C \sum_{n=0}^{+\infty} R^{-\alpha} \, (1+\lambda_n)^{\alpha} \, (1 + |R - \lambda_n|)^{-k} \, \left|(1+\lambda_n)^{-\alpha} \, Y_n f(\mathbf{o})\right| \, .$$

Observe that $\sum_{n=0}^{+\infty} R^{-\alpha} \, (1+\lambda_n)^{\alpha} \, (1 + |R - \lambda_n|)^{-k} < C < +\infty$, with C independent on R. If $\{(1+\lambda_n)^{-\alpha} Y_n f(\mathbf{o})\} \to 0$, then also $\{B_R f(\mathbf{o})\} \to 0$. Hence (2) implies (1). □

1.3 Pointwise localization

In the Euclidean case the critical index for pointwise localization of Bochner-Riesz means of function in $L^p(\mathbb{R}^d)$ is $\alpha = (d-1)/2$ for every $1 \le p \le +\infty$. This was proved by Bochner in [7]. The difference between the Euclidean case and the compact rank one symmetric spaces case is in the antipodal manifolds. Antipodal points play an important role in determining the critical indices. In the next theorem we compute the critical index for each space when $f(x)$ is in $L^p(\mathcal{M})$, $1 \le p \le +\infty$, using the fact that

$$Y_n f(x) = \int_{\mathcal{M}} Z_n(x,y) \, f(y) \, dy$$

and that the zonal spherical functions $Z_n(x,y)$ are polynomials.

Theorem 1.6. *Assume that:*

$$-\infty < \gamma < +\infty, \quad \alpha \ge 0, \quad 1 \le p \le +\infty \, .$$

Also assume that

(1) $\mathcal{M} = \mathbb{S}^d$ *and*

$$\begin{cases} \alpha + \gamma \ge d/p - 1 & \text{if} \quad p < 2d/(d+1) \, , \\ \alpha + \gamma > (d-1)/2 & \text{if} \quad p = 2d/(d+1) \, , \\ \alpha + \gamma \ge (d-1)/2 & \text{if} \quad p > 2d/(d+1) \, . \end{cases}$$

(2) $\mathcal{M} = P^d(\mathbb{R})$ *and, for every* p,

$$\alpha + \gamma \ge (d-1)/2 \, .$$

(3) $\mathcal{M} = P^d(\mathbb{C})$ *and*

$$\begin{cases} \alpha+\gamma \ \geq \ (d-4)/2 \ + \ 2/p & \textit{if} \quad p < 4/3\,, \\ \alpha+\gamma \ > \ (d-1)/2 & \textit{if} \quad p = 4/3\,, \\ \alpha+\gamma \ \geq \ (d-1)/2 & \textit{if} \quad p > 4/3\,. \end{cases}$$

(4) $\mathcal{M} = P^d(\mathbb{H})$ *and*

$$\begin{cases} \alpha+\gamma \ \geq \ (d-6)/2 \ + \ 4/p & \textit{if} \quad p < 8/5\,, \\ \alpha+\gamma \ > \ (d-1)/2 & \textit{if} \quad p = 8/5\,, \\ \alpha+\gamma \ \geq \ (d-1)/2 & \textit{if} \quad p > 8/5\,. \end{cases}$$

(5) $\mathcal{M} = P^{16}(Cay)$ *and*

$$\begin{cases} \alpha+\gamma \ \geq \ 3 \ + \ 8/p & \textit{if} \quad p < 16/9\,, \\ \alpha+\gamma \ > \ 15/2 & \textit{if} \quad p = 16/9\,, \\ \alpha+\gamma \ \geq \ 15/2 & \textit{if} \quad p > 16/9\,. \end{cases}$$

Under the above assumptions, if $f(x)$ *is in* $L^p(\mathcal{M})$ *and if* $G^\gamma f(x) = 0$ *in an open set* Ω*, then for every* $x \in \Omega$

$$\lim_{R\to+\infty} \{S_R^\alpha G^\gamma f(x)\} \ = \ 0\,.$$

Proof. Fix $x \in \mathcal{M}$, $\varepsilon > 0$ and assume that $G^\gamma f(y) = 0$ if $|x-y| \leq \varepsilon$, and decompose $S_R^\alpha G^\gamma f(x)$ into $A_R G^\gamma f(x) + B_R G^\gamma f(x)$ as in the previous section. By Lemma 1.3, $A_R G^\gamma f(x) = 0$. Since $B_R G^\gamma f(x)$ converges to zero when $f(x)$ is a test function, in order to prove the theorem it then suffices to show that these linear functionals are uniformly bounded on $L^p(\mathcal{M})$. Recall that

$$B_R G^\gamma f(x) = \int_{\mathcal{M}} f(y) \left(\sum_{n=0}^{+\infty} (m_R(\lambda_n) - m_R * \psi(\lambda_n)) \ \left(1+\lambda_n^2\right)^{-\gamma/2} \ Z_n(x,y) \right) dy.$$

The norms of the functionals on $L^p(\mathcal{M})$ are the norms on $L^q(\mathcal{M})$, $1/p + 1/q = 1$, of the associate kernels. In particular, if $G^\gamma f(y) = 0$ when $|x-y| \leq \varepsilon$,

$$\begin{aligned} &|B_R G^\gamma f(x)| \\ &\leq \|f\|_{L^p} \left\{ \int_{|x-y|\geq\varepsilon} \left| \sum_{n=0}^{+\infty} (m_R(\lambda_n) - m_R * \psi(\lambda_n)) \ \left(1+\lambda_n^2\right)^{-\gamma/2} \ Z_n(x,y) \right|^q dy \right\}^{\frac{1}{q}} \\ &\leq \|f\|_{L^p} \sum_{n=0}^{+\infty} |m_R(\lambda_n) - m_R * \psi(\lambda_n)| \ \left(1+\lambda_n^2\right)^{-\gamma/2} \left\{ \int_{|x-y|\geq\varepsilon} |Z_n(x,y)|^q \ dy \right\}^{\frac{1}{q}}\,. \end{aligned}$$

By Lemma 1.4,

$$|m_R(\lambda_n) - m_R * \psi(\lambda_n)| \left(1+\lambda_n^2\right)^{-\gamma/2} \leq C\, R^{-\alpha} \, (1+|R-\lambda_n|)^{-k} \, (1+\lambda_n)^{-\gamma} \ .$$

This implies that in the above sum only a finite number of terms come into play, the ones with $\lambda_n \approx R$. Then the theorem follows from the estimates for $\left\{ \int_{|x-y|\geq\varepsilon} |Z_n(x,y)|^q \, dy \right\}^{1/q}$ in Lemma 1.7 below. □

Lemma 1.7. *Let $\varepsilon > 0$ and define*

$$Z_{\varepsilon,q}(x) \;=\; \left\{ \int_{|x-y|\geq\varepsilon} |Z_n(x,y)|^q \, dy \right\}^{\frac{1}{q}} .$$

(1) If $\mathcal{M} = \mathbb{S}^d$, then

$$Z_{\varepsilon,q}(x) \leq \ C \begin{cases} (1+n)^{d-1-d/q} & \textit{if} \quad q > 2d/(d-1)\,, \\ (1+n)^{(d-1)/2} \, (\log(2+n))^{(d-1)/(2d)} & \textit{if} \quad q = 2d/(d-1)\,, \\ (1+n)^{(d-1)/2} & \textit{if} \quad q < 2d/(d-1)\,. \end{cases}$$

(2) If $\mathcal{M} = P^d(\mathbb{R})$ then, for every q,

$$Z_{\varepsilon,q}(x) \leq \ C \ (1+n)^{(d-1)/2} \ .$$

(3) If $\mathcal{M} = P^d(\mathbb{C})$, then

$$Z_{\varepsilon,q}(x) \leq \ C \begin{cases} (1+n)^{d/2-2/q} & \textit{if} \quad q > 4\,, \\ (1+n)^{(d-1)/2} \, (\log(2+n))^{1/4} & \textit{if} \quad q = 4\,, \\ (1+n)^{(d-1)/2} & \textit{if} \quad q < 4\,. \end{cases}$$

(4) If $\mathcal{M} = P^d(\mathbb{H})$, then

$$Z_{\varepsilon,q}(x) \leq \ C \begin{cases} (1+n)^{(d+2)/2-4/q} & \textit{if} \quad q > 8/3\,, \\ (1+n)^{(d-1)/2} \, (\log(2+n))^{3/8} & \textit{if} \quad q = 8/3\,, \\ (1+n)^{(d-1)/2} & \textit{if} \quad q < 8/3\,. \end{cases}$$

(5) If $\mathcal{M} = P^{16}(Cay)$, then

$$Z_{\varepsilon,q}(x) \leq \ C \begin{cases} (1+n)^{11-8/q} & \textit{if} \quad q > 16/7\,, \\ (1+n)^{15/2} \, (\log(2+n))^{7/16} & \textit{if} \quad q = 16/7\,, \\ (1+n)^{15/2} & \textit{if} \quad q < 16/7\,. \end{cases}$$

Proof. The zonal spherical functions $Z_n(x,y)$ are radial around x and, with the notation $|x-y|=t$ and $Z_n(x,y)=Z_n(\cos t)$, an integration in polar coordinates gives

$$\left\{\int_{|x-y|\geq\varepsilon}|Z_n(x,y)|^q\ dy\right\}^{\frac{1}{q}} = \left\{\int_\varepsilon^\pi |Z_n(\cos t)|^q\ \mathcal{A}(t)\,dt\right\}^{\frac{1}{q}} .$$

If $\mathcal{M}=\mathbb{S}^d$, then $\mathcal{A}(t)=C\,(\sin t)^{d-1}$. Moreover, if $0<\varepsilon\leq t\leq\pi$, by parts (2) and (3) of Lemma 1.1,

$$|Z_n(\cos t)|\ \leq\ C\,(1+n)^{d-1}\ (1+n(\pi-t))^{-(d-1)/2} .$$

Hence

$$\begin{aligned}&\left\{\int_\varepsilon^\pi |Z_n(\cos t)|^q\ \mathcal{A}(t)\,dt\right\}^{\frac{1}{q}}\\ \leq&\ C\,(1+n)^{d-1-\frac{d}{q}}\ +\ C\,(1+n)^{\frac{d-1}{2}}\left\{\int_{1/n}^{\pi-\varepsilon} t^{-\frac{d-1}{2}q+d-1}\,dt\right\}^{\frac{1}{q}}\\ \leq&\ C\begin{cases}(1+n)^{d-1-d/q} & \text{if}\ \ q>2d/(d-1)\,,\\ (1+n)^{(d-1)/2}\ (\log(2+n))^{(d-1)/(2d)} & \text{if}\ \ q=2d/(d-1)\,,\\ (1+n)^{(d-1)/2} & \text{if}\ \ q<2d/(d-1)\,.\end{cases}\end{aligned}$$

This proves the lemma for $\mathbb{S}^d$. The proof for projective spaces is similar. □

Examples. The indices in Theorem 1.6 are best possible when $\mathcal{M}$ is the sphere $\mathbb{S}^d$ or the real projective space $P^d(\mathbb{R})$ for every p. When $p=1$, this indices are best possible also for $P^d(\mathbb{C})$, $P^d(\mathbb{H})$ and $P^{16}(\mathrm{Cay})$. As shown by Bochner [7], the critical index for pointwise localization of Bochner-Riesz means of functions in $L^p(\mathbb{R}^d)$ is $\alpha=(d-1)/2$ for every $1\leq p\leq+\infty$. The Bochner-Riesz kernel in $\mathbb{R}^d$ is a Bessel function,

$$S_R^\alpha f(x)\ =\ \int_{\mathbb{R}^d}\pi^{-\alpha}\,\Gamma(\alpha+1)\,R^{d/2-\alpha}\,|y|^{-\alpha-d/2}\,J_{\alpha+d/2}\,(2\pi R|y|)\ f(x-y)\,dy\,.$$

Hence, by the asymptotic expansion of Bessel functions, $S_R^\alpha f(x)$ is approximated by

$$\pi^{-\alpha-1}\,\Gamma(\alpha+1)\,R^{\frac{d-1}{2}-\alpha}\int_{\mathbb{R}^d}|y|^{-\alpha-\frac{d+1}{2}}\cos\left(2\pi R|y|-\frac{(2\alpha+d+1)\pi}{4}\right)f(x-y)\,dy\,.$$

From this approximation it easily follows that a necessary condition for localization is the boundedness of the term $R^{(d-1)/2-\alpha}$, that is $\alpha\geq(d-1)/2$. This result of Bochner has been extended by Il'in [3] to spectral decompositions of self-adjoint elliptic operators: If $\alpha+\gamma<(d-1)/2$, then for any point $x\in\mathcal{M}$ there exists a function finite and in the Hölder class $\mathcal{C}^\alpha(\mathcal{M})$, which vanishes in a neighbourhood of x, and such that $\limsup_{R\to+\infty}\{|S_R^\alpha G^\gamma f(x)|\}=+\infty$. In particular, for every $1\le p\le+\infty$ the assumption $\alpha+\gamma\ge(d-1)/2$ is necessary for pointwise localization. By Theorem 1.5, another necessary condition for the pointwise Bochner-Riesz summability of $S_R^\alpha f(x)$ is that $\{\lambda_n^{-\alpha}Y_nf(x)\}\to0$ when $n\to\infty$. Fix $0<\varepsilon<d$, $\mathbf{o}\in\mathcal{M}$, and define

$$f(x)\;=\;G^\varepsilon(x,\mathbf{o})\;=\;\sum_{n=0}^{+\infty}\left(1+\lambda_n^2\right)^{-\varepsilon/2}\,Z_n(x,\mathbf{o})\,.$$

By Lemma 1.2, this function is smooth on $\mathcal{M}-\{\mathbf{o}\}$ and it behaves as $|x-\mathbf{o}|^{\varepsilon-d}$ when $x\to\mathbf{o}$. In particular, this function is in $L^p(\mathcal{M})$ for every $p<d/(d-\varepsilon)$. Finally observe that, when $|x-\mathbf{o}|=\pi$,

$$\left|(1+\lambda_n^2)^{-\varepsilon/2}\,Z_n(x,\mathbf{o})\right|\;\approx\;\begin{cases}(1+n)^{d-1-\varepsilon} & \text{if}\quad\mathcal{M}=\mathbb{S}^d\,,\\(1+n)^{(d-1)/2-\varepsilon} & \text{if}\quad\mathcal{M}=P^d(\mathbb{R})\,,\\(1+n)^{d/2-\varepsilon} & \text{if}\quad\mathcal{M}=P^d(\mathbb{C})\,,\\(1+n)^{(d+2)/2-\varepsilon} & \text{if}\quad\mathcal{M}=P^d(\mathbb{H})\,,\\(1+n)^{11-\varepsilon} & \text{if}\quad\mathcal{M}=P^{16}(\mathrm{Cay})\,.\end{cases}$$

The problem of convergence of eigenvalue expansions on compact Riemannian manifolds has also been studied in [9], [10], [40], [41], [42] and [49]. In these papers it is proved that localization for spherical sums may fail for piecewise smooth functions on three dimensional manifolds, the so-called Pinsky phenomenon, while it is proved that a sufficient condition for the pointwise Bochner-Riesz summability of order α for piecewise smooth function is $\alpha>(d-3)/2$.

1.4 Localization and Hausdorff dimension

The following theorem concerns the almost everywhere localization. It revisits and extends the almost everywhere localization result of Bastis [5] and Meaney [36].

Theorem 1.8. *Assume that:*

$$\varepsilon>0,\quad -\infty<\gamma<+\infty,\quad \alpha\ge0,\quad 1\le p\le2,\quad 0\le\beta\le\alpha+\gamma-(d-1)\left(\frac{1}{p}-\frac{1}{2}\right).$$

Then there exists a positive constant C with the following property: If $f(x)$ is in $L^p(\mathcal{M})$ and if $G^\gamma f(x) = 0$ in an open set Ω, then there exists $F(x)$ with $\|F\|_{L^p} \le C\|f\|_{L^p}$ and such that for all $x \in \Omega$ with distance $\{x, \partial\Omega\} > \varepsilon$,

$$\sup_{R>0} \{|S_R^\alpha G^\gamma f(x)|\} \le G^\beta F(x)\,.$$

First observe that from Lemma 1.3 we know that $A_R G^\gamma f(x) = 0$ if *distance* $\{x, \partial\Omega\} > \varepsilon$. Then we only need to control $B_R G^\gamma f(x)$, which can be factorized as

$$B_R G^\gamma f(x) = G^\beta \, B_R G^{\gamma-\beta} f(x)\,.$$

When $\beta \ge 0$ the operator G^β is positive, and this gives

$$\sup_{R>0} \{|B_R G^\gamma f(x)|\} \le G^\beta \left(\sup_{R>0} \left\{ \left| B_R G^{\gamma-\beta} f \right| \right\} \right)(x)\,.$$

Then, in order to prove the theorem, it suffices to prove that the maximal operator $\sup_{R>0} \left\{ |B_R G^{\gamma-\beta} f(x)| \right\}$ is bounded on $L^p(\mathcal{M})$. It suffices to consider two cases:

(1) $p = 1$ and $\mathrm{Re}(\alpha) - \beta + \gamma \ge (d-1)/2$;

(2) $p = 2$ and $\mathrm{Re}(\alpha) - \beta + \gamma \ge 0$.

The intermediate cases will follow by Stein's interpolation theorem for analytic families of operators (see [46], Chapter V).

Lemma 1.9. *If $f(x)$ is in $L^1(\mathcal{M})$ and $\mathrm{Re}(\alpha) - \beta + \gamma \ge (d-1)/2$, then*

$$\int_{\mathcal{M}} \sup_{R>0} \left\{ \left| B_R G^{\gamma-\beta} f(x) \right| \right\} dx \le C\,(1+|\alpha|)^h \int_{\mathcal{M}} |f(x)|\, dx\,.$$

Proof. Let $B_R G^{\gamma-\beta}(x,y)$ be the kernel associated to the operator $B_R G^{\gamma-\beta}$,

$$B_R G^{\gamma-\beta}(x,y) = \sum_{n=0}^{+\infty} \left(1+\lambda_n^2\right)^{(\beta-\gamma)/2} \left(m(\lambda_n) - m * \psi(\lambda_n)\right) Z_n(x,y)\,.$$

The maximal operator $\sup_{R>0} \left\{ \left| B_R G^{\gamma-\beta} f(x) \right| \right\}$ is dominated by

$$\sup_{R>0} \left\{ \left| B_R G^{\gamma-\beta} f(x) \right| \right\} \le \int_{\mathcal{M}} \sup_{R>0} \left\{ \left| B_R G^{\gamma-\beta}(x,y) \right| \right\} |f(y)|\, dy\,.$$

By Lemma 1.4,

$$\left| B_R G^{\gamma-\beta}(x,y) \right| \le C\,(1+|\alpha|)^h\, R^{-\mathrm{Re}(\alpha)} \sum_{n=0}^{+\infty} \frac{(1+\lambda_n^2)^{(\beta-\gamma)/2}\, |Z_n(x,y)|}{(1+|R-\lambda_n|)^k}\,.$$

By part (2) of Lemma 1.1, with $t = |x-y|$ and $0 \le t \le \pi/2$,

$$\begin{aligned}
&\left|B_R G^{\gamma-\beta}(x,y)\right| \\
&\le C\,(1+|\alpha|)^h\,R^{-\operatorname{Re}(\alpha)} \sum_{n=0}^{+\infty} \frac{(1+\lambda_n)^{\beta-\gamma}\,(1+\lambda_n)^{d-1}\,(1+\lambda_n t)^{-(d-1)/2}}{(1+|R-\lambda_n|)^k} \\
&= C\,(1+|\alpha|)^h\,R^{-\operatorname{Re}(\alpha)}\,t^{-(d-1)/2} \sum_{n=0}^{+\infty} \left(\frac{t+\lambda_n t}{1+\lambda_n t}\right)^{(d-1)/2} \frac{(1+\lambda_n)^{\beta-\gamma+(d-1)/2}}{(1+|R-\lambda_n|)^k} \\
&\le C\,(1+|\alpha|)^h\,R^{-\operatorname{Re}(\alpha)}\,t^{-(d-1)/2} \sum_{n=0}^{+\infty} \frac{(1+\lambda_n)^{\beta-\gamma+(d-1)/2}}{(1+|R-\lambda_n|)^k} \\
&\le C\,(1+|\alpha|)^h\,R^{-\operatorname{Re}(\alpha)+\beta-\gamma+(d-1)/2}\,t^{-(d-1)/2}\,.
\end{aligned}$$

This implies that, if $\operatorname{Re}(\alpha)-\beta+\gamma \ge (d-1)/2$ and $0 \le |x-y| \le \pi/2$,

$$\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta}(x,y)\right|\right\} \le C\,(1+|\alpha|)^h\,|x-y|^{-(d-1)/2}\,.$$

Similarly, by part (3) of Lemma 1.1, if $\operatorname{Re}(\alpha)-\beta+\gamma \ge (d-1)/2$ and $\pi/2 \le |x-y| \le \pi$,

$$\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta}(x,y)\right|\right\} \le \begin{cases} C\,(1+|\alpha|)^h\,(\pi-|x-y|)^{-(d-1)/2} & \text{if } \mathcal{M}=\mathbb{S}^d\,, \\ C\,(1+|\alpha|)^h & \text{if } \mathcal{M}=P^d(\mathbb{R})\,, \\ C\,(1+|\alpha|)^h\,(\pi-|x-y|)^{-1/2} & \text{if } \mathcal{M}=P^d(\mathbb{C})\,, \\ C\,(1+|\alpha|)^h\,(\pi-|x-y|)^{-3/2} & \text{if } \mathcal{M}=P^d(\mathbb{H})\,, \\ C\,(1+|\alpha|)^h\,(\pi-|x-y|)^{-7/2} & \text{if } \mathcal{M}=P^{16}(\mathrm{Cay})\,. \end{cases}$$

By these estimates, $\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta}(x,y)\right|\right\}$ is integrable with respect to x for every y, and

$$\begin{aligned}
&\int_{\mathcal{M}} \sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|\right\} dx \\
&\le \int_{\mathcal{M}} \left(\int_{\mathcal{M}} \sup_{R>0}\left\{\left|B_R G^{\gamma-\beta}(x,y)\right|\right\} dx\right) |f(y)|\,dy \\
&\le C \sup_{y\in\mathcal{M}} \left\{\int_{\mathcal{M}} \sup_{R>0}\left\{\left|B_R G^{\gamma-\beta}(x,y)\right| dx\right\}\right\} \int_{\mathcal{M}} |f(y)|\,dy\,.
\end{aligned}$$

Actually this proof shows that $\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|\right\}$ can be controlled by a fractional integral of order $(d-1)/2$ of $f(x)$. In particular, if $f(x) \in L^1(\mathcal{M})$, then $\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|\right\}$ is in $L^p(\mathcal{M})$ for all $p < 2d/(d+1)$. □

Lemma 1.10. *If $f(x)$ is in $L^2(\mathcal{M})$ and $\mathrm{Re}(\alpha)-\beta+\gamma\geq 0$, then*

$$\int_{\mathcal{M}} \sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|^2\right\} dx \;\leq\; C\,(1+|\alpha|)^{2h} \int_{\mathcal{M}} |f(x)|^2\, dx\,.$$

Proof. By Lemma 1.4,

$$\left|B_R G^{\gamma-\beta} f(x)\right|$$

$$= \left|\sum_{n=0}^{+\infty}\sum_{j=1}^{d_n}(1+\lambda_n^2)^{(\beta-\gamma)/2}\;(m_R(\lambda_n)-m_R*\psi(\lambda_n))\;\widehat{f}(n,j)\,Y_{n,j}(x)\right|$$

$$\leq\; C\,(1+|\alpha|)^h\,R^{-\mathrm{Re}(\alpha)}\sum_{n=0}^{+\infty}\frac{(1+\lambda_n^2)^{(\beta-\gamma)/2}}{(1+|R-\lambda_n|)^k}\left|\sum_{j=1}^{d_n}\widehat{f}(n,j)\,Y_{n,j}(x)\right|$$

$$\leq\; C\,(1+|\alpha|)^h\,R^{-\mathrm{Re}(\alpha)}\left\{\sum_{n=0}^{+\infty}\frac{(1+\lambda_n^2)^{\beta-\gamma}}{(1+|R-\lambda_n|)^{2k}}\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{+\infty}\left|\sum_{j=1}^{d_n}\widehat{f}(n,j)\,Y_{n,j}(x)\right|^2\right\}^{\frac{1}{2}}$$

$$\leq\; C\,(1+|\alpha|)^h\,R^{-\mathrm{Re}(\alpha)+\beta-\gamma}\left\{\sum_{n=0}^{+\infty}\left|\sum_{j=1}^{d_n}\widehat{f}(n,j)\,Y_{n,j}(x)\right|^2\right\}^{\frac{1}{2}}.$$

Hence, if $\mathrm{Re}(\alpha)-\beta+\gamma\geq 0$,

$$\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|\right\} \;\leq\; C\,(1+|\alpha|)^h\left\{\sum_{n=0}^{+\infty}\left|\sum_{j=1}^{d_n}\widehat{f}(n,j)\,Y_{n,j}(x)\right|^2\right\}^{\frac{1}{2}},$$

and

$$\int_{\mathcal{M}}\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|^2\right\} dx \;\leq\; C\,(1+|\alpha|)^{2h}\sum_{n=0}^{+\infty}\sum_{j=1}^{d_n}\left|\widehat{f}(n,j)\right|^2.$$

□

Lemma 1.11. *If $f(x)$ is in $L^p(\mathcal{M})$, $1\leq p\leq 2$, and $\alpha-\beta+\gamma\geq(d-1)\left(\frac{1}{p}-\frac{1}{2}\right)$, then*

$$\int_{\mathcal{M}}\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|^p\right\} dx \;\leq\; C\,(1+|\alpha|)^{ph}\int_{\mathcal{M}}|f(x)|^p\, dx\,.$$

Proof. The lemma follows from the previous two lemmas via Stein's interpolation theorem for analytic families of operators. □

By Theorem 1.8, since the set where a potential $G^\beta F(x) = +\infty$ has Hausdorff dimension at most $d - \beta p$, the set of point where localization fails has a small dimension.

Corollary 1.12. *Under the assumptions on p, α and γ in Theorem 1.8, if $G^\gamma f(x) = 0$ in an open set Ω, then the following hold:*

(1) If $\alpha + \gamma = (d-1)\left(\frac{1}{p} - \frac{1}{2}\right)$, then for almost every point in Ω,

$$\lim_{R\to+\infty} \{S_R^\alpha G^\gamma f(x)\} \;=\; 0\,;$$

(2) If $\alpha + \gamma > (d-1)\left(\frac{1}{p} - \frac{1}{2}\right)$, then

$$\lim_{R\to+\infty} \{S_R^\alpha G^\gamma f(x)\} \;=\; 0$$

at all points in this open set Ω, with possible exceptions in a set with Hausdorff dimension at most $\delta = d - p\left(\alpha + \gamma - (d-1)\left(\frac{1}{p} - \frac{1}{2}\right)\right)$.

Proof. The Bessel (β, p) capacity of a Borel set E in $\mathcal{M}$ is defined as

$$\mathcal{B}_{\beta,p}(E) \;=\; \inf \left\{ \|f\|_{L^p}^p \;:\; G^\beta f(x) \;\geq\; 1 \text{ on } E \right\}\,.$$

See [52], Section 2.6, for the definition on the Euclidean spaces $\mathbb{R}^d$ and the Appendix below. The definition and properties of Bessel capacity in a manifold are similar. For $\varepsilon > 0$ and $t > 0$, let

$$E \;=\; \left\{ x \in \Omega \cap \{\mathit{distance}\,\{x, \partial\Omega\} > \varepsilon\} \;:\; \limsup_{R\to+\infty} \{|S_R^\alpha G^\gamma f(x)|\} \;>\; t \right\}\,.$$

Theorem 1.8 and an approximation of $f(x)$ with test functions show that the (β, p) capacity of E is zero for every $\varepsilon > 0$ and $t > 0$. On the other hand, if a set has (β, p) capacity zero, then it also has $d - \beta p + \eta$ Hausdorff measure zero for every $\eta > 0$. This implies that the $d - \beta p$ Hausdorff dimension of the divergent set is zero. □

Examples. In [35] it is shown that there exists radial functions in $L^{2d/(d+1)}(\mathcal{M})$, vanishing on half of $\mathcal{M}$, with spherical harmonic expansions diverging almost everywhere on $\mathcal{M}$. Actually a small modification of the argument gives divergence everywhere. See [20] for the two dimensional case of the expansion in Legendre polynomials. On the other hand, by an application of the Rademacher-Menshov theorem on orthogonal series, in [33] it is shown that the spherical partial sums of functions in L^2 Sobolev spaces of positive order converge almost everywhere.

As mentioned at the end of the proof of Lemma 1.9, we suspect that some of the indexes in Theorem 1.8 and Corollary 1.12 can be improved.

1.5 Localization for square integrable functions

Theorem 1.13. *Assume that one of the following conditions holds:*

(1) $\alpha \geq 0\,,\qquad 0 \leq \alpha+\gamma < d/2\,,\qquad \delta = d-2(\alpha+\gamma);$

(2) $\alpha \geq 0\,,\qquad (d-1)/4 \leq \alpha+\gamma \leq (d-1)/2\,,\qquad \delta = d-2(\alpha+\gamma)-1;$

(3) $\alpha \geq 0\,,\qquad \alpha+\gamma \geq (d-1)/2\,,\qquad \delta = 0.$

Assume that $f(x)$ is in $L^2(\mathcal{M})$ and that $G^\gamma f(x) = 0$ in an open set Ω, and let $dv(x)$ a non-negative Borel measure with support in $\Omega \cap \{distance\,\{x, \partial\Omega\} > \varepsilon\}$ for some $\varepsilon > 0$. Then there exists a positive constant C such that

$$\int_{\mathcal{M}} \sup_{R>0} \{|S_R^\alpha G^\gamma f(x)|\}\, dv(x) \;\leq\; C\,\|f\|_{L^2} \left\{ \int_{\mathcal{M}}\int_{\mathcal{M}} \frac{dv(x)\, dv(y)}{|x-y|^\delta} \right\}^{\frac{1}{2}} .$$

If $G^\gamma f(x)$ vanishes in an open set Ω and if $\alpha+\gamma \geq 0$, by the positivity of the Bessel kernel, for every x in $\Omega \cap \{distance\,\{x, \partial\Omega\} > \varepsilon\}$ one has

$$\begin{aligned}
&\sup_{R>0} \{|S_R^\alpha G^\gamma f(x)|\} \\
&= \sup_{R>0} \{|B_R G^\gamma f(x)|\} \;=\; \sup_{R>0} \left\{\left|G^{\alpha+\gamma} B_R G^{-\alpha} f(x)\right|\right\} \\
&\leq\; G^{\alpha+\gamma} \left(\sup_{R>0} \left\{\left|B_R G^{-\alpha} f\right|\right\} \right)(x)\,.
\end{aligned}$$

By Lemma 1.10, if $\alpha \geq 0$, then

$$\int_{\mathcal{M}} \sup_{R>0} \left\{\left|B_R G^{-\alpha} f(x)\right|^2\right\} dx \;\leq\; C \int_{\mathcal{M}} |f(x)|^2\, dx\,.$$

Then part (1) of Theorem 1.13 follows from the following lemma and the estimate $G^{2(\alpha+\gamma)}(x,y) \leq C\,|x-y|^{2(\alpha+\gamma)-d}$ in Lemma 1.2.

Lemma 1.14. *For every $F(x) \in L^2(\mathcal{M})$, for every non-negative finite Borel measure $dv(x)$, and for every $\eta > 0$,*

$$\int_{\mathcal{M}} |G^\eta F(x)|\; dv(x) \;\leq\; \|F\|_{L^2} \left\{ \int_{\mathcal{M}}\int_{\mathcal{M}} G^{2\eta}(x,y)\, dv(x)\, dv(y) \right\}^{\frac{1}{2}} .$$

Proof. Since the Bessel kernel is positive, it follows that

$$|G^{\eta}F(x)| \;\leq\; G^{\eta}|F|(x)\,.$$

Then it suffices to assume $F(x) \geq 0$. We get

$$\int_{\mathcal{M}} G^{\eta}F(x)\,d\upsilon(x) \;=\; \int_{\mathcal{M}} \left(\sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left(1+\lambda_n^2\right)^{-\eta/2} \widehat{F}(n,j)\,Y_{n,j}(x) \right) d\upsilon(x)$$

$$\leq \left\{ \sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left|\widehat{F}(n,j)\right|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left(1+\lambda_n^2\right)^{-\eta} \left| \int_{\mathcal{M}} Y_{n,j}(x)\,d\upsilon(x) \right|^2 \right\}^{\frac{1}{2}}$$

$$= \|F\|_{L^2} \left\{ \int_{\mathcal{M}} \int_{\mathcal{M}} \left(\sum_{n=0}^{+\infty} \left(1+\lambda_n^2\right)^{-\eta} \sum_{j=1}^{d_n} Y_{n,j}(x)\,\overline{Y_{n,j}(y)} \right) d\upsilon(x)\,d\upsilon(y) \right\}^{\frac{1}{2}}$$

$$= \|F\|_{L^2} \left\{ \int_{\mathcal{M}} \int_{\mathcal{M}} G^{2\eta}(x,y)\,d\upsilon(x)\,d\upsilon(y) \right\}^{\frac{1}{2}}.$$

□

To prove part (2) and (3) of Theorem 1.13 it suffices to replace the maximal operator $\sup_{R>0}\{|S_R^{\alpha}G^{\gamma}f(x)|\}$ with a linearised version $g(x)B_{R(x)}G^{\gamma}f(x)$, where $g(x)$ and $R(x)$ are arbitrary Borel functions with $|g(x)| \leq 1$ and $R(x) \geq 1$. Moreover, possibly splitting the measure $d\nu(x)$ into a finite sum of measures with small support, one can assume that the diameter of the support of the measure is smaller than half of the diameter of the manifold $\mathcal{M}$. In particular, if a point x is in the support of the measure, then the antipodal points are far from this support. Set

$$g(x) \;=\; \frac{\overline{B_{R(x)}G^{\gamma}f(x)}}{\left|B_{R(x)}G^{\gamma}f(x)\right|}\,.$$

Then, with the notation $\widehat{B_R}(\lambda) = m_R(\lambda) - m_R * \psi(\lambda)$,

$$\int_{\mathcal{M}} \left|B_{R(x)}G^{\gamma}f(x)\right|\,d\upsilon(x)$$

$$= \int_{\mathcal{M}} g(x) \left(\sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left(1+\lambda_n^2\right)^{-\gamma/2} \widehat{B_{R(x)}}(\lambda_n)\,\widehat{f}(n,j)\,Y_{n,j}(x) \right) d\upsilon(x)$$

$$\leq \|f\|_{L^2} \left\{ \sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left(1+\lambda_n^2\right)^{-\gamma} \left| \int_{\mathcal{M}} g(x)\,\widehat{B_{R(x)}}(\lambda_n)\,Y_{n,j}(x)\,d\upsilon(x) \right|^2 \right\}^{\frac{1}{2}}.$$

Lemma 1.15. *Assume that $d\upsilon(x)$ is a nonnegative measure with support smaller than half of the diameter of the manifold $\mathcal{M}$, and that*

$$\delta = \begin{cases} d-2(\alpha+\gamma)-1 & \text{if} \quad (d-1)/4 \le \alpha+\gamma \le (d-1)/2\,, \\ 0 & \text{if} \quad \alpha+\gamma \ge (d-1)/2\,. \end{cases}$$

Also assume that $|g(x)| \le 1$. Then

$$\sum_{n=0}^{+\infty}\sum_{j=1}^{d_n}\left(1+\lambda_n^2\right)^{-\gamma}\left|\int_{\mathcal{M}} g(x)\,\widehat{B_{R(x)}}(\lambda_n)\,Y_{n,j}(x)\,d\upsilon(x)\right|^2 \le C\int_{\mathcal{M}}\int_{\mathcal{M}}\frac{d\upsilon(x)\,d\upsilon(y)}{|x-y|^{\delta}}.$$

Proof. By the addition formula $Z_n(x,y)=\sum_{j=1}^{d_n} Y_{n,j}(x)\overline{Y_{n,j}(y)}$, we can write

$$\sum_{n=0}^{+\infty}\sum_{j=1}^{d_n}\left(1+\lambda_n^2\right)^{-\gamma}\left|\int_{\mathcal{M}} g(x)\,\widehat{B_{R(x)}}(\lambda_n)\,Y_{n,j}(x)\,d\upsilon(x)\right|^2 =$$

$$\int_{\mathcal{M}}\int_{\mathcal{M}}\left(\sum_{n=0}^{+\infty}\left(1+\lambda_n^2\right)^{-\gamma}\,\widehat{B_{R(x)}}(\lambda_n)\,\widehat{B_{R(y)}}(\lambda_n)\,Z_n(x,y)\right)g(x)\,\overline{g(y)}\,d\upsilon(x)\,d\upsilon(y)\,.$$

Define

$$I(x,y) = \left|\sum_{n=0}^{+\infty}\left(1+\lambda_n^2\right)^{-\gamma}\,\widehat{B_{R(x)}}(\lambda_n)\,\widehat{B_{R(y)}}(\lambda_n)\,Z_n(x,y)\right|.$$

Using the estimates for $Z_n(x,y)$ in part (2) of Lemma 1.1 and the estimate on $\widehat{B_R}(\lambda)$ in Lemma 1.4, if $t=|x-y|$ with $0\le t\le \pi/2$,

$$\begin{aligned} I(x,y) &\le C\sum_{n=0}^{+\infty}\left(1+\lambda_n\right)^{d-2\gamma-1}\left(1+\lambda_n t\right)^{-(d-1)/2}\left|\widehat{B_{R(x)}}(\lambda_n)\right|\left|\widehat{B_{R(y)}}(\lambda_n)\right| \\ &\le C\,R(x)^{-\alpha}\,R(y)^{-\alpha}\sum_{n=0}^{+\infty}\frac{\left(1+\lambda_n\right)^{d-2\gamma-1}\left(1+\lambda_n t\right)^{-(d-1)/2}}{\left(1+|R(x)-\lambda_n|\right)^k\left(1+|R(y)-\lambda_n|\right)^k}\,. \end{aligned}$$

Observe that in the last sum only a finite number of terms come into play, the ones with $|R(x)-\lambda_n|\lesssim C$ and $|R(y)-\lambda_n|\lesssim C$. This gives

$$\begin{aligned} I(x,y) \le{}& C\,R(y)^{-\alpha}\,R(x)^{d-\alpha-2\gamma-1}\left(1+R(x)t\right)^{-(d-1)/2}\left(1+|R(x)-R(y)|\right)^{-k} \\ &+ C\,R(x)^{-\alpha}\,R(y)^{d-\alpha-2\gamma-1}\left(1+R(y)t\right)^{-(d-1)/2}\left(1+|R(x)-R(y)|\right)^{-k}\,. \end{aligned}$$

If k is large and if $R(x)$ and $R(y)$ are close to each other, $R(x)\le R(y)\le 3R(x)$, then $\left(1+|R(x)-R(y)|\right)^{-k}$ can be bounded by one and this gives

$$\begin{aligned} I(x,y) &\le C\,R(y)^{-\alpha}\,R(x)^{d-\alpha-2\gamma-1}\left(1+R(x)t\right)^{-(d-1)/2} \\ &\quad + C\,R(x)^{-\alpha}\,R(y)^{d-\alpha-2\gamma-1}\left(1+R(y)t\right)^{-(d-1)/2} \\ &\le C\,R(x)^{d-2\alpha-2\gamma-1}\left(1+R(x)t\right)^{-(d-1)/2}\,. \end{aligned}$$

If $R(x)$ and $R(y)$ are far from each other, $3R(x) < R(y)$, then $(1+|R(x)-R(y)|)^{-k}$ can be bounded by $R(y)^{-k}$ and this gives

$$\begin{aligned} I(x,y) \ &\leq C\, R(x)^{d-\alpha-2\gamma-1}\, R(y)^{-\alpha-k}\, (1+R(x)t)^{-(d-1)/2} \\ &\quad + C\, R(x)^{-\alpha}\, R(y)^{d-\alpha-2\gamma-1-k}\, (1+R(y)t)^{-(d-1)/2} \\ &\leq C\, R(x)^{d-2\alpha-2\gamma-1}\, (1+R(x)t)^{-(d-1)/2} \\ &\quad + C\, R(y)^{d-2\alpha-2\gamma-1}\, (1+R(y)t)^{-(d-1)/2}\,. \end{aligned}$$

In both cases, when $(d-1)/4 \leq \alpha+\gamma \leq (d-1)/2$,

$$\begin{aligned} I(x,y) \ &\leq\ C \sup_{R\geq 1} \left\{ R^{d-2\alpha-2\gamma-1}\, (1+Rt)^{-(d-1)/2} \right\} \\ &\leq\ C\, t^{-d+2\alpha+2\gamma+1} \sup_{R\geq 1} \left\{ (Rt)^{d-2\alpha-2\gamma-1}\, (1+Rt)^{-(d-1)/2} \right\} \\ &\leq\ C\, t^{-d+2\alpha+2\gamma+1}\,. \end{aligned}$$

A similar computation shows that, if $\alpha+\gamma \geq (d-1)/2$, then $I(x,y) \leq C$. □

This concludes the proof of Theorem 1.13. By this theorem, the Bochner-Riesz means cannot diverge on the supports of measures with finite energy. Hence, by the relation between energy, capacity, and dimension, these means cannot diverge on sets with large dimension. This implies the following.

Corollary 1.16. *Under the above assumptions on p, α, γ and δ, if $G^\gamma f(x) = 0$ in an open set Ω, then*

$$\lim_{R\to+\infty} \{S_R^\alpha G^\gamma f(x)\} \ =\ 0$$

at all points in the open set Ω, with possible exceptions in a set with Hausdorff dimension at most δ.

Proof. It suffices to show that the maximal function $\sup_{R>0}\{|S_R^\alpha G^\gamma f(x)|\}$ cannot be infinite on subset of Ω with Hausdorff dimension greater then δ. Consider $\tau, \sigma, \eta \in \mathbb{R}$ and recall that the τ-energy of a finite Borel measure υ on a metric space $\mathcal{M}$ is defined by

$$\int_{\mathcal{M}}\int_{\mathcal{M}} \frac{d\upsilon(x)\, d\upsilon(y)}{|x-y|^\tau}\,.$$

If $\sigma < \eta$, it follows directly from the properties of the Hausdorff measure that every set of dimension η has infinite σ-dimensional measure. Besides, by Frostman's Lemma (see [32], Theorem 8.17), there is a finite and nontrivial Borel measure supported on one of these sets with $\upsilon\{|x-p| < r\} \leq r^\sigma$ for each $p \in \mathcal{M}$ and $r > 0$. In particular, this measure has finite τ-energy for every $\tau < \sigma$. To obtain the corollary it then sufficient to apply Theorem 1.13 with $\delta < \tau < \sigma < \eta$. □

The above Theorem 1.13 and Corollary 1.16 extend to compact rank one symmetric spaces the results in [21] for the Euclidean spaces $\mathbb{R}^d$. It is likely that some of the above results can be further extended to eigenfunction expansions of elliptic differential operators on Riemannian manifolds.

Finally, we want to point out that while the first two theorems on pointwise localization are sharp, we do not expect that in the other theorems the indexes on the dimension of sets where localization may fail are best possible.

Chapter 2

Localization for Riesz Means on compact manifolds

2.1 The Laplace-Beltrami operator on compact manifolds

Let $\mathcal{M}$ be a smooth connected and compact Riemannian manifold of dimension $d \geq 2$. Assume that the Riemannian metric $g = (g_{ij})$ is of class $\mathcal{C}^\infty$ and indicate the canonical measure with $d\mu(x) = \sqrt{|\det g|}\, dx$. With respect to this measure, the Lebesgue space $L^2(\mathcal{M})$ contains all those measurable functions on $\mathcal{M}$ for which

$$\int_{\mathcal{M}} |f(x)|^2 \, d\mu(x) \; < \; +\infty \, .$$

On $L^2(\mathcal{M})$ one can define the usual inner product and the induced norm, given by

$$\langle f, g \rangle_{L^2} \; = \; \int_{\mathcal{M}} f(x) g(x) \, d\mu(x) \, ,$$

and

$$\|f\|_{L^2} \; = \; \sqrt{\langle f, f \rangle_{L^2}} \, .$$

With this inner product, $L^2(\mathcal{M})$ is an Hilbert space. We shall denote by $|x - y|$ the Riemannian distance between the two points x and y in $\mathcal{M}$.

The Laplace-Beltrami operator Δ_g associated with the metric g of $\mathcal{M}$ is given, in local coordinates $(x_1, x_2, \ldots, x_d)$, by

$$\Delta_g \; = \; -\frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{|\det g|} \, \frac{\partial}{\partial x_j} \right) \, ,$$

where the matrix (g^{ij}) is the inverse matrix of g. Δ_g is a symmetric, self-adjoint and positive definite elliptic operator. Furthermore, it follows from the compactness of $\mathcal{M}$ and the theory of elliptic partial differential equations that Δ_g has pure point spectrum. This means that $L^2(\mathcal{M})$ admits an orthonormal basis consisting of eigenfunctions $\varphi_\lambda(x)$ of Δ_g with associated eigenvalues λ^2:

$$\Delta_g \varphi_\lambda(x) \;=\; \lambda^2\, \varphi_\lambda(x)\,.$$

The set of eigenvalues is an infinite sequence $\{\lambda^2\}$ whose only accumulation point is at infinity and each eigenvalue occurs with finite multiplicity. It turns out that, in general, the spectrum and the eigenfunctions cannot be computed explicitly. The very few exceptions are manifolds like round spheres and flat tori (see [16] for some classical examples where the spectrum is known). However, it is possible to get estimate of the spectrum, and these estimation are related to the geometry of the manifold $(\mathcal{M}, g)$ we consider.

Asymptotically, we know how the spectrum of Δ_g behave. Let $N(R)$ be the number of eigenvalues, counted with multiplicity, less than R: $N(R) = \#\{\lambda^2 \leq R\}$. Then *Weyl's asymptotic formula* says that (see [28]):

$$N(R) \;\sim\; \frac{Vol(\mathcal{M})\, w_d}{(2\pi)^d}\, R^{d/2}\,,$$

as $R \to +\infty$, where w_d denotes the volume of the unit ball in $\mathbb{R}^d$. This was first proved by Weyl [51] for a bounded domain $\Omega \subset \mathbb{R}^3$. Written in a slightly different form it is known in physics as the Rayleigh-Jeans law. Raleigh [43] derived it for a cube. Garding [22] proved Weyl's law for a general elliptic operator on a domain in $\mathbb{R}^d$. For a closed Riemannian manifold this law was proved by Minakshisundaram and Pleijel [39].

Hörmander in [28] gives a sharp estimate for the remainder in Weyl's law. Suppose that $\varphi_\lambda(x)$ is an eigenfunction of Δ_g with associated eigenvalue $\lambda^2 \neq 0$. If one scales the metric by $g_{ij} \to \lambda^2 g_{ij}$, an elliptic equation with bounded coefficients is obtained. Also, a geodesic ball of radius $C\lambda^{-1}$ scales to a ball of radius C. Elementary local elliptic theory shows that the L^∞-norm of $\varphi_\lambda(x)$ is bounded by its L^2-norm relative to a scaled metric. Rescaling back to the original problem yields the estimate

$$\|\varphi_\lambda\|_{L^\infty} \;\leq\; C\,\lambda^{d/2}\,\|\varphi_\lambda\|_{L^2}\,.$$

Remarkably, Hörmander [28] proved that for eigenfunctions on compact manifolds one has

$$\|\varphi_\lambda\|_{L^\infty} \;\leq\; C\,\lambda^{(d-1)/2}\,\|\varphi_\lambda\|_{L^2}\,. \tag{2.1}$$

In Hörmander's proof, one constructs a parametrix for the fundamental solution of the wave equation. In particular, $(d-1)/2$ is the optimal power of λ. In fact, rotationally symmetric spherical harmonics on $\mathbb{S}^d$ illustrate the sharpness of (2.1).

2.2 Eigenfunction expansions and the Bochner-Riesz means

As in the previous chapter, we define the Fourier expansion of a square integrable function, and more generally of a tempered distribution, as

$$f(x) \;=\; \sum_{\lambda} \widehat{f}(\lambda)\, \varphi_\lambda(x)\,,$$

with

$$\widehat{f}(\lambda) \;=\; \int_{\mathcal{M}} f(x)\, \overline{\varphi_\lambda(x)}\, d\mu(x)\,.$$

These Fourier series converge in the metric of $L^2(\mathcal{M})$ and in the topology of distributions. However, if $f(x)$ is merely in $L^p(\mathcal{M})$, $1 \le p < 2$, we can only expect that $\sum_\lambda \widehat{f}(\lambda)\varphi_\lambda(x) \to f(x)$ pointwise after a suitable summation method has been applied. In this chapter, as in Chapter 1 and in most of the literature on eigenfunction expansions, only Riesz means will be considered.

Definition 2.1. The *Bochner-Riesz means* of order α of functions in $\mathcal{M}$ are defined by

$$S_R^\alpha f(x) \;=\; \sum_\lambda \left(1-\frac{\lambda^2}{R^2}\right)_+^\alpha \widehat{f}(\lambda)\, \varphi_\lambda(x) \;=\; \int_{\mathcal{M}} S_R^\alpha(x,y)\, f(y)\, d\mu(y)\,,$$

where

$$S_R^\alpha(x,y) \;=\; \sum_\lambda \left(1-\frac{\lambda^2}{R^2}\right)_+^\alpha \varphi_\lambda(x)\, \overline{\varphi_\lambda(y)}$$

is the *Bochner-Riesz kernel.*

We also recall the definition of the Bessel potentials and sum up the related properties.

Definition 2.2. Let $-\infty < \gamma < +\infty$ and $f(x) = \sum_\lambda \widehat{f}(\lambda)\varphi_\lambda(x)$ be a tempered distribution on $\mathcal{M}$. The *Bessel potential* $G^\gamma f(x)$ of $f(x)$ is that tempered distribution defined by

$$G^\gamma f(x) \;=\; \sum_\lambda \left(1+\lambda^2\right)^{-\gamma/2}\, \widehat{f}(\lambda)\, \varphi_\lambda(x)\,.$$

As before, with an abuse of notation we write $G^\gamma f(x)$ as an integral:

$$G^\gamma f(x) \;=\; \int_{\mathcal{M}} G^\gamma(x,y)\, f(y)\, dy\,,$$

with

$$G^\gamma(x,y) \;=\; \sum_\lambda \left(1+\lambda^2\right)^{-\gamma/2}\, \varphi_\lambda(x)\, \overline{\varphi_\lambda(y)}\,.$$

Indeed, if $\gamma > 0$ then $G^\gamma(x,y)$ is an integrable kernel, but when $\gamma = -2n$, with n integer, then G^γ is the differential operator $(1+\Delta)^n$.

Lemma 2.1. *If $\gamma > 0$, then the Bessel kernel $G^\gamma(x,y)$ is positive and integrable, and it is smooth in $\{|x-y| \neq 0\}$. Moreover, if $0 < \gamma < d$, then $G^\gamma(x,y) \approx |x-y|^{\gamma-d}$ when $|x-y| \to 0$.*

Proof. The same proof of Lemma 1.2 in Chapter 1 applies. □

The Rademacher-Menshov theorem plays an important role in the theory of orthogonal series. Suppose that we have a series in an arbitrary orthonormal system $\{\varphi_n(x)\}_{n=1}^{+\infty}$ on a finite measure space. The Rademacher-Menshov theorem states that the sequence $\{\log^2 n\}$ is a Weyl multiplier for the almost everywhere convergence with respect to the Lebesgue measure of this series, that is, if $\sum_{n=1}^{+\infty} |c(n)|^2 \log^2(n) < +\infty$, then $\sum_{n=1}^{+\infty} c(n)\varphi_n(x)$ converges for almost every x. As consequence of the Rademacher-Menshov theorem and of Weyl's estimates for the spectral function, Meaney in [33] proved the theorem below.

Theorem 2.2 (Meaney)**.** *Let $f(x)$ be a square integrable function of $\mathcal{M}$ and assume that*

$$\sum_\lambda |\widehat{f}(\lambda)|^2\, \log^2(1+\lambda) \;<\; +\infty\,.$$

Then, for almost every x in $\mathcal{M}$,

$$\lim_{R\to+\infty} \left\{ \sum_{\lambda<R} \widehat{f}(\lambda)\, \varphi_\lambda(x) \right\} \;=\; f(x)\,.$$

Another result concerning the almost everywhere convergence of the Bochner-Riesz means, is the following theorem.

Theorem 2.3. *Let $\alpha > 0$ and assume that*

$$\sum_\lambda |\widehat{f}(\lambda)|^2 \;<\; +\infty\,.$$

Then, for almost every x in $\mathcal{M}$,

$$\lim_{R\to+\infty}\left\{\sum_{\lambda<R}\left(1-\frac{\lambda^2}{R^2}\right)_+^\alpha \widehat{f}(\lambda)\,\varphi_\lambda(x)\right\} = f(x)\,.$$

This is a classical result when $\mathcal{M}$ is the Euclidean torus $\mathbb{T}^d$ (see [46]). For the general case see Hörmander [27], Theorem 6.4. In what follows we are looking for localization results that are somehow better than the above theorems.

2.3 Decomposition of the Bochner-Riesz kernel

As it was seen in Section 1.2, the main tool in our localization results is a decomposition of $S_R^\alpha(x,y)$ into a kernel with small support plus a remainder with small Fourier transform.

Let $\varepsilon>0$ and let $\psi(\lambda)$ be an even test function with cosine Fourier transform

$$\begin{cases} \widehat{\psi}(\tau) = 1 & \text{if} \quad |\tau|\le\varepsilon/2\,,\\ \widehat{\psi}(\tau) = 0 & \text{if} \quad |\tau|\ge\varepsilon\,.\end{cases}$$

Let $m_R=(1-\lambda^2/R^2)_+^\alpha$, then

$$\begin{aligned} S_R^\alpha f(x) &= \sum_\lambda m_R(\lambda)\,\widehat{f}(\lambda)\,\varphi_\lambda(x)\\ &= \sum_\lambda m_R*\psi(\lambda)\,\widehat{f}(\lambda)\,\varphi_\lambda(x) + \sum_\lambda \left(m_R(\lambda)-m_R*\psi(\lambda)\right)\,\widehat{f}(\lambda)\,\varphi_\lambda(x)\\ &= A_Rf(x) + B_Rf(x)\,.\end{aligned}$$

The associated kernels are given by

$$A_R(x,y) = \sum_\lambda m_R*\psi(\lambda)\,\varphi_\lambda(x)\,\overline{\varphi_\lambda(y)}\,,$$

$$B_R(x,y) = \sum_\lambda \left(m_R(\lambda)-m_R*\psi(\lambda)\right)\,\varphi_\lambda(x)\,\overline{\varphi_\lambda(y)}\,.$$

We can rewrite Lemma 1.3 and Lemma 1.4 in order to obtain the following properties: $A_R(x,y)$ has small support and the Fourier transform of the kernel $B_R(x,y)$ is small. For reasons that will be clearer in what follows, it is convenient to write some estimates also for the derivative of the quantity $m_R(\lambda)-m_R*\psi(\lambda)$.

Lemma 2.4. *The kernel $A_R(x,y)$ has support in $\{|x-y| \le \varepsilon\}$. In particular, if a tempered distribution $f(x)$ vanishes in an open set Ω, then for all $x \in \Omega$ with distance $(x, \partial\Omega) > \varepsilon$,*

$$A_R f(x) \;=\; 0\,.$$

Proof. One proceeds as in the proof of Lemma 1.3. □

Lemma 2.5. *Let $\alpha > -1$ real. For every $k > 0$ there exists $C > 0$ such that for every $R > 1$,*

(1) If $\alpha \ge 0$, then

$$|m_R(\lambda) - m_R * \psi(\lambda)| \;\le\; C\,R^{-\alpha}\;(1 + |R-\lambda|)^{-k}\;;$$

(2) If $-1 < \alpha < 0$, then

$$|m_R(\lambda) - m_R * \psi(\lambda)| \;\le\; \begin{cases} C\,R^{-\alpha}\;|R-\lambda|^{-k} & \text{if} \quad |R-\lambda| \ge 1 \\ C\,R^{-\alpha}\,|R-\lambda|^{\alpha} & \text{if} \quad |R-\lambda| \le 1 \end{cases}\;;$$

(3) If $\alpha \ge 1$, then

$$\left|\frac{d}{d\lambda}\left\{m_R(\lambda) - m_R * \psi(\lambda)\right\}\right| \;\le\; C\,R^{-\alpha}\;(1 + |R-\lambda|)^{-k}\;;$$

(4) If $0 < \alpha < 1$, then

$$\left|\frac{d}{d\lambda}\left\{m_R(\lambda) - m_R * \psi(\lambda)\right\}\right| \;\le\; \begin{cases} C\,R^{-\alpha}\;|R-\lambda|^{-k} & \text{if} \quad |R-\lambda| \ge 1 \\ C\,R^{-\alpha}\,|R-\lambda|^{\alpha} & \text{if} \quad |R-\lambda| \le 1 \end{cases}\,.$$

Moreover,

(5) For every $\alpha \ge 0$,

$$\int_{R-1}^{R+1} |d\left\{m_R(\lambda) - m_R * \psi(\lambda)\right\}| \;\le\; C\,R^{-\alpha}\,.$$

Proof. (1) One proceeds as in the proof of Lemma 1.4, with α real.

(2) When $|R-\lambda| \ge 1$ we proceed as in the proof of (1). For $|R-\lambda| \le 1$ we use the estimate:

$$\begin{aligned} |m_R(\lambda)| \;&=\; R^{-2\alpha}\;(\,R+\lambda\,)^{\alpha}\;(\,R-\lambda\,)_+^{\alpha} \\ &\le\; R^{-\alpha}\;(\,R-\lambda\,)_+^{\alpha}\;. \end{aligned}$$

Furthermore, if $|\lambda - R| \leq 1$,

$$|m_R * \psi(\lambda)| \;\leq\; \int_{\mathbb{R}} R^{-\alpha}\, [R - (\lambda - \tau)]_+^{\alpha}\; |\psi(\tau)|\; d\tau$$
$$\leq\; C\, R^{-\alpha}\,.$$

(3) First observe that

$$\frac{d}{d\lambda}\, \{m_R(\lambda) - m_R * \psi(\lambda)\} \;=\; \frac{d}{d\lambda} \left\{ \left(1 - \frac{\lambda^2}{R^2}\right)_+^{\alpha} \;-\; \psi * \left(1 - \frac{\lambda^2}{R^2}\right)_+^{\alpha} \right\}$$
$$=\; -2\alpha\lambda R^{-2} \left\{ \left(1 - \frac{\lambda^2}{R^2}\right)_+^{\alpha-1} \;-\; \psi * \left(1 - \frac{\lambda^2}{R^2}\right)_+^{\alpha-1} \right\}$$
$$-\, 2\alpha R^{-2}\; (\lambda\psi(\lambda)) * \left(1 - \frac{\lambda^2}{R^2}\right)_+^{\alpha-1}\,.$$

If $\alpha \geq 1$, then $(1 - \lambda^2/R^2)_+^{\alpha-1}$ is integrable. By part (1), for any $k > 0$, with the abuse of notation $k - 1 = k$, since k can be arbitrary large,

$$2\alpha\lambda R^{-2} \left| \left(1 - \frac{\lambda^2}{R^2}\right)_+^{\alpha-1} \;-\; \psi * \left(1 - \frac{\lambda^2}{R^2}\right)_+^{\alpha-1} \right|$$
$$\leq\; C\lambda R^{-2} \left(R^{-(\alpha-1)}\; (1 + |R - \lambda|)^{-k} \right)$$
$$\leq\; C\, R^{-\alpha}\; (1 + |R - \lambda|)^{-k}\,.$$

Since $\psi(\lambda)$ is even, $t\psi(t)$ has mean value zero. Hence

$$(\lambda\psi(\lambda)) * \left(1 - \frac{\lambda^2}{R^2}\right)_+^{\alpha-1} = \int_{\mathbb{R}} \tau\psi(\tau) \left\{ \left(1 - \frac{(\lambda-\tau)^2}{R^2}\right)_+^{\alpha-1} - \left(1 - \frac{\lambda^2}{R^2}\right)_+^{\alpha-1} \right\} d\tau.$$

Then, arguing as in part (1), one obtains the desired estimate.

(4) We can argue as in (3) and use part (2) to get the estimates for the two terms of the derivative of $m_R(\lambda) - m_R * \psi(\lambda)$.

(5) When $\alpha > 0$ the last part of the lemma immediately follows from (3) and (4). While if $\alpha = 0$ it suffices to note that the function $m_R(\lambda)$ has bounded total variation and

$$\int_{R-1}^{R+1} |d\, \{m_R(\lambda) - m_R * \psi(\lambda)\}| \;\leq\; \int_{R-1}^{R+1} |dm_R(\lambda)| \;+\; \int_{R-1}^{R+1} |d\,(m_R * \psi(\lambda))|$$
$$\leq\; 1 \;+\; \int_{\mathbb{R}} |\psi(\lambda)|\, d\lambda\,.$$

□

2.4 Pointwise and almost everywhere results

The following Theorem 2.7 generalizes to Bochner-Riesz means on a compact manifold $\mathcal{M}$ an analogue result of Meaney for spherical partial sums. Meaney in [38] showed the following.

Theorem 2.6 (Meaney). *Suppose that Ω is an open set in $\mathcal{M}$ and $f(x) \in L^2(\mathcal{M})$ has support disjoint from Ω. If $y \in \Omega$ and*

$$\sup_{0\le h\le 1} \left| \sum_{t\le\lambda\le t+h} \widehat{f}(\lambda)\varphi_\lambda(y) \right| \longrightarrow 0$$

as $t \to +\infty$, then the spherical partial sums of the eigenfunction expansion of $f(x)$ converges to zero at y:

$$\lim_{R\to+\infty} \left\{ \sum_{\lambda<R} \widehat{f}(\lambda)\varphi_\lambda(y) \right\} = 0\,.$$

The analogous theorem for compact symmetric spaces of rank one has been proved once again by Meaney in [36]. Observe that in this case, the eigenvalues are concentrated around an integer, hence the supremum in not necessary. The following theorem is a generalization to Bochner-Riesz means of order $\alpha \ge 0$ of Theorem 2.6, furthermore it gives an alternative proof when $\alpha = 0$.

Theorem 2.7. *Let $\alpha \ge 0$ and assume that $f(x)$ is a tempered distribution on $\mathcal{M}$, with eigenfunction expansion $\sum_\lambda \widehat{f}(\lambda)\varphi_\lambda(x)$. Also assume that $f(x) = 0$ for all x in a ball $\{|x-y| < \varepsilon\}$, with radius $\varepsilon > 0$ and centre y. If the following condition holds,*

$$\lim_{R\to+\infty} \left\{ R^{-\alpha} \sup_{0\le h\le 1} \left| \sum_{R\le\lambda\le R+h} \widehat{f}(\lambda)\varphi_\lambda(y) \right| \right\} = 0\,,$$

then

$$\lim_{R\to+\infty} \{S_R^\alpha f(y)\} = 0\,.$$

Proof. Since $f(x) = 0$ in $\{|x-y| < \varepsilon\}$, by Lemma 2.4, $A_R f(y) = 0$.

Let $\{\Lambda_j\}_{j=0}^{+\infty}$ be an increasing sequence tending to infinity and such that $\Lambda_0 = 0$ and $j - 1/3 < \Lambda_j < j + 1/3$ for $j = 1, 2, \ldots$. Furthermore assume that no Λ_j is an eigenvalue of the Laplace-Beltrami operator Δ. Then one can write

$$\begin{aligned} &B_R f(y) \\ &= (m_R(0) - m_R * \psi(0))\ \widehat{f}(0) + \sum_{j=0}^{+\infty} \left(\sum_{\Lambda_j<\lambda\le\Lambda_{j+1}} (m_R(\lambda) - m_R * \psi(\lambda))\ \widehat{f}(\lambda)\,\varphi_\lambda(y) \right). \end{aligned}$$

The first term tends to zero, because of $m_R(0) - m_R * \psi(0) \to 0$ as $R \to +\infty$. Define

$$F(t) \;=\; m_R(t) - m_R * \psi(t)\,,$$
$$G_j(t) \;=\; \sum_{\Lambda_j < \lambda \le t} \widehat{f}(\lambda)\, \varphi_\lambda(y)\,.$$

Observe that $F(t)$ is also a function of R and $G_j(t)$ is also a function of y, but y is fixed. Writing the above sums as a Riemann-Stieltjes integrals, one obtains:

$$\sum_{\Lambda_j < \lambda \le \Lambda_{j+1}} (m_R(\lambda) - m_R * \psi(\lambda))\; \widehat{f}(\lambda)\, \varphi_\lambda(y) \;=\; \int_{\Lambda_j}^{\Lambda_{j+1}} F(t)\, dG_j(t)$$
$$= F(\Lambda_{j+1})\, G_j(\Lambda_{j+1}) \;-\; \int_{\Lambda_j}^{\Lambda_{j+1}} G_j(t)\, dF(t)\,.$$

Recall that for every $k > 0$, by Lemma 2.5,

$$|F(t)| \;\le\; C\, R^{-\alpha}\; (1 + |R - t|)^{-k}$$

and, by hypothesis,

$$j^{-\alpha}\; |G_j(t)| \;\longrightarrow\; 0 \quad \text{if} \quad j \to +\infty\,.$$

Putting these facts together, for every $\delta > 0$ there exists $T > 0$ such that for every $j = 1, 2, \ldots$ and for every $R > T$,

$$|F(\Lambda_{j+1})\, G_j(\Lambda_{j+1})| \;\le\; \begin{cases} C\, j^\alpha\, R^{-\alpha-k} & \text{if } j \le R/2\,, \\ \delta\, j^\alpha\, R^{-\alpha}\; (1 + |R - j|)^{-k} & \text{if } j \ge R/2\,. \end{cases}$$

Then, for the boundary terms,

$$\sum_{j=0}^{+\infty} |F(\Lambda_{j+1})\, G_j(\Lambda_{j+1})|$$
$$\le\; C\, R^{-\alpha-k} \sum_{j \le R/2} j^\alpha \;+\; 2^\alpha\, \delta \sum_{R/2 < j \le 2R} (1 + |R - j|)^{-k} \;+\; 2^k\, \delta\, R^{-\alpha} \sum_{j > 2R} j^{\alpha-k}$$
$$\le\; C\, \left(R^{1-k} + \delta + \delta\, R^{1-k}\right)\,.$$

In particular, if R is sufficiently large, $R^{1-k} < \delta$, the above quantity is dominated by $C\delta$. Now we have just to estimate

$$\sum_{j=0}^{+\infty} \int_{\Lambda_j}^{\Lambda_{j+1}} |G_j(t)\, dF(t)|\,.$$

First consider $\alpha \geq 1$. In this case $dF(t)$ is absolutely continuous with respect to the Lebesgue measure, then

$$\begin{aligned}\int_{\Lambda_j}^{\Lambda_{j+1}} |G_j(t)|\,|dF(t)| &= \int_{\Lambda_j}^{\Lambda_{j+1}} |G_j(t)| \left|\frac{d}{dt}F(t)\right| dt \\ &\leq \sup_{\Lambda_j \leq t \leq \Lambda_{j+1}} \{|G_j(t)|\} \int_{\Lambda_j}^{\Lambda_{j+1}} \left|\frac{d}{dt}F(t)\right| dt\,.\end{aligned}$$

By part (3) of Lemma 2.5, for every $j = 1, 2, \ldots$, and with the abuse of notation $k - 1 = k$, since k can be arbitrary large,

$$\begin{aligned}&\sup_{\Lambda_j \leq t \leq \Lambda_{j+1}} \{|G_j(t)|\} \int_{\Lambda_j}^{\Lambda_{j+1}} \left|\frac{d}{dt}F(t)\right| dt \\ &\leq C \sup_{\Lambda_j \leq t \leq \Lambda_{j+1}} \left\{j^{-\alpha}\,|G_j(t)|\right\} (j/R)^{\alpha}\,(1 + |R - j|)^{-k}\,.\end{aligned}$$

Splitting the sum over j into three parts, one gets

$$\begin{aligned}&\sum_{j=0}^{+\infty} \int_{\Lambda_j}^{\Lambda_{j+1}} |G_j(t)\,dF(t)| \\ &\leq C \sum_{j=0}^{+\infty} \sup_{\Lambda_j \leq t \leq \Lambda_{j+1}} \left\{j^{-\alpha}\,|G_j(t)|\right\} (j/R)^{\alpha}\,(1 + |R - j|)^{-k} \\ &\leq C\,R^{-\alpha-k} \sum_{j<R/2} j^{\alpha} + C\,\delta \sum_{R/2 \leq j \leq 2R} (1 + |R - j|)^{-k} + C\,\delta\,R^{-\alpha} \sum_{j>2R} j^{\alpha-k} \\ &\leq C\,R^{1-k} + C\,\delta + C\,\delta\,R^{1-k}\,.\end{aligned}$$

In particular, if R is sufficiently large, $R^{1-k} < \delta$, the above quantity is dominated by $C\delta$.

When $0 < \alpha < 1$ one can use part (4) of Lemma 2.5 to get a similar estimate.

Finally, when $\alpha = 0$, $dF(t)$ is no more absolutely continuous, but part (5) of Lemma 2.5 applies. □

Now recall that in Chapter 1 we proved the following result.

Theorem 2.8 (Bastis - Meaney)**.** *Suppose that $\mathcal{M}$ is a compact rank one symmetric space. If $f(x) \in L^2(\mathcal{M})$ is zero almost everywhere on an open set $\Omega \subset \mathcal{M}$, then the spherical partial sums of its eigenfunction expansion converges to zero almost everywhere on Ω:*

$$\lim_{R \to +\infty} \left\{ \sum_{\lambda < R} \widehat{f}(\lambda)\varphi_\lambda(x) \right\} = 0 \quad a.e. \ in\ \Omega\,.$$

When $\mathcal{M}$ is a smooth connected and compact Riemannian manifold we are unable to give a complete generalization, but we can prove a sightly weaker result: If $\{R_j\}_{j=1}^{+\infty}$ is an increasing sequence which tends to infinity discretely, the almost everywhere localization for $S^0_{R_j}f(x)$ holds when $f(x) \in L^2(\mathcal{M})$. Note that, if the eigenvalues group together, as in the compact rank one symmetric space case, one obtains the full result of Bastis and Meaney.

Theorem 2.9. *Assume that $f(x)$ is in $L^2(\mathcal{M})$ and that $f(x) = 0$ in an open set Ω of $\mathcal{M}$. If $R_j - R_{j-1} > \delta > 0$ for every $j = 1, 2, \ldots,$ then for almost every x in Ω,*

$$\lim_{j \to +\infty} \{|S^0_{R_j}f(x)|\} = 0 .$$

Proof. It suffices to show that for every $\varepsilon > 0$ the almost everywhere localization holds in $\Omega_\varepsilon = \{x \in \Omega \ : \ distance(x, \partial\Omega) > \varepsilon\}$. As we have seen in Section 2.3, $S^\alpha_{R_j}f(x)$ can be decomposed into the sum of $A_{R_j}f(x)$ and $B_{R_j}f(x)$. By Lemma 2.4, $A_{R_j}f(x) = 0$ if $distance(x, \partial\Omega) > \varepsilon$. It then suffices to show that $B_{R_j}f(x) \to 0$ almost everywhere as $j \to +\infty$. Replace the usual maximal operator

$$\sup_{R>0} \{|B_R f(x)|\}$$

by the discrete maximal operator

$$\sup_{j \geq 1} \left\{\left|B_{R_j}f(x)\right|\right\} .$$

Then,

$$\begin{aligned}
\int_{\mathcal{M}} \sup_{j \geq 1} \left\{\left|B_{R_j}f(x)\right|^2\right\} dx &\leq \int_{\mathcal{M}} \left(\sum_{j=1}^{+\infty} \left|B_{R_j}f(x)\right|^2\right) dx \\
&= \sum_{j=1}^{+\infty} \left(\int_{\mathcal{M}} \left|B_{R_j}f(x)\right|^2 dx\right) \\
&= \sum_{j=1}^{+\infty} \left(\sum_{\lambda} \left|m_{R_j}(\lambda) - m_{R_j} * \psi(\lambda)\right|^2 |\widehat{f}(\lambda)|^2\right) \\
&= \sum_{\lambda} |\widehat{f}(\lambda)|^2 \left(\sum_{j=1}^{+\infty} \left|m_{R_j}(\lambda) - m_{R_j} * \psi(\lambda)\right|^2\right) .
\end{aligned}$$

Rememberer that $\alpha = 0$. By Lemma 2.5 and by the hypothesis $R_j - R_{j-1} > \delta$, for all $j = 1, 2, \ldots,$ there exists $C > 0$ such that, for every $k > 0$,

$$\sum_{j=1}^{+\infty} \left|m_{R_j}(\lambda) - m_{R_j} * \psi(\lambda)\right|^2 \leq C \sum_{j=1}^{+\infty} (1 + |R_j - \lambda|)^{-k} \leq C .$$

Then we can conclude that

$$\int_{\mathcal{M}} \left\{ \sum_{j=1}^{+\infty} \left| B_{R_j} f(x) \right|^2 \right\} dx \leq C \sum_{\lambda} |\widehat{f}(\lambda)|^2 .$$

□

2.5 Hausdorff dimension of divergent set in L^2-Sobolev spaces

Beurling and then Salem and Zygmund (see [53], Chapter XIII) studied the capacity of sets of divergence of one dimensional Fourier series of functions in Sobolev classes, and these results have been extended to multidimensional Fourier expansions on Eucledean spaces by Carbery and Soria in [13] and [14]. In particular, in [13], [14] and [15] it is stated that at the indices $\alpha = \gamma = 0$ localization may fail on sets of measure zero but of full dimension. The following result is an analogue of part (1) of Theorem 1.13 in the previous chapter.

Theorem 2.10. *Assume that the following conditions hold:*

$$\alpha \geq 0 , \qquad 0 \leq \alpha + \gamma < d/2 , \qquad \delta = d - 2(\alpha + \gamma) .$$

Assume that $f(x)$ is in $L^2(\mathcal{M})$ and that $G^\gamma f(x) = 0$ in an open set Ω, and let $d\upsilon(x)$ a non-negative Borel measure with support in $\Omega \cap \{distance\,(x, \partial\Omega) > \varepsilon\}$ for some $\varepsilon > 0$. Then there exists a positive constant C such that

$$\int_{\mathcal{M}} \sup_{R>0} \{ |S_R^\alpha G^\gamma f(x)| \} \, d\upsilon(x) \leq C \, \|f\|_{L^2} \left\{ \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{d\upsilon(x)\, d\upsilon(y)}{|x-y|^\delta} \right\}^{\frac{1}{2}} .$$

Proof. The proof of Theorem 2.10 is similar to that of Theorem 1.13, but a bit more delicate. Decompose S_R^α as $A_R + B_R$ as said before, the operator A_R is localized, then it suffices to consider B_R. We shall prove that, for every $\varepsilon > 0$ there exists $F \in L^2(\mathcal{M})$ such that

$$(1) \quad \sup_{R>0} \{ |B_R f(x)| \} \leq \int_{\mathcal{M}} |x-y|^{\alpha+\gamma-\varepsilon-d} \, F(y) \, dy , \tag{2.2}$$

$$(2) \quad \|F\|_{L^2} \leq C \int_{\mathcal{M}} \left| G^{-\gamma} f(z) \right|^2 dz . \tag{2.3}$$

Lemma 2.11. *Fix $\varepsilon > 0$. If $0 < \varepsilon < \alpha + \gamma$, then there exists $C > 0$ such that*

$$\sup_{R>1} \{|B_R f(x)|\} \le C\, G^{\alpha+\gamma-\varepsilon} \sup_{R>0} \left\{\left|\left(S_R^0 \circ G^{\varepsilon-\gamma}\right) f(x)\right|\right\} .$$

Proof. Define

$$P(\lambda) = \left(1+\lambda^2\right)^{\alpha/2} \left(m(\lambda) - m * \psi(\lambda)\right)$$
$$Q(t) = \left(S_t^0 \circ G^{\alpha}\right) f(x) = \sum_{\lambda<t} \left(1+\lambda^2\right)^{-\alpha/2} \widehat{f}(\lambda)\, \varphi_\lambda(x) .$$

Observe that $P(\lambda)$ is also a function of R and $Q(t)$ is also a function of x. Using the Riemann-Stieltjes integrals we can write

$$\begin{aligned} B_R f(x) &= \sum_{\lambda} \left(m(\lambda) - m * \psi(\lambda)\right) \widehat{f}(\lambda)\, \varphi_\lambda(x) \\ &= \sum_{\lambda} P(\lambda) \left(1+\lambda^2\right)^{-\alpha/2} \widehat{f}(\lambda)\, \varphi_\lambda(x) \\ &= \int_0^{+\infty} P(\tau)\, dQ(\tau) \\ &= -\int_0^{+\infty} Q(\tau)\, dP(\tau) . \end{aligned}$$

In particular,

$$\sup_{R>1} \{|B_R f(x)|\} \le \sup_{\tau>0} \{|Q(\tau)|\} \sup_{R>1} \left\{ \int_0^{+\infty} \left|\frac{d}{d\tau} P(\tau)\right| d\tau \right\} .$$

The integral of $dP(\tau)/d\tau$ can be bounded independently on R. Indeed

$$\begin{aligned} \frac{d}{d\tau} P(\tau) &= \frac{d}{d\tau} \left\{\left(1+\tau^2\right)^{\alpha/2} \left(m(\tau) - m * \psi(\tau)\right)\right\} \\ &= \alpha\tau \left(1+\tau^2\right)^{\alpha/2-1} \left(m(\tau) - m * \psi(\tau)\right) \\ &\quad - 2\alpha\tau R^{-2} \left(1+\tau^2\right)^{\alpha/2} \left[\left(1-\frac{\tau^2}{R^2}\right)_+^{\alpha-1} - \psi * \left(1-\frac{\tau^2}{R^2}\right)_+^{\alpha-1}\right] \\ &\quad - 2\alpha R^{-2} \left(1+\tau^2\right)^{\alpha/2} \left(\tau\psi(\tau)\right) * \left(1-\frac{\tau^2}{R^2}\right)_+^{\alpha-1} . \end{aligned}$$

Observe that if $\alpha > 0$ then $(1-\tau^2/R^2)_+^{\alpha-1}$ is integrable. Using Lemma 2.5, for any $k > 0$, we have the following estimates:

$$\begin{aligned} &\alpha \int_0^{+\infty} \tau \left(1+\tau^2\right)^{\alpha/2-1} \left|m(\tau) - m * \psi(\tau)\right| d\tau \\ &\le C\, R^{-\alpha} \int_0^{+\infty} \tau \left(1+\tau^2\right)^{\alpha/2-1} \left(1+|R-\tau|\right)^{-k} d\tau \le C\, R^{-1} , \end{aligned}$$

$$2\alpha R^{-2}\int_0^{+\infty}\tau\ \left(1+\tau^2\right)^{\alpha/2}\left|\left(1-\frac{\tau^2}{R^2}\right)_+^{\alpha-1}-\ \psi*\left(1-\frac{\tau^2}{R^2}\right)_+^{\alpha-1}\right|\,d\tau\ \leq\ C\,,$$

and, similarly,

$$2\alpha R^{-2}\int_0^{+\infty}\left(1+\tau^2\right)^{\alpha/2}\ |\tau\psi(\tau)|*\left(1-\frac{\tau^2}{R^2}\right)_+^{\alpha-1}\ d\tau\ \leq\ C\,R^{-1}\,.$$

Finally write $G^\alpha(x,y)=\left(G^{\alpha+\gamma-\varepsilon}\circ G^{\epsilon-\gamma}\right)(x,y)$. Under our assumptions $\alpha+\gamma-\varepsilon>0$, then $G^{\alpha+\gamma-\varepsilon}(x,y)$ is positive and we have

$$\sup_{\tau>0}\{|Q(\tau)|\}\ =\ \sup_{R>0}\left\{\left|\left(S_R^0\circ G^\alpha\right)f(x)\right|\right\}\ \leq\ G^{\alpha+\gamma-\varepsilon}\ \sup_{R>0}\left\{\left|\left(S_R^0\circ G^{\varepsilon-\gamma}\right)f(x)\right|\right\}\,.$$

□

The following Lemma is due to C. Meaney (see [33]).

Lemma 2.12. *Fix $\varepsilon>0$. Then there exists $C>0$ such that, for every tempered distribution $f(x)$,*

$$\int_{\mathcal{M}}\sup_{R>0}\left\{\left|\left(S_R^0\circ G^\varepsilon\right)f(x)\right|\right\}^2\,dx\ \leq\ C\,\|f\|_{L^2}^2\,.$$

Proof. The proof follows from the Rademacher-Menshov theorem on convergence of orthogonal expansions together with Weyl's estimates on the eigenvalues of elliptic differential operators. □

Lemma 2.13. *For every $F(x)\in L^2(\mathcal{M})$, for every non-negative finite Borel measure $d\upsilon(x)$ and for every $\eta>0$*

$$\int_{\mathcal{M}}|G^\eta F(x)|\ d\upsilon(x)\ \leq\ \|F\|_{L^2}\left\{\int_{\mathcal{M}}\int_{\mathcal{M}}G^{2\eta}(x-y)\,d\upsilon(x)\,d\upsilon(y)\right\}^{\frac{1}{2}}\,.$$

Proof. The proof of Lemma 1.14 applies. □

Now take $0<\varepsilon<\alpha+\gamma$, $\eta=\alpha+\gamma+\varepsilon$ and

$$F(x)\ =\ \sup_{R>0}\left\{\left|\left(S_R^0\circ G^{\varepsilon-\gamma}\right)f(x)\right|\right\}\,.$$

Recalling that for $0<2\eta<d$ the Bessel kernel $G^{2\eta}(x,y)$ blows up as $|x-y|^{2\eta-d}$ and using Lemma 2.11, Lemma 2.12 and Lemma 2.13, we have demonstrate (2.2) and (2.3). So we get the proof of Theorem 2.10. □

By Theorem 2.10, the maximal function $\sup_{R>0} \{|S_R^\alpha G^\gamma f(x)|\}$ cannot be infinite on the support of a measure of finite energy, and this implies the following.

Corollary 2.14. *Under the above assumptions on p, α, γ and δ, if $G^\gamma f(x) = 0$ in an open set Ω, then*

$$\lim_{R\to+\infty} \{S_R^\alpha G^\gamma f(x)\} = 0$$

at all points in the open set Ω, with possible exceptions in a set with Hausdorff dimension at most δ.

We are unable to give a complete extension of Theorem 1.13. This because we do not know precise estimates for the analogue of the zonal spherical functions on compact rank one symmetric spaces, i.e.

$$\sum_{\Lambda\leq\lambda<\Lambda+\varepsilon} \varphi_\lambda(x)\,\overline{\varphi_\lambda(y)}\,.$$

When $f(x) \in L^p(\mathcal{M})$ with $p < 2$ the pointwise localization is more difficult to study. Just to illustrate the problem, we recall that in [25] Hebisch has proved that in a general compact manifold the Bochner-Riesz means of order $\alpha > d/2$ of functions in $L^1(\mathcal{M})$ converges pointwise almost everywhere. We don't know if the critical index for the almost everywhere convergence or localization is $d/2$ as proved by Hebisch, or $(d-1)/2$ as suggested by the case of compact rank one symmetric spaces or the torus.

Chapter 3

Appendix: Bessel capacity and Hausdorff dimension

In this appendix we recall the notion of Bessel capacity which is critical in describing the appropriate class of null sets for the treatment of our problems of localizations. We refer to [52], Section 2.6, for more details.

The concept of capacity, as a theoretic measuring device, is one of the cornerstones of potential theory and it is intimately associated to the idea of a Sobolev function space, in much the same way that Lebesgue measure is related to the classical L^p spaces. The sets of capacity zero are the exceptional sets for representatives of the function spaces.

The notion of capacity has its origins in physics, where it measures the maximum amount of positive electric charge which can be carried by conductor while keeping the potential generated by the charge below a fixed threshold. The notion of capacity has been extended to nonlinear potentials, to various metric space settings, to the theory of stochastic processes and more.

For the purposes of this thesis we are interested in the Bessel capacity. As shown by Calderon's theorem (see [1], pg.13), every function $F(x)$ in a classical Sobolev space $W^{\gamma,p}(\mathcal{M})$ can be represented as a Bessel potential $F(x) = G^{\gamma} f(x)$, with $f(x) \in L^p(\mathcal{M})$. G^{γ}, $-\infty < \gamma < +\infty$, is the Bessel operator presented in Section 1.1 for the rank one symmetric spaces and in Section 2.2 for the general case. It is interesting to observe that the associated Bessel kernel $G^{\gamma}(x, y)$ is a positive and integrable function and Young's inequality for convolutions gives

$$\|G^{\gamma} f\|_{L^p} \le C \|f\|_{L^p}, \quad 1 \le p \le +\infty .$$

The behaviour of $G^\gamma(x,y)$ when $y \to x$ is of particular interest. We have as $|x-y| \to 0$ that $G^\gamma(x,y) \approx |x-y|^{\gamma-d}$. See Lemma 1.2 and Lemma 2.1 for the proof.

We now introduce the notion of capacity, which we develop in terms of the Bessel potentials.

Definition 3.1. Let $\beta > 0$ and $1 \leq p < +\infty$. The *Bessel* (β, p) *capacity* of a Borel set E in $\mathcal{M}$ is defined as

$$\mathcal{B}_{\beta,p}(E) \;=\; \inf\left\{ \|f\|_{L^p}^p \;:\; G^\beta f(x) \geq 1 \;\text{ on }\; E \right\}.$$

In case $\beta = 0$, we take $\mathcal{B}_{\beta,p}$ as the Lebesgue measure.

Suppose $p > 1$ and $\beta p < d$. Then it is essentially a consequence of Sobolev's inequality that there exists a constant C depending only on p, β and d such that

$$C^{-1}\, r^{d-\beta p} \;\leq\; \mathcal{B}_{\beta,p}(B(x,r)) \;\leq\; C\, r^{d-\beta p}\,,$$

for all balls $B(x,r)$ with $x \in \mathcal{M}$ and $r > 0$. This suggest that $\mathcal{B}_{\beta,p}$ and $H^{d-\beta p}$, the Hausdorff measure of dimension $d - \beta p$, are related. An important connection between the Bessel capacity $\mathcal{B}_{\beta,p}$ and H^r, the Hausdorff measure of dimension r, is given by the following theorem.

Theorem 3.1. *If $p > 1$ and $0 < \beta \leq d/p$, then $\mathcal{B}_{\beta,p}(E) = 0$ if $H^{d-\beta p}(E) < +\infty$. Conversely, if $\mathcal{B}_{\beta,p}(E) = 0$, then $H^{d-\beta p+\varepsilon}(E) = 0$ for every $\varepsilon > 0$.*

One can reads the proof for Euclidean spaces in [24], Theorem 7.1. The proof for the general case of a smooth, compact and connected Riemannian manifold is similar. See also [1]. As a consequence the Hausdorff dimension of sets with $\mathcal{B}_{\beta,p} = 0$ is at most $d - \beta p$.

From the definition of capacity it easily follows some elementary properties. The first is the weak type inequality: for any $f(x) \geq 0$ a.e. on $\mathcal{M}$

$$\mathcal{B}_{\beta,p}(\{G^\beta f(x) > t\}) \;\leq\; t^{-p} \int_{\mathcal{M}} f(x)^p \, dx\,.$$

Lemma 3.2. *For $0 \leq \beta < d$ and $1 < p < +\infty$, the following hold:*

(1) $\mathcal{B}_{\beta,p}(\emptyset) = 0$*;*

(2) If $E_1 \subset E_2$*, then* $\mathcal{B}_{\beta,p}(E_1) \leq \mathcal{B}_{\beta,p}(E_2)$*;*

(3) If $E_k \subset \mathcal{M},\ k = 1, 2, \ldots,$ *then*

$$\mathcal{B}_{\beta,p}\left(\bigcup_{k=1}^{+\infty} E_k\right) \le \sum_{k=1}^{+\infty} \mathcal{B}_{\beta,p}(E_k)\,.$$

A useful characterization of capacity is the following:

$$\mathcal{B}_{\beta,p}(E) = \inf_f \left\{\inf_{x\in E} G^\beta f(x)\right\}^{-p} = \left\{\sup_f \inf_{x\in E} G^\beta f(x)\right\}^{-p},$$

where $f(x) \in L^p(\mathcal{M}),\ f(x) \ge 0$ and $\|f\|_{L^p} \le 1$.

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