Knots and braids on the Sun

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Abstract: In this paper we show how new techniques of topological fluid mechanics and physical knot theory can be applied to estimate magnetic energy levels in solar physics. In particular, we show that magnetic energy stored in complex configurations of plasma loops present on the Sun can be quantified by geometric and topological information. These studies find important applications in the energetics of solar and stellar physics and in laboratory plasmas.
Keywords: topological magnetohydrodynamics, magnetic knots, braids.

1. SOLAR CORONAL STRUCTURES AND THE PHYSICS OF MAGNETIC KNOTS

Most of the energy emitted by the Sun that reaches us in the form of visible and invisible light and heating is generated by the intense activity of solar magnetic fields. Near the solar surface, and possibly in the entire convection zone just below the photosphere, these fields structure the plasma into coherent regions in the shape of giant arches, loops and flux tubes rooted in the solar surface and projected into the corona (see figure 1; Spruit & Roberts, 1983; Bray *et al.*, 1991). Twisted and braided magnetic loops are indeed basic structural elements of solar and stellar atmospheres. On the Sun, more than 90% of the magnetic flux outside sunspots is concentrated into intense flux tubes, which are responsible for a great part of energy emission from the solar corona. Many phenomena associated with solar activity, such as formation of sunspots, prominences and flares, are directly related to the presence of these structures. Solar loops, which are made visible through density and temperature inhomogeneities aligned with the field, have typical thickness of the order of 100–1500 km and can rise up to 400000 km in the corona, at temperatures of millions of degrees.

Direct observation of these structures reveals braided configurations with complex topology of the magnetic field. The magnetic lines of force appear to be highly tangled, and this makes it much more difficult to estimate the magnetic energy stored in the structure. In recent years, an alternative approach based on topological techniques borrowed from knot theory and applied to magnetohydrodynamics

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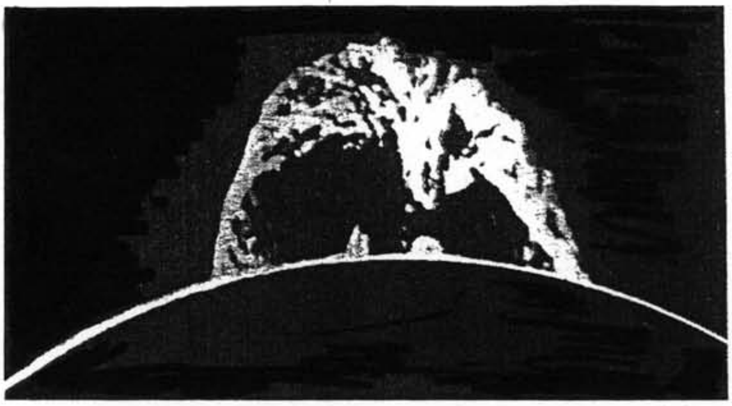

Figure 1. *Eruptive prominence as seen at the solar limb on the 13th of March, 1970. The full high of the prominence is estimated to be 370000 km and to rise at 160 km/s (from Giovanelli, 1984).*

(see Ricca & Berger, 1996; Ricca, 1998a; 1998b) has made it possible to provide new information on some important aspects of the energetics of astrophysical flows.

Mathematical models of coronal loops consider a collection of magnetic filaments, knotted and linked together to form arched braids (as in figure 2) embedded in a perfectly conducting plasma (i.e. in ideal conditions). Magnetic filaments interact with the surrounding medium through the Lorentz force and, as elastic strings, experience a tension proportional to their curvature. The curvature force is responsible for the dynamics of each filament in the medium. In ideal conditions, then, the topology of the magnetic field is 'frozen' in the fluid: this means that there are no reconnections of magnetic field lines taking place, and therefore knot and link types (and physical quantities, such as total energy and helicity) of the magnetic field distribution do not change with time. These quantities guide the magnetic relaxation of the structure towards states of minimum energy. Hence, two aspects influence the energetics of magnetic loops: a dynamical action of a curvature force (essentially related to the geometry of the magnetic field), and the type of knot and link of the magnetic configuration.

2. FROM MAGNETIC KNOTS TO BRAIDS

Knots and links are classified according to their topology. A useful measure of knot complexity is given by the minimum number of crossings $c_{\min}$ of a knot type, that is a topological invariant of the knot: its value does not change as the knot changes shape; $c_{\min}$ is determined by counting the number of crossings in the knot diagram, obtained by projecting the knot onto a plane so as that it shows the minimum

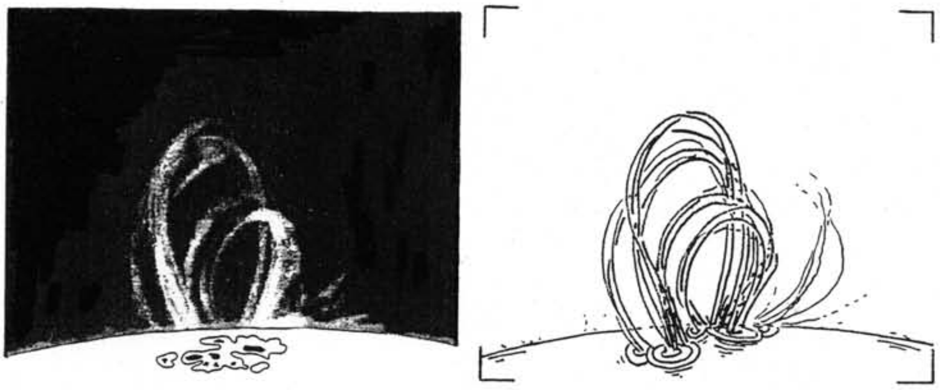

Figure 2. *Coronal loops in an active region (from Bray et al., 1991; on the left) and a sketch of the corresponding magnetic pattern (on the right). These loops are anchored to sunspots, whose locations (shown in the drawing below the picture on the left) are established with the aid of auxiliary data.*

possible number of crossings. The three knot types in the first row of figure 3, for example, are shown in their minimal projection (standard representation).

Of course a knot can be drawn to take any shape. Particularly interesting geometries are obtained when we take a standard knot and, by a series of continuous deformations, we change its shape to a closed braid with the least possible number c_o of crossings (minimal braids), as shown by the knots in the second row of figure 3. It is known (cf. Ricca, 1998) that c_o is bounded by

$$c_{\min} \leq c_o \leq c_{\min} + (c_{\min} - 1)(c_{\min} - 2) \; . \tag{1}$$

Minimal braids have interesting features: the curve goes around a centre as in an orbit-like pattern and without inflexions. An inflexion point (in isolation) is present when the intrinsic curvature of the knot vanishes: in a planar, S-shaped curve, for example, this point is visible at the change of concavity. The standard knots in the first row of figure 3 exhibit inflexional configurations, whereas their minimal braid configurations have none.

A geometric quantity useful to measure the average amount of coiling of a knot is the writhing number Wr. The writhe can be measured by the sum of the signed crossings of the knot diagram (not necessarily the minimal diagram) onto some projection plane, averaged over all projections, that is

$$Wr = < n_+(\boldsymbol{\nu}) - n_-(\boldsymbol{\nu}) > \; , \tag{2}$$

where $< \cdot >$ denotes averaging over all directions $\boldsymbol{\nu}$ of projection, and $n_\pm$ denotes the number of apparent $\pm$ crossings, from the direction of projection $\boldsymbol{\nu}$. For a nearly

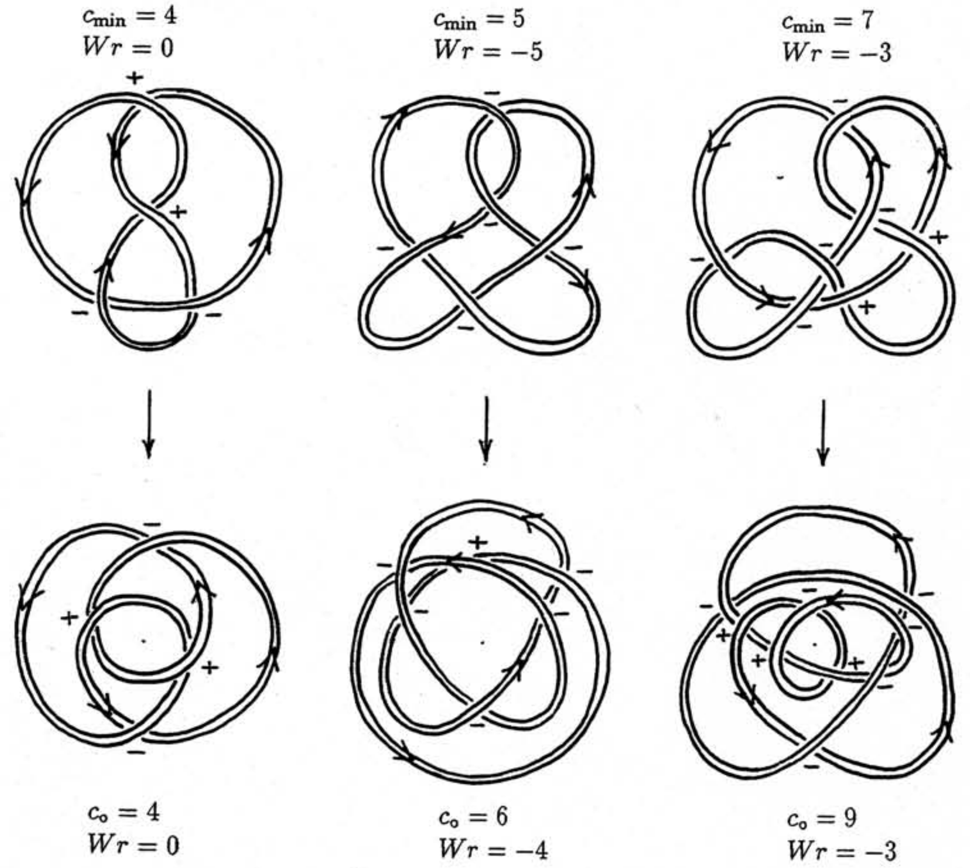

Figure 3. *Different knot types in standard representation (first row) can be transformed by a series of continuous deformations to minimal braids (second row).*

plane curve (except small indentations to allow crossings, as in the diagrams of Figure 3) the writhe can be directly estimated by the sum of the signed crossings. For a physical knot a change in writhe has physical implications and it is therefore a useful information.

Magnetic knots are dynamical objects that spontaneously tend to minimize their energy by re-arranging their shape. In a recent analysis (Ricca, 1997) it has been shown that twisted magnetic flux tubes are in disequilibrium in inflexional configurations (which are ubiquitous in astrophysical flows). Since the topology of the magnetic field is frozen and the Lorentz force drives the lines of force to an inflexion-free state, magnetic knots tend to remove inflexion points from their geometry to form closed braid configurations. Moreover, as the knot changes shape, the writhe value changes as well (remember that Wr is a geometric quantity that depends on the shape). Since in ideal conditions magnetic helicity is conserved, an

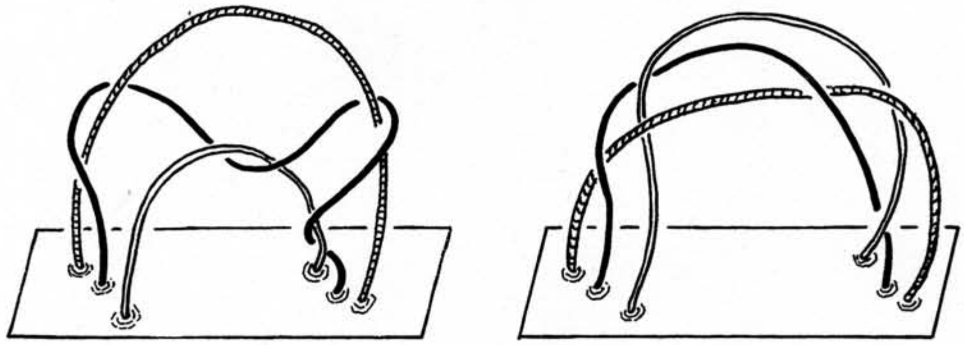

Figure 4. *A magnetic braid of 8 crossings (on the left) can relax to a minimal configuration (on the left) with* $c_o = c_{\min} = 6$.

increase in writhe is always accompanied by a decrease in the field lines twist, a mechanism that is favoured by the natural relaxation of the magnetic system.

3. MINIMAL BRAIDS AND ENERGY ESTIMATES

Is it possible to measure energy levels according to magnetic knot complexity? Recent studies on knot energy (Moffatt, 1990; Freedman & He, 1991; Berger, 1993) show that energy levels can be indeed related to crossing numbers, hence to knot complexity. Bounds on energy levels of magnetic knots are given by relations of the kind

$$E_{\min} \geq E_0 \equiv F(\Phi, V, N; c_{\min}) \; , \tag{3}$$

where $E_{\min}$ is the minimum energy level, E_0 is the ground state energy and $F(\cdot)$ gives the relationship between magnetic quantities such as total flux Φ, volume V, number of filaments N, and knot topology given by $c_{\min}$. Typically energy increases with $c_{\min}$ and therefore with knot complexity. The inequality sign, though, allows ample margin for error.

These estimates are based on calculations that provide limit values for global minima of energy, without taking into account any dynamic feature associated with the magnetic field. The competition between magnetic field distribution and the geometry of the structure may actually prevent the physical knot from reaching these limit values. We should remember that in ideal conditions magnetic knots relax their energy by moving according to the Lorentz force (a curvature force), that makes the lines of force shrinking. Since inflexional magnetic knots are forced to remove inflexions, there is a possibility that they simply cannot relax to minimal knots, if these exhibit inflexions. In this case they are forced to move to inflexion-free braids first, and then, by removing redundant crossings so as to reduce as much as possible their potential energy, to take the shape of minimal braids (that represent local minima for magnetic energy; see figure 4). Since minimal braids have $c_o \geq c_{\min}$, eq. (3) suggests that the least possible energy stored in these braids ($E_{o,\min}$) is likely to be higher than the theoretical limit given by E_0, i.e. $E_{o,\min} \geq E_{\min} \geq E_0$. Considerations based on application of the upper bound of inequality (1), then, makes us believe that the minimum amount of energy actually embodied in solar magnetic structures can be well above the theoretical limit E_0.

These considerations have important implications for astrophysical and terrestrial applications. Accurate estimates of minimum magnetic energy levels are useful to quantify the energy that can be released into heat during flares, microflares, and mass ejections from the Sun, and are also important for space applications and fusion physics in laboratory plasmas.

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