FINITE QUOTIENTS OF GALOIS PRO-p GROUPS AND RIGID FIELDS

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Abstract. For a prime number p, we show that if two certain canonical finite quotients of a finitely generated Bloch-Kato pro-p group G coincide, then G has a very simple structure, i.e., G is a p-adic analytic pro-p group (see Theorem A). This result has a remarkable Galois-theoretic consequence: if the two corresponding canonical finite extensions $F^{(3)}/F$ and $F^{\{3\}}/F$ of a field F – with F containing a primitive p-th root of unity – coincide, then F is p-rigid (see Corollary B). The proof relies only on group-theoretic tools, and on certain properties of Bloch-Kato pro-p groups. This paper will appear on the *Annales mathématiques du Québec*.

1. Introduction

Let p be a prime number, and let G be a pro-p group. The Frattini subgroup $\Phi(G)$ of G is the closed subgroup of G generated by the p-powers and the commutators of the elements of G. In particular, the quotient $G/\Phi(G)$ is an elementary abelian p-group. Let $\Phi_2(G)$ be the Frattini subgroup of the Frattini subgroup of G, i.e., $\Phi_2(G) = \Phi(\Phi(G))$.

Also, let $P_n(G)$, $n \geq 1$, denote the p-descending central series of G. In particular, one has $P_2(G) = \Phi(G)$ and $P_3(G) = \Phi(G)^p[G, \Phi(G)] \supseteq \Phi_2(G)$. For the class of finitely generated Bloch-Kato pro-p groups, we prove the following result.

Theorem A. *One has the equality $\Phi_2(G) = P_3(G)$ if, and only if, G is p-adic analytic.*

In this case the group G has a very simple structure, as it is meta-abelian and it is possible to provide an explicit presentation for G (cf. [13, Theorem 4.6]).

One has also the following Galois-theoretic consequence. Let F be a field containing a primitive p-th root of unity. By $F^\times$ we denote the (multiplicative) group of non-zero elements of F. We consider the Galois extension $F^{(3)}$ of F obtained by first taking $F^{(2)}$ to be the compositum over F of all extensions of F of degree p, and then taking $F^{(3)}$ to be the compositum over

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$F^{(2)}$ of all the extensions of $F^{(2)}$ of degree p that are Galois over F. We also denote by $F^{\{3\}}$ the compositum over $F^{(2)}$ of all extensions of $F^{(2)}$ of degree p (cf. [3, § 2.3]). Thus

$$F^{\{3\}} = \left(F^{(2)}\right)^{(2)}.$$

Then one may characterize those fields F with the property that $F^{(3)} = F^{\{3\}}$. In fact, from Theorem A we shall obtain the following result.

Corollary B. *Let F be a field containing a primitive p-th root of unity, and assume that the quotient $F^\times/(F^\times)^p$ is finite. (Assume further that $\sqrt{-1} \in F$ if $p = 2$). Then $F^{(3)} = F^{\{3\}}$ if, and only if, F is p-rigid;*

(For the definition of p-rigid field, see Section 4.)

Bloch-Kato pro-p groups were introduced in [2] and studied first in [13]. A Bloch-Kato pro-p group is a pro-p group which satisfies the conclusion of the Rost-Voevodsky theorem (formerly known as the Bloch-Kato conjecture), i.e., such that the cohomology ring of every closed subgroup of G with coefficients in the finite field $\mathbb{F}_p$ is a quadratic algebra over $\mathbb{F}_p$. For example, absolute Galois groups of fields which are pro-p and Galois groups of the maximal p-extension of certain fields are Bloch-Kato pro-p groups. Thus, a Bloch-Kato pro-p group is a very natural "candidate" for being realized as absolute Galois group, and this shows the relevance of Bloch-Kato pro-p groups for Galois theory.

The problem to characterize a field F yielding the equality

$$F^{(3)} = F^{\{3\}} \tag{1.1}$$

arises rather naturally, and the case when equality (1.1) holds is considered very significant in field theory. Indeed, such problem has been widely studied in the past: in the case $p = 2$ Corollary B was proved in [1, Theorem 3.1], with arguments which make use of Galois cohomology, and later in [8, Theorem A], with arguments relying on the theory of quadratic forms. For p odd, Corollary B was proved in [3, Theorem A], and the proof relies on certain properties of Bloch-Kato pro-p groups, together with an essential arithmetic argument (cf. [3, Theorem 4.3]).

The above results provide a motivation for the paper, as Theorem A is the "group-theoretic translation", and it is in fact more genaral, as it holds for Bloch-Kato pro-p groups, and not only for Galois groups of maximal p-extensions. Moreover, part of the interest of this result lies in the fact that the proof is purely group-theoretical, and it does not rely on results form field theory. Further, the proof makes use of the Zassenhaus filtration of pro-p groups, which is gaining increasing importance as tool for the study of Galois groups (see, e.g., [5] and [11]).

The paper is organized as follows. In the second section, we state a number of properties on pro-p groups and on their descending series. In section 3 we prove Theorem A, and in section 4 we provide the "arithmetic translation" of Theorem A, and we prove Corollary B.

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2. Preliminaries on pro-p groups

Throughout this paper, subgroups of pro-p groups are assumed to be closed (in the pro-p topology), and every generator is to be intended as topological generator. In particular, given two (closed) subgroups H_1 and H_2 of a pro-p group G, the subgroup $[H_1, H_2]$ is the (closed) subgroup of G generated by the commutators $[g_1, g_2]$, with $g_i \in H_i$ for $i = 1, 2$. Also, for a positive integer n, G^n denotes the (closed) subgroup of G generated by the n-powers of the elements of G.

For a finitely generated pro-p group G, let $\mathrm{d}(G)$ denote the minimal number of generators of G. In particular, $\mathrm{d}(G)$ is the dimension of the quotient $G/\Phi(G)$ as vector space over the finite field $\mathbb{F}_p$ (cf. [4, Prop. 1.14]). Then, one defines the rank of a pro-p group G to be the number

$$\mathrm{rk}(G) = \sup\{\mathrm{d}(H) \mid H \leq G \text{ closed}\} \in \mathbb{N} \cup \{\infty\}$$

(cf. [4, Definition 3.12]).

For a pro-p group G, the lower p-central series of G is the series $P_n = P_n(G)$, $n \geq 1$, of characteristic subgroups defined by $P_1 = G$ and

$$P_{n+1} = P_n^p[G, P_n].$$

In particular, one has that $P_2(G)$ is the Frattini subgroup $\Phi(G)$, and $[P_i, P_j] \leq P_{i+j}$ for every $i, j \geq 1$. Moreover, if G is finitely generated, then the lower p-central series is a base of neighbourhoods of 1 in G (cf. [4, Prop. 1.16]).

Definition. A pro-p group G is said to be *powerful* if G/G^p is abelian, if p is odd, or if G/G^4 is abelian, if $p = 2$.

In particular, one has the following (cf. [4, Theorems 3.6, 3.8]).

Proposition 2.1. *Let G be a powerful pro-p group.*

(1) $P_n(G) = G^{p^{n-1}}$ *for every* $n \geq 1$.
(2) *if G is finitely generated, then* $\mathrm{rk}(G) = \mathrm{d}(G)$.

Another important descending series of pro-p groups is the *Zassenhaus filtration*. For an arbitrary group G, the Zassenhaus filtration of G is the series $D_n = D_n(G)$, $n \geq 1$, of characteristic subgroups defined by $D_1 = G$ and

$$D_n = D_{\lceil n/p \rceil}^p \prod_{i+j=n} [D_i, D_j], \tag{2.1}$$

where $\lceil n/p \rceil$ is the least integer m such that $mp \geq n$. In particular, the Zassenhaus filtration is the fastest descending series starting at G such that $[D_i, D_j] \leq D_{i+j}$ and $D_i^p \leq D_{ip}$ for every $i, j \geq 1$. For computational purposes, one has the formula

$$D_n = \prod_{ip^h \geq n} \gamma_i(G)^{p^h}, \tag{2.2}$$

established by M. Lazard (cf. [4, Theorem 11.2]), where the $\gamma_i(G)$'s are the elements of the descending central series of G (i.e., $\gamma_1(G) = G$ and

$\gamma_{i+1}(G) = [G, \gamma_i(G)]$ for every $i \geq 1$). Thus, if G is a (pro-)p group, then $D_2(G)$ is the Frattini subgroup $\Phi(G)$.

For the Zassenhaus filtration of a pro-p group, one has the following remarkable result (cf. [4, Theorem 11.4]).

Theorem 2.2. *Let G be a finitely generated pro-p group. Then G has finite rank if, and only if, $D_n(G) = D_{n+1}(G)$ for some $n \geq 1$.*

Definition. A topological group G is a *p-adic analytic group* if G has the structure of analytic manifold over the field of p-adic numbers $\mathbb{Q}_p$ with the properties

(1) the multiplication function $G \times G \to G$ given by $(x, y) \mapsto xy$ is analytic;
(2) the inversion function $G \to G$ defined by $x \mapsto x^{-1}$ is analytic.

Powerful pro-p groups and p-adic analytic groups are tightly related. Indeed, a topological group G has the structure of a p-adic analytic group if, and only if, G has an open subgroup which is a powerful finitely generated pro-p group (cf. [4, Theorem 8.1]). In the case of Bloch-Kato pro-p groups, p-adic analytic groups have a rather simple structure, as stated by the following (cf. [13, Theorem 4.8]).

Theorem 2.3. *Let G be a finitely generated Bloch-Kato pro-p group, and assume furhter that G is torsion-free, if $p = 2$. The following are equivalent.*

(1) *G has finite rank.*
(2) *G is p-adic analytic.*
(3) *G is powerful.*
(4) *G has a presentation*

$$G = \left\langle \sigma, \tau_1, \ldots, \tau_d \;\middle|\; \sigma\tau_i\sigma^{-1} = \tau_i^{1+p^k}, \tau_i\tau_j = \tau_j\tau_i \;\forall\, i, j \right\rangle, \tag{2.3}$$

with $d = \mathrm{d}(G) - 1$, for some $k \geq 1$ ($k \geq 2$, if $p = 2$).

3. Proof of Theorem A

Lemma 3.1. *If G is a powerful Bloch-Kato group, then $\Phi_2(G) = P_3(G)$.*

Proof. Recall first that if G is a Bloch-Kato pro-p group, then every closed subgroup of G is again a Bloch-Kato pro-p group. By Proposition 2.1, one has $\Phi(G) = G^p$ and $P_3(G) = G^{p^2}$. Since $\mathrm{rk}(G)$ is finite, also $\mathrm{rk}(\Phi(G))$ is finite, thus $\Phi(G)$ is powerful by Theorem 2.3. Therefore,

$$\Phi_2(G) = \Phi(\Phi(G)) = \Phi(G)^p = G^{p^2},$$

and this yields the claim. □

Proof of Theorem A. Assume that G is a finitely generated p-adic analytic Bloch-Kato group. Then, the claim holds by Theorem 2.3 and Lemma 3.1.

Conversely, assume that $\Phi_2(G) = P_3(G)$. Since $[D_2, D_2] \leq D_4$ and $D_2^p \leq D_{2p}$, one has $\Phi_2(G) = D_2^p[D_2, D_2] \leq D_4$, as $\Phi(G) = D_2$. Moreover, one has the inclusion $\gamma_3(G) \leq P_3(G)$. Therefore, one has the chain of inclusions

$$\gamma_3(G) \leq P_3(G) = \Phi_2(G) \leq D_4. \tag{3.1}$$

We shall split the proof of this implication in three cases.

(1) Assume $p > 3$. By (2.2), one has

$$D_3 = \prod_{ip^h \geq 3} \gamma_i(G)^{p^h} = \gamma_3(G) \cdot G^p$$
$$\text{and} \quad D_4 = \prod_{ip^h \geq 4} \gamma_i(G)^{p^h} = \gamma_4(G) \cdot G^p.$$

Therefore, (3.1) implies

$$D_3(G) = \gamma_3(G) \cdot G^p \leq P_3(G) = \Phi_2(G) \cdot G^p \leq D_4,$$

as $G^p \leq D_4$. Thus, one has the equality $D_3 = D_4$. Hence, Theorem 2.2 implies that $\mathrm{rk}(G)$ is finite, and thus by Theorem 2.3 G is a p-adic analytic Bloch-Kato pro-p group.

(2) Assume $p = 2$. From (2.2) one obtains

$$D_3 = \prod_{i2^h \geq 3} \gamma_i(G)^{2^h} = \gamma_3(G) \cdot \gamma_2(G)^2 \cdot G^4$$
$$\text{and} \quad D_4 = \prod_{i2^h \geq 4} \gamma_i(G)^{2^h} = \gamma_4(G) \cdot \gamma_2(G)^2 \cdot G^4.$$

Therefore, (3.1) implies

$$D_3 = \gamma_3(G) \cdot \gamma_2(G)^2 \cdot G^4 \leq \Phi_2(G) \cdot \gamma_2(G)^2 \cdot G^4 \leq D_4,$$

as $\gamma_2(G)^2 G^4 \leq D_4$. Thus, one has the equality $D_3 = D_4$. Hence, Theorem 2.2 implies that $\mathrm{rk}(G)$ is finite, and thus by Theorem 2.3 G is a p-adic analytic Bloch-Kato pro-p group.

(3) Assume $p = 3$. By (2.2), one has

$$D_4 = \prod_{i3^h \geq 4} \gamma_i(G)^{3^h} = \gamma_4(G) \cdot \gamma_2(G)^3 \cdot G^9$$
$$\text{and} \quad D_5 = \prod_{i3^h \geq 5} \gamma_i(G)^{3^h} = \gamma_5(G) \cdot \gamma_2(G)^3 \cdot G^9.$$

Therefore, from (3.1) one obtains the chain of inclusions

$$\gamma_4(G) = [G, \gamma_3(G)] \leq [G, D_4] = [D_1, D_4] \leq D_5,$$

which implies

$$D_4 = \gamma_4(G) \cdot \gamma_2(G)^3 \cdot G^9 \leq D_5,$$

as $G^9, \gamma_2(G)^3 \leq D_5$. Thus, one has the equality $D_4 = D_5$. Hence, Theorem 2.2 implies that $\mathrm{rk}(G)$ is finite, and thus by Theorem 2.3 G is a p-adic analytic Bloch-Kato pro-p group.

This establishes the theorem. $\square$

Note that if G is a finitely generated pro-p group, then $\Phi_2(G)$ is an open subgroup of G. Thus, the quotient $G/\Phi_2(G)$ is finite, and one may reduce the equality $\Phi_2(G) = P_3(G)$ to a condition on finite p-groups, as done in [3, Corollary 4.15].

Corollary 3.2. *A finitely generated Bloch-Kato pro-p group G is p-adic analytic if, and only if, $\Phi(G)/\Phi_2(G)$ is contained in the centre of $G/\Phi_2(G)$.*

Proof. Assume that G is p-adic analytic. Then Theorem A yields the equality $\Phi_2(G) = P_3(G)$. Since $[G, P_2] = [P_1, P_2] \leq P_3$, one has $[G, \Phi(G)] \leq \Phi_2(G)$, and $\Phi(G)/\Phi_2(G)$ is central in $G/\Phi_2(G)$.

Conversely, assume that $\Phi(G)/\Phi_2(G)$ is central in $G/\Phi_2(G)$. Hence the commutator subgroup $[G, \Phi(G)]$ is contained in $\Phi_2(G)$. Since

$$\Phi(G)^p \leq \Phi_2(G) \quad \text{and} \quad P_3 = \Phi(G)^p[G, \Phi(G)],$$

it follows that $\Phi_2(G)$ contains $P_3(G)$, and thus the two subgroups are equal. Therefore G is p-adic analytic by Theorem A. $\square$

4. Proof of Corollary B

Throughout this section, a field F is always assumed to contain a primitive p-th root of unity (and also $\sqrt{-1}$, if $p = 2$). Also, $F^\times$ denotes the multiplicative group of non-zero elements of F, and $(F^\times)^p$ is the subgroup of p-powers of $F^\times$.

Definition. Let N denote the norm map $N\colon F(\sqrt[p]{a}) \to F$ of the p-cyclic extension $F(\sqrt[p]{a})/F$. An p-power-free unit $a \in F^\times$ is said to be *p-rigid* if

$$b \in N\left(F(\sqrt[p]{a})\right) \quad \text{if, and only if,} \quad b \in \bigcup_{k=0}^{p-1} a^k (F^\times)^p$$

for every $b \in F^\times \smallsetminus (F^\times)^p$. The field F is called *p-rigid* if every element of $F^\times \smallsetminus (F^\times)^p$ is p-rigid.

Recall from the Introduction that $F^{(2)} = F(\sqrt[p]{F})$ is the compositum over F of all extensions $F(\sqrt[p]{a})$ with $a \in F^\times$. Also,

- $F^{\{3\}} = F^{(2)}(\sqrt[p]{F^{(2)}})$ is the compositum over $F^{(2)}$ of all the extensions $F^{(2)}(\sqrt[p]{a})$ with $a \in (F^{(2)})^\times$;
- $F^{(3)}$ is the compositum over $F^{(2)}$ of all the extensions $F^{(2)}(\sqrt[p]{a})$ such that $F^{(2)}(\sqrt[p]{a})/F$ is Galois.

Therefore, both $F^{\{3\}}/F$ and $F^{(3)}/F$ are Galois extensions, and $F^{(3)} \subseteq F^{\{3\}}$ (cf. [3, § 2.3]).

Let G be the *maximal pro-p Galois group* of F, i.e.,

$$G = G_F(p) = \mathrm{Gal}(F(p)/F),$$

where $F(p)$ is the maximal p-extension of F. Recall that the maximal pro-p Galois group of a field containing a primitive p-th root of unity is a Bloch-Kato pro-p group (cf. [13, § 2]).

By Kummer theory, one has that the Galois group of $F^{(2)}/F$ is the quotient $G/\Phi(G)$. Note that G is finitely generated if, and only if, the quotient $F^\times/(F^\times)^p$ is finite (and in this case $\mathrm{d}(G) = \dim(F^\times/(F^\times)^p)$), as $G/\Phi(G)$ and $F^\times/(F^\times)^p$ are isomorphic as discrete groups of exponent p. Moreover,

$$\mathrm{Gal}(F^{(3)}/F) = G/P_3(G) \quad \text{and} \quad \mathrm{Gal}(F^{\{3\}}/F) = G/\Phi_2(G) \tag{4.1}$$

(cf. [3, § 4.1], see also [1, § 2]).

Remark 4.1. In the case $p = 2$, the Galois groups $\mathrm{Gal}(F^{(3)}/F)$ and $\mathrm{Gal}(F^{\{3\}}/F)$ are called W-group, resp. V-group, of the field F, for the relations with the Witt ring of F (cf. [10] and [1]).

Proof of Corollary B. Let G be the maximal pro-p Galois group $G_F(p)$. By hypothesis, G is finitely generated. Moreover, G is torsion free, since we are assuming that $\sqrt{-1} \in F$ for $p = 2$.

Assume first that the equality $F^{(3)} = F^{\{3\}}$ holds. Then, by (4.1) one has also the equality $\Phi_2(G) = P_3(G)$, and thus Theorem A implies that G is a p-adic analytic Bloch-Kato pro-p group, and Theorem 2.3 implies that G is powerful. Therefore, by [3, Proposition 3.8] the field F is p-rigid.

Conversely, assume that F is p-rigid. Then, again by [3, Proposition 3.8] the Galoi group G is powerful, and thus p-adic analytic by Theorem 2.3. Therefore, Theorem A implies the equality $\Phi_2(G) = P_3(G)$, and the equality $F^{(3)} = F^{\{3\}}$ follows by (4.1). □

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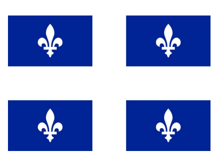

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