IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **47** (2014) 205501 (9pp) doi:10.1088/1751-8113/47/20/205501

On the groundstate energy spectrum of magnetic knots and links

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Received 27 December 2013, revised 26 March 2014
Accepted for publication 27 March 2014
Published 30 April 2014

Abstract
By using analytical results for the constrained minimum energy of magnetic knots we determine the influence of internal twist on the minimum magnetic energy levels of knots and links, and by using ropelength data from the RIDGERUNNER tightening algorithm (Ashton *et al* 2011 *Exp. Math.* **20** 57–90) we obtain the groundstate energy spectra of the first 250 prime knots and 130 prime links. The two spectra are found to follow an almost identical logarithmic law. By assuming that the number of knot types grows exponentially with the topological crossing number, we show that this generic behavior can be justified by a general relationship between ropelength and crossing number, which is in good agreement with former analytical estimates (Buck and Simon 1999 *Topol. Appl.* **91** 245–57, Diao 2003 *J. Knot Theory Ramifications* **12** 1–16). Moreover, by considering the ropelength averaged over a given knot family, we establish a new connection between the averaged ropelength and the topological crossing number of magnetic knots.

Keywords: ideal magnetohydrodynamics, magnetic knots and links , magnetic energy spectrum, tight knots and links, ropelength
PACS numbers: 02.10.Kn, 11.10.−z, 47.10.A, 52.30.Cv

(Some figures may appear in colour only in the online journal)

1. Magnetic knots and links in an ideal fluid

The search for possible relationships between energy and topology has a long history, which has its roots in Lord Kelvin's vortex atom theory and Tait's knot tabulation (see, for instance,

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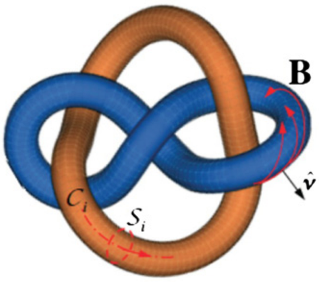

Figure 1. Magnetic link given by the linking of two magnetic flux tubes.

Kelvin's original papers reproduced in [9]). Recent advances in knot theory and topological fluid dynamics have triggered a renewed interest in this problem [14, 16]. Of particular relevance is the study of magnetic knots, as this study offers a good prototype for a variety of other problems, where mathematical techniques and results are not readily available. So, let us consider magnetic knots and links in an ideal, incompressible and perfectly conducting fluid in S^3 (i.e. $\mathbb{R}^3 \cup \{\infty\}$, simply connected). Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ be the fluid velocity, a smooth function of the position vector $\mathbf{x}$ and time t, with $\nabla \cdot \mathbf{u} = 0$ in S^3 and $\mathbf{u} = 0$ at infinity. The magnetic field $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ is frozen in the fluid and has finite energy of

$$\mathbf{B} \in \{\nabla \cdot \mathbf{B} = 0,\ \partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}),\ L_2\text{−norm}\}. \tag{1}$$

A magnetic knot is a magnetic flux tube prescribed by the knot type $\mathcal{K}$ and the magnetic field $\mathbf{B}$, defined on a *regular* tubular support $\mathcal{T}(\mathcal{K})$ centered on $\mathcal{K}$. We assume $\mathcal{K}$ to be a C^2-smooth, closed loop (i.e. a submanifold of S^3 homeomorphic to S^1), simple (i.e. non-self-intersecting) and parametrized by the arc-length s. The tube $\mathcal{T} = \mathcal{K} \otimes \mathcal{S}$, given by the cartesian product of $\mathcal{K}$ and the circular disk $\mathcal{S}$ is centered on the knot, whose total length is $L = L(\mathcal{K})$ (hence $s \in [0, L]$), local radius of curvature $\rho > 0$, and cross-sectional area $A = \pi R^2$ of radius $R > 0$.

The knot topology is identified by standard knot tabulation, such as that given in [19]. The knot is said to be *trivial*, if $\mathcal{K}$ (the *unknot*) bounds a smoothly embedded disk, or *essential*, if otherwise. A magnetic link is just a finite collection of magnetic knots (as in the example of figure 1).

Since the magnetic knot is a physical object, it is useful to introduce the volume $V(\mathcal{T})$, the magnetic flux Φ and the magnetic energy M. The total volume is given by $V = V(\mathcal{T}) = \pi R^2 L$, with a tubular boundary $\partial\mathcal{T}$ that is a magnetic surface, i.e.

$$\text{supp}(\mathbf{B}) := \mathcal{T}(\mathcal{K}), \qquad \mathbf{B} \cdot \hat{\boldsymbol{\nu}}_\perp = 0 \quad \text{on } \partial\mathcal{T}, \tag{2}$$

where $\hat{\boldsymbol{\nu}}_\perp$ is a unit normal to $\partial\mathcal{T}$. The existence and regularity of non-self-intersecting nested tori, the support of the magnetic field inside $\mathcal{T}$, is guaranteed by the tubular neighborhood theorem [20], provided $\rho \geqslant R$ all along $\mathcal{K}$. The magnetic flux Φ is defined by

$$\Phi = \int_{\mathcal{S}} \mathbf{B} \cdot \hat{\boldsymbol{\nu}}\, \mathrm{d}^2\mathbf{x}, \tag{3}$$

where now $\hat{\boldsymbol{\nu}}$ is the unit normal to $\mathcal{S}$; the magnetic energy M is given by

$$M = \frac{1}{2} \int_{V(\mathcal{T})} \|\mathbf{B}\|^2\, \mathrm{d}^3\mathbf{x}. \tag{4}$$

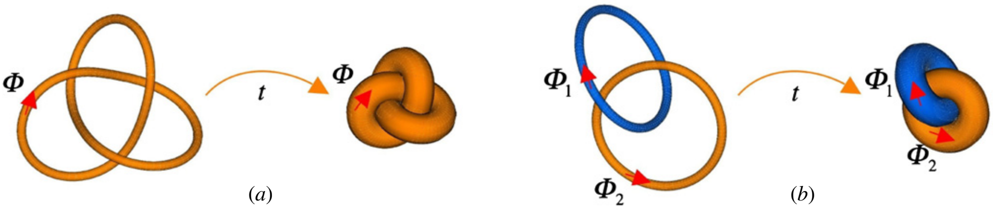

Figure 2. Ideal relaxation of magnetic flux tubes: (*a*) trefoil knot and (*b*) Hopf link.

2. The prototype problem

For a magnetic knot, whose field is confined to a single tube of *signature* (V, Φ), the combined action of magnetic stresses and Lorentz force induces the field lines to shrink like elastic bands, shortening the knot, while conserving volume and flux [11]. Magnetic energy gradually gets converted into kinetic energy and becomes dissipated by viscosity or other equivalent effects. As the relaxation progresses, the average cross-section increases proportionately and the tubular knot becomes thicker and tighter, until the knot topology prevents any further adjustment. The final state is reached when the relaxation comes to a complete stop (see figure 2). During this process the knot is also gradually deformed by the action of a signature-preserving flow (a diffeomorphism), which governs the relaxation from the initial configuration. Since the tight configuration of the end-state resembles that of an *ideal* knot of platonic features [21], magnetic relaxation also provides a physical mechanism to investigate geometric properties of ideal knots.

Let (r, ϑ_R, s) denote an *orthogonal*, curvilinear coordinate system centered on $\mathcal{K}$ (see [10]); $r \in [0, R]$ and $\vartheta_R \in [0, 2\pi]$ are the radial and azimuthal coordinates on the cross-sectional plane of $\mathcal{S}$, with origin O at $s = 0$ (where O is an inflexion-free point of $\mathcal{K}$), and $\vartheta_R = 0$ given by the direction of the principal normal to $\mathcal{K}$ at O. The metric is orthogonal, with scale factors $h_r = 1$, $h_{\vartheta_R} = r^2$, $h_s = 1 - cr\cos\vartheta$, where $c = c(s)$ is curvature,

$$\vartheta = \vartheta(\vartheta_R, s) = \vartheta_R - \int_0^s \tau(\bar{s})\,\mathrm{d}\bar{s}, \tag{5}$$

and $\tau = \tau(s)$ torsion. Orthogonality is ensured by equation (5), which provides the necessary correction to the standard azimuthal angle by the torsion contribution (see details in [10], section 3). The results presented below were derived by using this metric.

The magnetic field $\mathbf{B}$ may be decomposed into meridian and longitudinal components; that is

$$\mathbf{B} = (0, B_{\vartheta_R}(r), B_s(r)), \tag{6}$$

and in general we assume that the longitudinal field is far greater than the meridian field, i.e. $B_s \gg B_{\vartheta_R}$. This is consistent with the usual definition of a twisted flux tube, whose field lines wind around the knot axis in a longitudinal direction. By using the solenoidal condition $\nabla \cdot \mathbf{B} = 0$, the magnetic field can be expressed in terms of poloidal (meridian) and toroidal (longitudinal) fluxes Φ_P and Φ_T, i.e.

$$\mathbf{B} = \left(0, \frac{1}{L}\frac{\mathrm{d}\Phi_P}{\mathrm{d}r}, \frac{1}{2\pi r}\frac{\mathrm{d}\Phi_T}{\mathrm{d}r}\right) + \left(0, \frac{\partial\widetilde{\psi}}{\partial s}, -\frac{\partial\widetilde{\psi}}{\partial\vartheta_R}\right), \tag{7}$$

where the total field is given by the sum of an average field plus a fluctuating field with zero net flux, in terms of the flux function $\widetilde{\psi} = \widetilde{\psi}(r, \vartheta_R, s)$. The twist $h = \Phi_P/\Phi_T$ of the field lines provides the magnetic field *framing* given by $(2\pi)^{-1}$ times the turns of twist required to generate a poloidal field from a toroidal field, starting from $\Phi_P = 0$.

According to the process described above, knot topology dictates a lower bound on the relaxation of magnetic energy M, which must be bounded from below by a minimum $M_{\min} > 0$, which on dimensional grounds is given by (see [12])

$$M_{\min} = m(h)\Phi^2 V^{-1/3}, \tag{8}$$

where $m(h)$ is a positive, dimensionless function of the internal twist h. Of particular interest is the value of h for which $m(h)$ is minimal ($m_{\min}$). Here, a fundamental problem is this [13]:

Problem. Determine $m_{\min}$ for knots of minimum crossing number 3, 4, 5, ...

If $h = 0$ (a condition referred to as *zero-framing*), we can prove [15] that for zero-framed flux tubes we have $m(0) = (2/\pi)^{1/3} c_{\min}$; thus

$$M_{\min} = \left(\frac{2}{\pi}\right)^{1/3} c_{\min}\Phi^2 V^{-1/3}, \tag{9}$$

where $c_{\min}$ is the *topological crossing number* of the knot. Equation (9) establishes a correspondence between the minimum energy levels and topology, since $M_{\min} \propto c_{\min}$, a result of general validity, but still rather loose. From a direct inspection of the knot table (see, for instance, the standard tabulation in [19]) we have only one knot for $c_{\min} = 3$ (the trefoil) and $c_{\min} = 4$ (the four-crossing knot), but for all the other values of $c_{\min} > 4$ there are several distinct knot types, whose numbers grow exponentially for increasing values of $c_{\min}$ (2 for $c_{\min} = 5$, 3 for $c_{\min} = 6$, 7 for $c_{\min} = 7$, 21 for $c_{\min} = 8$, and so on). Hence, a natural question is to determine whether different knot types of the same $c_{\min}$-family have the same minimum energy level or not. This problem will be addressed in the following section.

3. Relaxation of magnetic energy and constrained minima

Let us consider the relaxation of a flux tube in some generality. Let $V_r = \pi r^2 L$ be the partial volume of the tubular neighborhood of radius r; the ratio of the partial to total volume is given by $V_r/V(\mathcal{T}) = (r/R)^2$. Now, let $f(r/R)$ be a monotonically increasing function of r/R; for example $f(r/R) = (r/R)^{\gamma}$, with $\gamma > 0$; $\gamma = 2$ gives the standard ratio of partial to total volume, which defines the *standard* flux tube. A detailed analysis of the relaxation of a magnetic flux tube with twist was made by Maggioni and Ricca (see [10], section 5). By using the orthogonal, curvilinear system (r, ϑ_R, s) and the magnetic field decomposition given by (7), a standard minimization of (4) was carried out, subject to the periodicity of ϑ_R and s. This led to the following result:

Theorem 1. *Let $\mathcal{K}$ be an essential magnetic knot with signature $\{V, \Phi\}$ and the magnetic field given by (7). By assuming that*

(i) $\{V, \Phi\}$ is invariant;
(ii) the circular cross-section is independent of s;
(iii) $\widetilde{\psi}$ is independent of s;
(iv) the knot length is independent of h,

the constrained minimization of magnetic energy yields

$$M^* = \left(\frac{\gamma^2 L^{*2}}{8(\gamma-1)V} + \frac{\gamma\pi h^2}{2L^*}\right)\Phi^2, \tag{10}$$

where L^* is the minimal tube length of the tight knot.

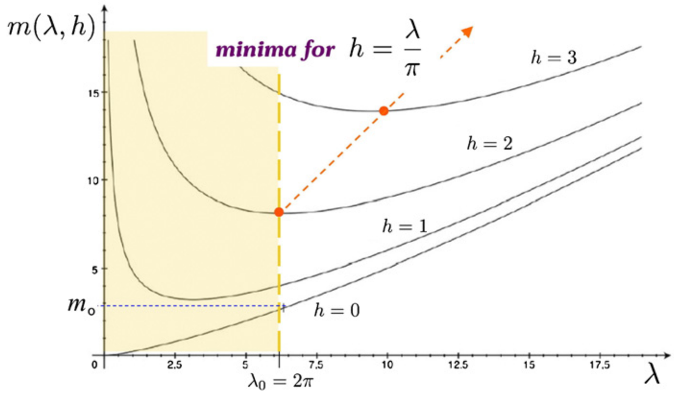

Figure 3. Influence of twist h on the energy function $m(\lambda, h)$, plotted against the ropelength λ, according to equation (13). The absolute minimum is given by the tight torus, for which $\lambda = \lambda_0 = 2\pi$ and $m_\circ \approx 2.70$.

For the standard flux tube case, we have $\gamma = 2$ and (10) reduces to

$$M^* = \left(\frac{L^{*2}}{2V} + \frac{\pi h^2}{L^*}\right)\Phi^2, \tag{11}$$

which is equivalent to the functional relation obtained by a scaling argument by Chui and Moffatt (see [6], equation (4.2), with coefficients left undetermined). Note that because of the constraints, for any given knot family we have $\langle M^* \rangle_{c_{\min}} \geqslant M_{\min}$, where angular brackets denote averaging over the number of knots of the same $c_{\min}$ family.

In order to investigate the relation between energy and knot topology, let us confine ourselves to the case of standard flux tubes. It is useful to rewrite equation (11) in terms of *ropelength*, a powerful indicator of knot complexity [3]: this is defined by $\lambda = L^*/R^*$, where L^* is the minimal length and R^* the maximal cross-sectional radius of the knot in tight configuration. In the case of the unknot, the least possible value of λ (say λ_0) is that given by the tight torus; hence $\lambda \geqslant \lambda_0 = 2\pi$. By using $V = \pi R^{*2} L^* =$ cst., and after some straightforward algebra, we have

$$M^* = \left(\frac{\lambda^{4/3}}{2\pi^{2/3}} + \frac{\pi^{4/3} h^2}{\lambda^{2/3}}\right)\Phi^2 V^{-1/3}. \tag{12}$$

By comparing (8) and (12), we can state (within the validity of the assumptions of the above theorem) that $m(h)$ can be re-written as

$$m(\lambda, h) = \frac{\lambda^{4/3}}{2\pi^{2/3}} + \frac{\pi^{4/3} h^2}{\lambda^{2/3}}, \tag{13}$$

showing explicitly the effect of ropelength and framing on the energy levels.

4. Groundstate energy spectra of knots and links

First, let us investigate the minima $m_{\min} = m_{\min}(h)$ by plotting (13) against λ for $h = 0, 1, 2, 3, \ldots$ (see figure 3). The absolute minimum $m_\circ$ corresponds to the zero-framed unknot (tight torus), given by $h = 0$ and $\lambda = \lambda_0 = 2\pi$: $m_\circ = (2\pi^2)^{1/3} \approx 2.70$. The groundstate energy of the zero-framed flux tubes provides the absolute minimum energy level; $m(h)$ remains a monotonic increasing function of λ for $h \leqslant 2$: at $\lambda_0 = 2\pi$ we have $m(h = 1) = 4.05$ and

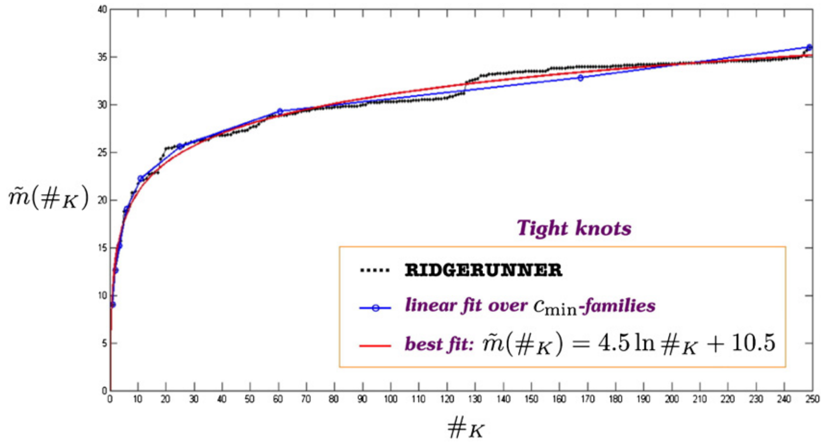

Figure 4. Energy spectrum $\tilde{m} = \tilde{m}(\#_K)$ of tight knots plotted against the knot number $\#_K$, given by the position of the knot K listed according to the increasing value of ropelength $\lambda_K = \lambda(\#_K)$. Best fit goodness: 95% confidence bounds, summed square of residuals (SSE) = 80.27, $R^2 = 0.98$, root mean squared error (RMSE) = 0.57.

$m(h = 2) = 8.11$. For $h \geqslant 2$ the energy minima are attained for $h = \lambda/\pi$; thus, by substituting the optimal value $\lambda = \pi h$ in (13), we have

$$m_{\min}(h) = \tfrac{3}{2}\pi^{2/3}h^{4/3} \quad (h \geqslant 2). \tag{14}$$

For $h > 2$ (and $\lambda \geqslant \lambda_0$) the functional dependence of $m(h)$ on λ ceases to be monotonic. It is interesting to note that the same $h^{4/3}$ power–law of equation (14) was also found by Chui and Moffatt by means of a scaling argument (see [6], p 206, equation (4.15)).

The minimum energy spectra of the first prime knots and links is determined by setting $h = 0$ in (13) and by using the ropelength data (λ_K) obtained by the RIDGERUNNER tightening algorithm [1] for each knot/link type K. A particularly simple expression is obtained by normalizing $m(\lambda_K, 0)$ with respect to the minimum energy value $m_\circ$ of the tight torus; thus, we have

$$\tilde{m}(K) = \frac{m(\lambda_K, 0)}{m_\circ} = \left(\frac{\lambda_K}{2\pi}\right)^{4/3}, \tag{15}$$

which gives the one-to-one relationship between minimum energy level and knot ropelength. Since the relation $\lambda_K = \lambda(K)$ is not known analytically, it must be reconstructed from numerical data. We take $\lambda_K = \lambda(\#_K)$, where $\#_K$ denotes the position of the knot/link K listed according to the increasing value of ropelength as given by RIDGERUNNER. Hence, instead of plotting energy levels as functions of the knot/link position according to standard knot tabulation, by taking $\lambda_K = \lambda(\#_K)$ in (15) we plot $\tilde{m} = \tilde{m}(\#_K)$, according to increasing ropelength data. The energy spectra are shown in figures 4 and 5 for the first 250 prime knots (up to ten crossings) and 130 prime links (up to nine crossings), respectively.

The curve, dotted with circles, is a result of the linear fit made over each $c_{\min}$ family, whilst the continuous curve is the best-fit interpolation over all available data. To the first decimal place, we find that the best-fit interpolations follow an almost identical logarithmic law, given by

$$\tilde{m}(\#_K) = a \ln \#_K + b, \tag{16}$$

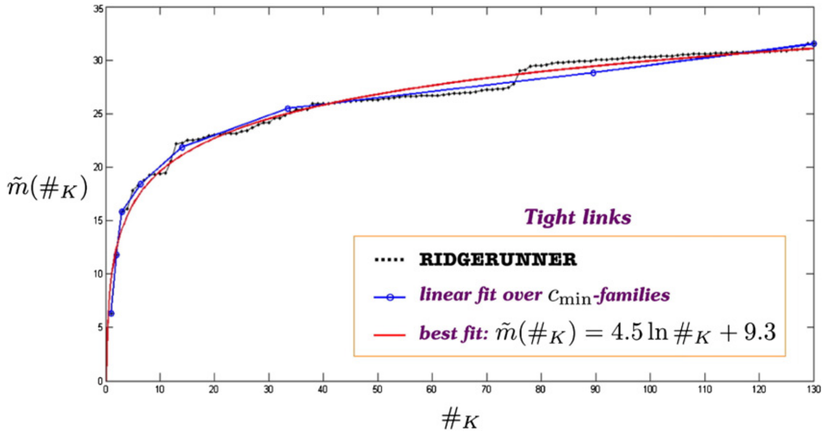

Figure 5. Energy spectrum $\tilde{m} = \tilde{m}(\#_K)$ of tight links plotted against the link number $\#_K$, given by the position of the link K listed according to increasing value of ropelength $\lambda_K = \lambda(\#_K)$. Best fit goodness: 95% confidence bounds, summed square of residuals (SSE) = 55.9, $R^2 = 0.98$, root mean squared error (RMSE) = 0.66.

where $a = 4.5$ and $b = b_K = 10.5$ for knots, $b = b_L = 9.3$ for links. This unexpected result is quite remarkable and calls for some justification. Ropelength is certainly an increasing function of topological complexity (given by $c_{\min}$), simply because an increasing number of crossings implies an increase in the minimal length necessary to tie a flux tube into a knot or a link. Results on ropelength bounds [3, 5, 7, 8] show that

$$O\left(c_{\min}^{3/4}\right) \leqslant \lambda_K \leqslant O(c_{\min} \ln^5 c_{\min}), \tag{17}$$

where $O(\cdot)$ denotes order of magnitude. From (15) we have that $\tilde{m}(\#_K) \propto [\lambda(\#_K)]^{4/3}$; by combining this with (16), we have

$$[\lambda(\#_K)]^{4/3} \propto a \ln \#_K + b. \tag{18}$$

Now, if we assume that the number of knots grows exponentially with $c_{\min}$ (a plausible assumption), then $\#_K \sim C^{c_{\min}}$ for some constant C. Hence, by (18) we have $[\lambda(\#_K)]^{4/3} \propto c_{\min}$, or

$$\lambda(\#_K) \propto c_{\min}^{3/4}, \tag{19}$$

providing a result that, if not true in full generality, is certainly in good agreement with the lower estimate given by (17). Furthermore, let us set (for simplicity) $V = \Phi = 1$ in (9), and define

$$\overline{m}(c_{\min}) \equiv \frac{M_{\min}}{m_\circ} = \frac{1}{\pi} c_{\min}. \tag{20}$$

We can then relate (9) to (15), and write

$$\langle \tilde{m}(K) \rangle_{c_{\min}} \geqslant \overline{m}(c_{\min}) = \frac{1}{\pi} c_{\min}, \tag{21}$$

since for any given K (as was remarked in section 2) $\tilde{m}(K)$ could be further decreased to the actual minimum by relaxing the constraints (i)–(iv) of theorem 1. By writing (15) in terms of $\#_K$ and substituting this into the above equation, we have

$$\langle \lambda(\#_K) \rangle_{c_{\min}} \geqslant 2\pi^{1/4} c_{\min}^{3/4}, \tag{22}$$

which gives a new relationship between ropelength, averaged over each $c_{\min}$ family, and $c_{\min}$. Note that the coefficient $2\pi^{1/4} \approx 2.66$ is independent of the knot family, and this result, in good agreement with (17), is the best analytical result valid for *any* $c_{\min}$ so far (see, for instance, [3]).

5. Conclusions

By using the analytical results for the constrained minimum energy of magnetic knots obtained by Maggioni and Ricca [10], we have established a general functional relationship between minimum energy levels of knots, links and internal twist h, given by an $h^{4/3}$ power law. In the case of standard flux tubes our result is in good agreement with an earlier result by Chui and Moffatt, [6] obtained by a scaling argument. Then, by using ropelength data from the RIDGERUNNER tightening algorithm developed by Ashton *et al* [1] we have computed the groundstate energy spectra of the first 250 prime knots and 130 prime links. We have shown that the two spectra follow an almost identical logarithmic law. By assuming that the number of knot types grows exponentially with the topological crossing number $c_{\min}$, we have shown that this generic behavior can be justified by the general relationship between ropelength and crossing number, which is independent of the number of components (knots or links). Moreover, by considering the ropelength averaged over a given knot family, we have established a new relation between this averaged ropelength and $c_{\min}^{3/4}$, valid for knots/links of *any* $c_{\min}$. However, as recent analytical work demonstrates [8], these results cannot be considered fully general and further improvements must be expected. In the context of magnetic relaxation, corrections are expected to come from a finer realization of the analytical constraints (for instance, by allowing the cross-section to adapt to an optimal shape) and from further improvements to the tightening procedure. In any case, our results demonstrate the great potential of magnetic energy methods to investigate and establish new relationships between the energy contents and topological properties of complex systems, and to study optimal properties of 3D–packing and global geometry. These results have useful applications in many disparate fields, from the study of structural complexity in physical and biological systems [4, 17], to applications in plasma physics and solar physics [18]. They may also provide a fresh insight into the ongoing search for fundamental aspects in the mass-energy relations of modern theoretical physics [2].

Acknowledgments

Useful comments from HK Moffatt and Y Diao on a preliminary version of this paper are gratefully acknowledged. RLR wishes to thank the Kavli Institute for Theoretical Physics at UC Santa Barbara for the kind hospitality and support, through the National Science Foundation under grant no. NSF PHY11-25915. The authors would also like to thank the 'Ennio De Giorgi' Mathematical Research Center in Pisa and the Isaac Newton Institute for Mathematical Sciences in Cambridge (UK), for hospitality and financial support.

References

[1] Ashton T, Cantarella J, Piatek M and Rawdon E 2011 Knot tightening by constrained gradient descent *Exp. Math.* **20** 57–90
[2] Buniy R V and Kephart T W 2005 Glueballs and the universal energy spectrum of tight knots and links *Int. J. Mod. Phys.* A **20** 1252–9
[3] Buck G and Simon J 1999 Thickness and crossing number of knots *Topol. Appl.* **91** 245–57

[4] Buck G and Simon J 2012 The spectrum of filament entanglement complexity and an entanglement phase transition *Proc. R. Soc. Lond.* A **468** 4024–40
[5] Cantarella J, Kusner R B and Sullivan J M 2002 On the minimum ropelength of knots and links *Invent. Math.* **150** 257–86
[6] Chui A Y K and Moffatt H K 1992 Minimum energy magnetic fields with toroidal topology *Topological Aspects of the Dynamics of Fluids and Plasmas* ed H K Moffatt *et al* (Dordrecht: Kluwer) pp 195–218
[7] Diao Y 2003 The lower bounds of the lengths of thick knots *J. Knot Theory and its Ramifications* **12** 1–16
[8] Diao Y, Ernst C, Por A and Ziegler U 2013 The ropelengths of knots are almost linear in terms of their crossing numbers (arXiv:0912.3282)
[9] Kauffman L H (ed) 1995 *Knots and Applications (Series on Knots and Everything* vol 6*)* (Singapore: World Scientific)
[10] Maggioni F and Ricca R L 2009 On the groundstate energy of tight knots *Proc. R. Soc. Lond.* A **465** 2761–83
[11] Moffatt H K 1985 Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology: part I. Fundamentals *J. Fluid. Mech.* **159** 359–78
[12] Moffatt H K 1990 The energy spectrum of knots and links *Nature* **347** 367–9
[13] Moffatt H K 2001 Some remarks on topological fluid mechanics *An Introduction to the Geometry and Topology of Fluid Flows* ed R L Ricca (Dordrecht: Kluwer) p 3
[14] Moffatt H K, Bajer K and Kimura Y (ed) 2013 *Topological Fluid Dynamics: Theory and Applications (Procedia IUTAM* vol 7*)* (Dordrecht: Elsevier)
[15] Ricca R L 2008 Topology bounds energy of knots and links *Proc. R. Soc. Lond.* A **464** 293–300
[16] Ricca R L (ed) 2009 *Lectures on Topological Fluid Mechanics (Springer-CIME Lecture Notes in Mathematics* vol 1973*)* (Heidelberg: Springer)
[17] Ricca R L 2009 Detecting structural complexity: from visiometrics to genomics and brain research *Mathknow* ed M Emmer and A Quarteroni (Berlin: Springer) pp 167–81
[18] Ricca R L 2013 New energy and helicity lower bounds for knotted and braided magnetic fields *Geophys. Astrophys. Fluid Dyn.* **107** 385–402
[19] Rolfsen D 1976 *Knots and Links* (Berkeley, CA: Publish or Perish)
[20] Spivak M 1979 *A Comprehensive Introduction to Differential Geometry* vol 1 (Houston, TX: Publish or Perish)
[21] Stasiak A, Katritch V and Kauffman L H 1998 *Ideal Knots (Series on Knots and Everything* vol 19*)* (Singapore: World Scientific)