GRADED LIE ALGEBRAS OF TYPE FP

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Abstract. It will be shown that every $\mathbb{N}$-graded Lie algebra generated in degree 1 of type FP with entropy less or equal to 1 must be finite-dimensional (cf. Thm. A). As a consequence every Koszul Lie algebra with entropy less or equal to 1 must be abelian (cf. Cor. C). These results are obtained from a generalized Witt formula (cf. Thm. D) for $\mathbb{N}$-graded Lie algebras of type FP and the analysis of necklace polynomials at roots of unity.

1. Introduction

Several deep results in group theory relate certain growth phenomena to the structure theory of a group, e.g., M. Gromov's celebrated theorem states that a finitely generated group has polynomial word growth if, and only if, it is virtually nilpotent (cf. [7]); while A. Lubotzky and A. Mann showed that a finitely generated pro-p group has polynomial subgroup growth if, and only if, it is p-adic analytic (cf. [14]). In a second paper together with D. Segal they achieved the beautiful result that a finitely generated residually finite (discrete) group has polynomial subgroup growth if, and only if, it is virtually soluble of finite rank (cf. [15]).

The main purpose of this paper is to establish an analogue of the just mentioned results in the context of $\mathbb{N}$-graded $\mathbb{F}$-Lie algebras which are generated in degree 1 and which are of finite type. Following [20] for such a Lie algebra $\mathbf{L} = \coprod_{k\geq 1} \mathbf{L}_k$ one calls the number

$$\mathbf{h}(\mathbf{L}) = \limsup_{n\to\infty} \sqrt[n]{\dim(\mathbf{L}_n)} \in \mathbb{R}_{\geq 1} \cup \{0, \infty\} \tag{1.1}$$

the *entropy* of $\mathbf{L}$. Moreover, a Lie algebra $\mathbf{L}$ is called to be *of type FP_∞*, if the trivial left $\mathbf{L}$-module $\mathbb{F}$ has a projective resolution $(P_\bullet, \partial_\bullet, \varepsilon)$, where P_k is a finitely generated projective left $\mathcal{U}(\mathbf{L})$-module for all k. Here $\mathcal{U}(\mathbf{L})$ denotes the *universal enveloping algebra* of $\mathbf{L}$. The Lie algebra $\mathbf{L}$ is said to be of *finite cohomological dimension* $\mathrm{cd}(\mathbf{L}) < \infty$, if $\mathbb{F}$ has a projective resolution $(P_\bullet, \partial_\bullet, \varepsilon)$ of finite length, i.e., there exists a positive integer n such that $P_k = 0$ for all $k \geq n$. If $\mathbf{L}$ is of type FP_∞ and of finite cohomological dimension, $\mathbf{L}$ is called to be *of type FP* (cf. [2, Chap. VIII.6]). The main result of this paper can be formulated as follows (cf. Thm. 3.7).

Theorem A. *Let $\mathbf{L}$ be an $\mathbb{N}$-graded Lie algebra generated in degree 1 and of type FP satisfying $\mathbf{h}(\mathbf{L}) \leq 1$. Then $\mathbf{L}$ is finite-dimensional and nilpotent, i.e., $\mathbf{h}(\mathbf{L}) = 0$.*

There are many examples of $\mathbb{N}$-graded Lie algebras of finite type which are generated in degree 1 and whose entropy is equal to 1, e.g., if $\mathbf{L}$ is infinite dimensional and filiform (cf. [17]) one has $\dim(\mathbf{L}_k) = 1$ for $k \geq 2$ and therefore $\mathbf{h}(\mathbf{L}) = 1$. The

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Lie algebras $\mathbf{L}$ constructed in [9] satisfy $\mathbf{h}(\mathbf{L}) = 1$ and the series $(\dim(\mathbf{L}_k))_{k\geq 1}$ has intermediate growth. As a consequence of Theorem A one obtains the following.

Corollary B. *Let $\mathbf{L}$ be an $\mathbb{N}$-graded finitely generated Lie algebra which is generated in degree 1 satisfying $\mathbf{h}(\mathbf{L}) = 1$. Then either $\mathrm{cd}(\mathbf{L}) = \infty$ or there exists $k \geq 2$ such that $\dim(H^k(\mathbf{L}, \mathbb{F})) = \infty$.*

An $\mathbb{N}$-graded Lie algebra $\mathbf{L}$ will be said to be *Koszul*, if $\mathcal{U}(\mathbf{L})$ is a Koszul algebra. Koszul algebras are a rather mysterious class of $\mathbb{N}_0$-graded associative algebras, and there are still many open questions. Theorem A provides an answer to one of these open questions (cf. [21, Chap. 7.1, Conj. 3]) in case that the Koszul algebra $\mathbf{A}$ happens to be the universal enveloping algebra of an $\mathbb{N}$-graded Lie algebra (cf. Prop. 4.2).

Corollary C. *Let $\mathbf{L}$ be a Koszul Lie algebra satisfying $\mathbf{h}(\mathbf{L}) \leq 1$. Then $\mathbf{L}$ is finite-dimensional and abelian.*

An $\mathbb{N}$-graded Lie algebra $\mathbf{L}$ of type FP has a *characteristic polynomial* (cf. (2.13))

$$\chi_{\mathbf{L}}(y) = \textstyle\prod_{1\leq i\leq n}(1 - \lambda_i y) \in \mathbb{C}[y] \tag{1.2}$$

which depends entirely on the cohomology of $\mathbf{L}$. The complex numbers λ_i will be called the *eigenvalues* of $\mathbf{L}$. For such a Lie algebra one has the following generalized Witt formula (cf. Thm. 3.4).

Theorem D. *Let $\mathbf{L}$ be an $\mathbb{N}$-graded Lie algebra of type FP, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the eigenvalues of $\mathbf{L}$. Then*

$$\dim(\mathbf{L}_k) = \textstyle\sum_{1\leq i\leq n} M_k(\lambda_i), \tag{1.3}$$

where $\mu\colon \mathbb{N} \to \mathbb{Z}$ is the Möbius function and $M_k(y) = \frac{1}{k}\sum_{j|k}\mu(k/j)\cdot y^j \in \mathbb{Q}[y]$ denotes the necklace polynomial of degree k.

Theorem A can be deduced from Theorem D and an analysis of necklace polynomials at roots of unity (cf. §3.3). For an infinite dimensional $\mathbb{N}$-graded Lie algebra $\mathbf{L}$ of type FP which is generated in degree 1 its entropy $\mathbf{h}(\mathbf{L})$ is the positive real root of the characteristic polynomial $\chi_{\mathbf{L}}(y)$ of maximal absolute value (cf. Prop. 2.6, Cor. 3.3). Some general results on the eigenvalues of $\chi_{\mathbf{L}}(y)$ can be obtained (cf. Remark 2.8, Fact 2.11), but several questions will remain unanswered (cf. Question 1 and 2). Since it seems that the Koszul property has been investigated only scarsely in the context of Lie algebras, we close this paper with a discussion of three classes of examples illustrating the diversity of this class of Lie algebras.

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2. Graded connected algebras of finite type

Throughout the paper $\mathbb{F}$ will denote a field. By $\mathbb{N}_0$ (resp. $\mathbb{N}$) we denote the set of non-negative (resp. positive) integers, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An $\mathbb{N}_0$-graded $\mathbb{F}$-vector space $V = \coprod_{k\geq 0} V_k$ is said to be of *finite type*, if $\dim(V_k) < \infty$ for all $k \geq 0$. For such a graded $\mathbb{F}$-vector space its *Hilbert series* is defined by

$$h_V(y) = \textstyle\sum_{k\geq 0}\dim(V_k)\cdot y^k \in \mathbb{Z}[\![y]\!], \tag{2.1}$$

e.g., V is finite-dimensional if, and only if, $h_V(y)$ is a polynomial.

2.1. **Graded connected algebras of finite type.** An $\mathbb{N}_0$-*graded* $\mathbb{F}$-*algebra* $\mathbf{A} = \coprod_{k\geq 0} \mathbf{A}_k$ is an associative $\mathbb{F}$-algebra and an $\mathbb{N}_0$-graded $\mathbb{F}$-vector space satisfying $\mathbf{A}_m \cdot \mathbf{A}_n \subseteq \mathbf{A}_{m+n}$ for all $m, n \geq 0$. From now we will omit the appearance of the field $\mathbb{F}$ in the notation.

An $\mathbb{N}_0$-graded algebra $\mathbf{A}$ is called to be *connected*, if $\mathbf{A}_0 = \mathbb{F} \cdot 1$, i.e., in this case one has a unique *augmentation* $\varepsilon\colon \mathbf{A} \to \mathbb{F}$ which is a homomorphism of $\mathbb{N}_0$-graded algebras. By $\mathbf{A}^+ = \coprod_{k\geq 1} \mathbf{A}_k = \ker(\varepsilon)$ we denote its *augmentation ideal*, and $\mathbb{F}$ will denote the $\mathbb{N}_0$-graded left $\mathbf{A}$-module whose representation is equal to ε.

For an $\mathbb{N}_0$-graded, connected algebra $\mathbf{A}$ of finite type let ${}_{\mathbf{A}}\mathbf{mod}^{\mathrm{ft}}$ denote the abelian category of $\mathbb{N}_0$-graded left $\mathbf{A}$-modules of finite type. In particular, if $0 \to M \to N \to Q \to 0$ is a short exact sequence in ${}_{\mathbf{A}}\mathbf{mod}^{\mathrm{ft}}$, one has

$$h_N(y) = h_M(y) + h_Q(y). \tag{2.2}$$

For $M \in \mathrm{ob}({}_{\mathbf{A}}\mathbf{mod}^{\mathrm{ft}})$ put $M_{\mathbf{A}} = M/\mathbf{A}^+M$. Then $M_{\mathbf{A}}$ is an $\mathbb{N}_0$-graded vector space of finite type, and one has a canonical projection $\pi_M\colon M \to M_{\mathbf{A}}$ of $\mathbb{N}_0$-graded left $\mathbf{A}$-modules. Moreover, $M = 0$ if, and only if, $M_{\mathbf{A}} = 0$.

Let $\sigma\colon M_{\mathbf{A}} \to M$ be a section of π_M in the category of $\mathbb{N}_0$-graded vector spaces, i.e., $\pi_M \circ \sigma = \mathrm{id}_{M_{\mathbf{A}}}$. The map σ induces a surjective homomorphism of $\mathbb{N}_0$-graded, left $\mathbf{A}$-modules $\tilde{\sigma}\colon \mathbf{A} \otimes M_{\mathbf{A}} \to M$, where $\otimes = \otimes_{\mathbb{F}}$, given by $\tilde{\sigma}(a \otimes m) = a\sigma(m)$, $a \in \mathbf{A}$, $m \in M_{\mathbf{A}}$. Moreover, if $M_j = 0$ for $0 \leq j \leq k$, then $\ker(\tilde{\sigma})_i = 0$ for $0 \leq i \leq k+1$.

The just mentioned procedure can be used to build up a projective resolution $(P_\bullet, \partial_\bullet, \varepsilon)$ of $\mathbb{F}$, i.e., one may define

(i) $P_0 = \mathbf{A}$,
(ii) $P_s = \mathbf{A} \otimes \ker(\partial_{s-1})_{\mathbf{A}}$ for $s \geq 1$, where we put $\partial_0 = \varepsilon$.

The mappings $\partial_s\colon P_s \to \ker(\partial_{s-1})$, $s \geq 1$, coincide with $\tilde{\sigma}_s$, where $\sigma_s\colon \ker(\partial_{s-1})_{\mathbf{A}} \to \ker(\partial_{s-1})$ is a section in the category of $\mathbb{N}_0$-graded vector spaces. For an $\mathbb{N}_0$-graded, connected algebra $\mathbf{A}$ its *cohomology algebra* $\mathrm{Ext}^{\bullet,\bullet}_{\mathbf{A}}(\mathbb{F}, \mathbb{F})$ is naturally bigraded. The first degree in the notation will correspond to the *homological degree*, while the second will refer to the *internal degree*. The construction of the projective resolution $(P_\bullet, \partial_\bullet, \varepsilon)$ has shown the following.

Proposition 2.1. *Let* $\mathbf{A}$ *be an* $\mathbb{N}_0$*-graded, connected algebra of finite type.*

(a) $\mathrm{Ext}^{s,t}_{\mathbf{A}}(\mathbb{F}, \mathbb{F}) \simeq (\ker(\partial_{s-1})_{\mathbf{A},t})^*$, *where* $\_^* = \mathrm{Hom}_{\mathbb{F}}(\_, \mathbb{F})$.
(b) $\dim(\mathrm{Ext}^{s,t}_{\mathbf{A}}(\mathbb{F}, \mathbb{F})) < \infty$, *and* $\mathrm{Ext}^{s,t}_{\mathbf{A}}(\mathbb{F}, \mathbb{F}) = 0$ *for* $t < s$.

One can collect the information provided by Proposition 2.1 in the power series

$$\begin{aligned} \tilde{\chi}_{\mathbf{A}}(x, y) &= \sum_{s,t\geq 0} \dim(\mathrm{Ext}^{s,t}_{\mathbf{A}}(\mathbb{F}, \mathbb{F})) \cdot (-x)^s \cdot y^t \quad &\in \mathbb{Z}[\![x, y]\!], \\ \chi_{\mathbf{A}}(y) = \tilde{\chi}_{\mathbf{A}}(1, y) &= \sum_{s,t\geq 0} (-1)^s \cdot \dim(\mathrm{Ext}^{s,t}_{\mathbf{A}}(\mathbb{F}, \mathbb{F})) \cdot y^t \quad &\in \mathbb{Z}[\![y]\!]. \end{aligned} \tag{2.3}$$

The power series $\chi_{\mathbf{A}}(y) \in \mathbb{Z}[\![y]\!]$ will be called the *characteristic power series* of the $\mathbb{N}_0$-graded, connected $\mathbb{F}$-algebra $\mathbf{A}$. Obviously, $\tilde{\chi}_{\mathbf{A}}(x, y)$ contains the complete information on the dimensions of the bi-graded cohomology algebra $\mathrm{Ext}^{\bullet,\bullet}_{\mathbf{A}}(\mathbb{F}, \mathbb{F})$. The information contained in $\chi_{\mathbf{A}}(y)$ is somehow weaker, but its importance is reflected in the following property.

Proposition 2.2. *Let* $\mathbf{A}$ *be an* $\mathbb{N}_0$*-graded, connected* $\mathbb{F}$*-algebra. Then one has*

$$\chi_{\mathbf{A}}(y)\cdot h_{\mathbf{A}}(y)=1 \qquad \text{in } \mathbb{Z}[\![y]\!]. \tag{2.4}$$

Proof. The acyclicity of the chain complex $(P_\bullet,\partial_\bullet)$ and Proposition 2.1 imply that

$$1=\sum_{s\geq 0}(-1)^s\cdot h_{P_s}(y)=h_{\mathbf{A}}(y)\cdot\sum_{s,t\geq 0}(-1)^s\cdot\dim(\mathrm{Ext}^{s,t}_{\mathbf{A}}(\mathbb{F},\mathbb{F}))\cdot y^t. \tag{2.5}$$

This yields the claim. □

2.2. **Cohomological finiteness conditions.** Let $\mathbf{A}$ be an $\mathbb{N}_0$-graded, connected algebra. The *cohomological dimension* $\mathrm{cd}(\mathbf{A})$ of $\mathbf{A}$ is defined by

$$\mathrm{cd}(\mathbf{A})=\min(\{\,n\in\mathbb{N}_0\mid \mathrm{Ext}^{n+1}_{\mathbf{A}}(\mathbb{F},M)=0 \text{ for all } M\in\mathrm{ob}({}_{\mathbf{A}}\mathbf{mod})\,\}\cup\{\infty\}), \tag{2.6}$$

where ${}_{\mathbf{A}}\mathbf{mod}$ denotes the abelian category of left $\mathbf{A}$-modules. Moreover, $\mathbf{A}$ is called to be *of type* FP_m, $1\leq m\leq\infty$, if there exists a partial projective resolution

$$P_m \xrightarrow{\partial_m} P_{m-1} \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_a} P_0 \xrightarrow{\varepsilon} \mathbb{F} \longrightarrow 0 \tag{2.7}$$

with $P_0,\dots,P_m$ being finitely generated. If $\mathbf{A}$ is of type FP_∞ and $\mathrm{cd}(\mathbf{A})<\infty$, then one calls $\mathbf{A}$ to be *of type FP* (cf. [2, Chap. 8.6]). From Proposition 2.1 one concludes the following.

Fact 2.3. *Let* $\mathbf{A}$ *be an* $\mathbb{N}_0$*-graded connected algebra of finite type.*

(a) $\mathrm{cd}(\mathbf{A})\leq d<\infty$ *if, and only if,* $\mathrm{Ext}^{d+1,\bullet}_{\mathbf{A}}(\mathbb{F},\mathbb{F})=0$.
(b) $\mathbf{A}$ *is of type* FP_∞ *if, and only if, for every* $s\geq 1$ *there exists* $m(s)\geq 0$ *such that* $\mathrm{Ext}^{s,t}_{\mathbf{A}}(\mathbb{F},\mathbb{F})=0$ *for all* $t\geq m(s)$.
(c) $\mathbf{A}$ *is of type FP if, and only if,* $\tilde{\chi}_{\mathbf{A}}(x,y)$ *is a polynomial.*

Remark 2.4. (a) For an $\mathbb{N}$-graded vector space V ($V_0=0$) the *tensor algebra*

$$\mathbf{T}(V)=\textstyle\coprod_{k\geq 0}\mathbf{T}_k(V),\qquad \mathbf{T}_0(V)=\mathbb{F}\cdot 1,\qquad \mathbf{T}_k(V)=\overbrace{V\otimes\cdots\otimes V}^{k\text{-times}} \tag{2.8}$$

is a connected $\mathbb{N}_0$-graded algebra with the grading induced by the grading of V. Moreover, one has a projective resolution

$$0\longrightarrow \mathbf{T}(V)\otimes V \xrightarrow{\partial_1} \mathbf{T}(V) \xrightarrow{\varepsilon} \mathbb{F}\longrightarrow 0\ , \tag{2.9}$$

where ∂_1 is given by multiplication. Thus, $\mathbf{T}(V)$ is of finite type if, and only if, V is of finite type, and $\mathbf{T}(V)$ is of type FP_1 if, and only if, V is finite-dimensional.
(b) Let $\mathbf{A}$ be an $\mathbb{N}_0$-graded, connected algebra, and let $\tau\colon \mathbf{A}^+_{\mathbf{A}}\to\mathbf{A}^+$ be a section in the category of $\mathbb{N}$-graded vector spaces. Then one has a unique homomorphism $\tau_\bullet\colon \mathbf{T}(\mathbf{A}^+_{\mathbf{A}})\to\mathbf{A}$ of $\mathbb{N}_0$-graded, connected algebras satisfying $\tau_1=\tau$. Moreover, $\tau_\bullet$ is surjective. Hence $\mathbf{A}$ is finitely generated if, and only if, $\mathbf{A}^+_{\mathbf{A}}$ is finite-dimensional. In particular, every finitely generated $\mathbb{N}_0$-graded, connected algebra is of finite type. Let $\partial^{\mathbf{A}}_1\colon \mathbf{A}\otimes\mathbf{A}^+_{\mathbf{A}}\to\mathbf{A}$ be given by $\partial^{\mathbf{A}}_1(a\otimes b)=a\tau(b)$, $a\in\mathbf{A}$, $b\in\mathbf{A}^+_{\mathbf{A}}$. Then

$$\mathbf{A}\otimes\mathbf{A}^+_{\mathbf{A}} \xrightarrow{\partial^{\mathbf{A}}_1} \mathbf{A} \xrightarrow{\varepsilon_{\mathbf{A}}} \mathbb{F}\longrightarrow 0. \tag{2.10}$$

is a minimal partial projective resolution, and from the minimality one concludes that $\mathrm{Tor}^{\mathbf{A}}_1(\mathbb{F},\mathbb{F})\simeq\mathbf{A}^+_{\mathbf{A}}$. Thus $\mathbf{A}$ is finitely generated if, and only if, $\mathbf{A}$ is of type FP_1.

(c) By construction, one has a commutative diagram

$$
\begin{tikzcd}
& & & \ker(\tau_\bullet) \arrow[dl, dashed, "j"'] \arrow[d] \arrow[ddll, dotted, bend right, "\rho"'] & & \\
& 0 \arrow[r] & \mathbf{T}(\mathbf{A}^+_{\mathbf{A}}) \otimes \mathbf{A}^+_{\mathbf{A}} \arrow[r, "\partial_1"] \arrow[d, "\tau_\bullet \otimes \mathrm{id}"] & \mathbf{T}(\mathbf{A}^+_{\mathbf{A}}) \arrow[r, "\varepsilon"] \arrow[d, "\tau_\bullet"] & \mathbb{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{r}(\tau_\bullet) \arrow[r] & \mathbf{A} \otimes \mathbf{A}^+_{\mathbf{A}} \arrow[r, "\partial_1^{\mathbf{A}}"] & \mathbf{A} \arrow[r, "\varepsilon_{\mathbf{A}}"] & \mathbb{F} \arrow[r] & 0
\end{tikzcd} \tag{2.11}
$$

with exact rows. The left $\mathbf{A}$-module $\mathbf{r}(\tau_\bullet) \simeq \mathrm{Tor}_1^{\mathbf{T}(\mathbf{A}^+_{\mathbf{A}})}(\mathbf{A}, \mathbb{F})$ is also called the *relation module* of the presentation $\tau_\bullet$. Since $\ker(\tau_\bullet) \subseteq \mathbf{T}(\mathbf{A}^+_{\mathbf{A}})^+$, there exists an injective homomorphism of left $\mathbf{T}(\mathbf{A}^+_{\mathbf{A}})$-modules $j\colon \ker(\tau_\bullet) \to \mathbf{T}(\mathbf{A}^+_{\mathbf{A}}) \otimes \mathbf{A}^+_{\mathbf{A}}$ making the diagram (2.11) commute. Let $\beta = (\tau_\bullet \otimes \mathrm{id}) \circ j$. As $\partial_1^{\mathbf{A}} \circ \beta = 0$, there exists a homomorphism of left $\mathbf{T}(\mathbf{A}^+_{\mathbf{A}})$-modules $\rho\colon \ker(\pi) \to \mathbf{r}(\tau_\bullet)$ making the diagram (2.11) commute. This homomorphism has the following property.

Proposition 2.5. *The homomorphism $\rho\colon \ker(\tau_\bullet) \to \mathbf{r}(\tau_\bullet)$ is surjective and induces an isomorphism $\bar{\rho}\colon \ker(\tau_\bullet)/\ker(\tau_\bullet)\mathbf{A}^+_{\mathbf{A}} \longrightarrow \mathbf{r}(\tau_\bullet)$ of left $\mathbf{A}$-modules.*

Proof. Let y be an element in $\mathbf{r}(\tau_\bullet)$, and let z be its canonical image in $\mathbf{A} \otimes \mathbf{A}^+_{\mathbf{A}}$. As $\tau_\bullet \otimes \mathrm{id}$ is surjective, there exists $w \in \mathbf{T}(\mathbf{A}^+_{\mathbf{A}}) \otimes \mathbf{A}^+_{\mathbf{A}}$ such that $(\tau_\bullet \otimes \mathrm{id})(w) = z$. By the commutativity of (2.11), $\partial_1(w) \in \ker(\tau_\bullet)$, and it is easy to verify that $\rho(\partial_1(w)) = y$. Hence ρ is surjective. By definition, $\ker(\rho) = \partial_1(\ker(\tau_\bullet \otimes \mathrm{id})) = \ker(\tau_\bullet)\mathbf{A}^+_{\mathbf{A}}$. $\square$

From Proposition 2.5 one concludes that one has isomorphisms

$$
\mathrm{Tor}_2^{\mathbf{A}}(\mathbb{F}, \mathbb{F}) \simeq \mathbf{r}(\tau_\bullet)_{\mathbf{A}} \simeq \ker(\tau_\bullet)/(\mathbf{A}^+_{\mathbf{A}} \ker(\tau_\bullet) + \ker(\tau_\bullet)\mathbf{A}^+_{\mathbf{A}}). \tag{2.12}
$$

Hence $\mathbf{A}$ is type FP_2 if, and only if, $\mathbf{A}$ is finitely presented. Note that one has an isomorphism $\mathrm{Ext}^{2,\bullet}_{\mathbf{A}}(\mathbb{F}, \mathbb{F}) \simeq \mathbf{r}(\tau_\bullet)^*_{\mathbf{A}}$, where $\_^* = \mathrm{Hom}_{\mathbb{F}}(\_, \mathbb{F})$.

2.3. **χ-finite algebras.** An $\mathbb{N}_0$-graded, connected algebra $\mathbf{A}$ will be called to be *χ-finite*, if it is of finite type and $\chi_{\mathbf{A}}(y)$ is a polynomial. By Fact 2.3, the class of χ-finite algebras contains the class of $\mathbb{N}_0$-graded, connected algebras which are of finite type and of type FP. Moreover, by Proposition 2.2, this class coincides with the class considered in [20, §2] of $\mathbb{N}_0$-graded, connected algebras of finite type with a linear recurrence relation.

Let $\mathbf{A}$ be a χ-finite algebra. The integer $\deg(\mathbf{A}) = \deg(\chi_{\mathbf{A}}(y))$ will be called the *degree* of $\mathbf{A}$. Let $K_{\mathbf{A}} = \mathbb{Q}(\chi_{\mathbf{A}})$ denote the splitting field of $\chi_{\mathbf{A}}(y)$ over $\mathbb{Q}$, and let $\iota\colon K_{\mathbf{A}} \to \mathbb{C}$ be a fixed complex embedding of $K_{\mathbf{A}}$ in the field of complex numbers. For simplicity we may assume that ι is given by inclusion. The numbers $\lambda_1, \ldots, \lambda_n \in K_{\mathbf{A}} \subseteq \mathbb{C}$, $n = \deg(\mathbf{A})$, satisfying

$$
\chi_{\mathbf{A}}(y) = (1 - \lambda_1 y) \cdots (1 - \lambda_n y) \tag{2.13}
$$

will be called the *eigenvalues* of $\mathbf{A}$, and the leading coefficient of $\chi_{\mathbf{A}}(y)$ times $(-1)^n$ will be called the *conductor* $c_{\mathbf{A}}$ of $\mathbf{A}$, i.e., one has $c_{\mathbf{A}} = \lambda_1 \cdots \lambda_n$. For $\mathbf{A} \neq \mathbb{F}$ put

$$
\lambda_{\max} = \max\{\, |\lambda_1|, \ldots, |\lambda_n| \,\} \in \mathbb{R}_{>0}. \tag{2.14}
$$

Obviously, $\lambda_{\max} \geq 1$, and one has the following property.

Proposition 2.6. *Let $\mathbf{A}$, $\mathbf{A} \neq \mathbb{F}$, be a χ-finite algebra. Then $\mathbf{h}(\mathbf{A}) = \lambda_{\max}$, and $\lambda_{\max}$ is an eigenvalue of $\mathbf{A}$. Moreover, one has $1 \leq \mathbf{h}(\mathbf{A}) < \infty$.*

Proof. Let $h = h_{\mathbf{A}}(y)$ denote the Hilbert series of $\mathbf{A}$, and let $\chi = \chi_{\mathbf{A}}(y)$ denote the characteristic polynomial of $\mathbf{A}$. Since χ is a polynomial of degree $n = \deg(\mathbf{A}) \geq 1$, χ can be interpreted as a holomorphic function $\chi\colon \mathbb{C} \to \mathbb{C}$. Thus, by (2.4), $h = h_{\mathbf{A}}$ defines a meromorphic function $h\colon \mathbb{C} \to \bar{\mathbb{C}}$. In particular, since $\lambda_1^{-1}, \cdots, \lambda_n^{-1} \in \mathbb{C}$ are the roots of χ, they are also the poles of h. Hence, if λ_j is an eigenvalue of $\mathbf{A}$ of maximal absolute value, λ_j^{-1} is a pole of h closest to 0. This implies that $\lambda_{\max}^{-1} = |\lambda_j^{-1}| = \rho$ coincides with the convergence radius of the power series h, i.e., $\lambda_{\max} = \limsup_{n\to\infty} \sqrt[n]{\dim(\mathbf{A}_n)} = \mathbf{h}(\mathbf{A})$. Since all coefficients of the power series h are non-negative, one has

$$|h(z)| \leq \textstyle\sum_{k\geq 0} \dim(\mathbf{A}_k) \cdot |z|^k = h(|z|) \tag{2.15}$$

for all $z \in \mathbb{C}$ with $|z| < \rho$. Hence $\rho = |\lambda_j^{-1}|$ is a pole of h and thus must coincide with one of the elements $\lambda_1^{-1}, \cdots, \lambda_n^{-1}$, i.e., there exists $i \in \{1, \ldots, n\}$ such that $\lambda_i = |\lambda_j| = \rho^{-1} = \mathbf{h}(\mathbf{A})$. As $\lambda_1 \cdots \lambda_n = c_{\mathbf{A}}$, one has $\lambda_{\max} \geq 1$. □

Proposition 2.7. *Let $\mathbf{A}$, $\mathbf{A} \neq \mathbb{F}$, be a χ-finite algebra satisfying $\mathbf{h}(\mathbf{A}) = 1$. Then all eigenvalues $\lambda_1, \ldots, \lambda_n$, $n = \deg(\mathbf{A})$, are roots of unity.*

Proof. By definition, $K_{\mathbf{A}}/\mathbb{Q}$ is a Galois extension. Let $G = \mathrm{Gal}(K_{\mathbf{A}}/\mathbb{Q})$ denote its Galois group. In particular, G acts on the set $\{\lambda_1, \ldots, \lambda_n\}$. Let $\iota_k\colon K_{\mathbf{A}} \to \mathbb{C}$, $1 \leq k \leq r$, $\iota_1 = r$, denote the different embeddings of $K_{\mathbf{A}}$ into the field of complex numbers, and let $|\_|_j = |\_| \circ \iota_j\colon K_{\mathbf{A}} \to \mathbb{R}_{\geq 0}$ denote the associated absolute values. Then one has a homomorphism of groups

$$\beta = \textstyle\coprod_{1\leq k\leq r} |\_|_j\colon K_{\mathbf{A}}^\times \longrightarrow \coprod_{1\leq k\leq r} \mathbb{R}_{>0}, \tag{2.16}$$

where $K_{\mathbf{A}}^\times$ denotes the multiplicative group of the field $K_{\mathbf{A}}$.

By construction, $f(y) = y^n \chi_{\mathbf{A}}(1/y) \in \mathbb{Z}[y]$ and $f(\lambda_j) = 0$ for all $j \in \{1, \ldots, n\}$. Hence $\lambda_j \in \mathcal{O}$, where $\mathcal{O}$ denotes the integral closure of $\mathbb{Z}$ in $K_{\mathbf{A}}$. As $\lambda_1 \cdots \lambda_n = c_{\mathbf{A}}$ and $\lambda_{\max} = 1$, one has $|c_{\mathbf{A}}| \leq 1$. Since $c_{\mathbf{A}}$ is a non-trivial integer, this implies that $c_{\mathbf{A}} \in \{\pm 1\}$, and therefore, $|\lambda_j| = 1$ for all $j \in \{1, \ldots, n\}$. Moreover, as $\lambda_1 \cdots \lambda_n \in \{\pm 1\}$, one has that $\lambda_j \in \mathcal{O}^\times$, where $\mathcal{O}^\times$ denotes the group of units in the ring $\mathcal{O}$. Since $K_{\mathbf{A}}/\mathbb{Q}$ is a Galois extension, for any $k \in \{1, \ldots, r\}$ there exists $g_k \in G$ such that $\iota_k = \iota \circ g_k$. Hence

$$|\lambda_j|_k = |g_k(\lambda_j)| = 1. \tag{2.17}$$

Hence $\lambda_j \in \ker(\beta) \cap \mathcal{O}^\times = \mu(K_{\mathbf{A}})$, where $\mu(K_{\mathbf{A}})$ denotes the group of roots of unity in the number field $K_{\mathbf{A}}$ (cf. [19, Chap. I, §7, Thm. 1]). This yields the claim. □

Remark 2.8. Let $\mathbf{A}$ be an $\mathbb{N}_0$-graded, connected algebra of finite type which is of type FP_∞. Then there is also another type of power series one studies in this context, i.e., the power series

$$p_{\mathbf{A}}(x) = \sum_{s\geq 0} (-1)^s \cdot \dim(H^s(\mathbf{A}, \mathbb{F})) \cdot x^s \in \mathbb{Z}[\![x]\!] \tag{2.18}$$

is called the *Poincaré series* of $\mathbf{A}$, i.e., one has $p_{\mathbf{A}}(x) = \tilde{\chi}_{\mathbf{A}}(x, 1)$ (cf. (2.3)). If $\mathbf{A}$ is additionally of type FP, then $p_{\mathbf{A}}(1) = \chi_{\mathbf{A}}(1)$ is also called the *Euler-Poincaré characteristic* of $\mathbf{A}$, i.e., $\mathbf{A}$ has Euler-Poincaré characteristic 0 if, and only if, 1 is an eigenvalue of $\mathbf{A}$.

2.4. **Graded algebras generated in degree** 1. Let V be a graded vector space concentrated in degree 1, i.e., $V_s = 0$ for $s \neq 1$. The $\mathbb{N}_0$-graded algebra $\mathbf{T}(V)$ has the property that for every $\mathbb{N}_0$-graded algebra $\mathbf{A}$ and for every homomorphism of vector spaces $\phi\colon V \to \mathbf{A}_1$ there exists a unique homomorphism of $\mathbb{N}_0$-graded algebras $\phi_\bullet\colon \mathbf{T}(V) \to \mathbf{A}$ such that $\phi_1 = \phi$. The $\mathbb{N}_0$-graded algebra $\mathbf{A}$ is said to be *generated in degree* 1, if $\mathrm{id}_\bullet\colon \mathbf{T}(\mathbf{A}_1) \to \mathbf{A}$ is surjective. In particular, such an $\mathbb{N}_0$-graded algebra must be connected, and multiplication induces a surjective map $\mathbf{A}_m \otimes \mathbf{A}_n \to \mathbf{A}_{n+m}$ for all $m, n \geq 0$, i.e., $\dim(\mathbf{A}_{m+n}) \leq \dim(\mathbf{A}_m) \cdot \dim(\mathbf{A}_n)$ for all $m, n \geq 0$. One has the following properties.

Fact 2.9. *Let* $\mathbf{A}$ *be an* $\mathbb{N}_0$*-graded, connected algebra.*

(a) $\mathbf{A}$ *is generated in degree* 1 *if, and only if,* $\mathrm{Ext}^{1,t}_{\mathbf{A}}(\mathbb{F},\mathbb{F}) = 0$ *for all* $t \geq 2$.
(b) *Suppose* $\mathbf{A}$ *is generated in degree* 1*. Then* $\mathbf{A}$ *is finitely generated if, and only if, it is of finite type.*
(c) *If* $\mathbf{A}$ *is finitely generated and generated in degree* 1*, then* $\lim_{k\to\infty} \sqrt[k]{\dim(\mathbf{A}_k)}$ *exists and is equal to* $\mathbf{h}(\mathbf{A})$*. Moreover,* $\mathbf{h}(\mathbf{A}) \leq \dim(\mathbf{A}_1)$.

Proof. (a) and (b) are straightforward. For (c) see [22, Part I, Prob. 98, p. 23]. □

2.5. **Quadratic algebras.** Let $\mathbf{A}$ be an $\mathbb{N}_0$-graded algebra which is generated in degree 1. Then $\mathbf{A}$ is said to be *quadratic*, if

$$\ker(\mathrm{id}_\bullet) = \langle \mathbf{r}_2(\mathbf{A})\rangle = \mathbf{T}(\mathbf{A}_1) \otimes \mathbf{r}_2(\mathbf{A}) \otimes \mathbf{T}(\mathbf{A}_1), \tag{2.19}$$

where $\mathbf{r}_2(\mathbf{A}) = \ker(\mathrm{id}_2) \subseteq \mathbf{A}_1 \otimes \mathbf{A}_1 = \mathbf{T}_2(\mathbf{A}_1)$. From Remark 2.4(c) one concludes the following.

Fact 2.10. *Let* $\mathbf{A}$ *be an* $\mathbb{N}_0$*-graded, connected algebra which is generated in degree* 1*. Then* $\mathbf{A}$ *is quadratic if, and only if,* $\mathrm{Ext}^{2,t}_{\mathbf{A}}(\mathbb{F},\mathbb{F}) = 0$ *for all* $t \geq 3$.

Let $\mathbf{A}$ be a quadratic algebra of finite type, and put

$$\mathbf{r}_2(\mathbf{A})^\perp = \{\, c \in \mathbf{A}_1^* \otimes \mathbf{A}_1^* \mid \langle c, a\rangle = 0 \text{ for all } a \in \mathbf{r}_2(\mathbf{A}) \,\}, \tag{2.20}$$

where $\_^* = \mathrm{Hom}_{\mathbb{F}}(\_,\mathbb{F})$, and $\langle \_,\_\rangle\colon (\mathbf{A}_1^* \otimes \mathbf{A}_1^*) \otimes \mathbf{A}_1 \otimes \mathbf{A}_1 \longrightarrow \mathbb{F}$ denotes the evaluation homomorphism. Then $\mathbf{A}^! = \mathbf{T}(\mathbf{A}_1^*)/\langle \mathbf{r}_2(\mathbf{A})^\perp\rangle$ is a quadratic algebra which is called the *quadratic dual of* $\mathbf{A}$. By construction, one has a natural isomorphism $(\mathbf{A}^!)^! \simeq \mathbf{A}$.

2.6. **Koszul algebras.** A quadratic algebra of finite type $\mathbf{A}$ is said to be *Koszul*, if $\mathrm{Ext}^{s,t}_{\mathbf{A}}(\mathbb{F},\mathbb{F}) = 0$ for $s \neq t$. By definition, $\mathbf{A}$ is of type FP_∞, and $\chi_{\mathbf{A}}(y) = p_{\mathbf{A}}(y)$ (cf. (2.18)). Koszul algebras were introduced by S.B. Priddy in [23], where he showed that for such algebras one has an isomorphism

$$\mathbf{A}^! \simeq \mathrm{diag}(\mathrm{Ext}^{\bullet,\bullet}_{\mathbf{A}}(\mathbb{F},\mathbb{F})); \tag{2.21}$$

in particular, $\chi_{\mathbf{A}}(y) = p_{\mathbf{A}}(y) = h_{\mathbf{A}^!}(-y)$. An $\mathbb{F}$-Koszul algebra is of finite cohomological dimension if, and only if, $\mathbf{A}$ is χ-finite. In this case one has $\deg(\mathbf{A}) = \mathrm{cd}(\mathbf{A})$. If $\mathbf{A}$ is an $\mathbb{F}$-Koszul algebra of finite cohomological dimension $d = \deg(\mathbf{A})$, then

$$\chi_{\mathbf{A}}(y) = p_{\mathbf{A}}(y) = 1 - b_1 \cdot y + b_2 \cdot y^2 + \cdots (-1)^d b_d \cdot y^d \tag{2.22}$$

for positive integers $b_j \geq 1$. Hence by R. Descartes' rule of signs one concludes the following:

Fact 2.11. *Let* $\mathbf{A}$ *be an Koszul algebra of finite cohomological dimension. Then every real eigenvalue of* $\mathbf{A}$ *must be positive. In particular,* -1 *is not an eigenvalue of* $\mathbf{A}$.

There exist Koszul algebras of finite cohomological dimension with non-real eigenvalues (cf. Ex. 4.1(d)). Nevertheless, the author could not find any example settling the following question.

Question 1. *Does there exist a Koszul algebra of finite cohomological dimension with an eigenvalue* λ *satisfying* $\mathrm{Re}(\lambda) < 0$*?*

By definition, every quadratic $\mathbb{F}$-algebra $\mathbf{A}$ of finite type satisfying $\mathrm{cd}(\mathbf{A}) \leq 2$ is Koszul. Such an algebra has two eigenvalues. By Proposition 2.6, one of it is a positive real number. Hence the other is real as well, and, by Fact 2.11, it is also positive. From this fact one concludes the following.

Corollary 2.12. *Let* $\mathbf{A}$ *be a Koszul algebra of finite cohomological dimension less or equal to* 2. *Then the eigenvalues are positive real numbers, and*

$$\dim(\mathrm{Ext}^{2,2}_{\mathbf{A}}(\mathbb{F},\mathbb{F})) \leq \frac{\dim(\mathrm{Ext}^{1,1}_{\mathbf{A}}(\mathbb{F},\mathbb{F}))^2}{4}. \tag{2.23}$$

Hence Koszul algebra of cohomological dimension less or equal to 2 satisfies the Golod-Shafarevich inequality (cf. [26, §I, App. 2]) in the opposite direction. Corollary 2.12 applied to right-angled Artin algebras (cf. §4.2.2) yields an alternative proof of W. Mantel's theorem on the number of edges in a triangle free graph (cf. [16]). This result is a special case of a more general result due to P. Turán (cf. [28]). The natural question is whether there exists an analogue of Turán's theorem also in the context of Koszul algebras.

Question 2. *Let* $\mathbf{A}$ *be a Koszul algebra of cohomological dimension* d. *Is it true that*

$$\dim(\mathrm{Ext}^{2,2}_{\mathbf{A}}(\mathbb{F},\mathbb{F})) \leq \frac{d-1}{2d} \cdot \dim(\mathrm{Ext}^{1,1}_{\mathbf{A}}(\mathbb{F},\mathbb{F}))^2? \tag{2.24}$$

3. Graded Lie algebras of finite type

Let $\mathbf{L} = \coprod_{k\geq 1} \mathbf{L}_k$ be an $\mathbb{N}$-graded Lie algebra, i.e., $[\mathbf{L}_n, \mathbf{L}_m] \subseteq \mathbf{L}_{n+m}$. Then its universal eneveloping algebra $\mathcal{U}(\mathbf{L})$ is an $\mathbb{N}_0$-graded, connected algebra. Moreover, $\mathbf{L}$ is of finite type if, and only if, $\mathcal{U}(\mathbf{L})$ is of finite type. We will say that $\mathbf{L}$ has one of the properties $\mathfrak{X}$ discussed in previous section, if $\mathcal{U}(\mathbf{L})$ has the property $\mathfrak{X}$, e.g., $\mathbf{L}$ is said to be *generated in degree* 1, if $[\mathbf{L},\mathbf{L}] = \coprod_{k\geq 2} \mathbf{L}_k$, and $\mathbf{L}$ is said to be *quadratic* (resp. *Koszul*), if $\mathcal{U}(\mathbf{L})$ is quadratic (resp. Koszul). Hence, if $\mathbf{L}$ is generated in degree 1, one has

$$\mathbf{L}_k = \overbrace{[\mathbf{L}_1, [\mathbf{L}_1, \ldots, [\mathbf{L}_1, \mathbf{L}_1]]]}^{k\text{-times}} \tag{3.1}$$

In particular, if $\mathbf{L}$ is of finite type, generated in degree 1 and $\mathbf{L}_k = 0$ for some $k > 1$, then $\mathbf{L}$ must be finite-dimensional. If $\mathcal{U}(\mathbf{L})$ is χ-finite (cf. §2.3), we will call $\mathbf{L}$ to be χ-finite, and will say that the eigenvalues $\lambda_1, \ldots, \lambda_n$, $n = \deg(\mathcal{U}(\mathbf{L}))$, of $\mathcal{U}(\mathbf{L})$ are also the eigenvalues of $\mathbf{L}$. In this case we also put $\deg(\mathbf{L}) = \deg(\mathcal{U}(\mathbf{L}))$. By Fact 2.3(c), if $\mathbf{L}$ is of type FP, then it is also χ-finite. For an $\mathbb{N}$-graded Lie algebra of finite type, we put $\chi_{\mathbf{L}}(y) = \chi_{\mathcal{U}(\mathbf{L})}(y) \in \mathbb{Z}[\![y]\!]$ (cf. (2.3)).

3.1. **The entropy of a graded Lie algebra.** Let $\mathbf{L}$ be an $\mathbb{N}$-graded Lie algebra of finite type. The *entropy*[1] $\mathbf{h}(\mathbf{L})$ of $\mathbf{L}$ is defined by

$$\mathbf{h}(\mathbf{L}) = \limsup_{k\to\infty} \sqrt[k]{\dim(\mathbf{L}_k)}. \tag{3.2}$$

In [1, Lemma 1], A.E. Bereznyĭ stated the following lemma.

Lemma 3.1 (A.E. Bereznyĭ)**.** *Let $\mathbf{L}$ be an $\mathbb{N}$-graded Lie algebra of finite type such that $\ell_k = \dim(\mathbf{L}_k) \geq 1$. Then $\mathbf{h}(\mathbf{L}) = \mathbf{h}(\mathcal{U}(\mathbf{L}))$.*

As A.E. Bereznyĭ gives a hint, but no complete proof of the lemma stated above, for the convenience of the reader we provide a short proof based on the argument given in [9, Proof of Thm. 1]. The proof will make use of the following property.

Proposition 3.2. *Let $(a_j)_{j\geq 1}$ be a sequence of positive integers, and let $s_k = \sum_{1\leq j\leq k} a_j$. If $\limsup_{k\to\infty} \sqrt[k]{a_k} = \lambda < \infty$, then $\limsup_{k\to\infty} \sqrt[k]{s_k} = \lambda$.*

Proof. It suffices to show that $\limsup_{k\to\infty} \sqrt[k]{s_k} \leq \lambda$. By hypothesis, $\lambda \geq 1$. For a given $\varepsilon > 0$ one has $a_k \leq (\lambda+\varepsilon)^k$ for all $k \geq N(\varepsilon)$. Hence there exists $c(\varepsilon) \in \mathbb{R}_{>0}$ such that

$$\begin{aligned} s_k &\leq c(\varepsilon) + \sum_{1\leq j\leq k} (\lambda+\varepsilon)^j = c(\varepsilon) + \frac{(\lambda+\varepsilon)^{k+1} - (\lambda+\varepsilon)}{\lambda - 1 + \varepsilon} \\ &\leq c(\varepsilon) + \frac{\lambda+\varepsilon}{\lambda-1+\varepsilon} \cdot (\lambda+\varepsilon)^k. \end{aligned} \tag{3.3}$$

Hence $\limsup_{k\to\infty} \sqrt[k]{s_k} \leq \lambda+\varepsilon$, and this yields the claim. □

Proof of Lemma 3.1. Let $h_{\mathbf{L}}(y) = \sum_{k\geq 1} \ell_k \cdot y^k$ denote the Hilbert series of $\mathbf{L}$, and let $h_{\mathbf{A}}(y)$ denote the Hilbert series of $\mathbf{A} = \mathcal{U}(\mathbf{L})$. Let $\sum_{k\geq 1} b_k \cdot y^k = \log(h_{\mathbf{A}}(y))$ be the formal power series obtained by substituting $u = 1 - h_{\mathbf{A}}(y)$ in the formal power series $\log(1-u) = -\sum_{j\geq 1} \frac{u^j}{j}$. Then, as $\exp(y) = \sum_{j\geq 0} y^j/j!$ has convergence radius equal to ∞, the formal power series $\log(h_{\mathbf{A}}(y))$ and $h_{\mathbf{A}}(y)$ have the same convergence radius, i.e., one has $\mathbf{h}(\mathbf{A}) = \limsup_{k\to\infty} \sqrt[k]{b_k}$. From the Poincaré-Birkhoff-Witt theorem (cf. [9, §1.2]), one concludes easily that $b_k = \sum_{d|k} \frac{\ell_{k/d}}{d}$. In particular, $\ell_k \leq b_k \leq \left(\sum_{d|k} \frac{1}{d}\right) \cdot s_k$, where $s_k = \sum_{1\leq j\leq k} \ell_j$. Moreover, by the unconditional Robin inequality (cf. [24]), one has $\limsup_{k\to\infty} \sqrt[k]{\sum_{d|k} \frac{1}{d}} = 1$. Hence Proposition 3.2 yields the claim. □

Applying Lemma 3.1 to $\mathbb{N}$-graded Lie algebras of finite type which are generated in degree 1 one obtains the following.

Corollary 3.3. *Let $\mathbf{L} \neq 0$ be a finitely generated Lie algebra which is generated in degree 1.*

(a) *If $\dim(\mathbf{L}) < \infty$, then $\mathbf{h}(\mathbf{L}) = 0$ and $\mathbf{h}(\mathcal{U}(\mathbf{L})) = 1$.*
(b) *If $\dim(\mathbf{L}) = \infty$, then $\mathbf{h}(\mathbf{L}) = \mathbf{h}(\mathcal{U}(\mathbf{L})) \geq 1$.*

In [20], the authors implicitly assume that every graded Lie algebra $\mathbf{L}$ under consideration is infinite dimensional. This is the reason why case (a) of Corollary 3.3 does never occur in their paper.

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[1]Following [6] one may consider $\mathbf{h}(\mathbf{L})$ also as the *exponential growth rate* of $\mathbf{L}$.

3.2. **A generalized Witt formula.** The *necklace polynomial*[2] of degree k, $k \geq 1$, is the polynomial given by

$$M_k(y) = \frac{1}{k} \sum_{j|k} \mu(k/j) \cdot y^j \in \mathbb{Q}[y], \tag{3.4}$$

where $\mu\colon \mathbb{N} \to \mathbb{Z}$ denotes the *Möbius function.* E.g., one has

$$\begin{aligned} M_1(y) &= y, \\ M_2(y) &= \frac{1}{2}(y^2 - y), \\ M_3(y) &= \frac{1}{3}(y^3 - y), \\ M_4(y) &= \frac{1}{4}(y^4 - y^2), \text{ etc.} \end{aligned} \tag{3.5}$$

In [30], E. Witt showed that the dimension of the k^{th}-homogeneous component of a Lie algebra $\mathbf{L}$ generated freely by r-elements is given by $M_k(r)$. This result can be generalized in the following way.

Theorem 3.4. *Let* $\mathbf{L}$ *be a* χ*-finite* $\mathbb{N}$*-graded Lie algebra, let* $n = \deg(\mathbf{L})$*, and let* $\lambda_1, \ldots, \lambda_n$ *denote the eigenvalues of* $\mathbf{L}$*. Then*

$$\dim(\mathbf{L}_k) = \textstyle\sum_{1 \leq i \leq n} M_k(\lambda_i). \tag{3.6}$$

Proof. Let $\ell_k = \dim(\mathbf{L}_k)$. By the Poincaré-Birkhoff-Witt theorem, (2.4) and the definition of the eigenvalues of $\mathbf{L}$, one has

$$h_{\mathcal{U}(\mathbf{L}_\bullet)}(y) = \textstyle\prod_{k \geq 1} (1 - y^k)^{-\ell_k} = \prod_{1 \leq i \leq n} (1 - \lambda_i y)^{-1}. \tag{3.7}$$

Applying $-\log(\_)$ on both sides and using the identity $-\log(1-u) = \sum_{j \geq 1} \frac{u^j}{j}$ for $u \in y\mathbb{C}[\![y]\!]$ one obtains

$$\sum_{k \geq 1} \ell_k \sum_{j \geq 1} \frac{y^{kj}}{j} = \sum_{m \geq 1} \frac{y^m}{m} \sum_{1 \leq i \leq n} \lambda_i^m = \sum_{m \geq 1} \frac{\mathbf{p}_m(\lambda_1, \ldots, \lambda_n)}{m} \cdot y^m, \tag{3.8}$$

where $\mathbf{p}_m(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leq i \leq n} \lambda_i^m$. Comparing coefficients of y^m on both sides yields

$$\sum_{k|m} k \cdot \ell_k = \mathbf{p}_m(\lambda_1, \ldots, \lambda_n). \tag{3.9}$$

Hence, by the Möbius inversion formula, one obtains

$$\ell_k = \frac{1}{k} \sum_{j|k} \mu(k/j) \cdot \mathbf{p}_j(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leq i \leq n} M_k(\lambda_i). \tag{3.10}$$

This yields the claim. □

Remark 3.5. (a) For the Lie algebra $\mathbf{L} = L\langle\mathfrak{X}\rangle$, $|\mathfrak{X}| = r$, generated freely by r elements, one has $\chi_{\mathbf{L}}(y) = 1 - r \cdot y$, i.e., $\deg(\mathbf{L}) = 1$ and r is the eigenvalue. Hence (3.10) coincides with Witt's formula in this case. There exists a generalization of Witt's formula in the case when $\mathfrak{X}$ is a graded set (cf. [8]).

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[2]The number of necklaces of length k made from r-colured beads was first studied by Col. C.P.N. Moreau in 1872 (cf. [18]). The integer $M_k(r)$ equals the number of aperiodic necklaces of length k made from r-coloured beads.

(b) Let $\mathbf{L} = L\langle\mathfrak{X}\rangle \oplus L\langle\mathfrak{Y}\rangle$, $|\mathfrak{X}| = r$, $|\mathfrak{Y}| = s$. Then $\chi_{\mathbf{L}}(y) = (1 - r \cdot y)(1 - s \cdot y)$, i.e., $\deg(\mathbf{L}) = 2$ and r and s are the eigenvalues. For this Lie algebra there exists also an explicit formula for the multi-graded homogeneous components (cf. [4]).
(c) Let $\mathbf{L}$ be a finite-dimensional $\mathbb{N}$-graded Lie algebra. Put $m_k = \sum_{j\in\mathbb{N}} \dim(\mathbf{L}_{kj})$ for $k \geq 1$. By Proposition 2.2 and (3.7), one has

$$\chi_{\mathbf{L}}(y) = \prod_{k\geq 1} \Phi_k(y)^{m_k}, \tag{3.11}$$

where $\Phi_k(y)$ denotes the k^{th}-cyclotomic polynomial of degree $\varphi(k)$, and $\varphi\colon \mathbb{N} \to \mathbb{Z}$ denotes Euler's φ-function, i.e., all eigenvalues of $\mathbf{L}$ are roots of unity. From (3.7) one concludes that

$$\deg(\mathbf{L}) = \sum_{k\geq 1} k \cdot \dim(\mathbf{L}_k). \tag{3.12}$$

(d) Let $\mathbf{L}$ be a filiform Lie algebra. Then

$$\chi_{\mathbf{L}}(y) = h_{\mathcal{U}(\mathbf{L})}(y)^{-1} = (1 - y) \cdot \phi(y), \tag{3.13}$$

where $\phi(y)$ is Euler's function. This Lie algebra is an example where $\mathbf{h}(\mathbf{L}) = 1$ holds, but the formal power series $\chi_{\mathbf{L}}(y)$ cannot be continued meromorphically to the whole complex plane.

3.3. **Necklace polynomials at roots of unity.** For a positive integer m let $\Xi_m \subseteq \mathbb{C}^*$ denote the set of primitive m^{th}-roots of unity in the field of complex numbers. The aim of this subsection is to compute the values of the functions $P_k\colon \mathbb{N} \to \mathbb{C}$, $C_k\colon \mathbb{N} \to \mathbb{C}$, $k \geq 1$, where

$$\begin{aligned} P_k(m) &= \textstyle\sum_{\xi\in\Xi_m} \xi^k, \\ C_k(m) &= \textstyle\sum_{\xi\in\Xi_m} M_k(\xi), \end{aligned} \tag{3.14}$$

for $m \in \mathbb{N}$. For a positive integer $k \in \mathbb{N}$ we define a function $\delta_k\colon \mathbb{N} \to \mathbb{C}$ by

$$\delta_k(m) = \begin{cases} m & \text{if } m|k, \\ 0 & \text{if } m\nmid k. \end{cases} \tag{3.15}$$

Obviously, $\delta_k(1) = 1$ for all $k \geq 1$. If m_1 and m_2 are positive coprime integers, then $m_1 m_2$ divides k if, and only if, m_1 divides k and m_2 divides k. Hence δ_k is a multiplicative arithmetic function. Moreover, δ_1 coincides with the unit under convolution "$*$". The following proposition gives a complete description of the function P_k, $k \geq 1$.

Proposition 3.6. *Let k be a positive integer.*

(a) *P_k is a multiplicative arithmetic function.*
(b) *$P_1 = C_1 = \mu$, where $\mu\colon \mathbb{N} \to \mathbb{Z}$ is the Möbius function.*
(c) *For $k \geq 1$ one has $P_k = \delta_k * \mu$.*
(d) *Let $m \geq 1$ with* $\gcd(m, k) = 1$. *Then $P_k(m) = \mu(m)$.*
(e) *Let $m \geq 1$ with* $\gcd(m, k) = 1$ *and assume that $k > 1$. Then $C_k(m) = 0$.*

Proof. (a) By definition, $P_k(1) = 1$ for all $k \geq 1$. If m_1 and m_2 are positive coprime integers, one has $\Xi_{m_1 m_2} = \Xi_{m_1}\Xi_{m_2}$. Thus $P_k(m_1 m_2) = P_k(m_1) \cdot P_k(m_2)$ showing that P_k is multiplicative.
(b) For every prime number p one has $\Phi_p(y) = \sum_{0\leq j\leq p-1} y^j$, where $\Phi_p(y) \in \mathbb{Z}[y]$ denotes the p^{th}-cyclotomic polynomial. Hence $P_1(p) = -1$. Moreover, for $\alpha \geq 2$

and p prime, the $(p^\alpha)^{th}$-cyclotomic polynomial $\Phi_{p^\alpha}(y) \in \mathbb{Z}[y]$ is given by

$$\Phi_{p^\alpha}(y) = \frac{y^{p^\alpha}-1}{y^{p^{\alpha-1}}-1} = y^{(p-1)p^{\alpha-1}} + y^{(p-2)p^{\alpha-1}} + \cdots + y^{p^{\alpha-1}} + 1. \tag{3.16}$$

Hence $P_1(p^\alpha) = \sum_{\xi \in \Xi_{p^\alpha}} \xi = 0$, and P_1 coincides with μ on all prime powers.
(c) Let $k \geq 2$, and let p^α be a prime power. We proceed by a case-by-case analysis.
Case 1: $\gcd(k, p^\alpha) = 1$. In this case $\_^k \colon \Xi_{p^\alpha} \to \Xi_{p^\alpha}$ is a bijection. Hence $P_k(p^\alpha) = P_1(p^\alpha) = \mu(p^\alpha)$. On the other hand, for $\gcd(k, p^\alpha) = 1$ one has

$$(\delta_k * \mu)(p^\alpha) = \sum_{0 \leq j \leq \alpha} \delta_k(p^j)\mu(p^{\alpha-j}) = \mu(p^\alpha). \tag{3.17}$$

Case 2: $k = p^\gamma \cdot \beta$, $\gcd(\beta, p) = 1$, $\alpha \leq \gamma$. In this case one has $\xi^k = 1$ for all $\xi \in \Xi_{p^\alpha}$, and thus $P_k(p^\alpha) = \varphi(p^\alpha)$. On the other hand,

$$(\delta_k * \mu)(p^\alpha) = \sum_{0 \leq j \leq \alpha} \delta_k(p^j)\mu(p^{\alpha-j}) = \sum_{0 \leq j \leq \alpha} p^j \mu(p^{\alpha-j}) = \varphi(p^\alpha). \tag{3.18}$$

Case 3: $k = p^\gamma \cdot \beta$, $\gcd(\beta, p) = 1$, $\gamma < \alpha$. As in Case 1, $\_^\beta \colon \Xi_{p^\alpha} \to \Xi_{p^\alpha}$ is a bijection, and $\_^{p^\gamma} \colon \Xi_{p^\alpha} \to \Xi_{p^{\alpha-\gamma}}$ is surjective with all fibers of cardinality p^γ. Hence $P_k(p^\alpha) = p^\gamma \mu(p^{\alpha-\gamma})$, i.e., $P_k(p^{\gamma+1}) = -p^\gamma$, and $P_k(p^\alpha) = 0$ for $\alpha > \gamma + 1$. On the other hand, for $0 \leq j \leq \alpha$, one has $\mu(p^{\alpha-j}) = 0$ unless $j = \alpha$ or $j = \alpha - 1$. Hence

$$(\delta_k * \mu)(p^\alpha) = \sum_{0 \leq j \leq \alpha} \delta_k(p^j)\mu(p^{\alpha-j}) = -\delta_k(p^{\alpha-1}) + \delta_k(p^\alpha). \tag{3.19}$$

By hypothesis, $\delta_k(p^\alpha) = 0$. Moreover, $\delta_k(p^{\alpha-1}) \neq 0$ if, and only if, $\gamma = \alpha - 1$. This yields the claim.
(d) By hypothesis, $P_k(m) = \sum_{d|m} \delta_k(d)\mu(m/d) = \mu(m)$.
(e) By hypothesis and (d),

$$C_k(m) = \frac{1}{k}\sum_{j|k} \mu(j) \cdot P_{k/j}(m) = \frac{1}{k}\mu(m)\sum_{j|k}\mu(j) = \frac{1}{k}\mu(m) \cdot \delta_1(k) = 0, \tag{3.20}$$

and hence the claim. □

3.4. **Graded Lie algebras with $\mathbf{h}(\mathcal{U}(\mathbf{L})) = 1$.** The main purpose of this subsection is to prove the following theorem and to discuss its consequences.

Theorem 3.7. *Let $\mathbf{L}$ be an $\mathbb{N}$-graded Lie algebra of finite type such that*
 (i) *$\mathbf{L}$ is generated in degree 1;*
 (ii) *$\mathbf{L}$ is χ-finite;*
 (iii) *$\mathbf{h}(\mathcal{U}(\mathbf{L})) = 1$.*

Then $\mathbf{L}$ is finite-dimensional and $\mathbf{h}(\mathbf{L}) = 0$.

Proof. By hypothesis (ii), $\mathcal{U}(\mathbf{L})$ is χ-finite. Let $n = \deg(\mathbf{L})$, and let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $\mathbf{L}$. Then, by hypothesis (iii) and Proposition 2.7, λ_i is a root of unity. Let $m_i = \mathrm{ord}(\lambda_i)$ denote its order in the multiplicative group $\mathbb{C}^\times$, let $\mathcal{M}(\mathbf{L}) = \{ m_i \mid 1 \leq i \leq n \}$, and let $n_i = |\{ 1 \leq j \leq n \mid \lambda_j = \lambda_i \}|$ denote their multiplicities. Let $\Xi_m \subseteq \mathbb{C}^\times$ denote the set of primitive m^{th}-roots of unity of $\mathbb{C}$. Since the absolute Galois group $G_\mathbb{Q}$ of $\mathbb{Q}$ acts on the set $\Lambda = \{ \lambda_i \mid 1 \leq i \leq n \}$, one has $\Xi_{m_i} \subseteq \Lambda$ for all $i \in \{1, \dots, n\}$, and $n_i = n_j$ if $m_i = m_j$. For $m = m_i \in \mathcal{M}(\mathbf{L})$ let $n(m) = n_i$. By Theorem 3.4, one has for all $k \geq 1$ that

$$\dim(\mathbf{L}_k) = \sum_{1\leq i\leq n} M_k(\lambda_i) = \sum_{m\in\mathcal{M}(\mathbf{L})} n(m) \cdot \sum_{\xi\in\Xi_m} M_k(\xi) = \sum_{m\in\mathcal{M}(\mathbf{L})} n(m) \cdot C_k(m) \tag{3.21}$$

(cf. (3.14)). For $k = 1+\prod_{m\in\mathcal{M}(\mathbf{L})} m$, one has that $\gcd(k, m) = 1$ for all $m \in \mathcal{M}(\mathbf{L})$. By Proposition 3.6(e), $C_k(m) = 0$ for all $m \in \mathcal{M}(\mathbf{L})$, and thus $\dim(\mathbf{L}_k) = 0$. As $\mathbf{L}$ is 1-generated, this shows that $\mathbf{L}$ is finite-dimensional (cf. (3.1)). □

From A.E. Bereznyĭ's lemma (cf. Lemma 3.1) one concludes the following:

Corollary 3.8. *Let* $\mathbf{L}$ *be a finitely generated* $\mathbb{N}$*-graded Lie algebra generated in degree* 1 *satisfying* $\mathbf{h}(\mathbf{L}) = 1$*. Then* $\mathbf{L}$ *is not of type FP.*

An alternative reformulation of Corollary 3.8 is the following.

Corollary 3.9. *Let* $\mathbf{L}$ *be a finitely generated* $\mathbb{N}$*-graded Lie algebra generated in degree* 1 *of type FP. Then either* $\mathbf{L}$ *is finite-dimensional and* $\mathbf{h}(\mathbf{L}) = 0$*, or* $\mathbf{h}(\mathbf{L}) > 1$.

4. Koszul Lie algebras

Let $\mathbf{L}$ be a Koszul Lie algebra. Then, by definition, $\mathcal{U}(\mathbf{L})$ is $\mathbb{N}_0$-graded, quadratic and of finite type. Moreover, for the cohomology algebra one has an isomorphism

$$H^\bullet(\mathbf{L}, \mathbb{F}) = \mathrm{diag}(\mathrm{Ext}^{\bullet,\bullet}_{\mathcal{U}(\mathbf{L})}(\mathbb{F}, \mathbb{F})) \simeq \mathcal{U}(\mathbf{L})^!. \tag{4.1}$$

Let $\mathbf{L}^{\mathrm{ab}} = \mathbf{L}/\coprod_{k\geq 2} \mathbf{L}_k$ denote the maximal abelian quotient of $\mathbf{L}$. Since $H^\bullet(\mathbf{L}, \mathbb{F})$ is a quadratic algebra, inflation $\iota^\bullet \colon H^\bullet(\mathbf{L}^{\mathrm{ab}}, \mathbb{F}) \to H^\bullet(\mathbf{L}, \mathbb{F})$ is a surjective homomorphism of algebras. As $H^\bullet(\mathbf{L}^{\mathrm{ab}}, \mathbb{F})$ is isomorphic to the exterior algebra $\Lambda(\mathbf{L}_1^*)$, one concludes the following fact (cf. [21, §7.1, Conj. 2]).

Fact 4.1. *Let* $\mathbf{L}$ *be a Koszul Lie algebra. Then* $\mathrm{cd}(\mathbf{L}) \leq \dim(\mathbf{L}_1)$ *and equality holds if, and only if,* $\mathbf{L}$ *is abelian.*

Proof. If $\ell = \dim(\mathbf{L}_1)$, then $H^{\ell+1}(\mathbf{L}^{\mathrm{ab}}, \mathbb{F}) = 0$. As $\iota^\bullet$ is surjective, this implies $H^{\ell+1}(\mathbf{L}, \mathbb{F}) = 0$. Hence $\mathrm{cd}(\mathbf{L}) \leq \ell$ (cf. Fact 2.3(a)). Assume that $\mathrm{cd}(\mathbf{L}) = \ell$. Then $\ker(\iota^\ell) = 0$. For any non-trivial element $x \in \Lambda_k(\mathbf{L}_1^*)$, there exists $y \in \Lambda_{\ell-k}(\mathbf{L}_1^*)$ such that $x \wedge y \neq 0$. Thus $\ker(\iota^\ell) = 0$ implies that $\iota^\bullet$ is injective, and hence an isomorphism. Therefore, $(\iota^\bullet)^! \colon \mathcal{U}(\mathbf{L}) \to \mathcal{U}(\mathbf{L}^{\mathrm{ab}})$ is an isomorphism, and this yields the claim. □

4.1. Koszul Lie algebras with entropy equal to 1.

From Theorem 3.7 one concludes the following result which is again an open problem for Koszul algebras in general (cf. [21, §7.1, Conj. 3]).

Proposition 4.2. *Let* $\mathbf{L}$ *be a Koszul Lie algebra satisfying* $\mathbf{h}(\mathcal{U}(\mathbf{L})) = 1$*. Then* $\mathbf{L}$ *is abelian.*

Proof. As $\mathcal{U}(\mathbf{L})$ is Koszul, $\mathcal{U}(\mathbf{L})$ is of type FP_∞ (cf. Fact 2.3(b) and (2.21)). Thus by Fact 4.1, $\mathbf{L}$ is of type FP and, therefore, χ-finite (cf. Fact 2.3(c)). Hence, by Theorem 3.7, $\mathbf{L}$ is finite-dimensional. Then

$$\chi_{\mathbf{L}}(y) = \prod_{k\geq 1} \Phi_k(y)^{m_k} \tag{4.2}$$

where $\Phi_k(y)$ is the k^{th}-cyclotomic polynomial, and $m_k = \sum_{j\geq 1}\dim(\mathbf{L}_{jk})$ (cf. Remark 3.5(c)). By Fact 2.11, $m_2 = 0$. Hence $\mathbf{L}_2 = 0$, and $\mathbf{L}$ is abelian. $\square$

4.2. Examples of Koszul Lie algebras.

4.2.1. *Quadratic* 1*-relator Lie algebras.* Let $\mathfrak{X}$ be a finite set of cardinality $m \geq 2$, and let $L\langle\mathfrak{X}\rangle$ denote the free $\mathbb{F}$-Lie algebra over the set $\mathfrak{X}$. Then $L\langle\mathfrak{X}\rangle$ is $\mathbb{N}$-graded, of finite type and generated in degree 1. Let $\mathbf{r} \in L_2\langle\mathfrak{X}\rangle \setminus \{0\}$ be a non-trivial homogeneous element of degree 2. Then putting

$$L\langle\,\mathfrak{X} \mid \mathbf{r}\,\rangle = L\langle\mathfrak{X}\rangle/\langle\mathbf{r}\rangle_{\mathrm{Lie}}, \tag{4.3}$$

where $\langle\mathbf{r}\rangle_{\mathrm{Lie}}$ denotes the Lie ideal generated by $\mathbf{r}$, one has an isomorphism

$$\mathcal{U}(L\langle\,\mathfrak{X} \mid \mathbf{r}\,\rangle) \simeq \mathcal{U}(L\langle\mathfrak{X}\rangle)/\langle\mathbf{r}\rangle_{\mathrm{alg}}, \tag{4.4}$$

where $\langle\mathbf{r}\rangle_{\mathrm{alg}}$ denotes the ideal in the associative algebra $\mathcal{U}(L\langle\mathfrak{X}\rangle)$ generated by $\mathbf{r}$. In particular, $L\langle\,\mathfrak{X} \mid \mathbf{r}\,\rangle$ is quadratic and of finite type. By J. Labute's theorem (cf. [12, Thm. 1]), $\mathrm{cd}(L\langle\,\mathfrak{X} \mid \mathbf{r}\,\rangle) = 2$. Hence $L\langle\,\mathfrak{X} \mid \mathbf{r}\,\rangle$ is a Koszul Lie algebra,

$$\chi_{\mathbf{L}}(y) = 1 - m\cdot y + y^2, \tag{4.5}$$

and

$$\mathbf{h}(L\langle\,\mathfrak{X} \mid \mathbf{r}\,\rangle) = \lambda_1 = \frac{1}{2}(m + \sqrt{m^2-4}), \qquad \lambda_2 = \frac{1}{2}(m - \sqrt{m^2-4}); \tag{4.6}$$

e.g., for $m = 2$, $L\langle\,\mathfrak{X} \mid \mathbf{r}\,\rangle$ is abelian. Moreover, for $\ell_k = \dim(L_k\langle\,\mathfrak{X} \mid \mathbf{r}\,\rangle)$ one has

$$\ell_k = \frac{1}{k}\sum_{j|k}\mu(k/j)\sum_{0\leq i\leq j/2}(-1)^i\frac{j}{j-i}\binom{j-i}{i}\cdot m^{j-2i} \tag{4.7}$$

(cf. [13, Eq. (1)]). One can use (4.7) to express $\mathbf{p}_k(\lambda_1,\lambda_2) = \lambda_1^k + \lambda_2^k$ as

$$\mathbf{p}_k(\lambda_1,\lambda_2) = \sum_{0\leq i\leq k/2}(-1)^i\frac{k}{k-i}\binom{k-i}{i}\cdot m^{k-2i}. \tag{4.8}$$

4.2.2. *Right-angled Artin Lie algebras.* Let $\Gamma = (\mathfrak{X}, \mathcal{E})$ be a finite loop-free graph with unoriented edges, i.e., $|\mathfrak{X}| < \infty$ and $\mathcal{E} \subseteq \mathcal{P}_2(\mathfrak{X})$, where $\mathcal{P}_2(\mathfrak{X})$ is the set of subsets of $\mathfrak{X}$ of cardinality 2. Then

$$L\langle\Gamma\rangle = L\langle\,\mathfrak{X} \mid xy - yx,\ \{x,y\}\in\mathcal{E}\,\rangle \tag{4.9}$$

will be called the *right-angled Artin Lie algebra* associated with Γ. Moreover,

$$\mathcal{U}(L\langle\Gamma\rangle) \simeq A\langle\Gamma\rangle = \mathbb{F}\langle\mathfrak{X}\rangle/\langle\,xy - yx,\ \{x,y\}\in\mathcal{E}\,\rangle_{\mathrm{alg}}, \tag{4.10}$$

where $\mathbb{F}\langle\mathfrak{X}\rangle$ denotes the free associative algebra over the set $\mathfrak{X}$. Thus $L\langle\Gamma\rangle$ is quadratic and of finite type, and, by R. Fröberg's theorem (cf. [5]), $L\langle\Gamma\rangle$ is Koszul. For a graph $\Gamma_\circ = (\mathfrak{X}_\circ, \mathcal{E}_\circ)$ let the *exterior algebra associated with* $\Gamma_\circ$ be given by

$$\Lambda\langle\Gamma_\circ\rangle = \Lambda\langle\mathfrak{X}_\circ\rangle/\langle\,x\wedge y \mid (x,y)\in\mathcal{E}_\circ\,\rangle_{\mathrm{alg}}, \tag{4.11}$$

where $\Lambda\langle\mathfrak{X}_\circ\rangle$ denotes the free exterior algebra over the set $\mathfrak{X}_\circ$. Then, by (2.21),

$$H^\bullet(L\langle\Gamma\rangle, \mathbb{F}) \simeq \Lambda\langle\Gamma^{\mathrm{op}}\rangle, \tag{4.12}$$

where $\Gamma^{\mathrm{op}} = (\mathfrak{X}, \mathcal{P}_2(\mathfrak{X})\setminus\mathcal{E})$ for $\Gamma = (\mathfrak{X}, \mathcal{E})$. In particular,

$$\chi_{L\langle\Gamma\rangle}(y) = \mathrm{Cl}_\Gamma(y) = 1 + \sum_{1\leq k\leq n}(-1)^k\cdot c_k(\Gamma)\cdot y^k, \tag{4.13}$$

where $c_k(\Gamma)$ denotes the number of *k-cliques* (= complete subgraphs with k vertices) in the graph Γ, and $n = \mathrm{cd}(L\langle\Gamma\rangle)$ coincides with the *clique number* of Γ, i.e., $\chi_{L\langle\Gamma\rangle}(-y)$ coincides with the *clique polynomial* of Γ which is equal to the *independence polynomial* of Γ^{op}. Therefore, $\mathrm{Cl}_\Gamma(y)$ will be called the *alternating clique polynomial* of Γ. Thus applying Corollary 2.12 for a triangle-free graph Γ, one obtains Mantel's theorem. Moreover, P. Turán's theorem (cf. [28]) shows that Question 2 has an affirmative answer for *right-angled Artin algebras* $A\langle\Gamma\rangle$ (cf. (4.10)).

Example 4.1. (a) If $\Gamma = (\mathfrak{X}, \emptyset)$ has no edges, $L\langle\Gamma\rangle$ is the free Lie algebra over the set $\mathfrak{X}$, and $\chi_{L\langle\Gamma\rangle}(y) = 1 - v \cdot y$, where $v = |\mathfrak{X}|$.
(b) If $\Gamma = (\mathfrak{X}, \mathcal{P}_2(\mathfrak{X}))$ is the complete graph on $v = |\mathfrak{X}|$ vertices, $L\langle\Gamma\rangle$ is abelian of dimension v, and $\chi_{L\langle\Gamma\rangle}(y) = (1-y)^v$.
(c) If $\Gamma = (\mathfrak{X}, \mathcal{E})$ is a finite tree, $e = |\mathcal{E}|$, one has $\chi_{L\langle\Gamma\rangle}(y) = (1 - e\cdot y)(1-y)$ (cf. [25, §I.2, Prop. 12]). In particular, $L\langle\Gamma\rangle$ has Euler-Poincaré characteristic equal to 0 (cf. Rem. 2.8).
(d) The following example was communicated to the author by P. Spiga. Let $\Gamma = \Gamma(C, S)$ denote the Cayley graph for the finite cyclic group $C = \mathbb{Z}/11\mathbb{Z}$, and let S be the symmetric generating system $S = \{\pm 2, \pm 3, \pm 5\}$. Then Γ has clique

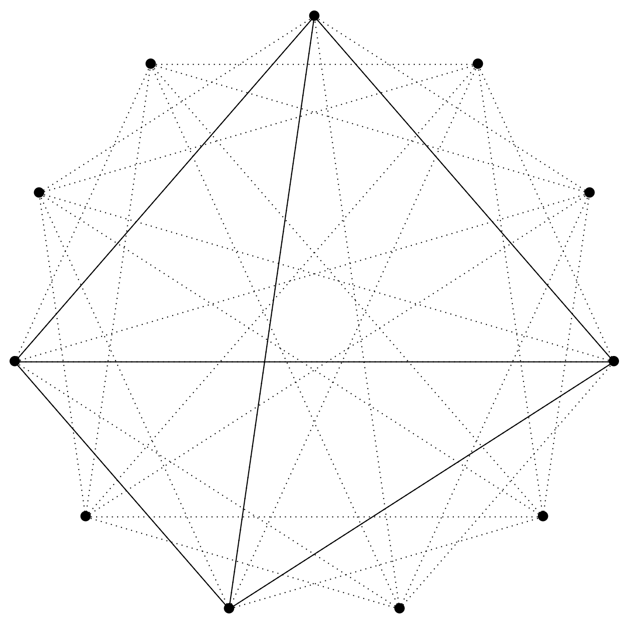

FIGURE 1. Spiga graph

number 4 and $\chi_{L\langle\Gamma\rangle}(y) = 1 - 11y + 33y^2 - 33y^3 + 11y^4$. Moreover, a numerical computation of the eigenvalues shows that

$$\begin{aligned} \mathbf{h}(L\langle\Gamma\rangle) = \lambda_1 &= 6.85317, & \lambda_3 &= 0.751697 + 0.205541\,i, \\ \lambda_2 &= 2.64361, & \lambda_4 &= 0.751697 - 0.205541\,i. \end{aligned} \tag{4.14}$$

In particular, not all eigenvalues of $L\langle\Gamma\rangle$ are real.

Remark 4.3. It was shown in [3] that if Γ^{op} is *claw-free*, then all eigenvalues of $\mathrm{Cl}_\Gamma(y)$ are real (and positive). Nevertheless, characterizing the finite graphs Γ for which all eigenvalues of $\mathrm{Cl}_\Gamma(y)$ are real seem to be an extremely difficult problem.

Let $G_\Gamma = \langle\, \mathfrak{X} \mid xyx^{-1}y^{-1},\ \{x,y\} \in \mathcal{E}\,\rangle$ denote the *right-angled Artin group* associated with Γ, and let $\mathrm{gr}_\bullet(G_\Gamma)$ denote the graded $\mathbb{Z}$-Lie algebra associated with the *lower central series* of G_Γ, i.e., $\mathrm{gr}_k(G_\Gamma) = \gamma_k(G_\Gamma)/\gamma_{k+1}(G_\Gamma)$, where $\gamma_1(G_\Gamma) = G_\Gamma$ and $\gamma_{k+1}(G_\Gamma) = [G_\Gamma, \gamma_k(G_\Gamma)]$ for $k \geq 1$. Then $\mathrm{gr}_k(G_\Gamma)$ is a torsion-free abelian group, and $L\langle\Gamma\rangle \simeq \mathbb{F} \otimes_{\mathbb{Z}} \mathrm{gr}_\bullet(G_\Gamma)$ (cf. [29]). In particular,

$$\mathrm{rk}_{\mathbb{Z}}(\mathrm{gr}_k(G_\Gamma)) = \textstyle\sum_{1\leq j\leq n} M_k(\lambda_j), \tag{4.15}$$

where n is the clique number of Γ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are the eigenvalues of the alternating clique polynomial, i.e., $\mathrm{Cl}_\Gamma(y) = \prod_{1\leq j\leq n}(1-\lambda_j y)$.

4.2.3. *Holonomy Lie algebras of supersolvable hyperplane arrangements.* Let X be a connected topological space having the homotopy type of a finite cell complex, and let

$$\alpha_2\colon H_2(X,\mathbb{F}) \to \Lambda_1(H_1(X,\mathbb{F})) \tag{4.16}$$

denote the mapping induced by the co-multiplication in $H_\bullet(X,\mathbb{F})$. Then

$$L^X = L[H_1(X,\mathbb{F})]/\langle \mathrm{im}(\alpha_2)\rangle_{\mathrm{Lie}}, \tag{4.17}$$

where $L[H_1(X,\mathbb{F})]$ is the free Lie algebra over the vector space $H_1(X,\mathbb{F})$ and $\mathrm{im}(\alpha_2)$ is identified with the corresponding subspace in $L_2[H_1(X,\mathbb{F})]$, is called the *holonomy Lie algebra* associated with X. By definition, L^X is of finite type and quadratic.

In case that $\{\, H_i \mid 1 \leq i \leq s \,\}$ is a finite set of hyperplanes in the complex vector space $\mathbb{C}^r$ and $X = \mathbb{C}^r \setminus \bigcup_{1\leq i\leq s} H_i$, T. Kohno has shown in [10] that $L^X_{\mathbb{C}}$ and $\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{gr}_\bullet(\pi_1(X,x_0))$ are canonically isomorphic. He also gave a presentation of the quadratic algebra $L^X_{\mathbb{C}}$. If $\{\, H_i \mid 1 \leq i \leq s \,\}$, $H_i \subseteq \mathbb{C}^{n+1}$ are the hyperplanes associated with the rootsystem of type A_n, he showed in [11] that $L^X_{\mathbb{C}}$ is a Koszul Lie algebra of cohomological dimension n with eigenvalues $1,\ldots,n$. In [27], B. Shelton and S. Yuzvinski extended his result to all supersoluble hyperplane arrangements.

References

[1] A. E. Bereznyĭ, *Discrete groups with subexponential growth of coefficients of the Golod-Šafarevič series*, Uspekhi Mat. Nauk **35** (1980), no. 5(215), 225–226. MR 595148 (82k:20064)

[2] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1982. MR 672956 (83k:20002)

[3] M. Chudnovsky and P. Seymour, *The roots of the independence polynomial of a clawfree graph*, J. Combin. Theory Ser. B **97** (2007), no. 3, 350–357. MR 2305888 (2008c:05130)

[4] D. Ž. Doković, *A generalization of Witt's formulae*, J. Algebra **111** (1987), no. 1, 262–278. MR 913209 (89e:17005)

[5] R. Fröberg, *Determination of a class of Poincaré series*, Math. Scand. **37** (1975), no. 1, 29–39. MR 0404254 (53 #8057)

[6] R. Grigorchuk and P. de la Harpe, *On problems related to growth, entropy, and spectrum in group theory*, J. Dynam. Control Systems **3** (1997), no. 1, 51–89. MR 1436550 (98d:20039)

[7] M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. (1981), no. 53, 53–73. MR 623534 (83b:53041)

[8] S-J. Kang and M-H. Kim, *Free Lie algebras, generalized Witt formula, and the denominator identity*, J. Algebra **183** (1996), no. 2, 560–594. MR 1399040 (97e:17042)

[9] A. A. Kirillov, M. L. Kontsevich, and A. I. Molev, *Algebras of intermediate growth*, Akad. Nauk SSSR Inst. Prikl. Mat. Preprint (1983), no. 39, 19, Translated in Selecta Math. Soviet. **9** (1990), no. 2, 137–153. MR 753873 (85e:17011)

[10] T. Kohno, *On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces*, Nagoya Math. J. **92** (1983), 21–37. MR 726138 (85d:14032)

[11] ______, *Série de Poincaré-Koszul associée aux groupes de tresses pures*, Invent. Math. **82** (1985), no. 1, 57–75. MR 808109 (87c:32015a)

[12] J. P. Labute, *Algèbres de Lie et pro-p-groupes définis par une seule relation*, Invent. Math. **4** (1967), 142–158. MR 0218495 (36 #1581)

[13] ______, *On the descending central series of groups with a single defining relation*, J. Algebra **14** (1970), 16–23. MR 0251111 (40 #4342)

[14] A. Lubotzky and A. Mann, *On groups of polynomial subgroup growth*, Invent. Math. **104** (1991), no. 3, 521–533. MR 1106747 (92d:20038)

[15] A. Lubotzky, A. Mann, and D. Segal, *Finitely generated groups of polynomial subgroup growth*, Israel J. Math. **82** (1993), no. 1-3, 363–371. MR 1239055 (95b:20051)

[16] W. Mantel, *Wiskundige opgaven, 10*, 1906, pp. 60–61.

[17] D. V. Millionshchikov, *Filiformal N-graded Lie algebras*, Uspekhi Mat. Nauk **57** (2002), no. 2(344), 197–198. MR 1918204

[18] C. Moreau, *Sur les permutations circulaires distinctes*, Nouvelles annales de mathématiques, journal des candidats aux écoles polytechnique et normale, Sér. 2 **11** (1872), 309–331.

[19] J. Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999, Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. MR 1697859 (2000m:11104)

[20] M. F. Newman, C. Schneider, and A. Shalev, *The entropy of graded algebras*, J. Algebra **223** (2000), no. 1, 85–100. MR 1738253 (2001c:16078)

[21] A. Polishchuk and L. Positselski, *Quadratic algebras*, University Lecture Series, vol. 37, American Mathematical Society, Providence, RI, 2005. MR 2177131 (2006f:16043)

[22] G. Pólya and G. Szegő, *Problems and theorems in analysis. Vol. I: Series, integral calculus, theory of functions*, Springer-Verlag, New York, 1972, Translated from the German by D. Aeppli, Die Grundlehren der mathematischen Wissenschaften, Band 193. MR 0344042 (49 #8782)

[23] S. B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60. MR 0265437 (42 #346)

[24] G. Robin, *Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann*, J. Math. Pures Appl. (9) **63** (1984), no. 2, 187–213. MR 774171 (86f:11069)

[25] J-P. Serre, *Trees*, Springer-Verlag, Berlin, 1980, Translated from the French by John Stillwell.

[26] ______, *Galois cohomology*, Springer-Verlag, Berlin, 1997, Translated from the French by Patrick Ion and revised by the author. MR 1466966 (98g:12007)

[27] B. Shelton and S. Yuzvinsky, *Koszul algebras from graphs and hyperplane arrangements*, J. London Math. Soc. (2) **56** (1997), no. 3, 477–490. MR 1610447 (99c:16044)

[28] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452. MR 0018405 (8,284j)

[29] R. D. Wade, *The lower central series of a right-angled artin group*, preprint, available at http://arxiv.org/abs/1109.1722, 2011.

[30] E. Witt, *Treue Darstellungen beliebiger Liescher Ringe*, J. Reine Angew. Math. **177** (1937), 152–160.

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