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Holography for six-dimensional theories A universal framework

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Abstract

It is hard to define quantum field theories in higher dimensions. For example, gauge theories are non-renormalizable in $d > 4$. In worse cases it is not even possible to give a lagrangian description to systems of degrees of freedom that are known to exist from indirect arguments, for example string theory.

It is the case of the popular $(2, 0)$, a six-dimensional conformal field theory living on the worldvolume of M5-branes. Despite its bizarre nature, attempts to understand this theory more deeply led to some really rewarding achievements. For example, compactifications on Riemann surfaces with punctures give a class of $\mathcal{N} = 2$ four-dimensional CFT's with amazing duality properties. Compactifications on three and four-manifolds also have interesting applications.

The subject of this thesis is the study of a class of six-dimensional CFT's that are less supersymmetric and even more mysterious, the $(1, 0)$ theories. Infinitely many and non-lagrangian, they are conjectured to exist on intersecting systems of NS5-, D6-, and D8-branes, but their holographic interpretation was lacking for a long time.

Collecting the results of a series of recent papers [1–5], we are finally able to give these CFT's a complete supergravity description. Their duals are all possible AdS₇ solutions in massive IIA, which are all known analytically. The internal space M_3 is topologically a three-sphere with SU(2) isometry, dual to the R-symmetry of the CFT.

We also classify all possible compactifications of these solutions, obtained wrapping the branes on two, three and four-manifolds of negative curvature, leading to AdS₅, AdS₄ and AdS₃ vacua. Moreover, complete flows connecting AdS₇ to the lower dimensional vacua can be constructed, suggesting the existence of a renormalization group flow between the six-dimensional CFT at high energies and CFT's in four, three, two dimensions at low energies. Even though these lower dimensional theories are not yet known, they can now be studied holographically. The AdS₄ solutions are also interesting as four-dimensional vacua with localized sources.

As a by-product, the compactification procedure is so universal it suggests to consider a reduction Ansatz of type IIA supergravity on M_3 . The resulting effective theory is minimal gauged supergravity in seven dimensions, a theory with sixteen supercharges and an SU(2) gauge field. It can be embedded in

type IIA in infinitely many ways, independently on the details of M_3 and thereby on the specific choice of brane configuration. This implies that the seven-dimensional theory is a sector common to all the $(1,0)$ theories and the $(2,0)$ itself. We can thus claim to have a universal consistent truncation for 6d/7d gauge/gravity duals.

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Chapter 1

Motivations

The study of field theories at strong coupling is a challenging subject. For example, understanding confinement in quantum chromodynamics (QCD) is still one of the most relevant open problems in physics. Although it is known that in the QCD vacuum the fundamental degrees of freedom (gluons and quarks) condensate into color singlets, no analytic proof exists. Some useful insights on this mechanism and on the possible behavior of field theories at low energy can be obtained introducing supersymmetry. This essentially amounts to enlarging the spacetime symmetry to also include the fermionic symmetries, which requires a proper completion of the Lagrangian by the addition of extra fields.

Supersymmetry makes the theory more constrained and manageable, especially at the quantum level, allowing to derive exact results. Increasing the amount of supersymmetry the constraints become stronger and stronger, so much so that in some cases it is possible to control the full renormalization group (RG) flow. An enlightening example is given by the Seiberg-Witten theory [6, 7], a supersymmetric extension of QCD with gauge group $SU(2)$. This theory is so under control that it is possible to determine an exact low-energy effective action. Enlarging further the amount to supersymmetry leads to $\mathcal{N} = 4$ super-Yang-Mills (SYM), the maximally supersymmetric extension of QCD. This theory is so constrained that the β function describing the RG flow vanishes, namely the theory is scale invariant also at the quantum level.

The aforementioned theories, like many other interesting supersymmetric field theories, can be realized explicitly in the realm of string theory. The string theory perspective can help a lot in understanding some crucial features, especially thanks to its many amazing duality properties.

Indeed, much of the progress seen in the last years in the subject of quantum field theory is tightly connected to the beautiful concept of *holography*. First introduced by 't Hooft and Susskind, the holographic principle states that the degrees of freedom of a $(p + 2)$ -dimensional quantum gravity are

much more reduced than we naively think, and will be comparable to those of quantum many body systems in $(p + 1)$ -dimensions [8, 9].

The first explicit realization of the holographic principle in string theory is due to Maldacena, and goes under the name of AdS/CFT correspondence [10]. The original statement of the correspondence is that type IIB string theory on $\text{AdS}_5 \times S^5$ background is dual to $\mathcal{N} = 4$ SYM in four dimensions. This conjecture generalizes to the statement that a theory of quantum gravity in $(p + 2)$ -dimensional Anti-de Sitter space is dual to a conformal field theory in $(p + 1)$ -dimensions, living on its boundary.

Perhaps the most striking feature of AdS/CFT is that it is a weak/strong duality, namely it relates string theory at weak coupling to a conformal field theory in the strong coupling regime, and vice versa. Crucially, string theory at weak coupling is described by ten-dimensional supergravity, which is much simpler and manageable. Supergravity can thus be very helpful to investigate some aspects of CFT's at strong coupling.

After its formulation, AdS/CFT has been understood more deeply with a precise map between the observables on the two sides and a prescription for comparing physical quantities and amplitudes [11, 12]. For example, dimensions of operators in conformal field theory are given by masses of particles in supergravity. Over the last twenty years much progress has been made in writing the so-called holographic dictionary.

Moreover, the correspondence has been extended to the non-conformal realm. Indeed, the duality between supergravity (and string theory) on AdS space and boundary conformal field theory also relates the thermodynamics of $\mathcal{N} = 4$ SYM to the thermodynamics of Schwarzschild black holes in Anti-de Sitter space. In this picture, quantum phenomena such as confinement and spontaneous symmetry breaking can be explained in terms of classical geometry [13].

Holography and black hole physics are deeply intertwined. The historical origin of the concept of holography is the beautiful analogy that was discovered between the laws of black hole dynamics and the laws of thermodynamics. In this context it was found that the entropy of a black hole is not proportional to its volume, but to its area of the event horizon Σ (the famous Bekenstein-Hawking formula [14, 15]). The precise statement of the AdS/CFT correspondence shed some light on this formula, at least for AdS black holes. In some cases, holography even allowed to re-derive the Bekenstein-Hawking entropy formula by counting black hole microstates. For black holes in three dimensions (the so-called BTZ black hole [16]) this was achieved using basic properties of conformal field theories in two dimensions [17]. It is thus understood that the AdS/CFT correspondence not only has nice field theory applications, but can also be useful to investigate some long standing problems in quantum gravity.

Another challenge which is deeply connected to holography in string theory is understanding higher dimensional quantum field theories. These are often non-renormalizable, so they only make sense as low-energy descrip-

tion of a more complete theory, much as the Fermi Lagrangian for weak interactions is described at high energies by the standard model [18,19]. Remarkably, string theory plays the role of ultraviolet completion for a large variety of interesting field theories in $d > 4$.

A relevant case is Yang-Mills in six dimensions, which is non-renormalizable and becomes strongly coupled at high energies. A possible alternative is to use a two-form gauge field instead of the usual vector. Its nonabelian formulation is still unclear, but string theory predicts that a super-conformal completion of such a field actually exists on the worldvolume of N coincident M5-branes.

This mysterious theory is expected to enjoy $(2,0)$ supersymmetry, but the corresponding Lagrangian is not known. Nonetheless its study led to important discoveries in the last decade. For example, its compactifications on Riemann surfaces with punctures produce a class of $\mathcal{N} = 2$ four-dimensional CFT's with beautiful duality properties [20,21] and a description in terms of a curve inspired by the Seiberg-Witten theory.

Another important class of six-dimensional CFT's can be realized in string theory, the so-called $(1,0)$ theories. Infinitely many and non-Lagrangian, these are the less supersymmetric cousins of the $(2,0)$. About twenty years ago they were conjectured to arise from complicated brane configurations in type IIA, involving D6-, NS5- and D8-branes [22], but their holographic description was lacking for a long time.

Understanding these theories holographically is thus an interesting challenge from the point of view of both for quantum field theory and string theory. According to the AdS/CFT dictionary, the supergravity description allows to count the degrees of freedom via the holographic free energy and to explore the spectrum of operators that are dual to masses of particles. Also, we can have some intuition on the possible lower dimensional CFT's arising as compactifications of the $(1,0)$ theories, and perhaps pave the way for a further improvement in the understanding of dualities in four dimensions.

Chapter 2

Introduction and summary of the results

In this section we try to give to the reader the key ingredients that are necessary to understand the results of our work. We first give a quick introduction to string theory and to its low-energy limit, supergravity, with an eye to the subject of dualities. After a quick review of D-branes and other types of fundamental degrees of freedom in string theory, we explain how the AdS/CFT correspondence was formulated using D3-branes. We finally move to holography for six-dimensional field theories, with a summary of our results.

STRING THEORY

String theory was formulated back in the 1970's with the goal of unifying in a single model all the four fundamental interaction in nature: Gravitational, electromagnetic, weak and strong. This unification is expected by theoretical physicists to take place at high energy scales, where gravitational quantum effects are supposed to become relevant. A natural scale is given by the Planck mass, which is of the order of 10^{19} GeV.

String theory is a theory of quantum gravity that arises from the quantization of one-dimensional objects, called strings, moving in spacetime sweeping out a surface called worldsheet. Roughly speaking, the string action is obtained by integrating the surface element swept by the string and multiplying it by a factor called string tension which is proportional to $(l_s)^{-2}$, where l_s is the string length, the only dimensionful parameter of the theory. This is nothing but the generalization of the action for a relativistic point-like particle with mass moving in spacetime along a trajectory. To be more precise one has to introduce an equivalent worldsheet action which can be quantized, and then complete this action to be supersymmetric by adding a fermionic part. We are not going to give any details, which can

be found in many textbooks like [23, 24]. We just stress that one could also try to generalize this action to p -dimensional extended objects sweeping out a $(p + 1)$ -dimensional worldvolume, however strings are very special in this sense because their two-dimensional worldsheet theories are renormalizable, which doesn't happen for $p \neq 1$.

To be more precise, the vibration modes of a string produce a number of quantum states including a massless spin-2 particle, a candidate to describe the graviton.

The fact that fundamental objects are extended rather than point-like particles constitutes a huge difference with respect to quantum field theories like the Standard Model: this is the key feature which allows to solve the problem of ultraviolet divergences.

In addition string theory shows a number of new aspects like supersymmetry, a spacetime symmetry mapping bosons into fermions, which extends the Poincaré algebra adding fermionic generators, called supercharges, that satisfy anti-commutation rules.

String theory also requires the presence of extra dimensions, more precisely it predicts the dimension of spacetime to be ten. In fact there are five different consistent supersymmetric string theories living in ten dimensions, which are related to one another through a web of dualities, each one being a different phase of a unique underlying theory living in eleven dimensions, known as *M-theory*, which at the state of art seems a possible candidate for a theory of everything.

SUPERGRAVITY as LOW ENERGY LIMIT

The mode expansion of the string is defined in terms of the string length l_s . In particular the masses of all states other than the massless ones become very large for $l_s \rightarrow 0$, which corresponds to taking the low-energy limit. In this limit, it is a good approximation to replace string theory with an effective theory describing the interactions of the massless modes only. This is known to be ten-dimensional supergravity, whose degrees of freedom are the graviton and its massless supersymmetric partners. More precisely, each possible formulation of string theory has different supergravity as low-energy limit.

In what follows we will be dealing with type IIA string theory, which is described by type IIA at low energy. The latter is essentially a supersymmetric model for gravity coupled to a scalar field ϕ called dilaton and to generalized gauge fields. In type IIA there are odd degree gauge fields C_1, C_3 with corresponding field strengths $\tilde{F}_2 = dC_1, \tilde{F}_4 = dC_3$, plus a two-form gauge field B with field strength $H = dB$. The bosonic part of the action reads:

$$\begin{aligned}
S_{IIA} = \frac{1}{(2\pi)^7 l_s^8} \int d^{10}x \sqrt{-|g|} & \left(e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{2}H^3 \right) - \frac{1}{2} \left(\tilde{F}_2^2 + \tilde{F}_4^2 \right) \right) \\
& - \frac{1}{(2\pi)^7 l_s^8} \int B \wedge \tilde{F}_4 \wedge \tilde{F}_4.
\end{aligned} \tag{2.0.1}$$

This action is non-renormalizable as it contains, for example, the Hilbert-Einstein term. However this is not a problem in the picture of string theory, where supergravity is not meant to be a fundamental theory, but simply a low-energy description.

Notice that the supergravity action contains a second parameter given by the vacuum expectation value of the dilaton $g_s = \langle e^\phi \rangle$ which can be arbitrary since there is no potential for ϕ . g_s determines the string coupling constant and controls string interactions and quantum corrections.

The second interesting ten-dimensional supergravity is type IIB, the low-energy effective theory for type IIB string theory. The bosonic field content is similar to that of type IIA, we just need to replace the odd degree form fields with even degree ones: C_0, C_2, C_4 , with corresponding field strengths $\tilde{F}_1, \tilde{F}_3, \tilde{F}_5$. We don't report the action of type IIB supergravity, which can be found in many textbooks, for example [23, 24].

Moving to the fermionic fields, both type II supergravities have two spin 3/2 gravitinos ψ_M^1, ψ_M^2 and two spin 1/2 dilatinos λ^1, λ^2 . The two gravitinos have the same chirality in type IIB and opposite chirality in type IIA, and the same works for the dilatinos. In other words type IIA supergravity is a chiral theory, while type IIB is not.

The supersymmetry transformations of type II supergravity are:

$$\begin{aligned}
\delta\psi_M^1 &= \left(D_M + \frac{1}{4}H_M \right) \epsilon^1 + \frac{e^\phi}{16} F \Gamma_M \Gamma \epsilon^2, \\
\delta\psi_M^2 &= \left(D_M - \frac{1}{4}H_M \right) \epsilon^2 - \frac{e^\phi}{16} \lambda(F) \Gamma_M \Gamma \epsilon^1, \\
\Gamma^M \delta\psi_M^1 - \delta\lambda^1 &= \left(D - \partial\phi - \frac{1}{4}H \right) \epsilon^1, \\
\Gamma^M \delta\psi_M^2 - \delta\lambda^2 &= \left(D - \partial\phi + \frac{1}{4}H \right) \epsilon^2,
\end{aligned} \tag{2.0.2}$$

where we have collected all the fluxes into a single mixed degree differential form $\tilde{F} = \sum_p \tilde{F}_p$, and we have defined $F \equiv e^{B \wedge} \tilde{F}$. More precisely the formal sum F contains both the electric fluxes F_p and their magnetic duals F_{10-p} . These are not independent, but they are related by the electro-magnetic duality constraint which reads: $F_p = \lambda \star F_{10-p}$, where the operator λ acts on a p -form as: $\lambda F_p = (-)^{Int(p/2)} F_p$.

All the quantities in Eq. (2.0.2) must be understood as bi-spinors, that is to say we have to apply to each differential form the so-called Clifford map,

that associates to dx^M the corresponding generator of the $SO(9,1)$ algebra Γ^M .

The supersymmetry infinitesimal generators ϵ^i are two Majorana fermions with the same chirality in type IIB and opposite chirality in type IIA. The covariant derivative which acts on the supersymmetry parameters contains the spin connection: $D_M = \partial_M + \frac{1}{4}\omega_M^{ab}\Gamma_{ab}$.

Supersymmetric solutions of the equations of motion, which means solutions which are invariant under supersymmetry transformations, can be found setting to zero all the variations in (2.0.2). This leads to four equations involving the fluxes F and H , the supersymmetry parameters ϵ^i , the dilaton ϕ and implicitly also the metric. Once one solves these equations together with the Bianchi identities for the fluxes: $(d - H \wedge)F = 0$, $dH = 0$, then the equations of motion follow automatically.

To be more precise, it is also possible to introduce an extra ingredient: sources that compatible with supersymmetry. This kind of sources are the so-called D-branes, which we will explain shortly.

M-THEORY and STRING DUALITIES

We saw in the last section that the supergravity action is obtained as a low-energy limit of string theory, and that this limit is implemented by sending to zero the string length l_s . We also saw that there is another dimensionless parameter in the action, the string coupling g_s . At this point one could ask what happens varying this parameter. The answer is that, at strong string coupling, string theory reveals a new phase described by a theory living in eleven dimensions, which is called M-theory.

The simplest way to explain this statement is to consider supergravity in eleven dimensions: there exists a unique eleven-dimensional maximal supergravity, which is also called the mother of all supergravities. The field content of this theory is surprisingly simple: the bosons are the graviton and a three-form gauge field A_3 with field strength $G_4 = dA_3$, and all the fermionic degrees of freedom are contained into a single gravitino.

The bosonic action of 11-dimensional supergravity, which reads

$$S = \frac{1}{(2\pi)^8 l_p^9} \left(\int d^{11}x \sqrt{-|g|} \left(R - \frac{1}{2} G_4^2 \right) - \frac{1}{6} \int A_3 \wedge G_4 \wedge G_4 \right), \quad (2.0.3)$$

is related to the actions of the various lower dimensional supergravities. The most direct connection is between 11-dimensional supergravity and type IIA supergravity: compactifying the eleventh dimension on a circle we get a spacetime which is topologically an S^1 fibred over a ten-dimensional manifold M_{10} :

$$\begin{array}{ccc} S^1 & \longrightarrow & M_{11} \\ & & \downarrow \\ & & M_{10} \end{array} \quad (2.0.4)$$

Indeed, applying to the action (2.0.3) a dimensional reduction along the S^1 we get exactly type IIA supergravity, with the following identification between the parameters of the two theories: $l_p = (g_s)^{1/3} l_s$, with l_p being the Planck length in eleven dimensions, the only parameter appearing in (2.0.3).

Upon reduction the graviton in eleven dimensions gives the graviton in ten dimensions, plus the dilaton ϕ and the gauge field C_1 , according to:

$$ds_{11}^2 = e^{-2\phi/3} ds_{10}^2 + e^{4\phi/3} (dz + C_1)^2, \quad (2.0.5)$$

while the generalized gauge field A_3 gives both B and C_3 . Finally, reducing the eleven-dimensional gravitino gives all the fermionic degrees of type IIA, with right chiralities. It thus seems reasonable to conjecture that:

$$\lim_{g_s \rightarrow \infty} \text{IIA} = 11\text{d sugra}. \quad (2.0.6)$$

In other words when string theory is strongly coupled it is better described by an eleven-dimensional theory, whose low-energy limit is given by eleven-dimensional supergravity. To be more precise one should also consider the whole spectrum of string theory, also containing non-perturbative extended objects called D-branes. We will give a brief introduction to these objects shortly. This correspondence between two apparently different theories is a first example of duality, called *S-duality*.

At this point one would like to derive type IIB string theory from the eleven-dimensional theory as well. This can be accomplished in two steps involving a second type of duality relating type IIA and type IIB, called *T-duality*. It is a perturbative duality relating type IIA string theory on $M_9 \times S_R^1$ to type IIB on $M_9 \times S_{1/R}^1$, where we used a compact notation to indicate the dual circles S_R^1 and $S_{1/R}^1$, which have radius R and $1/R$ respectively. Combining *T* and *S*-duality we get that M-theory on $M_9 \times T^2$ is dual to type IIB on $M_9 \times S^1$.

This is just a taste of dualities in string theory, which is a very complex and fascinating subject. It can be shown that all the five different string theories living in ten dimensions can be mapped into one another and also into M-theory via duality. This very exciting feature is the reason why M-theory is thought to be the unique underlying theory living in eleven dimensions which could describe a theory of everything. For a complete treatment of this subject see for example [23, 24].

For completeness, we have to mention that there exists a generalization of type IIA supergravity which will play an important role in this work. It is called massive type IIA supergravity, and it is obtained introducing a zero-form flux F_0 , the so-called *Romans mass*. F_0 is a special flux: it cannot be written as the derivative of a gauge potential, but it can be thought of as the dual of a spacetime-filling flux $F_{10} = dC_9$.

In absence of electric sources for F_{10} , the free-field equations would be $d \star F_{10} = 0$, which implies that the Romans mass is a constant, therefore

there are no propagating degrees of freedom associated to it. Still we can add F_0 to the Lagrangian and think of it as another free parameter.

As conjectured in [25], massive type IIA shows a striking feature: it cannot be strongly coupled. There is indeed a general argument which shows that the dilaton cannot be made arbitrarily large for a solution of type IIA supergravity with non-zero Romans mass, which means that there cannot be a lift to M-theory.

D-BRANES and M-BRANES

String theory has a variety of classical solutions corresponding to extended black holes, called *black p-branes*. These are charged massive objects that extend in p -spacelike directions and interact with the gravitational and gauge fields, just like charged black holes in general relativity. The equation of motion for a $(p+2)$ -form field strength in presence of a localized p -brane in type II supergravity is:

$$d \star F_{p+2} = \delta_{p\text{-brane}} . \quad (2.0.7)$$

As in electro-magnetism, we can measure the charge by the flux on a sphere surrounding the source. For an extended object with $(p+1)$ space-time directions, the transverse space is \mathbb{R}^{9-p} and the correct definition is $\int_{S^{8-p}} \star F_{p+2} = N$.

The most general p -brane solution in flat space can be found in many textbooks, for example [23, 24]; it has a singularity surrounded by an inner and an outer horizons, similarly to Kerr black holes. When the two horizons are coincident we speak of extremal p -branes, or *Dp-branes*. These are essentially a generalization of extremal black holes in general relativity. The corresponding type II supergravity solution can be written as

$$ds_{10}^2 = H(\rho)^{-1/2} ds_{\mathbb{R}^{1,p}}^2 + H(\rho)^{1/2} (d\rho^2 + \rho^2 ds_{S^{8-p}}^2) , \quad (2.0.8)$$

$$C_{p+1} = H(\rho) \text{vol}_{\mathbb{R}^{1,p}} , \quad (2.0.9)$$

$$e^\phi = g_s H(\rho)^{(3-p)/4} , \quad (2.0.10)$$

$$H(\rho) = 1 + \frac{c_p g_s N l_s^{7-p}}{\rho^{7-p}} , \quad (2.0.11)$$

where ρ is the radial direction in the transverse space \mathbb{R}^{9-p} and $\mathbb{R}^{1,p}$ parametrizes the so-called *worldvolume* of the Dp -brane. The symmetry of the solution is $\text{SO}(9-p) \times \text{ISO}(1,p)$, corresponding to rotations in the transverse space and to the Poincaré group in the worldvolume. The function $H(\rho)$ is an harmonic function in the transverse space to the Dp -brane, just like in the case of black holes in general relativity. The quantity c_p is a dimensional factor given by $c_p = (2\sqrt{\pi})^{5-p} \Gamma(\frac{7-p}{2})$.

This solution has a physical singularity at $\rho = 0$ where the source is located and two coincident horizons at: $(\rho_+)^{7-p} = c_p g_s N (l_s)^{7-p}$. In general relativity extremal black holes saturate the bound $M \geq |Q|$, coming from the

requirement for the absence of a naked singularity. In the case of Dp -branes we have saturation of analogous bound:

$$M \geq \frac{N}{(2\pi)^p g_s l_s^{p+1}}, \quad (2.0.12)$$

where M is the mass and N is the charge. It is not surprising that extremal solutions preserve part of the supersymmetry of the theory; this is also the case in four dimensions, where the extremal Reissner-Nordstrom black hole is a supersymmetric solution of $\mathcal{N} = 2$ supergravity which preserves half of the supersymmetry. Analogously the Dp -brane solution of type IIA supergravity preserves sixteen of the thirty-two supercharges.

The saturation of the aforementioned bound is what defines a so-called *BPS state*. BPS states are protected by supersymmetry, so they cannot disappear from the spectrum and they must be regarded as fundamental degrees of freedom.

Indeed, there are some extra degrees of freedom living on the worldvolume of D-branes. So far we have discussed these objects using the classical supergravity, but this description is only appropriate when the curvature of the brane geometry is small compared to the string scale, so that stringy corrections are negligible. Since the strength of the curvature is characterized by ρ_+ , this requires $\rho_+ \gg l_s$. To suppress string loop corrections, the effective string coupling g_s also needs to be kept small.

Alternatively, D-branes can be defined from the worldsheet perspective as planes on which open strings can end; the letter D is for Dirichlet, the type of boundary condition that one has to impose on some of the worldsheet scalars on these open strings.

By quantizing open strings ending on extended planes we indeed find massless excitations corresponding to a vector multiplet with sixteen supercharges. We also find a tower of massive string modes with squared masses of order $1/l_s^2$. The effective action for worldvolume fields and the interaction with the background can be determined by the open plus closed string perturbative expansion. Its bosonic part reads

$$S_{Dp} = \frac{1}{l_s^{p+1}} \int dx^{p+1} \left(e^{-\phi} \sqrt{\det(g + 2\pi l_s^2 f + B)} + e^{2\pi l_s^2 + B} \sum_k C_k |_{p+1} \right), \quad (2.0.13)$$

where f is the brane worldvolume field strength. The first term is called the Dirac-Born-Infeld action; If ϕ is constant and if $B = 0 = f$, this term is just the ordinary volume of the $(p + 1)$ -dimensional worldvolume. The second term is called the Wess-Zumino term and generalizes to an extended object the coupling $\int dx A$ of the electro-magnetic potential A to a point-like particle with one-dimensional worldline.

By expanding up to two-derivatives in the gauge fields this Lagrangian, we get an effective action for gauge fields: the $U(N)$ gauge theory in $(p + 1)$ -dimensions with sixteen supercharges, where N is the number of coincident

Dp -branes. We can thus say that multiple D-branes allow to realize non-abelian gauge theories in string theory. Historically, the p -branes were originally found as classical solutions to supergravity, and later it was pointed out by Polchinski that D-branes give their full string theoretical description.

It is worth giving a list of all possible types of branes that are allowed in string theory [26]:

$$\begin{array}{ll} \text{D0} & \text{D2} & \text{D4} & \text{D6} & \text{D8} & \text{(IIA)} , & (2.0.14) \\ \text{D1} & \text{D3} & \text{D5} & \text{D7} & \text{D9} & \text{(IIB)} . \end{array}$$

As we mentioned, from the point of view of supergravity a Dp -brane is an electric source for an F_{p+2} generalized field strength, as defined by Eq. (2.0.7). Analogously, according to electro-magnetic duality, a $D(6-p)$ -brane is a magnetic source, since: $d \star F_{8-p} = dF_{p+2} = \delta_{D(6-p)}$. Something special happens for D8-branes, which couple to a spacetime filling flux $F_{10} = \star F_0$; this is exceptional in that it doesn't have any propagating degrees of freedom, so this is a parameter, rather than a field. In the presence of D8-branes the Romans mass is piecewise constant, since it can jump in the points where the D8's are located, according to: $dF_0 = \delta_{D8}$.

There are also solitonic objects we didn't mention so far, like the NS5-brane. One way of understanding this object is as the dual to the fundamental string F1; the F1 is indeed an electric source for the two-form gauge field B through a coupling of the type $\int B$ in the string action, where the integral is taken on the string worldsheet. The NS5-brane can be understood as the magnetic dual of the fundamental string, that is to say as a magnetic source for the B-field according to: $dH = \delta_{NS5}$.

To complete the possible spectrum of membranes we still have to consider M-theory. As we mentioned, eleven-dimensional supergravity has a four-form flux G_4 ; analogously to what happens for D-branes, a two-dimensional membrane called M2-brane can act as electric source for this flux according to: $d \star G_4 = \delta_{M2}$. Vice versa, a five-dimensional membrane or M5-brane is the magnetic dual source.

Indeed, there exist supersymmetric solutions of eleven-dimensional supergravity with localized sources corresponding to these membranes. Here we only give the solution corresponding to a stack of N coincident M5-branes, which will be useful for later purposes. It reads

$$ds_{11}^2 = H(\rho)^{-1/3} ds_{\mathbb{R}^{1,5}}^2 + H(\rho)^{2/3} (d\rho^2 + \rho^2 ds_{S^4}^2) , \quad (2.0.15)$$

$$G_4 = N \text{vol}_{S^4} , \quad (2.0.16)$$

$$H(\rho) = 1 + \frac{\pi N l_p^3}{\rho^3} . \quad (2.0.17)$$

M-branes are the possible solitonic objects in M-theory. They are actually the fundamental degrees of freedom of M-theory, the analogue of the fundamental string F1 in string theory. Unfortunately, their sigma model is very hard to quantize, but we might think that if we were able to do so,

they would give rise to a quantum gravity theory that reduces to eleven-dimensional supergravity at low energies, just like superstrings give rise to supergravity at low energies in ten dimensions.

Indeed, it is easy to understand an M2-brane as the S-dual of a fundamental string: this is achieved wrapping the M2 along the eleventh compact direction in (2.0.4), so that it looks like a string to a ten-dimensional observer. Playing with S-duality it is possible to obtain all types of D-branes in type IIA string theory; for example if an M2 doesn't wrap the eleventh direction along which we reduce, it looks like a D2-brane in ten dimensions. Similarly, reducing an M5-brane to ten dimensions gives a D4 if the M5 is wrapped along x^{11} , an NS5 otherwise.

We are still missing two types of D-branes: D0's and D6's. States with k D0-branes can be understood as Kaluza-Klein tower of masses for the extra S^1 in (2.0.4). But what about a D6-brane? This is a magnetic source for F_2 whose gauge field A_1 describes the connection for the fibration (2.0.4), namely it becomes a part of the eleven-dimensional metric. This means the D6 itself becomes a feature of the manifold M_{11} , we can say that it lifts to pure geometry.

Finally there are D8-branes in type IIA supergravity, but these don't have a counterpart in eleven dimensions, since they are sources for the Romans mass F_0 , and we already mentioned that massive type IIA supergravity cannot lift to M-theory.

THE CORRESPONDENCE AdS/CFT

String theory provides us with a powerful tool for the study of conformal field theories in various dimensions: the AdS/CFT correspondence, an explicit realization of the so-called holographic principle. Holography claims that the degrees of freedom in $(p+2)$ -dimensional quantum gravity are much more reduced than we naively think, and will be comparable to those of quantum many body systems in $(p+1)$ -dimensions [8, 9]. This was essentially found by remembering that the entropy of a black hole is not proportional to its volume, but to its area of the event horizon Σ (the Bekenstein-Hawking formula [14, 15]):

$$S_{BH} = \frac{\text{Area}(\Sigma)}{4G_N},$$

where G_N is the Newton constant.

AdS/CFT [27] is an example of gauge gravity duality, that relates a theory of quantum gravity in Anti-de Sitter space in $(p+2)$ -dimensions to a conformal field theory in $(p+1)$ -dimensions living on its boundary. Crucially, AdS/CFT is a weak strong duality, that is to say it relates the gravity theory at weak coupling to a field theory in the strong coupling regime and vice versa. This means that it is somehow possible to use gravity to investigate strongly coupled CFT's through holography.

The original Maldacena conjecture states that $\mathcal{N} = 4$ super-Yang-Mills is dual to the type IIB string background $\text{AdS}_5 \times S^5$. $\mathcal{N} = 4$ SYM is the maximally supersymmetric extension of QCD; thanks to supersymmetry the theory is so constrained that the β function describing the renormalization group flow vanishes, so that the theory turns out to be scale invariant also at the quantum level. The conjecture arose from the observation that the two theories can be obtained by the same decoupling limit $l_s \rightarrow 0$, performed on the worldvolume theory and on the back-reacted metric in spacetime.

Let us briefly explain how. In the string theory realm, $\mathcal{N} = 4$ SYM with gauge group $U(N)$ can be realized as low-energy effective action for N parallel D3-branes in Type IIB. This can be seen expanding in powers of l_s the action for the coupled brane/bulk system (2.0.13), and taking the limit $l_s \rightarrow 0$. Indeed, it turns out that the interactions between the gauge fields living on the brane and the gravity degrees of freedom can be neglected, and the Lagrangian reduces to that of $\mathcal{N} = 4$ SYM.

On the other hand we saw that a D3-brane is a BPS solution of the equation of motion of type IIB supergravity, whose metric is given by Eq. (2.0.8) for $p = 3$. We can now perform the decoupling limit for this solution, which turns out to be equivalent to taking $\rho \rightarrow 0$ and zooming on the region where the branes sit. For this reason this procedure is called a *near-horizon* limit. We get

$$ds_{10}^2 = R^2 (ds_{\text{AdS}_5}^2 + ds_{S^5}^2), \quad R^2 = l_s^2 \sqrt{4\pi N g_s}. \quad (2.0.18)$$

What happened is that the radial direction ρ of the transverse space \mathbb{R}^6 joined the parallel directions to the brane to give the AdS_5 metric in Poincaré coordinates: $ds_{\text{AdS}_5}^2 = \frac{d\rho^2}{\rho^2} + \rho^2 ds_{\mathbb{R}^{1,3}}^2$. In this limit the physical singularity in $\rho = 0$ corresponding to the point where the D3 was located disappeared, but we are left with a constant five-form flux along the volume of AdS_5 . Crucially, the near horizon geometry has supersymmetry enhancement to thirty-two supercharges, the maximal possible amount.

We can understand the correspondence intuitively looking at the symmetries on the two sides. $\mathcal{N} = 4$ SYM is a superconformal theory with conformal symmetry $O(4,2)$, which coincides with the isometry group of AdS_5 on the gravity side. The R-symmetry group $SU(4)_R$ is mapped into the isometry group of the five-sphere $SO(6) \sim SU(4)$. Finally $\mathcal{N} = 4$ has 16 supercharges and 16 conformal supercharges, for a total of 32 supercharges which is the same amount of supersymmetry enjoyed by the $\text{AdS}_5 \times S^5$ solution in type IIB supergravity. If we want to use a compact expression, we can say that the symmetry on both sides is the superconformal group $SU(2,2|4)$.

In order to understand the correspondence more deeply, it is also important to compare the parameters in the two theories. We first have to identify the Yang-Mills coupling as: $4\pi g_s = g_{YM}^2$. A useful parameter for the study of the large N limit of field theories is the so-called t'Hooft coupling, which is defined as: $x = g_{YM}^2 N$. The parameters on the CFT side can be matched

with the two string parameters g_s and l_s . From Eq. (2.0.18), we get:

$$\frac{x}{N} = 4\pi g_s \quad , \quad \sqrt{x} = \frac{R^2}{l_s^2} . \quad (2.0.19)$$

The dual string theory is useful when it is weakly coupled, that is to say when it is described by type IIB supergravity. This happens in the combined limits $g_s \rightarrow 0$, the regime in which quantum loops corrections are suppressed, and $l_s/R \rightarrow 0$, which corresponds to neglecting higher derivatives terms in the supergravity Lagrangian and massive modes in the string expansion. This regime corresponds to the large N limit and the strong coupling $x \rightarrow \infty$ of the CFT. In other words we can say that AdS/CFT is a weak/strong duality. This is a crucial feature of this duality, which opens to the possibility of making computations for field theories at strong coupling using classical supergravity.

HOLOGRAPHY for SIX-DIMENSIONAL THEORIES

So far we have considered only one particular example of AdS/CFT correspondence, but many more explicit realizations of the duality can be constructed using D-branes and M-branes. Taking a near horizon limit is a general method for obtaining the gravity dual of the gauge theory living on a set of branes. This works every time the limit is able to decouple consistently brane and bulk physics. The simplest possibility is given by a stack of coincident branes in flat spacetime, but we can find many other examples with less supersymmetry by placing branes on non-trivial singularities [28] or considering more complicated sets of intersecting branes [29]. This allows to realize a large variety of interesting conformal field theories in the context of string theory.

The subject of this thesis is the holographic study of six-dimensional theories. The most famous example of CFT₆ is the one living on the worldvolume of N multiple M5-branes, which goes under the name of $(2, 0)$ theory. No Lagrangian description is available for it, but its impenetrability is only apparent. Indeed, much has been learned about this theory in the last decade, for example it is known that its degrees of freedom scale like N^3 .

Despite the difficulties that one finds in describing its gauge degrees of freedom, the holographic dual of the $(2, 0)$ theory is defined in a simple way as near horizon geometry of the M5-brane solution (2.0.15). Taking the limit $l_p \rightarrow 0$ we get the maximally supersymmetric vacuum $\text{AdS}_7 \times S^4$ of eleven-dimensional supergravity. This reads

$$ds_{11}^2 = R^2 \left(ds_{\text{AdS}_7}^2 + \frac{1}{4} ds_{S^4}^2 \right) , \quad R = 2l_p (\pi N)^{1/3} , \quad (2.0.20)$$

where $ds_{\text{AdS}_7}^2 = \frac{d\rho^2}{\rho^2} + \rho^2 ds_{\mathbb{R}^{1,5}}^2$. Again, the physical singularity in $\rho = 0$ where the branes were located disappeared. The solution also has a constant a four-form flux G_4 proportional to the volume of the four-sphere, with flux integer

N . The $SO(5)$ isometry of the four-sphere is dual to the $Sp(2)$ R-symmetry of the $(2, 0)$ theory.

It is very instructive for our purposes to reduce the M5-solution (2.0.20) to type IIA supergravity, according to the reduction Ansatz (2.0.5). To achieve this we parametrize the S^4 as a warped product of S^1 and S^3 , and then use Hopf coordinates for S^3 , that is to say we see the three-sphere as a non-trivial fibration of a circle over a two-sphere.

All in all, $ds_{S^4}^2 = d\alpha^2 + \sin^2\alpha ds_{S^3}^2 = d\alpha^2 + \sin^2\alpha (\frac{1}{4} ds_{S^2}^2 + (dz + A_1)^2)$, where $dA_1 = -\frac{1}{2} \text{vol}_{S^2}$. As shown in App. B, reducing along the vector ∂_z breaks half of the supersymmetry. The resulting geometry in ten dimensions is described by the metric:

$$\frac{R^3}{2} \sin \alpha \left(ds_{AdS_7}^2 + \frac{1}{4} \left(d\alpha^2 + \frac{1}{4} \sin^2 \alpha ds_{S^2}^2 \right) \right), \quad (2.0.21)$$

with two-form flux $F_2 \sim \text{vol}_{S^2}$ and three-form H proportional to the volume of the internal manifold M_3 . Notice that (2.0.21) might appear problematic for two reasons. First of all, the warping function goes to zero at the two poles $\alpha = 0, \alpha = \pi$; second, the internal metric is singular at poles because of the $1/4$ factor in front of $ds_{S^2}^2$. However these singularities can be interpreted physically as D6 and anti-D6 singularities! To see this let us expand the solution around the pole $\alpha = 0$; we get: $ds_{M_3}^2 \sim \alpha (d\alpha^2 + \frac{\alpha^2}{4} ds_{S^2}^2)$, which after the change of coordinates $\alpha = \rho^{1/2}$ gives the same type of singular behavior of the D6-brane solution of Eq. (2.0.8).

It is not surprising that a D6 is lifted to pure geometry in eleven dimensions, but what happened to the M5-branes? We mentioned that the S-dual of an M5-brane is an NS5-brane, whenever the M5 does not wrap the eleventh dimension. Indeed, the solution (2.0.21) has a non-vanishing H with flux integer N ; this signals the presence of N coincident NS5's which disappeared in the near horizon limit.

Summarizing, reducing the M5-solution to type IIA supergravity we got a new brane configuration involving both D6's and NS5's. More precisely, before the near horizon limit the configuration should be given by a stack of N coincident NS5-branes, with a D6-brane ending on them from both sides. The NS5's extend along the directions x_0, \dots, x_5 that parametrize $\mathbb{R}^{1,5}$ in Eq. (2.0.15), while the D6's also extend along the radial coordinate ρ of \mathbb{R}^5 , which becomes the radial coordinate of AdS_7 after the near horizon limit.

As conjectured long ago in [22], the degrees of freedom of the NS5-D6 system are described in the decoupling limit by a $(1, 0)$ six-dimensional conformal field theory, with R-symmetry group $Sp(1)$ corresponding to the $SO(3)$ isometry of the two-sphere.

Infinite more possible $(1, 0)$ theories can be engineered in type IIA. Perhaps the simplest generalization occurs when one introduces orbifold singularities [30–32]. From the holographic perspective, however, these theories are not very different: their dual is simply $AdS_7 \times S^4/\mathbb{Z}_k$ [33, 34]. When we reduce it to ten dimensions, we get the metric (2.0.21) rescaled by a factor

of $1/k$, which corresponds to two stacks of k D6's and k anti-D6's at the two poles. This brane configuration is represented in figure (2.1) together with a sketch of the internal manifold.

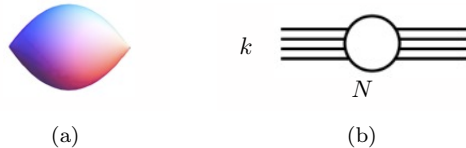


Figure 2.1: In (a), a sketch of the internal manifold M_3 in Eq. (2.0.21); the cusps represents the D6 stacks. In (b), the brane configuration whose near-horizon should originate Eq. (2.0.21). The dot represents a stack of N coincident NS5-branes; the horizontal lines represent k D6-branes ending on them.

More general theories with $(1, 0)$ supersymmetry can be obtained in the AdS/CFT setup via configurations involving NS5, D6, and also D8-branes. Indeed, the advantage of the IIA description is that one can modify the setup to add Dirichlet boundary conditions for the seven-dimensional gauge theory on the D6-branes, by having them end on two stacks of D8-branes. In the decoupling limit these configurations give rise to AdS₇ vacua in massive type IIA supergravity, which were found relatively recently: first numerically in [1], then analytically in [2]. Their interpretation as the duals of the $(1, 0)$ CFT's was given in [35]. Up to orbifolds and orientifolds, these are the most general AdS₇ solutions in perturbative type II supergravity. The internal space M_3 is an S^2 fibration over an interval, parametrized by a coordinate $r \in [r_-, r_+]$:

$$e^{2A} ds_{\text{AdS}_7}^2 + dr^2 + e^{2A} v^2 ds_{S^2}^2 . \quad (2.0.22)$$

A (the “warping”) and v are functions of r ; so is the dilaton ϕ . The S^2 has a round metric, and its isometry group is the SU(2) R-symmetry of the $(1, 0)$ CFT₆. It shrinks at the endpoints r_{\pm} of the interval. The fluxes have all the components compatible with the R-symmetry: $F_0, F_2 \propto \text{vol}_{S^2}$, $H \propto dr \wedge \text{vol}_{S^2}$. We will explain how to obtain this infinite class of analytic solutions in the next chapter; here we just give a simple example. The metric can be written as

$$\frac{n_{\text{D6}}}{F_0} \sqrt{\rho} \left(\frac{4}{3} ds_{\text{AdS}_7}^2 + \frac{d\rho^2}{4\rho(3-\rho)} + \frac{\rho}{3} \frac{(3-\rho)}{(12-\rho^2)} ds_{S^2}^2 \right) , \quad (2.0.23)$$

where $\rho \in [0, 3]$. Around $\rho = 0$ the metric behaves as $\sim 16\sqrt{\rho} ds_{\text{AdS}_7}^2 + \frac{1}{\sqrt{\rho}}(d\rho^2 + \rho^2 ds_{S^2}^2)$, which is the correct behavior near a stack of D6-branes wrapping AdS₇. On the other hand around $\rho = 3$, the internal metric turns into flat space: $d\tilde{\rho}^2 + \tilde{\rho}^2 ds_{S^2}^2$, after the change of coordinates $\tilde{\rho} = \sqrt{\rho - 3}$, so we get a regular point.



Figure 2.2: In (a), a sketch of the internal M_3 in Eq. (2.0.23); the cusp represents the single D6 stack. In (b), the brane configuration whose near-horizon should originate Eq. (2.0.23). The dot represents a stack of N NS5-branes; the horizontal lines represent n_{D6} D6-branes ending on them.

Having a holographic description for the $(1, 0)$ CFT $_6$'s makes it possible to investigate some aspects of these theories, which are still unknown in large part due to the lack of a Lagrangian description. For example, we can count their degrees of freedom.

A common way of estimating the number of degrees of freedom using holography in any dimension is to introduce a cut-off in AdS, and estimate the Bekenstein-Hawking entropy. In a warped compactification with non-constant dilaton, this is given by the integral: $\mathcal{F}_{0,6} = \int_{M_3} \text{vol}_{M_3} e^{5A-2\phi}$. This quantity can be computed explicitly.

In the case of the $(2, 0)$ theory, Eq (2.0.20), we reproduce the N^3 scaling behavior: $\mathcal{F}_{0,6} = \frac{128}{3} \pi^4 N^3$; for its \mathbb{Z}_k orbifold, this number is multiplied by k^2 . In the case of Fig. 2.2(a), using Eq. (2.0.23) we get $\mathcal{F}_{0,6} = \frac{512}{45} \pi^4 n_{D6}^2 N^3$. More general solutions, also involving D8-branes are described in chapter 3.

UNIVERSAL CONSISTENT TRUNCATION

Interesting phenomena arise upon compactification of six-dimensional theories to lower dimensions. For example, reducing the $(2, 0)$ theory on a T^2 gives again $\mathcal{N} = 4$ super-Yang-Mills. More interestingly, compactifying on a Riemann surface Σ_g , one breaks conformal invariance, but the resulting four-dimensional theory flows in the infrared to an $\mathcal{N} = 2$ conformal field theory. Theories obtained this way have interesting duality properties encoded by Σ_g [20, 36].

On the gravity side, the holographic duals of the compactifications of the $(2, 0)$ theory can be obtained by replacing AdS $_7$ with either AdS $_5 \times \Sigma_2$ [37, 38] or AdS $_4 \times \Sigma_3$ [39–41], where Σ_2 is a Riemann surface and Σ_3 is a compact quotient of hyperbolic space. In both these cases the internal S^4 is also distorted in a certain way.

It is natural to wonder whether a similar process can also be applied to the AdS $_7$ solutions of Eq. (2.0.22). This would indicate that the corresponding $(1, 0)$ CFTs give rise upon compactifications to CFTs in four and three dimensions, just like for the $(2, 0)$ theory. In recent work [3, 4] we found that

this can indeed be done.

In the process of doing so, we were able to find analytic expressions for the AdS₇ solutions themselves, and analytic maps ψ_5, ψ_4 from those to the AdS₅ \times Σ_2 and AdS₄ \times Σ_3 solutions. These maps are invertible and they can be composed. So in the end we have three new classes of infinitely many backgrounds with analytic expressions, holographically dual to CFTs in six, four, and three dimensions, with respectively eight, four, and two supercharges.

The map ψ_4 associates the AdS₇ metric, Eq. (2.0.22), to

$$\sqrt{\frac{5}{8}} \left[\frac{5}{8} e^{2A} \left(ds_{\text{AdS}_4}^2 + \frac{4}{5} ds_{\Sigma_3}^2 \right) + dr^2 + \frac{e^{2A} v^2}{1 - 6v^2} Ds_{S^2}^2 \right], \quad (2.0.24)$$

with Σ_3 a compact quotient of hyperbolic space with constant curvature. S^2 is now fibered over Σ_3 , in a way associated to its tangent bundle; even though the S^2 is still round, the total internal space has no isometries. The solution has now four supercharges; it is dual to an $\mathcal{N} = 1$ CFT₃. The fluxes acquire also components along Σ_3 .

Similarly, the map ψ_5 takes the metric, Eq. (2.0.22), to

$$\sqrt{\frac{3}{4}} \left[\frac{3}{4} e^{2A} \left(ds_{\text{AdS}_5}^2 + ds_{\Sigma_2}^2 \right) + dr^2 + \frac{e^{2A} v^2}{1 - 4v^2} Ds_{S^2}^2 \right], \quad (2.0.25)$$

with Σ_2 a Riemann surface. S^2 is fibered over Σ_2 via one of its U(1) isometries; the isometry group is now this U(1), which is the R-symmetry of the $\mathcal{N} = 1$ superalgebra of the CFT₄. Again the fluxes acquire components along Σ_2 .

This is how the AdS₇ solutions get mapped to AdS₄ and AdS₅ solutions. What is perhaps nicer than expected is that this map is universal. Namely, even though there are infinitely many AdS₇ solutions, the map to obtain the AdS₅ and AdS₄ metric is always the same. Moreover, the two maps are very similar to each other: they differ only by the value of certain numerical factors.

Indeed, this universality can be extended to a complete reduction Ansatz for massive type IIA supergravity on the internal manifold M_3 , where AdS₇ gets replaced by any seven-dimensional metric $g_{\mu\nu}$, and the internal space gets distorted in a way that depends on a single scalar parameter X according to the following formula:

$$X^{\frac{15}{2}} e^{2A} ds_7^2 + X^{\frac{5}{2}} \left(dr^2 + \frac{e^{2A} v^2}{1 + 16v^2(X^5 - 1)} Ds_{S^2}^2 \right), \quad (2.0.26)$$

where X takes different values for the three classes of vacua: $X^5 = \{1, \frac{3}{4}, \frac{5}{8}\}$, for AdS₇, AdS₅, AdS₄ respectively.

The resulting seven-dimensional effective theory has bosonic fields X and $g_{\mu\nu}$ themselves, together with a three-form potential, and an SU(2) gauge

field which is related to the fibration of the internal space over the seven external dimensions. This effective theory is the so-called minimal gauged supergravity in seven dimensions [42, 43], which describes the dynamics of (a gauged version of) the gravity multiplet with sixteen supercharges and has AdS_7 as maximally supersymmetric vacuum.

It is a subsector of the bigger “maximal” [44] theory, which describes the gravity multiplet with thirty-two supercharges and has gauge group $\text{SO}(5)$. Both theories can be obtained [45, 46] as consistent truncations from eleven dimensions.

Here we describe how to obtain the minimal theory from massive IIA, in *infinitely many ways*, corresponding to the infinite possible brane configurations involving NS5’s, D6’s and D8’s. In each of these reductions, the supersymmetric AdS_7 vacuum is one of the solutions (2.0.22). This is perhaps surprising, but the idea is that, in reducing, we are only using the ordinary differential equation (ODE) that the internal geometry has to solve in the vacuum, and not the details of the individual solution.

Vice versa, we can uplift to massive IIA any solution of the seven-dimensional supergravity, in infinitely many ways. For example, minimal gauged supergravity also has “Renormalization Group (RG) flow” solutions that connect the above AdS_5 and AdS_4 backgrounds to the AdS_7 maximally supersymmetric vacuum. This shows conclusively that the solutions (2.0.24, 2.0.25) are indeed dual to compactifications on Σ_2 and Σ_3 of the six-dimensional $(1, 0)$ CFT’s.

Minimal gauged supergravity also admits $\text{AdS}_3 \times \Sigma_4$ solutions, preserving $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry. In the latter case Σ_4 is a Kähler–Einstein manifold of negative constant curvature, while in the former case Σ_4 is (a compact quotient of) hyperbolic space \mathbb{H}^4 . The corresponding CFT duals are two-dimensional theories with $(0, 2)$ and $(0, 1)$ supersymmetry. Uplifting these solutions yields new AdS_3 solutions of massive IIA supergravity. On the field theory side, this implies that all the six-dimensional CFT’s of [22, 35, 47] can be compactified on four-manifolds Σ_4 to produce two-dimensional CFT’s.

The universal character of the truncation implies that supergravity in seven dimensions describes a sector common to all the six-dimensional $(1, 0)$ CFT’s engineered by NS5–D6–D8-brane intersections, including also the $(2, 0)$ theory itself, described by the original M-theory reduction of [45]. This is just a first step towards a more complete understanding of these mysterious theories. Beyond this common sector, discerning finer differences between the CFT₆’s would require more sophisticated reduction procedures, where one keeps more internal modes. For example, the Kaluza–Klein spectrum of the $\text{AdS}_7 \times M_3$ backgrounds, beyond the massless modes, can be used to analyze the spectrum of the dual operators.

Chapter 3

All AdS₇ solutions of type II supergravity

As we stressed in the introduction, the study of six-dimensional conformal field theories is a challenging subject. Even though they cannot be given a Lagrangian description, some examples are known to exist thanks to string theory. This is the case for the (2,0)-superconformal field theory living on the worldvolume of M5-branes.

This prompts the question of whether other non-trivial six-dimensional theories exist. There are in fact several other string theory constructions [22, 31, 33, 47] that would engineer such theories. Progress has also been made (see for example [48, 49]) in writing explicitly their classical actions.

Holography offers another way to establish the existence of superconformal theories in six dimensions, looking for supersymmetric AdS₇ solutions in string theory. In this section we explain how these solutions can be obtained. As we anticipated, M-theory has only an AdS₇ × S⁴ (which is holographically dual to the (2,0) theory) or an orbifold thereof. That leaves us with AdS₇ × M₃ in IIA with non-zero Romans mass $F_0 \neq 0$ (which cannot be lifted to M-theory) or in IIB. These solutions were classified in [1]; the result is that there are no such solutions in IIB, while infinitely many do exist in IIA with non-zero Romans mass F_0 .

The supersymmetry equations for a AdS₇ vacua can be derived using generalized geometry techniques, first introduced in four dimensions in [50] with the so-called *pure spinor* formalism. Analogous pure spinor equations were derived in [51] for Mink₆ × M₄. Viewing AdS₇ as a warped product of Mink₆ with a line allowed then to obtain a system valid for AdS₇ × M₃. A similar trick was applied in [52] to derive a system for AdS₅ × M₅ from Mink₄ × M₆. The AdS₇ system is written in terms of differential forms satisfying some algebraic constraints; mathematically, these constraints mean that the forms define a generalized identity × identity structure on $T_{M_3} \oplus T_{M_3}^*$, which in turn can be written in terms of a vielbein $\{e_a\}$ and some angles.

Unlike the previous cases, supersymmetry puts strong constraints on the geometry of the internal manifold without any assumption. Indeed, the equations determine explicitly the vielbein $\{e_a\}$ in terms of derivatives of our parameterization function. This gives a local, explicit form for the metric, without any Ansatz. The resulting metric describes an S^2 fibration over a one-dimensional space. This feature has a nice holographic interpretation: these solutions are dual to $(1, 0)$ superconformal theory has an $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ R-symmetry group, which should appear as the isometry group of the internal space M_3 .

The existence of a solution was then reduced to a system of coupled ODEs, which at first were studied numerically. Later on compactifications to AdS_4 and AdS_5 were considered in [3, 4]. Remarkably, the AdS_5 system of equations turned out much simpler than the ones in AdS_7 and AdS_4 ; so much so that it was possible to integrate it analytically! Meanwhile, it was possible to map the system of ODE's for the AdS_7 vacua to those in AdS_4 and AdS_5 . Making use of these maps, we were able to produce AdS_7 and AdS_4 solutions as well.

All in all, an infinite class of analytic AdS_7 solutions can be given. Their holographic interpretation was given in [35]; they are dual to the superconformal $(1, 0)$ theories in six dimensions, describing the degrees of freedom of brane systems involving NS5's, D6's and D8's first considered in [22].

3.1 Supersymmetry and pure spinors in $d = 3$

In this section, we give a system of pure spinor equations in three dimensions that is equivalent to preserved supersymmetry for solutions of the type $\mathrm{AdS}_7 \times M_3$. This was derived by a commonly-used trick: namely, by considering AdS_{d+1} as a warped product of Mink_d and \mathbb{R} .

3.1.1 Pure spinor equations for $\mathrm{AdS}_7 \times M_3$

Preserved supersymmetry for $\mathrm{Mink}_4 \times M_6$ was found [50] to be equivalent to the existence on M_6 of an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure satisfying certain differential equations reminiscent of generalized complex geometry [53, 54]. Similar methods can be useful in other dimensions. For $\mathrm{Mink}_6 \times M_4$ solutions, [51] found a system in terms of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure on M_4 , described by a pair of pure spinors $\phi^{1,2}$.

We would like to classify solutions of the type $\mathrm{AdS}_7 \times M_3$. These in general will have a metric

$$ds_{10}^2 = e^{2A} ds_{\mathrm{AdS}_7}^2 + ds_{M_3}^2, \quad (3.1.1)$$

where A is a warping function. A genuine AdS_7 solution is one where not only the metric is of the form (2.0.22), but where there are also no fields that break its $\mathrm{SO}(6,2)$ invariance. This can be easily achieved by additional

assumptions: for example, A should be a function of M_3 . The fluxes F and H , should now be forms on M_3 . For IIA, $F = F_0 + F_2 + F_4$: in order not to break $\text{SO}(6, 2)$, we impose $F_4 = 0$, since it would necessarily have a leg along AdS_7 ; for IIB, $F = F_1 + F_3$.

Moving to fermions, a decomposition of gamma matrices appropriate to seven-dimensional compactifications reads

$$\begin{aligned}\gamma_\mu^{(7+3)} &= \gamma_\mu^{(7)} \otimes 1 \otimes \sigma_2, \\ \gamma_{i+6}^{(7+3)} &= 1 \otimes \gamma_i \otimes \sigma_1,\end{aligned}\tag{3.1.2}$$

where $\gamma_\mu^{(7)}$, $\mu = 0, \dots, 6$, are a basis of seven-dimensional gamma matrices, and γ_i , $i = 1, 2, 3$, are a basis of gamma matrices in three dimensions (which in flat indices can be taken to be the Pauli matrices). For a supersymmetric solution of the form $\text{AdS}_7 \times M_3$, the supersymmetry parameters are of the form

$$\begin{aligned}\epsilon_1^{(7+3)} &= (\zeta \otimes \chi_1 + \zeta^c \otimes \chi_1^c) \otimes v_+, \\ \epsilon_2^{(7+3)} &= (\zeta \otimes \chi_2 \mp \zeta^c \otimes \chi_2^c) \otimes v_\mp.\end{aligned}\tag{3.1.3}$$

Here, $\chi_{1,2}$ are spinors on M_3 , with $\chi_{1,2}^c \equiv B_3 \chi_{1,2}^*$ their Majorana conjugates; a possible choice of B_3 is $B_3 = \sigma_2$. ζ is a spinor on AdS_7 , and $\zeta^c \equiv B_7 \zeta^*$ is its Majorana conjugate; there exists a choice of B_7 which is real and satisfies $B_7 \gamma_\mu = \gamma_\mu^* B_7$. (It also obeys $B_7 B_7^* = -1$, which is the famous statement that one cannot impose the Majorana condition in seven Lorentzian dimensions.) The ten-dimensional conjugation matrix can then be taken to be $B_{10} = B_7 \otimes B_3 \otimes \sigma_3$; the last factor in (3.1.3), v_\pm , are then spinors chosen in such a way as to give the $\epsilon_i^{(7+3)}$ the correct chirality, and to make them Majorana. They satisfy: $\sigma_3 v_\pm = \pm v_\pm$.

The presence of the cosmological constant in seven dimensions means that ζ is not constant, but rather that it satisfies the so-called Killing spinor equation, which for $R_{\text{AdS}} = 1$ reads

$$\nabla_\mu \zeta = \frac{1}{2} \gamma_\mu^{(7)} \zeta.\tag{3.1.4}$$

One class of solutions to this equation [55, 56] is simply of the form

$$\zeta_+ = \rho^{1/2} \zeta_+^0,\tag{3.1.5}$$

where here ρ is the radial coordinate of AdS_7 in Poincaré coordinates, which express AdS_7 as a warped product of Mink_6 and \mathbb{R} . ζ_+^0 is a spinor constant along Mink_6 and such that $\gamma_{\hat{\rho}} \zeta_+^0 = \zeta_+^0$ (the hat denoting a flat index).

Following the approach of generalized geometry, many of the data of a supergravity vacuum can be encoded in a pair of pure spinors ψ^1, ψ^2 , which are defined as the following bispinors:

$$\psi^1 = \chi_1 \otimes \chi_2^\dagger, \quad \psi^2 = \chi_1 \otimes \chi_2^{c\dagger}.\tag{3.1.6}$$

Using the Clifford map, which associates to each generator of the $SO(3)$ algebra σ^i a dx^i , it is possible to see bispinors as differential forms. Dealing with differential forms is much simpler than dealing with spinors, reason why the approach of generalized geometry is a very useful strategy to find supersymmetric vacua in any dimensions.

$\psi^{1,2}$ are differential forms on M_3 , but not just any forms. (3.1.6) imply that they should obey some algebraic constraints. Those constraints could be interpreted in a fancy way as saying that they define an identity \times identity structure on $T_{M_3} \oplus T_{M_3}^*$. However, three-dimensional spinorial geometry is simple enough that we can avoid such language: rather, in Sec. 3.1.2 we will give a parameterization that will allow us to solve all the algebraic constraints resulting from (3.1.6).

It was proved in [1] that the supersymmetry transformations of type II supergravity for an $AdS_7 \times M_3$ vacuum are equivalent to the following system of pure spinor equations:

$$d_H \text{Im}(e^{3A-\phi} \psi_{\pm}^1) = -2e^{2A-\phi} \text{Re} \psi_{\mp}^1, \quad (3.1.7a)$$

$$d_H \text{Re}(e^{5A-\phi} \psi_{\pm}^1) = 4e^{4A-\phi} \text{Im} \psi_{\mp}^1, \quad (3.1.7b)$$

$$d_H(e^{5A-\phi} \psi_{\pm}^2) = -4ie^{4A-\phi} \psi_{\mp}^2, \quad (3.1.7c)$$

$$\pm \frac{1}{8} e^{\phi} *_3 \lambda F = dA \wedge \text{Im} \psi_{\pm}^1 + e^{-A} \text{Re} \psi_{\mp}^1, \quad (3.1.7d)$$

$$dA \wedge \text{Re} \psi_{\mp}^1 = 0, \quad (3.1.7e)$$

$$(\psi_+^{1,2}, \overline{\psi_-^{1,2}}) = -\frac{i}{2}; \quad (3.1.7f)$$

again with the upper sign for IIA, and the lower for IIB.

These equations were actually derived from the analogous system for a $Mink_6 \times M_4$ solution, using the trick that a supersymmetric $AdS_7 \times M_3$ solution can be viewed as a supersymmetric $Mink_6 \times M_4$ solution. (3.1.7) can also be obtained directly from the ten-dimensional system in [57], but other equations also appear, and extra work is needed to show that those extra equations are redundant.

In (3.1.7) the cosmological constant of AdS_7 does not appear directly, since we have taken its radius to be one in (2.0.22). We did so because a non-unit radius can be reabsorbed in the factor e^{2A} in (3.1.1).

Before we can solve (3.1.7), we have to solve the algebraic constraints that follow from the definition of $\psi^{1,2}$ in (3.1.6); we will now turn to this problem.

3.1.2 Parameterization of the pure spinors

This subsection is taken from [1, Sec. 3]. We have just obtained a system of differential equations, (3.1.7), that is equivalent to supersymmetry for an $AdS_7 \times M_3$ solution. The $\psi^{1,2}$ appearing in that system are not arbitrary

forms; they should have the property that they can be rewritten as bispinors (via the Clifford map $dx^{i_1} \wedge \dots \wedge dx^{i_k} \mapsto \gamma^{i_1 \dots i_k}$) as in (3.1.6). In this section, we will obtain a parameterization for the most general set of $\psi^{1,2}$ that has this property. This will allow us to analyze (3.1.7) more explicitly in Sec. 3.2.

We will begin with a quick review of the case $\chi_1 = \chi_2$, and then show how to attack the more general situation where $\chi_1 \neq \chi_2$.

One spinor

We will use the Pauli matrices σ_i as three-dimensional gamma matrices, and use $B_3 = \sigma_2$ as a conjugation matrix (so that $B_3 \sigma_i = -\sigma_i^t B_3 = -\sigma_i^* B_3$). We will define

$$\chi^c \equiv B_3 \chi^* , \quad \bar{\chi} \equiv \chi^t B_3 ; \quad (3.1.8)$$

notice that $\chi^{c \dagger} = \chi^t B_3^\dagger = \bar{\chi}$.

We will now evaluate $\psi^{1,2}$ in (3.1.6) when $\chi_1 = \chi_2 \equiv \chi$; also, χ is normalized to one. Notice first a general point about the Clifford map $\alpha_k = \frac{1}{k!} \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \mapsto \not{\alpha}_k \equiv \frac{1}{k!} \alpha_{i_1 \dots i_k} \gamma^{i_1 \dots i_k}$ in three dimensions (and, more generally, in any odd dimension). Unlike what happens in even dimensions, the antisymmetrized gamma matrices $\gamma^{i_1 \dots i_k}$ are a redundant basis for bispinors. For example, we see that the slash of the volume form is a number: $\not{\text{vol}}_3 = \sigma^1 \sigma^2 \sigma^3 = i$. More generally we have

$$\not{\alpha} = -i * \not{\lambda} \alpha . \quad (3.1.9)$$

In other words, when we identify a form with its image under the Clifford map, we lose some information: we effectively have an equivalence $\alpha \cong -i * \lambda \alpha$. When evaluating $\psi^{1,2}$, we can give the corresponding forms as an even form, or as an odd form, or as a mix of the two.

Let us first consider $\chi \otimes \chi^\dagger$. We can choose to express it as an odd form. In its Fierz expansion, both its one-form part and its three-form part are a priori non-zero; we can parameterize them as

$$\chi \otimes \chi^\dagger = \frac{1}{2} (e_3 - i \text{vol}_3) . \quad (3.1.10)$$

(We can also write this in a mixed even/odd form as $\chi \otimes \chi^\dagger = \frac{1}{2} (1 + e_3)$; recall that the right hand sides have to be understood with a Clifford map applied to them.) e_3 is clearly a real vector, whose name has been chosen for later convenience. The fact that the three-form part is simply $-\frac{i}{2} \text{vol}_3$ follows from $\|\chi\| = 1$. Notice also that

$$e_3 \chi = \sigma_i \chi e_3^i = \sigma_i \chi \chi^\dagger \sigma^i \chi = \frac{1}{2} (-e_3 - 3i \text{vol}_3) \chi \quad \Rightarrow \quad e_3 \chi = \chi , \quad (3.1.11)$$

where we have used (3.1.10), and that $\sigma_i \alpha_k \sigma^i = (-)^k (3 - 2k) \alpha_k$ on a k -form. (3.1.11) also implies that e_3 has norm one.¹

Coming now to $\chi \otimes \bar{\chi}$, we notice that the three-form part in its Fierz expansion is zero, since $\bar{\chi} \chi = \chi^t B_3 \chi = 0$. The one-form part is now a priori no longer real; so we write

$$\chi \otimes \bar{\chi} = \frac{1}{2}(e_1 + ie_2). \quad (3.1.12)$$

Similar manipulations as in (3.1.11) show that $(e_1 + ie_2)\chi = 0$; using this, we get that

$$e_i \cdot e_j = \delta_{ij}. \quad (3.1.13)$$

In other words, $\{e_i\}$ is a vielbein, as notation would suggest.

Two spinors

We will now analyze the case with two spinors $\chi_1 \neq \chi_2$ (again both with norm one). We will proceed in a similar fashion as in [58, Sec. 3.1].

Our aim is to parameterize the bispinors $\psi^{1,2}$ in (3.1.6). Let us first consider their zero-form parts, $\chi_2^\dagger \chi_1$ and $\chi_2^{c\dagger} \chi_1$. The parameterization (3.1.11) can be applied to both χ_1 and χ_2 , resulting in two one-forms e_3^i . (This notation is a bit inconvenient, but these two one-forms will cease to be useful very soon.) Using then (3.1.10) twice, we see that

$$|\chi_2^\dagger \chi_1|^2 = \chi_2^\dagger \chi_1 \chi_1^\dagger \chi_2 = \text{Tr}(\chi_1 \chi_1^\dagger \chi_2 \chi_2^\dagger) = \frac{1}{4} \text{Tr}((1 + e_3^1)(1 + e_3^2)) = \frac{1}{2}(1 + e_3^1 \cdot e_3^2). \quad (3.1.14)$$

Similarly we have

$$|\chi_2^{c\dagger} \chi_1|^2 = \text{Tr}(\chi_1 \chi_1^{c\dagger} \chi_2 \chi_2^{c\dagger}) = \frac{1}{4} \text{Tr}((1 + e_3^1)(1 - e_3^2)) = 1 - |\chi_2^\dagger \chi_1|^2. \quad (3.1.15)$$

Both $|\chi_2^\dagger \chi_1|^2$ and $|\chi_2^{c\dagger} \chi_1|^2$ are positive and ≤ 1 . Thus we can parameterize $\chi_2^\dagger \chi_1 = e^{ia} \cos(\psi)$, $\chi_2^{c\dagger} \chi_1 = e^{ib} \sin(\psi)$. (The name of this angle should not be confused with the forms $\psi^{1,2}$.) By suitably multiplying χ_1 and χ_2 by two phases, we can assume $a = -\frac{\pi}{2}$ and $b = \frac{\pi}{2}$; we will reinstate generic values of these phases at the very end. Thus we have

$$\chi_2^\dagger \chi_1 = -i \cos(\psi), \quad \chi_2^{c\dagger} \chi_1 = i \sin(\psi). \quad (3.1.16)$$

Just as in [58, Sec. 3.1], we can now introduce

$$\chi_0 = \frac{1}{2}(\chi_1 - i\chi_2), \quad \tilde{\chi}_0 = \frac{1}{2}(\chi_1 + i\chi_2). \quad (3.1.17)$$

¹An alternative, perhaps more amusing, way of seeing this is to consider $\chi \otimes \chi^\dagger$ as a two-by-two spinorial matrix. It has rank one, which will be true if and only if its determinant is one. Using that $\det(A) = \frac{1}{2}(\text{Tr}(A)^2 - \text{Tr}(A^2))$ for 2×2 matrices, one gets easily that e_3 has norm one.

In three Euclidean dimensions, a spinor and its conjugate form a (pointwise) basis of the space of spinors. For example, χ_0 and χ_0^c are a basis. We can then expand $\tilde{\chi}_0$ on this basis. Actually, its projection on χ_0 vanishes, due to (3.1.16): $\chi_0^\dagger \tilde{\chi}_0 = \frac{i}{4}(\chi_1^\dagger \chi_2 + \chi_2^\dagger \chi_1) = 0$. With a few more steps we get

$$\tilde{\chi}_0 = \frac{\chi_0^{c\dagger} \tilde{\chi}_0}{\|\chi_0\|^2} \chi_0^c = \tan\left(\frac{\psi}{2}\right) \chi_0^c . \quad (3.1.18)$$

We can now invert (3.1.17) for χ_1 and χ_2 , and use (3.1.18). It is actually more symmetric-looking to define $\chi_0 \equiv \cos\left(\frac{\psi}{2}\right) \chi$, to get

$$\chi_1 = \cos\left(\frac{\psi}{2}\right) \chi + \sin\left(\frac{\psi}{2}\right) \chi^c , \quad \chi_2 = i \left(\cos\left(\frac{\psi}{2}\right) \chi - \sin\left(\frac{\psi}{2}\right) \chi^c \right) . \quad (3.1.19)$$

We have thus obtained a parameterization of two spinors χ_1 and χ_2 in terms of a single spinor χ and of an angle ψ . Let us count our parameters, to see if our result makes sense. A spinor χ of norm 1 accounts for 3 real parameters; ψ is one more. We should also recall we have rotated both $\chi_{1,2}$ by a phase at the beginning of our computation, to make things easier. We have a grand total of 6 real parameters, which is correct for two spinors of norm 1 in three dimensions.

We can now use the parameterization (3.1.19), and the bilinears (3.1.10), (3.1.12) previously obtained. As a result we get:

$$\chi_1 \otimes \chi_2^\dagger = -\frac{i}{2} [e_3 - i \sin(\psi) e_2 - i \cos(\psi) \text{vol}_3] . \quad (3.1.20)$$

A computation along these lines allows us to evaluate $\chi_1 \otimes \overline{\chi_2}$ as well. We can also reinstate at this point the phases of χ_1 and χ_2 , absorbing the overall factor $-i$. The bilinear in (3.1.20) is expressed as an odd form, but we can also express it as a mixed even/odd form. Recalling the definition (3.1.6), we get:

$$\psi^1 = \frac{e^{i\varphi_1}}{2} (\cos(\psi) + e_3) , \quad \psi^2 = \frac{e^{i\varphi_2}}{2} (\sin(\psi) + e_1 + i \cos(\psi) e_2) , \quad (3.1.21)$$

where the two- and three-form components are obtained via Hodge duality, Eq. (3.1.9). Notice that the normalization condition (3.1.7f) is satisfied automatically.

Armed with this parameterization, we will now attack the system (3.1.7) for $\text{AdS}_7 \times M_3$ solutions.

3.2 Constrained geometry on M_3

In the previous section, we have obtained the system (3.1.7), equivalent to supersymmetry for $\text{AdS}_7 \times M_3$ solutions. The $\psi_{\pm}^{1,2}$ appearing in that system

are not just any forms; they should have the property that they can be written as bispinors as in (3.1.6). Also, we have obtained a parameterization for the most general set of $\psi_{\pm}^{1,2}$ that fulfills that constraint; it is (3.1.21), where $\{e_i\}$ is a vielbein.

Thus we can now use (3.1.21) into the differential system (3.1.7), and explore its consequences.

3.2.1 Metric and Fluxes

We will start by looking at the equations in (3.1.7) that do not involve any fluxes. These are (3.1.7e), and the lowest-component form part of (3.1.7a), (3.1.7b) and (3.1.7c).

First of all, we can observe quite quickly that the IIB case cannot possibly work. (3.1.7a), (3.1.7b) and (3.1.7c) all have a zero-form part coming from their right-hand side, which, using (3.1.21), read respectively

$$\cos(\psi) \cos(\varphi_1) = 0, \quad \cos(\psi) \sin(\varphi_1) = 0, \quad \sin(\psi) e^{i\varphi_2} = 0. \quad (3.2.22)$$

These cannot be satisfied for any choice of ψ , φ_1 and φ_2 . Thus we can already exclude the IIB case.²

Having disposed of IIB so quickly, we will devote the rest of the analysis to IIA. Actually, we already know that we can get something new only with non-zero Romans mass, $F_0 \neq 0$. This is because for $F_0 = 0$ we can lift to an eleven-dimensional supergravity solution $\text{AdS}_7 \times N_4$. There, we only have a four-form flux G_4 at our disposal, and the only way not to break the $\text{SO}(6,2)$ invariance of AdS_7 is to switch it on along the internal four-manifold N_4 . This is the Freund–Rubin Ansatz, which requires N_4 to admit a Killing spinor. This means that the cone $C(N_4)$ over N_4 admits a covariantly constant spinor; but in five dimensions the only manifold with restricted holonomy is \mathbb{R}^5 (or one of its orbifolds, of the form $\mathbb{R}^4/\Gamma \times \mathbb{R}$). Thus we know already that all solutions with $F_0 = 0$ lift to $\text{AdS}_7 \times S^4$ (or $\text{AdS}_7 \times S^4/\Gamma$) in eleven dimensions. We will thus focus on $F_0 \neq 0$, and use the case $F_0 = 0$ as a control.

In IIA, the lowest-degree equations of (3.1.7a), (3.1.7b) and (3.1.7c) are one-forms; they are less dramatic than (3.2.22), but still rather interesting. Using (3.1.21), after some manipulations we get

$$\begin{aligned} e_1 &= -\frac{1}{4} e^A \sin(\psi) d\varphi_2, & e_2 &= \frac{1}{4} e^A (d\psi + \tan(\psi) d(5A - \phi)), \\ e_3 &= \frac{1}{4} e^A \left(-\cos(\psi) d\varphi_1 + \frac{\cot(\varphi_1)}{\cos(\psi)} d(5A - \phi) \right). \end{aligned} \quad (3.2.23)$$

²This quick death is reminiscent of the fate of $\text{AdS}_4 \times M_6$ with $\text{SU}(3)$ structure in IIB. The system in [50] has a zero-form equation and two-form equation coming from the right-hand side of its fluxless equation, which look like $\cos(\theta) = 0 = \sin(\theta)J$, where θ is an angle similar to ψ in (3.1.21). This is consistent with a no-go found with lengthier computations in [59].

Notice that (3.2.23) determine the vielbein. Usually (i.e. in other dimensions), the geometrical part of the differential system coming from supersymmetry gives the *derivative* of the forms defining the metric. In this case, the forms themselves are determined in terms of derivatives of the angles appearing in our parameterizations. This will allow us to give a more complete and concrete classification than is usually possible.

The metric $ds_{M_3}^2 = e_a e_a$, following from (3.2.23) looks quite complicated. However, it simplifies enormously after a proper redefinition of the variables. We first trade the two angles (ψ, φ_1) for (x, θ) as:

$$\sin \theta \equiv \frac{\sin \psi}{\sqrt{1-x^2}}, \quad x \equiv \cos(\psi) \sin(\varphi_1). \quad (3.2.24)$$

We then introduce a new coordinate r , defined as in terms of the warping function and the dilaton as: $dr = 4e^A \frac{\sqrt{1-x^2}}{4x - e^{A+\phi} F_0} dA$, so that the metric now reads

$$ds_{M_3}^2 = dr^2 + \frac{1}{16} e^{2A} (1-x^2) ds_{S^2}^2, \quad ds_{S^2}^2 = d\theta^2 + \sin^2(\theta) d\varphi^2. \quad (3.2.25)$$

Without any Ansatz, the metric has taken the form of a fibration of a round S^2 , with coordinates $\{\theta, \varphi = \pi - \varphi_2\}$, over an interval with coordinate r which has been defined in such a way that $g_{rr} = 1$. In other words, r measures the distance along the base of the S^2 fibration. All the remaining functions: A , x and ϕ have become functions of r .

Notice that none of the scalars appearing in (3.2.25) were originally intended as coordinates, but rather as functions in the parameterization of the pure spinors $\psi^{1,2}$. Usually, one would then need to introduce coordinates independently, and to make an Ansatz about how all functions should depend on those coordinates, sometimes imposing the presence of some particular isometry group in the process. Here, on the other hand, the functions we have introduced are suggesting themselves as coordinates to us rather automatically.

It is not hard to understand why this S^2 has emerged. The holographic dual of any solutions we might find is a (1,0) CFT in six dimensions. Such a theory would have SU(2) R-symmetry; an SU(2) isometry group should then appear naturally on the gravity side as well. This is what we are seeing in (3.2.25).

The fact that the S^2 in (3.2.25) is rotated by R-symmetry also helps to explain a possible puzzle about IIB. Often, given a IIA solution, one can produce a IIB one via T-duality along an isometry. All the Killing vectors of the S^2 in (3.2.25) vanish in two points; T-dualizing along any such direction would produce a non-compact solution in IIB, but still a valid one. But the IIB case died very quickly; there are no solutions, not even non-compact or singular ones. Here is how this puzzle is resolved. Since the SU(2) isometry group of the S^2 is an R-symmetry, supercharges transform as a doublet under

it (we will see this more explicitly in subsection 3.2.2). Thus even the strange IIB geometry produced by T-duality along a U(1) isometry of S^2 would not be supersymmetric.

Before we proceed in our analysis, some further constraints are coming from the purely geometrical equations. These involve the differentials of the functions x, A, ϕ . They can be summarized in the following two equations

$$x dx = (1 + x^2)d\phi - (5 + x^2)dA, \quad d\phi \wedge dA = 0, \quad (3.2.26)$$

where dA can also be expressed in terms of the new radial coordinate r in the same way we did to rewrite the local metric in (3.2.25). These two equations imply that both x and ϕ are functionally dependent on A , and thus on r .

So far we have analyzed the purely geometrical equations. We still have to look at equation (3.1.7d), which gives us the RR flux. First we compute F_0 , which gives

$$F_0 = 4xe^{-A-\phi} \frac{3 - \partial_A \phi}{5 - 2x^2 - \partial_A \phi}. \quad (3.2.27)$$

The Bianchi identity for F_0 says that it should be (piecewise) constant. It will thus be convenient to use this equation to eliminate $\partial_A \phi$ from our equations.

We move to the two-form flux F_2 , which is determined by the supersymmetry variations to be:

$$F_2 = \frac{1}{4} \sqrt{1 - x^2} e^{A-\phi} \left(-1 + \frac{F_0}{4} x e^{A+\phi} \right) \text{vol}_{S^2}, \quad (3.2.28)$$

where

$$\text{vol}_{S^2} = \sin(\theta) d\theta \wedge d\varphi \quad (3.2.29)$$

is the only two-form which is compatible with the SU(2) symmetry that naturally emerged in the metric (3.2.25).

For later purposes, it is useful to give a definition to the coefficients of the two-form flux, isolating the dependence on the Romans mass:

$$F_2 \equiv (p - qF_0) \text{vol}_{S^2}. \quad (3.2.30)$$

The function q will play a crucial role in the discussion of flux quantization. Moreover, we will see that the supersymmetry variations will eventually reduce to a single elementary differential equation for q . Nicely, q also has a nice geometrical interpretation:

$$q \equiv \frac{1}{4} e^A \sqrt{1 - x^2} = e^{-\phi} \text{radius}(S^2). \quad (3.2.31)$$

Finally, let us look at the three-form part of (3.1.7a), (3.1.7b) and (3.1.7c). One of them can be used to determine H :

$$H = -(6e^{-A} + xF_0 e^\phi) \text{vol}_{M_3}, \quad (3.2.32)$$

where vol_{M_3} is the volume form of the metric $ds_{M_3}^2$ in (3.2.25).

Our analysis of the fluxes is not over: we should of course now impose the equation of motion and the Bianchi identities. The equation of motion for F_2 , $d * F_2 + H * F_0 = 0$, follows automatically from (3.1.7d), much as it happens in the pure spinor system for $\text{AdS}_4 \times M_6$ solutions [50]. We should then impose the Bianchi identity for F_2 , which reads $dF_2 - HF_0 = 0$ (away from sources). This does not follow manifestly from (3.1.7d), but in fact it is a consequence of the explicit expressions (3.2.27, 3.2.28) and (3.2.32) above. When $F_0 \neq 0$, it also implies that the B field such that $H = dB$ can be locally written as

$$B_2 = \frac{F_2}{F_0} + b, \quad (3.2.33)$$

for a closed two-form b . As we show in the next section, using a gauge transformation, it can be assumed to be proportional (by a constant) to vol_{S^2} ; we then have that it is a constant, $\partial_r b = 0$. The equation of motion for H , which reads for us $d(e^{7A-2\phi} *_3 H) = e^{7A} F_0 *_3 F_2$ (again away from sources), is also automatically satisfied, as shown in general in [60].

Let us now sum up the results of our analysis of (3.1.7). Most of the supersymmetry equations determined some fields: the vielbein (3.2.23) and the fluxes (3.2.28), (3.2.32). There are still two genuine differential equations to be solved to obtain a supersymmetric solution, coming from (3.2.26, 3.2.27). These constraints can be rewritten as a coupled system of ordinary differential equations on the radial coordinate r . We will present and solve this system analytically in the next section.

Right now we would like to understand more clearly the induced geometry on the internal space. We start with a small detour to see how the R-symmetry $\text{SU}(2)$ emerges in the pure spinors $\psi^{1,2}$.

3.2.2 Spinors

We have just seen that the metric takes the particularly simple form (3.2.25) in coordinates (r, β, θ_2) ; the appearance of the S^2 is related to the $\text{SU}(2)$ R-symmetry group of the $(1, 0)$ holographic dual.

Since these coordinates are so successful with the metric, let us see whether they also simplify the pure spinors $\psi^{1,2}$. We can start by the zero-form parts of (3.1.21), which read

$$\psi_0^1 = ix + \sqrt{1-x^2} \cos(\theta), \quad \psi_0^2 = \sqrt{1-x^2} \sin(\theta) e^{i\varphi}. \quad (3.2.34)$$

Recalling that (θ, φ) are the polar coordinates on the S^2 (see the expression of $ds_{S^2}^2$ in (3.2.25)), we recognize in (3.2.34) the appearance of the $\ell = 1$ spherical harmonics

$$y^i = \{\sin(\theta) \cos(\varphi), \sin(\theta) \sin(\varphi), \cos(\theta)\}. \quad (3.2.35)$$

Notice that y^3 appears in $\psi^1 = \chi_1 \otimes \chi_2^\dagger$, while $y^1 + iy^2$ appears in $\psi^2 = \chi_1 \otimes \chi_2^{c\dagger}$. This suggests that we introduce a 2×2 matrix of bispinors. From

(3.1.3) we see that for IIA $\begin{pmatrix} \chi_1 \\ \chi_1^c \end{pmatrix}$ and $\begin{pmatrix} \chi_2 \\ -\chi_2^c \end{pmatrix}$ are both $SU(2)$ doublets, so that it is natural to define

$$\Psi = \begin{pmatrix} \chi_1 \\ \chi_1^c \end{pmatrix} \otimes (\chi_2^\dagger, -\chi_2^{c\dagger}) = \begin{pmatrix} \psi^1 & \psi^2 \\ -(-)^{\deg(\psi^2)*} & -(-)^{\deg(\psi^1)*} \end{pmatrix}, \quad (3.2.36)$$

where $(-)^{\deg}$ acts as \pm on a even (odd) form. The even form part can then be written as

$$\Psi_+^{ab} = i\text{Im}\psi_+^1 \text{Id}_2 + (\text{Re}\psi_+^2 \sigma_1 - \text{Im}\psi_+^2 \sigma_2 + \text{Re}\psi_+^1 \sigma_3), \quad (3.2.37a)$$

where σ_α are the Pauli matrices while the odd form part is

$$\Psi_-^{ab} = \text{Re}\psi_-^1 \text{Id}_2 + i(\text{Im}\psi_-^2 \sigma_1 + \text{Re}\psi_-^2 \sigma_2 + \text{Im}\psi_-^1 \sigma_3). \quad (3.2.37b)$$

(3.2.37) shows more explicitly how the R-symmetry $SU(2)$ acts on the bispinors Ψ^{ab} , which split between a singlet and a triplet. If we go back to our original system (3.1.7), we see that (3.1.7a), (3.1.7d), (3.1.7e) each behave as a singlet, while (3.1.7b), (3.1.7c) behave as a triplet — thanks also to the fact that the factor $e^{5A-\phi}$ appears in both those equations.

More concretely, (3.2.34) can now be written as

$$\Psi_0^{ab} = ix \text{Id}_2 + \sqrt{1-x^2} y^i \sigma_i; \quad (3.2.38a)$$

the one-form part reads

$$\Psi_1^{ab} = \sqrt{1-x^2} dr \text{Id}_2 + i \left[xy^i dr + \frac{1}{4} e^A \sqrt{1-x^2} dy^i \right] \sigma_i. \quad (3.2.38b)$$

The rest of Ψ^{ab} can be determined by (3.1.9): $\Psi_3^{ab} = -i*_3 \Psi_0^{ab} = -i\Psi_0^{ab} \text{vol}_3$, $\Psi_2^{ab} = -i*_3 \Psi_1^{ab}$. (The three-dimensional Hodge star can be easily computed from (3.2.25).)

So the $SU(2)$ symmetry is also manifest at the level of the pure spinors, once the geometrical constraints coming from the supersymmetry conditions have been imposed. In particular, the matrix Ψ is manifestly invariant under this symmetry. Going back to the definition (3.2.36), this implies that the symmetry acts on the doublets $(\chi_1, \chi_1^c)^t$ and $(\chi_2, -\chi_2^c)^t$ in the fundamental representation. From Eq. (3.1.3), we deduce that $(\xi, \xi^c)^t$ are also a doublet under the R-symmetry, in such a way to make the ten-dimensional spinors (ϵ^1, ϵ^2) invariant.

Having fixed the transformation properties of the full spinor Ansatz, we can now go further. Comparing the expression of the pure spinors on M_3 fixed by supersymmetry (3.2.38) and their definition (3.1.6) in terms of the basis spinors, we can a posteriori determine χ^1, χ^2 explicitly. We choose the following representation of the three-dimensional gamma matrices: $\gamma^1 = \sigma^3$, $\gamma^2 = \sigma^1$, $\gamma^3 = \sigma^2$. This choice is also motivated by the reduction of the

massless solution $\text{AdS}_7 \times S^4$ from eleven dimension, as described in App. B. With this choice, we get:

$$\chi_1 = -ie^{-i\frac{\pi}{2}\sigma_3} e^{i\frac{\alpha}{2}\sigma^3} \chi_{S^2}, \quad \chi_2 = e^{-i\frac{\alpha}{2}\sigma^3} \chi_{S^2}, \quad (3.2.39)$$

where we have introduced a new angle: $\alpha \equiv \arcsin x$. Perhaps not surprisingly, the two spinors turned out to be proportional to the S^2 Killing spinor, up to a unitary transformation that depends on the radial coordinate only. The Killing spinor on S^2 can be written as

$$\chi_{S^2} = e^{\frac{i\theta}{2}\sigma^1} e^{\frac{\varphi}{2}\sigma^{12}} \chi_0, \quad (3.2.40)$$

for some constant spinor χ_0 . It satisfies the Killing spinor equation on the two-sphere $\nabla_\mu \chi_{S^2} = \frac{i}{2} \gamma_\mu \chi_{S^2}$, with $\gamma_1 = \sigma_1, \gamma_2 = \sigma_2$ in flat indices, and the covariant derivative is defined with respect to the round metric in (3.2.25).

Nicely, in App. B we show that the explicit expression for the spinors of the massless solution is the same as the one we got here. In the massless case the angle α has a natural interpretation as coordinate parametrizing a circle S^1 inside S^4 in hyperspherical coordinates, Eq. (A.1.2).

3.2.3 Topology

We saw that the internal manifold M_3 is constrained by supersymmetry to be an S^2 fibration over an interval (3.2.25). The $\text{SU}(2)$ isometry group of the S^2 is to be identified holographically with the R-symmetry group of the $(1,0)$ -superconformal dual theory. For holographic applications, we would actually like to know whether the total space of the S^2 -fibration can be made compact.

A natural way of compactifying it would be making the two-sphere shrink at the two end points of the interval, say r_N and r_S ; the topology³ of M_3 would then be S^3 . The next question to be asked is how should the sphere shrink. In absence of sources, it has to shrink in a regular way, that is to say with a local behavior for the metric around $r \sim r_N$ of the type:

$$ds_{M_3}^2 \sim d\rho^2 + \rho^2 ds_{S^2}^2, \quad (3.2.41)$$

for ρ a function of $(r - r_N)$. In other words, a regular solution behaves locally like \mathbb{R}^3 near the pole.

Another possibility is to allow for sources located at the two poles, which would result in a physical singularity in the metric. Once again, the possible sources and thereby the ways the sphere can shrink are dictated by supersymmetry. We already mentioned in the introduction that the only sources that are compatible with supersymmetry are Dp -branes. Moreover, for an $\text{AdS}_7 \times M_3$ spacetime the sources cannot break either the AdS_7 isometries

³Another possible strategy would be for r to be periodically identified, so that the topology of M_3 would become $S^1 \times S^2$. As shown in [4, Eq.(4.24)], this is actually impossible as a consequence of the the supersymmetry equations.

or the S^2 ones. In type IIA this leaves us with two possibilities: a D6-brane extending along AdS_7 or a D8-brane extending along AdS_7 and wrapping the two-sphere.

In the case of a D6-brane, the local behavior for the metric around a pole can be deduced from the corresponding solution in flat space in Eq. (2.0.8) for $p = 6$. Around $\rho = 0$ the internal metric has the following behavior:

$$ds_{M_3}^2 \sim \rho^{-1/2}(d\rho^2 + \rho^2 ds_{S^2}^2) . \quad (3.2.42)$$

This is a first possibility for a singular behavior at poles in presence of a D6-brane.

Actually there is a further possible type of source that we haven't mentioned so far which is compatible with supersymmetry: the *orientifold plane*. An Op -plane is defined in string theory as fixed locus of a certain involution (a map whose square is the identity). Op -planes are not dynamical (unlike D-branes), and have a negative charge: $\text{charge}(Op) = -2^{p-5}\text{charge}(Dp)$. This makes them, in a sense, sources of anti-gravity.

The metric for an Op -plane can be obtained in a simple way from the metric for a Dp -brane. We are interested in O6-planes, the only ones that are compatible with the spacetime symmetry in the case of AdS_7 . The metric describing an O6 in flat space is very similar to that of a D6-brane (2.0.8), we just need to modify it to $H(\rho) = 1 - \frac{\rho_0}{\rho}$.

The function H has to be positive, so the O6 solution is defined only for $\rho > \rho_0$, ρ_0 being the point where the plane is located. Expanding around $\rho \sim \rho_0$ we get the following local behavior for the internal metric:

$$ds_{M_3}^2 \sim \tilde{\rho}^{1/2}(d\tilde{\rho}^2 + \rho_0^2 ds_{S^2}^2) , \quad (3.2.43)$$

where $\tilde{\rho} = \rho - \rho_0$. This is a second possible singular behavior at poles with a corresponding physical interpretation.

3.2.4 Quantization

For completeness we also have to look at the quantization conditions for the fluxes, which give further constraints. A D6-brane is a magnetic source for the two-form flux, so it makes sense to measure the induced charge integrating F_2 on a sphere at the pole surrounding the source as:

$$n_2 = \frac{1}{2\pi} \int_{S^2} \tilde{F}_2 , \quad (3.2.44)$$

where we have introduced the modified curvature: $\tilde{F}_2 \equiv F_2 - BF_0$. This formula can be seen as a quantization condition to be imposed on the local expression for the fluxes (3.2.28), some kind of generalization of a Dirac quantization condition for the magnetic monopole. If the integral is taken around the source the integer n_2 represents the number of D6-branes. The

quantization condition in presence of an O6-plane is Eq. (3.2.44) with $n_2 = -2$.

As we are looking for solutions with non-vanishing Romans mass, an analogous quantization condition needs to be imposed on this (piecewise) constant zero-form flux:

$$n_0 = 2\pi F_0 . \quad (3.2.45)$$

Finally, we saw that supersymmetry in AdS₇ also requires the presence of a non-vanishing three-form flux H , given by Eq. (3.2.32). Similarly to F_2 and F_0 , this flux obeys its own quantization condition which can be written as:

$$N = -\frac{1}{4\pi^2} \int_{M_3} H . \quad (3.2.46)$$

What is the interpretation of the integer N in terms of branes? As we already mentioned in the introduction, the solutions that we are about to present are dual to the (1,0) theories. In the AdS/CFT setup, these arise from brane configurations involving D6, D8 and NS5-branes, engineered according to [22]. What happens is that in the near horizon limit the NS5-branes disappear, leaving as a trace a non-vanishing three-form flux H . The flux integer N has thus a clear interpretation before the near horizon geometry as the number of coincident NS5-branes on which the D6 can end.

Actually, N is related to the number of D6-branes by a simple formula that can be derived from the Bianchi identities for F_2 , which in our situation read $dF_2 - HF_0 = \delta_{D6}$. Integrating this gives

$$n_2 = N n_0 , \quad (3.2.47)$$

which gives a general constraint for a brane configuration consisting of n_2 D6-branes ending on N NS5-branes.

Introducing D8-branes

Things get more involved if we also allow for the presence of D8-branes. As we mentioned, these are somehow special since they couple to a non-dynamical field like F_0 . Also, a stack of D8-branes can carry a worldvolume gauge field-strength f_2 (not to be confused with the RR field-strength F_2), which induces a D6-brane charge distribution on it. The field f_2 also obeys a quantization condition:

$$\mu = \frac{1}{2\pi} \int_{S^2} f_2 , \quad (3.2.48)$$

where μ is interpreted as the D6-brane charge induced on the worldvolume of the D8-brane; this is the Chern class of a gauge bundle, and as such it is an integer. D8-branes with the same μ will be stabilized by supersymmetry on top of each other.

The Bianchi identity for the RR fluxes in presence of a D6-D8 bound state reads $d\tilde{F} = \frac{1}{2\pi} n_8 e^{2\pi f_2} \delta_{D8}$, where $\tilde{F} = F_0 + \tilde{F}_2$ and n_8 is the number of

D8-branes in the stack. The zero-form part of this equation can be rewritten as: $dF_0 = \frac{1}{2\pi}n_8\delta_{D8}$, meaning that F_0 is piecewise constant and it can jump in the points where the D8's are located. Integrating across the stack gives:

$$\Delta n_0 = n_8 . \quad (3.2.49)$$

Moreover, integrating the two-form part of the Bianchi identity across the D8's gives $\Delta\tilde{F}_2 = n_8 f_2$. Taking into account (3.2.44, 3.2.48), this equation can be rewritten as the following constraints between the flux integers before and after the stack:

$$\Delta n_2 = \mu\Delta n_0 . \quad (3.2.50)$$

This is not yet enough to ensure that the full solution has a well defined behavior in the presence of D8-branes. Since learned that F_0 can jump according to (3.2.49), we have to make sure that all the functions entering the metric are continuous together with their first derivative. Discontinuities are only allowed in the second derivatives.

Imposing that the B field does not jump is trickier. First, recall that it can be written as $B = \frac{F_2}{F_0} + b$ as in (3.2.33), when $F_0 \neq 0$. Combining this with the flux quantization condition (3.2.44) we get:

$$b = -\frac{n_2}{2F_0}\text{vol}_{S^2} . \quad (3.2.51)$$

It is clear that such a term can jump across a D8 stack. Let us consider the F_2/F_0 term. Looking at the expression (3.2.28) for F_2 determined by the supersymmetry equations, we can rewrite the B -field as:

$$B_2 = \left(p - \frac{q}{F_0} - \frac{n_2}{2F_0} \right) \text{vol}_{S^2} . \quad (3.2.52)$$

Let us call (n_0, n_2) the flux integers on one side of the D8 stack, and $(\tilde{n}_0, \tilde{n}_2)$ the fluxes on the other side. Let us at first assume that both n_0 and \tilde{n}_0 are non-zero. Then, equating B on the two sides, we see that p cancels out, and we get the following constraint:

$$q|_{r=r_{D8}} = \frac{\tilde{n}_2 n_0 - n_2 \tilde{n}_0}{2(\tilde{n}_0 - n_0)} . \quad (3.2.53)$$

Here we understood how to impose continuity of the B -field. In general it would actually be allowed to jump by a gauge transformation. Indeed, B is not technically a two-form, but a 'connection on a gerbe', in the sense that it transforms non-trivially on chart intersections: on $U \cap U'$, $B_U - B_{U'}$ can be a 'small' gauge transformation $d\lambda$, for λ a 1-form, or more generally a 'large' gauge transformation, namely a two-form whose periods are integer multiples of $4\pi^2$. In our case, if we cover S^3 with two patches U_N and U_S , around the equator we can have $B_N - B_S = N\pi\text{vol}_{S^2}$. In this case

$\int_{S^3} H = B_N - B_S = N\pi \text{vol}_{S^2} = (4\pi^2)N$, in agreement with flux quantization for H . One way of taking care of this is to define

$$\hat{b}(r) \equiv \frac{1}{4\pi} \int_{S_r^2} B_2, \quad (3.2.54)$$

which has period π . Large gauge transformations can then be written as

$$\hat{b} \rightarrow \hat{b} + k\pi. \quad (3.2.55)$$

It is clear that the flux integer n_2 also transforms under large gauge transformation. According to equation (3.2.51) we get:

$$n_2 \rightarrow n_2 - kn_0. \quad (3.2.56)$$

Nicely, the constraint (3.2.53) is invariant under large gauge transformation. A practical use of this type of transformation is that we can use it to set to zero b where this is needed, for example when we have a regular pole. In this case the term F_2/F_0 is clearly regular as it is, without the addition of b ; this suggests that one should set $b = 0$. To make this more precise, consider the limit

$$\lim_{r \rightarrow 0} \int_{\Delta_r} H = \lim_{r \rightarrow 0} \int_{S_r^2} B_2, \quad (3.2.57)$$

where Δ_r is a three-dimensional ball such that $\partial\Delta_r = S_r^2$. In correspondence of a regular pole the right hand side of this equation is equal to $\int_{S^2} b$, which is constant. This constant signals a delta in H . We are thus forced to perform a large gauge transformation that takes $b \rightarrow 0$, which also implies from (3.2.51) that $n_2 \rightarrow 0$. Everything is thus consistent, since there are actually no D6-branes located at a regular pole.

Since (3.2.53) was found by imposing that B should be continuous, it looks easy to impose the condition on flux quantization. However, in presence of D8's one might encounter a region where $F_0 = 0$; generically such a region will exist. In such a region, (3.2.33) cannot be used; we have to resort to the expression for the B -field of the massless solution. This allows to write a general expression for the integral of H , as shown in [35, Eq.(4.7)].

$$N \equiv -\frac{1}{4\pi^2} \int H = (|\mu_n| + |\mu_{n+1}|) + \frac{1}{4\pi} e^{2A(x=0)} (|x_n| + |x_{n+1}|), \quad (3.2.58)$$

where x_n and x_{n+1} are the values of x_7 at the branes $D8_n$ and $D8_{n+1}$.

3.3 Analytic Solutions

We saw in the last section how strong the constraints imposed by supersymmetry are on the geometry of the internal manifold. Indeed, without any Ansatz the internal space M_3 is determined to be a fibration of a round S^2

over an interval with coordinate r , as defined by Eq. (3.2.25). What is left to be determined is the r -dependence of the functions entering the metric: the dilaton ϕ , the warping function A and the variable x . This is described by the following set of coupled ordinary differential equations (ODE's):

$$\begin{aligned}\partial_r \phi &= \frac{1}{4} \frac{e^{-A}}{\sqrt{1-x^2}} (12x + (2x^2 - 5)F_0 e^{A+\phi}) , \\ \partial_r x &= -\frac{1}{2} e^{-A} \sqrt{1-x^2} (4 + xF_0 e^{A+\phi}) , \\ \partial_r A &= \frac{1}{4} \frac{e^{-A}}{\sqrt{1-x^2}} (4x - F_0 e^{A+\phi}) .\end{aligned}\tag{3.3.59}$$

The existence of a supersymmetric solution of the form $\text{AdS}_7 \times M_3$ in IIA is reduced to solving this system. More precisely these equations can be derived from the requirement of F_0 being a constant, where the expression for F_0 is (3.2.27), plus the algebraic constraint between the differentials, (3.2.26). A third condition is given by the definition of the r coordinate in terms of the warping factor A , which was necessary to rewrite the internal metric as in (3.2.25). Combining these three conditions one gets to the system of ODE's (3.3.59).

Providing an analytic solutions to the system (3.3.59) required a long detour. At first this was studied numerically in [1], and a holographic interpretation to its solutions was given in [35]. Later on it was shown that the system of ODE's for an AdS_7 vacuum is in one to one correspondence with an analogous system derived in [4, Eq.(5.15)] for a class of AdS_5 solutions in type IIA supergravity. The latter was solved analytically, so analytic solutions were pulled back to AdS_7 as well. Actually, this correspondence shows the AdS_5 solutions of [4] can be interpreted as compactifications the AdS_7 solutions that we are discussing here. The map between the two systems is given by:

$$\begin{aligned}e^{\phi_7} &= \left(\frac{3}{4}\right)^{1/4} \frac{e^{\phi_5}}{\sqrt{1 - \frac{1}{4}x_5^2}} , & e^{A_7} &= \left(\frac{4}{3}\right)^{3/4} e^{A_5} , \\ x_7 &= \left(\frac{3}{4}\right)^{1/2} \frac{x_5}{\sqrt{1 - \frac{1}{4}x_5^2}} , & r_7 &= \left(\frac{4}{3}\right)^{1/4} r_5 .\end{aligned}\tag{3.3.60}$$

We can translate this map into words by saying that to any solution $\{\phi_5, x_5, A_5\}$ one can associate a solution $\{\phi_7, x_7, A_7\}$, where it is understood that the indices 7 and 5 label the dimension of the corresponding AdS factors. Remarkably, the AdS_5 system was reduced to a single second order ODE of the following form:

$$\partial_y(q^2) = \frac{2}{9} F_0 , \quad q \equiv -\frac{4y\sqrt{\beta}}{\partial_y \beta} .\tag{3.3.61}$$

The function q turns out to be the same function that we have defined in (3.2.31), with a clear geometric interpretation and a crucial role in the discussion of flux quantization. We have also introduced a new radial coordinate y , defined as $dy = \sqrt{\beta} e^{-3A_5} dr_5 = \left(\frac{4}{3}\right)^2 \sqrt{\beta} e^{-3A_7} dr_7$.

Notably, the differential equation (3.3.61) can be solved analytically in an elementary way for q^2 a linear function of $\frac{2}{9}F_0(y - y_0)$. The functions (ϕ_7, A_7, x_7) are then determined in terms of q through β in the following way:

$$e^{A_7} = \frac{2}{3} \left(-\frac{\beta'}{y}\right)^{1/4}, \quad x_7 = \sqrt{\frac{-y\beta'}{4\beta - y\beta'}}, \quad e^{\phi_7} = \frac{(-\beta'/y)^{5/4}}{12\sqrt{4\beta - y\beta'}}. \quad (3.3.62)$$

From now on we drop the index labeling the dimension of the AdS factor, and we go back to the original notation where (ϕ, A, x) are the functions entering the AdS₇ system (3.3.59). We are finally ready to present a fully general analytic solution to the supersymmetry equations for an AdS₇ × M₃ solution in type IIA supergravity.

3.3.1 General solution

Given a solution for β to the second order ODE, (3.3.61), all the functions entering the system (3.3.59) are also determined as (3.3.62). The AdS₇ metric itself can be rewritten in terms of β in a quite simple way:

$$ds_{10}^2 = \frac{4}{9} \sqrt{-\frac{\beta'}{y}} \left(ds_{AdS_7}^2 - \frac{1}{16} \frac{\beta' dy^2}{y\beta} + \frac{\beta}{16\beta - 4y\beta'} ds_{S^2}^2 \right). \quad (3.3.63)$$

Analogously, it is possible to give an expression for the dilaton and fluxes in full generality in terms of the function β only. We get:

$$\begin{aligned} F_2 &= y \frac{\sqrt{\beta}}{\beta'} \left(4 - \frac{F_0}{18y} \frac{(\beta')^2}{4\beta - y\beta'} \right) \text{vol}_{S^2}, \\ H &= -9 \left(-\frac{y}{\beta'}\right)^{1/4} \left(1 + \frac{F_0}{108y} \frac{(\beta')^2}{4\beta - y\beta'} \right) \text{vol}_{M_3}, \\ e^\phi &= \frac{(-\beta'/y)^{5/4}}{12\sqrt{4\beta - y\beta'}}. \end{aligned} \quad (3.3.64)$$

The simplest analytic solution to equation (3.3.61) is the one with $F_0 = 0$. As we show in the next subsection, the metric associated to this solution through (3.3.63) is the reduction of the AdS₇ × S₄/Z_k solution of eleven-dimensional supergravity. It has a stack of k D6-branes at $y = y_0$ and a stack of k anti-D6-branes at $y = -y_0$, both in correspondence of a double zero of the function β , which reads:

$$\beta = \frac{2}{k} (y^2 - y_0^2)^2. \quad (3.3.65)$$

Infinite more brane configurations are possible if we turn on a non-vanishing Romans mass $F_0 \neq 0$. Remarkably, it is possible to provide a fully general solution for β , depending on two parameters.

An accurate study in [4] shows that the behavior of the solution is well understood in terms of b_2 , a parameter related to the second derivative of the function β in y_0 . The second parameter is y_0 itself, which we will assume to be positive. The general solution can be written analytically as the following polynomial expression in y :

$$\beta = \frac{y_0^3}{b_2^3 F_0} \left(\sqrt{\hat{y}} - 6 \right)^2 \left(\hat{y} + 6\sqrt{\hat{y}} + 6b_2 - 72 \right)^2, \quad (3.3.66)$$

where we have defined $\hat{y} \equiv 2b_2 \left(\frac{y}{y_0} - 1 \right) + 36$. Expanding around $y \sim y_0$ we get $\beta \sim \frac{y_0}{F_0} b_2 (y - y_0)^2$, so we recognize b_2 as the second derivative of β in y_0 .

We can already understand what is going on in $y = y_0$. Plugging the expression for β in the ten-dimensional metric (3.3.63) and expanding around this point we get $ds_{M_3}^2 \sim \frac{dy^2}{\sqrt{y-y_0}} + (y-y_0)^{3/2} ds_{S^2}^2$. Upon defining $\rho = y - y_0$, we get precisely the local behavior corresponding to a D6 singularity, as defined by Eq. (3.2.42). So we learn that a double zero in β signals the presence of a D6 stack.

Nicely, the presence of a D6 stack in $y = y_0$ is a feature common to all of the massive solutions, which can then be classified according to the behavior at the second pole. Three distinct classes can be obtained varying the parameter b_2 :

- If $0 < b_2 < 12$, the solution is defined in the interval $y \in [y_1, y_0]$. The second pole is located at $y_1 = y_0 \left(\frac{27-2b_2-3\sqrt{81-2b_2}}{b_2} \right)$, with range $-y_0 < y_1 < -\frac{1}{2}y_0$. The function β has a double zero at both extrema, which means that the solutions within this class have two D6 stacks.

Unlike in the massless case, the number of D6 is not the same on the two sides. Let these numbers be (\tilde{n}_2, n_2) . We have to take into account the two constraints coming from the flux quantization condition (3.2.44) at both poles. These fix the parameters y_0 and b_2 in terms of the two integers in the following way:

$$y_0 = \frac{3n_2^2}{4F_0} \left(1 - \frac{\tilde{n}_2}{2n_2} - \frac{\tilde{n}_2^2}{2n_2^2} \right), \quad b_2 = 12 \left(1 - \frac{\tilde{n}_2}{2n_2} - \frac{\tilde{n}_2^2}{2n_2^2} \right). \quad (3.3.67)$$

Notice that these expressions are symmetric under $(\tilde{n}_2, n_2) \rightarrow (-\tilde{n}_2, -n_2)$. There are thus two possible ranges for the two flux integers that give rise to a meaningful solution, the first one given by:

$$\{n_2 > 0, -2n_2 < \tilde{n}_2 < -n_2 \cup 0 < \tilde{n}_2 < n_2\}.$$

The second possible range is obtained applying $(\tilde{n}_2, n_2) \rightarrow (-\tilde{n}_2, -n_2)$, that is to say considering the symmetric region with respect to the

origin in the plane parametrized by the two integers. We get:

$$\{n_2 < 0, n_2 < \tilde{n}_2 < 0 \cup -n_2 < \tilde{n}_2 < -2n_2\}.$$

Notice that in the massless case the only accessible region in the (\tilde{n}_2, n_2) plane was the line $\tilde{n}_2 = n_2$. Turning on a non-vanishing Romans mass allows to obtain infinite new solutions, so that a vast region in the plane (n_2, \tilde{n}_2) can now be filled (perhaps we should call it lattice instead of plane, since both n_2 and \tilde{n}_2 take integer values).

- If $12 < b_2 < 18$, the solution is defined for $y \in [y_1, y_0]$, where the second pole is located at $y_1 = y_0 \left(1 - \frac{18}{b_2}\right)$, with range $-\frac{1}{2}y_0 < y_1 < 0$. In this case we have a double zero at y_0 , and an O6 singularity (see (3.2.43)) at y_1 , corresponding to a local expansion of the form $\beta \sim \beta_0 + O(\sqrt{y - y_1})$.

We thus obtained a second class of solutions with one D6 stack at one end, and an O6 at the other extremum. Again we have to impose proper quantization conditions at poles, where the two-form field strength has flux integers $(-2, n_2)$. We get:

$$y_0 = \frac{3n_2^2}{4F_0} \left(1 + \frac{1}{1 + 2n_2}\right), \quad b_2 = 12 \left(1 + \frac{1}{1 + 2n_2}\right). \quad (3.3.68)$$

The allowed range for the flux integer is $n_2 \neq 0$.

- In the limiting case, $b_2 = 12$, the solution is defined for $y \in [-\frac{1}{2}y_0, y_0]$; in this case we have the usual double zero in $y = y_0$ and a single zero in $y = -\frac{1}{2}y_0$. The latter corresponds to a regular point, with local behavior for the metric defined by (3.2.41).

The solutions belonging to this class thus have only one stack of n_2 coincident D6-branes located at $y = y_0$, with a regular point on the other side. This results in the following quantization condition:

$$y_0 = \frac{3n_2^2}{4F_0}. \quad (3.3.69)$$

Notice that this constraint coincides with the more general one given in Eq. (3.3.67) for $\tilde{n}_2 = 0$. The allowed range for the flux integer is simply $n_2 \neq 0$.

Having concluded this preliminary analysis, we are now ready to start writing explicit solutions corresponding to these three classes. The nice result of the present analysis is that all possible AdS₇ solutions of massive type IIA supergravity are eventually classified by the values of two flux integers (\tilde{n}_2, n_2) , namely each solution corresponds to a point on a lattice.

3.3.2 Massless solution

It is straightforward to find a solution in the case where $F_0 = 0$. Indeed, a simple expression is available for β , given by Eq. (3.3.65). Plugging that expression into (3.3.63) gives the following internal space metric:

$$ds_{M_3}^2 = \frac{4}{9k} \frac{1}{\sqrt{y_0^2 - y^2}} \left(dy^2 + \frac{(y_0^2 - y^2)^2}{4y_0^2} ds_{S^2}^2 \right). \quad (3.3.70)$$

As we anticipated in the introduction, the solution with $F_0 = 0$ can be lifted to eleven dimensions to the $\text{AdS}_7 \times S^4/\mathbb{Z}_k$ corresponding to the near horizon geometry of a stack of M5-branes probing a $\mathbb{R}^5/\mathbb{Z}_k$ singularity. This can be seen applying a proper change of variable: $y = y_0 \cos \alpha$, provided the identification $y_0 = \frac{9}{32} R^3$. All in all, we get the following metric on the internal space:

$$ds_{M_3}^2 = \frac{R^3}{8k} \sin(\alpha) \left(d\alpha^2 + \frac{1}{4} \sin^2(\alpha) ds_{S^2}^2 \right). \quad (3.3.71)$$

As promised, this is nothing but the \mathbb{Z}_k quotient of the M5 near horizon geometry (2.0.21) reduced to ten dimensions.

At first sight Eq. (3.3.71) might appear problematic for two reasons. First of all, the warping function goes to zero at the two poles $\alpha = \{0, \pi\}$; second, the internal metric is singular at poles because of the $1/4$ factor in front of $ds_{S^2}^2$.

However these singularities can be interpreted physically as due to the presence of D6's and anti-D6's. To see this let us expand the solution around the pole $\alpha = 0$; we get: $ds_{M_3}^2 \sim \alpha(d\alpha^2 + \frac{\alpha^2}{4} ds_{S^2}^2)$, which after the change of coordinates $\alpha = \rho^{1/2}$ gives the same type of singular behavior of the D6-brane solution of Eq. (2.0.8).

The presence of D6's could actually be inferred more directly looking at the fluxes. Indeed, plugging the solution (3.3.65) into Eq. (3.3.64) we get:

$$F_2 = -\frac{1}{2} k \text{vol}_{S^2}, \quad H = -\frac{3}{32} \frac{R^3}{k} \sin^3(\alpha) d\alpha \wedge \text{vol}_{S^2}. \quad (3.3.72)$$

The integral of F_2 over the S^2 is constant and equal to $-2\pi k$. We can take the S^2 close to the north or the south pole, where it signals the presence of D6-brane charge. More precisely, there are k anti-D6-branes at the north pole and k D6-branes at the south pole, as represented in Fig. 2.1.

One crucial difference with the usual D6 behavior, however, is the presence of the three-form H . From (3.3.72) we see that it does not vanish near the D6. Rather, it diverges. Indeed, if we expand it around $\alpha = 0$, using the coordinate $\rho = \alpha^2$ that makes manifest the D6 behavior in the metric, we get: $H \sim \rho^{-\frac{1}{4}} \text{vol}_{M_3}$. We should remember, in any case, that this solution is non-singular in eleven dimensions; the diverging behavior is cured by M-theory, just like the divergence of the curvature at the poles, where the D6 stacks are located.

For completeness we also give the expression for the dilaton:

$$e^\phi = \left(\frac{R \sin \alpha}{2k} \right)^{3/2}. \quad (3.3.73)$$

Expanding around $\alpha = 0$ this behaves like $e^\phi \sim \rho^{3/4}$, which is precisely what we expected from the D6-brane solution in flat space of Eq. (2.0.8).

This solution, and its brane interpretation, is shown in Fig. 2.1.

3.3.3 One D6 stack

The simplest example of solution with $F_0 \neq 0$ is the one with a single D6 stack, corresponding to the general massive solution (3.3.66) for the choice $b_2 = 12$. The resulting solution is $\beta = \frac{4}{F_0}(y - y_0)^2(2y + y_0)$, which has a single zero at $y = -y_0/2$ and a double zero at $y = y_0$. The metric on the internal space takes the following simple form:

$$ds_{M_3}^2 = \frac{1}{\sqrt{3F_0}} \frac{1}{\sqrt{(y_0 + 2y)^2(y_0 - y)}} \left(dy^2 + \frac{1}{3} \frac{(y_0 - y)^2(y_0 + 2y)^2}{2y_0^2 + 2yy_0 - y^2} ds_{S^2}^2 \right), \quad (3.3.74)$$

where, consistently with Eq. (3.3.69), we choose both y_0 and F_0 to be positive. Equivalently one could choose them to be both negative.

We find it clearer to present the solution in terms of a new radial coordinate ρ , defined as $y = y_0(1 - \frac{\rho}{2})$, with range $\rho \in [0, 3]$. As a result, $\beta = \frac{y_0^3}{F_0} \rho^2(3 - \rho)$. Taking into account the quantization condition (3.3.69), the metric on the total takes the following form:

$$ds_{10}^2 = \frac{n_2}{F_0} \sqrt{\rho} \left(\frac{4}{3} ds_{\text{AdS}_7}^2 + \frac{d\rho^2}{4\rho(3 - \rho)} + \frac{\rho}{3} \frac{(3 - \rho)}{(12 - \rho^2)} ds_{S^2}^2 \right). \quad (3.3.75)$$

Around $\rho = 0$ the metric behaves as $\sim 16\sqrt{\rho} ds_{\text{AdS}_7}^2 + \frac{1}{\sqrt{\rho}}(d\rho^2 + \rho^2 ds_{S^2}^2)$, which is the correct behavior near a stack of D6-branes wrapping AdS_7 . On the other hand around $\rho = 3$, the internal metric turns into flat space: $d\tilde{\rho}^2 + \tilde{\rho}^2 ds_{S^2}^2$, after the change of coordinates $\tilde{\rho} = \sqrt{\rho - 3}$, so we get a regular point.

We can convince ourselves that this is the case by looking at the flux F_2 , which can be read off from Eq. (3.3.64). We get:

$$F_2 = \frac{n_2(3 - \rho)^{3/2}(6 - \rho)}{\sqrt{3}(-12 + \rho^2)} \text{vol}_{S^2}. \quad (3.3.76)$$

Near $\rho = 0$ this behaves as $F_2 \sim -\frac{n_2}{2} \text{vol}_{S^2}$, which is precisely the behavior for a stack of n_2 D6-branes, as we just saw in (3.3.72). On the other hand, near the regular point $\rho = 3$ the F_2 vanishes, as it should since no source is localized there. We can thus see that the quantization condition works correctly. This brane configuration and a sketch for the internal space is represented in Fig. 2.2.

It is also possible to write a compact expression for the three-form flux:

$$H = \sqrt{\frac{3F_0}{n_2\sqrt{\rho}}} \left(\frac{36 + 4\rho - 5\rho^2}{-12 + \rho^2} \right) \text{vol}_{M_3} . \quad (3.3.77)$$

The behavior at the two extrema is the one we would expect: around $\rho = 0$ it shows a divergent behavior $H \sim \rho^{-1/4} \text{vol}_{M_3}$, which is precisely the same type of divergence we had in the massless case in presence of a D6-brane. Around the regular pole $\rho = 3$ it goes to a constant value.

Finally, the dilaton is determined as:

$$e^{4\phi} = \frac{16\rho^3}{n_2^2 F_0^2 (12 - \rho^2)^2} , \quad (3.3.78)$$

which goes to a constant value at the regular point $\rho = 3$ and goes to zero at the second pole as $e^\phi \sim \rho^{3/4}$, which is the correct behavior near a D6 singularity, as defined by the D6 solution in flat space (2.0.8).

This solution, and its brane interpretation, is shown in Fig. 2.2.

3.3.4 O6/D6

The next possibility is to have a massive solution with a D6 stacks at one pole and an orientifold plane at the other. This type of solutions belong to the class $12 < b_2 < 18$. The D6 stack is located at $y = y_0$, the O6 sits at $y = y_1$. The flux integers at the two poles are $(-2, n_2)$.

Again, instead of discussing this class in full generality, we find it clearer to present a single solution specifying the flux integer to be $n_2 = 6$. In other words we discuss here a solution with an O6 plane at one side, and a stack of six D6-branes on the other side. According to formula (3.3.68), this choice is equivalent to fixing the parameters as: $(y_0, b_2) = (\frac{243}{8F_0}, \frac{27}{2})$. As a result, we get that the O6 plane is located at $y_1 = -\frac{1}{3}y_0$.

The local behavior at poles can be better understood introducing a new radial coordinate ρ with range $\rho \in [0, 2]$, defined by $y = \frac{y_0}{3}(\rho^2 - 1)$. In terms of the new coordinate we have $\beta = \frac{8y_0^3}{27F_0}(2 - \rho)^2(1 + \rho)^4$. The resulting internal space metric is

$$ds_{M_3}^2 = \frac{1}{F_0} \sqrt{\frac{27\rho}{(2 - \rho)(1 + \rho)^2}} \left(d\rho^2 + \frac{(2 - \rho)^2(1 + \rho)}{(9 - 3\rho)} ds_{S^2}^2 \right) . \quad (3.3.79)$$

Around $\rho = 0$ it behaves as $\sim \rho^{1/2}(d\rho^2 + \frac{4}{9}ds_{S^2}^2)$, which is the correct behavior for an O6 singularity, as defined by equation (3.2.43). Around $\rho = 2$ the internal metric behaves as $\sim \frac{1}{\sqrt{\rho-2}}(d\rho^2 + (\rho - 2)^2 ds_{S^2}^2)$, which signals the presence of a D6 stack.

Let us check that the flux F_2 is properly quantized. It takes the following form:

$$F_2 = \frac{3}{\rho - 3} \text{vol}_{S^2} . \quad (3.3.80)$$

Near $\rho = 2$ we get $F_2 \sim -3\text{vol}_{S^2}$, which is precisely the behavior for a stack of six D6-brane. On the other hand, near $\rho = 0$ we get $F_2 \sim -\text{vol}_{S^2}$, which corresponds to an O6-plane.

We can also check that the flux H behaves correctly. The general expression is:

$$H = \frac{\sqrt{F_0}}{\rho^{3/4}} \frac{(4 - 15\rho + 5\rho^2)}{(3 - \rho)(6 + 9\rho - 3\rho^3)^{1/4}} \text{vol}_{M_3} . \quad (3.3.81)$$

Around the D6 singularity $\rho = 2$ we get $H \sim (\rho - 2)^{-1/4} \text{vol}_{M_3}$, while in correspondence of the O6-plane $\rho = 0$ it shows a different divergent behavior: $H \sim \rho^{-3/4} \text{vol}_{M_3}$.

Finally, the dilaton is determined as:

$$e^{4\phi} = \frac{16(2 - \rho)^3}{27F_0^2(3 - \rho)^2\rho^3} , \quad (3.3.82)$$

which reproduces the correct D6 behavior $e^\phi \sim (\rho - 2)^{3/4}$ around $\rho = 2$, while at the pole $\rho = 0$ where the orientifold plane is located it goes like $e^\phi \sim \rho^{-3/4}$. This singular behavior is precisely what we expect from the solution for an O6 in flat space. The latter is very similar to the D6-brane solution (2.0.8), provided a change of sign in the harmonic function H , which now reads: $H(\rho) = 1 - \frac{\rho_0}{\rho}$. Expanding the corresponding expression for the dilaton around $\rho = \rho_0$ we get $e^\phi \sim (\rho - \rho_0)^{-3/4}$, in agreement with the present analysis.

A sketch describing the geometry of the internal manifold for the D6/O6 solution is represented in Fig. 4.1.

3.3.5 Two D6 stacks

The last class of solutions is the one with two D6 stacks, corresponding to $0 < b_2 < 12$. The two stacks are located at y_1 and y_0 , with flux integers (\tilde{n}_2, n_2) . Such a situation would not be possible in the massless case, where the flux integers on the two sides are equal. However, as we already stressed, adding a non-vanishing Romans mass allows to obtain infinite more solutions.

Again, instead of discussing this class in full generality, we find it clearer to give a single example, specifying the flux integers to be $(\tilde{n}_2, n_2) = (1, 2)$. In other words we present a solution with a stack of two D6-branes at one pole and a single D6 at the other pole. According to formula (3.3.67), this fixes the two parameters as: $(y_0, b_2) = (\frac{15}{8F_0}, \frac{15}{2})$. As a result, the second stack is located at $y_1 = -\frac{4}{5}y_0$.

We introduce a new coordinate ρ with range $\rho \in [1, 2]$, defined as $y = \frac{y_0}{5}(3\rho^2 - 7)$. As a result $\beta = \frac{216y_0^3}{125F_0^3}(2 - \rho)^2(1 - \rho)^2(3 + \rho)^2$, and the metric reads

$$ds_{M_3}^2 = \frac{1}{F_0} \sqrt{\frac{3\rho}{(7\rho - 6 - \rho^3)}} \left(d\rho^2 + \frac{(7\rho - 6 - \rho^3)^2}{(49 - 72\rho + 42\rho^2 - 3\rho^4)} ds_{S^2}^2 \right) . \quad (3.3.83)$$

Around $\rho = 1$ this behaves as $\sim \frac{1}{\sqrt{\rho-1}}(d\rho^2 + (\rho-1)^2 ds_{S^2}^2)$, which is the correct behavior near a stack of D6-branes. The same happens around $\rho = 2$.

We can check that number of D6 at poles are $(\tilde{n}_2 = 1, n_2 = 2)$. As always this can be done looking at the two-form flux, which has the following expression:

$$F_2 = \frac{(21 - 49\rho + 27\rho^2 - 7\rho^3)}{(49 - 72\rho + 42\rho^2 - 3\rho^4)} \text{vol}_{S^2} . \quad (3.3.84)$$

Near $\rho = 1$ this behaves as $F_2 \sim \frac{-1}{2} \text{vol}_{S^2}$, which is precisely the behavior for a single D6-brane, as we just saw in (3.3.72). On the other hand, near $\rho = 2$ we get $F_2 \sim \text{vol}_{S^2}$, which corresponds to a stack of two D6-branes.

It is also possible to write a rational expression for H :

$$H = \frac{\sqrt{F_0} \sqrt{3}}{\rho^{3/4}} \frac{(84 + 49\rho - 252\rho^2 + 182\rho^3 - 15\rho^5)}{(-49 + 72\rho - 42\rho^2 + 3\rho^4)(-6 + 7\rho - \rho^3)^{1/4}} \text{vol}_{M_3} . \quad (3.3.85)$$

This doesn't look particularly enlightening, nonetheless we can check that the singular behavior near the two poles $\rho \sim 1$ and $\rho \sim 2$ is the expected one in presence of a D6 stack, that is to say $H \sim (\rho - 1)^{-1/4} \text{vol}_{M_3}$ and $H \sim (\rho - 2)^{-1/4} \text{vol}_{M_3}$.

Finally, the dilaton is determined as:

$$e^{4\phi} = \frac{48}{F_0^2 \rho^3} \frac{(7\rho - 6 - \rho^3)^3}{(49 - 72\rho + 42\rho^2 - 3\rho^4)^2} , \quad (3.3.86)$$

which goes like $e^\phi \sim (\rho - 1)^{3/4}$ around $\rho = 1$ and like $e^\phi \sim (\rho - 2)^{3/4}$ around $\rho = 2$, a further confirm that we are in presence of two D6 stacks.

3.4 Solutions with D8

So far we presented various types of solutions with localized D6-branes and orientifold planes. One can also obtain metrics with arbitrary numbers of D8-branes. This is achieved by gluing together copies of the metrics we have obtained so far, and by tuning properly the parameters in such a way to respect a set of constraints that we have already worked out in subsection 3.2.4.

Let us briefly review these constraints. Like for any type of source, the D8's backreaction will give rise to a singularity. In particular, they give rise to a jump in the Romans mass F_0 , which turns out to be piecewise constant according to Eq. (3.2.49). The flux integer n_2 can also jump across a D8, as described by Eq. (3.2.50). This is mainly due to the fact that a D8-brane carries a D6-brane charge μ on its worldvolume.

These two conditions are not enough to ensure that the full solution has a well defined behavior. Indeed, despite the jump in the Romans mass, the metric has to be made continuous, imposing suitable conditions in the points where the stacks of D8's are located. Continuity of the metric requires

imposing that the function β is continuous with its first derivative β' across the stack.

Also, there is an extra constraint coming from the requirement of continuity for the B field across the D8-stack. We already analyzed this condition in detail, and we were able to translate it into a simple equation for the function q , (3.2.31).

3.4.1 One D8 stack

To begin with we present a solution with a single D8 stack. This can be obtained by gluing two metrics of the type (3.3.74). Since the quantization conditions will change in presence of D8's, it is better to stick to the original coordinate $y = y_0(1 - \frac{\rho}{2})$. We will assume

$$y_0 > 0, \quad F_0 > 0; \quad \tilde{y}_0 < 0, \quad \tilde{F}_0 < 0. \quad (3.4.87)$$

Let us call (n_0, n_2) and $(\tilde{n}_0, \tilde{n}_2)$ the flux integers before and after the D8. For simplicity let us also assume $\tilde{n}_2 = 0$, so that no large gauge transformations are needed on that side. As we remarked in subsection 3.2.4, $\Delta n_2 = \tilde{n}_2 - n_2 = -n_2$ should be an integer multiple of $\Delta n_0 = \tilde{n}_0 - n_0 = n_8$, according to $\Delta n_2 = \mu \Delta n_0, \mu \in \mathbb{Z}$. To take care of flux quantization, it is enough to also demand that $n_2 = N n_0$, for N integer.

According to the discussion at the end of subsection 3.2.4, in this case at the North Pole we get $\hat{b} = -\pi N$; since this is an integer multiple of π , it can be brought to zero by using large gauge transformations. Together, the conditions we have imposed determine

$$\tilde{n}_0 = n_0 \left(1 - \frac{N}{\mu} \right). \quad (3.4.88)$$

Putting together two copies of (3.3.74), we can write the following expression for the internal space metric:

$$ds_{M_3}^2 = \begin{cases} \frac{1}{\sqrt{3F_0}} \frac{1}{\sqrt{(y_0 + 2y)^2(y_0 - y)}} \left(dy^2 + \frac{1}{3} \frac{(y_0 - y)^2(y_0 + 2y)^2}{2y_0^2 + 2yy_0 - y^2} ds_{S^2}^2 \right), \\ \frac{1}{\sqrt{-3\tilde{F}_0}} \frac{1}{\sqrt{(\tilde{y}_0 + 2y)^2(y - \tilde{y}_0)}} \left(dy^2 + \frac{1}{3} \frac{(\tilde{y}_0 - y)^2(\tilde{y}_0 + 2y)^2}{2\tilde{y}_0^2 + 2y\tilde{y}_0 - y^2} ds_{S^2}^2 \right), \end{cases} \quad (3.4.89)$$

where the first line corresponds to the range $-\frac{y_0}{2} < y < y_{D8}$, and the second line to $y_{D8} < y < -\frac{\tilde{y}_0}{2}$. Our job is not over yet, since we still have to impose continuity of the metric and the dilaton (or, equivalently, continuity of β and β'), plus the analogue condition for the B field, Eq. (3.2.53). Taking these three conditions into account amounts to fixing the positions of the two regular points and the position of the D8 stack as follows:

$$\begin{aligned} y_0 &= 3F_0\pi^2(N^2 - \mu^2), & \tilde{y}_0 &= -3F_0\pi^2(N - \mu)(2N - \mu), \\ y_{D8} &= 3F_0\pi^2(N - 2\mu)(N - \mu). \end{aligned} \quad (3.4.90)$$

The allowed range for the two integers N and μ can be obtained by imposing the following set of constraints: $\{y_0 > 0, \tilde{y}_0 < 0, -\frac{1}{2}y_0 < y_{D8} < -\frac{1}{2}\tilde{y}_0\}$. As a result, we get the bound $N > \mu$.

This solution, and its brane interpretation, is shown in Fig. 3.1.

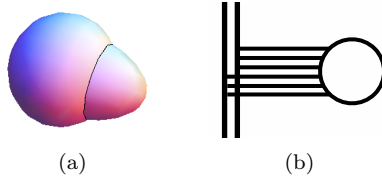


Figure 3.1: In (a), a sketch of the internal M_3 in the solution with one D8-brane stack, represented by a “crease”. In (b), the corresponding brane configuration. The vertical lines represent the D8-branes; the stack has $n_0 = 2$ branes with $|\mu| = 3$.

3.4.2 Two D8 stacks

We can also consider a configuration with two D8 stacks. We will take it to be symmetric, in the sense that the flux integers before the first D8 stack will be $(n_0, 0)$, between the two stacks $(0, n_2 = -k < 0)$, and after the second stack $(-n_0, 0)$. Notice that we have a massless region in the middle, which implies a more involved quantization condition for the flux H , described by Eq. (3.2.58).

A proper solution describing this brane configuration can be obtained gluing together the metrics in three different regions. The metric in the two external regions is described by Eq. (3.3.74), while for the massless region between the two D8 stacks we must consider Eq. (3.3.70).

Again we will assume $y_0 > 0$; the positions of the two D8 stacks will be $y_{D8} > 0$ and $y_{D8'} = -y_{D8} < 0$. The result is the following:

$$ds_{M_3}^2 = \begin{cases} \frac{1}{\sqrt{3F_0}} \frac{1}{\sqrt{(y_0 + 2y)^2(y_0 - y)}} \left(dy^2 + \frac{1}{3} \frac{(y_0 - y)^2(y_0 + 2y)^2}{2y_0^2 + 2yy_0 - y^2} ds_{S^2}^2 \right), \\ \frac{4}{9k} \frac{1}{\sqrt{\tilde{y}_0^2 - y^2}} \left(dy^2 + \frac{(\tilde{y}_0^2 - y^2)^2}{4\tilde{y}_0^2} ds_{S^2}^2 \right), \\ \frac{1}{\sqrt{3F_0}} \frac{1}{\sqrt{(-y_0 + 2y)^2(y_0 + y)}} \left(dy^2 + \frac{1}{3} \frac{(y_0 + y)^2(-y_0 + 2y)^2}{2y_0^2 - 2yy_0 - y^2} ds_{S^2}^2 \right), \end{cases} \quad (3.4.91)$$

where the first line corresponds to the external region $-\frac{y_0}{2} < y < -y_{D8}$, the middle line to the region between the two stacks $-y_{D8} < y < y_{D8}$, and the last line to the second exterior region $y_{D8} < y < \frac{y_0}{2}$.

We now have three unknowns: \tilde{y}_0 , y_0 , y_{D8} . Continuity of β and β' this time only imposes one condition; we then have (3.2.53) and the condition (3.2.58). We get

$$\begin{aligned} y_0 &= \frac{9}{2}k\pi(N - \mu) , & y_{D8} &= \frac{9}{4}k\pi(N - 2\mu) , \\ \tilde{y}_0 &= \frac{9}{4}k\pi^2 \sqrt{N^2 - \frac{4}{3}\mu^2} , \end{aligned} \tag{3.4.92}$$

where in this case $\mu = \frac{k}{n_0}$. Notice that in this case the bound in [35, Eq.(4.10)] (which can also be found by (3.2.58)) implies $N > 2\mu$, which is precisely the condition that one obtains imposing $\{y_0 > 0, y_{D8} > 0, \tilde{y}_0 > 0\}$.

This solution, and its brane interpretation, is represented in Fig. 3.2.

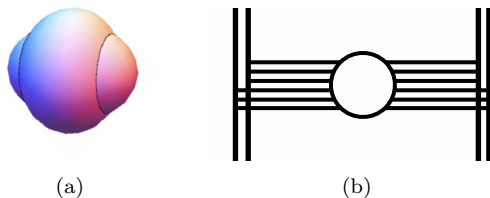


Figure 3.2: In (a), a sketch of the internal M_3 in the solution with two D8-brane stacks, represented by two “creases”. In (b), the corresponding brane configuration. The vertical lines represent the D8-branes; each stack has $n_0 = 2$ branes with $|\mu| = 3$.

It would now be possible to produce solutions with a larger number of D8’s. It is in fact possible to introduce an arbitrary number of them, although there are certain constraints on their numbers and their D6 charges [35, Sec. 4]. The most general solution can be labeled by the choice of two Young diagrams; there is also a one-to-one correspondence with the brane configurations in [22, 47]. One can in fact think of the AdS₇ solutions as a particular near-horizon limit of the brane configurations. For more details, see [35]. For these more general solutions, we expect to have to glue together not only pieces of the solution in subsection 3.3.3 and of the massless solution, but also pieces of the more complicated solutions described by the general form of the function β in (3.3.66), like those described in subsections 3.3.4 and 3.3.5.

3.5 Field theory interpretation

In this section we have found infinitely many new AdS₇ solutions in massive IIA, describing the near horizon geometries of brane configurations involving NS5-, D6- and D8-branes. The importance of these brane configurations was

known since a long time [22], and now they finally have a proper supergravity description.

The importance of these solutions is mainly due to their holographic interpretation: they are dual to the (1,0) conformal field theories in six dimensions, living on the aforementioned bound states of branes. At the level of the symmetries, the correspondence maps the $\text{Sp}(1)$ R-symmetry group on the field theory side to the $\text{SU}(2)$ isometries of the internal space of the supergravity solutions.

As we anticipated in the introduction, very little is known about the (1,0) theories since they lack a Lagrangian description. Having found their holographic duals can thus be crucial to extract some information on these mysterious theories.

Here, we will limit ourselves to pointing out a couple of preliminary results about the number of degrees of freedom.

A common way of estimating the number of degrees of freedom using holography in any dimension is to introduce a cut-off in AdS, and estimate the Bekenstein–Hawking entropy (see for example [61, Sec. 3.1.3]). This leads to $\frac{R_{\text{AdS}_7}^5}{G_{N,7}}$ in AdS_7 , where $G_{N,7}$ is Newton’s constant in seven dimensions. The latter can be computed as $\frac{1}{g_s^2} \text{vol}_{10-d}$. In a warped compactification with non-constant dilaton, both R_{AdS} and g_s are non-constant, and should be integrated over the internal space. In our case, for AdS_7 this leads to

$$\mathcal{F}_{0,6} \equiv \int e^{5A-2\phi} \text{vol}_{M_3} . \quad (3.5.93)$$

These can be thought of as the coefficient in the thermal partition function, $\mathcal{F} = \mathcal{F}_{0,6} VT^6$, where V is the volume of space and T is temperature. These computations however are basically the same for the coefficients in the Weyl anomaly, at least at leading order (i.e. in the supergravity approximation).

We have not computed $\mathcal{F}_{0,6}$ in full generality for the (1,0) theories. This would now be possible in principle, since the analytic expressions are now known. Here we present the results corresponding to some particularly relevant brane configurations.

One first example is the solution described by the metric (3.3.75). The corresponding brane configuration according to the identification in [35] consists in k D6’s ending on $N = \frac{k}{n_0}$ NS5-branes; see figure 3.3(a). We get

$$\mathcal{F}_{0,6} = \frac{512}{45} k^2 \pi^4 N^3 , \quad (3.5.94)$$

which reassuringly goes like N^3 . (By way of comparison, for the massless case one gets $\mathcal{F}_{0,6} = \frac{128}{3} k^2 \pi^4 N^3$.)

We also computed $\mathcal{F}_{0,6}$ for the solution (3.4.91), which has two D8’s and a massless region between them. The corresponding brane configuration would be N NS5-branes in the middle with $k = \mu n_0$ D6’s sticking out of them, ending on n_0 D8-branes both on the left and on the right; see figure 3.3(b).

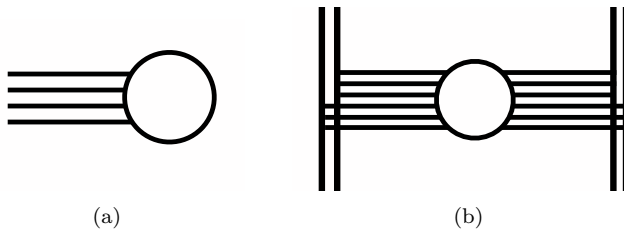


Figure 3.3: Brane configurations for two sample theories. The circles represent stacks of N NS5-branes; the horizontal lines represent D6-branes; the vertical lines represent D8-branes. In the second case, on each side we have $n_0 = 2$ D8-branes; $|\mu| = 3$ D6-branes end on each, for a total of $k = n_0|\mu| = 6$.

This case was considered in [35, Sec. 5], where approximate expressions for $\mathcal{F}_{0,6}$ were computed, using perturbation theory around the massless limit. Using (3.4.91) we can now obtain the exact result:

$$\mathcal{F}_{0,6} = \frac{128}{3} k^2 \pi^4 \left(N^3 - 4N\mu^2 + \frac{16}{5} \mu^3 \right). \quad (3.5.95)$$

This agrees with [35, Sec. 5], but is now exact. Recall that $\mu = \frac{k}{n_0}$; since this number can be large, the second and third term are also large, and are not competing with stringy corrections. Stringy corrections will modify this result with terms linear in N and probably in μ .

Chapter 4

AdS₄ compactifications of AdS₇ solutions in type IIA supergravity

In many string theoretic constructions, the presence of extended sources such as D-branes or O-planes is a crucial ingredient. In compactifications, for examples, O-planes are thought to be important to overcome no-go arguments that forbid de Sitter (or even Minkowski with non-trivial flux) compactifications [37, 62, 63]. However, in most cases these sources back-react on the metric in a way which destroys whatever symmetries were previously present, and makes it prohibitively hard to find a full solution to the equations of motion.

To overcome this problem, sources are often “smeared” over the internal space: namely, they are assumed to occur in a continuous distribution with varying positions, much like the individual electrons on a charged piece of conductor. While this is fine for D-branes, it is incompatible with the definition of an O-plane, which must in fact lie at the fixed locus of an involution. When the smearing trick is performed on O-planes, it is usually done with the hope that it might be a good indicator of whether a non-smeared solution exists. It is hence interesting to find solutions with localized (i.e. non smeared) sources, even ones where the cosmological constant is negative. Although there already exists one family of supersymmetric AdS₄ solutions with localized sources, in type IIB supergravity [29], such examples remain rare.

In this chapter, we are going to present a class of infinitely many new supersymmetric AdS₄ solutions with localized sources, in type IIA supergravity

with Romans mass parameter F_0 . As an example:

$$\frac{(1+\rho)}{F_0} \sqrt{\frac{15(2-\rho)}{8\rho}} \left(\frac{5}{2} ds_{\text{AdS}_4}^2 + 2ds_{\Sigma_3}^2 + \frac{3\rho d\rho^2}{(2-\rho)(1+\rho)^2} + \frac{2\rho(2-\rho)Ds_{S^2}^2}{(6-2\rho+\rho^2)} \right), \quad (4.0.1)$$

with $\rho \in [0, 2]$, Σ_3 a compact hyperbolic three-manifold, and $Ds_{S^2}^2$ the round S^2 metric fibred over Σ_3 in a certain way.¹ This has a stack of D6-branes at $\rho = 2$, and a localized orientifold plane at $\rho = 0$, so that the topology of the space M_3 described by ρ and the S^2 is that of an S^3 . We will also present analytic solutions with a single D6 stack and a regular point on the other extremum of the interval, solutions with two D6 stacks, and with D8-branes. Moreover, we will present numerical solutions where Σ_3 can be replaced with an S^3 , and also where sources can even be absent; in particular we will have a family of completely regular solutions with topology $\text{AdS}_4 \times S^3 \times S^3$, but different from the one in [64].

As we anticipated in the introduction, the various classes of AdS_4 analytic solutions that we will present were found as compactifications of the AdS_7 solutions described the previous chapter. In view of the issues explained above with localized branes, it was indeed interesting to ask whether those findings could be somehow transported to four dimensions. (In a series of interesting papers [65–68], an $\text{AdS}_7 \times M_3$ setup similar to [1] was examined to understand the differences between localized and smeared branes.) For this, we needed to somehow replace AdS_7 with $\text{AdS}_4 \times \Sigma_3$, where Σ_3 is some new compact three-manifold.

The holographic duals of the AdS_7 solutions in [1, 35] were argued in [35] to be CFT_6 's arising from NS5-D6-D8-brane configurations studied long ago [22, 47]. Replace AdS_7 with $\text{AdS}_4 \times \Sigma_3$ sounds like compactifying the CFT_6 to a CFT_3 , on a three-manifold Σ_3 . This is more commonly done from a CFT_6 to a CFT_4 , thus replacing AdS_7 with $\text{AdS}_5 \times \Sigma_2$. A famous example is the Maldacena-Nuñez solution [37], which is dual to a “twisted” compactification of the $(2, 0)$ theory on a Riemann surface. But it is also possible to compactify the $(2, 0)$ theories on a hyperbolic three-manifold: the solution dual to this is in fact even older, going back to [39] (later being lifted to eleven dimensions in [40, 41]). In that case the study of twisted compactification on three-manifolds led to the formulation of the so-called 3d-3d correspondence of [69].

¹The word “fibred” has different meanings in different contexts. In this paper, we will use the topological meaning of the word. Namely, there is a fibre bundle E_5 whose fibre is S^2 and whose base is Σ_3 ; the connection terms in $Ds_{S^2}^2$ (see (4.2.13) below) signal that the bundle is topologically non-trivial. The interval I parameterized by y is not topologically fibred, but it can formally be included in a bigger fibre bundle with fibre M_3 and base Σ_3 . Sometimes one wants to refine the definition of fibration by including the metric data; even for a space $M_1 \times M_2$ that is topologically a product, one sometimes says that M_1 is fibred over M_2 if the metric on M_1 depends on the coordinates on M_2 . In this second sense, we should rather say that the whole fibre bundle E_5 is itself fibred over the interval I .

Perhaps not surprisingly, we found that the same happens for the $(1, 0)$ theories, that is to say they naturally compactify on three-dimensional Einstein manifolds of negative curvature. This might eventually lead to a generalization of the aforementioned correspondence. (Notice however that supersymmetry is lower, namely $\mathcal{N} = 1$.) It would thus be interesting to understand what the resulting CFT₃'s are; some useful information can now be extracted from our solutions using holographic techniques. For example there exists a universal way of counting their degrees of freedom, that are related to the degrees of freedom of the corresponding CFT₆ via a simple formula given by Eq. (4.3.75).

4.1 Supersymmetry and pure spinors in $d = 6$

The AdS₄ solutions that we will present in this chapter were found making use of generalized geometry techniques, to so-called pure spinor formalism. It was developed after the discovery of Calabi-Yau compactifications, when the importance of having a geometric interpretation of the data of a given vacuum became evident. In the Calabi-Yau case it was possible to solve the supersymmetry equations for $\mathcal{N} = 2$ Minkowski vacua without fluxes in terms of an SU(3) structure on the internal manifold. Such a structure can be expressed either in terms of one spinor η , or equivalently in terms of a pair of forms (J, Ω) living on the internal space M_6 .

In general working with differential forms turns out to be much simpler than working with spinors, especially in more complicated situations like the study of flux compactifications. As understood in [50], that the data of any $\mathcal{N} = 1$ four-dimensional vacuum can be encoded into a pair of polyforms Φ_{\pm} , the *pure spinors*, that define an SU(3) × SU(3) structure on the internal manifold M_6 . Following this approach, the supersymmetry equations can be rewritten as an elegant set of differential equations for Φ_{\pm} , the so-called pure spinor equations.

We will now give a quick review of the essentials of the pure spinor formalism, specifying it to the case of AdS₄ compactifications in type IIA supergravity. For a more complete introduction to pure spinors see for example [70].

4.1.1 Pure spinor equations for AdS₄ × M₆

A warped AdS₄ compactification is a spacetime of the form

$$ds_{10}^2 = e^{2A} ds_{\text{AdS}_4}^2 + ds_6^2, \quad (4.1.2)$$

where ds_6^2 is the metric on the internal space M_6 , and A is a function of M_6 called *warping*. In general M_6 can be fibered non-trivially over AdS₄. All the other fields are constrained by the requirement that the full solution has to preserve the AdS₄ isometries. The dilaton can be a function of the

internal space only; the various fluxes can only have components along the internal space, apart from F_4 that is allowed to have a component along the volume form of the external space.

Moving to fermions, the topology of $M_{10} = M_4 \times M_6$ induces a decomposition of the spinor bundle $\Sigma_{10} = \Sigma_4 \times \Sigma_6$. The corresponding decomposition for the ten-dimensional gamma matrices is the following:

$$\begin{aligned}\gamma_\mu^{(4+6)} &= \gamma_\mu^{(4)} \otimes 1, \\ \gamma_{i+3}^{(4+6)} &= \gamma^{(4)} \otimes \gamma_i,\end{aligned}\tag{4.1.3}$$

where $\gamma_\mu^{(4)}$, $\mu = 0, \dots, 3$, are a basis of four-dimensional gamma matrices, and γ_i , $i = 1, \dots, 6$, are a basis of gamma matrices in six dimensions.

In type IIA supergravity the two ten-dimensional supersymmetry parameters have opposite chirality, and they can be parametrized as follows:

$$\epsilon^1 = \zeta_+ \eta_+^1 + c.c., \quad \epsilon^2 = \zeta_+ \eta_-^2 + c.c.,\tag{4.1.4}$$

where ζ_+ is a four-dimensional spinor and η^1, η^2 are a two spinors on the internal space with opposite chirality.

We can define the pure spinors Φ_\pm as the following pair of bispinors on the internal space:

$$\Phi_- \equiv \eta_+^1 \otimes (\eta_-^2)^\dagger, \quad \Phi_+ \equiv \eta_+^1 \otimes (\eta_+^{2c})^\dagger.\tag{4.1.5}$$

As usual we can apply the Clifford map $dx^{i_1} \wedge \dots \wedge dx^{i_k} \rightarrow \gamma^{i_1 \dots i_k}$ in order to transform bispinors into forms. Indeed, in this equation the subscript \pm denotes even/odd forms. From the point of view of generalized geometry, the pair of polyforms Φ_\pm can be understood as an $SU(3) \times SU(3)$ structure on the internal space.

Remarkably, under the AdS_4 compactification Ansatz the complicated set of supersymmetry equations of type IIA supergravity, Eq. (2.0.2), can be rewritten as the following elegant set of equations:

$$d_H \Phi_+ = -2e^{-A} \text{Re} \Phi_-, \quad \mathcal{J}_+ \cdot d_H (e^{-3A} \text{Im} \Phi_-) = -5e^{-4A} \text{Re} \Phi_+ + F, \quad d_H F = \delta,\tag{4.1.6}$$

that go under the name of pure spinor equations [50, 71]. The differential operator d_H is defined as $d_H \equiv d - H \wedge$, and \mathcal{J}_+ is an algebraic operator associated in a certain way to Φ_+ . This operator is reviewed for example [71], and more concretely in [72, Sec. 5].

4.1.2 Parametrization of the pure spinors

We already mentioned that the pair of polyforms Φ_\pm can be understood as defining an $SU(3) \times SU(3)$ structure on the internal space. Actually, in some cases it is more convenient to write the pure spinors in terms of a so-called $SU(2)$ structure on M_6 . This is given by a complex one-form z , plus

a complex two-form ω and a real two-form j satisfying the following set of algebraic constraints:

$$\omega \wedge \bar{\omega} = j^2, \quad \omega^2 = 0. \quad (4.1.7)$$

The parametrization of Φ_{\pm} in terms of the $SU(2)$ structure goes under the name of *Dielectric Ansatz*:

$$e^{-b_0} \Phi_+ = \rho e^{i\theta} e^{-iJ_\psi}, \quad e^{-b_0} \Phi_- = \rho \tan \psi z \wedge e^{i\omega_\psi}, \quad (4.1.8)$$

where ψ is the angle between the two spinors $\eta^{1,2}$, and ρ is a real number that determines the norm of the pure spinors. We have also defined the forms:

$$J_\psi \equiv \frac{1}{\cos \psi} j + \frac{i}{2} z \wedge \bar{z}, \quad \omega_\psi \equiv \frac{1}{\sin \psi} \left(\text{Re} \omega + i \frac{\text{Im} \omega}{\cos \psi} \right), \quad b_0 = \tan \psi \text{Im} \omega, \quad (4.1.9)$$

where the real two-form b_0 is called the *intrinsic* b -field associated to the pair Φ_{\pm} . One can always obtain a pure spinor pair with vanishing intrinsic b_0 by the action of a so-called b -transform:

$$\Phi_{\pm} \rightarrow \Phi_{\pm}^0 = e^{-b_0 \wedge} \Phi_{\pm}. \quad (4.1.10)$$

This operation turns out to be a symmetry of the pure spinor equations provided that also the physical NS three-form flux H and the *internal*² RR flux $F = \sum_k F_{2k}$ are transformed to the corresponding auxiliary fluxes given by:

$$H^0 = H - db_0, \quad F^0 = e^{-b_0} F. \quad (4.1.11)$$

It is easy to see that one can equivalently solve the pure spinor equations (4.1.6) for the set of auxiliary fields $\{\Phi_{\pm}^0, F^0, H^0\}$ and then perform an inverse b -transform (4.1.10) to get the physical fluxes.

Using the parametrization (4.1.8), the pure spinor equations have been rewritten in a more manageable form in [72, Sec. 5.2]. More concretely, the system (4.1.6) was reduced to the action of the operator $J_\psi^{-1} \lrcorner$, whose action on a form consists in contracting it with the bivector J_ψ^{-1} with inverse J_ψ in (4.1.9). This operator is analyzed in detail in appendix C.

4.2 Compactification Ansatz

In this section we specify the discussion to the case that we are interested in, namely to AdS_4 solutions that arise as compactifications of the AdS_7 solutions dual to the $(1,0)$ theories, that we described in detail in the previous chapter.

²We mean by this the flux with no legs along AdS_4 ; this determines via Hodge duality the external flux, namely the one with legs along AdS_4 .

The field theory perspective can help us a lot in formulating a proper compactification Ansatz for the bosonic and fermionic fields. Indeed, inspired by the AdS₄ solution [39, 41], that describe the compactification of the (2, 0) theory on a maximally symmetric three-manifold Σ_3 , we were able to formulate an analogous Ansatz which would be holographically dual to compactifying the (1, 0) CFT₆ on Σ_3 .

The crucial point is that the compactification procedure has to preserve some supersymmetry. An old strategy consists in a partial “twist” of the theory. Roughly speaking, fields with an R-symmetry index are considered to be sections of a certain R-symmetry bundle E , which is then chosen such that $E \otimes S$ (with S the spinor bundle) admits a global section. This global section (which can then taken to be constant, up to a gauge transformation) is then a preserved supercharge. For brane theories, often the procedure also has a geometrical interpretation: E can be interpreted as the normal bundle to the brane [73]. Thus the twisting corresponds roughly to how one wraps the brane.

More concretely, the twisting procedure will have to reflect that for us the internal manifold M_6 will be a fibration of M_3 over Σ_3 , as represented by the following graph:

$$\begin{array}{ccc} M_3 & \hookrightarrow & M_6 \\ & & \downarrow \\ & & \Sigma_3 . \end{array} \quad (4.2.12)$$

In the next three subsections we will describe the details of this fibration, first at the level of the metric, then at the level of the spinors, and finally at the level of pure spinors.

4.2.1 Metric

There is a natural way of fibering the round two-sphere in M_3 over the three-manifold Σ_3 , which amounts to twisting its SU(2) symmetry by mixing it with the SU(2) local Lorentz group on the tangent bundle to Σ_3 . For this purpose it is useful to write the S^2 metric in terms of the spherical harmonics (3.2.35) as $ds_{S^2}^2 = dy^i dy^i$.

In these coordinates, the fibered metric on the two-sphere is obtained replacing the ordinary derivative dy^i with an SU(2) covariant derivative Dy^i as follows:

$$Ds_{S^2}^2 = Dy^i Dy^i , \quad Dy^i = dy^i + \epsilon^{ijk} y^j \mathcal{A}^k , \quad (4.2.13)$$

where we introduced a triplet of vectors \mathcal{A}^i that describe the connection of the SU(2) bundle. These are forms on the base space Σ_3 , that are related to the spin connection by

$$\mathcal{A}^i = \frac{1}{2} \epsilon^{ijk} \omega^{jk} . \quad (4.2.14)$$

The covariant volume form of the two-sphere is associated to the metric (4.2.13) according to:

$$\text{vol}_2 \equiv \frac{1}{2} \epsilon^{ijk} y^i D y^{jk} . \quad (4.2.15)$$

Notice that we are assuming that the shape of the two-sphere does not get distorted in the compactification procedure. This assumption is also confirmed by a detailed study of the already existing $\text{AdS}_4 \times \Sigma_3$ compactification in eleven-dimensional supergravity, which can be reduced to ten dimensions preserving $\mathcal{N} = 1$ supersymmetry, as described in appendix A.

We will also assume Σ_3 to be a *maximally symmetric space*, or a quotient thereof. In three dimensions, this constraint implies that the two-form Riemann tensor, defined as $R^{ij} \equiv \frac{1}{2} R_{\mu\nu}^{ij} dx^{\mu\nu}$, is related to the orthonormal frame $\{e^i\}$ by: $R^{ij} = \frac{R}{6} e^{ij}$, where R is the Ricci scalar and $e^{ij} \equiv e^i \wedge e^j$.³ Under this assumptions, we can determine the curvature associated to the $\text{SU}(2)$ bundle to be the following two-form on Σ_3 :

$$\mathcal{F}_2^i \equiv d\mathcal{A}^i - \frac{1}{2} \epsilon^{ijk} \mathcal{A}^j \mathcal{A}^k = \frac{R}{12} \epsilon^{ijk} e^{jk} . \quad (4.2.16)$$

Our Ansatz for the ten-dimensional metric is obtained from the AdS_7 one (3.1.1, 3.2.25) by replacing AdS_7 with $\text{AdS}_4 \times \Sigma_3$, and by fibering the internal space according to (4.2.13). We also expect some deformations, so we introduce extra warpings which we assume to depend on the radial coordinate only. All in all:⁴

$$ds_{10}^2 = e^{2A} ds_{\text{AdS}_4}^2 + g^2 ds_{\Sigma_3}^2 + dr^2 + f^2 Ds_{S^2}^2 , \quad (4.2.17)$$

where $ds_{\Sigma_3}^2 = e^i e^i$ and the fibered metric on the two-sphere was defined in (4.2.13).

Notice that the $\text{SO}(3)$ symmetry of the original AdS_7 solution is broken by the twisting procedure, consistently with our compactification solutions being $\mathcal{N} = 1$ AdS_4 vacua. Indeed, even though the $\text{SO}(3)$ remains as a symmetry of the fiber, it is not a symmetry of the full solution.⁵ The ten-dimensional metric (4.2.17) is now invariant under a simultaneous $\text{SO}(3)$

³We are using the same notation for the vielbein on Σ_3 that we introduced in the previous section for the vielbein on M_3 , which however was determined by supersymmetry to be (3.2.23). From now $\{e^i\}$ will label the triad on Σ_3 without any source of confusion.

⁴Notice that we are using the same name as for the AdS_7 solutions (3.1.1, 3.2.25) for the warping factor A and the radial coordinate r , but of course these are in principle different quantities in AdS_4 . Actually, we will be able to find a map between the AdS_7 solutions and the AdS_4 compactifications, as described in subsection 4.3.1. This map will fix the four-dimensional warping A_4 in terms of the corresponding seven-dimensional one A_7 , and the radial coordinate r_4 will be determined in terms of r_7 as well. Here and in the rest of this thesis we prefer to drop the indices that label the dimension of the AdS factor when they are not necessary, to make the expressions more readable.

⁵If E is the total space of an F -fibration over a base space B , the isometries of B are promoted to isometries of E , but often the isometries of F are not. To see this, write the metric on E as $ds_E^2 = g_{ij}^F D x^i D x^j + g_B$, where x^i and g_{ij}^F are the coordinates and metric on F , and $D x^i \equiv dx^i + A^i$; A^i is a connection on B , which takes values in the

local Lorentz transformation on Σ_3 , and an identical $\text{SO}(3)$ rotation acting on the S^2 . This “diagonal” $\text{SO}(3)$ acts on the vielbein e^i of Σ_3 and on the y^i in (3.2.35) as:

$$e^i \rightarrow O^{ij} e^j, \quad y^i \rightarrow O^{ij} y^j. \quad (\text{SO}(3)_D) \quad (4.2.18)$$

This “twisted symmetry” will also play a crucial role in formulating our Ansatz for the supersymmetry parameters in the next section.

It is thus worth to make a list of the differential forms on the internal space that are invariant under this transformation. Not surprisingly, the polyforms Φ_{\pm} will turn out to belong to this class. Starting from one-forms, there are only two of them that are $\text{SO}(3)_D$ invariant:

$$\{dr, y^i e^i\}. \quad (4.2.19)$$

A third possible candidate is vanishing: $y^i Dy^i = 0$. Making use of the Cartan structure equation, that under the identification (4.2.14) reads $de^i = \epsilon^{ijk} e^j \mathcal{A}^k$, it is easy to see that: $d(y^i e^i) = Dy^i e^i$. This suggests that the subspace of invariant forms is closed under derivation, which is indeed the case.

Moving on to two-forms the structure becomes richer as there are five $\text{SO}(3)_D$ invariant combinations living on M_6 that are given by:

$$\{\text{vol}_2, y^i \mathcal{F}_2^i, e^i Dy^i, e^i \star Dy^i, dr y^i e^i\}, \quad (4.2.20)$$

where $\star Dy^i$ are defined as the Hodge dual forms of Dy^i on the two-sphere, where the Hodge operator is computed with respect to the volume (4.2.15), that is to say: $\star Dy^i \equiv -\epsilon^{ijk} y^j Dy^k$.

For later purposes, it is useful to compute the exterior derivatives of these forms. We already know that $e^i Dy^i$ is an exact form, being equal to $-d(y^i e^i)$. It is also possible to identify a two-form which is closed but not exact; since $d(y^i \mathcal{F}_2^i) = -d(\text{vol}_2) = \mathcal{F}_2^i Dy^i$, it follows that:

$$d(\text{vol}_2 + y^i \mathcal{F}_2^i) = 0. \quad (4.2.21)$$

In the space spanned by $\text{SO}(3)_D$ invariant two-forms, this is the only non-trivial closed form. To complete the list of derivation rules, we also have: $d(e^i \star Dy^i) = 2y^i e^i (\text{vol}_2 - y^i \mathcal{F}_2^i)$.

Finally it is worth noticing that there exist a triplet of forms that are a natural candidate for a possible $\text{SU}(2)$ structure on M_6 . Indeed, the following algebraic relations are valid:

$$(e^i Dy^i)^2 = (e^i \star Dy^i)^2 = (\text{vol}_2 - \frac{6}{R} y^i \mathcal{F}_2^i)^2, \quad (4.2.22)$$

space of isometries of F . Now it can be shown that an isometry ξ of F preserves the total metric g^E if and only if $d\xi + [\xi, A] = 0$, where the bracket is the Lie bracket of vectors on F ; in other words, if ξ is a covariantly constant section of the bundle $\text{ad}(E)$, the adjoint bundle associated to E . If $F = S^1$, the Lie bracket vanishes and one can take ξ to be constant over B . With more complicated F 's, $\text{ad}(E)$ is often non-trivial and does not have a non-trivial global section; thus ξ cannot be promoted to an isometry of E .

meaning that there are three two-forms that square to the same four-form and that are orthogonal to each other, just like the forms $\{j, \text{Re}\omega, \text{Im}\omega\}$ in formula (4.1.7). In fact, as we will see shortly, the pure spinors will turn out to be defined in terms of these forms.

4.2.2 Spinors

In the first chapter of this thesis we derived an explicit expression for the supercharges of the $\text{AdS}_7 \times M^3$ solutions. In particular, we saw how the $\text{SU}(2)$ isometry of the S^2 , which is the R-symmetry of the solution, acts on the internal and external spinors. This is implemented by having (ζ_c) transform as a doublet, and at the same time the internal spinors:

$$\chi^{1a} \equiv \begin{pmatrix} \chi^1 \\ \chi^{1c} \end{pmatrix} \in 2, \quad \chi^{2a} \equiv \begin{pmatrix} \chi^2 \\ -\chi^{2c} \end{pmatrix} \in 2, \quad (4.2.23)$$

where we introduced the $\text{SU}(2)$ spinor index: $a = \{1, 2\}$.

We now want to decompose the AdS_7 supercharges in a way which is appropriate to describe an AdS_4 compactification. This is easily accomplished with $\zeta_{\text{AdS}_7} \rightarrow \zeta_{\text{AdS}_4} \otimes \tilde{\chi}$, where $\tilde{\chi}$ is a complex spinor on the three-manifold Σ_3 and the Killing spinor on AdS_4 is a real non-chiral spinor that we can write as: $\zeta_{\text{AdS}_4} = \zeta + \zeta^*$.

The corresponding gamma matrices decomposition is: $\gamma_\mu^{(7)} = \gamma_\mu^{(4)} \otimes 1$, $\gamma_{i+3}^{(7)} = \gamma^{(4)} \otimes \tilde{\gamma}^i$, where $\tilde{\gamma}^i$ is a representation of the $\text{SO}(3)$ algebra associated to the tangent space of Σ_3 . The charge conjugation matrix on this basis is $B^{(7)} = 1 \otimes i\sigma^2$.

If we now plug this decomposition into the ten-dimensional gamma matrices (3.1.2), we immediately realize that a change of basis is needed in order to get a proper $10 = 4 + 6$ representation of the type of Eq. (4.1.3). We thus rotate the ten-dimensional spinors according to $\epsilon \rightarrow O\epsilon$, where the change of basis is parametrized by a matrix of the form: $O = \frac{1}{\sqrt{2}}(1 + i\rho)$, where $\rho^2 = 1$ in such a way that $O^{-1} = O^* = \frac{1}{\sqrt{2}}(1 - i\rho)$. The corresponding transformation law for the gamma matrices is $\Gamma \rightarrow O\Gamma O^{-1}$, which amounts to: $\Gamma \rightarrow \Gamma$ if Γ and ρ commute, and to: $\Gamma \rightarrow i\rho\Gamma$ if Γ and ρ anticommute. The charge conjugation matrix transforms as $B \rightarrow OB(O^*)^{-1}$.

A proper choice is $\rho = \gamma^{(4)} \otimes 1 \otimes 1 \otimes \sigma^2$, which leads to our final $4 + 3 + 3$ gamma matrices representation:

$$\begin{aligned} \Gamma_\mu^{(4+3+3)} &= i\gamma_\mu^{(4)}\gamma_\mu^{(4)} \otimes 1 \otimes 1 \otimes 1, \\ \Gamma_{i+3}^{(4+3+3)} &= \gamma^{(4)} \otimes \tilde{\gamma}^i \otimes 1 \otimes \sigma^2, \\ \Gamma_{i+6}^{(4+3+3)} &= \gamma^{(4)} \otimes 1 \otimes \gamma^i \otimes \sigma^3, \end{aligned} \quad (4.2.24)$$

where the index $i = \{1, 2, 3\}$ runs over both the manifold Σ_3 where the branes are wrapped and on M_3 , and γ^i and $\tilde{\gamma}^i$ are in principle two different representations of the $\text{SO}(3)$ algebra. In this basis chirality and charge conjugation are represented as: $\Gamma = \gamma^{(4)} \otimes 1 \otimes 1 \otimes (-\sigma^1)$, $B = 1 \otimes i\sigma^2 \otimes i\sigma^2 \otimes \sigma^3$.

The resulting transformed supercharges are:

$$\epsilon_+^1 = \zeta_+(\tilde{\chi}\chi^1 + \tilde{\chi}^c\chi^{1c})w_+ + \text{c.c.}, \quad \epsilon_-^2 = \zeta_+(\tilde{\chi}\chi^2 - \tilde{\chi}^c\chi^{2c})w_- + \text{c.c.}, \quad (4.2.25)$$

where w_{\pm} are eigenvectors of $-\sigma_1$, namely $w_{\pm} = \frac{1}{\sqrt{2}}(v^{\pm} \mp v^{-})$.

As we anticipated in the previous section, it is very convenient to rewrite our spinor Ansatz in such a way as to make the twisted symmetry (4.2.18) manifest. We already know from (4.2.23) the transformation properties for χ^1 and χ^2 , and we also know that the AdS_4 spinor ζ has to be invariant, since we are looking for $\mathcal{N} = 1$ solutions.

Therefore it seems natural to assume that the spinor $\tilde{\chi}$ living on Σ_3 transforms under local Lorentz transformation on Σ_3 in such a way as to compensate the variation of χ^a under S^2 isometry. To this aim we introduce a new $\text{SU}(2)$ doublet $\tilde{\chi}^a \equiv \begin{pmatrix} \tilde{\chi} \\ \tilde{\chi}^c \end{pmatrix}$ transforming in the $\bar{2}$.⁶ At this point it is crucial to notice that both $\tilde{\chi}$ and its conjugate also carry a spacetime spinor index $\alpha = \{1, 2\}$. Here we are assuming that they also transform as a doublet, which amounts to imposing a condition on them, the twisting condition:

$$U^{*ab}\tilde{\chi}^{b\alpha} = \tilde{\chi}^{a\beta}U^{\beta\alpha}. \quad (4.2.26)$$

This constraint is solved by choosing the spinor to be equal to the epsilon tensor:⁷

$$\tilde{\chi}^{a\alpha} = \epsilon^{a\alpha} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.2.28)$$

Remarkably, this expression coincides with the explicit form for the twisted spinor which is given in the $\text{AdS}_4 \times \Sigma_3$ solutions of seven-dimensional gauged supergravity in [39].

To summarize, we achieved the goal of rewriting the two six-dimensional internal spinors of Eq. (4.1.4) in a form which is manifestly invariant under the twisted $\text{SU}(2)$ symmetry:

$$\eta_+^1 = \tilde{\chi}^a\chi^{1a}w_+, \quad \eta_-^2 = \tilde{\chi}^a\chi^{2a}w_-, \quad (4.2.29)$$

which also implies that the four-dimensional spinor ζ is a singlet, as we expect for an $\mathcal{N} = 1$ vacuum in four dimensions.

⁶Of course the $\text{SU}(2)$ representations 2 and its conjugate $\bar{2}$ are equivalent. What we want to highlight here is that if χ^a transform as $\chi^a \rightarrow U^{ab}\chi^b$ then $\tilde{\chi}^a$ has to transform as $\tilde{\chi}^a \rightarrow U^{*ab}\tilde{\chi}^b$ in such a way to make the product $\tilde{\chi}^a\chi^a$ invariant.

⁷We can be a bit more explicit by choosing a representation for the gamma matrices on the tangent space to Σ_3 to be $\tilde{\sigma}^i = -(\sigma^i)^*$. We then define the spinor rotation matrix with respect to the euclidean rotation matrix in the following way:

$$O^{ij}\sigma^j = U^\dagger\sigma^i U, \quad O^{ij}\tilde{\sigma}^j = U^T\tilde{\sigma}^i U^*. \quad (4.2.27)$$

This identity implies that the spinor $\tilde{\chi}$, which is defined with respect to the generators in the $\tilde{\sigma}^i$ representation, transforms under local Lorentz transformation as $\tilde{\chi}^\alpha \rightarrow (U^T)^{\alpha\beta}\tilde{\chi}^\beta$. We also want $\tilde{\chi}$ and its conjugate to transform as a doublet under the same symmetry: $\tilde{\chi}^a \rightarrow U^{*ab}\tilde{\chi}^b$. Then the full spinor along Σ_3 is represented by a 2×2 matrix $\tilde{\chi}^{a\alpha}$, which gets constrained by setting the two transformation laws to be equivalent; this leads to (4.2.26).

4.2.3 Pure spinors on $M_6 = \Sigma_3 \times M_3$

In the last section we formulated a compactification Ansatz for both the metric and the spinors, now we move our attention to the pure spinor Ansatz.

We already introduced a possible parametrization for Φ_{\pm} in terms of a triplet of forms $\{z, j, \omega\}$, defining a so-called SU(2) structure on M_6 . That parametrization, (4.1.8), is valid in general. Our goal is now to determine those forms explicitly for the particular case of our interest, namely for an internal manifold given by the non-trivial bundle $M_3 \rightarrow \Sigma_3$. Not surprisingly, our final Ansatz will be naturally written in terms of the set of $SO(3)_{\mathbb{D}}$ invariant forms of subsection 4.2.1.

Also, understanding the $6 = 3 + 3$ splitting of the internal space at the level of pure spinors will require some generalized geometry techniques in three dimensions, which we already reviewed in section 3.1.

Pure spinors on Σ_3 and M_3

The first step is formulating an Ansatz for the bispinors living on the three-manifolds Σ_3 and M_3 . We will later be able to reassemble the three-dimensional pure spinors into the six-dimensional ones.

We already know from the spinor Ansatz (4.2.25) that there is a crucial difference between the two three-dimensional factors: namely, we have one single spinor $\tilde{\chi}$ on Σ_3 , while we have two spinors χ^1, χ^2 on M_3 .

We start from the space Σ_3 . It makes sense to define three-dimensional bispinors in a similar way as in Eq. (3.1.6). Namely, on Σ_3 :

$$\tilde{\psi}_1 = \tilde{\chi} \otimes \tilde{\chi}^\dagger, \quad \tilde{\psi}_2 = \tilde{\chi} \otimes \tilde{\chi}^{c\dagger}. \quad (4.2.30)$$

Following the approach of subsection 3.2.2, it is much more convenient to organize the bispinors in the following 2×2 matrix:

$$\tilde{\Psi} \equiv \begin{pmatrix} \tilde{\chi} \\ \tilde{\chi}^c \end{pmatrix} \otimes (\tilde{\chi}, \tilde{\chi}^c)^\dagger = \begin{pmatrix} \tilde{\psi}_1 & \tilde{\psi}_2 \\ -(-)^{\deg(\tilde{\psi}_2)^*} & (-)^{\deg(\tilde{\psi}_1)^*} \end{pmatrix}, \quad (4.2.31)$$

where $(-)^{\deg}$ acts as \pm on even (odd) forms. The advantage of this choice is that now we can expand these 2×2 hermitean matrices on the basis $\tilde{\sigma}^\mu = (1, \tilde{\sigma}^i)$, where $\tilde{\sigma}^i = -(\sigma^i)^*$ are the SU(2) generators in the $\bar{2}$ representation.

We can then use the explicit parametrization for $\tilde{\psi}^1$ and $\tilde{\psi}^2$ given in Eq. (3.1.21). In this case we have one single spinor $\tilde{\chi}$ on Σ_3 , so we can simplify those expressions assuming $\psi = 0$, $\theta_1 = 0$, $\theta_2 = 0$; the result is very compact and elegant:

$$\tilde{\Psi}_0 = 1, \quad \tilde{\Psi}_1 = -e^i \tilde{\sigma}^i. \quad (4.2.32)$$

The subscript indicates the degree of the forms. The remaining components of $\tilde{\Psi}$ are determined via Hodge duality as $\tilde{\Psi}_2 = -i \star_3 \tilde{\Psi}_1$, $\tilde{\Psi}_3 = -i \star_3 \tilde{\Psi}_0$. Notice that the expressions (4.2.32) are automatically covariant under the SO(3) of local Lorentz transformations even before solving the supersymmetry equations; with some abuse of language, we will say that they are

covariant “off-shell”. Indeed if we perform a local Lorentz transformation $e^i \rightarrow O^{ij} e^j$, it is clear that this can be traded with $\tilde{\sigma}^i \rightarrow (O^T)^{ij} \tilde{\sigma}^j = U^* \tilde{\sigma}^i U^T$, namely the matrix $\tilde{\Psi}$ transforms covariantly as $\tilde{\Psi} \rightarrow U^* \tilde{\Psi} U^T$.

Things are a bit more complicated on M_3 where we have two spinors χ_1, χ_2 . What happens is that the expression for Ψ is not automatically covariant under the $SU(2)$ that rotates the S^2 . However, as shown in (3.2.38), it became covariant “on shell”, meaning after solving the supersymmetry equations. This in effect means that the analysis there started with random spinors on the S^2 , and that imposing supersymmetry also required them to be Killing spinors when restricted to the S^2 .

Here we will just assume the $SU(2)$ covariance from the start. We will simply take the expression for the bispinors (3.2.38) and covariantize it by replacing dy^i with $Dy^i = dy + \epsilon^{ijk} y^j \mathcal{A}^k$. We must also substitute the radius of the two-sphere for the AdS_7 solution with the AdS_4 one, that is to say: $\frac{1}{4} e^A \sqrt{1-x^2} \rightarrow f$. All in all, we get:⁸

$$\Psi_0 = ix \, 1 + \sqrt{1-x^2} \, y^i \sigma^i, \quad \Psi_1 = \sqrt{1-x^2} dr \, 1 + i (xy^i dr + f \, Dy^i) \sigma^i. \quad (4.2.33)$$

Again the remaining components are determined by covariantizing the Hodge duals $\Psi_2 = -i \star_3 \Psi_1$, $\Psi_3 = -i \star_3 \Psi_0$.

The matrix Ψ now transforms covariantly under the diagonal symmetry in (4.2.18), which we can trade for

$$\sigma^i \rightarrow (O^T)^{ij} \sigma^j = U \sigma^i U^\dagger. \quad (4.2.34)$$

This implies $\Psi \rightarrow U \Psi U^\dagger$.

Assembling the pure spinors on Σ_3 and M_3

We will now assemble the pure spinors (4.2.33) and (4.2.32) that we have found on Σ_3 and M_3 , and find expression for the six-dimensional pure spinors (4.1.5).

We start from the odd form Φ_- , that we rewrite as:

$$\begin{aligned} \Phi_- &= \eta_+^1 \otimes \eta_-^{2\dagger} = \sum_{k=0}^6 \frac{1}{8k!} \eta_-^{2\dagger} \gamma_{M_k \dots M_1} \eta_+^1 dx^{M_1 \dots M_k} \\ &= \frac{1}{8} \sum_{q=0}^3 \sum_{k=0}^3 \frac{1}{q! k!} \eta_-^{2\dagger} \gamma_{M_q \dots M_1} \gamma_{\tilde{M}_k \dots \tilde{M}_1} \eta_+^1 dx^{\tilde{M}_1 \dots \tilde{M}_k} dx^{M_1 \dots M_q}. \end{aligned} \quad (4.2.35)$$

We now plug into this formula the spinor Ansatz (4.2.29), together with the

⁸Like in the case of formula (4.2.17), we gave the quantity x the same name that it had for the AdS_7 solutions. These two quantities are in principle different, and indeed we will see how x_4 can be determined in terms of x_7 making use of a supersymmetric map described in subsection 4.3.1.

explicit gamma matrix representation given in (4.2.24), and get:⁹

$$\Phi_- = \frac{1}{2} \left[(\tilde{\chi}^b \otimes \tilde{\chi}^{a\dagger})_+ (\chi_1^b \otimes \chi_2^{a\dagger})_- + i (\tilde{\chi}^b \otimes \tilde{\chi}^{a\dagger})_- (\chi_1^b \otimes \chi_2^{a\dagger})_+ \right]. \quad (4.2.36)$$

Comparing this expression with the 3d bispinor matrices we defined in (4.2.31), we realize that we can write more compactly

$$\Phi_- = \frac{1}{2} \text{tr} \left(\tilde{\Psi}_+^T \Psi_- + i \tilde{\Psi}_-^T \Psi_+ \right). \quad (4.2.37)$$

An analogous expression can be obtained for the even pure spinor Φ_+ :

$$\Phi_+ = i \eta_+^1 \otimes (\eta_+^{2c})^\dagger = \frac{1}{2} \text{tr} \left(\tilde{\Psi}_-^T \Psi_- - i \tilde{\Psi}_+^T \Psi_+ \right), \quad (4.2.38)$$

where the i factor in the definition is chosen in order to get a real zero-form part Φ_0 . Notice that the pure spinors Φ_\pm are naturally invariant under the twisted symmetry (4.2.18), as they should be. This can be seen by assembling the transformation rules we found for $\tilde{\Psi}$ and Ψ in the previous section:

$$\Psi \rightarrow U \Psi U^\dagger, \quad \tilde{\Psi} \rightarrow U^* \tilde{\Psi} U^T \implies \Phi_\pm \text{ invariant}. \quad (4.2.39)$$

The next step is plugging into Φ_\pm the explicit expressions for the matrices Ψ^{ab} and $\tilde{\Psi}^{ab}$ we gave in (4.2.33) and (4.2.32). As expected, the pure spinors turn out to be naturally expressed in terms of the twisted forms that we introduced in subsection 4.2.1, decorated properly with the warping functions f, g that we introduced in the internal metric. In particular they can be written in the dielectric form (4.1.8), where the complex one-form z is given by: $z = dr + ig y^i e^i$, and the two-forms $\{j, \omega\}$ are

$$j = -fg e^i \star D y^i, \quad \text{Re} \omega = -fg e^i D y^i, \quad \text{Im} \omega = f^2 \text{vol}_2 - \frac{6g^2}{R} y_i \mathcal{F}_2^i. \quad (4.2.40)$$

From (4.2.22) we see that these forms satisfy the set of algebraic constraints that define an $\text{SU}(2)$ structure, Eq. (4.1.7).¹⁰

We also get the following identification: $x = \cos \psi$, where ψ is the angle in (4.2.33). This provides a natural interpretation of the variable x in terms of the angle between the two six-dimensional spinors.

Finally, we also get a vanishing phase $\theta = 0$, which means that we are in the special case considered in [72, Sec. 5.2]. The pure spinor equations were

⁹(4.2.36) is obtained after some manipulations that involve computing the quantity $w_- \sigma_3^q \sigma_2^k w_+$, which is equal to 0 if $q+k$ is even, to 1 if k is even and q is odd, and to i if k is odd and q even.

¹⁰There are other linear combinations of the triplet of forms (4.2.22) that satisfy (4.1.7). One can see that the coefficient of j and ω along $dr \wedge y^i e^i$ has to vanish; the remaining coefficients describe a set of quadratic equations, which can be interpreted as describing a frame $\{\text{Re} \omega, \text{Im} \omega, j\}$ in a four-dimensional space of signature (3, 1). This might lead to a more general class of solutions, which however would not be interpreted as compactifications of the AdS_7 solutions of [1].

analyzed in detail there; (5.16)–(5.18) in that paper give the constraints on the geometry and the fluxes in terms of the $SU(2)$ structure $\{z, j, \omega\}$. Recall that before using the equations in that form we have to transform Φ_{\pm} to the corresponding pair Φ_{\pm}^0 with vanishing intrinsic b -field, as in (4.1.10). We also have to rescale the pure spinors as $\Phi_{\pm} \rightarrow e^{3A-\phi}\Phi_{\pm}$, which amounts to fixing their norm in (4.1.8) to $\rho = e^{3A-\phi} \cos \psi$ as in [72, Eq. 2.2]. Using the results of appendix C, after some work the supersymmetry equations reduce to a coupled system of ODE's which we now proceed to give.

4.2.4 The system of ODEs

Using the pure spinor Ansatz we formulated in the last section we were able to reduce the supersymmetry equations to a coupled system of five ODEs, which is necessary and sufficient to find an AdS_4 solution. We will write this general system later, in Sec. 4.4.

Indeed, our compactification Ansatz is not over, since we are now going to impose a certain constraint, which turns out to further simplify the supersymmetry equations.

Originally we found a simplification by noticing empirically that many solutions to the general system of ODE had a constant ratio between the functions g and e^A in (4.2.17), which are the “radii” of Σ_3 and of AdS_4 respectively. We thus assumed that ge^{-A} is constant.

As usual for a dynamical system, if one imposes a constraint one needs to worry about possible “secondary constraints”; in our case, we need to check what happens when we impose $\partial_r(ge^{-A}) = 0$. We do get a secondary constraint: it turns out that $\frac{fe^{-A}}{\sqrt{1-x^2}}$ needs to be constant as well. In principle we could get now a third constraint as well, but imposing compatibility with the general system of ODE's (4.4.77) of this second constraint we simply end up fixing both constants.

This procedure actually only works when Σ_3 has Ricci scalar $R < 0$; without loss of generality we then fix $R = -6$. The result is then

$$f = \frac{2}{5}e^A\sqrt{1-x^2}, \quad g = \frac{2}{\sqrt{5}}e^A. \quad (4.2.41)$$

In other words, within the five-dimensional space spanned by the parameters $\{f, g, A, x, \phi\}$, we have found a three-dimensional subspace that is left invariant by the flow, with the restriction that Σ_3 has to be an Einstein space of negative curvature.

A posteriori this assumption is quite natural, and indeed it was later found very useful for the AdS_5 solutions of [4] as well. A rough justification is as follows. The holographic dual of putting a CFT_6 on $\mathbb{R}^3 \times \Sigma_3$ would consist in replacing $ds_{AdS_7}^2 = \frac{d\rho^2}{\rho^2} + \rho^2 ds_{\mathbb{R}^6}^2$ with $\frac{d\rho^2}{\rho^2} + \rho^2(ds_{\mathbb{R}^3}^2 + ds_{\Sigma_3}^2)$. In the IR, if this leads to a CFT_3 , one would expect that the ρ^2 in front of

$ds_{\Sigma_3}^2$ somehow disappears; our Ansatz is somehow that it does not also get multiplied by a further function of Σ_3 , or worse.

The general system now simplifies quite a bit; after eliminating f and g using (4.2.41), it only involves the warping factor A , the dilaton ϕ and the angle between the two six-dimensional spinors $x = \cos\psi$. Moreover, two equations become redundant. The system then becomes

$$\begin{aligned}\partial_r\phi &= \frac{1}{8} \frac{e^{-A}}{\sqrt{1-x^2}} (21x - 6x^3 + 2(5-2x^2)F_0 e^{A+\phi}), \\ \partial_r x &= \frac{1}{4} e^{-A} \sqrt{1-x^2} (3x^2 - 8 + 2xF_0 e^{A+\phi}), \\ \partial_r A &= \frac{1}{8} \frac{e^{-A}}{\sqrt{1-x^2}} (5x + 2e^{A+\phi}F_0).\end{aligned}\tag{4.2.42}$$

Notice that this system looks very similar to the corresponding BPS equations in AdS₇ given in (3.3.59). As we will see shortly, this similarity can be made more explicit. Also, with the constraints (4.2.41), the full ten-dimensional metric (4.2.17) becomes

$$ds_{10}^2 = e^{2A} \left(ds_{\text{AdS}_4}^2 + \frac{4}{5} ds_{\Sigma_3}^2 \right) + dr^2 + \frac{4(1-x^2)}{25} e^{2A} Ds_{S^2}^2.\tag{4.2.43}$$

Notice the similarity with the general form of the AdS₇ metric (3.2.25).

Let us also give the form of the fluxes here. Their general expression, for the choice (4.2.41), looks quite simple:

$$\begin{aligned}F_2 &= -q (\text{vol}_2 + y^i \mathcal{F}_2^i) - \frac{2F_0}{5} x q e^{A+\phi} \text{vol}_2, \\ F_4 &= -\frac{2}{5} q e^A dr \mathcal{F}_2^i \star Dy^i + \frac{2}{5} x q^2 e^{A+\phi} y^i \mathcal{F}_2^i \text{vol}_2, \\ H &= \frac{2}{5} e^A dr y^i \mathcal{F}_2^i + \frac{2}{5} x q e^{A+\phi} \mathcal{F}_2^i Dy^i - \left(\frac{3(x^2-3)}{2} e^{-A} + x F_0 e^\phi \right) \text{vol}_{M_3}.\end{aligned}\tag{4.2.44}$$

These expressions are again very similar to the fluxes for the AdS₇ solutions of subsection 3.2.1. There, F_2 only had a component along the volume of the S^2 , H only had a component along the volume of the internal manifold M_3 , which is defined according to the internal metric in (4.2.43). In analogy with the AdS₇ case we have defined:

$$q \equiv \frac{2}{5} \sqrt{1-x^2} e^{A-\phi} = \text{radius}(S^2) e^{-\phi}.\tag{4.2.45}$$

It is not by chance that we gave the two quantities the same name. Indeed, as we are about to explain, they will turn out to be the same function.

4.3 Analytic solutions

In this section we give an infinite class of analytic solutions for the AdS₄ × Σ_3 compactifications of the AdS₇ solutions of type IIA supergravity. These

are dual to compactifications of the $(1,0)$ CFT_6 on a three-manifold Σ_3 to some $\mathcal{N} = 1$ CFT_3 . Also, Σ_3 is constrained to be a maximally symmetric and has negative curvature. In the compactification procedure part of the supersymmetry is preserved imposing a twisting condition that breaks the original $\text{SU}(2)$ R-symmetry completely.

These analytic solutions describe near horizon geometries of brane systems involving NS5's, D6's and D8's wrapped on a the compact three-manifold Σ_3 . It is also possible to include orientifold planes. Beyond their field theory interpretation, these solutions are clearly also interesting as four-dimensional vacua with localized sources.

As in the case of the AdS_7 solutions of [1] that we presented in the previous chapter, we were able to solve the BPS equations analytically after the discovery of a surprising map between the AdS_7 system and the one obtained after the compactification to AdS_4 . This map was then generalized to a second map between the AdS_7 system and the one describing AdS_5 compactifications. The latter was then solved analytically in [4]. Using the maps it was finally possible to produce analytic solutions also in seven and four dimensions.

4.3.1 Supersymmetric maps

We have noticed already a few similarities with the AdS_7 solutions of [1]. In particular, the ODE system in seven dimensions (3.3.59) looks very similar to the one we obtained compactifying to four dimensions, (4.2.42). Remarkably, the two systems are mapped into each other by:

$$\begin{aligned} e^{A_4} &= \left(\frac{5}{8}\right)^{3/4} e^{A_7}, & e^{\phi_4} &= \left(\frac{8}{5}\right)^{1/4} \frac{e^{\phi_7}}{\sqrt{1 + \frac{3}{5}x_7^2}}, \\ r_4 &= \left(\frac{5}{8}\right)^{1/4} r_7, & x_4 &= \left(\frac{8}{5}\right)^{1/2} \frac{x_7}{\sqrt{1 + \frac{3}{5}x_7^2}}. \end{aligned} \tag{4.3.46}$$

Actually the requirement that the two systems are mapped into each other leaves one parameter free, which we fixed by requiring that the function q , defined in AdS_7 as (3.2.31), transforms into the analogue q of (4.2.45) in AdS_4 . This quantity played a crucial role in the discussion of flux quantization of the AdS_7 solution. The identification $q_7 = q_4$, will essentially imply that flux quantization works in the same way in four dimensions.

Acting with the map (4.3.46) on the metric Ansatz (4.2.43) allows to define a one to one correspondence between an AdS_7 solution of the form (3.2.25) and its AdS_4 compactification according to:

$$\begin{aligned} e^{2A} ds_{\text{AdS}_7}^2 + dr^2 + \frac{1-x^2}{16} e^{2A} ds_{S^2}^2 &\rightarrow \\ \sqrt{\frac{5}{8}} \left[\frac{5}{8} e^{2A} \left(ds_{\text{AdS}_4}^2 + \frac{4}{5} ds_{\Sigma_3}^2 \right) + dr^2 + \frac{1-x^2}{2(5+3x^2)} e^{2A} Ds_{S^2}^2 \right]. \end{aligned} \tag{4.3.47}$$

For completeness we also have to mention that the map also implies a change of sign in the Romans mass, that is to say we find that $F_0^{(4)} = -F_0^{(7)}$. In what follows we will drop the annoying index labeling the dimension of the AdS factors, and we will stick to the convention that $F_0 = F_0^{(7)}$, to make the comparison easier with the AdS₇ solutions we presented previously.

The map (4.3.46) also inspired a similar map for the AdS₇ to AdS₅ compactifications on Riemann surfaces (3.3.60). Combining the two maps we get:

$$\begin{aligned} e^{A_4} &= \left(\frac{5}{6}\right)^{3/4} e^{A_5}, & e^{\phi_4} &= \left(\frac{6}{5}\right)^{1/4} \frac{e^{\phi_5}}{\sqrt{1 + \frac{x_5^2}{5}}}, \\ r_4 &= \left(\frac{5}{6}\right)^{1/4} r_5, & x_4 &= \left(\frac{6}{5}\right)^{1/2} \frac{x_5}{\sqrt{1 + \frac{x_5^2}{5}}}. \end{aligned} \tag{4.3.48}$$

This allows to associate to each AdS₅ solution an AdS₄ one, and vice versa. Once again the map has one free parameter which is fixed by requiring that $q_4 = \frac{2}{5}e^{A-\phi}\sqrt{1-x^2}$ transforms into $q_5 = \frac{1}{3}e^{A-\phi}\sqrt{1-x^2}$. With this choice q is a universal quantity for the AdS₇ solutions and all of their compactifications:

$$q_4 = q_5 = q_7. \tag{4.3.49}$$

In summary, the result of this section is that there is a one-to-one correspondence between solutions of the reduced BPS system (4.2.42) and solutions of the BPS system for AdS₇ solutions in [1, Eq.(4.17)]. Moreover, [4, Sec. 5.2] establishes that there is a one-to-one correspondence of AdS₇ solutions with AdS₅ \times Σ_2 solutions, with Σ_2 a Riemann surface:

$$\text{AdS}_4 \times \Sigma_3 \leftrightarrow \text{AdS}_7 \leftrightarrow \text{AdS}_5 \times \Sigma_2. \tag{4.3.50}$$

All in all, using these maps three infinite classes of analytic solutions can be found in AdS₇, AdS₅ and AdS₄, describing the holographic duals of the (1, 0) CFT₆'s and their compactifications on Riemann surfaces Σ_g to $\mathcal{N} = 1$ CFT₄'s and on hyperbolic three-manifolds Σ_3 to $\mathcal{N} = 1$ CFT₃'s.

This universality hides something deeper. As we will show in the next chapter, it can be generalized to a full reduction Ansatz for type IIA supergravity on the internal manifold M_3 . Remarkably, the reduction leads to minimal gauged supergravity in seven dimensions, a theory with sixteen supercharges and an SU(2) gauge field. This works independently on the details of M_3 , that is to say independently on the particular choice of brane configurations.

4.3.2 General solution

We now present the infinite class of AdS₄ analytic solutions, taking advantage of the complete analysis that we already performed in section 3.3 for the AdS₇ solutions.

Combining the map (4.3.46) and Eq. (3.3.62), any AdS₄ vacuum can be expressed in terms of a single function β , the same function we introduced to describe the AdS₇ solutions, whose general expression is given by Eq. (3.3.66).

Given a β , the ten-dimensional metric is completely determined as:¹¹

$$ds_{10}^2 = \frac{4}{9} \sqrt{-\frac{5\beta'}{8y}} \left(\frac{5}{8} ds_{AdS_4}^2 + \frac{1}{2} ds_{\Sigma_3}^2 - \frac{1}{16} \frac{\beta' dy^2}{y\beta} + \frac{\beta Ds_{S^2}^2}{10\beta - 4y\beta'} \right). \quad (4.3.52)$$

Analogously, it is possible to give an expression for the dilaton and fluxes in full generality in terms of the function β only. We get:

$$\begin{aligned} F_2 &= 4y \frac{\sqrt{\beta}}{\beta'} (\text{vol}_2 + y^i \mathcal{F}_2^i) - \frac{F_0}{9} \left(\frac{\beta' \sqrt{\beta}}{5\beta - 2y\beta'} \right) \text{vol}_2, \\ F_4 &= \frac{1}{9} dy \mathcal{F}_2^i \star Dy^i + \frac{4}{9} \left(\frac{y\beta}{5\beta - 2y\beta'} \right) y^i \mathcal{F}_2^i \text{vol}_2, \\ H &= \frac{\beta'}{36y\sqrt{\beta}} y^i \mathcal{F}_2^i dy - \left(\frac{\frac{1}{9}\sqrt{\beta}\beta'}{5\beta - 2y\beta'} \right) \mathcal{F}_2^i Dy^i + \left(\frac{27\beta - \frac{36}{5}y\beta' - \frac{F_0}{12y}\beta'^2}{\left(\frac{-\beta'}{10y}\right)^{1/4} (5\beta - 2y\beta')} \right) \text{vol}_{M_3}, \\ e^\phi &= \left(\frac{5}{2} \right)^{1/4} \frac{(-\beta'/y)^{5/4}}{12\sqrt{5\beta - 2y\beta'}}, \end{aligned} \quad (4.3.53)$$

where everything is written in terms of the set of SO(3)_D invariant forms which were defined in subsection 4.2.1.

One last step is needed. In fact, before we are able to claim that the map (4.3.50) is also a correspondence between solutions, we should also check that flux quantization is respected by it. Fortunately, thanks to the map (which leaves q invariant) the quantization conditions involving F_2 and B work essentially the same.

We can still write $B = \frac{F_2}{F_0} + b$, for b a closed two-form, which in AdS₇ was proportional to vol_{S^2} . In compactifying to AdS₄ the sphere gets fibered over Σ_3 , so it makes sense to define a covariant volume, (4.2.15), which is no longer closed.

As we explained in subsection 4.2.1, we can construct a new closed two-form by adding a term which has legs along Σ_3 . This form is: $(\text{vol}_2 +$

¹¹To be more precise, plugging the solution (3.3.62) for the AdS₇ system into (4.3.46) we get the following solution for the AdS₄ one:

$$e^{A_4} = \frac{5^{3/4}}{6} \left(-\frac{\partial_y \beta}{2y} \right)^{1/4}, \quad x_4 = \sqrt{\frac{-2y\partial_y \beta}{5\beta - 2y\partial_y \beta}}, \quad e^{\phi_4} = \left(\frac{5}{2} \right)^{1/4} \frac{(-\partial_y \beta/y)^{5/4}}{12\sqrt{5\beta - 2y\partial_y \beta}}. \quad (4.3.51)$$

Plugging this into the general expressions for the AdS₄ metric (4.2.43) we get Eq. (4.3.52). Similarly, the final expressions for the fluxes in (4.3.53) are obtained plugging the solution (4.3.51) into (4.2.44).

$y^i \mathcal{F}_2^i$). The quantization condition (3.2.51) is then simply replaced by $b = -\frac{n_2}{2F_0}(\text{vol}_2 + y^i \mathcal{F}_2^i)$, and everything else works the same. Even the subtle formula for the quantization of H in presence of D8's, Eq. (3.2.58), is left unchanged by the map.

What is new with respect to the AdS₇ solutions is the presence of a four-form flux F_4 , so we have to check that this is properly quantized. It is better to consider the modified flux $\tilde{F}_4 = F_4 - B \wedge F_2 + \frac{1}{2} B^2 F_0$. This can be written in terms of a gauge potential as $\tilde{F}_4 = dC_3$, for

$$C_3 = \frac{1}{2F_0} \left(q^2 - \frac{n_2^2}{4} \right) \mathcal{F}_2^i \star Dy^i, \quad (4.3.54)$$

where n_2 is the flux integer for \tilde{F}_2 as defined by Eq. (3.2.44). Near a regular point, regularity of B and F_2 implies that n_2 should be zero, and that $q \rightarrow 0$. Moreover, one can see from (4.2.42) that q starts linearly in the radial coordinate, so that in the end $C_3 \sim r^2 y^i e^i e^j Dy^j = x^i e^i e^j Dx^j$, where now the $x^i \equiv ry^i$ are coordinates on \mathbb{R}^3 ; so \tilde{C}_3 is a regular form, and \tilde{F}_4 has no periods in this case. In presence of sources, the discussion changes a bit. Flux quantization now requires the flux integrals to be integer for cycles that do not intersect the sources. We can take such cycles to be at fixed y ; then the only relevant term in \tilde{F}_4 is proportional to the form: $\text{vol}_2 \wedge y^i \mathcal{F}_2^i$, whose integral vanishes because $\int_{S^2} y^i = 0$.

All in all we do not have extra conditions to be imposed with respect to the AdS₇ solutions. This means that the classification of the possible analytic solutions works exactly in the same way as described in section (3.3.1), including the formulas for the quantization of the parameters $\{b_2, y_0\}$. This means that the AdS₄ vacua are also classified by the flux integers (\tilde{n}_2, n_2) at the two poles.

We can thus have AdS₄ solutions with one D6 stack at one pole and a regular point at the other, solutions with two D6 stacks, and solutions with one D6 stack and an orientifold plane at the opposite extremum. These provide interesting examples of four-dimensional vacua with localized sources.

4.3.3 Massless solution

It is straightforward to find a solution in the case where $F_0 = 0$. Indeed, a simple expression is available for β , given by Eq. (3.3.65). Plugging that expression into (4.3.52) gives the following internal space metric:

$$ds_{M_3}^2 = \sqrt{\frac{5}{8}} \frac{4}{9k} \frac{1}{\sqrt{y_0^2 - y^2}} \left(dy^2 + \frac{2(y_0^2 - y^2)^2}{5y_0^2 + 3y^2} Ds_{S^2}^2 \right). \quad (4.3.55)$$

As shown in App. A, this solution can be lifted to eleven dimensions to the AdS₄ solution corresponding to the near horizon geometry of a stack of M5-branes wrapped on Σ_3 . This can be seen applying a proper change of

variable $y = y_0 \cos \alpha$, and with the following identification: $y_0 = \frac{9}{32} R^3$. All in all, we get the following metric on the internal space:

$$ds_{M_3}^2 = \sqrt{\frac{5}{8}} \frac{R^3}{8k} \sin(\alpha) \left(d\alpha^2 + \frac{\sin^2(\alpha)}{10 + 6 \cos^2 \alpha} Ds_{S^2}^2 \right). \quad (4.3.56)$$

The behavior at poles of this metric is precisely the same as in the AdS₇ case, we have two singularities at the poles $\alpha = 0, \alpha = \pi$, that can be interpreted physically as D6 and anti-D6 singularities.

The presence of D6's can also be deduced looking at the fluxes. Indeed, plugging the solution (3.3.65) into Eq. (4.3.53) we get:

$$F_2 = -\frac{k}{2} (\text{vol}_2 + y^i \mathcal{F}_2^i), \quad (4.3.57)$$

$$F_4 = \frac{9R^3 \sin \alpha}{32} \mathcal{F}_2^i \star Dy^i d\alpha + \frac{R^3 \cos \alpha \sin^2 \alpha}{8(5 + 3 \cos^2 \alpha)} y^i \mathcal{F}_2^i \text{vol}_2. \quad (4.3.58)$$

The integral of F_2 over the S^2 is constant and equal to $-2\pi k$. We can take the S^2 close to the north or the south pole, where it signals the presence of D6-brane charge. More precisely, there are k anti-D6-branes at the north pole and k D6-branes at the south pole. The flux F_4 is regular at both poles, where both its components $\rightarrow 0$.

The three-form flux H has the following expression:

$$H = \frac{R^3 \sin \alpha}{16k} y^i \mathcal{F}_2^i d\alpha + \frac{R^3 \cos \alpha \sin^2 \alpha}{4k(5 + 3 \cos^2 \alpha)} \mathcal{F}_2^i Dy^i, \quad (4.3.59)$$

$$- \frac{3R^3(31 + \cos 2\alpha) \sin^3 \alpha}{8k(13 + 3 \cos 2\alpha)^2} d\alpha \text{vol}_2.$$

Around the poles where the D6 stacks are located, the first two components $\rightarrow 0$, while the third one shows the same divergent behavior that we had for the AdS₇ massless solution. Indeed, if we expand it around $\alpha = 0$ and introduce the coordinate $\rho = \alpha^2$, we get: $H \sim \rho^{-\frac{1}{4}} \text{vol}_{M_3}$. We should remember, in any case, that this solution is non-singular in eleven dimensions; the diverging behavior is cured by M-theory, just like the divergence of the curvature at the poles, where the D6 stacks are located.

A possible expression for the B field is

$$B = -\frac{R^3 \cos \alpha}{16k} y^i \mathcal{F}_2^i - \frac{R^3 \cos \alpha (9 - \cos^2 \alpha)}{16k(5 + 3 \cos^2 \alpha)} \text{vol}_2. \quad (4.3.60)$$

We actually used this expression in checking that (3.2.58) is also the correct flux quantization condition for AdS₄.

For completeness we also give the expression for the dilaton:

$$e^{4\phi} = \left(\frac{5}{8}\right)^{\frac{1}{2}} \frac{R^3 \sin^3 \alpha}{5 + 3 \cos^2 \alpha}. \quad (4.3.61)$$

Expanding around $\alpha = 0$ we get $e^\phi \sim \rho^{3/4}$, which is precisely what we expected from the D6-brane solution in flat space of Eq. (2.0.8).

4.3.4 One D6 stack

Again, the simplest example of solution with $F_0 \neq 0$ is the one with a single D6 stack, corresponding to the general massive solution (3.3.66) for the choice $b_2 = 12$. The resulting β has a single zero at $y = -y_0/2$ and a double zero at $y = y_0$. The metric on the internal space has the following simple form:

$$ds_{M_3}^2 = \sqrt{\frac{5}{24F_0}} \frac{1}{\sqrt{(y_0+2y)^2(y_0-y)}} \left(dy^2 + \frac{4}{3} \frac{(y_0-y)^2(y_0+2y)^2}{5y_0^2+5yy_0-2y^2} Ds_{S^2}^2 \right). \quad (4.3.62)$$

In analogy with what we did for the AdS₇ solution, we introduce a new radial coordinate $\rho = 2(1 - \frac{y}{y_0})$, with range $\rho \in [0, 3]$. Taking into account the quantization condition (3.3.69), the metric on the total space becomes:

$$ds_{10}^2 = \frac{n_2}{F_0} \sqrt{\frac{5\rho}{8}} \left(\frac{5}{6} ds_{\text{AdS}_4}^2 + \frac{2}{3} ds_{\Sigma_3}^2 + \frac{d\rho^2}{4\rho(3-\rho)} + \frac{2\rho(3-\rho)}{3(24-9\rho+\rho^2)} Ds_{S^2}^2 \right). \quad (4.3.63)$$

Around $\rho = 0$ the internal metric behaves as $\frac{1}{\sqrt{\rho}}(d\rho^2 + \rho^2 Ds_{S^2}^2)$, showing a physical singularity corresponding to a stack of D6-branes. On the other hand, around $\rho = 3$ the internal metric turns into flat space: $d\tilde{\rho}^2 + \tilde{\rho}^2 Ds_{S^2}^2$, after the change of coordinates $\tilde{\rho} = \sqrt{\rho-3}$. So we get a regular point.

The Ramond-Ramond fluxes can be expressed as:

$$\begin{aligned} F_2 &= -\frac{n_2}{2} \sqrt{1 - \frac{\rho}{3}} (\text{vol}_2 + y^i \mathcal{F}_2^i) + n_2 \sqrt{1 - \frac{\rho}{3}} \left(\frac{(2-\rho)\rho}{24-9\rho+\rho^2} \right) \text{vol}_2, \\ F_4 &= \frac{3n_2^2}{8F_0} \mathcal{F}_2^i \star Dy^i d\rho + \frac{n_2^2}{6F_0} \frac{\rho(6-5\rho+\rho^2)}{(24-9\rho+\rho^2)} y^i \mathcal{F}_2^i \text{vol}_2. \end{aligned} \quad (4.3.64)$$

Near $\rho = 0$ the two-form flux has a component $F_2 \sim -\frac{n_2}{2} \text{vol}_2$, which signals the presence a stack of n_2 D6-branes, as in (3.3.72). On the other hand, near the regular point F_2 vanishes, as it should since no source is localized there.

We can thus see that the quantization conditions that we have imposed on the AdS₇ solutions work correctly also in AdS₄.

Moving to the four-form flux, the first term goes to a constant value at both poles, while the second term goes to zero linearly near the D6 stack $\rho = 0$ and quadratically at the regular point $\tilde{\rho} = 0$.

The expression for the three-form flux is a bit more involved, but still it reproduces the expected behavior. We have:

$$\begin{aligned} H &= -\frac{n_2}{18} \sqrt{1 - \frac{\rho}{3}} y^i \mathcal{F}_2^i d\rho + \frac{n_2}{F_0} \sqrt{1 - \frac{\rho}{3}} \frac{\rho(2-\rho)}{(24-9\rho+\rho^2)} \mathcal{F}_2^i Dy^i \\ &\quad + \left(\frac{18F_0^2}{125n_2^2\rho} \right)^{1/4} \left(\frac{144-29\rho+\rho^2}{24-9\rho+\rho^2} \right) \text{vol}_{M_3}. \end{aligned} \quad (4.3.65)$$

The first term goes to a constant value both around the D6 stack $\rho = 0$ and at the regular point $\tilde{\rho} = 0$. The second term goes to zero linearly at

both extrema, using coordinates ρ and $\bar{\rho}$ respectively. The third term is the relevant one, and it reproduces the same behavior that we had in AdS₇: around $\rho = 0$ it is divergent: $H \sim \rho^{-1/4} \text{vol}_{M_3}$, around the regular pole it goes to a constant value.

Finally, the dilaton is determined as:

$$e^{4\phi} = \frac{40\rho^3}{n_2^2 F_0^2 (24 - 9\rho + \rho^2)^2}, \quad (4.3.66)$$

which goes to a constant value at the regular point $\rho = 3$ and goes to zero as $e^\phi \sim \rho^{3/4}$ at $\rho = 0$, as expected near a D6 stack, as defined by the D6 solution in flat space (2.0.8).

4.3.5 O6/D6

The solution that we present here is particularly relevant as a rare example of four-dimensional vacuum with a localized orientifold plane. It belongs to the second class of analytic solutions, with one D6 stack is located at $y = y_0$ and an orientifold plane at $y = y_1$. Our analysis will proceed in the exact same way as in subsection 3.3.4 for the corresponding AdS₇ solution.

Again, instead of discussing this class in full generality, we just give a single example corresponding to the choice of flux integer $n_2 = 6$. In other words we present a solution with an O6 plane at one side, and a stack of six D6-branes on the other side. Infinite more possibilities can be obtained varying n_2 , which is allowed to take any integer value except for zero.

We introduce a radial coordinate ρ defined by $y = \frac{y_0}{3}(\rho^2 - 1)$, with range $\rho \in [0, 2]$. The internal space metric corresponding to this solution takes the form:

$$ds_{M_3}^2 = \frac{3}{F_0} \sqrt{\frac{15\rho}{8(2-\rho)(1+\rho)^2}} \left(d\rho^2 + \frac{2(2-\rho)^2(1+\rho)^2}{3(6-2\rho+\rho^2)} Ds_{S^2}^2 \right). \quad (4.3.67)$$

Around $\rho = 0$ this behaves as $\sim \rho^{1/2}(d\rho^2 + \frac{4}{9}Ds_{S^2}^2)$, which is the correct behavior for an O6 plane, as defined by equation (3.2.43). Around $\rho = 2$ it goes like $\sim \frac{1}{\sqrt{\rho-2}}(d\rho^2 + (\rho-2)^2 Ds_{S^2}^2)$, so we get the usual D6 stack. The complete ten-dimensional metric is the one given in the introduction, Eq. (4.0.1).

The Ramond-Ramond fluxes take the following form:

$$F_2 = -\frac{3}{2}\rho (\text{vol}_2 + y^i \mathcal{F}_2^i) - \frac{3(2-\rho)(1-\rho^2)}{6-2\rho+\rho^2} \text{vol}_2, \quad (4.3.68)$$

$$F_4 = -\frac{9\rho}{4F_0} \mathcal{F}_2^i \star Dy^i d\rho - \frac{9\rho(2-\rho)(1-\rho^2)}{2F_0(6-2\rho+\rho^2)} y^i \mathcal{F}_2^i \text{vol}_2. \quad (4.3.69)$$

Near $\rho = 2$ the two-form flux behaves as $F_2 \sim -3(\text{vol}_2 + y^i \mathcal{F}_2^i)$, which integrated on the two-sphere reproduces the behavior for a stack of six D6-branes. On the other hand, near $\rho = 0$ we get $F_2 \sim -\text{vol}_2$, which corresponds to the charge of an O6.



Figure 4.1: A sketch of the internal M_3 in the solution with an orientifold plane at one pole, and one D6 stack at the other pole.

The four-form flux has a first term that goes to zero where the orientifold is located and to a constant in correspondence of the D6 stack. The second term goes to zero linearly at both extrema.

Again the expression for the three-form flux H is more involved. It reads:

$$\begin{aligned}
 H = & -\frac{3}{2F_0}y^i\mathcal{F}_2^i d\rho - \frac{3(2-\rho)(1-\rho^2)}{F_0(6-2\rho+\rho^2)}\mathcal{F}_2^i Dy^i \\
 & + \left(\frac{2F_0^2(5\rho)^{-3}}{6+9\rho-3\rho^2}\right)^{1/4} \left(\frac{20+26\rho-2\rho^2+\rho^3}{6-2\rho+\rho^2}\right)\text{vol}_{M_3} .
 \end{aligned}
 \tag{4.3.70}$$

The first term goes to a constant value at both poles, the second term is constant near the orientifold plane and vanishes linearly around the D6 stack $\rho = 2$. The relevant term is the one proportional to the internal volume, the third one. Around the D6 singularity $\rho \sim 2$ it behaves as $\sim (\rho-2)^{-1/4}\text{vol}_{M_3}$, while in correspondence of the O6 plane $\rho \sim 0$ it shows a different divergent behavior: $\sim \rho^{-3/4}\text{vol}_{M_3}$. These are the same type of singularities that we found for the corresponding AdS₇ solution.

Finally, the dilaton is determined as:

$$e^{4\phi} = \frac{40(2-\rho)^3(1+\rho)^2}{27F_0^2\rho^3(6-2\rho+\rho^2)^2} ,
 \tag{4.3.71}$$

which reproduces the correct D6 behavior $e^\phi \sim (\rho-2)^{3/4}$ around $\rho = 2$, while at the pole $\rho = 0$ where the orientifold plane is located it goes like $e^\phi \sim \rho^{-3/4}$. As described at the end of subsection 3.3.4, this singular behavior is precisely what we expect in presence of an O6 source.

4.3.6 Two D6 stacks

The last possibility we are left to explore is to have a massive solution with two D6 stacks localized at the two poles $y = y_1$ and $y = y_0$, with corresponding flux integers (\tilde{n}_2, n_2) . Our analysis will proceed in the exact same way as in subsection 3.3.5 for the corresponding AdS₇ solution.

We do not discuss the most general solution, which would look rather complicated. We find it clearer to give a simple example, the one corresponding to the choice of flux integers $(\tilde{n}_2, n_2) = (1, 2)$. In other words we

present a solution with a stack of two D6-branes at one pole and a single D6 at the other pole. According to formula (3.3.67), this particular choice of flux integers translates into the following quantization condition on the two parameters: $(y_0, b_2) = (\frac{15}{8F_0}, \frac{15}{2})$. The first stack is located at $y = y_0$, the second at $y_1 = -\frac{4}{5}y_0$.

We introduce a new radial coordinate ρ defined in terms of y as $y = \frac{y_0}{5}(3\rho^2 - 7)$, with range $\rho \in [1, 2]$. As a result, we get:

$$ds_{M_3}^2 = \frac{1}{F_0} \sqrt{\frac{15\rho}{8(7\rho - 6 - \rho^3)}} \left(d\rho^2 + \frac{2(7\rho - 6 - \rho^3)^2}{(98 - 90\rho + 21\rho^2 + 3\rho^4)} ds_{S^2}^2 \right). \quad (4.3.72)$$

Around $\rho = 1$ this goes like $\sim \frac{1}{\sqrt{\rho-1}}(d\rho^2 + (\rho-1)^2 ds_{S^2}^2)$, which is the correct behavior near a stack of D6-branes. The same happens around $\rho = 2$.

The Ramond-Ramond fluxes are:

$$\begin{aligned} F_2 &= -\frac{1}{2}\rho (\text{vol}_2 + y^i \mathcal{F}_2^i) + \left(\frac{42 - 49\rho - 18\rho^2 + 28\rho^3 - 3\rho^5}{98 - 90\rho + 21\rho^2 + 3\rho^4} \right) \text{vol}_2, \\ F_4 &= -\frac{\rho}{4F_0} \mathcal{F}_2^i \star D y^i d\rho + \frac{\rho}{2F_0} \left(\frac{42 - 49\rho - 18\rho^2 + 28\rho^3 - 3\rho^5}{98 - 90\rho + 21\rho^2 + 3\rho^4} \right) y^i \mathcal{F}_2^i \text{vol}_2. \end{aligned} \quad (4.3.73)$$

Near $\rho = 1$, the two-form flux has a component $\sim \frac{1}{2}\text{vol}_2$, which signals the presence a single D6-brane. Analogously, near $\rho = 2$ we get a component $\sim \text{vol}_{S^2}$, corresponding to a stack of two D6-branes.

The first term in F_4 goes o a constant value at both poles, while the second term goes to zero linearly near each D6 stack. We do not report the three-form flux for this solution, which we however checked to behave correctly.

Finally, the dilaton is:

$$e^{4\phi} = \frac{120(-6 + 7\rho - \rho^3)^3}{F_0^2 \rho^3 (98 - 90\rho + 21\rho^2 + 3\rho^4)^2}, \quad (4.3.74)$$

which goes like $e^\phi \sim (\rho-1)^{3/4}$ around $\rho = 1$ and like $e^\phi \sim (\rho-2)^{3/4}$ around $\rho = 2$. This is a further confirm that we are in presence of two D6 stacks at the two poles.

4.3.7 Field theory interpretation

Let us summarize the solutions in this section, and make a few comments about their field theory interpretation.

We have found an infinite class of $AdS_4 \times M_6$ solutions, where M_6 is a fibration of M_3 over Σ_3 ; M_3 is topologically $\cong S^3$, while Σ_3 is a compact quotient of hyperbolic space. These solutions are in one-to-one correspondence (4.3.46) with the AdS₇ solutions of [1]. In particular, the metric on

our M_3 is related to the internal manifolds in those AdS₇ solutions in the simple way (4.3.47). It is a fibration of a round S^2 over an interval, and as such it has SO(3) isometry group.

Our main aim was to find AdS₄ solutions dual to twisted compactifications of the (1,0) CFT₆ dual to the AdS₇ solutions. Because of the fibration structure of our solutions (which was part of our Ansatz), and of the one-to-one correspondence (which came out as a result), the solutions we found seem to be exactly what we were looking for.

We can contrast our solutions with the known massless one [41], this time from a field theory perspective. For the $\mathcal{N} = 1$ AdS₄ solution of eleven-dimensional supergravity (A.2.5), the internal space has SO(4) = SU(2)_L × SU(2)_R symmetry, and twisting mixes the SU(2)_L factor with the SU(2) of local Lorentz transformations on Σ_3 , leaving an SU(2) which is a flavor symmetry. There is no R-symmetry because the CFT₃ is only $\mathcal{N} = 1$ supersymmetric.

For our solutions (and indeed for the ten-dimensional reduction of the massless solution, studied in App. A), the isometry of the internal space is already just SU(2); twisting mixes it with the SU(2) of Σ_3 , so that in the end we have no flavor or R-symmetry. (Again this is in no contradiction with the fact that the CFT₃ is only $\mathcal{N} = 1$.) From the point of view of the gravity solution, the metric (4.2.43) has an S^2 factor, but the fact that it is non-trivially fibred means that the total space does not have SO(3) isometries: the presence of the connection breaks it. Even looking at the fluxes (4.2.44), we see that they contain components which break the SO(3) of the S^2 . Indeed, they are written in terms of a set of forms that are invariant under “twisted symmetry” (4.2.18), which of course cannot be considered an isometry: it is a mix of a local Lorentz transformation (which happens point by point on Σ_3) and of an internal rotation.

As an application, it is possible to count the number of degrees of freedom of the CFT₃, which parallels a similar observation in [4]. One can count the number of degrees of freedom of a CFT_d via the coefficient $\mathcal{F}_{0,d}$ in the free energy $\mathcal{F}_d = \mathcal{F}_{0,d} T^d \text{Vol}$, where T is the temperature. Holographically this evaluates to the integral of $e^{5A-2\phi}$ over M_3 for the CFT₆, and over M_6 for the CFT₃. Using the map (4.3.46), one finds easily that

$$\mathcal{F}_{0,3} = \left(\frac{5}{8}\right)^4 \text{Vol}(\Sigma_3) \mathcal{F}_{0,6} . \quad (4.3.75)$$

In other words, the ratio of degrees of freedom is universal. Since the AdS₇ solutions are now analytic, one can evaluate $\mathcal{F}_{0,6}$ explicitly; the results are given in section 3.5. This might help find the CFT₃.

However, the CFT₃'s are only $\mathcal{N} = 1$ supersymmetric, and have no flavor symmetry. For this reason, perhaps our solutions are more interesting as gravity solutions with localized sources; this was indeed our initial motivation. With this in mind, we will now return to our original system (4.4.77), and see if we can find more interesting solutions, irrespectively of their field

theory interpretation.

4.4 Attractor solutions

In the last section we obtained a very large set of analytic solutions, in one-to-one correspondence with the AdS₇ solutions of [1] and the AdS₅ solutions of [4], with a clear holographic interpretation. From the point of view of the geometry, those solutions are characterized by the requirement that the symmetric space Σ_3 needs to be hyperbolic.

In this section we will present another set of solutions, which depend on a larger number of parameters; we call them “attractor solutions”, for reasons that will become clear. They are only known numerically. They exist for all values (positive, null, negative) of the curvature of Σ_3 , although a positive sign appears to be preferred. As stated in the introduction, this solution does not appear to be the same as the AdS₄ \times $S^3 \times S^3$ solution of [64]. (One might also speculate of some duality to the IIB AdS₄ \times $S^3 \times S^3$ solution of [74], but we could not find any immediate relationship.)

The first sign that this class will be larger is that we will not impose the constraints (4.2.41) any longer. So we will have to revert to the original system of five ODEs. It will also not be possible any more to simplify the form of the metric like we did in (4.2.43), and we will have to keep the original form

$$ds_{10}^2 = e^{2A} ds_{\text{AdS}_4}^2 + g^2 ds_{\Sigma_3}^2 + dr^2 + f^2 Dy^i Dy^i. \quad (4.4.76)$$

4.4.1 The system of ODEs

The result of our analysis is that the pure spinor equations reduce, under the compactification Ansatz of section 4.2, to a system of five coupled ODE’s in five variables: the three warping functions in the metric $\{f, g, A\}$, the dilaton ϕ , and the angle between the two six-dimensional spinors $x = \cos \psi$. All of these functions depend on the radial coordinate r only. The system reads

$$\begin{aligned} \left(\frac{fg^2 e^{-A}}{\cos \psi} \right)' &= \frac{Rf^2 + 6g^2}{6 e^A \cos^2 \psi}, \\ (g e^{-A})' &= \frac{e^A (Rf^2 + 6g^2 \sin^2 \psi) - 12fg^2 \sin \psi}{6fg e^{2A} \cos \psi}, \\ (f e^{-\phi})' &= \frac{12fg^2 (e^A \sin \psi - f)}{e^A (Rf^2 - 6g^2 \sin^2 \psi)} F_0, \\ (g e^{3A})' &= \frac{g e^{3A} \cos \psi}{f} + \frac{12g^3 e^{2A+\phi} (f - e^A \sin \psi)}{(Rf^2 - 6g^2 \sin^2 \psi)} F_0, \\ \left(\frac{g^3 e^{3A}}{f^2} \right)' &= \frac{g e^{3A} R \cos \psi}{2f} - \frac{2g^3 e^{2A+\phi} (6fg^2 (\cos^2 \psi - 3) + e^A \sin \psi (Rf^2 + 12g^2))}{f^2 (Rf^2 - 6g^2 \sin^2 \psi)} F_0. \end{aligned} \quad (4.4.77)$$

We already mentioned in the previous section that a certain three-dimensional submanifold of the space of parameters is invariant under (4.4.77); this submanifold is defined by the set of constraints (4.2.41). On that submanifold the system was then reduced to a much more manageable system, so it was possible to solve it analytically.

Another interesting comparison can be made. Indeed, notice that in the massless limit $F_0 \rightarrow 0$ the first, fourth and fifth equation of this system reproduce the analogous equations in 11d supergravity given by [41, Eq.(9.71)–(9.73)]. The third fixes the function f in terms of the dilaton and the second is solved imposing the on shell constraints (4.2.41).

It so happens that the Bianchi identities for the fluxes are automatically satisfied. So (4.4.77) is the complete system we need to satisfy in order to find an AdS_4 solution.

Moreover, given a solution of (4.4.77), one can always find another rescaled solution for which the curvature and string coupling are both small, so that the supergravity approximation we are using in this paper is justified. This can be done by using the transformations [35, Eq.(4.2)–(4.3)]; the first is $F_0 \rightarrow nF_0$, $\phi \rightarrow \phi - \log n$, which is a symmetry of (4.4.77); the second has to be supplemented with transformation law for f and g :

$$(A, f, g, \phi, x, r) \rightarrow (A + \Delta A, e^{\Delta A} f, e^{\Delta A} g, \phi - \Delta A, x, e^{\Delta A} r) . \quad (4.4.78)$$

In what follows we will give some example of numerical solutions to (4.4.77), without giving the details of our analysis here. We refer to [3, Sec. 5] for a more complete description.

4.4.2 Numeric solutions

We studied numerically the system (4.4.77) with all three boundary conditions we discussed in subsection 3.2.3, corresponding to D6 singularity, O6 singularity and regular point. We allowed the manifold Σ_3 to have positive, null and negative curvature.

In what follows we will present the possible types of solutions corresponding to $\Sigma_3 = S^3$, but the behavior is essentially the same for T^3 and \mathbb{H}_3 . We expected to have to perform some fine-tuning in order to obtain a physical solution, arriving at one of the same three boundary conditions at the other pole. Indeed one often ends up at the other pole with a singularity that we cannot interpret physically, where numerically one sees $f \sim r^{1/3}$, $g \sim r^{-1/3}$, $e^A \sim r^{-1/3}$.

Even more often, however, one in fact ends up more or less automatically at the other pole with a regular point. This happens for a large *open set* in the space of the free parameters allowed by the possible boundary conditions (two for the regular boundary condition, three for the D6 and O6). In most other cases, one has instead to perform a number of fine tunings. In the present case, the regular point appears to be an attractor.

We show some examples of numerical solutions in figures 4.2 and 4.3(a). In all these cases, we started from the left with the relevant perturbative solution corresponding to a D6 stack, an O6 plane, or a regular point, and continued numerically. The solution then ends by itself in a point where $f = 0$ and the other functions go to constant values. Some solutions appear to display one or more mild kinks on the way to the attractor; one might worry about their effect on the curvature, but recall how (4.4.78) can be used to make the curvature as small as one wishes.

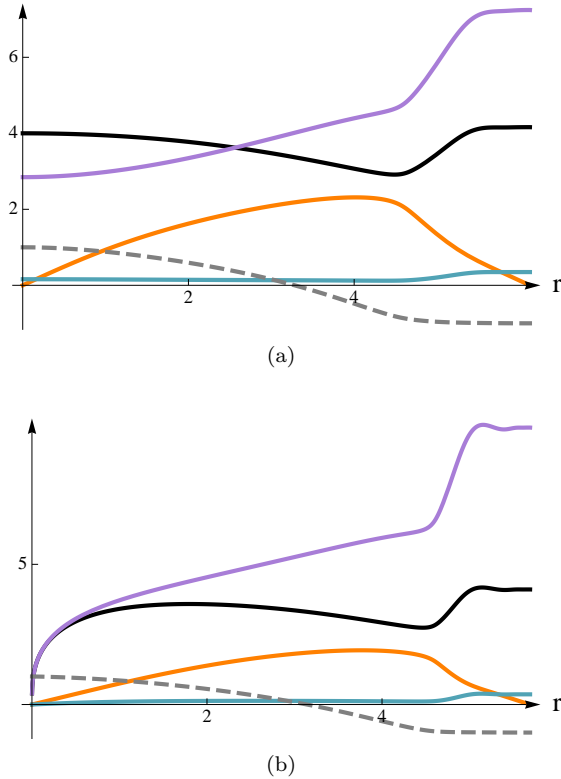


Figure 4.2: Massive attractor solutions. In (a) we see a solution with two regular poles, and $n_0 = -10$ (as usual, $F_0 = \frac{n_0}{2\pi}$). We plot f (orange), e^ϕ (green), e^A (black), g (purple), $x = \cos \psi$ (dashed). In (b) a solution with a stack of $n_2 = 10$ D6-branes at the north pole (left), and a regular point at the south pole (right); again $n_0 = -10$, and $N = -\frac{1}{4\pi^2} \int H = -1$. In both cases, $R = 6$, so $\Sigma_3 = S^3$.

It also appears to be equally easy to obtain solutions with D8-branes. Their position is again fixed by (3.2.53), and the attractor mechanism appears again at the south pole. An example is given in 4.3(b).

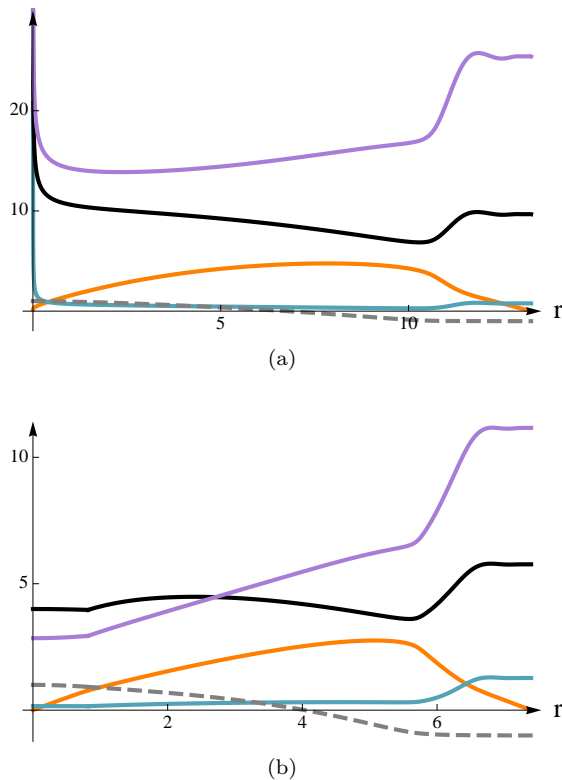


Figure 4.3: Massive attractor solutions. In (a) we see a solution with an O6 at the north pole (left), and a regular point at the south pole (right). In (b) a solution with two regular poles with a D8 stack in the middle (which is the sharp kink towards $r \sim 1$, most visible in the black and purple lines). In both cases, $R = 6$, so $\Sigma_3 = S^3$.

The O6 case in particular would appear promising to obtain examples with “separation of scales”. In AdS_4 compactifications, the Kaluza–Klein scale m_{KK} is usually of the same order of the cosmological constant Λ , which is obviously unphysical. One might object that the negative sign of Λ is even more unphysical. However, sometimes one manages to modify the AdS vacuum by adding some extra ingredient, which turns the cosmological constant positive [75]; the lack of separation of scales might then be inherited by the resulting de Sitter as well.¹² The presence of this phenomenon would also be interesting from the point of view of the CFT dual, since it would imply the presence of a large gap in operator dimensions. A few examples have been put forward where there is separation of scales (see for example [76–78]), but

¹²We thank T. Van Riet for interesting discussions on this point.

they usually rely on the smeared O6 we mentioned in the introduction to this chapter (although see [79] and the strategy in [80]). With the simplest solution of figure 4.3(a), which only has a single O6, we have not been able to achieve separation of scales, but by combining it with the other ingredients (D8-branes, and perhaps D6-branes at the other end) it might be possible. It would be interesting to explore this further.

Chapter 5

Universal consistent truncation for 6d/7d gauge/gravity duals

Let us briefly summarize our original motivations, and the results that we have obtained so far on holography for six-dimensional conformal field theories.

Conformal field theories in dimensions higher than four are still comparatively mysterious; there is usually no Lagrangian description. This is the case for example for the $(2, 0)$ -supersymmetric theory living on the world-volume of coincident M5-branes. Some indirect information can be obtained by compactifying the theory. Reducing it on a T^2 gives $\mathcal{N} = 4$ super-Yang–Mills. Reducing it on a Riemann surface produces a vast “class S” of four-dimensional theories with very interesting duality properties [20, 21, 81]. One can similarly compactify down to three [69] and to two [82] dimensions.

Similar phenomena occur with different six-dimensional CFT’s. Perhaps the simplest generalization of the $(2, 0)$ theory occurs when one introduces orbifold singularities [30–32]; the study of their compactifications on Riemann surfaces is just starting [83–85]. From the holographic perspective, however, these theories are not very different from the $(2, 0)$ theory: their dual is simply $\text{AdS}_7 \times S^4/\mathbb{Z}_k$ [33, 34].

However, an interesting further generalization can be obtained via NS5–D6–D8-brane systems [22, 47].¹ This class consists of $(1, 0)$ CFT’s which are non-Lagrangian, but which can be described by a quiver on a “tensor branch”. Their holographic duals were found relatively recently: first numerically in [1], then analytically in [2]. Their interpretation as the duals of the CFT’s described above was given in [35]. Up to orbifolds and orientifolds, these are the most general AdS_7 solutions in perturbative type II

¹One can engineer six-dimensional field theories also in F-theory [86–88].

supergravity.

Although the compactifications of these theories to lower dimensions are not yet known, they can already be studied holographically: the corresponding AdS₅ and AdS₄ solutions were found respectively in [4] and [3]. These solutions are similar in spirit to the duals of the compactifications of the (2, 0) theory [37–40]: namely, AdS₇ gets replaced by AdS₅ × Σ₂ or AdS₄ × Σ₃, and the internal space gets distorted in a certain way. What is perhaps nicer than expected is that this distortion is “universal”. Namely, even though there are infinitely many AdS₇ solutions, the map to obtain the AdS₅ and AdS₄ metric is always the same, as described in section 4.3.1. Moreover, the two maps are very similar to each other: they differ only by the value of certain numerical factors.

In the last part of this thesis, we greatly extend this universality [5]. We promote the maps to a more general Ansatz, where AdS₇ gets replaced by any seven-dimensional metric $g_{\mu\nu}$, and the internal space gets distorted in a way that depends on a single scalar parameter X . This Ansatz in fact becomes nothing but a reduction to a seven-dimensional effective theory. Its bosonic fields are X and $g_{\mu\nu}$ themselves, together with a three-form potential, and an SU(2) gauge field which is related to the fibration of the internal space over the seven external dimensions.

This effective theory is the so-called minimal gauged supergravity in seven dimensions [42, 43], which describes the dynamics of (a gauged version of) the gravity multiplet with sixteen supercharges. It is a subsector of the bigger “maximal” [44] theory, which describes the gravity multiplet with thirty-two supercharges and has gauge group SO(5). Both theories can be obtained [45, 46] as consistent truncations from eleven dimensions.

Here we find that the minimal theory can also be obtained from massive IIA, in *infinitely many ways*. In each of these reductions, the supersymmetric AdS₇ vacuum is one of the solutions in [1, 2]. This is perhaps surprising, but the idea is that, in reducing, we are only using the ordinary differential equation (ODE) that the internal geometry has to solve in the vacuum, and not the details of the individual solution. Moreover, since our reduction procedure consists in comparing equations of motion, we have a direct proof that these are all consistent truncations of massive IIA.

Thus we can uplift to massive IIA any solution of the seven-dimensional supergravity, in infinitely many ways. For example, the theory has AdS₅ × Σ₂ [37]² and AdS₄ × Σ₃ [39] solutions. They uplift to those of [3, 4]. In this sense we are explaining and extending the universality noticed in those papers. Minimal gauged supergravity also has “Renormalization Group (RG) flow” solutions that connect the above backgrounds to the AdS₇ maximally supersymmetric vacuum. This shows conclusively that the solutions of [3, 4] are indeed dual to compactifications on Σ₂ and Σ₃ of the six-dimensional (1, 0) CFT’s.

²This solution was actually obtained in the maximal theory, with SO(5) gauge group, but it is possible to show that it survives in the minimal theory.

Minimal gauged supergravity also admits $\text{AdS}_3 \times \Sigma_4$ solutions, preserving $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry. In the latter case Σ_4 is a Kähler–Einstein manifold of negative constant curvature, while in the former case Σ_4 is (a compact quotient of) hyperbolic space \mathbb{H}^4 . The corresponding CFT duals are two-dimensional $(0, 2)$ and $(0, 1)$ CFTs. Uplifting these solutions yields new AdS_3 solutions of massive IIA supergravity. On the field theory side, this implies that all the six-dimensional CFT’s of [22, 35, 47] can be compactified on four-manifolds Σ_4 to produce two-dimensional CFT’s.

Finally, minimal gauged supergravity has a second vacuum, which is not supersymmetric. This means that there are also non-supersymmetric analytical AdS_7 solutions in massive IIA. It would be interesting to analyze them further, for example by comparing them with the numerical non-supersymmetric solutions of [68].

5.1 A seven-dimensional effective theory

There exist many examples of consistent truncations of eleven-dimensional supergravity and type IIA supergravities to lower dimensional gauged supergravities. The oldest example is due to De Wit and Nicolai [89], who were able to formulate a complete reduction Ansatz for eleven-dimensional supergravity on S^7 to maximal gauged supergravity in four dimensions. The latter has $\text{SO}(8)$ R-symmetry, which corresponds to the isometry of the internal space upon which the reduction is performed. It has an AdS_4 vacuum that is lifted to eleven dimensions to the $\text{AdS}_4 \times S^7$ solution describing the near horizon geometry of a stack of M2-branes in flat space.

Analogously, reducing eleven-dimensional supergravity on S^4 with a proper Ansatz leads to maximal gauged supergravity in seven dimensions, a theory with $\text{SO}(5)$ gauge group and AdS_7 vacuum [46]. This lifts to eleven dimensions to the maximally supersymmetric background corresponding to the near horizon geometry of a stack of M5-branes, described in App. A.

We now wonder if a similar phenomenon can take place for the class of AdS_7 solutions that we have been dealing with so far, that describe near horizon geometries of more complicated systems of intersecting NS5-D6-D8-branes. In other words we ask if a seven-dimensional effective theory can exist that is obtained as consistent truncation of massive type IIA supergravity on a three-manifold M_3 , of the topology of a three-sphere with singularities at poles. What is certainly different with respect to the above mentioned cases is that we have infinite possible choices of M_3 , one for each brane configuration. This makes our attempt even more challenging.

As a first step we have to identify a seven-dimensional supergravity that could play the role of effective theory. Such identification is possible analyzing the symmetries that it is expected to enjoy. A common feature to all of the AdS_7 solutions is that the internal space M_3 has $\text{SU}(2)$ isometry group, corresponding to a round two-sphere. This suggests that, if a consistent truncation exists to a seven-dimensional theory, this is bound to have

SU(2) R-symmetry. Also, it must have sixteen supercharges, since this is the amount of supersymmetry of our vacua in type IIA supergravity.

Happily, a theory with the desired features exists and goes under the name of *minimal gauged supergravity* in seven dimensions [42]. It describes the dynamics of an $\mathcal{N} = 1$ gravity multiplet with gauged SU(2) R-symmetry.

The bosonic fields are the graviton $g_{\mu\nu}$, a triplet of one-forms \mathcal{A}^i , $i = 1, 2, 3$ transforming in the adjoint representation of SU(2), a single scalar X and a three-form \mathcal{A}_3 . The corresponding Lagrangian is³

$$\begin{aligned} \mathcal{L} = & R - 5X^{-2} * dX \wedge dX - V(X) * 1 - \frac{1}{2}X^4 * \mathcal{F}_4 \wedge \mathcal{F}_4 \\ & - \frac{1}{2}X^{-2} * \mathcal{F}_2^i \wedge \mathcal{F}_2^i + \frac{1}{2}\mathcal{F}_2^i \wedge \mathcal{F}_2^i \wedge \mathcal{A}_3 - \frac{g}{2\sqrt{2}}\mathcal{F}_4 \wedge \mathcal{A}_3 , \end{aligned} \quad (5.1.1)$$

where $\mathcal{F}_2^i = d\mathcal{A}^i - \frac{1}{2}g\epsilon^{ijk}\mathcal{A}^j \wedge \mathcal{A}^k$ and $\mathcal{F}_4 = d\mathcal{A}_3$ are the field strengths of \mathcal{A}^i and \mathcal{A}_3 respectively, and $V(X)$ is the scalar potential defined in terms of the coupling constant g as⁴

$$V(X) = \frac{g^2}{X^8} \left(\frac{1}{4} - 2X^5 - 2X^{10} \right) . \quad (5.1.3)$$

This potential has two extrema: a maximum at $X^5 = 1$ and a minimum at $X^5 = \frac{1}{2}$; only the former is supersymmetric [43], and the corresponding solution is the maximally supersymmetric AdS₇ vacuum.

Varying the lagrangian (5.1.1) one gets the following set of equations of motion for the scalar and gauge fields:

$$\begin{aligned} 0 = & d(X^{-1} *_7 dX) + \frac{1}{5}g^2(X^{-8} - 3X^{-3} + 2X^2)\text{vol}_7 \\ & - \frac{1}{5}X^4 *_7 \mathcal{F}_4 \wedge \mathcal{F}_4 + \frac{1}{10}X^{-2} *_7 \mathcal{F}_2^i \wedge \mathcal{F}_2^i , \end{aligned} \quad (5.1.4a)$$

$$0 = d(X^4 *_7 \mathcal{F}_4) + \frac{1}{\sqrt{2}}g\mathcal{F}_4 - \frac{1}{2}\mathcal{F}_2^i \wedge \mathcal{F}_2^i , \quad (5.1.4b)$$

$$0 = D(X^{-2} *_7 \mathcal{F}_2^i) - \mathcal{F}_2^i \wedge \mathcal{F}_4 , \quad (5.1.4c)$$

³A canonically normalized kinetic term for the scalar field can be obtained after the redefinition $X = e^{\varphi/\sqrt{10}}$. The resulting scalar φ is rescaled by a factor of $\frac{1}{\sqrt{2}}$ with respect to the original paper. The same rescaling also applies to the form fields.

⁴In the original paper there is one more parameter h called *topological mass*, and a more general potential which can be written using our normalizations as:

$$V(\varphi) = 2h^2 X^{-8} - 4\sqrt{2}hgX^{-3} - 2g^2 X^2 . \quad (5.1.2)$$

The constant h here is also rescaled by a factor of $\frac{1}{4}$. Whenever the ratio h/g is positive, this scalar potential has two extrema a maximum at $X^{-5} = \frac{1}{2\sqrt{2}}\frac{g}{h}$ and a minimum at $X^{-5} = \frac{1}{\sqrt{2}}\frac{g}{h}$; only the former is supersymmetric [43]. Without loss of generality we fixed this parameter in terms of the coupling constant as: $h = \frac{g}{2\sqrt{2}}$.

There is a dual formulation of the theory with a two- instead of a three-form. In this case, the topological mass and the corresponding term in the Lagrangian are absent and the scalar potential has no critical points. In [90] it was shown that this version can be embedded in ten-dimensional type I supergravity.

plus the Einstein equations for the graviton:

$$0 = R_{\mu\nu} - 5X^{-2}\partial_\mu X\partial_\nu X - \frac{1}{20}g^2(X^{-8} - 8X^{-3} - 8X^2)g_{\mu\nu} - \frac{1}{2}X^{-2}\left(\mathcal{F}_{2\mu}^i \cdot \mathcal{F}_{2\nu}^i - \frac{1}{5}\mathcal{F}_2^{i2}g_{\mu\nu}\right) - \frac{1}{2}X^4\left(\mathcal{F}_{4\mu} \cdot \mathcal{F}_{4\nu} - \frac{3}{5}\mathcal{F}_4^2g_{\mu\nu}\right). \quad (5.1.5)$$

The fermionic fields of minimal gauged supergravity in seven dimensions are the gravitino $\psi_{\mu a}$, $\mu = 1, \dots, 7$ and the dilatino λ_a . They are symplectic-Majorana spinors transforming as SU(2) doublets; $a = 1, 2$ is the symplectic-Majorana/SU(2) index. The supersymmetry variations of the fermions read

$$\begin{aligned} \delta_\xi \psi_{\mu a} = & (\nabla_\mu + ig(\mathcal{A}_\mu)_a{}^b)\xi_b + \frac{i}{10\sqrt{2}}X^{-1}(\gamma_\mu^{\alpha_1\alpha_2} - 8\delta_\mu^{\alpha_1}\gamma^{\alpha_2})(\mathcal{F}_{2\alpha_1\alpha_2})_a{}^b\xi_b \\ & + \frac{1}{160}X^2(\gamma_\mu^{\alpha_1\alpha_2\alpha_3\alpha_4} - \frac{8}{3}\delta_\mu^{\alpha_1}\gamma^{\alpha_2\alpha_3\alpha_4})\mathcal{F}_{4\alpha_1\alpha_2\alpha_3\alpha_4}\xi_a + m\gamma_\mu\xi_a, \end{aligned} \quad (5.1.6a)$$

$$\begin{aligned} \delta_\xi \lambda_a = & \frac{\sqrt{5}}{2}X^{-1}\not{\partial}X\xi_a - \frac{i}{\sqrt{10}}X^{-1}(\not{\mathcal{F}}_2)_a{}^b\xi_b + \frac{1}{2\sqrt{5}}X^2\not{\mathcal{F}}_4\xi_a \\ & - \sqrt{5}(m + \frac{g}{4\sqrt{2}}X^{-4})\xi_a, \end{aligned} \quad (5.1.6b)$$

where the gauge fields carry adjoint SU(2) index according to $(\mathcal{A})_a{}^b = \frac{1}{2}\mathcal{A}^i(\sigma^i)_a{}^b$ and analogously their field strengths obey $(\mathcal{F}_2)_a{}^b = \frac{1}{2}\mathcal{F}_2^i(\sigma^i)_a{}^b$, σ^i being the Pauli matrices. For simplicity we also defined the following scalar tensor

$$m = -\frac{g}{5\sqrt{2}}\left(X + \frac{1}{4X^4}\right). \quad (5.1.7)$$

Having identified minimal gauged supergravity as a possible effective theory, we must now show how this can be obtained as a consistent truncation of massive type IIA supergravity on M_3 . This requires the formulation of a complete reduction Ansatz for the fermions and bosons living in ten dimensions.

5.2 Consistent truncation of type IIA supergravity on M_3

In this section we present the Ansatz for the Kaluza-Klein reduction of massive IIA supergravity on M_3 , to the seven-dimensional minimal gauged supergravity. Our approach to verifying the consistency of the reduction (or truncation) is to substitute the Ansatz into the ten-dimensional equations of motion and show that these are satisfied provided that the seven-dimensional equations of motion are satisfied. Vice versa, any solution of the lower dimensional theory can be uplifted on M_3 to an exact solution of the higher dimensional theory. This is described in subsection 5.2.2.

In subsection 5.2.3 we take a further step and show that any *supersymmetric* solution of the seven-dimensional theory uplifts to a solution that also

preserves supersymmetry. We provide a decomposition Ansatz for the ten-dimensional supersymmetry parameters and require that the supersymmetry variations of the fermion fields of IIA supergravity vanish. This condition yields a set of equations for the seven-dimensional part of the supersymmetry parameters: it is exactly the set of equations one obtains by setting to zero the supersymmetry variations of the fermion fields of the seven-dimensional minimal gauged supergravity. Vice versa, any spinor ξ_a such that the lower dimensional supersymmetry transformations (5.1.6) vanish can be uplifted so that the higher dimensional supersymmetry transformations vanish as well.

5.2.1 Motivation

Our starting point is the universality observed in the AdS_7 solutions of massive type IIA supergravity dual to the $(1, 0)$ theories and their twisted compactifications.

First, although there are infinitely many AdS_7 solutions corresponding to infinite possible brane configurations, they all share a few fundamental features. The internal space M_3 is an S^2 -fibration over an interval, whose coordinate we called r . The S^2 shrinks at the two endpoints of this interval, so that M_3 has the topology of an S^3 with $\text{SU}(2)$ isometry.

Second, there is a universal way of compactifying these solutions to $\text{AdS}_5 \times \Sigma_2$ and $\text{AdS}_4 \times \Sigma_3$, based on the existence of a one to one map between the system of ODE's describing the AdS_7 vacua and the corresponding systems in lower dimensions. Again, even though there are infinitely many AdS_7 solutions, the map to obtain the AdS_5 and AdS_4 solution is always the same.

Moreover, the two maps look very similar to each other and they only differ for some numerical factor that parametrize the distortion of the internal space. This similarity can be made more explicit by writing a single formula that describes both maps in a unified way. For an $\text{AdS}_7 \rightarrow \text{AdS}_{7-d} \times \Sigma_d$ compactification we have

$$\begin{aligned} e^A &\rightarrow X^{\frac{15}{4}} e^A, & x &\rightarrow \frac{x}{\sqrt{w}}, \\ e^\phi &\rightarrow X^{\frac{5}{4}} \frac{e^\phi}{\sqrt{w}}, & r &\rightarrow X^{\frac{5}{4}} r, \end{aligned} \tag{5.2.8}$$

where the functions $\{A, \phi, x\}$ parametrize the AdS_7 solutions and their dependence on the radial coordinate r is ruled by the system of ODE's (3.3.59).

Also, X is a numerical factor that takes values $X^5 = \{1, \frac{3}{4}, \frac{5}{8}\}$ for the AdS_7 , AdS_5 and AdS_4 solutions respectively. The distortion of the internal manifold is encoded into a single warping function w , defined in terms of X as

$$w \equiv X^5(1 - x^2) + x^2. \tag{5.2.9}$$

As a result of this analysis, it is possible to write a single formula describing the three classes of solutions in type IIA supergravity that we have

discussed so far. We have:

$$ds_{10}^2 = X^{\frac{15}{2}} e^{2A} ds_7^2 + X^{\frac{5}{2}} ds_{M_3}^2, \quad ds_{M_3}^2 = dr^2 + \frac{1-x^2}{16w} e^{2A} Ds_{S^2}^2, \quad (5.2.10)$$

$$ds_7^2 = \begin{cases} ds_{\text{AdS}_7}^2 \\ ds_{\text{AdS}_5}^2 + \frac{1}{3} ds_{\Sigma_2}^2 \\ ds_{\text{AdS}_4}^2 + \frac{4}{5} ds_{\Sigma_3}^2 \end{cases}, \quad X^5 = \begin{cases} 1 \\ 3/4 \\ 5/8 \end{cases},$$

where $ds_{\Sigma_2}^2$ and $ds_{\Sigma_3}^2$ are metrics of unit radius. The $SU(2)$ covariant metric on the two-sphere can be written in angular coordinates in terms of the three Killing vectors: $K_1 = \cot \theta \cos \psi \partial_\psi + \sin \psi \partial_\theta$, $K_2 = \cot \theta \sin \psi \partial_\psi - \cos \psi \partial_\theta$ and $K_3 = -\partial_\psi$. We get

$$Ds_{S^2}^2 = (d\theta + K_i^\theta \mathcal{A}^i)^2 + \sin^2 \theta (d\psi + K_i^\psi \mathcal{A}^i)^2. \quad (5.2.11)$$

The S^2 is fibered over Σ^2 or Σ^3 , with the $U(1)$ spin connection of Σ^2 twisting a $U(1)$ isometry inside the full $SU(2)$ isometry of S^2 in the first case, and the $SU(2)$ spin connection of Σ^3 twisting the whole isometry in the second.

As a result, the field strengths $\mathcal{F}_2^i = d\mathcal{A}^i - \frac{1}{2} \epsilon^{ijk} \mathcal{A}^j \wedge \mathcal{A}^k$ are determined to be $\mathcal{F}_2^i = -e^1 e^2 \delta^{i3}$ for the AdS_5 solutions and $\mathcal{F}_2^i = -\frac{1}{2} \epsilon^{ijk} e^j e^k$ in AdS_4 , where e^i are the vielbein on Σ_2 and Σ_3 respectively. The \mathcal{F}_2^i 's are instead vanishing for the AdS_7 solutions.

We mentioned that there exist infinite possible analytic solutions for each class of vacua, each one corresponding to a different brane configuration. Indeed, according to the analysis of section 3.3, a fully general solution to the supersymmetry variations for the AdS_7 vacua can be given in terms of a single function β . This function depends essentially on two flux integers (\tilde{n}_2, n_2) , that count the number of coincident D6-branes located at the two poles in the internal space where the S^2 shrinks. For every choice of integers we have an analytic solution for β , Eq. (3.3.66, 3.3.67).

As a result, the d -dimensional metric (5.2.10) can be written in terms of β and the constant parameter X as:

$$ds_{10}^2 = \frac{4}{9} \sqrt{-\frac{X^5 \beta'}{y}} \left(X^5 ds_7^2 - \frac{1}{16} \frac{\beta' dy^2}{y\beta} + \frac{\beta}{16X^5 \beta - 4y\beta'} Ds_{S^2}^2 \right), \quad (5.2.12)$$

where y is a new radial coordinate defined as: $dy = \left(\frac{4}{3}\right)^2 \sqrt{\beta} e^{-3A} dr$.

So far we have focused on bosonic fields only. Remarkably, with some effort we were also able to derive explicit expressions for the spinors χ^1 and χ^2 on the internal manifold M_3 , both for the AdS_7 solutions and for their AdS_4 compactifications. Perhaps not surprisingly, it turns out that the spinors

(5.2.13) also transform according to the map (5.2.8), so that it is possible to write a general expression for the internal spinors that reads

$$\chi_1 = -ie^{-i\frac{\pi}{2}\sigma_3}e^{i\frac{\alpha}{2}\sigma_3}\chi_{S^2}, \quad \chi_2 = e^{-i\frac{\alpha}{2}\sigma_3}\chi_{S^2}, \quad (5.2.13)$$

where the angle α is a function of the radial coordinate r , defined as:

$$\alpha = \arcsin\left(\frac{x}{\sqrt{w}}\right). \quad (5.2.14)$$

To conclude the present analysis, it is worth to remind the holographic interpretation of the three classes of solutions (5.2.10). The AdS_7 solutions are dual to the $(1, 0)$ theories in six dimensions, while the ones in AdS_5 and AdS_4 are dual to their twisted compactifications to $\mathcal{N} = 1$ CFT_4 's and to $\mathcal{N} = 1$ CFT_3 's respectively.

The supergravity description allows to give an estimate of the number of degrees of freedom of these conformal field theories. Indeed, the holographic free energy $\mathcal{F}_{0,6}$ of the $(1, 0)$ theories can be computed explicitly, as shown in section 3.5 for some brane configurations. Nicely, this quantity also has nice transformation properties under the map (5.2.8), so it is possible to write a universal formula for the free energies of the lower dimensional CFT's:

$$\mathcal{F}_{0,6-d} = X^{20} \text{Vol}(\Sigma_d) \mathcal{F}_{0,6}. \quad (5.2.15)$$

Having listed all these properties, there is now enough evidence that a consistent truncation of massive type IIA supergravity on M_3 to a seven-dimensional theory should be possible. We can thus proceed in formulating a complete reduction Ansatz.

5.2.2 Bosonic Ansatz

As a start, an Ansatz for the ten-dimensional metric can be easily deduced from (5.2.10). We just need to introduce some normalization factors that are necessary to obtain the correct seven-dimensional theory. So we write

$$\ell^{-1} ds_{10}^2 = \frac{1}{8} g^2 X^{-\frac{1}{2}} e^{2A} ds_7^2 + X^{\frac{5}{2}} ds_{M_3}^2, \quad (5.2.16)$$

where the internal space metric is expressed in terms of the warping function w as in Eq. (5.2.10). We have rescaled the full metric by a factor $\ell \equiv \frac{8\sqrt{2}}{g}$, which depends on an extra parameter g that will turn out to be the coupling constant in seven dimensions.

The covariantized metric $Ds_{S^2}^2$ is defined in the same way as in (5.2.11), with the only difference that the gauge fields \mathcal{A}^i should also be rescaled by a factor of g . We can switch from angular coordinates to spherical harmonics according to (3.2.35), and rewrite the metric on the two-sphere as

$$Ds_{S^2}^2 \equiv Dy^i Dy^i, \quad Dy^i \equiv dy^i + \epsilon^{ijk} y^j g \mathcal{A}^k. \quad (5.2.17)$$

The Ansatz for the dilaton Φ is again dictated by the map (4.3.46). After a proper rescaling, it reads

$$e^{2\Phi} = \ell \frac{X^{\frac{5}{2}}}{w} e^{2\phi} . \quad (5.2.18)$$

Here and in what follows, ϕ is the dilaton for the AdS₇ solution, whose dependence on the radial coordinate r is described by the system of ODE's (3.3.59).

Formulating an Ansatz for the reduction of the fluxes takes some more effort. A lot of information can be extracted from the expressions of the fluxes of the AdS₄ solutions (4.2.44), or better from their transformed under the map (4.3.46). However, this is not enough to construct a fully general Ansatz.

For example there might be extra terms proportional to the derivative of X , the candidate to become the scalar field of minimal gauged supergravity in seven dimensions. This type of terms would be vanishing for all the solutions that we have considered so far, since X takes constant values.

We must also include terms proportional to the four-form $\mathcal{F}_4 = d\mathcal{A}_3$ appearing in the seven-dimensional Lagrangian (5.1.1). Such terms are also not present in our compactification solutions, since they would break their symmetry.

How to construct the extra terms? Some intuition can be gained from [45], where it is shown how to embed minimal gauged supergravity in seven dimensions into eleven-dimensional supergravity. This implicitly tells us how to embed it in type IIA supergravity with vanishing Romans mass, according to the standard 11d to 10d reduction. In particular, reducing to type IIA the four-form field strength [45, Eq. 8] helped us formulate a complete Ansatz for the ten-dimensional fluxes.

Collecting our knowledge, the Ansatz for the Neveu-Schwarz potential B is

$$\ell^{-1}B = e^{2A} \frac{x\sqrt{1-x^2}}{16w} \text{vol}_2 - \frac{1}{2}e^A dr \wedge (A_1 - \frac{1}{2}y^i \mathcal{A}^i) , \quad (5.2.19)$$

where the covariant volume form is defined with respect to the metric (5.2.11) as $\text{vol}_2 \equiv \frac{1}{2}\epsilon^{ijk}y^i Dy^{jk}$, and A_1 is defined via $dA_1 = -\frac{1}{2}\text{vol}_{S^2}$. As shown in App. A, this is nothing but the connection on the bundle $S^1 \rightarrow S^2$, describing S^3 in Hopf coordinates.

Given (5.2.19), the three-form flux $H = dB$ reads

$$\begin{aligned} \ell^{-1}H &= \left\{ (2 - 6X^5 + 4X^{10})x^2 - 2X^5 - 4X^{10} \right\} w^{-1}e^{-A}\text{vol}_{M_3} \\ &\quad - X^5 w^{-1} \ell F_0 e^\phi x \text{vol}_{M_3} - \frac{1}{16} w^{-1} e^{2A} x \sqrt{1-x^2} g \mathcal{F}_2^i \wedge D y^i \\ &\quad - \frac{1}{4} e^A dr \wedge y^i g \mathcal{F}_2^i - \frac{5}{16} X^4 w^{-2} e^{2A} x (1-x^2)^{\frac{3}{2}} dX \wedge \text{vol}_2 , \end{aligned} \quad (5.2.20)$$

where the volume of the internal manifold vol_{M_3} is defined with respect to the metric in Eq. (5.2.10).

The Ansätze for the Ramond-Ramond fluxes are

$$F_2 = -q (\text{vol}_2 + y^i g \mathcal{F}_2^i) + \frac{1}{16} w^{-1} \ell F_0 e^{2A} x \sqrt{1-x^2} \text{vol}_2 , \quad (5.2.21a)$$

$$\begin{aligned} \ell^{-1} F_4 &= -\frac{q}{2} e^A dr \wedge X^4 g^2 *_7 \mathcal{F}_4 - \ell^{-1} \frac{1}{2} e^{3A-\phi} x \mathcal{F}_4 \\ &\quad - \frac{q}{16} w^{-1} e^{2A} x \sqrt{1-x^2} y^i g \mathcal{F}_2^i \wedge \text{vol}_2 - \frac{q}{4} e^A dr \wedge \epsilon^{ijk} g \mathcal{F}_2^i \wedge y^j Dy^k , \end{aligned} \quad (5.2.21b)$$

where q was defined in (3.2.31). F_2 and F_4 must satisfy the Bianchi identities in the form

$$dF_2 - H F_0 = 0 , \quad dF_4 - H \wedge F_2 = 0 . \quad (5.2.22)$$

A way to see that this is the case for the above expressions is to note that

$$F_2 - B F_0 = dC_1 , \quad (5.2.23a)$$

$$F_4 - \frac{1}{2F_0} F_2 \wedge F_2 = dC_3 , \quad (5.2.23b)$$

where

$$C_1 = 2q(A_1 - \frac{1}{2} y^i \mathcal{A}^i) , \quad (5.2.24a)$$

$$C_3 = -\frac{q^2}{2F_0} (\epsilon^{ijk} g \mathcal{F}_2^i y^j Dy^k + g^2 \omega_3) - \frac{1}{2} e^{3A-\phi} x \mathcal{A}_3 . \quad (5.2.24b)$$

$\omega_3 \equiv \mathcal{A}^i \wedge \mathcal{F}_2^i + \frac{1}{6} g \epsilon^{ijk} \mathcal{A}^i \wedge \mathcal{A}^j \wedge \mathcal{A}^k$, satisfying $d\omega_3 = \mathcal{F}_2^i \wedge \mathcal{F}_2^i$. In deriving (5.2.23b) one has to take into account the ‘‘odd dimensional self-duality’’ equation [91]

$$X^4 *_7 \mathcal{F}_4 = -\frac{1}{\sqrt{2}} g \mathcal{A}_3 + \frac{1}{2} \omega_3 . \quad (5.2.25)$$

Armed with this bosonic Ansatz, we can now proceed in the reduction to seven dimensions. What we will actually do is to show that it is possible to derive the equations of motion of the seven-dimensional theory by plugging our Ansatz into the equations of motion of massive type IIA supergravity.

We haven’t written these equations explicitly anywhere in this thesis, so it is worth to spend a few lines to fix the notation and conventions that we are adopting.

We employ the democratic formulation [92] of type II supergravity and work in the string frame. The equations of motion of the fluxes are

$$(d + H \wedge) * F = 0 , \quad d(e^{-2\Phi} * H) - \frac{1}{2} \sum_p * F_p \wedge F_{p-2} = 0 , \quad (5.2.26)$$

where $F \equiv \sum_{p=0,2,4,6,8,10} F_p$. The Einstein equations are

$$R_{MN} + 2\nabla_M \nabla_N \Phi - \frac{1}{2} H_M \cdot H_N - \frac{1}{4} e^{2\Phi} F_M \cdot F_N = 0 , \quad (5.2.27)$$

where $F_M \cdot F_N \equiv \frac{1}{(p-1)!} \sum_p (F_p)_M^{M_1 \dots M_{p-1}} (F_p)_{NM_1 \dots M_{p-1}}$ and similarly for $H_M \cdot H_N$. Finally the dilaton equation is

$$\nabla^2 \Phi - (\nabla \Phi)^2 + \frac{1}{4} R - \frac{1}{8} H^2 = 0 . \quad (5.2.28)$$

Happily, substituting the Ansätze into the flux and dilaton equations of motion, we were able to reproduce the set of equations (5.1.4) of minimal gauged supergravity.

In particular, (5.1.4b) and (5.1.4c) come from the equations of motion of F_4 and F_2 respectively, while both equations of motion of H and Φ give rise to (5.1.4a).

In order to reduce the Einstein equations, we had to compute the Riemann and subsequently the Ricci tensor via the curvature two-form $R^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B$; the spin connection $\omega^A{}_B$ is that of the orthonormal frame introduced in [5, App. A]. After a lengthy calculation we find that the ten-dimensional Einstein equations, upon using (5.1.4a), reduce to the Einstein equations in seven dimensions, Eq. (5.1.5).

5.2.3 Fermionic Ansatz

In this section we take a further step. We prove that the reduction of massive type IIA supergravity to minimal gauged supergravity in seven dimensions also works at the level of supersymmetry. In other words we rederive the supersymmetry variations of the seven-dimensional theory starting from the ten-dimensional ones, after formulating a complete spinor Ansatz.

A proper decomposition for the ten-dimensional supersymmetry parameters is

$$\epsilon_1 = (\xi \otimes \chi_1 + \xi^c \otimes \chi_1^c) \otimes |\uparrow\rangle, \quad \epsilon_2 = (\xi \otimes \chi_2 - \xi^c \otimes \chi_2^c) \otimes |\downarrow\rangle, \quad (5.2.29)$$

where the factors $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenvalues of the matrix σ_3 which carry the chirality of the full spinor, according to the following decomposition for the ten-dimensional gamma matrices:

$$\begin{aligned} \gamma_\mu^{(7+3)} &= \gamma_\mu^{(7)} \otimes 1 \otimes \sigma_2, \\ \gamma_{i+6}^{(7+3)} &= 1 \otimes \sigma_i \otimes \sigma_1. \end{aligned} \quad (5.2.30)$$

Here $\gamma_\mu^{(7)}$, $\mu = 0, \dots, 6$, are a basis of seven-dimensional gamma matrices, and σ_i , $i = 1, 2, 3$, are the Pauli matrices. Chirality is represented in this basis by the matrix $\Gamma^{(7+3)} = 1 \otimes 1 \otimes \sigma_3$.

The present decomposition (5.2.29) is analogous to the spinor decomposition (3.1.3) for the AdS₇ solutions, with the difference that here ξ is any seven-dimensional spinor, which coincides with the AdS₇ Killing spinor (3.1.5) only on the vacuum. Still, the seven-dimensional spinor maintains its transformation properties under the SU(2) R-symmetry, that is to say ξ and its conjugate ξ^c transform as a doublet: $\xi_a \equiv (\xi, \xi^c)$.

Moreover, explicit expressions for the internal spinors are already available, Eq. (5.2.13). We simply rescale by a factor of $e^{A/2}$, the resulting normalization being: $\|\chi^1\|^2 = \|\chi^2\|^2 = e^A$. The transformation properties under the SU(2) R-symmetry are also known: we can define two doublets χ_a^1 and χ_a^2 , according to (4.2.23).

We can now plug this Ansatz into the supersymmetry variations of massive type IIA supergravity. In our conventions, the dilatino variations read:

$$\begin{aligned}\delta\Psi_{1M} &= (\nabla_M - \frac{1}{4}H_M) \epsilon_1 - \frac{1}{16}e^\Phi F\Gamma_M\epsilon_2, \\ \delta\Psi_{2M} &= (\nabla_M + \frac{1}{4}H_M) \epsilon_2 - \frac{1}{16}e^\Phi\lambda(F)\Gamma_M\epsilon_1,\end{aligned}\quad (5.2.31)$$

where fermion fields with a subscript 1 have positive chirality, whereas fermion fields with a subscript 2 have negative chirality. The suppressed indices of the fluxes are contracted with anti-symmetric products of gamma matrices. λ is an operator acting on a p -form as $\lambda(F_p) = (-1)^{\lfloor \frac{p}{2} \rfloor} F_p$, where the square brackets denote the integer part of $\frac{p}{2}$. The supersymmetry transformations of the dilatini are

$$\begin{aligned}\delta\lambda_1 &= (\partial\Phi - \frac{1}{2}H) \epsilon_1 - \frac{1}{16}e^\Phi\Gamma^M F\Gamma_M\epsilon_2, \\ \delta\lambda_2 &= (\partial\Phi + \frac{1}{2}H) \epsilon_2 - \frac{1}{16}e^\Phi\Gamma^M\lambda(F)\Gamma_M\epsilon_1.\end{aligned}\quad (5.2.32)$$

Setting the dilatino variations (5.2.32) to zero with our spinor Ansatz gives

$$0 = \frac{5}{2}X^{-1}\not{\partial}X\xi_a + \frac{1}{2}X^2\mathcal{F}_4\xi_a - \frac{i}{\sqrt{2}}X^{-1}(\mathcal{F}_2^i)_a{}^b\xi_b - \frac{1}{\sqrt{2}}g(X^{-4}-X)\xi_a, \quad (5.2.33)$$

whereas setting the gravitino variations (5.2.31) to zero amounts to the above equation for the internal components and to

$$\begin{aligned}0 &= (\nabla_\mu + ig(\mathcal{A}^i{}_\mu)_a{}^b)\xi_b + \frac{i}{10\sqrt{2}}X^{-1}(\gamma_\mu^{\alpha_1\alpha_2} - 8\delta_\mu^{\alpha_1}\gamma^{\alpha_2})(\mathcal{F}_{2\alpha_1\alpha_2}^i)_a{}^b\xi_b \\ &+ \frac{1}{160}X^2(\gamma_\mu^{\alpha_1\alpha_2\alpha_3\alpha_4} - \frac{8}{3}\delta_\mu^{\alpha_1}\gamma^{\alpha_2\alpha_3\alpha_4})\mathcal{F}_{4\alpha_1\alpha_2\alpha_3\alpha_4}\xi_a + m\gamma_\mu\xi_a,\end{aligned}\quad (5.2.34)$$

for the external ones. These constraints on ξ_a are no other than those that one obtains by setting to zero the supersymmetry variations of minimal gauged supergravity in seven dimensions, Eq. (5.1.6).

Thus, preserved supersymmetry in seven dimensions guarantees preserved supersymmetry in ten.

5.3 New solutions of type IIA supergravity

In this section we discuss supersymmetric solutions of seven-dimensional minimal gauged supergravity that uplift to new solutions of massive IIA in ten dimensions via the formulas presented in the previous section.

In particular we focus on those solutions that are relevant for the holographic description of the $(1, 0)$ theories and their compactifications.

The nicest result in this sense is the existence of two AdS₃ solutions which uplift to *new* AdS₃ solutions of massive type IIA supergravity, with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry. More precisely, for each of the two solutions we can associate an infinite class of analytic solutions in ten dimensions. These are dual to twisted compactification of the $(1, 0)$ theories on four-manifolds of negative curvature.

We also recover AdS₅ and AdS₄ solutions which uplift to the ten-dimensional ones presented in the previous chapters. What is new about these vacua from the seven-dimensional perspective is that they are connected to the vacuum of the theory via holographic RG-flow. Indeed, interpolating solutions have been constructed for all the lower dimensional Anti-de Sitter vacua, included the AdS₃ ones.

The existence of these interpolating solutions is a further proof that the original solutions of type IIA supergravity of [3, 4] are dual to twisted compactifications of six-dimensional (1, 0) theories on Σ_2 and Σ_3 manifolds of negative curvature.

5.3.1 Interpolating solutions

$\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric AdS₅ \times \mathbb{H}^2 solutions were first found in [37], in a certain truncation of the maximal gauged supergravity in seven dimensions, keeping two scalars and two U(1) gauge vector fields. In the case of the $\mathcal{N} = 1$ solution, the two scalars and the two gauge vector fields are set to be equal and thus the solution can also be embedded⁵ in the minimal theory of section 5.1.

The AdS₅ \times \mathbb{H}^2 geometry is a subset of warped product geometries

$$ds_7^2 = e^{2f_1(r)}(dr^2 + ds_{\mathbb{R}^{3,1}}^2) + e^{2f_2(r)}ds_{\mathbb{H}^2}^2, \quad (5.3.35)$$

with a boundary condition for f_1 and f_2 as $r \rightarrow 0$, $f_1 \sim f_2 \sim \log r$. That is, asymptotically or in the UV the metric approaches AdS₇ with an $\mathbb{R}^{3,1} \times \mathbb{H}^2$ boundary. In order to preserve supersymmetry, the U(1) gauge field is identified with the spin connection of \mathbb{H}^2 while f_1 and f_2 (as well as the scalar) are subject to a set of ODEs — these can be found in [37, Eq. (27)].

The latter admit an AdS₅ \times \mathbb{H}^2 solution, which (in our language) reads

$$ds_7^2 = \frac{8}{g^2} X^8 \left(ds_{\text{AdS}_5}^2 + \frac{1}{3} ds_{\mathbb{H}^2}^2 \right), \quad X^5 = \frac{3}{4}, \quad (5.3.36)$$

with the field strength of the U(1) gauge field $g\mathcal{F}_2^i = -\text{vol}_{\mathbb{H}^2} \delta^{i3}$, while the three-form potential is equal to zero. In [38], it was shown numerically (within a broader context) that the AdS₅ \times \mathbb{H}^2 solution arises as the IR fixed point of an RG flow that connects it to the AdS₇ region.

From the ten-dimensional perspective it makes more sense to consider compact quotients of \mathbb{H}^2 , obtained modding out by discrete subgroups of the isometry group $\text{PSI}(2, \mathbb{R})$, so as to obtain a Riemann surface Σ_2 of genus $g \geq 2$. Indeed, the ten-dimensional lifts of the seven-dimensional solution (5.3.36) describe near horizon geometries for systems of NS5-D6-D8-branes wrapped on Riemann surfaces of [4].

⁵The translation between the languages of [37, appendix 7.3] and section 5.1 is: $m \equiv \frac{g}{\sqrt{2}}$, $\lambda_1 = \lambda_2 = -\phi/2 \equiv \frac{\varphi}{2\sqrt{10}}$.

An $\mathcal{N} = 1$ supersymmetric $\text{AdS}_4 \times \mathbb{H}^3$ solution of seven-dimensional minimal gauged supergravity was found long ago in [39]. The metric and the scalar field of the solution read

$$ds_7^2 = \frac{8}{g^2} X^8 \left(ds_{\text{AdS}_4}^2 + \frac{4}{5} ds_{\mathbb{H}^3}^2 \right), \quad X^5 = \frac{5}{8}. \quad (5.3.37)$$

The $\text{SU}(2)$ gauge field is identified with the $\text{SU}(2)$ spin connection ω^{ij} of \mathbb{H}^3 via $g\mathcal{A}^i = \frac{1}{2}\epsilon^{ijk}\omega^{jk}$. The field strength is then $g\mathcal{F}_2^i = \frac{1}{2}\epsilon^{ijk}R^{jk}$, where R^{jk} is the curvature two-form of the spin connection, while the three-form potential is zero.

Again, one can quotient \mathbb{H}^3 by a discrete subgroup of its isometry group $\text{PSU}(2, \mathbb{C})$ to obtain a compact three-manifold Σ_3 of constant negative curvature. The resulting $\text{AdS}_4 \times \Sigma_3$ solution lifts to the ten-dimensional solutions of [3] presented in the previous chapter.

It was shown numerically [40] — in an analogous analysis to that for the $\text{AdS}_5 \times \mathbb{H}^2$ solution — that the solution (5.3.37) also arises as the IR fixed point of an “RG flow geometry”,

$$ds_7^2 = e^{2f_1(r)}(dr^2 + ds_{\mathbb{R}^{2,1}}^2) + e^{2f_2(r)}ds_{\mathbb{H}^3}^2, \quad (5.3.38)$$

with $f_1 \sim f_2 \sim \log r$ in the UV and the corresponding values for the $\text{AdS}_4 \times \mathbb{H}^3$ solution in the IR.

The existence of the above RG flow solutions in the seven-dimensional minimal gauged supergravity, in conjunction with the consistent truncation of massive IIA supergravity presented in this chapter, shows that the AdS_5 and AdS_4 solutions of [3, 4] are connected to the AdS_7 ones of [1] by RG flows. This ultimately proves that the solutions of [3, 4] are dual to compactifications of six-dimensional $(1, 0)$ theories on Σ_2 and Σ_3 manifolds of negative curvature.

5.3.2 AdS_3 solutions

We now turn to the supersymmetric AdS_3 solutions. The first one is $\text{AdS}_3 \times \mathbb{H}^4$ preserving two (real) supercharges. The metric and the scalar field of the solution read

$$ds_7^2 = \frac{2}{g^2} X^{-2} \left(ds_{\text{AdS}_3}^2 + \frac{4}{7} ds_{\mathbb{H}^4}^2 \right), \quad X^5 = \frac{7}{12}. \quad (5.3.39)$$

The $\text{SU}(2)$ gauge field equals the self-dual part of the $\text{SO}(4)$ spin connection of \mathbb{H}^4 .

$$g\mathcal{A}^i = \frac{1}{2}\epsilon^{ijk}\omega^{jk} + \omega^{i4}. \quad (5.3.40)$$

The field strength is then $g\mathcal{F}_2^i = \frac{1}{2}\epsilon^{ijk}R^{jk} + R^{i4}$. Finally, the four-form flux is proportional to the volume of \mathbb{H}^4 :

$$\mathcal{F}_4 = \frac{3\sqrt{2}}{g^3} \text{vol}_{\mathbb{H}^4}. \quad (5.3.41)$$

The second one is $\text{AdS}_3 \times M_4$, where M_4 is Kähler–Einstein of constant negative curvature -4 (for example $\mathbb{H}^2 \times \mathbb{H}^2$), preserving four supercharges. The metric and the scalar field of the solution read

$$ds_7^2 = \frac{2}{g^2} X^{-2} (ds_{\text{AdS}_3}^2 + \frac{4}{3} ds_{M_4}^2) , \quad X^5 = \frac{4}{3} . \quad (5.3.42)$$

Only a $U(1) \subset SU(2)$ gauge field is non-zero and is identified with the center $U(1)$ component of the $U(2)$ spin connection of M_4 , or equivalently with the Kähler connection on the canonical bundle of M_4 . Taking the spin connection of the center $U(1)$ to be the truncation of the self-dual part of the spin connection we can write

$$g\mathcal{A}^i = (\omega^{12} + \omega^{34})\delta^{i3} . \quad (5.3.43)$$

The field strength is then identified with the Ricci form of M_4 . Finally, the four-form flux is proportional to the volume of M_4 :

$$\mathcal{F}_4 = \frac{\sqrt{2}}{g^3} \text{vol}_{M_4} . \quad (5.3.44)$$

Using the Ansatz presented in the previous section, these two AdS_3 vacua lift to two new infinite classes of analytic solutions of massive type IIA supergravity, with $\mathcal{N} = 2$ and $\mathcal{N} = 1$.

In addition, the above AdS_3 solutions were also found in [93] as the IR fixed points of RG flows constructed in certain truncations of the maximal seven-dimensional gauged supergravity. When uplifted to M-theory, the $\text{AdS}_3 \times M_4$ solution arises from M5-branes wrapping Kähler four-cycles in Calabi–Yau four-folds while the $\text{AdS}_3 \times \mathbb{H}^4$ one from M5-branes wrapping Cayley four-cycles in manifolds of $\text{Spin}(7)$ holonomy. The scalar and gauge field sector of the truncations can be identified with the corresponding ones of the minimal theory, while the three-form potential sector is formulated in a dual frame, via (5.2.25). The $\text{AdS}_3 \times M_4$ solution was also constructed with different methods in [94].

5.3.3 Field theory interpretation

Let us conclude with a few words on the field theory duals of the two AdS_3 solutions we described in this section, or better of the two infinite classes of solutions that are obtained via their ten-dimensional lift, which describe near horizon geometries for intersecting systems of NS5-D6-D8-branes wrapped on four-manifolds.

In analogy with the AdS_5 and AdS_4 solutions, these two classes of solutions are naturally interpreted as dual to twisted compactifications of the $(1, 0)$ theories.

In the case of the $\mathcal{N} = 1$ AdS_3 compactification, (5.3.39), the $SU(2)$ R-symmetry of the original AdS_7 solution is completely broken by the gauge

fields (5.3.40). Since no R-symmetry is left, the dual field theory should be a two-dimensional (0, 1) CFT.

The second AdS_3 solution, (5.3.42), enjoys $\mathcal{N} = 2$ supersymmetry. In this case only a $U(1)$ gauge field is switched on; its commutant in $SU(2)_R$ is the $U(1)$ itself. This signals that the IIA uplift still has a $U(1)$ isometry; this is the R-symmetry of the dual theory, which should then be a (0, 2) CFT_2 this time. It would be interesting to study these theories, perhaps generalizing [82].

Unfortunately, they are not known. Nonetheless, our results might help find them. Indeed, using the AdS/CFT dictionary it is possible to count their degrees of freedom through an object called holographic free energy. This is easily computed using the formalism described in this thesis, and in particular specifying formula (5.2.15) to the case of the above AdS_3 solutions. The corresponding two classes of CFT_2 's have free energy

$$\mathcal{F}_{0,2} = \frac{3}{56} \text{vol}(\Sigma_4) \mathcal{F}_{0,6} , \quad (5.3.45)$$

in the $\mathcal{N} = 1$ case, and

$$\mathcal{F}_{0,2} = \frac{3}{128} \text{vol}(\Sigma_g) \text{vol}(\Sigma_{\tilde{g}}) \mathcal{F}_{0,6} , \quad (5.3.46)$$

for the $\mathcal{N} = 2$ case, where we specified M_4 to be the product of two Riemann surfaces of genus $g, \tilde{g} \geq 2$. Their volume can be computed using the Gauss-Bonnet theorem.

The quantity $\mathcal{F}_{0,6}$ is the free energy of the corresponding (1, 0) theory. We have computed this quantity explicitly for some particularly relevant configuration of branes, see section 3.5. It shows the expected N^3 scaling behavior, with $O(N)$ corrections. An analogous computation can be repeated for all the possible brane configurations. In the case represented in Fig. 2.1, corresponding to the reduction to ten dimensions of the geometry describing M5-branes wrapped on four-manifolds, a precise holographic matching between the free energy computed in supergravity and the central charge of the corresponding CFT_2 's has been performed in [95].

Chapter 6

Conclusions

The AdS/CFT correspondence is the most powerful tool that physicists are equipped with to study conformal field theories. In this work we took advantage of this explicit realization of the holographic principle in string theory to investigate a class of non-Lagrangian six-dimensional theories.

Recently, important discoveries on this research field have been accomplished. The study of the theory living on the worldvolume of multiple M5-branes, the famous $(2,0)$, led to great achievements both from the stringy perspective and for quantum field theory. It has been found that its twisted compactifications produce a vast “class S” of $\mathcal{N} = 2$ four-dimensional theories, with beautiful duality properties [20, 21, 81]. These arise as the IR limit of M5-branes wrapping a Riemann surface with punctures. Interesting duality properties were also unveiled studying compactifications on hyperbolic three-manifolds [69]. This goes under the name of 3d-3d correspondence.

The $(2,0)$ are just a subset of the CFT’s in six dimensions. Another important class is given by the less supersymmetric $(1,0)$, which are infinitely many and even more mysterious. They are known to exist from string theory arguments, as they are expected to describe the degrees of freedom of intersecting systems of NS5-D6-D8-branes [22].

Thanks to our work, we finally have a corresponding complete holographic description. Their supergravity duals are the AdS₇ vacua in massive type IIA supergravity [1, 2]. The internal space has the topology of an S^3 with only SU(2) isometry and physical singularities in the points where brane sources are located.

The supergravity description allows to extract some important information. For example we can give a rough estimate of their degrees of freedom computing the holographic free energy. An analytic result is now available for this quantity; it shows the expected N^3 scaling with $O(N)$ corrections.

Thanks to our efforts, a complete holographic description for the twisted compactifications of the $(1,0)$ theories is also known. They turn out to naturally compactify on Einstein manifolds of negative curvature.

Indeed, four distinct classes of $\text{AdS}_{7-d} \times \Sigma_d$ solutions arise wrapping the above mentioned brane systems on compact quotients of hyperbolic manifolds of dimension $d = \{2, 3, 4\}$. These are in one to one correspondence to the AdS_7 vacua via a surprising map that relates the BPS equations in different dimensions [3]. According to this map, the internal manifold M_3 gets distorted in a universal way parametrized by a constant X that takes a different value for each class.

In the compactification procedure, supersymmetry is partially preserved with a twist of the $\text{SU}(2)$ isometry of the internal space with the local Lorentz group of Σ_d .

The AdS_5 solutions have a residual $\text{U}(1)$ isometry, corresponding to the R-symmetry of the dual CFT_4 's which enjoy $\mathcal{N} = 1$ supersymmetry [4]. The AdS_4 ones have no residual R-symmetry [3]. Their holographic duals are then some $\mathcal{N} = 1$ supersymmetric theories in three dimensions. Besides their field theoretical interpretation, our AdS_4 solutions are clearly also interesting as four-dimensional vacua with localized sources, especially if we consider that orientifold planes can enter the game. Finally, two classes of AdS_3 solutions exist that are dual to CFT_2 's with $(0, 1)$ and $(0, 2)$ supersymmetry [5].

None of these lower dimensional CFT's are known, but they are expected to sit at the IR point of an RG-flow connecting them to the corresponding $(1, 0)$ theory in the UV. We managed to reproduce these flows holographically with four classes of interpolating solutions.

Moreover, we were also able to compute their free energy. As a result we found that the ratio between the degrees of freedom of the six-dimensional theory and those of the CFT's in lower dimensions is constant, and it is proportional to the volume of Σ_d , according to formula (5.2.15).

The universality observed in the compactification procedure can be greatly extended: it can be promoted to a complete reduction Ansatz of type IIA supergravity on the internal manifold M_3 . The resulting effective theory is minimal gauged supergravity in seven dimensions, a theory with a single scalar and $\text{SU}(2)$ R-symmetry.

We mentioned that there are infinite possible geometries for M_3 , each one corresponding to a specific system of intersecting branes. Remarkably, the reduction works independently on the particular choice of brane configuration. In other words it is possible to reduce type IIA supergravity to seven dimensions in infinitely many ways. This is perhaps surprising, but the idea is that, in reducing, we are only using the ordinary differential equation (ODE) that the internal geometry has to solve in the vacuum, and not the details of the individual solution [5].

The universal character of this truncation implies that supergravity in seven dimensions describes a sector common to all the six-dimensional $(1, 0)$ CFT's engineered by NS5-D6-D8-brane intersections, including also the $(2, 0)$ theory itself, described by the original M-theory reduction.¹

¹A similar "common sector" phenomenon is witnessed in five dimensions, where it was found that for every AdS_5 solutions of IIB there is a consistent truncation down to

Beyond this common sector, discerning finer differences between the CFT_6 's would require more sophisticated reduction procedures, where one keeps more internal modes. These might be gravity modes, or they could come from the D6- and D8-branes which are present in all the IIA vacua of [1, 2]. In both cases, one would end up coupling the minimal theory to vector multiplets.²

Via the gauge/gravity duality, our work paves the way for a broader study of the aforementioned six-dimensional field theories. Asymptotically locally Anti-de Sitter solutions of seven-dimensional gauged supergravity can probe regions away from the superconformal fixed point. The Kaluza–Klein spectrum of the $\text{AdS}_7 \times M_3$ backgrounds, beyond the massless modes, can be used to analyze the spectrum of the dual operators.

Finally, since the minimal seven-dimensional gauged supergravity can also be embedded in M-theory [45], lessons learned from the more extensively studied $\text{AdS}_7/\text{CFT}_6$ correspondence stemming from the dynamics of M5-branes can guide us in the study of its $(1, 0)$ cousin in the massive IIA theory. It would be particularly interesting to include punctures. This might lead to some generalization of the correspondence between CFT_6 and CFT_4 similar to the celebrated class S theories [20].

minimal five-dimensional supergravity [96]. In the same paper, it was conjectured that this phenomenon should hold in any dimensions; our results prove their conjecture in dimension seven. For certain internal manifolds, it is possible to excite more modes and get bigger theories, e.g. for Sasaki–Einstein reductions [97].

² [98] argues however that the massive IIA vacua cannot be truncated either to the maximal theory, with gauge group $\text{SO}(5)$, nor to a theory with gauge group $\text{SO}(4)$ [99] (which can be obtained as reduction from M-theory [100, 101]).

Appendix A

Massless solutions from 11d

A.1 M5-branes in flat spacetime

In this appendix we describe in detail the reduction to ten dimension of the M-theory background describing the near horizon geometry of M5-branes. This provides a first example of gravity dual of a $(1, 0)$ CFT₆ in type IIA supergravity, and can also pave our way in understanding the possible features of more general solutions in ten dimensions.

The $(2, 0)$ theory on the M5 worldvolume is dual to the $\text{AdS}_7 \times S^4$ background:

$$ds_{11}^2 = R^2 \left(ds_{\text{AdS}_7}^2 + \frac{1}{4} ds_{S^4}^2 \right) . \quad (\text{A.1.1})$$

There are two different coordinate systems on S^4 that are appropriate to study two different types of AdS_4 compactifications of this solution on three-manifold, with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry. For the $\mathcal{N} = 1$ compactification, it is convenient to write the S^4 as:

$$ds_{S^4}^2 = d\alpha^2 + \sin^2 \alpha ds_{S^3}^2 , \quad (\text{A.1.2})$$

where the metric on the S^3 can be written in terms of the Maurer–Cartan forms as $ds_{S^3}^2 = \frac{1}{4} \sigma^i \sigma^i$, with $d\sigma^i = \frac{1}{2} \epsilon^{ijk} \sigma^j \sigma^k$. Alternatively we can choose Hopf coordinates, which are already appropriate to study the reduction to ten dimensions:

$$ds_{S^3}^2 = \frac{1}{4} ds_{S^2}^2 + (dz + A_1)^2 , \quad (\text{A.1.3})$$

where the connection $A_1 = \cos(\theta) d\varphi$, is such that $dA_1 = -\frac{1}{2} \text{vol}_{S^2}$. The transformation rules between these two sets of coordinates is given in detail later on in this appendix.

We can reduce the M5 bear horizon geometry to type IIA supergravity, according to the Ansatz (2.0.5). We get the following ten-dimensional metric:

$$ds_{10}^2 = \frac{R^3}{2k} \sin(\alpha) \left[ds_{AdS_7}^2 + \frac{1}{4} \left(d\alpha^2 + \frac{1}{4} \sin^2(\alpha) ds_{S^2}^2 \right) \right]. \quad (\text{A.1.4})$$

The dilaton is determined by $e^{2\phi/3} = \frac{R}{2k} \sin(\alpha)$. In App. B we show that reducing along the vector ∂_z preserves half of the original supersymmetry, so that this ten-dimensional solution has sixteen supercharges. Indeed, it is dual to a (1, 0) theory. Also, the residual isometry of the internal space in (A.1.4) is $SU(2)$, the symmetry of the two-sphere, which is dual to the $Sp(1)$ R-symmetry of the CFT_6 .

As shown in detail in subsection 3.3.2, the internal space is singular at the two poles $\{\alpha = 0, \alpha = \pi\}$, where a D6-brane and an anti-D6 brane are located. The presence of these sources can be easily inferred looking at the two-form flux $F_2 = -\frac{1}{2} \text{vol}_{S^2}$, whose integral over S^2 is equal to -2π .

We can actually generalize this construction a bit by considering the orbifold S^4/\mathbb{Z}_k , where \mathbb{Z}_k is taken to be a subgroup of the $U(1)$ generated by ∂_z . This is equivalent to multiplying the $(dz + A_1)$ term in (A.1.4) by a factor of $1/k^2$. The corresponding solution in ten dimensions and its field theory interpretation are described in Fig. (2.1). It again has two physical singularities at the poles, where two stacks of k D6-branes are located.

A.2 M5-branes wrapped on three-manifolds

Two types of compactifications on three-manifolds of this fully BPS background have been considered in the literature, preserving $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry in four dimensions. The $\mathcal{N} = 1$ solution corresponds to breaking of the $SO(5)$ isometry group of the S^4 to $SO(4)$, while in the $\mathcal{N} = 2$ case the subgroup preserved is $SO(3) \times SO(2)$. These will be the isometry groups of the fiber metric; the fact that the S^4 is fibred over Σ_3 will break the isometry group further, down to a flavor $SU(2)$ in the $\mathcal{N} = 1$ case and down to $SO(2)$ (which is then the R-symmetry group) in the $\mathcal{N} = 2$ case.

Geometrically, the $\mathcal{N} = 1$ solution can be thought of as arising when one wraps an M5 stack on a submanifold $\mathbb{R}^3 \times \Sigma_3 \subset \mathbb{R}^4 \times$ a G_2 manifold; supersymmetry demands Σ_3 to be an ‘‘associative’’ submanifold. In this case, four of the five scalars transverse to the M5 span directions in the G_2 manifold, corresponding to the $SO(4)$; these scalars will be ‘‘twisted’’, meaning that they are really sections of the normal bundle. The remaining scalar represents the transverse direction inside the \mathbb{R}^4 , and is not fibred. The $\mathcal{N} = 2$ solution, on the other hand, arises when wrapping an M5 stack on a submanifold $\mathbb{R}^3 \times \Sigma_3 \subset \mathbb{R}^5 \times$ Calabi–Yau₆; supersymmetry demands Σ_3 to be a ‘‘special Lagrangian’’ submanifold. In this case, three scalars are inside

the CY_6 , and two trivial ones are in the flat directions; this corresponds to the $SO(2) \times SO(3)$.

Compactifications on hyperbolic three-manifolds Σ_3 were studied in [41], lifting an earlier solution in [39], preserving either four or eight supercharges.¹ After wrapping the M5 on Σ_3 , which corresponds to replacing AdS_7 with $AdS_4 \times \Sigma_3$, the metric of the S^4 will be deformed in such a way that the original $SO(5)$ isometry will be broken to the subgroups mentioned above. Part of the residual symmetry gets mixed with the local Lorentz group of the three-manifold where the M5 is wrapped, meaning that a subspace of S^4 which is left untouched by the supersymmetric deformation gets fibered over Σ_3 . In the $\mathcal{N} = 1$ case, the S^4 metric (A.1.2) gets deformed in such a way as to preserve the shape of the S^3 :

$$ds^2(S_{\mathcal{N}=1}^4) = d\alpha^2 + \frac{8 \sin^2 \alpha}{5 + 3 \cos^2 \alpha} Ds_{S^3}^2 . \quad (A.2.5)$$

The upper case on $Ds_{S^3}^2$ means that the S^3 is now fibered over Σ_3 . In terms of the Maurer–Cartan forms:

$$Ds_{S^3}^2 = \frac{1}{4} \mu^i \mu^i , \quad (A.2.6)$$

where $\mu^i = \sigma^i - \mathcal{A}^i$, and the \mathcal{A}^i are related to the spin connection on the base space Σ_3 by $\mathcal{A}^i = \frac{1}{2} \epsilon^{ijk} \omega^{jk}$.

It would be useful to rewrite this metric in Hopf coordinates, in view of the reduction to ten dimensions to be performed along the Hopf fiber. In order to do this we need to compute the components of the one-forms μ^i along the S^2 . This is achieved by introducing parallel and orthogonal projectors:

$$P_{\parallel}^{ij} = \delta^{ij} - y^i y^j , \quad P_{\perp}^{ij} = y^i y^j , \quad (A.2.7)$$

which satisfy $P_{\parallel} + P_{\perp} = 1$, where y^i are the spherical harmonics that parametrize the S^2 , (3.2.35). The corresponding decomposition for the one-forms μ^i is the following:

$$\mu^i = \epsilon^{ijk} y^j Dy^k + 2y^i Dz . \quad (A.2.8)$$

z is the coordinate on the Hopf fiber; we introduced $SU(2)$ covariant derivatives $Dy^i = dy^i + \epsilon^{ijk} y^j \mathcal{A}^k$ and $Dz = dz + A_1 - \frac{1}{2} y^k \mathcal{A}^k$. Applying this decomposition to (A.2.6) we can finally obtain the S^3 fibered metric in Hopf coordinates:

$$Ds_3^2 = Dz^2 + \frac{1}{4} Ds_{S^2}^2, \quad Ds_{S^2}^2 = Dy^i Dy^i , \quad (A.2.9)$$

¹Punctures along Σ_3 can also be introduced; they were studied in the probe approximation in [102].

The complete metric describing the $\mathcal{N} = 1$ AdS₄ twisted compactification of (A.1.1) can finally be rewritten as follows:

$$m^2 ds_{11}^2 = \left(\frac{5 + 3 \cos^2 \alpha}{8} \right)^{1/3} \left(ds_{\text{AdS}_4}^2 + \frac{4}{5} ds_{\Sigma_3}^2 + \frac{2}{5} ds^2(S_{\mathcal{N}=1}^4) \right), \quad (\text{A.2.10})$$

where the three-manifold Σ_3 is constrained by supersymmetry to be a (compact quotient of) a maximally symmetric space of negative curvature, with Ricci scalar R normalized to -6 . The constant m is fixed in terms of the radius of the AdS₇ solution by the relation $m^3 R^3 = \left(\frac{8}{5}\right)^2$.

We can reduce along the direction z to ten dimensions preserving supersymmetry. The ten-dimensional metric reads

$$ds_{10}^2 = \left(\frac{5}{8}\right)^{\frac{3}{2}} \frac{R^3}{2} \sin \alpha \left(ds_{\text{AdS}_4}^2 + \frac{4}{5} ds_{\Sigma_3}^2 + \frac{2}{5} d\alpha^2 + \frac{4}{5} \frac{\sin^2 \alpha}{3 \cos^2 \alpha + 5} Ds_{S^2}^2 \right). \quad (\text{A.2.11})$$

An accurate analysis reveals that at the two poles we have a D6 and an anti-D6 stack wrapping Σ_3 . This ten-dimensional solution is dual to an $\mathcal{N} = 1$ CFT₃ with no R-symmetry obtained by twisting the $(1, 0)$ theory dual to (A.1.4) on the three-manifold Σ_3 . Indeed, this solution has no internal isometry since the $SU(2)$ is completely broken by the twisting procedure. More details on this solution, explicit expressions for the dilaton and the fluxes are given in subsection 3.3.2.

Another possibility would be reducing to ten dimensions the $\mathcal{N} = 2$ AdS₄ solution of [41]. However it was shown in [3] that this cannot be done without breaking supersymmetry.

Appendix B

Massless spinors from 11d

B.1 Killing spinor on S^4

In this appendix we will follow [1, App. B]. The $\text{AdS}_7 \times S^4$ is a familiar Freund–Rubin solution; the flux is taken to be proportional to the internal volume form, $G_4 = g \text{vol}_{S^4}$. The eleven-dimensional supersymmetry transformation reads: $\nabla_M \epsilon_{11} + \frac{1}{144} G_{NPQR} (\gamma^{NPQR}{}_M - 8 \gamma^{NPQ} \delta_M^R) \epsilon_{11} = 0$; decomposing $\epsilon_{11} = \sum_{a=1}^4 \zeta_a \otimes \eta_a + \text{c.c.}$, and using (3.1.4), one reduces the requirement of supersymmetry (for $R_{\text{AdS}} = 1$) to taking $g = 3/4$, and to the equation

$$(\nabla_m - \frac{1}{2} \gamma \gamma^m) \eta = 0 \tag{B.1.1}$$

on S^4 , where we have used the standard decomposition for the eleven-dimensional gamma matrices: $\Gamma_\mu^{(7+4)} = \gamma_\mu^{(7)} \otimes \gamma$, $\Gamma_{m+6}^{(7+4)} = 1 \otimes \gamma^m$.

This is an alternative form of the Killing spinor equation; it was solved in [103] in any dimension. However, we are using different coordinates, adapted to the S^1 reduction used in App. A; we will here solve (B.1.1) again, using more or less the same method.

The idea is to start from the easiest components of the equation, and to work one’s way to the more complicated ones. Our coordinates in section A are $\{\alpha, z, \theta, \varphi\}$, with z being the reduction coordinate and θ and φ are spherical coordinates on S^2 .

Our vielbein, as defined by Eq. (A.1.2), reads: $e^1 = d\alpha$, $e^2 = \frac{1}{2} \sin(\alpha) d\theta$, $e^3 = \frac{1}{2} \sin(\alpha) \sin(\theta) d\varphi$, $e^4 = \frac{1}{2} \sin(\alpha) (dz + \cos(\theta) d\varphi)$. We begin with the α component of (B.1.1):

$$\partial_\alpha \eta = \frac{1}{2} \gamma \gamma_1 \eta \quad \Rightarrow \quad \eta = e^{\frac{1}{2} \alpha \gamma \gamma_1} \eta_1 . \tag{B.1.2}$$

The next component we use is

$$\left(\partial_\theta - \frac{1}{4} \cos(\alpha) \right) \eta = \frac{1}{4} \sin(\alpha) \gamma \gamma_2 \eta . \tag{B.1.3}$$

This can be manipulated as follows:

$$0 = \left(\partial_\theta - \frac{1}{4} e^{\alpha\gamma\gamma_1} \gamma_{12} \right) \eta = e^{\frac{1}{2}\alpha\gamma\gamma_1} \left(\partial_\theta - \frac{1}{4} \gamma_{12} \right) \eta_1 \Rightarrow \eta_1 = e^{\frac{1}{4}\theta\gamma_{12}} \eta_2 . \quad (\text{B.1.4})$$

We proceed in a similar way for the two remaining coordinates; the details are complicated, and we omit them here. The final result is

$$\eta_{S^4} = \exp \left[\frac{\alpha}{2} \gamma\gamma_1 \right] \exp \left[\frac{\theta}{4} \gamma_{12} + \frac{\theta - \pi}{4} \gamma_{34} \right] \exp \left[\frac{z + \varphi}{4} \gamma_{13} + \frac{z - \varphi}{4} \gamma_{24} \right] \eta_0 . \quad (\text{B.1.5})$$

where η_0 is a constant spinor.

In order to reduce this spinor to ten dimensions along the z direction, we have to impose the condition $\partial_z \eta = 0$, which is easily achieved imposing the projection $(\gamma_{13} + \gamma_{24})\eta_0 = 0$, which is equivalent to $\gamma\eta_0 = -\eta_0$. This projection keeps only half of the components, those with negative chirality, so that the solution is half BPS in ten dimensions.

B.2 Reduction to M_3

We now choose the following decomposition for the 4d gamma matrices:

$$\gamma^1 = \sigma^3 \otimes \sigma^1, \quad \gamma^2 = \sigma^1 \otimes \sigma^1, \quad \gamma^3 = \sigma^2 \otimes \sigma^1, \quad \gamma^4 = 1 \otimes \sigma^3, \quad (\text{B.2.6})$$

where σ^i are the Pauli matrices. The condition $\gamma\eta_0 = -\eta_0$ is easily solved by $\eta_0 = (\chi_0, -i\chi_0)$. With some more effort, the full S^4 Killing spinor (B.1.5) turns out to admit a natural decomposition in terms of a Killing spinor on the two-sphere which is left untouched by the reduction to ten dimensions, Eq. (A.1.4). We get:

$$\eta_{S^4} = \begin{pmatrix} -i e^{\frac{i}{2}(\alpha-\pi)\sigma^3} \chi_{S^2} \\ e^{-\frac{i}{2}\alpha\sigma^3} \chi_{S^2} \end{pmatrix}, \quad (\text{B.2.7})$$

where χ_{S^2} is the S^2 Killing spinor that can be written explicitly as $\chi_{S^2} = e^{\frac{i\theta}{2}\sigma^1} e^{\frac{\varphi}{2}\sigma^{12}} \tilde{\chi}_0$, for a new constant spinor which is related to the old one by a simple unitary transformation: $\tilde{\chi}_0 = \frac{1}{2}(1 - i\sigma^1)(1 + i\sigma^3)\chi_0$. Notice that the spinor dependence on the coordinate α is factorized in an overall unitary transformation.

The gamma matrix representation that we chose is already appropriate for the reduction from eleven to ten dimensions of the spinor (B.2.7). Indeed, chirality in ten dimensions is given by the eigenvalues of γ^4 , which in our basis is $\gamma^4 = 1 \oplus -1$. The spinor η decomposes as (χ^1, χ^2) , or equivalently as $\eta = \chi^1 \otimes v_+ + \chi^2 \otimes v_-$, where v_\pm are σ^3 eigenvectors and the two spinors on M_3 are given by

$$\chi_1 = -i e^{-i\frac{\pi}{2}\sigma_3} e^{i\frac{\alpha}{2}\sigma^3} \chi_{S^2}, \quad \chi_2 = e^{-i\frac{\alpha}{2}\sigma^3} \chi_{S^2}. \quad (\text{B.2.8})$$

So we succeeded in showing how the $\text{AdS}_7 \times S^4$ background can be reduced to ten dimensions preserving half of the supersymmetry. We also got a very elegant expression for the internal spinors in ten dimensions, which make manifest their transformation property under the residual isometry group $\text{SU}(2)$ of the two-sphere, dual to the $\text{Sp}(1)$ R-symmetry of the $(1, 0)$ theory.

Appendix C

J_ψ^{-1} Operator

In [72, Sec. 5.2], the pure spinor equations (4.1.6) were massaged for the particular case needed in this thesis. All we need now is to compute the action of the $J_\psi^{-1} \lrcorner$ operator on the two- and four-forms defined in subsection 4.2.1. J_ψ^{-1} is a bi-vector defined as the inverse of the two-form J_ψ entering the dielectric expression (4.1.8), which for our class of solutions can be expanded as: $J_\psi = j_1 e^i \star Dy^i + j_2 dr y^i e^i$, with coefficients $j_2 = -\frac{fg}{\cos \psi}$ and $j_3 = g$.

It is natural to choose $f^i \equiv j_2 Dy^i - j_3 y^i dr$ as basis of one-forms on M_3 and the vielbein e^i as basis on Σ_3 , so that we can write J_ψ as:

$$J_\psi = e^i \wedge f^i . \quad (\text{C.0.1})$$

Equivalently, the inverse operator can be expanded on the dual basis of vectors as:

$$J_\psi^{-1} \lrcorner = F^i \lrcorner E^i \lrcorner , \quad (\text{C.0.2})$$

where the basis of forms and dual vectors satisfy:

$$F^i \lrcorner f^j = \delta^{ij} , \quad F^i \lrcorner e^j = 0 , \quad E^i \lrcorner f^j = 0 , \quad E^i \lrcorner e^j = \delta^{ij} . \quad (\text{C.0.3})$$

We now compute the dual vectors to be:

$$F^i = \frac{1}{j_2} v^i - \frac{1}{j_3} y^i dr , \quad E^i = E_0^i - v^j y^k (E_0^i \lrcorner \omega^{jk}) . \quad (\text{C.0.4})$$

E_0^i are the dual vectors to e^i on the base space satisfying $E_0^i \lrcorner e^j = \delta^{ij}$. The vectors v^i are given by

$$v^1 = \cos \theta \cos \varphi \partial_\theta - \frac{\sin \varphi}{\sin \theta} \partial_\varphi , \quad v^2 = \cos \theta \sin \varphi \partial_\theta + \frac{\cos \varphi}{\sin \theta} \partial_\varphi , \quad v^3 = -\sin \theta \partial_\theta ; \quad (\text{C.0.5})$$

they satisfy $v^i \lrcorner Dy^j = \delta^{ij} - y^i y^j$. (They also happen to be conformal Killing vectors on S^2 : $\mathcal{L}_{v^i} g_{S^2} = -2y^i g_{S^2}$.)

It is now straightforward to compute the action of J_ψ^{-1} on the set of twisted two-forms:

$$\begin{aligned} J_\psi^{-1} \lrcorner \text{vol}_2 &= 0, & J_\psi^{-1} \lrcorner e^i D y^i &= \frac{2}{j_2}, & J_\psi^{-1} \lrcorner dr y^i e^i &= \frac{1}{j_3}, \\ J_\psi^{-1} \lrcorner \tilde{e}^i \star D y^i &= 0, & J_\psi^{-1} \lrcorner y^i \mathcal{F}_2^i &= 0. \end{aligned} \quad (\text{C.0.6})$$

We finally compute the action of J_ψ^{-1} on some the four-forms, which are also needed in the pure spinor equations:

$$\begin{aligned} J_\psi^{-1} \lrcorner dr y^i e^i \text{vol}_2 &= \frac{1}{j_3} \text{vol}_2, & J_\psi^{-1} \lrcorner y^i \mathcal{F}_2^i \text{vol}_2 &= \frac{R}{6j_2} e^i D y^i, \\ J_\psi^{-1} \lrcorner dr \text{vol}_\Sigma &= \frac{1}{j_3} y^i \mathcal{F}_2^i. \end{aligned} \quad (\text{C.0.7})$$

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