THE COXETER COMPLEX AND THE EULER CHARACTERISTIC OF A HECKE ALGEBRA

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Abstract. For any Hecke algebra $\mathcal{H} = \mathcal{H}_q(W,S)$ associated to a Coxeter group (W,S) and a distinguished element $q \in R$ of a commutative ring with unit R we introduce a finite chain complex of left $\mathcal{H}$-modules $(C_\bullet, \partial_\bullet)$ which reflects many properties of the Coxeter complex of (W,S), i.e., it is acyclic if (W,S) is non-spherical (cf. Thm. A), and $\mathcal{H}$ is of type FP under suitable conditions on the distinguished element $q \in R$ (cf. Prop. B). There exists a canonical trace function $\tilde{\mu}\colon \mathcal{H} \to R$ (cf. Prop. 5.1). This trace function $\tilde{\mu}$ evaluated on the Hattori-Stallings rank of $(C_\bullet, \partial_\bullet)$ can be considered as the Euler characteristic $\chi_{\mathcal{H}}$ of $\mathcal{H}$. It will be shown that for generic values of q the Euler characteristic coincides with the reciprocal of the Poincaré series of (W,S) evaluated in q (cf. Thm. C).

1. Introduction

For any commutative ring R with unit, and any distinguished element $q \in R$ one may define an *R-Hecke algebra* $\mathcal{H} = \mathcal{H}_q(W,S)$ associated to any *Coxeter group* (W,S). This algebra can be seen as a deformation of the R-group algebra of the Coxeter group (W,S). It particular, it comes equipped with an *antipodal map* $\_^{\natural}\colon \mathcal{H} \to \mathcal{H}^{\mathrm{op}}$, an *augmentation* $\varepsilon_q\colon \mathcal{H} \to R$, and an R-basis $\mathcal{B} = \{\, T_w \mid w \in W\,\}$. Moreover, $\varepsilon_q(T_w) = q^{\ell(w)}$ for $w \in W$, where $\ell\colon W \to \mathbb{N}_0$ denotes the *length function* on (W,S). The *Poincaré series* of (W,S) is given by

$$p_{(W,S)}(t) = \sum_{w \in W} t^{\ell(w)} \in \mathbb{Z}[\![t]\!]. \tag{1.1}$$

It is well known (cf. [4, Chap. IV, §1, Ex. 25 and 26]) that $p_{(W,S)}(t)$ is a rational function in t. Moreover, if (W,S) is *spherical* then $p_{(W,S)}(t) \in \mathbb{Z}[t]$ is just a polynomial with integer coefficients. The left $\mathcal{H}$-module R_q, which is as R-module isomorphic to R and which action is given by $h.r = \varepsilon_q(h)r$ for $h \in \mathcal{H}$, $r \in R_q$, can be seen as the trivial $\mathcal{H}$-module.

The main purpose of this paper is to introduce a chain complex of left $\mathcal{H}$-modules $C = (C_\bullet, \partial_\bullet)$ concentrated in degrees 0 to $|S|-1$, which can be seen as the module theoretic analogue of the *Coxeter complex* associated to (W,S) (cf. [1, Chap. 3]). It is canonical up to the choice of a total ordering of the finite set S. The most significant properties of the chain complex C can be summarized as follows (cf. §3).

Theorem A. *Let (W,S) be a Coxeter group with $2 \le |S| < \infty$, and let C be the Coxeter complex of the R-Hecke algebra $\mathcal{H} = \mathcal{H}_q(W,S)$.*

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(a) *If* (W,S) *is spherical, then* $H_k(C) = 0$ *unless* $k = 0$ *or* $k = |S| - 1$. *Moreover,* $H_0(C) \simeq R_q$ *and* $H_{|S|-1}(C) \simeq R_{-1}$.
(b) *If* (W,S) *is non-spherical then* C *is acyclic with* $H_0(C) \simeq R_q$.

A left $\mathbf{A}$-module M of an associative R-algebra $\mathbf{A}$ is called to be *of type* FP, if it has a finite, projective resolution $(P_\bullet, \partial^P_\bullet, \varepsilon_M)$. From Theorem A one may deduce a sufficient criterion on the distinguished element $q \in R$ ensuring the FP-property of the trivial $\mathcal{H}$-module R_q (cf. Prop. 5.4).

Proposition B. *Suppose that for the distinguished element* $q \in R$ *one has*

$$p_{(W_I,I)}(q) \in R^\times \tag{1.2}$$

for every spherical parabolic subgroup (W_I, I) *of* (W,S). *Then* R_q *is of type* FP.

Here $R^\times \subset R$ denotes the group of invertible elements in R. If $q \in R$ satisfies (1.2), one may define the *Hattori–Stallings rank* r_{R_q} of the trivial left $\mathcal{H}$-module $R_q \in \mathcal{H}/[\mathcal{H},\mathcal{H}]$ by a standard procedure (cf. [5, Chap. IX]). Moreover, every R-Hecke algebra $\mathcal{H}$ has a canonical trace function $\tilde{\mu}\colon \mathcal{H} \to R$ (cf. Prop. 5.1). We will define the *Euler characteristic* of $\mathcal{H}$ by $\chi_{\mathcal{H}} = \mu(r_{R_q})$, where μ is the induced map on $\mathcal{H}/[\mathcal{H},\mathcal{H}]$. For generic values of q the Euler characteristic of $\mathcal{H}$ can be computed explicitly (cf. §5).

Theorem C. *If* $q \in R$ *satisfies* (1.2)*, then* $p_{(W,S)}(q) \in R^\times$ *and*

$$\chi_{\mathcal{H}} = p_{(W,S)}(q)^{-1}. \tag{1.3}$$

It might look surprising that the Poincaré series of a Coxeter group can be recovered from the representation theory of the associated Hecke algebra. On the other hand the alternating minus signs in the formula which is usually used to calculate the series explicitly (cf. [8, §5.12]) suggest that its reciprocal value might be an Euler characteristic of something. In the case that $R = R_\circ[\![q]\!]$ for some commutative ring with unit $R_\circ$ there is another interesting phenomenon. Obviously, $q \in R$ satisfies (1.2), and $p_{(W,S)}(q)$ can be rewritten as

$$p_{\mathcal{H}} = p_{(W,S)}(q) = \sum_{T_w \in \mathcal{B}} \varepsilon_q(T_w) \in R \tag{1.4}$$

interpreting $p_{\mathcal{H}}$ as a series associated to $(\mathcal{H}, \varepsilon_q, \mathcal{B})$. Then one has the identity $p_{\mathcal{H}} \cdot \chi_{\mathcal{H}} = 1$. In fact, a similar identity is known for a *Koszul algebra* $\mathbf{A}_\bullet$ defined over a field F, i.e., in this case one has

$$h_{\mathbf{A}_\bullet}(t) \cdot h_{H^{\bullet,\bullet}(A_\bullet,F)}(-t) = 1 \tag{1.5}$$

where $h_{\mathbf{A}_\bullet}(t)$ (resp. $h_{H^{\bullet,\bullet}(A_\bullet,F)}(t)$) denotes the Hilbert series of the graded F-algebra $\mathbf{A}_\bullet$ (resp. $H^{\bullet,\bullet}(A_\bullet,F)$) (cf. [9, §2, p. 22, Cor. 2.2]). It would be interesting to know whether there exist other types of generic $R_\circ[\![q]\!]$-algebras $(\mathbf{A}, \_^\natural, \varepsilon, \mathcal{B})$ satisfying the identity $p_{\mathbf{A}} \cdot \chi_{\mathbf{A}} = 1$.

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2. Coxeter groups and Hecke algebras

2.1. **Coxeter groups.** A *Coxeter graph* Γ is a finite combinatorial graph[1] with non-oriented edges $\mathfrak{e}$ labelled by positive integers $m(\mathfrak{e}) \geq 3$ or infinity. The *Coxeter group* (W, S) associated to Γ consists of the group W generated by the set of involutions $S = \{ s_v \mid v \in \mathfrak{V}(\Gamma) \}$ subject to the relations $(s_v s_w)^{m(\mathfrak{e})} = 1$, where $\mathfrak{e} = \{v, w\} \in \mathfrak{E}(\Gamma)$ is an edge of label $m(\mathfrak{e}) < \infty$, and the commutation relations $s_v s_w = s_w s_v$ whenever $\{v, w\} \notin \mathfrak{E}(\Gamma)$. The *length function* on W with respect to S will be denoted by $\ell \colon W \to \mathbb{N}_0$. Since $S = S^{-1}$ is a set of involutions, $\ell(w) = \ell(w^{-1})$, and it is well known that a longest element $w_0 \in W$ exists if, and only if, W is finite. In this case it is unique and has the property that $\ell(w_0 x) = \ell(w_0) - \ell(x)$ for all $x \in W$. A Coxeter group which is finite is called *spherical*, and *non-spherical* otherwise.

For a subset $I \subseteq S$ let W_I be the corresponding parabolic subgroup, i.e., W_I is the subgroup of W generated by I. It is isomorphic to the Coxeter group associated to the Coxeter subgraph Γ' based on the vertices $\{ v \in \mathfrak{V}(\Gamma) \mid s_v \in I \}$. The length function of W restricted to W_I coincides with the intrinsic length function of the Coxeter group (W_I, I). Put

$$W^I = \{ w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in I \}, \tag{2.1}$$

and let ${}^I W = (W^I)^{-1}$, i.e.,

$${}^I W = \{ w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in I \}. \tag{2.2}$$

For $w \in W$ the *right ascent set* is given by

$$A^\rho(w) = \{ s \in S \mid \ell(ws) > \ell(w) \}. \tag{2.3}$$

One has the following properties (cf. [8, §5.12]).

Proposition 2.1. *Let (W, S) be a Coxeter group, let $w \in W$ and let $I \subseteq S$.*

(a) *There exist a unique element $w_I \in W_I$ and a unique element $w^I \in W^I$ such that $w = w^I w_I$. Moreover, $\ell(w) = \ell(w^I) + \ell(w_I)$.*
(b) *There exist a unique element ${}_I w \in W_I$ and a unique element ${}^I w \in {}^I W$ such that $w = {}_I w\, {}^I w$. Moreover, $\ell(w) = \ell({}_I w) + \ell({}^I w)$.*
(c) *W^I and ${}^I W$ are sets of coset representatives, distinguished in the sense that the decomposition is length-additive.*
(d) *The element $w^I \in W^I$ is the unique shortest element in wW_I.*
(e) *Let $y \in W^I$ and $u \in W_I$. Then $(yu)^I = y$, $(yu)_I = u$, and $\ell(yu) = \ell(y) + \ell(u)$.*
(f) *For $s \in S$ one has $W = {}^{\{s\}}W \sqcup s({}^{\{s\}}W)$, where $\sqcup$ denotes disjoint union.*
(g) *Let $I \subseteq J \subseteq S$. Then $W^J \subseteq W^I$. In particular, $W^S = \{1\}$ and $W^\emptyset = W$.*
(h) *$A^\rho(w) = \bigcup_{w \in W^I} I = \max\{ I \mid w \in W^I \}$.*
(i) *The element w is contained in W^I if, and only if, $I \subseteq A^\rho(w)$. In particular, $\ell(w^I) \lneq \ell(w)$ if, and only if, $I \not\subseteq A^\rho(w)$.*

2.2. **Hecke algebras.** Let R be a commutative ring with unit and with a distinguished element $q \in R$.[2] The *R-Hecke algebra* $\mathcal{H} = \mathcal{H}_q(W, S)$ associated to

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[1] In this context the graph $\emptyset$ with empty vertex set is also considered as a Coxeter graph.

[2] For certain types it is also possible to consider multiple parameter Hecke algebras. This will be discussed in [11].

(W, S) and q is the unique associative R-algebra which is a free R-module with basis $\{ T_w \mid w \in W \}$ subject to the relations

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ (q-1)T_w + qT_{sw} & \text{if } \ell(sw) < \ell(w), \end{cases} \tag{2.4}$$

for $s \in S$, $w \in W$. In particular, one has a canonical isomorphism $\mathcal{H}_1 \simeq R[W]$, where $R[W]$ denotes the R-group algebra of W. The R-algebra $\mathcal{H}$ comes equipped with an antipodal map $\_^\natural \colon \mathcal{H} \to \mathcal{H}^{\mathrm{op}}$, $T_w^\natural = T_{w^{-1}}$, i.e., $\_^\natural$ is an isomorphism satisfying $\_^{\natural\natural} = \mathrm{id}_{\mathcal{H}}$ (cf. [8, Chap. 7.3, Ex. 1]).

For $I \subseteq S$ we denote by $\mathcal{H}_I$ the corresponding parabolic subalgebra, i.e., the $\mathcal{H}$-subalgebra of $\mathcal{H}$ generated by $\{ T_s \mid s \in I \}$ which coincides with the R-module spanned by $\{ T_w \mid w \in W_I \}$. For further details see [8, Chap. 7].

2.3. **$\mathcal{H}$-modules.** Any R-algebra homomorphism $\lambda \in \mathrm{Hom}_{R\text{-alg}}(\mathcal{H}, R)$ defines a 1-dimensional left $\mathcal{H}$-module R_λ, i.e., for $T_w \in \mathcal{H}$, $w \in W$, and $r \in R_\lambda$ one has $T_w.r = \lambda(T_w)r$. Note that the relations (2.4) force $\lambda(T_s) \in \{-1, q\}$ for all $s \in S$. Moreover, for $s, s' \in S$ and $m(s, s')$ odd, one has $\lambda(T_s) = \lambda(T_{s'})$. There are two particular R-algebra homomorphisms ε_q, $\varepsilon_{-1} \in \mathrm{Hom}_{R\text{-alg}}(\mathcal{H}, R)$, given by $\varepsilon_q(T_s) = q$, $\varepsilon_{-1}(T_s) = -1$, $s \in S$. One may consider ε_q as the *augmentation* and ε_{-1} as the *sign-character*. Note that $\varepsilon_q(T_w) = q^{\ell(w)}$ and $\varepsilon_{-1}(T_w) = (-1)^{\ell(w)}$, and therefore $\varepsilon_q(T_w) = \varepsilon_q(T_w^\natural)$ and $\varepsilon_{-1}(T_w) = \varepsilon_{-1}(T_w^\natural)$ for all $w \in W$. For short we put $R_q = R_{\varepsilon_q}$, $R_{-1} = R_{\varepsilon_{-1}}$, and use also the same notation for the restriction of these modules to any parabolic subalgebra.

For $I \subseteq S$ let $\mathcal{H}^I = \mathrm{span}_R\{ T_w \mid w \in W^I \} \subseteq \mathcal{H}$. Multiplication in $\mathcal{H}$ induces a canonical map of right $\mathcal{H}_I$-modules $\mathcal{H}^I \otimes_R \mathcal{H}_I \longrightarrow \mathcal{H}$. Let $y \in W^I$ and $u \in W_I$. As $\ell(yu) = \ell(y) + \ell(u)$ (cf. Prop. 2.1(e)), one has $T_y T_u = T_{yu}$. This shows that this map is an isomorphism. In particular, $\mathcal{H}$ is a projective and thus a flat right $\mathcal{H}_I$-module. This implies that

$$\mathbf{ind}_I^S = \mathbf{ind}_{\mathcal{H}_I}^{\mathcal{H}} = \mathcal{H} \otimes_{\mathcal{H}_I} \_ \colon \mathcal{H}_I\text{-mod} \longrightarrow \mathcal{H}\text{-mod} \tag{2.5}$$

is an exact functor mapping projectives to projectives. Moreover, one has the following.

Fact 2.2. *The canonical map $c_I \colon \mathcal{H}^I \to \mathbf{ind}_I^S R_q$ given by $c_I(T_w) = T_w \eta_I$, where $\eta_I = T_1 \otimes 1 \in \mathbf{ind}_I^S R_q$ and $w \in W^I$, is an isomorphism of R-modules. Moreover, for $w \in W$, one has $T_w \eta_I = \varepsilon_q(T_{w_I}) T_{w^I} \eta_I$.*

Proposition 2.3. *Let I be a subset of S such that W_I is finite. Put $\tau_I = \sum_{w \in W_I} T_w$. Then one has the following:*

(a) $\tau_I^2 = p_{(W_I, I)}(q) \tau_I$.

Moreover if $p_{(W_I, I)}(q) \in R^\times$ is invertible in R, let $e_I = (p_{(W_I, I)}(q))^{-1} \tau_I$. Then:

(b) *The element e_I is a central idempotent in $\mathcal{H}_I$ satisfying $e_I^\natural = e_I$.*
(c) *The left ideal $\mathcal{H}e_I$ is a finitely generated, projective, left $\mathcal{H}$-module isomorphic to $\mathbf{ind}_I^S R_q$.*
(d) $T_w e_I = \varepsilon_q(T_{w_I}) T_{w^I} e_I$.

Proof. For $s \in I$ put $X_s = \sum_{w \in {}^{\{s\}}(W_I)} T_w$. Then $\tau_I = (T_1 + T_s) X_s$ (cf. Prop. 2.1(f)) and therefore

$$T_s \tau_I = T_s (T_1 + T_s) X_s = [T_s + qT_1 + (q-1)T_s] X_s = q(T_1 + T_s) X_s = \varepsilon_q(T_s) \tau_I.$$

This shows (a). Part (b) is an immediate consequence of (a), and the first part of (c) follows from the decomposition of the regular module $\mathcal{H} = \mathcal{H}e_I \oplus \mathcal{H}(T_1 - e_I)$. The canonical map $\pi\colon \mathcal{H} \to \mathbf{ind}_I^S R_q$, $\pi(T_w) = T_w\eta_I$, is a surjective morphism of $\mathcal{H}$-modules with $\ker(\pi) = \mathcal{H}(T_1 - e_I)$. This yields the second part of (c). Part (d) follows from part (b) and Proposition 2.1(a). $\square$

Proposition 2.4. *Let W be finite with longest element w_0. Assume further that $p_{(W,S)}(q) \in R^\times$ and let*

$$z = \big(p_{(W,S)}(q)\big)^{-1} \sum_{w\in W} \varepsilon_{-1}(T_w)\varepsilon_q(T_{w_0w})T_w \in \mathcal{H}.$$

Then one has the following.

(a) *For $w \in W$ one has $T_w z = \varepsilon_{-1}(T_w)z$, i.e., $\mathcal{H}z$ is isomorphic to R_{-1} as $\mathcal{H}$-module.*
(b) *The element $z \in \mathcal{H}$ is a central idempotent satisfying $z^\natural = z$.*
(c) *The left ideal $\mathcal{H}z$ is a finitely generated, projective, left $\mathcal{H}$-module.*

Proof. Let $\alpha(w) = (p_{(W,S)}(q))^{-1}\varepsilon_{-1}(T_w)\varepsilon_q(T_{w_0w})$. Then

$$\begin{aligned} T_s z = \sum_{w\in W}\alpha(w)T_sT_w &= \sum_{w\in {}^{\{s\}}W}\alpha(w)T_{sw} + \sum_{w\notin {}^{\{s\}}W}\alpha(w)(qT_{sw} + (q-1)T_w) \\ &= \underbrace{\sum_{w\in {}^{\{s\}}W}\alpha(w)T_{sw}}_{A} + \underbrace{\sum_{v\in {}^{\{s\}}W}-\alpha(v)T_v}_{B} + \underbrace{\sum_{w\notin {}^{\{s\}}W}\alpha(w)(q-1)T_w}_{C}. \end{aligned}$$

Here we used the fact that for $w \notin {}^{\{s\}}W$ it follows that $\ell(sw) < \ell(w)$. Hence for $v = sw$, one has $\ell(v) = \ell(w) - 1$ and therefore $q\alpha(w) = -\alpha(v)$.

For $w \in {}^{\{s\}}W$ and $y = sw \notin {}^{\{s\}}W$ one has $\ell(y) = \ell(w)+1$. Hence $\alpha(w) = -q\alpha(y)$ and A can be rewritten as $\sum_{v\notin {}^{\{s\}}W} -q\alpha(v)T_v$. Then

$$A + C = \sum_{x\notin {}^{\{s\}}W}\alpha(x)\left[-q + (q-1)\right]T_x = -\sum_{x\notin {}^{\{s\}}W}\alpha(x)T_x.$$

This yields (a).

It is easy to check that $z^\natural = z$. Thus, by (a), $z \in Z(\mathcal{H})$. Moreover,

$$z^2 = (p_{(W,S)}(q))^{-1}\sum_{w\in W}\varepsilon_{-1}(T_w)\varepsilon_q(T_{w_0w})T_w.z = (p_{(W,S)}(q))^{-1}\sum_{w\in W}\varepsilon_q(T_{w_0w})z = z.$$

This shows (b), and (c) is a direct consequence of (b). $\square$

3. The Coxeter complex of a Hecke algebra

3.1. **The sign map.** Let "$<$" be a total order (which is supposed to be fixed throughout) on the finite set S. Then one has a *sign-map*

$$\operatorname{sgn}\colon S \times \mathcal{P}(S) \longrightarrow \{\pm 1\}, \quad \operatorname{sgn}(s, I) = (-1)^{|\{\, t\in S\setminus I \mid t<s \,\}|}, \tag{3.1}$$

where $\mathcal{P}(S)$ denotes the set of subsets of S. One has:

$$\operatorname{sgn}(s, I \sqcup \{t\}) = \operatorname{sgn}(s, I) \qquad \text{for } t \geq s, \tag{3.2}$$

$$\operatorname{sgn}(s, I \sqcup \{t\}) = -\operatorname{sgn}(s, I) \qquad \text{for } t < s, \tag{3.3}$$

$$\operatorname{sgn}(s, I \setminus \{t\}) = \operatorname{sgn}(s, I) \qquad \text{for } t \geq s, \tag{3.4}$$

$$\operatorname{sgn}(s, I \setminus \{t\}) = -\operatorname{sgn}(s, I) \qquad \text{for } t < s. \tag{3.5}$$

Moreover, the following holds.

Fact 3.1. *If $I \subseteq S$ and $s, t \notin I$, $s \neq t$, then*

$$\operatorname{sgn}(t, I) \operatorname{sgn}(s, I \sqcup \{t\}) + \operatorname{sgn}(s, I) \operatorname{sgn}(t, I \sqcup \{s\}) = 0. \tag{3.6}$$

Proof. Note that either $s < t$ or $t < s$. By (3.2) and (3.3), the left-hand side of (3.6) reduces in the first case to

$$\operatorname{sgn}(t, I) \operatorname{sgn}(s, I) + \operatorname{sgn}(s, I)(-\operatorname{sgn}(t, I)) = 0;$$

while in the second case one has

$$\operatorname{sgn}(t, I)(-\operatorname{sgn}(s, I)) + \operatorname{sgn}(s, I) \operatorname{sgn}(t, I) = 0. \qquad \square$$

3.2. **Induction.** Let I and J be subsets of S such that $I \subseteq J \subseteq S$. As induction is the left adjoint to restriction one has a natural isomorphism

$$\phi\colon \operatorname{Hom}_{\mathcal{H}_I}(R_q, R_q) \longrightarrow \operatorname{Hom}_{\mathcal{H}_J}(\mathbf{ind}_I^J(R_q), R_q) \tag{3.7}$$

given by $\phi(\alpha)(h \otimes r) = h.\alpha(r)$, $h \in \mathbf{ind}_I^J(R_q)$, $r \in R_q$. Put $b_I^J = \phi(\mathrm{id}_{R_q})$, and let

$$\begin{aligned} d_I^J = \mathbf{ind}_J^S(b_I^J)\colon \mathbf{ind}_I^S(R_q) &\longrightarrow \mathbf{ind}_J^S(R_q), \\ d_I^J(T_w \otimes_{\mathcal{H}_I} r) &= T_w \otimes_{\mathcal{H}_J} r. \end{aligned} \tag{3.8}$$

3.3. **The Coxeter complex.** For a subset $I \subseteq S$ put $\deg(I) = |S| - |I| - 1$, thus $\deg(I) \in \{-1, \ldots, |S| - 1\}$. For a non-negative integer k let C_k be the left $\mathcal{H}$-module

$$C_k = \coprod_{\substack{I \subseteq S \\ \deg(I) = k}} \mathbf{ind}_I^S R_q. \tag{3.9}$$

The differential $\partial_k\colon C_k \to C_{k-1}$ is defined to be the map

$$\partial_k = \sum_{\substack{I, J \subseteq S \\ \deg(I) = k, \\ \deg(J) = k-1}} \partial_{I,J}, \tag{3.10}$$

where

$$\partial_{I,J} = \begin{cases} \operatorname{sgn}(s, I) d_I^J & \text{if } J = I \sqcup \{s\} \\ 0 & \text{if } J \not\supseteq I, \end{cases} \tag{3.11}$$

and d_I^J is given as in (3.8). Obviously, $\partial_k\colon C_k \to C_{k-1}$ are mappings of left $\mathcal{H}$-modules, and $C_k = 0$ for $k > |S| - 1$.

Remark 3.2. Let $C = (C_\bullet, \partial_\bullet)$ be defined as in (3.9), (3.10) and (3.11).
(a) One may apply the definition also in degree -1, i.e., $C_{-1} = \mathbf{ind}_S^S R_q \simeq R_q$ and one has also a map $\varepsilon = \partial_0\colon C_0 \to R_q$, where $\partial_0 = \sum_{s \in S} \operatorname{sgn}(s, S \setminus \{s\}) d_{S \setminus \{s\}}^S$.
(b) In degree $|S| - 1$, $C_{|S|-1} = \mathbf{ind}_\emptyset^S R_q \simeq {}^{\mathrm{reg}}\mathcal{H}$ coincides with the regular left $\mathcal{H}$-module.

(c) By Fact 2.2, every element of C_k can be written uniquely as a finite R-linear combination of monomials $T_w\eta_I$, where $I \subseteq S$, $\deg(I) = k$ and $w \in W^I$.
(d) The set $\{ T_w\eta_I \mid I \subseteq S,\ \deg(I) = k,\ w \in W^I \}$ is the standard R-basis of C_k.

Proposition 3.3. *For all k one has $\partial_{k-1} \circ \partial_k = 0$. In particular, $(C_\bullet, \partial_\bullet)$ is a chain complex of left $\mathcal{H}$-modules.*

Proof. If $I \subseteq S$ and $\deg I = k$ then

$$\begin{aligned}
\partial_{k-1}\partial_k(\eta_I) &= \partial_{k-1}\left(\sum_{s\in S\setminus I} \operatorname{sgn}(s, I)\eta_{I\sqcup\{s\}}\right)\\
&= \sum_{s\in S\setminus I} \operatorname{sgn}(s, I) \sum_{t\in S\setminus(I\sqcup\{s\})} \operatorname{sgn}(t, I \sqcup \{s\})\eta_{I\sqcup\{s,t\}}\\
&= \sum_{\substack{s,t\in S\setminus I\\ t\neq s}} \operatorname{sgn}(s, I) \operatorname{sgn}(t, I \sqcup \{s\})\eta_{I\sqcup\{s,t\}}\\
&= \sum_{\substack{s,t\in S\setminus I\\ t<s}} \left[\operatorname{sgn}(s, I) \operatorname{sgn}(t, I \sqcup \{s\}) + \operatorname{sgn}(t, I) \operatorname{sgn}(s, I \sqcup \{t\})\right] \eta_{I\sqcup\{s,t\}},
\end{aligned}$$

which vanishes by Fact 3.1. □

From now on $C = (C_\bullet, \partial_\bullet)$ will be called the *Coxeter complex* of $\mathcal{H}$. The following property will turn out to be useful for our purpose.

Proposition 3.4. *Let $h = \sum_{\deg(I)=k} \sum_{w\in W^I} \alpha(w, I)T_w\eta_I \in C_k$, $k \geq 0$, and*

$$\partial_k(h) = \sum_{\deg(J)=k-1} \sum_{v\in W^J} \beta(v, J)\, T_w\eta_J.$$

Then one has, for $J \subseteq S$, $\deg(J) = k - 1$ and $v \in W^J$,

$$\beta(v, J) = \sum_{t\in J} \sum_{x\in W_J^{J\setminus\{t\}}} \operatorname{sgn}(t, J \setminus \{t\})\, \alpha(vx, J \setminus \{t\})\, \varepsilon_q(T_x). \tag{3.12}$$

In particular, if $(\bar{w}, \bar{I})$ is such that $\bar{w} \in W^{\bar{I}}$, $\alpha(\bar{w}, \bar{I}) \neq 0$ and $\alpha(w, I) = 0$ for all $w \in W$ with $\ell(w) > \ell(\bar{w})$ and $\deg(I) = k$, then

$$\beta(\bar{w}, J) = \sum_{t\in J} \operatorname{sgn}(t, J \setminus \{t\})\, \alpha(\bar{w}, J \setminus \{t\}). \tag{3.13}$$

Proof. For $I \subset J \subseteq S$ one has $W_J = W_J^I W_I$. As $W^J \subset W^I$, one concludes that $W^I = W^J W_J^I$ (cf. Prop. 2.1). Hence

$$\begin{aligned}
\partial_k(h) &= \sum_{\deg(I)=k} \sum_{w\in W^I} \alpha(w, I) \sum_{t\in S\setminus I} \operatorname{sgn}(t, I)\, T_w\eta_{I\sqcup\{t\}}\\
&= \sum_{\deg(J)=k-1} \sum_{t\in J} \sum_{w\in W^{J\setminus\{t\}}} \operatorname{sgn}(t, J \setminus \{t\})\, \alpha(w, J \setminus \{t\})T_w\eta_J
\end{aligned}$$

and thus by the previous remark

$$= \sum_{\deg(J)=k-1} \sum_{v\in W^J} \sum_{t\in J} \operatorname{sgn}(t, J \setminus \{t\}) \sum_{x\in W_J^{J\setminus\{t\}}} \alpha(vx, J \setminus \{t\})\, T_{vx}\eta_J$$

Thus by Fact 2.2 one concludes that

$$= \sum_{\deg(J)=k-1} \sum_{v\in W^J} \sum_{t\in J} \operatorname{sgn}(t, J\setminus\{t\}) \sum_{x\in W_J^{J\setminus\{t\}}} \alpha(vx, J\setminus\{t\})\, \varepsilon_q(T_x)\, T_v \eta_J$$

This yields (3.12), and (3.13) is a direct consequence of (3.12). □

Proposition 3.5. *If W is infinite, $\partial_{|S|-1}$ is injective; while for W finite and $p_{(W,S)}(q) \in R^\times$ one has $\ker(\partial_{|S|-1}) = \mathcal{H}z \simeq R_{-1}$ (cf. Prop. 2.4).*

Proof. Put $\partial = \partial_{|S|-1}$. Let $\zeta = \sum_{w\in W} \beta(w) T_w \eta_\emptyset \in \ker\partial \subseteq C_{|S|-1}$. Proposition 2.1(f) yields

$$\begin{aligned} 0 = \partial(\zeta) &= \sum_{w\in W} \beta(w) \sum_{s\in S} \operatorname{sgn}(s,\emptyset) T_w \eta_{\{s\}} \\ &= \sum_{s\in S} \operatorname{sgn}(s,\emptyset) \left(\sum_{w\in W^{\{s\}}} \beta(w) T_w \eta_{\{s\}} + \sum_{v\in W^{\{s\}}} \beta(vs) T_v T_s \eta_{\{s\}} \right) \\ &= \sum_{s\in S} \operatorname{sgn}(s,\emptyset) \sum_{x\in W^{\{s\}}} (\beta(x) + \beta(xs)q) T_x \eta_{\{s\}}. \end{aligned}$$

Hence one must have

$$\beta(x) + q\beta(xs) = 0 \quad \text{for all } s\in S \text{ and } x\in W^{\{s\}}. \tag{3.14}$$

Suppose W is infinite and that there exists $x_0 \in W$ such that $\beta(x_0) \neq 0$. Then —because W is infinite— there exists a sequence of elements $(x_k)_{k\in\mathbb{N}}$, $x_k \in W$ such that $x_{k+1} = x_k s_k$, $s_k \in W$ and $\ell(x_{k+1}) = \ell(x_k) + 1$. In particular $x_k \in W^{\{s_k\}}$. By induction and (3.14), one concludes that $\beta(x_k) \neq 0$ for all $k \in \mathbb{N}$, a contradiction, and this shows that $\partial_{|S|-1}$ is injective in this case.

Let W be finite with longest element w_0. Then by (3.14) and induction, one concludes that $\beta(x) = (-q)^{\ell(w_0 x)} \beta(w_0)$ for all $x \in W$. In particular, for $b = \varepsilon_{-1}(T_{w_0}) p_{(W,S)}(q) \beta(w_0) \in R$ one verifies easily that $\zeta = bz\eta_\emptyset$. This yields the claim. □

3.4. **Acyclicity of the Coxeter complex.** Throughout this subsection we will assume that $|S| \geq 2$. Let "$\preceq$" be the lexicographic order on $\mathbb{N}_0\times\mathbb{N}_0$, i.e., $(\mathbb{N}_0\times\mathbb{N}_0, \preceq)$ is a well-ordered set. For $k \in \{-1,\ldots,|S|-2\}$ and $h \in C_k \setminus \{0\}$ put

$$h = \sum_{\deg(I)=k} \alpha_I \eta_I, \qquad \alpha_I = \sum_{w\in W^I} \alpha(w,I) T_w \in \mathcal{H}^I,$$

where $\alpha(w,I) \in R$ (cf. Fact 2.2). Then the following are well-defined:

$$\begin{aligned} \operatorname{supp}(h) &= \{ (w,I) \mid I\subseteq S, \deg(I)=k,\, w\in W^I,\, \alpha(w,I)\neq 0\}, \\ \lambda(h) &= \max\{ \ell(w) \mid (w,I) \in \operatorname{supp}(h) \} \in \mathbb{N}_0, \\ \nu(h) &= |\{ (w,I) \in \operatorname{supp}(h) \mid \ell(w) = \lambda(h) \}| \in \mathbb{N}_0. \end{aligned}$$

Obviously, for $h, h' \in C_k$, $h \neq h'$, and $r \in R$ with $rh \neq 0$, one has

$$(\lambda,\nu)(h-h') \preceq \max\{(\lambda,\nu)(h), (\lambda,\nu)(h')\} \quad \text{and} \quad (\lambda,\nu)(rh) \preceq (\lambda,\nu)(h). \tag{3.15}$$

For short we put $\Omega_k = \ker\partial_k \setminus \operatorname{im}\partial_{k+1}$, and define

$$\Delta_k = (\lambda,\nu)|_{\Omega_k} \colon \Omega_k \to \mathbb{N}_0 \times \mathbb{N}_0.$$

Proof of Theorem A. Obviously, $\partial_0\colon C_0 \to C_{-1}$ is surjective. Suppose that for $k \in \{0, \dots, |S|-2\}$ the set $\Omega = \Omega_k$ is non-empty, and put $\Delta = \Delta_k$. As $(\mathbb{N}_0 \times \mathbb{N}_0, \preceq)$ is well-ordered, there exists a unique minimal element $\min \Delta \in \mathrm{im}(\Delta) \subseteq \mathbb{N}_0 \times \mathbb{N}_0$. Let $h \in \Omega$ be such that $\Delta(h) = \min \Delta$. As Ω does not contain zero, $h \neq 0$. Hence there exists a pair $(\bar{w}, \bar{I}) \in \mathrm{supp}(h)$ such that $\ell(\bar{w}) = \lambda(h)$. Let $A = A^\rho(\bar{w})$ (cf. (2.3)). By Proposition 2.1(h), one has to distinguish two cases.

Case 1: $\bar{I} = A$**.** By the hypothesis on k, $\bar{I} \neq \emptyset$. Choose any element $\bar{s} \in \bar{I}$, and let $\bar{J} = \bar{I} \setminus \{\bar{s}\}$. Then one has $\deg(\bar{J}) = k+1$, and by Proposition 2.1(g), $\bar{w} \in W^{\bar{J}}$. Hence $T_{\bar{w}}\eta_{\bar{J}}$ is an element of the standard basis of C_{k+1}, and

$$\partial_{k+1}(T_{\bar{w}}\eta_{\bar{J}}) = \sum_{\substack{\deg(I)=k \\ I=\bar{J}\sqcup\{s\}}} \mathrm{sgn}(s, \bar{J})\varepsilon_q(T_{\bar{w}_I})T_{\bar{w}^I}\eta_I.$$

Since $\bar{I} = \bar{J} \sqcup \{\bar{s}\}$ and $\bar{w}^{\bar{I}} = \bar{w}$, one has

$$\mathrm{sgn}(\bar{s}, \bar{J})\partial_{k+1}(T_{\bar{w}}\eta_{\bar{J}}) = T_{\bar{w}}\eta_{\bar{I}} + \overbrace{\sum_{s\in S\setminus \bar{I}} \mathrm{sgn}(\bar{s}, \bar{J})\,\mathrm{sgn}(s, \bar{J})\varepsilon_q(T_{\bar{w}_{\bar{J}\sqcup\{s\}}})T_{\bar{w}^{\bar{J}\sqcup\{s\}}}\eta_{\bar{J}\sqcup\{s\}}}^{X}.$$

For $s \in S \setminus \bar{I}$, one has $\bar{J} \sqcup \{s\} \not\subseteq \bar{I} = A$. Thus the elements $\bar{w}^{\bar{J}\sqcup\{s\}}$ are of shorter length than $\bar{w}$ (cf. Prop. 2.1(i)). Hence, if $X \neq 0$, then $\lambda(X) \lneq \ell(\bar{w}) = \lambda(h)$. Put

$$h' = h - \alpha(\bar{w}, \bar{I})\,\mathrm{sgn}(\bar{s}, \bar{J})\partial_{k+1}(T_{\bar{w}}\eta_{\bar{J}}) \in \ker(\partial_k). \tag{3.16}$$

As $h \notin \mathrm{im}(\partial_{k+1})$, one has also $h' \notin \mathrm{im}(\partial_{k+1})$. Hence $h' \in \Omega$. Moreover, by (3.15), $\Delta(h') \preceq \Delta(h)$. Thus the minimality of $\Delta(h)$ implies that $\Delta(h') = \Delta(h)$. In particular, $\lambda(h') = \lambda(h)$. However, by construction,

$$\{\, (w, I) \in \mathrm{supp}(h') \mid \ell(w) = \lambda(h') \,\} = \{\, (w, I) \in \mathrm{supp}(h) \mid \ell(w) = \lambda(h) \,\} \setminus \{\, (\bar{w}, \bar{I}) \,\},$$

and thus $\nu(h') < \nu(h)$, a contradiction, showing that Case 1 is impossible.

Case 2: $\bar{I} \subsetneq A$**.** For the chosen $(\bar{w}, \bar{I})$ define the disjoint sets

$$\begin{aligned}
\mathcal{A} &= \{\, (\bar{w}, I) \mid I \subseteq A,\ \deg(I) = k \,\},\\
\mathcal{B} &= \{\, (w, I) \mid \ell(w) = \lambda(h),\ w \neq \bar{w},\ I \subseteq A^\rho(w),\ \deg(I) = k \,\},\\
\mathcal{C} &= \{\, (w, I) \mid \ell(w) \lneq \lambda(h),\ I \subseteq A^\rho(w).\ \deg(I) = k \,\}.
\end{aligned}$$

Then $\mathrm{supp}(h) \subseteq \mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{C}$. Let $h = h_{\mathcal{A}} + h_{\mathcal{B}} + h_{\mathcal{C}}$ be the corresponding additive decomposition of h (cf. Fact 2.2). Then $h_{\mathcal{A}} \neq 0$, $\lambda(h_{\mathcal{A}}) = \lambda(h)$, $\lambda(h_{\mathcal{B}}) \leq \lambda(h)$, and $\Delta(h_{\mathcal{C}}) \prec \Delta(h)$.

If $\bar{I} \subsetneq J \subseteq A$ with $\deg(J) = k-1$, the element $T_{\bar{w}}\eta_J$ is an element of the standard R-basis of C_{k-1}. By hypothesis, the coefficient of $\partial_k(h)$ on $T_{\bar{w}}\eta_J$ equals 0. Thus by the maximality of $\ell(\bar{w})$ and Proposition 3.4, one has

$$\sum_{t\in J} \mathrm{sgn}(t, J \setminus \{t\})\,\alpha(\bar{w}, J \setminus \{t\}) = 0. \tag{3.17}$$

Let

$$\phi = \sum_{\substack{I\subseteq A \\ \deg(I)=k}} \alpha(\bar{w}, I)\eta_I, \tag{3.18}$$

i.e., $T_{\bar{w}}\phi = h_{\mathcal{A}}$. Define D_k, $k \geq -1$, to be the R-submodule

$$D_k = \mathrm{span}_R(\{\eta_I \mid I \subseteq A,\ \deg(I) = k\}) \subseteq C_k,$$

and let $d_k\colon D_k \to D_{k-1}$, $k \geq 0$, be the R-linear map given by

$$d_k(\eta_I) = \sum_{t \in A \setminus I} \operatorname{sgn}(t, I)\eta_{I \sqcup \{t\}};$$

one easily sees that $d_k d_{k+1} = 0$ for all k (cf. Fact 3.1). Hence $(D_\bullet, d_\bullet)$ is a chain complex. Moreover, for $I \subseteq A$, $\deg(I) = k$, one has

$$(\partial_k - d_k)(\eta_I) = \sum_{t \in S \setminus A} \operatorname{sgn}(t, I)\, \eta_{I \sqcup \{t\}}. \tag{3.19}$$

The chain complex of R-modules $(D_\bullet, d_\bullet)$ concentrated in degrees $k \geq -1$ is contractible (as $(D_k, d_k)_{k \geq 0}$ coincides with the singular chain complex of an $(|A| - 1)$-dimensional simplex with coefficients in R). Thus there exist homomorphisms of R-modules $\sigma_k\colon D_k \to D_{k+1}$, $k \geq -1$, satisfying $d_{k+1}\sigma_k + \sigma_{k-1}d_k = \mathrm{id}_{D_k}$. Hence for $\psi \in \ker d_k$, one has $d_{k+1}(\sigma_k(\psi)) = \psi$. Moreover, by (3.17)

$$\begin{aligned} d_k(\phi) &= \sum_{t \in A \setminus I} \operatorname{sgn}(t, I)\, \alpha(\bar{w}, I)\, \eta_{I \sqcup \{t\}} \\ &= \sum_{\substack{J \subseteq A \\ \deg(J) = k-1}} \sum_{t \in J} \operatorname{sgn}(t, J \setminus \{t\})\, \alpha(\bar{w}, J \setminus \{t\})\, \eta_J = 0 \end{aligned} \tag{3.20}$$

Claim 3.5.1. *For all $(w, I) \in \operatorname{supp}(h_{\mathcal{A}} - T_{\bar{w}}\partial_{k+1}(\sigma_k(\phi)))$ one has $\ell(w) < \ell(\bar{w})$.*

Proof of Claim 3.5.1. Note that $h_{\mathcal{A}} = T_{\bar{w}}\phi$. Since $d_k(\phi) = 0$ (cf. (3.20)), one has $d_{k+1}(\sigma_k(\phi)) = \phi$. Thus by the previous remark

$$h_{\mathcal{A}} - T_{\bar{w}}(\partial_{k+1}(\sigma_k(\phi))) = T_{\bar{w}}(\phi - \partial_{k+1}(\sigma_k(\phi))) = T_{\bar{w}}(d_{k+1} - \partial_{k+1})(\sigma_k(\phi)).$$

By (3.19), $(d_{k+1} - \partial_{k+1})(\sigma_k(\phi))$ is an R-linear combination of elements η_I with $I \not\subseteq A$, $\deg(I) = k$. As $I \not\subseteq A$, one has $\bar{w}_I \neq 1$ (cf. Prop. 2.1(i)), and therefore, $\ell(\bar{w}^I) < \ell(\bar{w})$. This yields the claim. □

Note that $h_0 = h - T_{\bar{w}}\partial_{k+1}(\sigma_k(\phi)) \in \Omega$. As

$$\mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{C} = \{\, (w, I) \mid \ell(w) \leq \ell(\bar{w}),\ \deg(I) = k,\ w \in W^I \,\}, \tag{3.21}$$

one concludes from Claim 3.5.1 that

$$h_1 = h_{\mathcal{A}} - T_{\bar{w}}(\partial_{k+1}(\sigma_k(\phi))) \in \operatorname{span}_R\{\, T_w\eta_I \mid (w, I) \in \mathcal{C} \,\}. \tag{3.22}$$

In particular, $\lambda(h_0) \leq \lambda(h)$. Thus by the minimality of $\min \Delta$ one must have $\lambda(h_0) = \lambda(h)$. But in this case one has by construction that $\nu(h_0) < \nu(h)$, a contradiction, showing that Case 2 is impossible. From this one concludes that $\Omega = \emptyset$; in particular, $\ker \partial_k = \operatorname{im} \partial_{k+1}$, for all $k \in \{\, -1, \dots, |S| - 1 \,\}$. Hence Proposition 3.5 completes the proof of the theorem. □

4. Traces and Euler characteristics

Throughout this section R will denote a commutative ring with unit. Without further mentioning we will always assume that an associative R-algebra $\mathbf{A}$ contains a unit $1 \in \mathbf{A}$. An R-linear isomorphism $\_^\natural\colon \mathbf{A} \to \mathbf{A}^{\mathrm{op}}$ will be called an *antipode*, if $\_^{\natural\natural} = \mathrm{id}_{\mathbf{A}}$. If $(\mathbf{A}, \_^\natural)$ is an R-algebra with antipode then $\varepsilon \in \operatorname{Hom}_{R\text{-alg}}(\mathbf{A}, R)$ will be called an *augmentation* if $\varepsilon(a) = \varepsilon(a^\natural)$ for all $a \in \mathbf{A}$.

4.1. **Trace functions.** Let $\mathbf{A}$ be an associative R-algebra with unit. A homomorphism of R-modules $\tau\colon \mathbf{A} \to R$ satisfying $\tau(ab) = \tau(ba)$ for all $a, b \in \mathbf{A}$ is called a *trace function* on $\mathbf{A}$. Let $[\mathbf{A}, \mathbf{A}] = \mathrm{span}_R(\{\, ab - ba \mid a, b \in \mathbf{A}\,\})$, and let $\underline{\mathbf{A}}$ denote the R-module $\mathbf{A}/[\mathbf{A}, \mathbf{A}]$[3]. Then $\underline{\mathbf{A}}^* = \mathrm{Hom}_R(\underline{\mathbf{A}}, R)$ is the R-module of all trace functions of $\mathbf{A}$. The following elementary property will be useful for our purpose.

Proposition 4.1. *Let $(\mathbf{A}, \_^\natural, \varepsilon)$ be an augmented, associative R-algebra with antipode, and let $\mathcal{B} \subset \mathbf{A}$ be a free generating system of the R-module $\mathbf{A}$ with the following properties:*

(a) $1 \in \mathcal{B}$;
(b) $\mathcal{B}^\natural = \mathcal{B}$;
(c) *the symmetric R-bilinear form*

$$\langle \_, \_ \rangle\colon \mathbf{A} \times \mathbf{A} \longrightarrow R, \qquad \langle a, b\rangle = \delta_{a,b}\varepsilon(a), \qquad a, b \in \mathcal{B}, \tag{4.1}$$

where $\delta_{.,.}$ denotes Kronecker's δ-function, satisfies

$$\langle ab, c\rangle = \langle b, a^\natural c\rangle \qquad \text{for all } a, b, c \in \mathbf{A}. \tag{4.2}$$

Then $\tilde{\mu} \in \mathrm{Hom}_R(\mathbf{A}, R)$ given by $\tilde{\mu}(a) = \langle 1, a\rangle$, $a \in \mathbf{A}$, is a trace function.

Proof. By definition, one has for all $a, b \in \mathbf{A}$ that $\langle a^\natural, b^\natural\rangle = \langle a, b\rangle$. Hence

$$\tilde{\mu}(ab - ba) = \langle 1, ab\rangle - \langle 1, ba\rangle = \langle a^\natural, b\rangle - \langle b^\natural, a\rangle = 0. \tag{4.3}$$

for all $a, b \in \mathbf{A}$. This yields the claim. □

Remark 4.2. Let $(\mathbf{A}, \_^\natural, \varepsilon, \mathcal{B})$ be an augmented, associative R-algebra with antipode containing an R-basis $\mathcal{B} \subset \mathbf{A}$ satisfying the hypothesis of Proposition 4.1. Then the induced map $\mu \in \mathrm{Hom}_R(\underline{\mathbf{A}}, R)$ can be seen as the *canonical trace function* associated to $(\mathbf{A}, \_^\natural, \varepsilon, \mathcal{B})$.

4.2. **Hattori-Stallings trace maps.** For a finitely generated, projective, left $\mathbf{A}$-module P let $P^* = \mathrm{Hom}_{\mathbf{A}}(P, \mathbf{A})$. Then P^* carries canonically the structure of a right $\mathbf{A}$-module, and it is also finitely generated and projective. One has a canonical isomorphism $\gamma_P\colon P^* \otimes_{\mathbf{A}} P \longrightarrow \mathrm{End}_{\mathbf{A}}(P)$ given by $\gamma_P(p^* \otimes p)(q) = p^*(q)p$, $p^* \in P^*$, $p, q \in P$ (cf. [5, Chap. I, Prop. 8.3]). The *evaluation map* $\mathrm{ev}_P\colon P^* \otimes_{\mathbf{A}} P \to \underline{\mathbf{A}}$ is given by $\mathrm{ev}_P(p^* \otimes p) = p^*(p) + [\mathbf{A}, \mathbf{A}]$. The map

$$\mathrm{tr}_P = \mathrm{ev}_P \circ \gamma_P^{-1}\colon \mathrm{End}_{\mathbf{A}}(P) \longrightarrow \underline{\mathbf{A}} \tag{4.4}$$

is called the *Hattori–Stallings trace map on P* and $r_P = \mathrm{tr}_P(\mathrm{id}_P) \in \underline{\mathbf{A}}$ the *Hattori–Stallings rank* of P (cf. [10], [5, Chap. IX.2]). In particular, tr_P is R-linear, and for $f, g \in \mathrm{End}_{\mathbf{A}}(P)$ one has

$$\mathrm{tr}_P(f \circ g) = \mathrm{tr}_P(g \circ f). \tag{4.5}$$

From the elementary properties of the evaluation map one concludes that if P_1 and P_2 are two finitely generated projective left $\mathbf{A}$-modules, one has

$$r_{P_1 \oplus P_2} = r_{P_1} + r_{P_2}. \tag{4.6}$$

Let $e \in \mathbf{A}$, $e = e^2$, be an idempotent in the R-algebra $\mathbf{A}$. Then $\mathbf{A}e$ is a finitely generated, projective, left $\mathbf{A}$-module, and

$$r_{\mathbf{A}e} = e + [\mathbf{A}, \mathbf{A}]. \tag{4.7}$$

[3] In the standard literature (cf. [3], [2], [5]) this R-module is denoted by $T(\mathbf{A})$.

4.3. **Finite, projective chain complexes.** A chain complex $P = (P_\bullet, \partial^P_\bullet)$ of left $\mathbf{A}$-modules is called *finite* if $\{\, k \in \mathbb{Z} \mid P_k \neq 0 \,\}$ is finite and P_k is finitely generated for all $k \in \mathbb{Z}$. Moreover, P will be called *projective*, if P_k is projective for all k.

For $P = (P_\bullet, \partial^P_\bullet)$ and $Q = (Q_\bullet, \partial^Q_\bullet)$ finite, projective chain complexes of left $\mathbf{A}$-modules we denote by $(\underline{\mathrm{Hom}}_{\mathbf{A}}(P,Q)_\bullet, d_\bullet)$ the chain complex of right $\mathbf{A}$-modules

$$\underline{\mathrm{Hom}}_{\mathbf{A}}(P,Q)_k = \prod_{j=i+k} \mathrm{Hom}_{\mathbf{A}}(P_i, Q_j), \tag{4.8}$$

with differential given by

$$(d_k(f_k))_{i,j-1} = \partial^Q_j \circ f_{i,j} - (-1)^k f_{i-1,j-1} \circ \partial^P_i, \tag{4.9}$$

for $f_k = \sum_{j=i+k} f_{i,j}$. In particular, $f_0 = \sum_{i\in\mathbb{Z}} f_{i,i} \in \underline{\mathrm{Hom}}_{\mathbf{A}}(P,Q)_0$ is a chain map of degree 0 if, and only if, $f_0 \in \ker(d_0)$, and f_0 is homotopy equivalent to the 0-map if, and only if, $f_0 \in \mathrm{im}(d_1)$ (cf. [5, Chap. I]). Put $\underline{\mathrm{Ext}}^{\mathbf{A}}_0(P,Q) = H_0(\underline{\mathrm{Hom}}_{\mathbf{A}}(P,Q))$.

Let $B = (B_\bullet, \partial^B_\bullet)$ be a finite, projective chain complex of right $\mathbf{A}$-modules. Then $(B \underline{\otimes}_{\mathbf{A}} P, \partial^{\underline{\otimes}}_\bullet)$ denotes the complex

$$\begin{aligned} (B \underline{\otimes}_{\mathbf{A}} P)_k &= \coprod_{i+j=k} B_i \otimes_{\mathbf{A}} P_j, \\ \partial^{\underline{\otimes}}_{i+j}(b_i \otimes p_j) &= \partial^B_i(b_i) \otimes p_j + (-1)^i b_i \otimes \partial^P_j(p_j). \end{aligned} \tag{4.10}$$

Let $\mathbf{A}[\![0]\!]$ denote the chain complex of left $\mathbf{A}$-modules concentrated in degree 0 with $\mathbf{A}[\![0]\!]_0 = \mathbf{A}$, and let $\underline{\mathbf{A}}[\![0]\!]$ denote the chain complex of R-modules concentrated in degree 0 with $\underline{\mathbf{A}}[\![0]\!]_0 = \underline{\mathbf{A}}$. Then $P^{\circledast} = (P^{\circledast}_\bullet, \partial^{P^{\circledast}}_\bullet) = (\underline{\mathrm{Hom}}_{\mathbf{A}}(P, \mathbf{A}[\![0]\!])_\bullet, d_\bullet)$,

$$\begin{aligned} P^{\circledast}_k &= \mathrm{Hom}_{\mathbf{A}}(P_{-k}, \mathbf{A}), \\ \partial^{P^{\circledast}}_k(p^*_k)(p_{1-k}) &= (-1)^{k+1} p^*_k(\partial^P_{1-k}(p_{1-k})). \end{aligned} \tag{4.11}$$

is a finite, projective complex of right $\mathbf{A}$-modules. Note that the differential of the complex is chosen in such a way that the *standard evaluation mapping*

$$\begin{aligned} \mathrm{ev}_P \colon P^{\circledast} \underline{\otimes}_{\mathbf{A}} P &\longrightarrow \underline{\mathbf{A}}[\![0]\!], \\ \mathrm{ev}_{s,t}(p^*_s \otimes p_t) &= \delta_{s+t,0}\ p^*_s(p_t), \end{aligned} \tag{4.12}$$

is a mapping of chain complexes. However, the natural isomorphism

$$\begin{aligned} \gamma\colon\ \underline{\mathrm{Hom}}_{\mathbf{A}}(\_{}_1, \mathbf{A}[\![0]\!]) \underline{\otimes}_{\mathbf{A}} \_{}_2 &\longrightarrow \underline{\mathrm{Hom}}_{\mathbf{A}}(\_{}_1, \_{}_2) \\ \gamma_{s,t}(p^*_s \otimes_{\mathbf{A}} q_t)(x_{-s}) &= (-1)^{st}\, p^*_s(x_{-s}) q_t \end{aligned} \tag{4.13}$$

comes equipped with a non-trivial sign (cf. [5, Chap. I, Prop. 8.3(b) and Chap. VI, §6, Ex. 1]). In this context the *Hattori–Stallings trace map* is given by

$$\mathrm{tr}_P = H_0(\mathrm{ev}_P \circ \gamma^{-1}_{P,P})\colon\ \underline{\mathrm{Ext}}^{\mathbf{A}}_0(P,P) \longrightarrow \underline{\mathbf{A}}. \tag{4.14}$$

It has the following properties:

Proposition 4.3. *Let $P = (P_\bullet, \partial^P_\bullet)$ be a finite, projective complex of left $\mathbf{A}$-modules, and let $[f], [g] \in \underline{\mathrm{Ext}}^{\mathbf{A}}_0(P,P)$, $f = \sum_{k\in\mathbb{Z}} f_k$, be homotopy classes of chain maps of degree* 0*. Then*

(a) $\mathrm{tr}_P([f]) = \sum_{k\in\mathbb{Z}} (-1)^k \,\mathrm{tr}_{P_k}(f_k)$;
(b) $\mathrm{tr}_P([f] \circ [g]) = \mathrm{tr}_P([g] \circ [f])$.

(c) *Let $Q = (Q_\bullet, \partial^Q_\bullet)$ be another finite, projective complex of left $\mathbf{A}$-modules which is homotopy equivalent to P, i.e., there exist chain maps $\phi\colon P \to Q$, $\psi\colon Q \to P$, which composites are homotopy equivalent to the respective identity maps. Let $[h] \in \underline{\mathrm{Ext}}^{\mathbf{A}}_0(Q,Q)$ such that $[\phi]\circ[f] = [h]\circ[\phi]$. Then $\mathrm{tr}_P([f]) = \mathrm{tr}_Q([h])$.*

Proof. Part (a) is a direct consequence of (4.13), and (b) follows from (a) and (4.5). The left hand side quadrangle in the diagram

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{\mathbf{A}}(P,P) \arrow[d, "\phi\circ\_\circ\psi"'] & P^{\circledast} \underline{\otimes}_{\mathbf{A}} P \arrow[l, "\gamma"'] \arrow[r, "\mathrm{ev}_P"] \arrow[d, "\psi^{\circledast}\otimes\phi"] & \underline{\mathbf{A}}[\![0]\!] \arrow[d, equal] \\ \underline{\mathrm{Hom}}_{\mathbf{A}}(Q,Q) & Q^{\circledast} \underline{\otimes}_{\mathbf{A}} Q \arrow[l, "\gamma"'] \arrow[r, "\mathrm{ev}_Q"] & \underline{\mathbf{A}}[\![0]\!] \end{tikzcd} \tag{4.15}$$

commutes, and the right hand side quadrangle commutes up to homotopy equivalence. This yields claim (c). ☐

Let $P = (P_\bullet, \partial^P_\bullet)$ be a finite, projective complex of left $\mathbf{A}$-modules. Then one defines the *Hattori–Stallings rank* of P by

$$r_P = \mathrm{tr}_P([\mathrm{id}_P]) = \textstyle\sum_{k\in\mathbb{Z}}(-1)^k r_{P_k}. \in \underline{\mathbf{A}} \tag{4.16}$$

Proposition 4.3 implies that if $Q = (Q_\bullet, \partial^Q_\bullet)$ is another finite, projective, complex of left $\mathbf{A}$-modules which is homotopy equivalent to P then $r_P = r_Q$.

Let $\mathcal{K}(\mathbf{A})$ denote the additive category the objects of which are finite, projective chain complexes of left $\mathbf{A}$-modules. Morphisms $\mathrm{Hom}_{\mathcal{K}(\mathbf{A})}(P,Q) = \underline{\mathrm{Ext}}^{\mathbf{A}}_0(P,Q)$ are given by the homotopy classes of chain maps of degree 0. In particular, $\mathcal{K}(\mathbf{A})$ is a *triangulated category* and distinguished triangles are triangles isomorphic to the cylinder/cone triangles (cf. [7], [12, Chap. 10]). Thus, if

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1] \tag{4.17}$$

is a distinguished triangle in $\mathcal{K}(\mathbf{A})$, one has $r_B = r_A + r_C$.

4.4. **Modules of type** FP. A left $\mathbf{A}$-module M is called of *type* FP, if it has a resolution $(P_\bullet, \partial^P_\bullet, \varepsilon_M)$, such that $P = (P_\bullet, \partial^P_\bullet)$ is a finite, projective complex of left $\mathbf{A}$-modules. For such an $\mathbf{A}$-module one defines the Hattori–Stallings rank by $r_M = r_P \in \underline{\mathbf{A}}$. The comparison theorem in homological algebra implies that this element is well defined.

An augmented R-algebra $(\mathbf{A}, \varepsilon)$ is called of *type* FP, if the left $\mathbf{A}$-module $R_\varepsilon = R$, $a.r = \varepsilon(a)r$, $a \in \mathbf{A}$, $r \in R$, is of type FP. Let $(\mathbf{A}, \_^\natural, \varepsilon)$ be an augmented, associative R-algebra with antipode, and let $\mathcal{B} \subset \mathbf{A}$ be a free basis of $\mathbf{A}$ as R-module such that

(a) $\mathbf{A}$ is of type FP, and
(b) $\mathcal{B}$ satisfies the hypothesis of Proposition 4.1.

Then one defines the *Euler characteristic* of $(\mathbf{A}, \_^\natural, \varepsilon, \mathcal{B})$ by

$$\chi_{\mathbf{A}} = \chi_{(\mathbf{A},\_^\natural,\varepsilon,\mathcal{B})} = \mu(r_{R_\varepsilon}) \in R, \tag{4.18}$$

where $\mu\colon \underline{\mathbf{A}} \to R$ denotes the canonical trace function (cf. Remark 4.2). The following property will be useful for our purpose.

Proposition 4.4. *Let $C = (C_\bullet, \partial^C_\bullet)$ be a chain complex of left $\mathbf{A}$-modules concentrated in non-negative degrees with the following properties:*

(a) *C is acyclic, i.e., $H_k(C) = 0$ for $k \in \mathbb{Z}$, $k \neq 0$;*

(b) *C is finitely supported, i.e., $C_k = 0$ for almost all $k \in \mathbb{Z}$;*
(c) *C_k is of type* FP *for all $k \in \mathbb{Z}$.*

Then $H_0(C)$ is of type FP*, and one has*

$$r_{H_0(C)} = \textstyle\sum_{k\geq 0}(-1)^k r_{C_k} \in \underline{\mathbf{A}}. \tag{4.19}$$

Proof. Let $\ell(C) = \min\{\, n \geq 0 \mid C_{n+j} = 0 \text{ for all } j \geq 0 \,\}$ denote the length of C. We proceed by induction on $\ell(C)$. For $\ell(C) = 1$, there is nothing to prove. Suppose the claim holds for chain complexes D, $\ell(D) \leq \ell - 1$, satisfying the hypothesis (a)–(c), and let C be a complex satisfying (a)–(c) with $\ell(C) = \ell$. Let $C^\wedge$ be the chain complex coinciding with C in all degrees $k \in \mathbb{Z} \setminus \{0\}$ and satisfying $C^\wedge_0 = 0$. Then $C^\wedge[-1]$ satisfies (a)–(c) and $\ell(C^\wedge[-1]) \leq \ell - 1$. Then by induction, $M = H_1(C^\wedge) = H_0(C^\wedge[-1])$ is of type FP, and $r_M = \sum_{k\geq 1}(-1)^{k+1} r_{C_k}$. By construction, one has a short exact sequence of left $\mathbf{A}$-modules $0 \to M \xrightarrow{\alpha} C_0 \to H_0(C) \to 0$. Let $(P_\bullet, \partial^P_\bullet, \varepsilon_M)$ be a finite, projective resolution of M, and let $(Q_\bullet, \partial^Q_\bullet, \varepsilon_{C_0})$ be a finite, projective resolution of C_0. By the comparison theorem in homological algebra, there exists a chain map $\alpha_\bullet \colon P_\bullet \to Q_\bullet$ inducing α. Let $\mathrm{Cone}(\alpha_\bullet)$ denote the mapping cone of $\alpha_\bullet$. Then $(\mathrm{Cone}(\alpha_\bullet), \tilde{\partial}_\bullet, \varepsilon_*)$ is a finite, projective resolution of $H_0(C)$, i.e., $H_0(C)$ is of type FP. Moreover, by the remark following (4.17) one has

$$r_{H_0(C)} = r_{\mathrm{Cone}(\alpha_\bullet)} = r_Q - r_P = r_{C_0} - r_M. \tag{4.20}$$

This yields the claim. □

4.5. **Induction.** Let $\mathbf{B} \subseteq \mathbf{A}$ be an R-subalgebra of $\mathbf{A}$. The canonical injection $j \colon \mathbf{B} \to \mathbf{A}$ induces a canonical map

$$\mathrm{tr}_{\mathbf{B}/\mathbf{A}} \colon \underline{\mathbf{B}} \to \underline{\mathbf{A}}. \tag{4.21}$$

Induction $\mathbf{ind}_{\mathbf{B}}^{\mathbf{A}} = \mathbf{A} \otimes_{\mathbf{B}} \_$ is a covariant additive right-exact functor mapping finitely generated projective left $\mathbf{B}$-modules to finitely generated projective left $\mathbf{A}$-modules. Moreover, if $\mathbf{A}$ is a flat right $\mathbf{B}$-module, then $\mathbf{ind}_{\mathbf{B}}^{\mathbf{A}}$ is exact. Let P be a finitely generated left $\mathbf{B}$-module, and let $Q = \mathbf{ind}_{\mathbf{B}}^{\mathbf{A}}(P)$. Then one has a canonical map $\iota \colon P \to Q$, $\iota(p) = 1 \otimes p$, which is a homomorphism of left $\mathbf{B}$-modules. As induction is left adjoint to restriction, every map $f \in \mathrm{End}_{\mathbf{B}}(P)$ induces a map $\iota_\circ(f) = (\iota \circ f)_* \in \mathrm{End}_{\mathbf{A}}(Q)$.

Let $P^* = \mathrm{Hom}_{\mathbf{B}}(P, \mathbf{B})$ and $Q^* = \mathrm{Hom}_{\mathbf{A}}(Q, \mathbf{A})$. Then for $f \in \mathrm{Hom}_{\mathbf{B}}(P, \mathbf{B})$ one has an induced map $\iota_*(f) = (j \circ f)_* \in Q^*$ making the diagram

$$\begin{tikzcd} \mathrm{End}_{\mathbf{B}}(P) \arrow[d, "\iota_\circ"'] & P^* \otimes_{\mathbf{B}} P \arrow[l, "\gamma_P"'] \arrow[r, "\mathrm{ev}_P"] \arrow[d, "\iota_* \otimes \iota"] & \underline{\mathbf{B}} \arrow[d, "\mathrm{tr}_{\mathbf{B}/\mathbf{A}}"] \\ \mathrm{End}_{\mathbf{A}}(Q) & Q^* \otimes_{\mathbf{A}} Q \arrow[l, "\gamma_Q"'] \arrow[r, "\mathrm{ev}_Q"] & \underline{\mathbf{A}} \end{tikzcd} \tag{4.22}$$

commute. This shows the following.

Proposition 4.5. *Let $\mathbf{B} \subseteq \mathbf{A}$ be an R-subalgebra of $\mathbf{A}$ such that $\mathbf{A}$ is a flat right $\mathbf{B}$-module, and let M be a left $\mathbf{B}$-module of type* FP*. Then $\mathbf{ind}_{\mathbf{B}}^{\mathbf{A}}(M)$ is of type* FP*, and one has*

$$r_{\mathbf{ind}_{\mathbf{A}}^{\mathbf{B}}(M)} = \mathrm{tr}_{\mathbf{B}/\mathbf{A}}(r_M). \tag{4.23}$$

Let $(\mathbf{A}, \_^\natural, \varepsilon, \mathcal{B})$ be an augmented, associative, R-algebra with antipode and a distinguished R-basis $\mathcal{B}$ satisfying the hypothesis of Proposition 4.1. Let $\mathbf{B} \subseteq \mathbf{A}$ be an R-subalgebra of $\mathbf{A}$ such that

(i) $\mathbf{A}$ is a flat right $\mathbf{B}$-module;
(ii) $\mathbf{B}^\natural = \mathbf{B}$;
(iii) $\mathcal{C} = \mathcal{B} \cap \mathbf{B}$ is an R-basis of $\mathbf{B}$.

Let $\mu_{\mathbf{A}} : \underline{\mathbf{A}} \to R$ and $\mu_{\mathbf{B}} : \underline{\mathbf{B}} \to R$ denote the associated canonical traces. Then one has a commutative diagram

$$\begin{tikzcd} \underline{\mathbf{B}} \arrow[rr, "\mathrm{tr}_{\mathbf{B}/\mathbf{A}}"] \arrow[dr, "\mu_{\mathbf{B}}"'] & & \underline{\mathbf{A}} \arrow[dl, "\mu_{\mathbf{A}}"] \\ & R & \end{tikzcd} \tag{4.24}$$

From this one concludes the following direct consequence of Proposition 4.5.

Corollary 4.6. *Let $(\mathbf{A}, \_^\natural, \varepsilon, \mathcal{B})$ be an augmented, associative, R-algebra with antipode and a distinguished R-basis $\mathcal{B}$ satisfying the hypothesis of Proposition 4.1, and let $\mathbf{B} \subseteq \mathbf{A}$ be an R-subalgebra satisfying* (i)-(iii). *Let M be a left $\mathbf{B}$-module of type FP. Then $\mu_{\mathbf{B}}(r_M) = \mu_{\mathbf{A}}(r_{\mathbf{ind}_{\mathbf{B}}^{\mathbf{A}}(M)})$.*

5. The Euler characteristic of a Hecke algebra

5.1. **The canonical trace of a Hecke algebra.** Let $\mathcal{H} = \mathcal{H}_q(W, S)$ be the R-Hecke algebra associated to the finitely generated Coxeter group (W, S), and let $\mathcal{B} = \{ T_w \mid w \in W \}$. Then $\_^\natural : \mathcal{H} \to \mathcal{H}^{\mathrm{op}}$, $T_w^\natural = T_{w^{-1}}$, is an anti-automorphism of $\mathcal{H}$ satisfying $\_^{\natural\natural} = \mathrm{id}_{\mathcal{H}}$ (cf. [8, Chap. 7.3, Ex. 1]) and $\varepsilon_q(a^\natural) = \varepsilon_q(a)$ for all $a \in \mathcal{H}$. One has the following property.

Proposition 5.1. *Let $\mathcal{H}$ be the Hecke algebra associated to the finitely generated Coxeter group (W, S). Then the R-bilinear map $\langle \_, \_ \rangle : \mathcal{H} \times \mathcal{H} \to R$ associated to $(\mathcal{H}, \mathcal{B}, \_^\natural, \varepsilon)$ satisfies* (4.2). *In particular, $\tilde{\mu}_{\mathcal{B}} = \langle T_1, \_ \rangle$ is a trace function.*

Proof. By Proposition 4.1, one has to show that

$$\langle T_u T_v, T_w \rangle = \langle T_v, T_{u^{-1}} T_w \rangle \qquad \text{for all } u, v, w \in W. \tag{5.1}$$

Using induction one easily concludes that it suffices to show (5.1) in the case that $u = s \in S$. In this case one has:

$$\lambda = \langle T_s T_v, T_w \rangle = \begin{cases} \delta_{sv,w} \varepsilon_q(T_{sv}) & \text{if } \ell(sv) > \ell(v) \\ (q-1)\delta_{v,w}\varepsilon_q(T_v) + q\delta_{sv,w}\varepsilon_q(T_{sv}) & \text{if } \ell(sv) < \ell(v) \end{cases} \tag{5.2}$$

and

$$\rho = \langle T_v, T_s T_w \rangle = \begin{cases} \delta_{v,sw} \varepsilon_q(T_v) & \text{if } \ell(sw) > \ell(w) \\ (q-1)\delta_{v,w}\varepsilon_q(T_v) + q\delta_{v,sw}\varepsilon_q(T_v) & \text{if } \ell(sw) < \ell(w) \end{cases} \tag{5.3}$$

We proceed by a case-by-case analysis.

Case 1: $\ell(sv) > \ell(v)$ and $\ell(sw) > \ell(w)$. Suppose that $\lambda \neq 0$. Then $sv = w$, but $\ell(w) = \ell(sv) > \ell(v) = \ell(sw)$, a contradiction. Hence $\lambda = 0$. Reversing the rôles of v and w yields $\lambda = \rho = 0$ and thus the claim.

Case 2: $\ell(sv) > \ell(v)$ and $\ell(sw) < \ell(w)$. Then, $v \neq w$. If $\lambda \neq 0$, then $sv = w$. Hence $\ell(w) = \ell(sv) = \ell(v)+1$, and $\lambda = \varepsilon_q(T_w) = \varepsilon_q(T_s)\varepsilon_q(T_v)$. On the other hand,

$\rho = (q-1)\delta_{v,w}\varepsilon_q(T_v) + q\delta_{v,sw}\varepsilon_q(T_v) = q\varepsilon_q(T_v) = \lambda$. If $\lambda = 0$, then $sv \neq w$. Hence $\rho = (q-1)\delta_{v,w}\varepsilon_q(T_v) + q\delta_{v,sw}\varepsilon_q(T_v) = 0 = \lambda$.

Case 3: $\ell(sv) < \ell(v)$ and $\ell(sw) > \ell(w)$. Reversing the rôles of v and w one can transfer the proof for Case 2 verbatim.

Case 4: $\ell(sv) < \ell(v)$ and $\ell(sw) < \ell(w)$. Suppose that $sv = w$, or, equivalently, $v = sw$. Then $\ell(sv) < \ell(v) = \ell(sw) < \ell(w)$, a contradiction. Hence $sv \neq w$ and $v \neq sw$. Thus $\lambda = \rho$. This completes the proof. □

Remark 5.2. The trace function $\tilde{\mu}\colon \mathcal{H} \to R$ can be seen as the *canonical trace function* on $\mathcal{H}$. It is straight forward to verify that for Hecke algebras of type A_n, B_n or D_n this trace function coincides with the Jones–Ocneanu trace evaluated in 0 (cf. [6]).

5.2. **Properties of the Coxeter complex.** Let (W,S) be a finite Coxeter group, and let $q \in R$ be such that $p_{(W,S)}(q) \in R^{\times}$. Then $R_q \simeq \mathcal{H}e_S$ (cf. Prop. 2.3); in particular, R_q is a projective left $\mathcal{H}$-module. This shows that for any Coxeter group (W,S) and $I \subseteq S$ such that W_I is finite, $\mathbf{ind}_{\mathcal{H}_I}^{\mathcal{H}}(R_q)$ is a finitely generated, projective, left $\mathcal{H}$-module. As a consequence one has the following (cf. [8, §6.8]):

Proposition 5.3. *Let (W,S) be a finitely generated Coxeter group, which is either affine, or compact hyperbolic and let $q \in R$ be such that $p_{(W_I,I)}(q) \in R^{\times}$ for any proper parabolic subgroup (W_I,I). Then the Coxeter complex $(C_\bullet, \partial_\bullet, \varepsilon)$ is a finite projective resolution of R_q.*

In the general case one has the following:

Proposition 5.4. *Let (W,S) be a finitely generated Coxeter group, and let $q \in R$ be such that $p_{(W_I,I)}(q) \in R^{\times}$ for any finite parabolic subgroup (W_I,I). Then $(C_\bullet, \partial_\bullet)$ is a chain complex of left $\mathcal{H}$-modules of type* FP*; in particular, R_q is a left $\mathcal{H}$-module of type* FP.

Proof. By hypothesis and the previously mentioned remark, $\mathbf{ind}_{\mathcal{H}_I}^{\mathcal{H}}(R_q)$ is a finitely generated projective $\mathcal{H}$-module for any finite parabolic subgroup (W_I,I). We proceed by induction on $d = |S|$. For $|S| \leq 2$, there is nothing to prove. Assume that the claim holds for all Coxeter groups (W_J,J) with $|J| < d$, and that $|S| = d$. By induction, for $K \subsetneq S$, R_q is a left $\mathcal{H}_K$-module of type FP. Hence $\mathbf{ind}_{\mathcal{H}_K}^{\mathcal{H}}(R_q)$ is a left $\mathcal{H}$-module of type FP. Thus C_k is a left $\mathcal{H}$-module of type FP for $0 \leq k \leq d-1$. If (W,S) is spherical, then R_q is a finitely generated, projective, left $\mathcal{H}$-module by the first remark. If (W,S) is non-spherical, $(C_\bullet, \partial_\bullet)$ is acyclic. Hence R_q is a left $\mathcal{H}$-module of type FP by Proposition 4.4. This completes the proof. □

5.3. **The Euler characteristic of a Hecke algebra.** Proposition 5.4 has the following consequence.

Proposition 5.5. *Let (W,S) be a finitely generated, non-spherical Coxeter group, and let $q \in R$ be such that $p_{(W_I,I)}(q) \in R^{\times}$ for any finite parabolic subgroup (W_I,I). Then*

$$r_{R_q} = \sum_{I \subsetneq S} (-1)^{|S \setminus I|-1} r_{\mathbf{ind}_I^S(R_q)}. \tag{5.4}$$

Proof. By (4.19) and (4.23), one has

$$r_{R_q} = \sum_{0 \leq k < |S|} (-1)^k r_{C_k} = \sum_{I \subsetneq S} (-1)^{|S \setminus I|-1} r_{\mathbf{ind}_I^S(R_q)}. \tag{5.5}$$

This yields the claim. □

Proof of Theorem C. If (W,S) is spherical, $R_q \simeq \mathcal{H}e_S$ where e_S is given as in Proposition 2.3. Hence, as $r_{R_q} = e_S + [\mathcal{H},\mathcal{H}]$ (cf. (4.7)), one has $\chi_{\mathcal{H}} = \mu(r_{R_q}) = p_{(W,S)}(q)^{-1}$.

If (W,S) is non-spherical, we proceed by induction on $|S|$. Proposition 4.5, Corollary 4.6 and Proposition 5.5 imply that

$$\begin{aligned}\chi_{\mathcal{H}} = \mu_{\mathcal{H}}(r_{R_q}) &= \sum_{I\subsetneq S}(-1)^{|S\setminus I|-1}\mu_{\mathcal{H}}(r_{\mathbf{ind}_I^S(R_q)})\\ &= \sum_{I\subsetneq S}(-1)^{|S\setminus I|-1}\mu_{\mathcal{H}_I}(R_q)\end{aligned}$$

and thus by induction

$$= \sum_{I\subsetneq S}(-1)^{|S\setminus I|-1}p_{(W_I,I)}(q)^{-1}. \tag{5.6}$$

It is well-known that the alternating sum (5.6) is equal to $p_{(W,S)}(q)^{-1}$ (cf. [8, Prop. 5.12]). □

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