

Nonlinear Schrödinger equations without compatibility conditions on the potentials

Michela Guida*, Sergio Rolando†

Abstract

We study the existence of nonnegative solutions (and ground states) to nonlinear Schrödinger equations in \mathbb{R}^N with radial potentials and super-linear or sub-linear nonlinearities. The potentials satisfy power type estimates at the origin and at infinity, but no compatibility condition is required on their growth (or decay) rates at zero and infinity. In this respect our results extend some well known results in the literature and we also believe that they can highlight the role of the sum of Lebesgue spaces in studying nonlinear equations with weights.

MSC (2010): Primary 35J60; Secondary 35J05, 35J20, 46E30

Keywords: Nonlinear Schrödinger equation, unbounded or decaying potentials, sum of weighted Lebesgue spaces, ground states

1 Introduction and main results

We study the semilinear elliptic equation

$$-\Delta u + V(|x|)u = K(|x|)f(u) \quad \text{in } \mathbb{R}^N, \quad N \geq 3, \quad (1)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $f(0) = 0$ and V, K satisfy the following assumptions:

(V) $V : (0, +\infty) \rightarrow [0, +\infty)$ is a continuous function such that

$$\liminf_{r \rightarrow 0^+} \frac{V(r)}{r^{a_0}} > 0 \quad \text{and} \quad \liminf_{r \rightarrow +\infty} \frac{V(r)}{r^a} > 0 \quad \text{for some } a_0, a \in \mathbb{R};$$

(K) $K : (0, +\infty) \rightarrow (0, +\infty)$ is a continuous function such that

$$\limsup_{r \rightarrow 0^+} \frac{K(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r \rightarrow +\infty} \frac{K(r)}{r^b} < \infty \quad \text{for some } b_0, b \in \mathbb{R}.$$

*Dipartimento di Matematica “Giuseppe Peano”, Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy. e-mail: michela.guida@unito.it

†Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Via Roberto Cozzi 53, 20125 Milano, Italy. e-mail: sergio.rolando@unito.it

The authors are members of the Gruppo Nazionale di Alta Matematica (INdAM).

More precisely, we are interested in finding nontrivial nonnegative radial solutions in the following weak sense (see also Remark 7.2): we call *radial solution* to Eq. (1) any $u \in H_{V,r}^1$ such that

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla h \, dx + \int_{\mathbb{R}^N} V(|x|) u h \, dx = \int_{\mathbb{R}^N} K(|x|) f(u) h \, dx \quad \text{for all } h \in H_{V,r}^1, \quad (2)$$

where

$$H_{V,r}^1 = H_{V,r}^1(\mathbb{R}^N) := \{u \in H_V^1(\mathbb{R}^N) : u(x) = u(|x|)\} \quad (3)$$

is the radial subspace of

$$H_V^1 = H_V^1(\mathbb{R}^N) := \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(|x|) u^2 \, dx < \infty \right\}. \quad (4)$$

Here $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$, $2^* := 2N/(N-2)$, denotes the usual Sobolev space, which identifies with the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the L^2 norm of the gradient. Of course, $u(x) = u(|x|)$ means that u is invariant under the action on H_V^1 of the orthogonal group of \mathbb{R}^N .

By well known arguments, the nonnegative weak solutions to Eq. (1) lead to special solutions (*solitary waves* and *solitons*) for several nonlinear field theories, such as nonlinear Schrödinger and Klein-Gordon equations, which arise in many branches of mathematical physics, such as nonlinear optics, plasma physics, condensed matter physics and cosmology (see e.g. [5, 12, 24]). In this respect, since the early studies of [15, 17, 19, 20], Eq. (1) has been massively addressed in the mathematical literature, recently focusing on the case of V possibly vanishing at infinity, that is, $\liminf_{|x| \rightarrow \infty} V(|x|) = 0$ (some first results on such a case can be found in [2, 10, 13, 14]; for more recent bibliography, see e.g. [1, 6, 11, 16, 21] and the references therein).

The most recent and general existence results for radial solutions to Eq. (1) under assumptions **(V)** and **(K)**, unifying and extending the previously existing ones, are contained in [22] and [21], which respectively concern the case of super-linear and sub-linear nonlinearities.

The result of [22], rewritten in a suitable form for comparing with our results, is Theorem 1 below, which uses the following notation. For every $a_0 \in \mathbb{R}$, set

$$\underline{b}(a_0) := \begin{cases} -\infty & \text{if } a_0 < -(2N-2) \\ \min\{a_0, -2\} & \text{if } a_0 \geq -(2N-2). \end{cases} \quad (5)$$

Then, for $a, b, a_0 \in \mathbb{R}$ and $b_0 > \underline{b}(a_0)$, define the functions

$$\underline{q} = \underline{q}(a, b, a_0, b_0) := \begin{cases} \max\left\{2, 2\frac{N+b}{N-2}\right\} & \text{if } a \leq -2, b_0 > \min\{-2, a_0\} \\ \max\left\{2, 2\frac{N+b}{N-2}, 2\frac{2N-2+2b_0-a_0}{2N-2+a_0}\right\} & \text{if } a \leq -2, b_0 \leq a_0 < -(2N-2) \\ \max\left\{2, 2\frac{2N-2+2b-a}{2N-2+a}\right\} & \text{if } a > -2, b_0 > \min\{-2, a_0\} \\ \max\left\{2, 2\frac{2N-2+2b-a}{2N-2+a}, 2\frac{2N-2+2b_0-a_0}{2N-2+a_0}\right\} & \text{if } a > -2, b_0 \leq a_0 < -(2N-2) \end{cases}$$

and

$$\bar{q} = \bar{q}(a_0, b_0) := \begin{cases} +\infty & \text{if } a_0 < -(2N-2) \text{ or } a_0 = -(2N-2) < b_0 \\ 2\frac{2N-2+2b_0-a_0}{2N-2+a_0} & \text{if } -(2N-2) < a_0 < -2, b_0 > a_0 \\ 2\frac{N+b_0}{N-2} & \text{if } a_0 \geq -2, b_0 > -2. \end{cases}$$

Observe that one always has $\underline{q} \geq 2$ and $\bar{q} > 2$.

Theorem 1 ([22, Theorem 5]). Assume **(V)**, **(K)** with $a_0, a, b \in \mathbb{R}$ and $b_0 > \underline{b}(a_0)$. Assume furthermore that $q < \bar{q}$. Then Eq. (1) has a nonnegative nontrivial radial solution for every continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying:

$$(\mathbf{f}_1) \sup_{t>0} \frac{|f(t)|}{t^{q-1}} < +\infty \text{ for some } q \in (\underline{q}, \bar{q});$$

$$(\mathbf{f}_2) \exists \theta > 2 \text{ such that } 0 < \theta F(t) \leq f(t)t \text{ for all } t \in \mathbb{R}.$$

In (\mathbf{f}_2) and everywhere in the following, we denote $F(t) := \int_0^t f(s) ds$.

Remark 2. To be precise, instead of (\mathbf{f}_1) , the growth condition used in [22, Theorem 5] is

$$\sup_{t \in \mathbb{R}} \frac{|f(t)|}{|t|^{q_1-1} + |t|^{q_2-1}} < +\infty \text{ for some } q_1, q_2 \in (\underline{q}, \bar{q}), \quad (6)$$

but the difference between (6) and (\mathbf{f}_1) is not essential. Indeed, we can just let $t > 0$ in (6) because we deal with nonnegative solutions, and the use of a sum of powers is a standard generalization of (\mathbf{f}_1) .

In order to recall the existence result of [21], we need some further notation. Define the following subsets of \mathbb{R}^2 :

$$\begin{aligned} \mathcal{A}_1 &:= \{(a, b) : \max\{-\frac{N+2}{2}, \frac{a-2}{2}\} \leq b < -2\}, & \mathcal{B}_1 &:= \{(a_0, b_0) : \max\{-\frac{N+2}{2}, \frac{a_0-2}{2}\} < b_0 \leq -2\}, \\ \mathcal{A}_2 &:= \{(a, b) : -\frac{N+2}{2} \leq b < \min\{-2, \frac{a-2N-2}{4}\}\}, & \mathcal{B}_2 &:= \{(a_0, b_0) : -\frac{N+2}{2} < b_0 \leq -2 \leq a_0\}, \\ \mathcal{A}_3 &:= \{(a, b) : a \leq -2, -\frac{N+2}{2} < b < \frac{a-2}{2}\}, & \mathcal{B}_3 &:= \{(a_0, b_0) : a_0 < -2, -\frac{N+2}{2} < b_0 \leq \frac{a_0-2}{2}\}, \\ \mathcal{A}_4 &:= \{(a, b) : b \leq -\frac{N+2}{2}, \frac{a-2N-2}{4} \leq b < \frac{a-2}{2}\}, & \mathcal{B}_4 &:= \{(a_0, b_0) : b_0 < -\frac{N+2}{2}, \frac{a_0-2N-2}{4} < b_0 \leq \frac{a_0-2}{2}\}, \\ \mathcal{A}_5 &:= \{(a, b) : a > -2, \frac{a-2N-2}{4} \leq b < \frac{a-2}{2}\}, & \mathcal{B}_5 &:= \{(a_0, b_0) : b_0 \geq -2, \frac{a_0-2N-2}{4} < b_0 \leq \frac{a_0-2}{2}\}, \\ \mathcal{B} &:= \mathcal{B}_1 \cup \dots \cup \mathcal{B}_5, & \mathcal{B}_6 &:= \{(a_0, b_0) : \frac{a_0-2}{2} < b_0 \leq \frac{a_0-2N-2}{4}\}. \end{aligned}$$

Then, for $(a, b) \in \mathcal{A}_1 \cup \dots \cup \mathcal{A}_5$ and $(a_0, b_0) \in \mathcal{B} \cup \mathcal{B}_6$, define the functions

$$\underline{q} = \underline{q}(a, b, a_0, b_0) := \begin{cases} 2\frac{N+b}{N-2} & \text{if } (a, b) \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3, (a_0, b_0) \in \mathcal{B} \\ \max\left\{2\frac{N+b}{N-2}, 4\frac{N+b_0}{2N-2+a_0}\right\} & \text{if } (a, b) \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3, (a_0, b_0) \in \mathcal{B}_6 \\ 4\frac{N+b}{2N-2+a} & \text{if } (a, b) \in \mathcal{A}_4 \cup \mathcal{A}_5, (a_0, b_0) \in \mathcal{B} \\ \max\left\{4\frac{N+b}{2N-2+a}, 4\frac{N+b_0}{2N-2+a_0}\right\} & \text{if } (a, b) \in \mathcal{A}_4 \cup \mathcal{A}_5, (a_0, b_0) \in \mathcal{B}_6 \end{cases}$$

and

$$\bar{q} = \bar{q}(a_0, b_0) := \begin{cases} 2\frac{N+b_0}{N-2} & \text{if } (a_0, b_0) \in \mathcal{B}_1 \cup \mathcal{B}_2 \\ 4\frac{N+b_0}{2N-2+a_0} & \text{if } (a_0, b_0) \in \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \\ 2 & \text{if } (a_0, b_0) \in \mathcal{B}_6. \end{cases}$$

Observe that one always has $1 \leq \underline{q} < 2$ and $1 < \bar{q} \leq 2$.

Theorem 3 ([21, Theorem 1.3]). *Assume $(\mathbf{V}), (\mathbf{K})$ with $(a, b) \in \mathcal{A}_1 \cup \dots \cup \mathcal{A}_5$ and $(a_0, b_0) \in \mathcal{B} \cup \mathcal{B}_6$. Assume furthermore that $\underline{q} < \bar{q}$. Then Eq. (1) with $f(u) = |u|^{q-2}u$ has a nonnegative nontrivial radial solution provided that $q \in (\underline{q}, \bar{q})$.*

The spirit of the above Theorems 1 and 3 is essentially the following: a compatibility condition between the behaviours of the potentials at zero and at infinity is required ($\underline{q} < \bar{q}$ or $\underline{q} < \bar{q}$) and a solution is then provided if the nonlinearity grows compatibly with the potentials (q between \underline{q}, \bar{q} or \underline{q}, \bar{q}).

Here we still require some compatibility between the nonlinearity and the potentials, but we remove any compatibility assumption between how the potentials behave at zero and infinity, getting existence results that contain and extend Theorems 1 and 3.

In order to state our results, we need some preliminary notations, which essentially consist in defining two intervals \mathcal{I}_1 and \mathcal{I}_2 which will provide a way of expressing the compatibility between the nonlinearity and the potentials required in order to get existence. In this respect, such intervals play the same role of the limiting exponents $\underline{q}, \bar{q}, \underline{q}, \bar{q}$ of Theorems 1 and 3, but in an unified way for both the cases of super-linear and sub-linear nonlinearities (see assumptions (\mathbf{f}_3) and (\mathbf{f}_7) below). On a more technical level, \mathcal{I}_1 and \mathcal{I}_2 are the exact ranges of exponents for which we can prove the compactness result given in Lemma 10 of Section 2.

For every $a_0 \in \mathbb{R}$, define

$$b_*(a_0) := \begin{cases} -\infty & \text{if } a_0 < -(2N-2) \\ \min \left\{ a_0, -\frac{N-a_0}{2}, -\frac{N+2}{2} \right\} & \text{if } a_0 \geq -(2N-2). \end{cases}$$

Recalling definition (5) of \underline{b} , observe that $b_*(a_0) \leq \underline{b}(a_0)$ for every $a_0 \in \mathbb{R}$ (precisely: $b_* = \underline{b}$ for $a_0 \leq -N$ and $b_* < \underline{b}$ for $a_0 > -N$). Then, for $a_0 \in \mathbb{R}$ and $b_0 > b_*(a_0)$, define the functions

$$q_*(a_0, b_0) := \begin{cases} \max \left\{ 1, 2\frac{N+b_0}{N+a_0}, 2\frac{2N-2+2b_0-a_0}{2N-2+a_0} \right\} & \text{if } a_0 < -(2N-2) \\ \max \left\{ 1, 2\frac{N+b_0}{N+a_0} \right\} & \text{if } -(2N-2) \leq a_0 < -N \\ 1 & \text{if } a_0 \geq -N, \end{cases}$$

$$q^*(a_0, b_0) := \begin{cases} +\infty & \text{if } a_0 \leq -(2N-2) \\ 2\frac{2N-2+2b_0-a_0}{2N-2+a_0} & \text{if } -(2N-2) < a_0 \leq -N \\ \min \left\{ 2\frac{N+b_0}{N+a_0}, 2\frac{2N-2+2b_0-a_0}{2N-2+a_0} \right\} & \text{if } -N < a_0 < -2 \\ 2\frac{N+b_0}{N-2} & \text{if } a_0 \geq -2 \end{cases}$$

and the interval

$$\mathcal{I}_1 = \mathcal{I}_1(a_0, b_0) := (q_*(a_0, b_0), q^*(a_0, b_0)). \quad (7)$$

Note that $b_0 > b_*(a_0)$ is equivalent to $q_*(a_0, b_0) < q^*(a_0, b_0)$, i.e., $\mathcal{I}_1 \neq \emptyset$. Finally, for every $a, b \in \mathbb{R}$, define the function

$$q_{**}(a, b) := \begin{cases} \max \left\{ 1, 2\frac{N+b}{N-2} \right\} & \text{if } a \leq -2 \\ \max \left\{ 1, 2\frac{N+b}{N+a}, 2\frac{2N-2+2b-a}{2N-2+a} \right\} & \text{if } a > -2 \end{cases}$$

and the interval

$$\mathcal{I}_2 = \mathcal{I}_2(a, b) := (q_{**}(a, b), +\infty). \quad (8)$$

In order to ease the visualization of the intervals \mathcal{I}_1 and \mathcal{I}_2 , the graphs of the functions $q_*(a_0, \cdot)$, $q^*(a_0, \cdot)$ and $q_{**}(a, \cdot)$, with a_0 and a fixed to different meaningful values, are plotted in Figures 1-8 below.

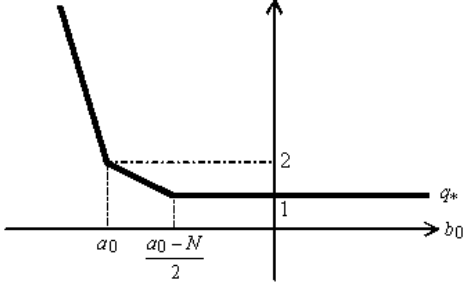


Fig.1. $q_*(a_0, \cdot)$ for $a_0 < -(2N - 2)$
 $(q^*(a_0, \cdot) = +\infty)$

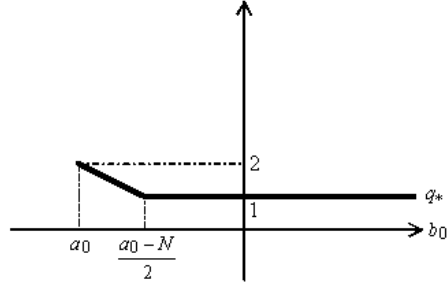


Fig.2. $q_*(a_0, \cdot)$ for $a_0 = -(2N - 2)$
 $(q^*(a_0, \cdot) = +\infty)$

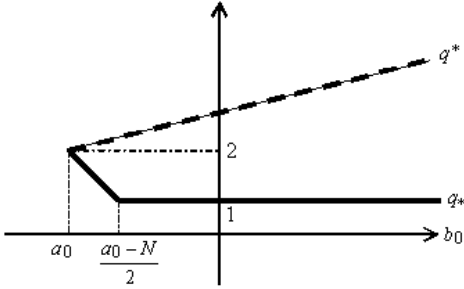


Fig.3. $q_*(a_0, \cdot)$ and $q^*(a_0, \cdot)$
for $-(2N - 2) < a_0 < -N$

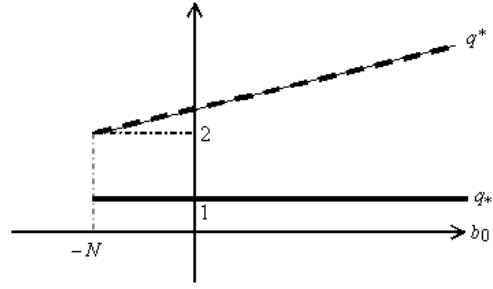


Fig.4. $q_*(a_0, \cdot)$ and $q^*(a_0, \cdot)$
for $a_0 = -N$

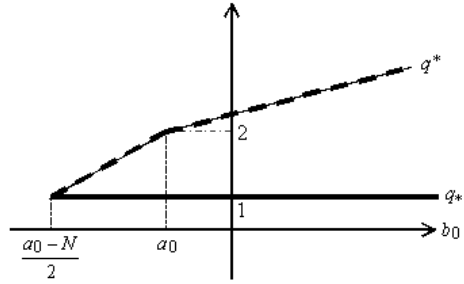


Fig.5. $q_*(a_0, \cdot)$ and $q^*(a_0, \cdot)$
for $-N < a_0 < -2$

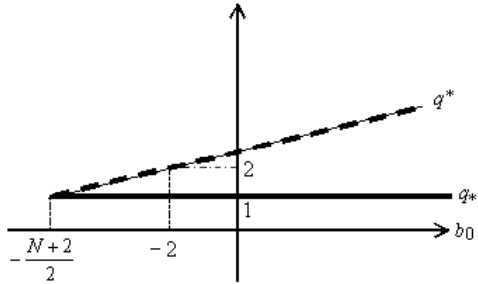


Fig.6. $q_*(a_0, \cdot)$ and $q^*(a_0, \cdot)$
for $a_0 \geq -2$

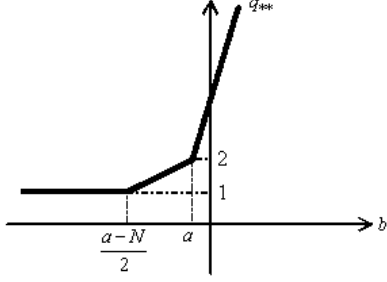


Fig.7. $q_{**}(a, \cdot)$ for $a \leq -2$

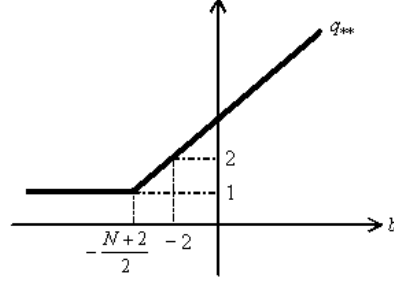


Fig.8. $q_{**}(a, \cdot)$ for $a > -2$

Notice that:

- both \mathcal{I}_1 and \mathcal{I}_2 are contained in $(1, +\infty)$ for every $a_0, a, b \in \mathbb{R}$ and $b_0 > b_*(a_0)$;
- $\mathcal{I}_1 \cap (2, +\infty) \neq \emptyset$ if and only if $a_0 \in \mathbb{R}$ and $b_0 > \underline{b}(a_0)$;
- $\mathcal{I}_2 \cap (2, +\infty) \neq \emptyset$ for every $a, b \in \mathbb{R}$;
- $\mathcal{I}_1 \cap (1, 2) \neq \emptyset$ if and only if $a_0 \in \mathbb{R}$ and $b_0 > \min \left\{ a_0, -\frac{N-a_0}{2}, -\frac{N+2}{2} \right\}$;
- $\mathcal{I}_2 \cap (1, 2) \neq \emptyset$ if and only if $a \in \mathbb{R}$ and $b < \max \{ a, -2 \}$.

Our main existence result for super-linear nonlinearities is the following theorem. A related result, concerning ground state solutions, will be given in Section 3.

Theorem 4. *Assume **(V)**, **(K)** with $a_0, a, b \in \mathbb{R}$ and $b_0 > \underline{b}(a_0)$. Then Eq. (1) has a nonnegative nontrivial radial solution for every continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying:*

$$(\mathbf{f}_3) \sup_{t>0} \frac{|f(t)|}{\min \{t^{q_1-1}, t^{q_2-1}\}} < +\infty \text{ for some } q_1, q_2 > 2 \text{ such that } q_1 \in \mathcal{I}_1, q_2 \in \mathcal{I}_2;$$

$$(\mathbf{f}_4) \exists \theta > 2 \text{ such that } 0 \leq \theta F(t) \leq f(t)t \text{ for all } t > 0;$$

$$(\mathbf{f}_5) \exists t_0 > 0 \text{ such that } F(t_0) > 0.$$

If $K(|\cdot|) \in L^1(\mathbb{R}^N)$, the same result holds with (\mathbf{f}_4) - (\mathbf{f}_5) replaced by the weaker condition:

$$(\mathbf{f}_6) \exists \theta > 2 \text{ and } \exists t_0 > 0 \text{ such that } 0 < \theta F(t) \leq f(t)t \text{ for all } t \geq t_0.$$

Observe that, as soon as we can take $q_1 = q_2$ in (\mathbf{f}_3) , i.e., when $\mathcal{I}_1 \cap \mathcal{I}_2 \neq \emptyset$, Theorem 4 only requires that the nonlinearity satisfies a single power growth condition, i.e., a condition of the form

$$|f(t)| \leq (\text{const.}) t^{q-1} \quad \text{with } q \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap (2, +\infty).$$

Indeed, such a condition is equivalent to (\mathbf{f}_3) , because: (i) it obviously implies (\mathbf{f}_3) with $q_1 = q_2 = q$; (ii) assuming for instance that (\mathbf{f}_3) holds with $q_1 \leq q_2$, one has $\min \{t^{q_1-1}, t^{q_2-1}\} \leq t^{q-1}$ for every $t > 0$ and $q \in [q_1, q_2]$, and one can find such a q which also belongs to $\mathcal{I}_1 \cap \mathcal{I}_2 \cap (2, +\infty)$.

Moreover, assuming $b_0 > \underline{b}(a_0)$, it is not difficult to check that $\mathcal{I}_1 \cap \mathcal{I}_2 \neq \emptyset$ if and only if $\underline{q} < \bar{q}$ (where \underline{q} and \bar{q} are the exponents of Theorem 1) and, in this case, one has

$$\mathcal{I}_1 \cap \mathcal{I}_2 \cap (2, +\infty) = (q, \bar{q}).$$

Therefore, the case $\mathcal{I}_1 \cap \mathcal{I}_2 \neq \emptyset$ is exactly the case in which the potentials behave compatibly at zero and infinity, and, in such a case, since (\mathbf{f}_3) becomes equivalent to (\mathbf{f}_1) , the first part of Theorem 4 gives exactly Theorem 1 (up to the fact that the pair (\mathbf{f}_4) - (\mathbf{f}_5) is a slightly weaker condition than (\mathbf{f}_2) , which is a rather technical generalization and it is not what we want to emphasize here). Note that \mathcal{I}_1 and \mathcal{I}_2 only depend on a_0, b_0 and a, b respectively, so that $\mathcal{I}_1 \cap \mathcal{I}_2 \neq \emptyset$ means a link between the potential rates at zero and infinity (precisely: $q_{**}(a, b) < q^*(a_0, b_0)$).

The case $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$, instead, is the case without compatibility (and amounts to $\bar{q} \leq \underline{q}$), so that Theorem 1 does not apply and Theorem 4 is a new result (both in its first and second part), concerning nonlinearities that satisfy the double power growth condition (\mathbf{f}_3) with $q_1 \neq q_2$. For the reader convenience, we explain this case in the following corollary.

Corollary 5. *Assume $(\mathbf{V}), (\mathbf{K})$ with $a_0 > -(2N - 2)$, $b_0 > \min\{a_0, -2\}$ and one of the following alternatives:*

$$a \leq -2, \quad b \geq \max \left\{ 2 \frac{(N-2)b_0 - (N-1)(a_0+2)}{2N-2+a_0}, b_0 \right\} \quad (9)$$

or

$$b > a > -2, \quad \frac{b-a}{2N-2+a} \geq \max \left\{ \frac{b_0 - a_0}{2N-2+a_0}, \frac{b_0+2}{2(N-2)} \right\}. \quad (10)$$

Then Eq. (1) has a nonnegative nontrivial radial solution for every continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (\mathbf{f}_4) , (\mathbf{f}_5) and

$$\sup_{t>0} \frac{|f(t)|}{\min\{t^{q_1-1}, t^{q_2-1}\}} < +\infty$$

for some

$$2 < q_1 < \max \left\{ 2 \frac{N+b_0}{N-2}, 2 \frac{2N-2+2b_0-a_0}{2N-2+a_0} \right\} \quad \text{and} \quad q_2 > \min \left\{ 2 \frac{N+b}{N-2}, 2 \frac{2N-2+2b-a}{2N-2+a} \right\}. \quad (11)$$

If $K(|\cdot|) \in L^1(\mathbb{R}^N)$, the same result holds with (\mathbf{f}_4) - (\mathbf{f}_5) replaced by (\mathbf{f}_6) .

The interested reader can check that Corollary 5 is exactly the case of Theorem 4 with $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$. We just observe that, under the assumptions of the corollary, one explicitly has

$$\begin{aligned} \max \left\{ \frac{b_0 - a_0}{2N-2+a_0}, \frac{b_0+2}{2(N-2)} \right\} &= \begin{cases} \frac{b_0 - a_0}{2N-2+a_0} & \text{if } a_0 < -2 \\ \frac{b_0+2}{2(N-2)} & \text{if } a_0 \geq -2, \end{cases} \\ \max \left\{ 2 \frac{(N-2)b_0 - (N-1)(a_0+2)}{2N-2+a_0}, b_0 \right\} &= \begin{cases} 2 \frac{(N-2)b_0 - (N-1)(a_0+2)}{2N-2+a_0} & \text{if } a_0 < -2 \\ b_0 & \text{if } a_0 \geq -2, \end{cases} \\ \max \left\{ 2 \frac{N+b_0}{N-2}, 2 \frac{2N-2+2b_0-a_0}{2N-2+a_0} \right\} &= \begin{cases} 2 \frac{2N-2+2b_0-a_0}{2N-2+a_0} & \text{if } a_0 < -2 \\ 2 \frac{N+b_0}{N-2} & \text{if } a_0 \geq -2, \end{cases} \end{aligned} \quad (12)$$

$$\min \left\{ 2 \frac{N+b}{N-2}, 2 \frac{2N-2+2b-a}{2N-2+a} \right\} = \begin{cases} 2 \frac{N+b}{N-2} & \text{if (9) holds} \\ 2 \frac{2N-2+2b-a}{2N-2+a} & \text{if (10) holds,} \end{cases} \quad (13)$$

where (12) and (13), which are the right hand sides of inequalities (11), respectively coincide with $q^*(a_0, b_0)$ and $q_{**}(a, b)$, or equivalently with $\bar{q}(a_0, b_0)$ and $\underline{q}(a, b, a_0, b_0)$, and satisfy $q^*(a_0, b_0) \leq q_{**}(a, b)$.

As far as sub-linear nonlinearities are concerned, we will prove the following result.

Theorem 6. *Assume (V), (K) with $a_0, a \in \mathbb{R}$, $b_0 > \min\{a_0, -\frac{N-a_0}{2}, -\frac{N+2}{2}\}$ and $b < \max\{a, -2\}$. Then Eq. (1) has a nonnegative nontrivial radial solution for every continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying:*

$$(f_7) \sup_{t>0} \frac{|f(t)|}{\min\{t^{q_1-1}, t^{q_2-1}\}} < +\infty \text{ for some } q_1, q_2 < 2 \text{ such that } q_1 \in \mathcal{I}_1, q_2 \in \mathcal{I}_2;$$

$$(f_8) \exists \theta < 2 \text{ such that } \liminf_{t \rightarrow 0^+} \frac{F(t)}{t^\theta} > 0.$$

In contrast with the case of Theorem 4 with respect to Theorem 1, Theorem 6 extends Theorem 3 in many directions (other than the obvious fact that Theorem 6 concerns more general nonlinearities than the pure power $f(u) = |u|^{q-2}u$). Such improvements are described by the following remarks, where the set

$$\mathcal{P} = \left\{ (a, b, a_0, b_0) : b_0 > \min\left\{a_0, -\frac{N-a_0}{2}, -\frac{N+2}{2}\right\}, b < \max\{a, -2\}, \mathcal{I}_1 \cap \mathcal{I}_2 \neq \emptyset \right\}$$

is used. Notice that \mathcal{P} is the set of the potential rates a, b, a_0, b_0 for which Theorem 6 concerns single power nonlinearities, in the sense that, if $(a, b, a_0, b_0) \in \mathcal{P}$, then (f₇) is equivalent to

$$|f(t)| \leq (\text{const.}) t^{q-1} \quad \text{with } q \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap (1, 2)$$

(cf. the discussion after Theorem 4).

- The set \mathcal{P} is strictly larger than the set of the potential rates for which Theorem 3 applies, i.e., the set

$$\mathcal{P}_1 = \left\{ (a, b, a_0, b_0) : (a, b) \in \mathcal{A}_1 \cup \dots \cup \mathcal{A}_5, (a_0, b_0) \in \mathcal{B}_1 \cup \dots \cup \mathcal{B}_6, \underline{q} < \bar{q} \right\}.$$

For example, if $(a, b) \in \mathcal{A}_2$ and $(a_0, b_0) \in \mathcal{B}_1$ with $a_0 < -(2N-2)$ and $b \geq b_0$, then one has

$$\underline{q} = 2\frac{N+b}{N-2}, \quad \bar{q} = 2\frac{N+b_0}{N-2}, \quad q_* = 1, \quad q^* = +\infty, \quad q_{**} = 1 \quad (14)$$

and therefore $\mathcal{I}_1 \cap \mathcal{I}_2 \cap (1, 2) = (1, 2)$ and $\underline{q} \geq \bar{q}$, so that $(a, b, a_0, b_0) \in \mathcal{P}$ but $(a, b, a_0, b_0) \notin \mathcal{P}_1$.

Other examples are given by those points $(a, b, a_0, b_0) \in \mathcal{P}$ for which at least one of the exponents \underline{q} and \bar{q} is undefined, i.e., $(a, b, a_0, b_0) \notin (\mathcal{A}_1 \cup \dots \cup \mathcal{A}_5) \times (\mathcal{B}_1 \cup \dots \cup \mathcal{B}_6)$. For instance, if

$$\frac{a-2}{2} \leq b < a, \quad a_0 < -(2N-2) \quad \text{and} \quad b_0 > -2,$$

then both \underline{q} and \bar{q} are undefined, while we get $q_* = 1$, $q^* = +\infty$, $q_{**} = 2\frac{N+b}{N+a}$ and therefore $\mathcal{I}_1 \cap \mathcal{I}_2 \cap (1, 2) = (2\frac{N+b}{N+a}, 2)$.

This means that Theorem 6 gives existence results to Eq. (1) with power type nonlinearities (even with $f(u) = |u|^{q-2}u$) for more potentials than the ones allowed by Theorem 3, and exactly for those potentials V, K satisfying (V), (K) with $(a, b, a_0, b_0) \in \mathcal{P} \setminus \mathcal{P}_1$. The explicit description of the whole set $\mathcal{P} \setminus \mathcal{P}_1$ is left to the interested reader. We just observe that the above examples show that $\mathcal{P} \setminus \mathcal{P}_1$ contains both points for which $\underline{q} \geq \bar{q}$ and points for which \underline{q} and \bar{q} are undefined.

- If $(a, b, a_0, b_0) \in \mathcal{P}_1$, the interval $\mathcal{I}_1 \cap \mathcal{I}_2 \cap (1, 2)$ can be strictly larger than (\underline{q}, \bar{q}) (depending on a, b, a_0, b_0). For example, if we take $(a, b) \in \mathcal{A}_1$ and $(a_0, b_0) \in \mathcal{B}_2$ with $b < b_0$, then

$$\underline{q} = 2\frac{N+b}{N-2}, \quad \bar{q} = 2\frac{N+b_0}{N-2}, \quad q_* = 1, \quad q^* = 2\frac{N+b_0}{N-2}, \quad q_{**} = 2\frac{N+b}{N-2}$$

and therefore

$$\mathcal{I}_1 \cap \mathcal{I}_2 \cap (1, 2) = \left(1, 2\frac{N+b}{N-2}\right) \cap \left(2\frac{N+b}{N-2}, +\infty\right) \cap (1, 2) = \left(2\frac{N+b}{N-2}, 2\frac{N+b_0}{N-2}\right) = (\underline{q}, \bar{q}).$$

But if we take $(a, b) \in \mathcal{A}_2$ and $(a_0, b_0) \in \mathcal{B}_1$ with $a_0 < -(2N-2)$ and $-\frac{N-2}{2} < b < b_0 < -2$, then we have (14) as before and therefore $\mathcal{I}_1 \cap \mathcal{I}_2 \cap (1, 2) = (1, 2)$ strictly contains $(\underline{q}, \bar{q}) = (2\frac{N+b}{N-2}, 2\frac{N+b_0}{N-2})$.

This means that there are potentials for which Theorem 3 applies but Theorem 6 gives a wider range of power type nonlinearities for which Eq. (1) admits solutions, and exactly those potentials V, K satisfying $(\mathbf{V}), (\mathbf{K})$ with a, b, a_0, b_0 such that $\mathcal{I}_1 \cap \mathcal{I}_2 \cap (1, 2) \setminus (\underline{q}, \bar{q})$ is nonempty. We leave to the interested reader the explicit description of the set $\mathcal{I}_1 \cap \mathcal{I}_2 \cap (1, 2) \setminus (\underline{q}, \bar{q})$, as a, b, a_0, b_0 vary.

- If a, b, a_0, b_0 are such that Theorem 6 applies with $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$, then one can check that Theorem 3 does not apply, so that Theorem 6 gives new existence results, concerning double power nonlinearities that satisfy the growth condition (\mathbf{f}_7) with $q_1 \neq q_2$. The explicit description of the set of the potential rates $a_0, a \in \mathbb{R}, b_0 > \min\{a_0, -\frac{N-a_0}{2}, -\frac{N+2}{2}\}$ and $b < \max\{a, -2\}$ for which $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$ is left to the interested reader.

Theorems 4 and 6 will be proved in Section 2 by variational methods, as a first application of the compactness results of [7]. Other applications will be given in [8], where Eq. (1) will be studied with more general potentials (not necessarily continuous and possibly not satisfying power type estimates at the origin and at infinity) and nonlinearities (e.g., the presence of an additional forcing term is considered), also dealing with the case of bounded and exterior domains. A version of Theorem 4 without the *Ambrosetti-Rabinowitz condition* (\mathbf{f}_4) will be given in [18].

The proof of Theorems 4 and 6 will be achieved here by the same techniques used in [22, 21] for proving Theorems 1 and 3, namely, respectively, the application of the Mountain Pass Theorem and the global minimization on $H_{V,r}^1$ of the Euler functional associated to the equation. The main difference between our arguments and the ones of [22, 21] is that the single power growth assumption required on the nonlinearity in Theorems 1 and 3 only allows to exploit the compact embedding of $H_{V,r}^1$ into the weighted Lebesgue space $L_K^q := L^q(\mathbb{R}^N, K(|x|) dx)$, while the double power growth assumptions (\mathbf{f}_3) and (\mathbf{f}_7) allow us to use the more general compact embedding [7] of $H_{V,r}^1$ into the sum space $L_K^{q_1} + L_K^{q_2}$ (see Section 2 for some recollections on such a space). The fact that, for $q_1 = q_2 = q$, the space $L_K^{q_1} + L_K^{q_2}$ becomes L_K^q and the double power growth assumption becomes the single power one reflects on a technical level the already discussed fact that Theorems 4 and 6 contain Theorems 1 and 3 and extend them to a wider class of potentials, avoiding any compatibility requirement between their behaviours at the origin and at infinity.

For all the considerations expounded in this introduction, we believe that the double power growth assumption and the related sum space $L_K^{q_1} + L_K^{q_2}$ are the “right” tools for studying problems like (1), i.e., problems on the whole space in which some weights are present and both their behaviours at zero and at infinity affect the solutions (for a different use of the sum of Lebesgue spaces in nonlinear problems, see [4, 25]).

We conclude the section with some remarks and examples of nonlinearities satisfying our assumptions.

Remark 7.

1. Under the same assumptions of Theorem 4, if f is also odd and satisfies

$$\inf_{t>0} \frac{f(t)}{\min\{t^{q_1-1}, t^{q_2-1}\}} > 0 \quad (15)$$

(with the same exponents of hypothesis (\mathbf{f}_3)), then Eq. (1) has infinitely many radial solutions. Similarly, if the same assumptions of Theorem 6 hold and if f is also odd, then Eq. (1) has infinitely many radial solutions. These results rely on the variational theory of symmetric functionals and we refer the reader to the analogous results of [8] for a detailed proof.

2. The solutions found in both Theorems 4 and 6 also satisfy (2) for all $h \in H_V^1$, since, under the hypotheses of the theorems, the symmetric criticality type results of [8] apply.
3. The continuity of V and K is not essential to Theorems 4 and 6, and may be replaced by some weaker integrability assumptions. We refer the interested reader again to [8] for a generalization of Theorems 4 and 6 in this direction.

Example 8. The more obvious function with a double power growth is $f(t) = \min\{|t|^{q_1-2}t, |t|^{q_2-2}t\}$, which also satisfies (\mathbf{f}_4) (with $\theta = \min\{q_1, q_2\}$) if $q_1, q_2 > 2$, and (\mathbf{f}_8) (with $\theta = \max\{q_1, q_2\}$) if $q_1, q_2 < 2$. Another model example is

$$f(t) = \frac{|t|^{q_2-2}t}{1 + |t|^{q_2-q_1}} \quad \text{with } q_1 \leq q_2,$$

for which (\mathbf{f}_4) holds (with $\theta = q_1$) if $q_1 > 2$ and (\mathbf{f}_8) holds (with $\theta = q_2$) if $q_2 < 2$. Note that both these functions are odd and also satisfy (15). Moreover, both of them become $f(t) = |t|^{q-2}t$ if $q_1 = q_2 = q$. Other examples of nonlinearities satisfying $\sup_{t>0} |f(t)| / \min\{t^{q_1-1}, t^{q_2-1}\} < +\infty$ are

$$f(t) = \frac{|t|^{q_1+q-1} - |t|^{q_2-1}}{1 + |t|^q}, \quad f(t) = \frac{|t|^{q_2-1+\varepsilon}}{1 + |t|^{q_2-q_1+2\varepsilon}} \ln |t|$$

(the latter extended at 0 by continuity) with $1 < q_1 \leq q_2 < q_1 + q$ and $\varepsilon > 0$, which do not satisfy (\mathbf{f}_4) or (\mathbf{f}_8) , but satisfy (\mathbf{f}_6) if $q_1 > 2$ and ε is small enough (precisely: $\varepsilon < q_1 - 2$).

2 Proof of Theorems 4 and 6

Let $N \geq 3$ and let V, K be as in $(\mathbf{V}), (\mathbf{K})$ with $a_0, a, b \in \mathbb{R}$ and $b_0 > b_*(a_0)$. Recall the definition (3) of $H_{V,r}^1$, which is a Hilbert space with respect the following inner product and related norm:

$$(u | v) := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^N} V(|x|) uv \, dx, \quad \|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} V(|x|) u^2 \, dx \right)^{1/2}. \quad (16)$$

Denote by $L_K^q(\mathbb{R}^N) := L^q(\mathbb{R}^N, K(|x|) \, dx)$ the usual Lebesgue space with respect to the measure $K(|x|) \, dx$ (dx stands for the Lebesgue measure on \mathbb{R}^N) and consider the sum space

$$L_K^{q_1} + L_K^{q_2} := \{u_1 + u_2 : u_1 \in L_K^{q_1}(\mathbb{R}^N), u_2 \in L_K^{q_2}(\mathbb{R}^N)\}, \quad 1 < q_i < \infty.$$

From [9], we recall that such a space is a Banach space with respect to the norm

$$\|u\|_{L_K^{q_1} + L_K^{q_2}} := \inf_{u_1 + u_2 = u} \max \left\{ \|u_1\|_{L_K^{q_1}(\mathbb{R}^N)}, \|u_2\|_{L_K^{q_2}(\mathbb{R}^N)} \right\}$$

and can be characterized as the set of the measurable mappings $u : \mathbb{R}^N \rightarrow \mathbb{R}$ for which there exists a measurable set $E \subseteq \mathbb{R}^N$ such that $u \in L_K^{q_1}(E) \cap L_K^{q_2}(E^c)$.

Recall the definitions (7) and (8) of the intervals $\mathcal{I}_1 = \mathcal{I}_1(a_0, b_0)$ and $\mathcal{I}_2 = \mathcal{I}_2(a, b)$.

Lemma 9. *For every $q_1 \in \mathcal{I}_1$ and $q_2 \in \mathcal{I}_2$ one has $\lim_{R \rightarrow 0^+} \mathcal{S}_1(R) = \lim_{R \rightarrow +\infty} \mathcal{S}_2(R) = 0$, where*

$$\mathcal{S}_1(R) := \sup_{u \in H_{V,r}^1, \|u\|=1} \int_{B_R} K(|x|) |u|^{q_1} dx, \quad \mathcal{S}_2(R) := \sup_{u \in H_{V,r}^1, \|u\|=1} \int_{\mathbb{R}^N \setminus B_R} K(|x|) |u|^{q_2} dx.$$

Proof. It follows from the results of [7], and precisely from Theorem 4 (apply with $\alpha_\infty = b$, $\beta_\infty = 0$, $\gamma_\infty = -a$ if $a > -2$ and $\gamma_\infty = 2$ if $a \leq -2$) and Theorem 5 (apply with $\alpha_0 = b_0$, $\beta_0 = 0$, $\gamma_0 = -a_0$ if $a_0 < -2$ and $\gamma_0 = 2$ if $a_0 \geq -2$). ■

Lemma 10. *The space $H_{V,r}^1$ is compactly embedded into $L_K^{q_1} + L_K^{q_2}$ for every $q_1 \in \mathcal{I}_1$ and $q_2 \in \mathcal{I}_2$.*

Proof. It readily follows from Lemma 9 above and Theorem 1 of [7]. ■

Now assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function for which there exist $q_1 \in \mathcal{I}_1$, $q_2 \in \mathcal{I}_2$ and $M > 0$ such that

$$|f(t)| \leq M \min \left\{ |t|^{q_1-1}, |t|^{q_2-1} \right\} \quad \text{for all } t \in \mathbb{R}. \quad (17)$$

Set $F(t) := \int_0^t f(s) ds$ and define the functional

$$I(u) := \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} K(|x|) F(u) dx \quad \text{for every } u \in H_{V,r}^1. \quad (18)$$

Lemma 11. *I is a C^1 functional on $H_{V,r}^1$ and its Fréchet derivative $I'(u)$ at any $u \in H_{V,r}^1$ is given by*

$$I'(u)h = \int_{\mathbb{R}^N} \nabla u \cdot \nabla h dx + \int_{\mathbb{R}^N} V(|x|) u h dx - \int_{\mathbb{R}^N} K(|x|) f(u) h dx, \quad \forall h \in H_{V,r}^1.$$

Proof. It follows from Lemma 10 above and the results of [9] about Nemytskiĭ operators on the sum of Lebesgue spaces. Indeed, by [9, Proposition 3.8], condition (17) implies that the functional

$$u \in L_K^{q_1} + L_K^{q_2} \mapsto \int_{\mathbb{R}^N} K(|x|) F(u) dx$$

is of class C^1 with Fréchet derivative at any $u \in L_K^{q_1} + L_K^{q_2}$ is given by

$$h \in L_K^{q_1} + L_K^{q_2} \mapsto \int_{\mathbb{R}^N} K(|x|) f(u) h dx.$$

The result then ensues by the continuous embedding $H_{V,r}^1 \hookrightarrow L_K^{q_1} + L_K^{q_2}$ given by Lemma 10. ■

By Lemma 11, the problem of finding radial solutions to Eq. (1) clearly reduces to the problem of finding critical points of $I : H_{V,r}^1 \rightarrow \mathbb{R}$.

For future reference, we observe here that, by condition (17), there exists $\tilde{M} > 0$ such that

$$|F(t)| \leq \tilde{M} \min \{ |t|^{q_1}, |t|^{q_2} \} \quad \text{for all } t \in \mathbb{R}. \quad (19)$$

Lemma 12. *There exist two constants $c_1, c_2 > 0$ such that*

$$I(u) \geq \frac{1}{2} \|u\|^2 - c_1 \|u\|^{q_1} - c_2 \|u\|^{q_2} \quad \text{for all } u \in H_{V,r}^1. \quad (20)$$

Proof. By Lemma 9, fix $R_2 > R_1 > 0$ such that $\mathcal{S}_1(R_1), \mathcal{S}_2(R_2) < 1$. Then, by [7, Lemma 1] and the continuous embedding $H_{V,r}^1 \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow L_{\text{loc}}^2(\mathbb{R}^N)$, there exists a constant $c_{R_1, R_2} > 0$ such that

$$\int_{B_{R_2} \setminus B_{R_1}} K(|x|) |u|^{q_1} dx \leq c_{R_1, R_2} \|u\|^{q_1} \quad \text{for all } u \in H_{V,r}^1.$$

Therefore, by (19) and the definitions of \mathcal{S}_1 and \mathcal{S}_2 , for every $u \in H_{V,r}^1$ we get

$$\begin{aligned} \left| \int_{\mathbb{R}^N} K(|x|) F(u) dx \right| &\leq \tilde{M} \int_{\mathbb{R}^N} K(|x|) \min\{|u|^{q_1}, |u|^{q_2}\} dx \\ &\leq \tilde{M} \left(\int_{B_{R_1}} K(|x|) |u|^{q_1} dx + \int_{B_{R_2}^c} K(|x|) |u|^{q_2} dx + \int_{B_{R_2} \setminus B_{R_1}} K(|x|) |u|^{q_1} dx \right) \\ &\leq \tilde{M} \left(\|u\|^{q_1} \int_{B_{R_1}} K(|x|) \frac{|u|^{q_1}}{\|u\|^{q_1}} dx + \|u\|^{q_2} \int_{B_{R_2}^c} K(|x|) \frac{|u|^{q_2}}{\|u\|^{q_2}} dx + c_{R_1, R_2} \|u\|^{q_1} \right) \\ &\leq \tilde{M} (\|u\|^{q_1} \mathcal{S}_1(R_1) + \|u\|^{q_2} \mathcal{S}_2(R_2) + c_{R_1, R_2} \|u\|^{q_1}). \end{aligned}$$

This yields (20). ■

Lemma 13. *Assume $f(t) = 0$ for all $t < 0$. If f satisfies (\mathbf{f}_4) , or $K(|\cdot|) \in L^1(\mathbb{R}^N)$ and f satisfies (\mathbf{f}_6) , then the functional $I : H_{V,r}^1 \rightarrow \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\}$ be a sequence in $H_{V,r}^1$ such that $\{I(u_n)\}$ is bounded and $I'(u_n) \rightarrow 0$ in the dual space of $H_{V,r}^1$. Hence

$$\frac{1}{2} \|u_n\|^2 - \int_{\mathbb{R}^N} K(|x|) F(u_n) dx = O(1) \quad \text{and} \quad \|u_n\|^2 - \int_{\mathbb{R}^N} K(|x|) f(u_n) u_n dx = o(1) \|u_n\|.$$

If f satisfies (\mathbf{f}_4) , then we have $\theta F(t) \leq f(t)t$ for all $t \in \mathbb{R}$ (because $f(t) = 0$ for $t < 0$) and therefore we get

$$\frac{1}{2} \|u_n\|^2 + O(1) = \int_{\mathbb{R}^N} K(|x|) F(u_n) dx \leq \frac{1}{\theta} \int_{\mathbb{R}^N} K(|x|) f(u_n) u_n dx = \frac{1}{\theta} \|u_n\|^2 + o(1) \|u_n\|,$$

which implies that $\{\|u_n\|\}$ is bounded, since $\theta > 2$. If $K(|\cdot|) \in L^1(\mathbb{R}^N)$ and f satisfies (\mathbf{f}_6) , then we have $\theta F(t) \leq f(t)t$ for all $|t| \geq t_0$ (because $f(t) = 0$ for $t < 0$) and

$$\begin{aligned} \int_{\{|u_n| \geq t_0\}} K(|x|) f(u_n) u_n dx &= \int_{\mathbb{R}^N} K(|x|) f(u_n) u_n dx - \int_{\{|u_n| < t_0\}} K(|x|) f(u_n) u_n dx \\ &\leq \int_{\mathbb{R}^N} K(|x|) f(u_n) u_n dx + \int_{\{|u_n| < t_0\}} K(|x|) |f(u_n) u_n| dx \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^N} K(|x|) f(u_n) u_n dx + M \int_{\{|u_n| < t_0\}} K(|x|) \min\{|u_n|^{q_1}, |u_n|^{q_2}\} dx \\
&\leq \int_{\mathbb{R}^N} K(|x|) f(u_n) u_n dx + M \min\{t_0^{q_1}, t_0^{q_2}\} \int_{\{|u_n| < t_0\}} K(|x|) dx \\
&\leq \int_{\mathbb{R}^N} K(|x|) f(u_n) u_n dx + M \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)},
\end{aligned}$$

so that, by (19), we get

$$\begin{aligned}
\frac{1}{2} \|u_n\|^2 + O(1) &= \int_{\mathbb{R}^N} K(|x|) F(u_n) dx = \int_{\{|u_n| < t_0\}} K(|x|) F(u_n) dx + \int_{\{|u_n| \geq t_0\}} K(|x|) F(u_n) dx \\
&\leq \tilde{M} \int_{\{|u_n| < t_0\}} K(|x|) \min\{|u_n|^{q_1}, |u_n|^{q_2}\} dx + \frac{1}{\theta} \int_{\{|u_n| \geq t_0\}} K(|x|) f(u_n) u_n dx \\
&\leq \tilde{M} \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)} + \frac{1}{\theta} \int_{\mathbb{R}^N} K(|x|) f(u_n) u_n dx + \frac{M}{\theta} \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)} \\
&= \left(\tilde{M} + \frac{M}{\theta} \right) \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)} + \frac{1}{\theta} \|u_n\|^2 + o(1) \|u_n\|.
\end{aligned}$$

This yields again that $\{\|u_n\|\}$ is bounded. Now, since the embedding $H_{V,r}^1 \hookrightarrow L_K^{q_1} + L_K^{q_2}$ is compact (see Lemma 10) and the functional $u \mapsto \int_{\mathbb{R}^N} K(|x|) F(u) dx$ is of class C^1 on $L_K^{q_1} + L_K^{q_2}$ (see the proof of Lemma 11), it is a standard exercise to conclude that $\{u_n\}$ has a strongly convergent subsequence in $H_{V,r}^1$. ■

We can now conclude the proof of Theorem 4.

Proof of Theorem 4. Assume all the hypotheses of the theorem and assume also that $f(t) = 0$ for all $t < 0$. This additional hypothesis is not restrictive, since the theorem concerns *nonnegative* solutions and all its assumptions still hold true if we replace $f(t)$ with $f(t) \chi_{\mathbb{R}_+}(t)$ (where $\chi_{\mathbb{R}_+}$ is the characteristic function of $\mathbb{R}_+ = (0, +\infty)$).

Thanks to Lemma 11, the theorem is proved if we find a nontrivial nonnegative critical point of $I : H_{V,r}^1 \rightarrow \mathbb{R}$.

To this end, we want to apply the Mountain-Pass Theorem [3]. From (20) of Lemma 12 we deduce that, since $q_1, q_2 > 2$, there exists $\rho > 0$ such that

$$\inf_{u \in H_{V,r}^1, \|u\| = \rho} I(u) > 0 = I(0).$$

Therefore, taking into account Lemmas 11 and 13, the Mountain Pass Theorem applies if we show that $\exists \bar{u} \in H_{V,r}^1$ such that $\|\bar{u}\| > \rho$ and $I(\bar{u}) < 0$. In order to prove this, from condition (\mathbf{f}_6) (which holds in any case, since it also follows from (\mathbf{f}_4) and (\mathbf{f}_5)), we infer that

$$F(t) \geq \frac{F(t_0)}{t_0^\theta} t^\theta \quad \text{for all } t \geq t_0.$$

Then we fix a radial nonnegative function $u_0 \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ such that the set $\{x \in \mathbb{R}^N : u_0(x) \geq t_0\}$ has positive Lebesgue measure. We now distinguish the case of assumptions (\mathbf{f}_4) and (\mathbf{f}_5) from the case

of $K(|\cdot|) \in L^1(\mathbb{R}^N)$. In the first one, we have $F(t) \geq 0$ for all $t \in \mathbb{R}$ (recall that $f(t) = 0$ for $t < 0$) and $F(t_0) > 0$, so that for every $\lambda > 1$ we get

$$\begin{aligned} \int_{\mathbb{R}^N} K(|x|) F(\lambda u_0) dx &\geq \int_{\{\lambda u_0 \geq t_0\}} K(|x|) F(\lambda u_0) dx \geq \lambda^\theta \frac{F(t_0)}{t_0^\theta} \int_{\{\lambda u_0 \geq t_0\}} K(|x|) u_0^\theta dx \\ &\geq \lambda^\theta \frac{F(t_0)}{t_0^\theta} \int_{\{u_0 \geq t_0\}} K(|x|) u_0^\theta dx \geq \lambda^\theta F(t_0) \int_{\{u_0 \geq t_0\}} K(|x|) dx > 0. \end{aligned}$$

Since $\theta > 2$, this gives

$$\lim_{\lambda \rightarrow +\infty} I(\lambda u_0) \leq \lim_{\lambda \rightarrow +\infty} \left(\frac{\lambda^2}{2} \|u_0\|^2 - \lambda^\theta F(t_0) \int_{\{u_0 \geq t_0\}} K(|x|) dx \right) = -\infty. \quad (21)$$

If $K(|\cdot|) \in L^1(\mathbb{R}^N)$, assumption (\mathbf{f}_6) still gives $F(t_0) > 0$ and from (19) we infer that

$$F(t) \geq -\tilde{M} \min\{t_0^{q_1}, t_0^{q_2}\} \quad \text{for all } 0 \leq t \leq t_0.$$

Therefore, arguing as above about the integral over $\{\lambda u_0 \geq t_0\}$, for every $\lambda > 1$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} K(|x|) F(\lambda u_0) dx &= \int_{\{\lambda u_0 < t_0\}} K(|x|) F(\lambda u_0) dx + \int_{\{\lambda u_0 \geq t_0\}} K(|x|) F(\lambda u_0) dx \\ &\geq -\tilde{M} \min\{t_0^{q_1}, t_0^{q_2}\} \int_{\{\lambda u_0 < t_0\}} K(|x|) dx + \lambda^\theta F(t_0) \int_{\{u_0 \geq t_0\}} K(|x|) dx, \end{aligned}$$

which implies

$$\lim_{\lambda \rightarrow +\infty} I(\lambda u_0) \leq \lim_{\lambda \rightarrow +\infty} \left(\frac{\lambda^2}{2} \|u_0\|^2 + \tilde{M} \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)} - \lambda^\theta F(t_0) \int_{\{u_0 \geq t_0\}} K(|x|) dx \right) = -\infty.$$

So, in any case, we can take $\bar{u} = \lambda u_0$ with λ sufficiently large and the Mountain-Pass Theorem provides the existence of a nontrivial critical point $u \in H_{V,r}^1$ for I . Since $f(t) = 0$ for $t < 0$ implies $I'(u)u_- = -\|u_-\|^2$ (where $u_- \in H_{V,r}^1$ is the negative part of u), one concludes that $u_- = 0$, i.e., u is nonnegative. ■

For concluding also the proof of Theorem 6, we prove one more lemma.

Lemma 14. *If (\mathbf{f}_8) holds, then the functional $I : H_{V,r}^1 \rightarrow \mathbb{R}$ takes negative values.*

Proof. By assumption (\mathbf{f}_8) , fix $m > 0$ and $t_0 > 0$ such that $F(t) \geq mt^\theta$ for all $0 \leq t \leq t_0$. Fix a nonzero radial function $u_0 \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ such that $0 \leq u_0 \leq t_0$. Then, for every $0 < \lambda < 1$ we get that $\lambda u_0 \in H_{V,r}^1$ satisfies $0 \leq \lambda u_0 \leq t_0$ and therefore

$$I(\lambda u_0) = \frac{1}{2} \|\lambda u_0\|^2 - \int_{\mathbb{R}^N} K(|x|) F(\lambda u_0) dx \leq \frac{\lambda^2}{2} \|u_0\|^2 - \lambda^\theta m \int_{\mathbb{R}^N} K(|x|) u_0^\theta dx,$$

where $\int_{\mathbb{R}^N} K(|x|) u_0^\theta dx > 0$ (recall that $K > 0$ everywhere). Since $\theta < 2$, this implies $I(\lambda u_0) < 0$ for λ sufficiently small. ■

Proof of Theorem 6. Assume all the hypotheses of the theorem and assume also that f is odd. This additional hypothesis is not restrictive, since the theorem concerns *nonnegative* solutions and all its assumptions still hold true if we replace $f(t)$ with $f(|t|) \operatorname{sgn}(t)$ (where sgn is the sign function).

Since $q_1, q_2 \in (1, 2)$, the inequality (20) of Lemma 12 readily implies that the functional $I : H_{V,r}^1 \rightarrow \mathbb{R}$ is bounded from below and coercive, so that

$$\mu := \inf_{u \in H_{V,r}^1} I(u)$$

is a finite value. Therefore, thanks to Lemma 11, the theorem is proved if we show that μ is attained by a nonnegative minimizer, which cannot be trivial, since $I(0) = 0$ and $\mu < 0$, by Lemma 14.

To this end, let $\{u_n\}$ be any minimizing sequence for μ . Since f is odd, $I(u)$ is even and therefore $\{|u_n|\}$ is still a minimizing sequence, so that, up to replacing u_n with $|u_n|$, we may assume $u_n \geq 0$. Since $\{u_n\}$ is bounded in $H_{V,r}^1$ (by the coercivity of I) and the embedding $H_{V,r}^1 \hookrightarrow L_K^{q_1} + L_K^{q_2}$ is compact (by Lemma 10), up to a subsequence we can assume that there exists $u \in H_{V,r}^1$ such that:

$$u_n \rightharpoonup u \quad \text{in } H_{V,r}^1, \quad u_n \rightarrow u \quad \text{in } L_K^{q_1} + L_K^{q_2}, \quad u_n \rightarrow u \quad \text{almost everywhere in } \mathbb{R}^N$$

(the almost everywhere convergence follows, for instance, from the continuous embedding $H_{V,r}^1 \hookrightarrow D^{1,2}(\mathbb{R}^N)$ and the fact that, up to a subsequence, weak convergence in $D^{1,2}(\mathbb{R}^N)$ implies almost everywhere convergence). Then $u_n \geq 0$ implies $u \geq 0$ and, thanks to the weak lower semi-continuity of the norm and to the continuity of the functional $v \mapsto \int_{\mathbb{R}^N} K(|x|) F(v) dx$ on $L_K^{q_1} + L_K^{q_2}$ (see the proof of Lemma 11 above), u satisfies

$$\|u\|^2 \leq \liminf_{n \rightarrow \infty} \|u_n\|^2 \quad \text{and} \quad \int_{\mathbb{R}^N} K(|x|) F(u) dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(|x|) F(u_n) dx.$$

This implies

$$I(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} K(|x|) F(u) dx \leq \lim_{n \rightarrow \infty} \left(\frac{1}{2} \|u_n\|^2 - \int_{\mathbb{R}^N} K(|x|) F(u_n) dx \right) = \mu$$

and therefore we conclude $I(u) = \mu$. ■

3 Existence of a ground state

In this section we give a version of Theorem 4 which ensures the existence of a *radial ground state* of Eq. (1), by which, assuming that the Euler functional I defined in (18) is of class C^1 on $H_{V,r}^1$ (as in Lemma 11), we mean a radial solution $u \neq 0$ such that

$$I(u) = \min_{v \in \mathcal{N}} I(v) \quad \text{where} \quad \mathcal{N} := \{v \in H_{V,r}^1 \setminus \{0\} : I'(v)v = 0\}$$

(\mathcal{N} is the Nehari manifold). As I is often called the “action” or “energy” functional associated to the equation, a radial ground state u is in fact a *least action* or *least energy* solution (among the nontrivial radial ones), since every radial solution $v \neq 0$ belongs to \mathcal{N} and therefore $I(u) \leq I(v)$. Of course, the solution found in Theorem 6 is itself a radial ground state, since it is a global minimizer of I on $H_{V,r}^1$.

The result we will prove is the following theorem. Observe that all its assumptions are satisfied by both the first two nonlinearities of Example 8, with $q_1, q_2 > 2$, $q_1 \in \mathcal{I}_1$, $q_2 \in \mathcal{I}_2$.

Theorem 15. *Under the same assumptions of the first part of Theorem 4 (i.e., the part with (\mathbf{f}_3) - (\mathbf{f}_5)), if f also satisfies*

$$(\mathbf{f}_9) \quad \frac{f(t)}{t} \text{ is a strictly increasing function on } (0, +\infty),$$

then Eq. (1) has a nonnegative radial ground state.

In proving Theorem 15, we will use the following variant of the Mountain Pass Theorem and an adaptation of well known arguments involving the Nehari manifold (see e.g. [23, Chapter 4]).

Lemma 16. *Let X be a real Banach space and let $J \in C^1(X; \mathbb{R})$. Assume that there exist $\rho > 0$ and $\bar{v} \in X$ such that*

$$\inf_{v \in X, \|v\|_X = \rho} J(v) > \inf_{v \in X, \|v\|_X \leq \rho} J(v) = J(0) = 0 > J(\bar{v}) \quad \text{and} \quad \|\bar{v}\|_X > \rho.$$

If J satisfies the Palais-Smale condition, then the minimax level

$$c := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J(\gamma(t)) > 0, \quad \Gamma := \{\gamma \in C([0, 1]; X) : \gamma(0) = 0, J(\gamma(1)) < 0\}, \quad (22)$$

is a critical value for J .

Proof. It follows for instance from [23, Theorem 2.9] (apply with $M = [0, 1]$, $M_0 = \{0, 1\}$ and $\Gamma_0 = \{\gamma \in C(M_0; X) : \gamma(0) = 0, J(\gamma(1)) < 0\}$). ■

Proof of Theorem 15. Assume all the hypotheses of the theorem. As in the proof of Theorem 4, we assume additionally that $f(t) = 0$ for all $t < 0$ and we deduce that I satisfies the assumptions of Lemma 16 (with $X = H_{V,r}^1$ and $J = I$), so that there exists $u \in H_{V,r}^1$ such that $I'(u) = 0$, $u \geq 0$ and $I(u) = c > 0$, where c is the minimax level (22). Since $u \neq 0$ is a critical point for I , we have that $u \in \mathcal{N}$ and therefore $c = I(u) \geq \nu := \inf_{v \in \mathcal{N}} I(v)$. Hence the theorem is proved if we show that $\nu \geq c$, which implies $I(u) = \nu$ with $u \in \mathcal{N}$.

To this end, we take any $v \in \mathcal{N}$ and first observe that it cannot be $v \leq 0$ almost everywhere. Otherwise, since $v \in \mathcal{N}$ implies $\|v\|^2 = \int_{\mathbb{R}^N} K(|x|) f(v) v dx$ (recall Lemma 11) and we have $f(t) = 0$ for $t \leq 0$, we would get the contradiction $\|v\| = 0$.

Then we show that $I(v) = \max_{t \geq 0} I(tv) > 0$. For this, we define $g_v(t) := I(tv)$, $t \geq 0$, and argue by the following three steps.

- $t = 1$ is a critical point for g_v , since $g'_v(t) = I'(tv)v$ for all t and $I'(v)v = 0$.
- $t = 1$ is the only critical point of g_v on $(0, +\infty)$. Indeed, if $t_2 > t_1 > 0$ are critical points for g_v , then we have $I'(t_1v)v = I'(t_2v)v = 0$, i.e.,

$$t_1 \|v\|^2 - \int_{\mathbb{R}^N} K(|x|) f(t_1v) v dx = t_2 \|v\|^2 - \int_{\mathbb{R}^N} K(|x|) f(t_2v) v dx = 0$$

(recall Lemma 11), which implies

$$0 = \int_{\mathbb{R}^N} K(|x|) \left(\frac{f(t_2v)}{t_2} - \frac{f(t_1v)}{t_1} \right) v dx = \int_{\{v>0\}} K(|x|) \left(\frac{f(t_2v)}{t_2v} - \frac{f(t_1v)}{t_1v} \right) v^2 dx$$

(recall that $f(t) = 0$ for $t \leq 0$). The last integrand is nonnegative by assumption (\mathbf{f}_9) and therefore we get $f(t_2v)/(t_2v) = f(t_1v)/(t_1v)$ almost everywhere on $\{x : v(x) > 0\}$ (because $K(|x|), v^2(x) > 0$ and the set $\{x : v(x) > 0\}$ has positive measure). Again by assumption (\mathbf{f}_9) , this implies $t_2v = t_1v$ almost everywhere on $\{x : v(x) > 0\}$ and therefore we conclude $t_2 = t_1$.

- g_v has a maximum point on $(0, +\infty)$, in which $g_v > 0$. Indeed, letting $\delta > 0$ be such that the set $\{x : v(x) \geq \delta\}$ has positive measure (which exists because $\{x : v(x) > 0\}$ has positive measure), we have $F(\delta) > 0$ (because (\mathbf{f}_4) , (\mathbf{f}_9) and $f(0) = 0$ imply $F(t) > 0$ for $t > 0$) and from (\mathbf{f}_4) we deduce that $F(t) \geq F(\delta)\delta^{-\theta}t^\theta$ for all $t \geq \delta$, so that, arguing as for (21), we get

$$\lim_{t \rightarrow +\infty} I(tv) \leq \lim_{t \rightarrow +\infty} \left(\frac{t^2}{2} \|v\|^2 - t^\theta F(\delta) \int_{\{v(x) \geq \delta\}} K(|x|) dx \right) = -\infty. \quad (23)$$

This, together with $g_v(0) = 0$ and $g_v(t) > 0$ for $t > 0$ small enough (which follows from (20) of Lemma 12, where $q_1, q_2 > 2$), yields the claim.

As a result, the maximum point of g_v on $(0, +\infty)$ must be the unique critical point $t = 1$ and we conclude that $I(v) = \max_{t \geq 0} I(tv) > 0$.

Now, using (23) again, we observe that there exists $t_v > 0$ satisfying $I(tv) < 0$ for all $t \geq t_v$, so that the path $\gamma_v(t) := tt_v v$, $t \in [0, 1]$, is such that $\gamma_v \in \Gamma$ and

$$\max_{0 \leq t \leq 1} I(\gamma_v(t)) = \max_{0 \leq t \leq 1} I(tt_v v) = \max_{0 \leq t \leq t_v} I(tv) = \max_{t \geq 0} I(tv) = I(v).$$

Hence

$$I(v) = \max_{0 \leq t \leq 1} I(\gamma_v(t)) \geq \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)) = c$$

and therefore, as $v \in \mathcal{N}$ is arbitrary, we get $\nu \geq c$. ■

References

- [1] ALVES C.O., SOUTO M.A.S., *Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity*, J. Differential Equations **254** (2013), 1977-1991.
- [2] AMBROSETTI A., FELLI V., MALCHIODI A., *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Eur. Math. Soc. **7** (2005), 117-144.
- [3] AMBROSETTI A., RABINOWITZ P.H., *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349-381.
- [4] AZZOLLINI A., D'AVENIA P., POMPONIO A., *Quasilinear elliptic equations in \mathbb{R}^N via variational methods and Orlicz-Sobolev embeddings*, Calc. Var. Partial Differential Equations **49** (2014), 197-213.
- [5] BADIALE M., BENCI V., ROLANDO S., *Solitary waves: physical aspects and mathematical results*, Rend. Sem. Math. Univ. Pol. Torino **62** (2004), 107-154.
- [6] BADIALE M., GUIDA M., ROLANDO S., *A nonexistence result for a nonlinear elliptic equation with singular and decaying potential*, Commun. Contemp. Math., in press (DOI: 10.1142/S0219199714500242).

- [7] BADIALE M., GUIDA M., ROLANDO S., *Compactness and existence results in weighted Sobolev spaces of radial functions. Part I: Compactness*, Cal. Var. Partial Differential Equations, in press (DOI: 10.1007/s00526-015-0817-2).
- [8] BADIALE M., GUIDA M., ROLANDO S., *Compactness and existence results in weighted Sobolev spaces of radial functions. Part II: Existence*, in preparation.
- [9] BADIALE M., PISANI L., ROLANDO S., *Sum of weighted Lebesgue spaces and nonlinear elliptic equations*, NoDEA, Nonlinear Differ. Equ. Appl. **18** (2011), 369-405.
- [10] BADIALE M., ROLANDO S., *Elliptic problems with singular potentials and double-power nonlinearity*, Mediterr. J. Math. **2** (2005), 417-436.
- [11] BADIALE M., ROLANDO S., *Nonlinear elliptic equations with subhomogeneous potentials*, Nonlinear Anal. **72** (2010), 602-617.
- [12] BENCI V., FORTUNATO D., *Variational methods in nonlinear field equations. Solitary waves, hydro-morphic solitons and vortices*, Springer Monographs in Mathematics, Springer, Cham, 2014.
- [13] BENCI V., GRISANTI C.R., MICHELETTI A.M., *Existence and non existence of the ground state solution for the nonlinear Schrödinger equation with $V(\infty) = 0$* , Topol. Methods Nonlinear Anal. **26** (2005), 203-220.
- [14] BENCI V., GRISANTI C.R., MICHELETTI A.M., *Existence of solutions for the nonlinear Schrödinger equation with $V(\infty) = 0$* , Contributions to nonlinear analysis, Progr. Nonlinear Differential Equations Appl., vol. 66, Birkhäuser, Basel, 2006.
- [15] BERESTYCKI H., LIONS P.L., *Nonlinear Scalar Field Equations, I - II*, Arch. Rational Mech. Anal. **82** (1983), 313-379.
- [16] CATRINA F., *Nonexistence of positive radial solutions for a problem with singular potential*, Adv. Nonlinear Anal. **3** (2014), 1-13.
- [17] FLOER A., WEINSTEIN A., *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal. **69** (1986), 397-408.
- [18] GUIDA M., ROLANDO S., *On the existence of bounded Palais-Smale sequences and applications to nonlinear equations without superlinearity assumptions*, in preparation.
- [19] RABINOWITZ P.H., *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270-291.
- [20] STRAUSS W.A., *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), 149-172.
- [21] SU J., TIAN R., *Weighted Sobolev type embeddings and coercive quasilinear elliptic equations on \mathbb{R}^N* , Proc. Amer. Math. Soc. **140** (2012), 891-903.
- [22] SU J., WANG Z.-Q., WILLEM M., *Weighted Sobolev embedding with unbounded and decaying radial potentials*, J. Differential Equations **238** (2007), 201-219.

- [23] WILLEM M., *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, vol. 24, Birkhäuser, Boston, 1996.
- [24] YANG Y., *Solitons in field theory and nonlinear analysis*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2001.
- [25] ZHANG G., FU H., *Ground states for a modified capillary surface equation in weighted Orlicz-Sobolev space*, Electron. J. Differential Equations **2015** (2015), No. 85, 1-18.