

UNIVERSITÀ DEGLI STUDI MILANO - BICOCCA

Dottorato in Matematica Pura e Applicata - XVII ciclo



**General smile asymptotics and
a multiscaling stochastic volatility model**

Ph.D. Supervisor:
Professor
Francesco Caravenna
Università degli studi
Milano-Bicocca

Candidate:
Corbetta Jacopo
705879

4 March 2015

Acknowledgments

I would like to thank my supervisor Francesco Caravenna for his guidance and support, I owe my deep gratitude to him for introducing me to this fascinating field, for his contagious and enthusiastic way of working and for all the things he has taught me I thank Paolo Dai Pra for agreeing to act as referee. I am very grateful to Fabio Bellini, Friedrich Hubalek and Carlo Sgarra for accepting to be part of the thesis committee.

I want to express my special thanks to all my Ph.D. colleagues at the University of Milano–Bicocca for the great time we have spent together. I am also grateful to the personnel of both departments for the stimulating ambience.

Last but not least, I want to express all my gratitude to my friends and to my family. Grazie per essermi stati vicini in questi anni meravigliosi

Preface

In this thesis we discuss several aspects of the implied volatility surface. We first derive some model independent results, linking tail probabilities to option price and implied volatility. We then apply these results to a specific stochastic volatility model, obtaining a complete picture of the asymptotic volatility smile for bounded maturity.

In Chapter 1 we present an extended summary of all the results obtained in this thesis. The details are contained in the following chapters, that are structured as follows.

In Chapter 2 we show that, under general conditions satisfied by many models, the probability tails of the log-price under the risk-neutral measure determine the behavior of European option prices and of the implied volatility, in the regime of either extremes strike (with bounded maturity) or short maturity. Our results provide a powerful extension of previous work by Benaim and Friz [BF09]. We discuss the application to some popular models, including Carr-Wu finite moment logstable moment, Heston's model and Merton's jump diffusion model.

In Chapter 3 we devote ourselves to the analysis of the implied volatility for a specific model, that has been recently shown by Andreoli, Caravenna, Dai Pra and Posta [ACDP12] to reproduce the multiscaling of moments and clustering of volatility observed in many financial series. Based on Chapter 2, this amounts to give sharp estimates on the tails of the log-price distribution. Although the moment generating function of the log-price is not known explicitly, we show that the tails can be well estimated via Large Deviation techniques, notably the Gärtner-Ellis theorem.

In Chapter 4 we propose a possible enrichment of the model, adding jumps to the log-price in order to take account of the so called leverage effect. We prove some basic results and we describe a natural one-parameter family of martingale measures for this enriched model. We also show that the price of European options can be expressed through a generalization of the celebrated Hull&White formula, by averaging the usual Black&Scholes formula with respect to both a random volatility and a random spot price.

Finally, in Chapter 5 we describe a numerical algorithm to price European option under the enriched model presented in Chapter 4, exploiting the generalized Hull&White formula. The algorithm uses a stratification method in order to improve the speed. Some preliminary results on the calibration of the model with real data, taken from the DAX index, are presented and discussed.

Contents

1	Overview of the thesis	1
1.1	Chapter 2. General smile asymptotics with bounded maturity	1
1.2	Chapter 3. The asymptotic smile of a multiscaling stochastic volatility model	2
1.3	Chapter 4. Enriching the model and pricing	3
1.4	Chapter 5. Simulation and numerics	3
2	General smile asymptotics with bounded maturity	5
2.1	Introduction	5
2.2	Examples	13
2.3	From option price to implied volatility	22
2.4	From tail probability to option price	28
2.5	Miscellanea	36
3	The asymptotic smile of a multiscaling stochastic volatility model	39
3.1	The model	39
3.2	Main results (I): implied volatility	43
3.3	Main results (II): tail probability and option price	45
3.4	Preliminary results	47
3.5	Proof of Theorem 3.2.1	48
3.6	Proof of Theorem 3.3.1	50
3.7	Proof of Theorem 3.3.2	50
3.8	Proof of Theorem 3.3.3	59
3.9	Numerical results	63
4	Enriching the model and pricing	67
4.1	An enriched version of the model	67
4.2	Pricing under the enriched model	70
4.3	Proof of the properties of the enriched model	72
4.4	Proof of the results on the pricing	77
5	Simulation and numerics	79
5.1	The Monte Carlo methods	79
5.2	The stratification method applied to our model	81
5.3	Calibration	83

A	The code	87
A.1	The C code	87
A.2	The MATLAB [®] code	96
B	Known properties of the model	101
B.1	Main properties of the model in the stationary setting	102
B.2	Deterministic τ_0	103
B.3	Proof in the stationary setting	105
B.4	Proof in the non-stationary setting	110

Chapter 1

Overview of the thesis

In this thesis we obtain asymptotic results on the asymptotic behavior of option prices, and of the related *implied volatility*, for a stochastic volatility model which exhibits multiscaling of moments, recently introduced in [ACDP12]. These results are described in Chapters 3 and 4, while some numerical investigations are presented in Chapter 5.

Our approach is based on explicit formulas that link the asymptotic behavior of the implied volatility to the *tail probability* of the log-price. These results, presented in Chapter 2, provide a powerful extension of previous work by Benaim and Friz [BF09] and are of independent interest, since they can be applied to a wide family of models.

We now give a short overview of the content of each chapter.

1.1 Chapter 2. General smile asymptotics with bounded maturity

The price of a European option is typically expressed in terms of the Black&Scholes *implied volatility* $\sigma_{\text{imp}}(\kappa, t)$, cf. [Gat06], where κ denotes the log-strike and t the maturity. Benaim and Friz [BF09] have provided explicit conditions on the log-return distribution to obtain the asymptotic behavior of $\sigma_{\text{imp}}(\kappa, t)$, in the special regime $|\kappa| \rightarrow \infty$ for fixed $t > 0$.

Here we strengthen and extend their results, allowing κ and t to vary simultaneously along an arbitrary curve, such that either $|\kappa| \rightarrow \infty$ with bounded t , or $t \rightarrow 0$ with arbitrary κ . Our results are organized as follows.

- First we provide universal formulas that link the asymptotic behavior of the implied volatility $\sigma_{\text{imp}}(\kappa, t)$ to that of the call $c(\kappa, t)$ and put $p(\kappa, t)$ option prices, cf. §2.1.2.
- Then we show that the asymptotic behavior of $c(\kappa, t)$ and $p(\kappa, t)$ can be linked explicitly to the tail probabilities $\bar{F}_t(\kappa) := \mathbb{P}(X_t > \kappa)$ and $F_t(\kappa) := \mathbb{P}(X_t \leq \kappa)$, where X_t denotes the risk-neutral log-return, cf. §2.1.3.

The main results are Theorems 2.1.1, 2.1.5 and 2.1.11.

As a consequence, whenever enough information on the tail probabilities is available, it is possible to write down explicitly the asymptotic behavior of the implied volatility. We illustrate this fact in Section 2.2, determining the complete asymptotic profile of the implied volatility with bounded maturity for the model of Carr&Wu, cf. Theorem 2.2.1. We also discuss the application of our results to the Heston and Merton models.

The application of our results to the multiscaling stochastic volatility model introduced in [ACDP12] is the subject of Chapter 3.

1.2 Chapter 3. The asymptotic smile of a multiscaling stochastic volatility model

In this chapter we apply the results of Chapter 2 to a stochastic volatility model that exhibits multiscaling of moments, recently introduced in [ACDP12]. Very briefly, the model can be described as follows: under the risk-neutral measure, the price $(S_t)_{t \geq 0}$ evolves according to the stochastic differential equation

$$\frac{dS_t}{S_t} = \sigma_t dB_t, \quad (1.2.1)$$

where $(B_t)_{t \geq 0}$ is a Brownian motion and $(\sigma_t)_{t \geq 0}$ is an *independent* process, function of three real parameters $D \in (0, \frac{1}{2})$, V and $\lambda \in (0, \infty)$, defined as follows: denoting by $(N_t)_{t \geq 0}$ a Poisson process independent of $(B_t)_{t \geq 0}$ of rate λ , with jump times $0 < \tau_1 < \tau_2 < \dots$,

$$\sigma_t = V \frac{\lambda^{D-\frac{1}{2}}}{\Gamma(2D)} (t - \tau_{N_t})^{D-\frac{1}{2}}. \quad (1.2.2)$$

In words, the volatility σ_t explodes at each jump time of the Poisson process (note that τ_{N_t} is the epoch of the last jump of the Poisson process before time t) after which it decays as an inverse power, with exponent tuned by D (see Figure 3.1 on page 40 in Chapter 3).

The properties of this model (under the historical measure) have been investigated in [ACDP12], and it was shown that interesting features emerge, namely:

- **Heavy tails:** the distribution of the log-price $X_t := \log(S_t/S_0)$ is asymptotically Gaussian for large time t , but asymptotically *heavy tailed for short time*.
- **Multiscaling of moments:** as $\Delta t \downarrow 0$, the moments $E(|X_{t+\Delta t} - X_t|^q)$ of the log-price rescale as $(\Delta t)^{A(q)}$, where $A(q) = \frac{q}{2}$ (as one would naively guess) only up to a critical moment $q < q^* := (\frac{1}{2} - D)^{-1} \in (2, \infty)$, while for $q > q^*$ one has $A(q) < \frac{q}{2}$.
- **Clustering of volatility:** the covariance between $|X_{t+h} - X_t|$ and $|X_{t+\Delta t+h} - X_{t+\Delta t}|$ decays exponentially fast for large Δt , but slower (polynomially) for $\Delta t = O(1)$.

In Chapter 3 we derive the asymptotic behavior of option prices and of the related implied volatility, showing that it displays interesting features. For instance, despite the price having continuous paths, the out-of-the-money implied volatility *diverges* in the small-maturity limit, with an explicit limiting shape displaying a very pronounced smile. More precisely, the following asymptotic formula holds both in the *deep out-of-the-money* regime ($|\kappa| \rightarrow \infty$ for fixed $t > 0$) and in the *short maturity* regime ($t \downarrow 0$ for fixed $\kappa \neq 0$):

$$\sigma_{\text{imp}}(\kappa, t) \sim A \left(\frac{|\kappa|/t}{\sqrt{\log(|\kappa|/t)}} \right)^{\frac{1-2D}{2-2D}}$$

where $A \in (0, \infty)$ is an explicit constant (depending on the parameters of the model).

We refer to Theorem 3.2.1 for the complete results about the asymptotics of the implied volatility, which cover a wide range of regimes for (κ, t) . The corresponding estimates for the tail probability and for the option price are given in Theorems 3.3.1, 3.3.2 and 3.3.3.

1.3 Chapter 4. Enriching the model and pricing

The model considered in Chapter 3 is a stochastic volatility model, cf. (1.2.1), in which the volatility process $(\sigma_t)_{t \geq 0}$ is *independent* of the Brownian motion $(B_t)_{t \geq 0}$ that drives the evolution of the price. It is well-known [RT96] that for such models the implied volatility is always a symmetric function of the log-strike, i.e. $\sigma_{\text{imp}}(-\kappa, t) = \sigma_{\text{imp}}(\kappa, t)$.

For this reason, in order to take into account the so-called *leverage effect*, we enrich the model introducing a jump component in the log-price, using the same Poisson process $(N_t)_{t \geq 0}$ that drives the evolution of the volatility $(\sigma_t)_{t \geq 0}$, cf. (1.2.2). The intuitive meaning is that shocks in the market, represented by jumps in the Poisson process, determine both an increase in the volatility and a jump in the price.

We thus introduce a further parameter $\varrho \in \mathbb{R}$, which represent the jump size in the log-price at shock-times: under the historical measure, the log-price X_t evolves as

$$dX_t = \sigma_t dX_t + \varrho d(N_t - \lambda t).$$

where $(\sigma_t)_{t \geq 0}$ is the same process as in (1.2.2). It is natural to wonder whether and how the properties of the original model are modified by the addition of jumps. We prove that we still have the *clustering of volatility* and the *heavy tails distribution* for small time, however the *multiscaling of moments* disappears, as we discuss in §4.1.

We then describe a natural one-parameter family of martingale measures for this enriched model (as well for the original model) indexed by the intensity $\tilde{\lambda} \in (0, \infty)$ of the Poisson process, which can be modified arbitrarily with respect to the original value $\lambda \in (0, \infty)$. Renaming $\tilde{\lambda}$ as λ for simplicity, under the enriched model the price evolves by

$$\frac{dS_t}{S_t} = \sigma_t dB_t + \varrho d(N_t - \lambda t). \tag{1.3.1}$$

Even though in the enriched model the volatility process $(\sigma_t)_{t \geq 0}$ is *not* independent of the Brownian motion $(B_t)_{t \geq 0}$, the price of European options can be expressed through a generalization of the celebrated Hull&White formula, i.e. by averaging the usual Black&Scholes formula with to both a *random volatility* and a *random spot price*, cf. Theorem 4.2.2. This allows for a *fast Monte Carlo evaluation of option prices*, as we discuss in Chapter 5.

1.4 Chapter 5. Simulation and numerics

Exploiting the generalized Hull&White formula described in Chapter 4, in Chapter 5 we describe a numerical algorithm to price European option under the enriched model, which uses a *stratification method* in order to improve the speed. The actual code for the both the *C* and the MATLAB[®] languages is given in Appendix A. Some preliminary results on the calibration of the model with real data from the DAX index are presented.

Chapter 2

General smile asymptotics with bounded maturity

In this chapter we provide explicit conditions on the distribution of risk-neutral log-returns which yield sharp asymptotic estimates on the implied volatility smile. These conditions extend previous results of Benaim and Friz [BF09] and are valid in great generality, both for extreme strike (with arbitrary bounded maturity, possibly varying with the strike) and for small maturity (with arbitrary strike, possibly varying with the maturity). Applications to popular models as the Carr-Wu finite moment logstable model, Merton's jump diffusion model, and Heston's model are discussed.

2.1 Introduction

The price of a European option is typically expressed in terms of the Black&Scholes *implied volatility* $\sigma_{\text{imp}}(\kappa, t)$, cf. [Gat06], where κ denotes the log-strike and t the maturity. Benaim and Friz [BF09] provide explicit conditions on the log-return distribution to obtain the asymptotic behavior of $\sigma_{\text{imp}}(\kappa, t)$, in the special regime $\kappa \rightarrow \pm\infty$ for fixed $t > 0$. In this chapter we strengthen and extend their results, *allowing κ and t to vary simultaneously along an arbitrary curve* such that either $|\kappa| \rightarrow \infty$ with bounded t , or $t \rightarrow 0$ with arbitrary κ .

This flexibility allows to determine *the asymptotics of $\sigma_{\text{imp}}(\kappa, t)$ as a surface*, when (κ, t) vary in open regions of the plane. We illustrate this fact in Section 2.2, where we apply our results to some concrete models (see Remarks 2.2.2, 2.2.4 and 2.2.5 below).

Our results are organized as follows.

- First we provide universal formulas that link the asymptotic behavior of the implied volatility $\sigma_{\text{imp}}(\kappa, t)$ to that of the call $c(\kappa, t)$ and put $p(\kappa, t)$ option prices, cf. §2.1.2.
- Then we show that the asymptotic behavior of the option prices $c(\kappa, t)$ and $p(\kappa, t)$ can be linked explicitly to the *tail probabilities* $\bar{F}_t(\kappa) := \mathbb{P}(X_t > \kappa)$ and $F_t(\kappa) := \mathbb{P}(X_t \leq \kappa)$, where X_t denotes the risk-neutral log-return, cf. §2.1.3.

Combining these results, whenever enough information on the tail probabilities is available, it is possible to write down explicitly the asymptotic behavior of the implied volatility.

2.1.1 The setting

We consider a generic stochastic process $(X_t)_{t \geq 0}$ representing the log-price of an asset, normalized by $X_0 := 0$. We work under the risk-neutral measure, that is (assuming zero interest rate) the price process $(S_t := e^{X_t})_{t \geq 0}$ is a martingale. European call and put options, with maturity $t > 0$ and a log-strike $\kappa \in \mathbb{R}$, are priced respectively

$$c(\kappa, t) = \mathbb{E}[(e^{X_t} - e^\kappa)^+], \quad p(\kappa, t) = \mathbb{E}[(e^\kappa - e^{X_t})^+], \quad (2.1.1)$$

and are linked by the *call-put parity* relation:

$$c(\kappa, t) - p(\kappa, t) = 1 - e^\kappa. \quad (2.1.2)$$

In all of our results, *we take limits along an arbitrary family (or “path”) of values of (κ, t)* . It is immaterial whether this is a sequence $((\kappa_n, t_n))_{n \in \mathbb{N}}$ or a curve $((\kappa_s, t_s))_{s \in [0, \infty)}$, therefore we omit subscripts. Without loss of generality, we assume that all the κ 's have the same sign (just consider separately the subfamilies with positive and negative κ 's). To simplify notation, we only consider positive families $\kappa \geq 0$ and give results for both κ and $-\kappa$.

Our main interest is for families of values of (κ, t) such that

$$\text{either } \kappa \rightarrow \infty \text{ with bounded } t, \quad \text{or } t \rightarrow 0 \text{ with arbitrary } \kappa \geq 0. \quad (2.1.3)$$

Note that (2.1.3) gathers many interesting regimes, namely:

- (1) $\kappa \rightarrow \infty$ and $t \rightarrow \bar{t} \in (0, \infty)$;
- (2) $\kappa \rightarrow \infty$ and $t \rightarrow 0$;
- (3) $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$ and $t \rightarrow 0$;
- (4) $\kappa \rightarrow 0$ and $t \rightarrow 0$.

Remarkably, while regime (4) needs to be handled separately, regimes (1)-(2)-(3) will be analyzed at once, as special instances of the case “ κ is bounded away from zero”.

Whenever (2.1.3) holds, one has (see §2.5.1)

$$c(\kappa, t) \rightarrow 0, \quad p(-\kappa, t) \rightarrow 0, \quad (2.1.4)$$

but relation (2.1.4) is more general, as it can be satisfied also when $t \rightarrow \infty$. Except for the results in §2.1.2, which are valid in complete generality under (2.1.4), we stick to the case of bounded t (we refer to [Te09, JKM13] for results in the regime $t \rightarrow \infty$).

A key quantity of interest is the *implied volatility* $\sigma_{\text{imp}}(\kappa, t)$ of the model, defined as the value of the volatility parameter $\sigma \in [0, \infty)$ that plugged into the Black&Scholes formula yields the given call and put prices $c(\kappa, t)$ and $p(\kappa, t)$ (see §2.3.2-§2.3.3 below). Note that $\sigma_{\text{imp}}(\kappa, t) = 0$ if $c(\kappa, t) = 0$ and, likewise, $\sigma_{\text{imp}}(-\kappa, t) = 0$ if $p(-\kappa, t) = 0$. Consequently, to avoid trivialities, we focus on families of (κ, t) such that $c(\kappa, t) > 0$ and $p(-\kappa, t) > 0$.

Throughout the chapter, we write $f(\kappa, t) \sim g(\kappa, t)$ to mean $f(\kappa, t)/g(\kappa, t) \rightarrow 1$. Let us recall a useful standard device, referred to as *subsequence argument*: to prove an asymptotic relation, such as e.g. $f(\kappa, t) \sim g(\kappa, t)$, along a given family of values of (κ, t) , it suffices to

show that from every subsequence one can extract a further sub-subsequence along which the relation holds. As a consequence, *in the proofs we may always assume that all quantities of interest have a (possibly infinite) limit*, e.g. $\kappa \rightarrow \bar{\kappa} \in [0, \infty]$ and $t \rightarrow \bar{t} \in [0, \infty)$, because this is always true extracting a suitable subsequence.

2.1.2 From option price to implied volatility

We first show that, whenever the option prices $c(\kappa, t)$ or $p(-\kappa, t)$ vanish, they determine the asymptotic behavior of the implied volatility through *explicit universal formulas*.

We need to introduce some notation. Denote by $\phi(\cdot)$ and $\Phi(\cdot)$ respectively the density and distribution function of a standard Gaussian (see (2.3.1) below), and define the function

$$D(z) := \frac{1}{z}\phi(z) - \Phi(-z), \quad \forall z > 0. \quad (2.1.5)$$

As we shown in §2.3.1 below, D is a smooth and strictly decreasing bijection from $(0, \infty)$ to $(0, \infty)$. Its inverse $D^{-1} : (0, \infty) \rightarrow (0, \infty)$ is also smooth, strictly decreasing and satisfies

$$D^{-1}(y) \sim \sqrt{2(-\log y)} \quad \text{as } y \downarrow 0, \quad D^{-1}(y) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{y} \quad \text{as } y \uparrow \infty. \quad (2.1.6)$$

The following theorem, proved in Section 2.3, describes the link between option price and implied volatility asymptotics, extending Benaim and Friz [BF09, Lemma 3.3]. As we discuss in Remark 2.1.3 below, it overlaps with recent results by Gao and Lee [GL14].

Theorem 2.1.1 (From option price to implied volatility). *Consider an arbitrary family of values of (κ, t) with $\kappa \geq 0$ and $t > 0$, such that $c(\kappa, t) \rightarrow 0$, resp. $p(-\kappa, t) \rightarrow 0$.*

- Case of κ bounded away from zero (i.e. $\liminf \kappa > 0$).

$$\begin{aligned} \sigma_{\text{imp}}(\kappa, t) &\sim \left(\sqrt{\frac{-\log c(\kappa, t)}{\kappa}} + 1 - \sqrt{\frac{-\log c(\kappa, t)}{\kappa}} \right) \sqrt{\frac{2\kappa}{t}}, \quad \text{resp.} \\ \sigma_{\text{imp}}(-\kappa, t) &\sim \left(\sqrt{\frac{-\log p(-\kappa, t)}{\kappa}} - \sqrt{\frac{-\log p(-\kappa, t)}{\kappa}} - 1 \right) \sqrt{\frac{2\kappa}{t}}. \end{aligned} \quad (2.1.7)$$

- Case of $\kappa \rightarrow 0$, with $\kappa > 0$.

$$\begin{aligned} \sigma_{\text{imp}}(\kappa, t) &\sim \frac{1}{D^{-1}\left(\frac{c(\kappa, t)}{\kappa}\right)} \frac{\kappa}{\sqrt{t}}, \quad \text{resp.} \\ \sigma_{\text{imp}}(-\kappa, t) &\sim \frac{1}{D^{-1}\left(\frac{p(-\kappa, t)}{\kappa}\right)} \frac{\kappa}{\sqrt{t}}. \end{aligned} \quad (2.1.8)$$

- Case of $\kappa = 0$.

$$\sigma_{\text{imp}}(0, t) \sim \sqrt{2\pi} \frac{c(0, t)}{\sqrt{t}} = \sqrt{2\pi} \frac{p(0, t)}{\sqrt{t}}. \quad (2.1.9)$$

Note that Theorem 2.1.1 requires *no assumption on the model*: the link between $\sigma_{\text{imp}}(\kappa, t)$ and the option prices $c(\kappa, t)$ and $p(\kappa, t)$ is *universal*, being essentially a statement about the inversion of Black&Scholes formula (see Theorem 2.3.3 for an explicit reformulation).

Remark 2.1.2. The formulas in Theorem 2.1.1 become more explicit when additional information on the asymptotic behavior of $c(\kappa, t)$ and $p(-\kappa, t)$ is available. For instance, whenever $\frac{-\log c(\kappa, t)}{\kappa} \rightarrow \infty$, resp. $\frac{-\log p(-\kappa, t)}{\kappa} \rightarrow \infty$, formula (2.1.7) reduces to

$$\sigma_{\text{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t(-\log c(\kappa, t))}}, \quad \text{resp.} \quad \sigma_{\text{imp}}(-\kappa, t) \sim \frac{\kappa}{\sqrt{2t(-\log p(-\kappa, t))}}. \quad (2.1.10)$$

Likewise, using the estimates in (2.1.6), formula (2.1.8) can be rewritten as follows:

$$\sigma_{\text{imp}}(\kappa, t) \sim \begin{cases} \frac{\kappa}{\sqrt{2t(-\log(c(\kappa, t)/\kappa))}} & \text{if } \frac{c(\kappa, t)}{\kappa} \rightarrow 0; \\ \frac{\kappa}{D^{-1}(a)\sqrt{t}} & \text{if } \frac{c(\kappa, t)}{\kappa} \rightarrow a \in (0, \infty); \\ \sqrt{2\pi} \frac{c(\kappa, t)}{\sqrt{t}} & \text{if } \frac{c(\kappa, t)}{\kappa} \rightarrow \infty, \text{ or if } \kappa = 0, \end{cases} \quad (2.1.11)$$

and analogously for $\sigma_{\text{imp}}(-\kappa, t)$, just replacing $c(\kappa, t)$ by $p(-\kappa, t)$.

It is interesting to observe that the first relation in (2.1.10) coincides with the first line of (2.1.11) when $\kappa \rightarrow 0$ slowly enough, more precisely when

$$-\log \kappa = o(-\log c(\kappa, t)), \quad (2.1.12)$$

so that $-\log(c(\kappa, t)/\kappa) \sim -\log c(\kappa, t)$. This means that *relations (2.1.7) and (2.1.8) match at the boundary* of their respective domain of validity. On the other hand, when $\kappa \rightarrow 0$ fast enough so that (2.1.12) fails, relation (2.1.7) must be replaced by (2.1.8)-(2.1.9).

Remark 2.1.3. Taking the square of both sides of the first line of (2.1.7), one can rewrite it as

$$\frac{\sigma_{\text{imp}}(\kappa, t)^2 t}{\kappa} \sim \psi\left(\frac{-\log c(\kappa, t)}{\kappa}\right), \quad \text{with} \quad \psi(x) := 2 - 4[\sqrt{x^2 + x} - x],$$

which is the key formula in Lemma 3.3 of Benaim and Friz [BF09] (who considered the regime $\kappa \rightarrow \infty$ for fixed t and made some additional assumptions). Theorem 2.1.1 provides a substantial extension, allowing for *any regime of* (κ, t) and making no extra assumptions.

We point out that equation (2.1.7) in full generality has been recently proved by Gao and Lee [GL14] (extending previous results of Lee [Le04], Roper and Rutkowski [RR09], Gulisashvili [Gu10]). Actually, Gao and Lee *prove much more than* (2.1.7), since their approach provides explicit estimates for the error and allows to obtain higher order asymptotics. On the other hand, in [GL14] condition (2.1.12) is assumed (cf. equation (4.2) therein), which means that all regimes in which $\kappa \rightarrow 0$ “fast enough” are excluded from their analysis.

Summarizing, our Theorem 2.1.1 provides a simple and comprehensive account of first order asymptotics for the implied volatility as a function of the option price, which can be applied to *any* family of (κ, t) such that $c(\kappa, t) \rightarrow 0$, resp. $p(-\kappa, t) \rightarrow 0$ (with no restriction such as (2.1.12)). This is especially useful for the results in the next subsection, which cover all possible regimes of (κ, t) with bounded t . For these reasons, despite the overlap with [GL14], we give a complete and self-contained proof of Theorem 2.1.1 in Section 2.3.

2.1.3 From tail probability to option price

For Theorem 2.1.1 to be concretely useful, one needs to control the asymptotic behavior of $c(\kappa, t)$ and $p(-\kappa, t)$. We are going to show that this can be extracted from the asymptotic behavior of the *tail probabilities* of the risk-neutral log-price $(X_t)_{t \geq 0}$, defined by:

$$\bar{F}_t(\kappa) := \mathbb{P}(X_t > \kappa), \quad F_t(-\kappa) := \mathbb{P}(X_t \leq -\kappa). \quad (2.1.13)$$

We need to distinguish two regimes for (κ, t) , namely when the tail probabilities converge to a strictly positive limit (*typical deviations*) or when they vanish (*atypical deviations*).

Atypical deviations. We first focus on families of values of (κ, t) such that

$$\bar{F}_t(\kappa) \rightarrow 0, \quad \text{resp.} \quad F_t(-\kappa) \rightarrow 0. \quad (2.1.14)$$

We stress that this includes regimes (1), (2) and (3) on page 6, and also regime (4) provided $\kappa \rightarrow 0$ sufficiently slow. We need to formulate a regularity assumption on the decay of $\bar{F}_t(\kappa)$, resp. $F_t(-\kappa)$, which is a natural generalization of the *regular variation* condition of Benaim and Friz [BF09] (see Remark 2.1.7 below for more details).

Hypothesis 2.1.4 (Regular decay of tail probability). *The family of values of (κ, t) with $\kappa > 0$, $t > 0$ satisfies (2.1.14), and for every $\varrho \in [1, \infty)$ the following limit exists in $[0, +\infty]$:*

$$I_+(\varrho) := \lim \frac{\log \bar{F}_t(\varrho\kappa)}{\log \bar{F}_t(\kappa)}, \quad \text{resp.} \quad I_-(\varrho) := \lim \frac{\log F_t(-\varrho\kappa)}{\log F_t(-\kappa)}, \quad (2.1.15)$$

and moreover

$$\lim_{\varrho \downarrow 1} I_+(\varrho) = 1, \quad \text{resp.} \quad \lim_{\varrho \downarrow 1} I_-(\varrho) = 1. \quad (2.1.16)$$

(The limits in (2.1.15) are taken along the given family of values of (κ, t) .)

Further assumptions on $I_{\pm}(\cdot)$, depending on the regime of κ , will be required below, coupled to suitable *moment conditions*, that we state here for convenience.

- Given $\eta \in (0, \infty)$, the first moment condition is

$$\limsup \mathbb{E}[e^{(1+\eta)X_t}] < \infty, \quad (2.1.17)$$

where the limsup is taken along the given family of values of (κ, t) (however, only t enters in this relation). Note that if $t \leq T$ it suffices to require that

$$\mathbb{E}[e^{(1+\eta)X_T}] < \infty, \quad (2.1.18)$$

because $(e^{(1+\eta)X_t})_{t \geq 0}$ is a submartingale and hence $\mathbb{E}[e^{(1+\eta)X_t}] \leq \mathbb{E}[e^{(1+\eta)X_T}]$.

- Always for $\eta \in (0, \infty)$, the second moment condition is

$$\limsup \mathbb{E} \left[\left| \frac{e^{X_t} - 1}{\kappa} \right|^{1+\eta} \right] < \infty, \quad (2.1.19)$$

along the given family of values of (κ, t) . Note that for $\eta = 1$ this simplifies to

$$\exists C \in (0, \infty) : \quad \mathbb{E}[e^{2X_t}] \leq 1 + C\kappa^2. \quad (2.1.20)$$

The following theorems, proved in §2.4.1, give the link between tail probabilities and option prices. Due to different assumptions, we present separately the results on $c(\kappa, t)$ and $p(-\kappa, t)$.

Theorem 2.1.5 (Right-tail atypical deviations). *Consider a family of values of (κ, t) with $\kappa > 0$, $t > 0$ such that Hypothesis 2.1.4 is satisfied by the right tail probability $\bar{F}_t(\kappa)$.*

- Case of κ bounded away from zero and t bounded away from infinity ($\liminf \kappa > 0$, $\limsup t < \infty$). Let the moment condition (2.1.17) hold for every $\eta > 0$, or alternatively let it hold only for some $\eta > 0$ but in addition assume that

$$I_+(\varrho) \geq \varrho, \quad \forall \varrho \geq 1. \quad (2.1.21)$$

Then

$$\log c(\kappa, t) \sim \log \bar{F}_t(\kappa) + \kappa, \quad (2.1.22)$$

which yields, by Theorem 2.1.1,

$$\sigma_{\text{imp}}(\kappa, t) \sim \left(\sqrt{\frac{-\log \bar{F}_t(\kappa)}{\kappa}} - \sqrt{\frac{-\log \bar{F}_t(\kappa)}{\kappa} - 1} \right) \sqrt{\frac{2\kappa}{t}}. \quad (2.1.23)$$

In the special case when $-\log \bar{F}_t(\kappa)/\kappa \rightarrow \infty$, assumption (2.1.21) can be relaxed to

$$\lim_{\varrho \rightarrow \infty} I_+(\varrho) = \infty, \quad (2.1.24)$$

relation (2.1.22) reduces to

$$\log c(\kappa, t) \sim \log \bar{F}_t(\kappa), \quad (2.1.25)$$

and (2.1.23) simplifies to

$$\sigma_{\text{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t(-\log \bar{F}_t(\kappa))}}. \quad (2.1.26)$$

- Case of $\kappa \rightarrow 0$ and $t \rightarrow 0$. Let the moment condition (2.1.19) hold for every $\eta > 0$, or alternatively let it hold only for some $\eta > 0$ but in addition assume (2.1.24). Then

$$\log(c(\kappa, t)/\kappa) \sim \log \bar{F}_t(\kappa), \quad (2.1.27)$$

which yields, by Theorem 2.1.1, precisely the same asymptotics (2.1.26) for $\sigma_{\text{imp}}(\kappa, t)$.

Theorem 2.1.6 (Left-tail atypical deviations). *Consider a family of values of (κ, t) with $\kappa > 0$, $t > 0$ such that Hypothesis 2.1.4 is satisfied by the left tail probability $F_t(-\kappa)$.*

- Case of κ bounded away from zero and t bounded away from infinity ($\liminf \kappa > 0$, $\limsup t < \infty$). With no moment condition and no extra assumption on $I_-(\cdot)$, one has

$$\log p(-\kappa, t) \sim \log F_t(-\kappa) - \kappa, \quad (2.1.28)$$

which yields, by Theorem 2.1.1,

$$\sigma_{\text{imp}}(-\kappa, t) \sim \left(\sqrt{\frac{-\log F_t(-\kappa)}{\kappa} + 1} - \sqrt{\frac{-\log F_t(-\kappa)}{\kappa}} \right) \sqrt{\frac{2\kappa}{t}}. \quad (2.1.29)$$

In the special case when $-\log F_t(-\kappa)/\kappa \rightarrow \infty$, relation (2.1.28) reduces to

$$\log p(-\kappa, t) \sim \log F_t(-\kappa), \quad (2.1.30)$$

and (2.1.29) simplifies to

$$\sigma_{\text{imp}}(-\kappa, t) \sim \frac{\kappa}{\sqrt{2t(-\log F_t(-\kappa))}}. \quad (2.1.31)$$

- Case of $\kappa \rightarrow 0$ and $t \rightarrow 0$. Let the moment condition (2.1.19) hold for every $\eta > 0$, or alternatively let it hold only for some $\eta > 0$ but in addition assume that

$$\lim_{\varrho \uparrow \infty} I_-(\varrho) = \infty. \quad (2.1.32)$$

Then

$$\log(p(-\kappa, t)/\kappa) \sim \log F_t(-\kappa), \quad (2.1.33)$$

which yields, by Theorem 2.1.1, precisely the same asymptotics (2.1.31) for $\sigma_{\text{imp}}(-\kappa, t)$.

Remark 2.1.7. Let us compare our Hypothesis 2.1.4 with the key assumption of Benaim and Friz [BF09], the *regular variation of the tail probabilities*, i.e. there exist $\alpha > 0$ and a slowly varying function[†] $L(\cdot) = L_t(\cdot)$ such that, as $\kappa \rightarrow \infty$ for fixed $t > 0$,

$$\log \bar{F}_t(\kappa) \sim -L(\kappa) \kappa^\alpha, \quad \text{resp.} \quad \log F_t(-\kappa) \sim -L(\kappa) \kappa^\alpha. \quad (2.1.34)$$

If (2.1.34) holds, conditions (2.1.15) and (2.1.16) are satisfied, with $I_\pm(\varrho) = \varrho^\alpha$. Remarkably, in the special regime $\kappa \rightarrow \infty$ with fixed t , conditions (2.1.15) and (2.1.16) are actually *equivalent* to (2.1.34), by [BGT89, Theorem 1.4.1]. Thus Hypothesis 2.1.4 is a natural extension of the regular variation assumption of Benaim and Friz, when the maturity t is allowed to vary.

Remark 2.1.8. The assumptions for left-tail asymptotics in Theorem 2.1.6 are weaker than those for right-tail asymptotics in Theorem 2.1.5. For instance, the left-tail condition $\mathbb{E}[e^{-\eta X_T}] < \infty$ required in [BF09, Theorem 1.2] is not needed, allowing to treat the case of a *polynomially decaying* left tail, like in the Carr-Wu model described in Section 2.2.

[†]A positive function $L(\cdot)$ is slowly varying if $\lim_{x \rightarrow \infty} L(\varrho x)/L(x) = 1$ for all $\varrho > 0$.

Remark 2.1.9. We stress that the “special case” conditions

$$-\frac{\log \bar{F}_t(\kappa)}{\kappa} \rightarrow \infty, \quad \text{resp.} \quad -\frac{\log F_t(-\kappa)}{\kappa} \rightarrow \infty, \quad (2.1.35)$$

are always fulfilled if $t \rightarrow 0$ and κ is bounded away from infinity (say $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$).

For families of (κ, t) with $\kappa \rightarrow \infty$, conditions (2.1.35) are satisfied if $\limsup \mathbb{E}[e^{(1+\eta)X_t}] < \infty$, resp. $\limsup \mathbb{E}[e^{-\eta X_t}] < \infty$, for every $\eta \in (0, \infty)$, by Markov’s inequality (see (2.4.5) below).

Typical deviations. We next focus on the case when $\kappa \rightarrow 0$ and $t \rightarrow 0$ in such a way that the tail probability $\bar{F}_t(\kappa)$, resp. $F_t(-\kappa)$ converges to a strictly positive limit. To deal with this regime, we make the following natural assumption.

Hypothesis 2.1.10 (Small time scaling). *There is a positive function $(\gamma_t)_{t>0}$ with $\lim_{t \downarrow 0} \gamma_t = 0$ such that X_t/γ_t converges in law as $t \downarrow 0$ to some random variable Y :*

$$\frac{X_t}{\gamma_t} \xrightarrow[t \downarrow 0]{d} Y. \quad (2.1.36)$$

Note that (2.1.36) is a condition on the tail probabilities: for all $a \geq 0$ with $\mathbb{P}(Y = a) = 0$,

$$\bar{F}_t(a\gamma_t) \rightarrow \mathbb{P}(Y > a), \quad F_t(a\gamma_t) \rightarrow \mathbb{P}(Y \leq a). \quad (2.1.37)$$

In particular, the limits in (2.1.37) are strictly positive for every $a \geq 0$ if the support of the law of Y is unbounded from above and below.

We can finally state the following result, proved in §2.4.2 below.

Theorem 2.1.11 (Right- and left-tail typical deviations). *Assume that Hypothesis 2.1.10 is satisfied, and moreover the moment condition (2.1.19) holds for some $\eta > 0$ with $\kappa = \gamma_t$:*

$$\exists \eta > 0 : \quad \limsup_{t \rightarrow 0} \mathbb{E} \left[\left| \frac{e^{X_t} - 1}{\gamma_t} \right|^{1+\eta} \right] < \infty. \quad (2.1.38)$$

Then the random variable Y in (2.1.36) is in L^1 and satisfies $\mathbb{E}[Y] = 0$.

For any family of values of (κ, t) such that

$$t \rightarrow 0 \quad \text{and} \quad \frac{\kappa}{\gamma_t} \rightarrow a \in [0, \infty),$$

assuming that $\mathbb{P}(Y > a) > 0$, resp. $\mathbb{P}(Y < -a) > 0$, one has

$$c(\kappa, t) \sim \gamma_t \mathbb{E}[(Y - a)^+], \quad \text{resp.} \quad p(-\kappa, t) \sim \gamma_t \mathbb{E}[(Y + a)^-]. \quad (2.1.39)$$

This yields, by Theorem 2.1.1,

$$\sigma_{\text{imp}}(\pm\kappa, t) \sim C_{\pm}(a) \frac{\gamma_t}{\sqrt{t}}, \quad \text{with} \quad C_{\pm}(a) = \begin{cases} \frac{a}{D^{-1}(\frac{\mathbb{E}[(Y \mp a)^{\pm}]}{a})} & \text{if } a > 0, \\ \sqrt{2\pi} \mathbb{E}[Y^{\pm}] & \text{if } a = 0. \end{cases} \quad (2.1.40)$$

2.1.4 Discussion and structure of the chapter

Theorems 2.1.5, 2.1.6 and 2.1.11 are useful because their assumptions, involving asymptotics for the tail probabilities $\overline{F}_t(\kappa)$ and $F_t(-\kappa)$, can be verified for concrete models (see Section 2.2 for some examples). The difference between the regimes of typical and atypical deviations can be described as follows:

- for typical deviations, the key assumption is Hypothesis 2.1.10, which concerns the *weak convergence* of X_t , cf. (2.1.36)-(2.1.37);
- for atypical deviations, the key assumption is Hypothesis 2.1.4, which concerns the *large deviations* properties of X_t , cf. (2.1.15)-(2.1.16).

In particular, it is worth stressing that Hypothesis 2.1.4 requires sharp asymptotics only for *the logarithm of the tail probabilities* $\log \overline{F}_t(\kappa)$ and $\log F_t(-\kappa)$, and not for the tail probabilities themselves, which would be a considerably harder task (out of reach for many models). As a consequence, Hypothesis 2.1.4 can often be checked through the celebrated *Gärtner-Ellis Theorem* [DZ98, Theorem 2.3.6], which yields sharp asymptotics on $\log \overline{F}_t(\kappa)$ and $\log F_t(-\kappa)$ under suitable conditions on the moment generating function of X_t .

The rest of the chapter is structured as follows.

- In Section 2.2 we apply our results to the finite moment logstable model of Carr-Wu, determining the complete asymptotic behavior of the implied volatility smile for bounded maturity. We then discuss the Heston model and the Merton model, while a stochastic volatility model recently introduced in [ACDP12] will be discussed in Chapter 3.
- In Section 2.3 we prove Theorem 2.1.1.
- In Section 2.4 we prove Theorems 2.1.5, 2.1.6 and 2.1.11.
- Finally, a few technical points have been deferred to Section 2.5.

2.2 Examples

We apply our main results to some models: the Carr-Wu model in §2.2.1, the Heston model in §2.2.2 and the Merton model in §2.2.3.

2.2.1 Carr-Wu's Finite Moment Logstable Model

Carr and Wu [CW04] consider a model where the log-strike X_t has characteristic function

$$\mathbb{E}[e^{iuX_t}] = e^{t[iu\mu - |u|^\alpha \sigma^\alpha (1 + i \operatorname{sign}(u) \tan(\frac{\pi\alpha}{2}))]}, \quad (2.2.1)$$

where $\sigma \in (0, \infty)$, $\alpha \in (1, 2]$, while $\mu := \sigma^\alpha / \cos(\frac{\pi\alpha}{2})$ in the risk-neutral measure, cf. [CW04, Proposition 1]. The moment generating function of X_t is

$$\mathbb{E}[e^{\lambda X_t}] = \begin{cases} e^{[\lambda\mu - \frac{(\lambda\sigma)^\alpha}{\cos(\frac{\pi\alpha}{2})}]t} & \text{if } \lambda \geq 0, \\ +\infty & \text{if } \lambda < 0. \end{cases} \quad (2.2.2)$$

Note that as $\alpha \rightarrow 2$ one recovers Black&Scholes model with volatility $\sqrt{2}\sigma$, cf. §2.3.2 below.

Let Y denote a random variable with characteristic function

$$\mathbb{E}[e^{iuY}] = e^{-|u|^\alpha(1+i\operatorname{sign}(u)\tan(\frac{\pi\alpha}{2}))}, \quad (2.2.3)$$

i.e. Y has a strictly stable law with index α and skewness parameter $\beta = -1$, and $\mathbb{E}[Y] = 0$. Applying Theorems 2.1.5, 2.1.6 and 2.1.11, we obtain the following complete characterization of the volatility smile asymptotics with bounded maturity for this model.

Theorem 2.2.1 (Smile asymptotics of Carr-Wu model). *The following asymptotics hold.*

- *Atypical deviations. Consider any family of (κ, t) such that*

$$0 < t \leq T \quad \text{for some fixed } T < \infty, \text{ and } \frac{\kappa}{t^{1/\alpha}} \rightarrow \infty. \quad (2.2.4)$$

(This includes, in particular, the regimes (1), (2), (3) on page 6, as well as part of regime (4).) Then one has the right-tail asymptotics

$$\sigma_{\text{imp}}(\kappa, t) \sim B_\alpha \left(\frac{\kappa}{t} \right)^{-\frac{2-\alpha}{2(\alpha-1)}}, \quad \text{where } B_\alpha := \frac{(\alpha\sigma)^{\frac{\alpha/2}{\alpha-1}}}{\sqrt{2} |\cos(\frac{\pi\alpha}{2})|^{\frac{1/2}{\alpha-1}}}, \quad (2.2.5)$$

and the left-tail asymptotics

$$\sigma_{\text{imp}}(-\kappa, t) \sim \left(\sqrt{\frac{\log \frac{\kappa^\alpha}{t}}{\kappa} + 1} - \sqrt{\frac{\log \frac{\kappa^\alpha}{t}}{\kappa}} \right) \sqrt{\frac{2\kappa}{t}}, \quad (2.2.6)$$

which can be made more explicit as follows:

$$\sigma_{\text{imp}}(-\kappa, t) \sim \begin{cases} \sqrt{\frac{2\kappa}{t}} & \text{if } \frac{\kappa}{\log \frac{1}{t}} \rightarrow \infty, \\ \frac{\sqrt{1+a}-1}{\sqrt{a}} \sqrt{\frac{2\kappa}{t}} & \text{if } \frac{\kappa}{\log \frac{1}{t}} \rightarrow a \in (0, \infty), \\ \frac{\kappa}{\sqrt{2t \log \frac{\kappa^\alpha}{t}}} & \text{if } \frac{\kappa}{\log \frac{1}{t}} \rightarrow 0. \end{cases} \quad (2.2.7)$$

- *Typical deviations. For any family of (κ, t) with*

$$t \rightarrow 0, \quad \frac{\kappa}{t^{1/\alpha}} \rightarrow a \in [0, \infty), \quad (2.2.8)$$

one has

$$\sigma_{\text{imp}}(\pm\kappa, t) \sim C_\pm(a) t^{\frac{2-\alpha}{2\alpha}}, \quad \text{with } C_\pm(a) := \begin{cases} \frac{a}{D^{-1}(\frac{\mathbb{E}[(\sigma Y \mp a)^\pm]})} & \text{if } a > 0, \\ \sqrt{2\pi} \sigma \mathbb{E}[Y^\pm] & \text{if } a = 0. \end{cases} \quad (2.2.9)$$

Remark 2.2.2 (Surface asymptotics for the Carr-Wu model). The fact that relations (2.2.5) and (2.2.6) hold for *any* family of (κ, t) satisfying (2.2.4) yields interesting consequences. In fact, for any $T \in (0, \infty)$ and $\varepsilon > 0$, we claim that there exists $M = M(\varepsilon, T) \in (0, \infty)$ such that the following inequalities hold *for all* (κ, t) *in the region* $\mathcal{A}_{T,M} := \{0 < t \leq T, \kappa > Mt^{1/\alpha}\}$:

$$(1 - \varepsilon)B_\alpha \left(\frac{\kappa}{t} \right)^{-\frac{2-\alpha}{2(\alpha-1)}} \leq \sigma_{\text{imp}}(\kappa, t) \leq (1 + \varepsilon)B_\alpha \left(\frac{\kappa}{t} \right)^{-\frac{2-\alpha}{2(\alpha-1)}}; \quad (2.2.10)$$

analogous inequalities can be written for $\sigma_{\text{imp}}(-\kappa, t)$, using relations (2.2.6)-(2.2.7). Likewise, in the typical deviations regime, by (2.2.9), we claim that for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ such that *for all* (κ, t) *in the region* $\mathcal{B}_\delta := \{0 < t < \delta, 0 \leq \kappa < \delta\}$ one has

$$(1 - \varepsilon)C_\pm \left(\frac{\kappa}{t^{1/\alpha}} \right) t^{\frac{2-\alpha}{2\alpha}} \leq \sigma_{\text{imp}}(\pm\kappa, t) \leq (1 + \varepsilon)C_\pm \left(\frac{\kappa}{t^{1/\alpha}} \right) t^{\frac{2-\alpha}{2\alpha}}, \quad (2.2.11)$$

Relations like (2.2.10) and (2.2.11) provide *uniform approximations* of the volatility surface $\sigma_{\text{imp}}(\kappa, t)$ that hold *for* (κ, t) *in open regions of the plane*, and not only along “thin lines”.

The proof of the above relations is simple. Let us focus on (2.2.10), for definiteness, and assume by contradiction that there exist $T, \varepsilon \in (0, \infty)$ such that for every $M \in (0, \infty)$ relation (2.2.10) *fails* for some $(\kappa_M, t_M) \in \mathcal{A}_{T,M}$; then the family $((\kappa_M, t_M))_{M \in (0, \infty)}$ satisfies (2.2.4) but (2.2.5) does not hold, contradicting Theorem 2.2.1.

Proof of Theorem 2.2.1. If we set

$$Y_t := \frac{X_t - \mu t}{\sigma t^{1/\alpha}}, \quad (2.2.12)$$

it follows by (2.2.1) that Y_t has the same distribution as Y in (2.2.3), because

$$\mathbb{E}[e^{iuY_t}] = \mathbb{E}[e^{iuY}] = e^{-|u|^\alpha(1+i\text{sign}(u)\tan(\frac{\pi\alpha}{2}))}. \quad (2.2.13)$$

It follows by (2.2.12) that

$$\frac{X_t}{t^{1/\alpha}} \xrightarrow[t \downarrow 0]{d} \sigma Y, \quad (2.2.14)$$

hence Hypothesis 2.1.10 is satisfied with $\gamma_t := t^{1/\alpha}$.

It is well-known that Y has a density which is strictly positive everywhere, hence $\mathbb{P}(Y > a) > 0$ and $\mathbb{P}(Y < -a) > 0$ for all $a \in \mathbb{R}$. We also note that the right tail of Y has a super-exponential decay: as $\kappa \rightarrow \infty$

$$\log \mathbb{P}(Y > k) \sim -\tilde{B}_\alpha \kappa^{\alpha/(\alpha-1)} \quad \text{where} \quad \tilde{B}_\alpha := \frac{\alpha-1}{\alpha} \left(\frac{|\cos(\frac{\pi\alpha}{2})|}{\alpha} \right)^{1/(\alpha-1)}, \quad (2.2.15)$$

cf. [CW04, Property 1 and references therein]. On the other hand the left tail is polynomial: there exists $c = c_\alpha \in (0, \infty)$ such that

$$\mathbb{P}(Y \leq -\kappa) \sim \frac{c}{\kappa^\alpha}, \quad \text{hence} \quad \log \mathbb{P}(Y \leq -\kappa) \sim -\alpha \log \kappa. \quad (2.2.16)$$

Recalling that $\bar{F}_t(\kappa) := \mathbb{P}(X_t > \kappa)$ and $F_t(-\kappa) := \mathbb{P}(X_t \leq \kappa)$, by (2.2.12) we can write

$$\bar{F}_t(\kappa) = \mathbb{P}\left(Y > \frac{k - \mu t}{\sigma t^{1/\alpha}}\right), \quad F_t(-\kappa) = \mathbb{P}\left(Y \leq \frac{-k - \mu t}{\sigma t^{1/\alpha}}\right), \quad (2.2.17)$$

hence we can transfer the estimates (2.2.15) and (2.2.16) to X_t .

Henceforth we consider separately the regimes of atypical deviations (2.2.4), and that of typical deviations (2.2.8). Note that it is easy to check that (2.2.6) is equivalent to (2.2.7).

Atypical deviations

Let us fix an arbitrary family of values of (κ, t) satisfying (2.2.4). Then also $\kappa/t \rightarrow \infty$ (because $\alpha > 1$ and $0 < t \leq T$), hence

$$\frac{\kappa - \mu t}{\sigma t^{1/\alpha}} \sim \frac{\kappa}{\sigma t^{1/\alpha}} \rightarrow \infty, \quad \frac{-\kappa - \mu t}{\sigma t^{1/\alpha}} \sim \frac{-\kappa}{\sigma t^{1/\alpha}} \rightarrow -\infty.$$

By (2.2.15), (2.2.16) and (2.2.17) we then obtain

$$\log \bar{F}_t(\kappa) \sim -\tilde{B}_\alpha \left(\frac{\kappa}{\sigma t^{1/\alpha}}\right)^{\alpha/(\alpha-1)}, \quad \log F_t(-\kappa) \sim -\log \frac{\kappa^\alpha}{t}. \quad (2.2.18)$$

Let us now check the assumptions of Theorem 2.1.5.

- The first relation in (2.2.18) shows that Hypothesis 2.1.4 is satisfied by the right tail $\bar{F}_t(\kappa)$, with $I_+(\varrho) = \varrho^{\alpha/(\alpha-1)}$. Note that $I_+(\varrho) \geq \varrho$ for all $\varrho \geq 1$, since $\alpha > 1$, hence also condition (2.1.21) is satisfied.
- Condition (2.1.17) is satisfied because (2.1.18) holds for all $T > 0$ and $\eta > 0$, by (2.2.2).
- It remains to check condition (2.1.19). As we prove below, for all $\eta \in (0, \alpha - 1)$ and $T > 0$ there are constants $A, B, C \in (0, \infty)$, depending on η, T and on the parameters α, σ , such that for all $0 < t \leq T$ and $\kappa \geq 0$ the following inequality holds:

$$\mathbb{E}\left[\left|\frac{e^{X_t} - 1}{\kappa}\right|^{1+\eta}\right] \leq A\left(\left(\frac{t^{1/\alpha}}{\kappa}\right)^B + C\right). \quad (2.2.19)$$

In particular, since $\kappa/t^{1/\alpha} \rightarrow \infty$ by assumption (2.2.4), condition (2.1.19) is satisfied.

Applying Theorem 2.1.5, since $-\log \bar{F}_t(\kappa)/\kappa \rightarrow \infty$ by the first relation in (2.2.18), the asymptotic behavior of $\sigma_{\text{imp}}(\kappa, t)$ is given by (2.1.26), which by (2.2.18) coincides with (2.2.5).

Next we want to apply Theorem 2.1.6. By the second relation in (2.2.18), Hypothesis 2.1.4 is satisfied by the left tail $F_t(-\kappa)$, with $I_-(\varrho) \equiv 1$. If κ is bounded away from zero, the asymptotic behavior of $\sigma_{\text{imp}}(\kappa, t)$ is given by (2.1.29), which by (2.2.18) yields precisely (2.2.6).

If $\kappa \rightarrow 0$ we cannot apply directly Theorem 2.1.6, because the moment condition (2.1.19) is satisfied only for some $\eta > 0$, and condition (2.1.32) is not satisfied, since $I_-(\varrho) \equiv 1$. However, we can show that (2.1.33) still holds by direct estimates. By (2.1.1)

$$p(-\kappa, t) = \mathbb{E}[(e^{-\kappa} - e^{X_t})\mathbf{1}_{\{X_t < -\kappa\}}] \geq \mathbb{E}[(e^{-\kappa} - e^{X_t})\mathbf{1}_{\{X_t < -2\kappa\}}] \geq (e^{-\kappa} - e^{-2\kappa})F_t(-2\kappa),$$

and since $(e^{-\kappa} - e^{-2\kappa}) = e^{-2\kappa}(e^{\kappa} - 1) \geq e^{-2\kappa}\kappa$, we can write by (2.2.18) (recall that $\kappa \rightarrow 0$)

$$\log(p(-\kappa, t)/\kappa) \geq -2\kappa - \log \frac{(2\kappa)^\alpha}{t} \sim -\log \frac{\kappa^\alpha}{t}. \quad (2.2.20)$$

Next we give a matching upper bound on $p(-\kappa, t)$. Since $\mu t \leq \kappa$ eventually (recall that $\kappa/t^{1/\alpha} \rightarrow \infty$, hence $\kappa/t \rightarrow \infty$), by (2.2.17) and (2.2.16) we obtain, for all $y \geq 1$

$$F_t(-\kappa y) \leq \mathbb{P}\left(Y \leq -\frac{2\kappa y}{\sigma t^{1/\alpha}}\right) \leq c' \frac{t}{\kappa^\alpha y^\alpha},$$

for some $c' = c'_{\alpha, \sigma, \mu} \in (0, \infty)$. Then by Fubini's theorem

$$\begin{aligned} p(-\kappa, t) &= \mathbb{E}[(e^{-\kappa} - e^{X_t})\mathbf{1}_{\{X_t < -\kappa\}}] = \mathbb{E}\left[\int_{\kappa}^{\infty} e^{-x} \mathbf{1}_{\{x < -X_t\}} dx\right] = \int_{\kappa}^{\infty} e^{-x} F_t(-x) dx \\ &= \kappa \int_1^{\infty} e^{-\kappa y} F_t(-\kappa y) dy \leq c' \kappa \frac{t}{\kappa^\alpha} \int_1^{\infty} \frac{1}{y^\alpha} dy =: c'' \kappa \frac{t}{\kappa^\alpha}, \end{aligned}$$

hence

$$\log(p(-\kappa, t)/\kappa) \leq \log c'' - \log \frac{\kappa^\alpha}{t} \sim -\log \frac{\kappa^\alpha}{t}.$$

This relation, together with (2.2.20), yields

$$\log(p(-\kappa, t)/\kappa) \sim -\log \frac{\kappa^\alpha}{t}.$$

Since $\kappa/t^{1/\alpha} \rightarrow \infty$, this shows that we are in the regime when $\kappa \rightarrow 0$ and $p(-\kappa, t)/\kappa \rightarrow 0$. We can thus apply equation (2.1.8) in Theorem 2.1.1, which recalling Remark 2.1.2 simplifies as the first line in (2.1.11) (with $p(-\kappa, t)$ instead of $c(\kappa, t)$), yielding

$$\sigma_{\text{imp}}(-\kappa, t) \sim \frac{\kappa}{\sqrt{2t(-\log(p(-\kappa, t)/\kappa))}} \sim \frac{\kappa}{\sqrt{2t \log \frac{\kappa^\alpha}{t}}},$$

hence (2.2.6) is proved also when $\kappa \rightarrow 0$ (cf. the last line of (2.2.7)).

Typical deviations.

Let us fix an arbitrary family of values of (κ, t) satisfying (2.2.8). Relation (2.2.19) for $\kappa = \gamma_t = t^{1/\alpha}$ shows that condition (2.1.38) is satisfied, and Hypothesis 2.1.10 holds by (2.2.14). We can then apply Theorem 2.1.11, and relation (2.1.40) gives precisely (2.2.9). \square

Proof of (2.2.19). Since $|\frac{e^x-1}{x}| \leq 1$ if $x < 0$ and $|\frac{e^x-1}{x}| \leq e^x$ if $x \geq 0$, we have $|\frac{e^x-1}{x}| \leq 1 + e^x$ for all $x \in \mathbb{R}$. If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, Young's inequality $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$ yields

$$\left| \frac{e^{X_t} - 1}{\kappa} \right| = \left| \frac{X_t}{\kappa} \right| \left| \frac{e^{X_t} - 1}{X_t} \right| \leq \frac{1}{p} \left(\frac{|X_t|}{\kappa} \right)^p + \frac{1}{q} (1 + e^{X_t})^q.$$

Noting that $(a + b)^r \leq 2^{r-1}(a^r + b^r)$ for $r \geq 1$, by Hölder's inequality, and denoting by $c = c_{p, \eta}$ a suitable constant depending only on p, η , we can write

$$\left| \frac{e^{X_t} - 1}{\kappa} \right|^{1+\eta} \leq c \left(\frac{|X_t|^{p(1+\eta)}}{\kappa^{p(1+\eta)}} + 1 + e^{q(1+\eta)X_t} \right).$$

Given $0 < \eta < \alpha - 1$, we fix $p = p_{\eta, \alpha} > 1$ such that $B := p(1 + \eta) < \alpha$. (Note that B depends only on η, α .) Moreover, it follows by (2.2.12) that

$$\mathbb{E}[|X_t|^B] = (\sigma t^{1/\alpha})^B \mathbb{E}[|Y|^B] (1 + O(t^{B(1-1/\alpha)})),$$

and note that $\mathbb{E}[|Y|^B] < \infty$, because Y has finite moments of all orders strictly less than α , cf. [CW04, Property 1]. Since for $t \leq T$ one has $\mathbb{E}[e^{q(1+\eta)X_t}] \leq \mathbb{E}[e^{q(1+\eta)X_T}] < \infty$, by (2.2.2), relation (2.2.19) is proved. \square

2.2.2 The Heston Model

Given the parameters $\lambda, \vartheta, \eta, \sigma_0 \in (0, \infty)$ and $\varrho \in [-1, 1]$, the Heston model [Hes93] is a stochastic volatility model $(S_t)_{t \geq 0}$ defined by the following SDEs

$$\begin{cases} dS_t = S_t \sqrt{V_t} dW_t^1, \\ dV_t = -\lambda(V_t - \vartheta) dt + \eta \sqrt{V_t} dW_t^2, \\ X_0 = 0, \quad V_0 = \sigma_0, \end{cases}$$

where $(W_t^1)_{t \geq 0}$ and $(W_t^2)_{t \geq 0}$ are standard Brownian motions with $\langle dW_t^1, dW_t^2 \rangle = \varrho dt$.

Unlike the Carr-Wu model, here S_t displays explosion of moments, i.e. $\mathbb{E}[S_t^p] = \infty$ for $p > 1$ large enough (note that $\mathbb{E}[S_t] = 1$, since $(S_t)_{t \geq 0}$ is a martingale). In particular for any fixed $t \geq 0$ we define the explosion moment $p^*(t)$ as

$$p^*(t) := \sup\{p > 0 : \mathbb{E}[S_t^p] < \infty\},$$

so that $\mathbb{E}[S_t^p] < \infty$ for $p < p^*(t)$ while $\mathbb{E}[S_t^p] = \infty$ for $p > p^*(t)$. The behavior of the explosion moment $p^*(t)$ is described in the following Lemma, proved below.

Lemma 2.2.3. *If $\varrho = -1$, then $p^*(t) = +\infty$ for every $t \geq 0$.*

If $\varrho > -1$, then $p^(t) \in (1, +\infty)$ for every $t > 0$. Moreover, as $t \downarrow 0$*

$$p^*(t) \sim \frac{C}{t},$$

where

$$C = C(\varrho, \eta) := \begin{cases} \frac{2}{\eta \sqrt{1 - \varrho^2}} \left(\arctan \frac{\sqrt{1 - \varrho^2}}{\varrho} + \pi 1_{\varrho < 0} \right) & \text{if } \varrho < 1 \\ \frac{2}{\eta} & \text{if } \varrho = 1 \end{cases}. \quad (2.2.21)$$

The asymptotic behavior of the implied volatility $\sigma_{\text{imp}}(\kappa, t)$ for the Heston model is known in the regimes of large strike (with fixed maturity) and small maturity (with fixed strike).

- In [BF08], Benaim and Friz show that for fixed $t > 0$, when $\kappa \rightarrow +\infty$

$$\sigma_{\text{imp}}(\kappa, t) \underset{\kappa \uparrow \infty}{\sim} \frac{\sqrt{2\kappa}}{\sqrt{t}} \left(\sqrt{p^*(t)} - \sqrt{p^*(t) - 1} \right), \quad (2.2.22)$$

based on the estimate (cf. also [AP07])

$$-\log \mathbb{P}(X_t > \kappa) \underset{\kappa \uparrow \infty}{\sim} p^*(t) \kappa. \quad (2.2.23)$$

- In [FJ09], Forde and Jacquier have proved that for any fixed $\kappa > 0$, as $t \downarrow 0$

$$\sigma_{\text{imp}}(\kappa, t) \underset{t \downarrow 0}{\sim} \frac{\kappa}{\sqrt{2\Lambda^*(\kappa)}}, \quad (2.2.24)$$

where $\Lambda^*(\cdot)$ is the Legendre transform of the function $\Lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{\infty\}$ given by

$$\Lambda(p) := \begin{cases} \frac{\sigma_0 p}{\eta \left(\sqrt{1 - \varrho^2} \cot \left(\frac{1}{2} \eta p \sqrt{1 - \varrho^2} \right) - \varrho \right)} & \text{if } p < C, \\ \infty & \text{if } p \geq C, \end{cases} \quad (2.2.25)$$

where C is the constant in (2.2.21). Their analysis is based on the estimate

$$-\log \mathbb{P}(X_t \geq \kappa) \underset{t \downarrow 0}{\sim} \frac{1}{t} \Lambda^*(\kappa), \quad (2.2.26)$$

obtained by showing that the log-price $(X_t)_{t \geq 0}$ in the Heston model satisfies a large deviations principle as $t \downarrow 0$, with rate $1/t$ and good rate function $\Lambda^*(\kappa)$.

We first note that the asymptotics (2.2.22) and (2.2.24) follow easily from our Theorem 2.1.5, plugging the estimates (2.2.23) and (2.2.26) into relations (2.1.23) and (2.1.26), respectively.

We also observe that the estimates (2.2.22) and (2.2.24) match, in the following sense: if we take the limit $t \rightarrow 0$ of the right hand side of (2.2.22) (i.e. we first let $\kappa \uparrow +\infty$ and then $t \downarrow 0$ in $\sigma_{\text{imp}}(\kappa, t)$), we obtain

$$(2.2.22) \underset{t \downarrow 0}{\sim} \frac{\sqrt{2\kappa}}{\sqrt{t}} \frac{1}{2\sqrt{p^*(t)}} \underset{t \downarrow 0}{\sim} \frac{\sqrt{2\kappa}}{\sqrt{t}} \frac{1}{2\sqrt{\frac{C}{t}}} = \frac{\sqrt{\kappa}}{\sqrt{2C}}. \quad (2.2.27)$$

If, on the other hand, we take the limit $\kappa \uparrow 0$ of the right hand side of (2.2.24) (i.e. we first let $t \downarrow 0$ and then $\kappa \uparrow +\infty$ in $\sigma_{\text{imp}}(\kappa, t)$), since $\Lambda^*(\kappa) \sim C\kappa$,[†] we obtain

$$(2.2.24) \underset{\kappa \uparrow +\infty}{\sim} \frac{\kappa}{\sqrt{2C\kappa}} = \frac{\sqrt{\kappa}}{\sqrt{2C}}, \quad (2.2.28)$$

which coincides with (2.2.27). Analogously, also the estimates (2.2.23) and (2.2.26) match.

It is then natural to conjecture that, for *any* family of values of (κ, t) such that $\kappa \uparrow +\infty$ and $t \downarrow 0$ jointly, one should have

$$\log \mathbb{P}(X_t \geq \kappa) \sim -C \frac{\kappa}{t}, \quad (2.2.29)$$

where C is the constant in (2.2.21). If this holds, applying Theorem 2.1.5, relation (2.1.26) yields

$$\sigma_{\text{imp}}(\kappa, t) \sim \frac{\sqrt{\kappa}}{\sqrt{2C}}, \quad (2.2.30)$$

providing a smooth interpolation between (2.2.22) and (2.2.24).

[†]This is because $\Lambda(p) \uparrow +\infty$ as $p \uparrow C$, hence the slope of $\Lambda^*(\kappa)$ converges to C as $\kappa \rightarrow \infty$.

Remark 2.2.4 (Surface asymptotics for the Heston model). If (2.2.30) holds for any family of values of (κ, t) with $\kappa \rightarrow \infty$ and $t \rightarrow 0$, it follows that for every $\varepsilon > 0$ there exists $M = M(\varepsilon) \in (0, \infty)$ such that the following inequalities hold:

$$(1 - \varepsilon) \frac{\sqrt{\kappa}}{\sqrt{2C}} \leq \sigma_{\text{imp}}(\kappa, t) \leq (1 + \varepsilon) \frac{\sqrt{\kappa}}{\sqrt{2C}},$$

for all (κ, t) in the region $\mathcal{A}_{T,M} := \{0 < t \leq \frac{1}{M}, \kappa > M\}$, as it follows easily by contradiction (cf. Remark 2.2.2 for a similar argument).

Proof of Lemma 2.2.3. Given any number $p > 1$ we define the explosion time $T^*(p)$ as

$$T^*(p) := \sup\{t > 0 : \mathbb{E}[S_t^p] < \infty\}.$$

Note that if $T^*(p) = t \in (0, +\infty)$ then $p^*(t) = p$. By [AP07] (see also [FK09])

$$T^*(p) = \begin{cases} +\infty & \text{if } \Delta(p) \geq 0, \chi(p) < 0, \\ \frac{1}{\sqrt{\Delta(p)}} \log \left(\frac{\chi(p) + \sqrt{\Delta(p)}}{\chi(p) - \sqrt{\Delta(p)}} \right) & \text{if } \Delta(p) \geq 0, \chi(p) > 0, \\ \frac{2}{\sqrt{-\Delta(p)}} \left(\arctan \frac{\sqrt{-\Delta(p)}}{\chi(p)} + \pi 1_{\chi(p) < 0} \right) & \text{if } \Delta(p) < 0, \end{cases} \quad (2.2.31)$$

where

$$\chi(p) := \varrho \eta p - \lambda, \quad \Delta(p) := \chi^2(p) - \eta^2(p^2 - p),$$

Observe that if $\varrho = -1$, then $\chi(p) = -\eta p - \lambda < 0$ and $\Delta(p) = \lambda^2 + p(2\eta\lambda + \eta^2) \geq 0$, which implies $T^*(p) = +\infty$ for every $p > 1$, or equivalently $p^*(t) = +\infty$ for every $t > 0$.

On the other hand, since

$$\Delta(p) = \varrho^2 \eta^2 p^2 + \lambda^2 - 2\eta\varrho\lambda p - \eta^2 p^2 + \eta^2 p = \eta^2 p^2 (\varrho^2 - 1) + p(\eta^2 - 2\eta\varrho\lambda) + \lambda^2,$$

we observe that if $\varrho \neq 1$, then $\Delta p < 0$ as $p \rightarrow +\infty$, which implies

$$\begin{aligned} T^*(p) &\underset{p \uparrow \infty}{\sim} \frac{2}{p(\eta\sqrt{1-\varrho^2})} \left(\arctan \frac{\eta p \sqrt{1-\varrho^2}}{\varrho \eta p} + \pi 1_{\varrho < 0} \right) \\ &= \frac{1}{p} \frac{2}{\eta\sqrt{1-\varrho^2}} \left(\arctan \frac{\sqrt{1-\varrho^2}}{\varrho} + \pi 1_{\varrho < 0} \right). \end{aligned} \quad (2.2.32)$$

In particular this leads to the conclusion that, if $|\varrho| \neq 1$, then

$$p^*(t) \underset{t \downarrow 0}{\sim} \frac{C}{t}$$

where C was defined in (2.2.21).

It remains to study the case $\varrho = 1$, in which $\chi(p) > 0$ for every p . We have two possibilities: if $\eta > 2\lambda$ then $\Delta(p) > 0$ when $p \rightarrow +\infty$, and so by (2.2.31)

$$T^*(p) \underset{p \uparrow \infty}{\sim} \frac{1}{\sqrt{p(\eta^2 + 2\eta\lambda)}} \log \left(1 + 2 \frac{\sqrt{p(\eta^2 + 2\eta\lambda)}}{\eta p - \sqrt{p(\eta^2 + 2\eta\lambda)}} \right) \sim \frac{2}{\eta} \frac{1}{p}.$$

On the other hand, if $\eta < 2\lambda$, then $\Delta(p) < 0$ when $p \rightarrow \infty$ and so

$$T^*(p) \underset{p \uparrow \infty}{\sim} \frac{2}{\sqrt{p(2\eta\lambda - \eta^2)}} \left(\arctan \frac{\sqrt{p(2\eta\lambda - \eta^2)}}{p\eta} \right) \sim \frac{2}{\eta} \frac{1}{p}.$$

Finally if $\eta = 2\lambda$, $\Delta(p) = \lambda^2$, and so

$$T^*(p) = \frac{1}{\lambda} \log \left(1 + \frac{2\lambda}{\eta p - 2\lambda} \right) \underset{p \uparrow \infty}{\sim} \frac{2}{\eta} \frac{1}{p}.$$

In all the cases we obtain $p^*(t) \underset{t \downarrow 0}{\sim} \frac{2}{\eta} \frac{1}{t}$, in agreement with (2.2.21). \square

2.2.3 Merton's Jump Diffusion Model

Consider a model [M76] where the log-return X_t has an infinitely divisible distribution, whose moment generating function is given by

$$\mathbb{E}[\exp(zX_t)] = \exp \left(t \left\{ z\mu + \frac{1}{2}z^2\sigma^2 + \lambda \left(e^{z\alpha + z^2\frac{\delta^2}{2}} - 1 \right) \right\} \right), \quad (2.2.33)$$

where $\mu, \alpha \in \mathbb{R}$ and $\sigma, \lambda, \delta \in (0, \infty)$ are fixed parameters.

Benaim and Fritz [BF09] observed that for fixed $t > 0$, as $\kappa \rightarrow \infty$,

$$\log \mathbb{P}(X_t \geq \kappa) \sim -\frac{\kappa}{\delta} \sqrt{2 \log \frac{\kappa}{t}}, \quad (2.2.34)$$

deducing that

$$\sigma_{\text{imp}}^2(\kappa, t) \sim \frac{\kappa}{2t} \frac{\delta}{\sqrt{2 \log \frac{\kappa}{t}}}. \quad (2.2.35)$$

Remarkably, formula (2.2.35) holds for any family of (κ, t) such that t is bounded, say $0 < t \leq T$, and $\kappa \gg \sqrt{\log \frac{1}{t}}$, by our Theorem 2.1.5, because the asymptotic relation (2.2.34) also holds for any such family (we thank Stefan Gerhold for this observation). In fact, for any $c \in (1, \infty)$ such that $\mathbb{E}[e^{cX_t}] < \infty$, we can write

$$\mathbb{P}(X_t \geq \kappa) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\mathbb{E}[e^{sX_t}] e^{-\kappa s}}{s} ds.$$

The asymptotic evaluation of this integral can be done by saddle point methods: the relevant estimate for the saddle point \hat{s} , taken from [FGY14],[†] reads as follows:

$$\hat{s} = \frac{\sqrt{2 \log \kappa}}{\delta} - \frac{\mu}{\delta^2} + \mathcal{O} \left(\frac{\log \log \kappa}{\sqrt{\log \kappa}} \right),$$

which gives precisely (2.2.34):

$$\log \mathbb{P}(X_t \geq \kappa) \sim -\kappa \hat{s} + \log M_{X_t}(\hat{s}) \sim -\frac{\kappa}{\delta} \sqrt{2 \log \frac{\kappa}{t}} + \frac{\kappa}{\delta \sqrt{2 \log \frac{\kappa}{t}}} \sim -\frac{\kappa}{\delta} \sqrt{2 \log \frac{\kappa}{t}}.$$

[†]The formula for \hat{s}^2 at the end of the section ‘‘The Merton Model’’ in [FGY14] contains a misprint, since the term $-\log(\lambda T \delta^2)$ should be replaced by $-\log(\lambda \delta^2)$. We also refer to [GMZ14] for the special case $\kappa \rightarrow \infty$ with fixed t , with a more detailed computation.

Remark 2.2.5 (Surface asymptotics for the Merton model). In analogy with Remark 2.2.4, since formula (2.2.35) holds for any family of (κ, t) with t bounded and $\kappa \gg \sqrt{\log \frac{1}{t}}$, for every $\varepsilon > 0$ there exists $M = M(\varepsilon) \in (0, \infty)$ such that the following inequalities hold for all (κ, t) in the region $\mathcal{A}_{T,M} := \{0 < t \leq T, \kappa > Mt\}$:

$$(1 - \varepsilon) \frac{\kappa}{2t} \frac{\delta}{\sqrt{2 \log \frac{\kappa}{t}}} \leq \sigma_{\text{imp}}^2(\kappa, t) \leq (1 + \varepsilon) \frac{\kappa}{2t} \frac{\delta}{\sqrt{2 \log \frac{\kappa}{t}}}.$$

2.3 From option price to implied volatility

In this section we prove Theorem 2.1.1. We start with some background on Black&Scholes model and on related quantities.

2.3.1 Mills ratio

We let Z be a standard Gaussian random variable and denote by ϕ and Φ its density and distribution functions:

$$\phi(z) := \frac{\text{P}(Z \in dz)}{dz} = \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}, \quad \Phi(z) := \text{P}(Z \leq z) = \int_{-\infty}^z \phi(t) dt. \quad (2.3.1)$$

The Mills ratio $U : \mathbb{R} \rightarrow (0, \infty)$ is defined by

$$U(z) := \frac{1 - \Phi(z)}{\phi(z)} = \frac{\Phi(-z)}{\phi(z)}, \quad \forall z \in \mathbb{R}. \quad (2.3.2)$$

The next lemma summarizes the main properties of U that will be used in the sequel.

Lemma 2.3.1. *The function U is smooth, strictly decreasing, strictly convex and satisfies*

$$U'(z) \sim -\frac{1}{z^2} \quad \text{as } z \uparrow \infty. \quad (2.3.3)$$

Proof. Since $\Phi'(z) = \phi(z)$ and ϕ is an analytic function, U is also analytic. Since $\phi'(z) = -z\phi(z)$, one obtains

$$U'(z) = zU(z) - 1, \quad U''(z) = U(z) + zU'(z) = (1 + z^2)U(z) - z. \quad (2.3.4)$$

Recalling that $U(z) > 0$, these relations already show that $U'(z) < 0$ and $U''(z) > 0$ for all $z \leq 0$. For $z > 0$, the following bounds hold [S54, eq. (19)], [P01, Th. 1.5]:

$$\frac{z}{z^2 + 1} = \frac{1}{z + \frac{1}{z}} < U(z) < \frac{1}{z + \frac{1}{z+\frac{2}{z}}} = \frac{z^2 + 2}{z^3 + 3z}, \quad \forall z > 0. \quad (2.3.5)$$

Applying (2.3.4) yields $U''(z) > 0$ and $-\frac{1}{1+z^2} < U'(z) < -\frac{1}{3+z^2}$ for all $z > 0$, hence (2.3.3). \square

We recall that the smooth function $D : (0, \infty) \rightarrow (0, \infty)$ was introduced in (2.1.5). Since

$$D'(z) = -\frac{1}{z^2}\phi(z) < 0, \quad (2.3.6)$$

$D(\cdot)$ is a strictly decreasing bijection (note that $\lim_{z \downarrow 0} D(z) = \infty$ and $\lim_{z \rightarrow \infty} D(z) = 0$). Its inverse $D^{-1} : (0, \infty) \rightarrow (0, \infty)$ is then smooth and strictly decreasing as well. Writing $D(z) = \phi(z)(\frac{1}{z} - U(z))$, it follows by (2.3.5) that $\frac{1}{z} - U(z) \sim \frac{1}{z^3}$ as $z \uparrow \infty$, hence

$$D(z) \sim \frac{1}{z^3}\phi(z) \sim \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}z^3} \quad \text{as } z \uparrow \infty, \quad D(z) \sim \frac{1}{z}\phi(0) = \frac{1}{\sqrt{2\pi}z} \quad \text{as } z \downarrow 0.$$

It follows easily that $D^{-1}(\cdot)$ satisfies (2.1.6).

2.3.2 Black&Scholes formula

The Black&Scholes model is defined by a risk-neutral log-price $(X_t := \sigma B_t - \frac{1}{2}\sigma^2 t)_{t \geq 0}$, where $(B_t)_{t \geq 0}$ is a standard Brownian motion and the parameter $\sigma \in (0, \infty)$ represents the volatility. The Black&Scholes formula for the price of a normalized European call is $\mathbf{C}_{\text{BS}}(\kappa, \sigma\sqrt{t})$, where κ is the log-strike, t is the maturity and we define

$$\mathbf{C}_{\text{BS}}(\kappa, v) := \mathbb{E}[(e^{vZ - \frac{1}{2}v^2} - e^\kappa)^+] = \begin{cases} (1 - e^\kappa)^+ & \text{if } v = 0, \\ \Phi(d_1) - e^\kappa \Phi(d_2) & \text{if } v > 0, \end{cases} \quad (2.3.7)$$

where Φ is defined in (2.3.1), and we set

$$\begin{cases} d_1 = d_1(\kappa, v) := -\frac{\kappa}{v} + \frac{v}{2}, \\ d_2 = d_2(\kappa, v) := -\frac{\kappa}{v} - \frac{v}{2}, \end{cases} \quad \text{so that} \quad \begin{cases} d_2 = d_1 - v, \\ d_2^2 = d_1^2 + 2\kappa. \end{cases} \quad (2.3.8)$$

Note that $\mathbf{C}_{\text{BS}}(\kappa, v)$ is a continuous function of (κ, v) . Since $e^\kappa \phi(d_2) = \phi(d_1)$, for all $v > 0$ one easily computes

$$\frac{\partial \mathbf{C}_{\text{BS}}(\kappa, v)}{\partial v} = \phi(d_1) > 0, \quad \frac{\partial \mathbf{C}_{\text{BS}}(\kappa, v)}{\partial \kappa} = -e^\kappa \Phi(d_2) < 0,$$

hence $\mathbf{C}_{\text{BS}}(\kappa, v)$ is strictly increasing in v and strictly decreasing in κ (see Figure 2.1). It is also directly checked that for all $\kappa \in \mathbb{R}$ and $v \geq 0$ one has

$$\mathbf{C}_{\text{BS}}(\kappa, v) = 1 - e^\kappa + e^\kappa \mathbf{C}_{\text{BS}}(-\kappa, v). \quad (2.3.9)$$

In the following key proposition, proved in Section 2.5.2, we show that when $\kappa \geq 0$ the Black&Scholes call price $\mathbf{C}_{\text{BS}}(\kappa, v)$ vanishes precisely when $v \rightarrow 0$ or $d_1 \rightarrow -\infty$ (or, more generally, in a combination of these two regimes, when $\min\{d_1, \log v\} \rightarrow -\infty$). We also provide sharp estimates for each regime, that will play a crucial role in the sequel.

Proposition 2.3.2. *For any family of values of (κ, v) with $\kappa \geq 0$, $v > 0$, one has*

$$\mathbf{C}_{\text{BS}}(\kappa, v) \rightarrow 0 \quad \text{if and only if} \quad \min\{d_1, \log v\} \rightarrow -\infty, \quad (2.3.10)$$

that is, $\mathbf{C}_{\text{BS}}(\kappa, v) \rightarrow 0$ if and only if from any subsequence of (κ, v) one can extract a sub-subsequence along which either $d_1 \rightarrow -\infty$ or $v \rightarrow 0$. Moreover:

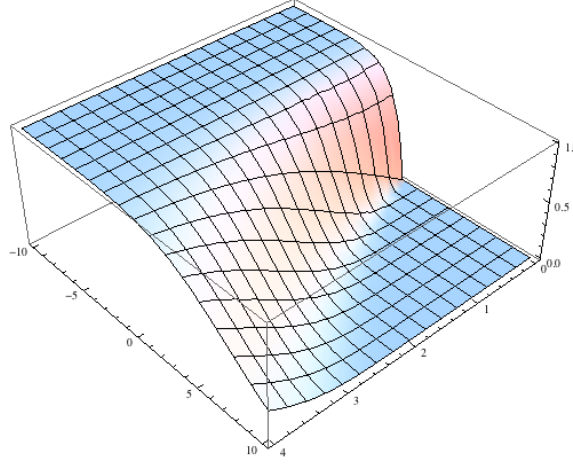


Figure 2.1: A plot of $(\kappa, v) \mapsto \mathbf{C}_{\text{BS}}(\kappa, v)$, for $\kappa \in [-10, 10]$ and $v \in [0, 4]$.

- if $d_1 := -\frac{\kappa}{v} + \frac{v}{2} \rightarrow -\infty$, then

$$\mathbf{C}_{\text{BS}}(\kappa, v) \sim \phi(d_1) \frac{v}{-d_1(-d_1 + v)}; \quad (2.3.11)$$

- if $v \rightarrow 0$, then

$$\mathbf{C}_{\text{BS}}(\kappa, v) \sim -U'(-d_1) \phi(d_1) v; \quad (2.3.12)$$

where $\phi(\cdot)$ and $U(\cdot)$ are defined in (2.3.1) and (2.3.2).

2.3.3 Implied volatility

Since the function $v \mapsto \mathbf{C}_{\text{BS}}(\kappa, v)$ is a strictly increasing bijection from $[0, \infty)$ to $[(1 - e^\kappa)^+, 1)$, it admits an inverse function $c \mapsto \mathbf{V}_{\text{BS}}(\kappa, c)$, defined by

$$\mathbf{C}_{\text{BS}}(\kappa, \mathbf{V}_{\text{BS}}(\kappa, c)) = c. \quad (2.3.13)$$

By construction, $\mathbf{V}_{\text{BS}}(\kappa, \cdot)$ is a strictly increasing bijection from $[(1 - e^\kappa)^+, 1)$ to $[0, \infty)$. We will mainly focus on the case $\kappa \geq 0$, for which $\mathbf{V}_{\text{BS}}(\kappa, \cdot) : [0, 1) \rightarrow [0, \infty)$.

Consider an arbitrary model, with a risk-neutral log-price $(X_t)_{t \geq 0}$, and let $c(\kappa, t)$ be the corresponding price of a normalized European call option, cf. (2.1.1). Since $z \mapsto (z - e^\kappa)^+$ is a convex function, one has $c(\kappa, t) \geq (\mathbf{E}[e^{X_t}] - e^\kappa)^+ = (1 - e^\kappa)^+$ by Jensen's inequality; since $(z - e^\kappa)^+ < z^+$, one has $c(\kappa, t) < \mathbf{E}[e^{X_t}] = 1$. Having shown that $c(\kappa, t) \in [(1 - e^\kappa)^+, 1)$, one defines the *implied volatility* $\sigma_{\text{imp}}(\kappa, t)$ of the model as the unique value of $\sigma \in [0, \infty)$ for which the Black&Scholes call price $\mathbf{C}_{\text{BS}}(\kappa, \sigma\sqrt{t})$ equals $c(\kappa, t)$. Equivalently, by (2.3.13),

$$\sigma_{\text{imp}}(\kappa, t) := \frac{\mathbf{V}_{\text{BS}}(\kappa, c(\kappa, t))}{\sqrt{t}}. \quad (2.3.14)$$

It is now convenient to rewrite Theorem 2.1.1 more transparently in terms of the function \mathbf{V}_{BS} . To this purpose, inspired by (2.1.2), let us define a new variable $p = p(\kappa, c)$ by

$$p := c - (1 - e^\kappa). \quad (2.3.15)$$

Theorem 2.3.3. *Consider a family of values of (κ, c) , such that either $\kappa \geq 0$, $c \in (0, 1)$ and $c \rightarrow 0$, or alternatively $\kappa \leq 0$, $p \in (0, 1)$ and $p \rightarrow 0$, where p is defined in (2.3.15).*

- *If κ bounded away from zero ($\liminf |\kappa| > 0$), one has*

$$\mathbb{V}_{\text{BS}}(\kappa, c) \sim \begin{cases} \sqrt{2(-\log c + \kappa)} - \sqrt{2(-\log c)} & \text{if } \kappa > 0, \\ \sqrt{2(-\log p)} - \sqrt{2(-\log p + \kappa)} & \text{if } \kappa < 0. \end{cases} \quad (2.3.16)$$

- *If κ is bounded away from infinity ($\limsup |\kappa| < \infty$), one has*

$$\mathbb{V}_{\text{BS}}(\kappa, c) \sim \begin{cases} \frac{\kappa}{D^{-1}(\frac{c}{\kappa})} & \text{if } \kappa > 0, \\ \sqrt{2\pi} c = \sqrt{2\pi} p & \text{if } \kappa = 0, \\ \frac{-\kappa}{D^{-1}(\frac{p}{-\kappa})} & \text{if } \kappa < 0, \end{cases} \quad (2.3.17)$$

where $D^{-1}(\cdot)$ is the inverse of the function $D(\cdot)$ defined in (2.1.5), and satisfies (2.1.6).

We give the proof in a moment (see §2.3.4 below), *restricting to the case $\kappa \geq 0$* , because the complementary case $\kappa \leq 0$ follows by symmetry, as we now briefly discuss. It follows by (2.3.9) and (2.3.13) that for all $k \in \mathbb{R}$ and $c \in [(1 - e^\kappa)^+, 1)$ one has

$$\mathbb{V}_{\text{BS}}(\kappa, c) = \mathbb{V}_{\text{BS}}(-\kappa, 1 - e^{-\kappa} + e^{-\kappa}c) = \mathbb{V}_{\text{BS}}(-\kappa, e^{-\kappa}p),$$

where we recall that p is defined in (2.3.15). As a consequence, in the case $\kappa \leq 0$, replacing κ by $-\kappa$ and c by $e^{-\kappa}p$ in the first line of (2.3.16), one obtains the second line of (2.3.16). Performing the same replacements in the first line of (2.3.17) yields

$$\mathbb{V}_{\text{BS}}(\kappa, c) \sim \frac{-\kappa}{D^{-1}(e^{-\kappa} \frac{p}{-\kappa})},$$

which is slightly different with respect to the third line of (2.3.17). However, the discrepancy is only apparent, because we claim that $D^{-1}(e^{-\kappa} \frac{p}{-\kappa}) \sim D^{-1}(\frac{p}{-\kappa})$. This is checked as follows: if $\kappa \rightarrow 0$, then $e^{-\kappa} \frac{p}{-\kappa} \sim \frac{p}{-\kappa}$; if, on the other hand, $\kappa \rightarrow \bar{\kappa} \in (-\infty, 0)$, since $p \rightarrow 0$ by assumption, the first relation in (2.1.6) yields $D^{-1}(e^{-\kappa} \frac{p}{-\kappa}) \sim \sqrt{2(-\log(\frac{p}{-\bar{\kappa}}) + \bar{\kappa})} \sim \sqrt{2(-\log(\frac{p}{-\bar{\kappa}}))} \sim D^{-1}(\frac{p}{-\bar{\kappa}})$, as requested. (For more details, see the end of the proof of Theorem 2.3.3, cf. (2.3.26) and the following lines.)

In conclusion, it suffices to prove Theorem 2.3.3 in the case $\kappa \geq 0$, and Theorem 2.1.1 follows.

2.3.4 Proof of Theorem 2.3.3 for $\kappa \geq 0$.

We prove separately relations (2.3.16) and (2.3.17).

Proof of (2.3.16). We fix a family of values of (κ, c) with $c \rightarrow 0$ and κ bounded away from zero, say $\kappa \geq \delta$ for some fixed $\delta > 0$. Our goal is to prove that relation (2.3.16) holds. If we set $v := \mathbb{V}_{\text{BS}}(\kappa, c)$, by definition (2.3.13) we have $\mathbb{C}_{\text{BS}}(\kappa, v) = c \rightarrow 0$.

Let us first show that $d_1 := -\frac{\kappa}{v} + \frac{v}{2} \rightarrow -\infty$. By Proposition 2.3.2, $\mathbf{C}_{\text{BS}}(\kappa, v) \rightarrow 0$ implies $\min\{d_1, \log v\} \rightarrow -\infty$, which means that every subsequence of values of (κ, c) admits a further sub-subsequence along which either $d_1 \rightarrow \infty$ or $v \rightarrow 0$. The key point is that $v \rightarrow 0$ implies $d_1 \rightarrow -\infty$, because $d_1 \leq -\frac{\delta}{v} + \frac{v}{2}$ (recall that $\kappa \geq \delta$). Thus $d_1 \rightarrow -\infty$ along every sub-subsequence, which means that $d_1 \rightarrow -\infty$ along the whole family of values of (κ, c) .

Since $d_1 \rightarrow -\infty$, we can apply relation (2.3.11). Taking log of both sides of that relation, recalling the definition (2.3.1) of ϕ and the fact that $\mathbf{C}_{\text{BS}}(\kappa, v) = c$, we can write

$$\log c \sim -\frac{1}{2}d_1^2 - \log \sqrt{2\pi} + \log \frac{v}{-d_1(-d_1 + v)}. \quad (2.3.18)$$

We now show that the last term in the right hand side is $o(d_1^2)$ and can therefore be neglected. Note that $-d_1 \geq 1$ eventually, because $d_1 \rightarrow -\infty$, hence

$$\log \frac{v}{-d_1(-d_1 + v)} \leq \log \frac{v}{1 + v} \leq 0.$$

Since $v \mapsto \frac{-d_1 + v}{v}$ is decreasing for $-d_1 > 0$, in case $v \geq -d_1$ one has

$$\left| \log \frac{v}{-d_1(-d_1 + v)} \right| = \log \frac{-d_1(-d_1 + v)}{v} \leq \log(-2d_1) = o(d_1^2).$$

On the other hand, recalling that $d_1 \leq -\frac{\delta}{v} + \frac{v}{2}$, in case $v < -d_1$ one has $d_1 \leq -\frac{\delta}{v} - \frac{d_1}{2}$, which can be rewritten as $v \geq \frac{2\delta}{-3d_1}$ and together with $v < -d_1$ yields

$$\left| \log \frac{v}{-d_1(-d_1 + v)} \right| = \log \frac{-d_1(-d_1 + v)}{v} \leq \log \frac{-d_1(-d_1 - d_1)}{\frac{2\delta}{-3d_1}} = \log \left(\frac{3(-d_1)^3}{2\delta} \right) = o(d_1^2).$$

In conclusion, (2.3.18) yields $\log c \sim -\frac{1}{2}d_1^2$, that is there exists $\gamma = \gamma(\kappa, c) \rightarrow 0$ such that $(1 + \gamma) \log c = -\frac{1}{2}d_1^2$, and since $\log c \leq 0$ we can write

$$(1 + \gamma)|\log c| = \frac{1}{2}d_1^2 = \frac{1}{2} \left(\frac{\kappa^2}{v^2} + \frac{v^2}{4} - \kappa \right).$$

This is a second degree equation in v^2 , whose solutions (both positive) are

$$v^2 = 2\kappa \left[1 + 2 \frac{(1 + \gamma)|\log c|}{\kappa} \pm 2 \sqrt{\left(\frac{(1 + \gamma)|\log c|}{\kappa} \right)^2 + \frac{(1 + \gamma)|\log c|}{\kappa}} \right]. \quad (2.3.19)$$

Since $d_1 \rightarrow -\infty$, eventually one has $d_1 < 0$: since $d_1 = -\frac{\kappa}{v} + \frac{v}{2} = -\frac{1}{2v}(\sqrt{2\kappa} - v)(\sqrt{2\kappa} + v)$, it follows that $v^2 < 2\kappa$, which selects the “-” solution in (2.3.19). Taking square roots of both sides of (2.3.19) and recalling that $v = \mathbf{V}_{\text{BS}}(\kappa, c)$ yields the equality

$$\mathbf{V}_{\text{BS}}(\kappa, c) = \sqrt{2(1 + \gamma)|\log c| + 2\kappa} - \sqrt{2(1 + \gamma)|\log c|}, \quad (2.3.20)$$

as one checks squaring both sides of (2.3.20).

Finally, since $\gamma \rightarrow 0$, it is quite intuitive that relation (2.3.20) yields (2.3.16). To prove this fact, we observe that by (2.3.20) we can write

$$\frac{\mathbf{V}_{\text{BS}}(\kappa, c)}{\sqrt{2|\log c| + 2\kappa} - \sqrt{2|\log c|}} = f_\gamma \left(\frac{\kappa}{|\log c|} \right), \quad (2.3.21)$$

where for fixed $\gamma > -1$ we define the function $f_\gamma : [0, \infty) \rightarrow (0, \infty)$ by

$$f_\gamma(x) := \frac{\sqrt{1+\gamma+x} - \sqrt{1+\gamma}}{\sqrt{1+x} - 1} \quad \text{for } x > 0, \quad f_\gamma(0) := \lim_{x \downarrow 0} f_\gamma(x) = \frac{1}{\sqrt{1+\gamma}}.$$

By direct computation, when $\gamma > 0$ (resp. $\gamma < 0$) one has $\frac{d}{dx} f_\gamma(x) > 0$ (resp. < 0) for all $x > 0$. Since $\lim_{x \rightarrow +\infty} f_\gamma(x) = 1$, it follows that for every $x \geq 0$ one has $f_\gamma(0) \leq f_\gamma(x) \leq 1$ if $\gamma > 0$, while $1 \leq f_\gamma(x) \leq f_\gamma(0)$ if $\gamma < 0$; consequently, for any γ ,

$$\frac{1}{\sqrt{1+|\gamma|}} \leq f_\gamma(x) \leq \frac{1}{\sqrt{1-|\gamma|}}, \quad \forall x \geq 0,$$

which yields $\lim_{\gamma \rightarrow 0} f_\gamma(x) = 1$ uniformly over $x \geq 0$. By (2.3.21), relation (2.3.16) is proved. \square

Proof of (2.3.17). We now fix a family of values of (κ, c) with $c \rightarrow 0$ and κ bounded away from infinity, say $0 \leq \kappa \leq M$ for some fixed $M \in (0, \infty)$, and we prove relation (2.3.17).

We set $v := \mathbf{V}_{\text{BS}}(\kappa, c)$ so that $\mathbf{C}_{\text{BS}}(\kappa, v) = c \rightarrow 0$, cf. (2.3.13). (Note that $v > 0$, because $c > 0$ by assumption.) Applying Proposition 2.3.2 we have $\min\{d_1, \log v\} \rightarrow -\infty$, i.e. either $d_1 \rightarrow -\infty$ or $v \rightarrow 0$ along sub-subsequences. However, this time $d_1 \rightarrow -\infty$ implies $v \rightarrow 0$, because $d_1 \geq -\frac{M}{v} + \frac{v}{2}$ (recall that $\kappa \leq M$), which means that $v \rightarrow 0$ along the whole given family of values of (κ, c) .

Since $v \rightarrow 0$, relation (2.3.12) yields

$$c \sim -U'(-d_1) \phi(d_1) v. \quad (2.3.22)$$

Let us focus on $U'(-d_1)$: recalling that $d_1 = -\frac{\kappa}{v} + \frac{v}{2}$ and $v \rightarrow 0$, we first show that

$$U'(-d_1) \sim U'\left(\frac{\kappa}{v}\right). \quad (2.3.23)$$

By a subsequence argument, we may assume that $\frac{\kappa}{v} \rightarrow \varrho \in [0, \infty]$, and we recall that $v \rightarrow 0$:

- if $\varrho < \infty$, $U'(-d_1)$ and $U'(\frac{\kappa}{v})$ converge to $U'(\varrho) \neq 0$, hence $U'(-d_1)/U'(\frac{\kappa}{v}) \rightarrow 1$;
- if $\varrho = \infty$, $-d_1$ and $\frac{\kappa}{v}$ diverge to ∞ and (2.3.3) yields $U'(-d_1)/U'(\frac{\kappa}{v}) \sim (\frac{\kappa}{v})/(-d_1) \rightarrow 1$.

The proof of (2.3.23) is completed. Next we observe that, again by $v \rightarrow 0$,

$$\phi(-d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\kappa^2}{v^2} + \frac{v^2}{2} - \kappa)} \sim e^{\frac{1}{2}\kappa} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\kappa^2}{v^2}} = e^{\frac{1}{2}\kappa} \phi\left(\frac{\kappa}{v}\right).$$

We can thus rewrite (2.3.22) as

$$c \sim -U'\left(\frac{\kappa}{v}\right) \phi\left(\frac{\kappa}{v}\right) e^{\frac{1}{2}\kappa} v. \quad (2.3.24)$$

If $\kappa = 0$, recalling (2.3.4) we obtain $c \sim \phi(0)v = \frac{1}{\sqrt{2\pi}}v$, which is the second line of (2.3.17).

Next we assume $\kappa > 0$. By (2.3.4), (2.3.2) and (2.1.5), for all $z > 0$ we can write

$$-U'(z) \phi(z) = -\phi(z)(zU(z) - 1) = \phi(z) - z\Phi(-z) = zD(z),$$

hence (2.3.24) can be rewritten as

$$c \sim \kappa e^{\frac{1}{2}\kappa} D\left(\frac{\kappa}{v}\right), \quad \text{i.e.} \quad (1 + \gamma)c = \kappa e^{\frac{1}{2}\kappa} D\left(\frac{\kappa}{v}\right),$$

for some $\gamma = \gamma(\kappa, c) \rightarrow 0$. Recalling that $v = \mathbf{V}_{\text{BS}}(\kappa, c)$, we have shown that

$$\mathbf{V}_{\text{BS}}(\kappa, c) = \frac{\kappa}{D^{-1}\left(\frac{(1+\gamma)c}{\kappa e^{\frac{1}{2}\kappa}}\right)}. \quad (2.3.25)$$

We now claim that

$$D^{-1}\left(\frac{(1+\gamma)c}{\kappa e^{\frac{1}{2}\kappa}}\right) \sim D^{-1}\left(\frac{c}{\kappa}\right). \quad (2.3.26)$$

By a subsequence argument, we may assume that $\frac{c}{\kappa} \rightarrow \eta \in [0, \infty]$ and $\kappa \rightarrow \bar{\kappa} \in [0, M]$.

- If $\eta \in (0, \infty)$, then $\bar{\kappa} = 0$ (recall that $c \rightarrow 0$) hence $(1 + \gamma)c/(\kappa e^{\frac{1}{2}\kappa}) \rightarrow \eta$; then both sides of (2.3.26) converge to $D^{-1}(\eta) \in (0, \infty)$, hence their ratio converges to 1.
- If $\eta = \infty$, then again $\bar{\kappa} = 0$, hence $(1 + \gamma)c/(\kappa e^{\frac{1}{2}\kappa}) \rightarrow \infty$: since $D^{-1}(y) \sim \frac{1}{\sqrt{2\pi}}y^{-1}$ as $y \rightarrow \infty$, cf. (2.1.6), it follows immediately that (2.3.26) holds.
- If $\eta = 0$, then $(1 + \gamma)c/(\kappa e^{\frac{1}{2}\kappa}) \rightarrow 0$: since $D^{-1}(y) \sim \sqrt{2|\log y|}$ as $y \rightarrow 0$, cf. (2.1.6),

$$D^{-1}\left(\frac{(1+\gamma)c}{\kappa e^{\frac{1}{2}\kappa}}\right) \sim \sqrt{2\left|\left(\log \frac{c}{\kappa}\right) + \left(\log \frac{1+\gamma}{e^{\frac{1}{2}\kappa}}\right)\right|} \sim \sqrt{2\left|\log \frac{c}{\kappa}\right|},$$

because $|\log \frac{c}{\kappa}| \rightarrow \infty$ while $|\log[(1 + \gamma)/e^{\frac{1}{2}\kappa}]| \rightarrow \frac{1}{2}\bar{\kappa} \in [0, \frac{M}{2}]$, hence (2.3.26) holds.

Having proved (2.3.26), we can plug it into (2.3.25), obtaining precisely the first line of (2.3.17). This completes the proof of Theorem 2.3.3. \square

2.4 From tail probability to option price

In this section we prove Theorems 2.1.5, 2.1.6 and 2.1.11.

2.4.1 Proof of Theorem 2.1.5 and 2.1.6

We prove Theorem 2.1.5 and 2.1.6 at the same time. We recall that the tail probabilities $\bar{F}_t(\kappa)$, $F_t(-\kappa)$ are defined in (2.1.13). Throughout the proof, we fix a family of values of (κ, t) with $\kappa > 0$ and $0 < t < T$, for some fixed $T \in (0, \infty)$, such that Hypothesis 2.1.4 is satisfied.

Extracting subsequences, we may distinguish three regimes for κ :

- if $\kappa \rightarrow \infty$ our goal is to prove (2.1.22), resp. (2.1.28);
- if $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$ our goal is to prove (2.1.25), resp. (2.1.30), because in this case, plainly, one has $-\log \bar{F}_t(\kappa)/\kappa \rightarrow \infty$, resp. $-\log F_t(-\kappa)/\kappa \rightarrow \infty$, by (2.1.14);
- if $\kappa \rightarrow 0$, our goal is to prove (2.1.27), resp. (2.1.33).

Of course, each regime has different assumptions, as in Theorem 2.1.5 and 2.1.6.

Step 0. Preparation. It follows by conditions (2.1.15) and (2.1.16) that

$$\forall \varepsilon > 0 \quad \exists \varrho_\varepsilon \in (1, \infty) : \quad I_\pm(\varrho_\varepsilon) < 1 + \varepsilon, \quad (2.4.1)$$

therefore for every $\varepsilon > 0$ one has eventually

$$\begin{aligned} \log \bar{F}_t(\varrho_\varepsilon \kappa) &\geq (1 + \varepsilon) \log \bar{F}_t(\kappa), \quad \text{resp.} \\ \log F_t(-\varrho_\varepsilon \kappa) &\geq (1 + \varepsilon) \log F_t(-\kappa), \end{aligned} \quad (2.4.2)$$

where the inequality is “ \geq ” instead of “ \leq ”, because both sides are negative quantities.

We stress that $\bar{F}_t(\kappa) \rightarrow 0$, resp. $F_t(-\kappa) \rightarrow 0$, by (2.1.14), hence

$$\log \bar{F}_t(\kappa) \rightarrow -\infty, \quad \text{resp.} \quad \log F_t(-\kappa) \rightarrow -\infty. \quad (2.4.3)$$

Moreover, we claim that in any of the regimes $\kappa \rightarrow \infty$, $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$ and $\kappa \rightarrow 0$ one has

$$\log \bar{F}_t(\kappa) + \kappa \rightarrow -\infty. \quad (2.4.4)$$

This follows readily by (2.4.3) if $\kappa \rightarrow 0$ or $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$. If $\kappa \rightarrow \infty$ we argue as follows: by Markov’s inequality, for $\eta > 0$

$$\bar{F}_t(\kappa) \leq \mathbb{E}[e^{(1+\eta)X_t}]e^{-(1+\eta)\kappa}, \quad (2.4.5)$$

hence

$$\log \bar{F}_t(\kappa) + \kappa \leq -\eta\kappa + \log \mathbb{E}[e^{(1+\eta)X_t}].$$

Since in the regime $\kappa \rightarrow \infty$ we assume that the moment condition (2.1.17) holds for some or every $\eta > 0$, the term $\log \mathbb{E}[e^{(1+\eta)X_t}]$ is bounded from above, hence eventually

$$\log \bar{F}_t(\kappa) + \kappa \leq -\frac{\eta}{2} \kappa, \quad (2.4.6)$$

which proves relation (2.4.4).

The rest of the proof is divided in four steps, in each of which we prove lower and upper bounds on $c(\kappa, t)$ and $p(-\kappa, t)$, respectively.

Step 1. Lower bounds on $c(\kappa, t)$. We are going to prove sharp lower bounds on $c(\kappa, t)$, that will lead to relations (2.1.22), (2.1.25) and (2.1.27).

By (2.1.1) and (2.4.1), for every $\varepsilon > 0$ we can write

$$c(\kappa, t) \geq \mathbb{E}[(e^{X_t} - e^\kappa) \mathbf{1}_{\{X_t > \varrho_\varepsilon \kappa\}}] \geq (e^{\varrho_\varepsilon \kappa} - e^\kappa) \bar{F}_t(\varrho_\varepsilon \kappa), \quad (2.4.7)$$

and applying (2.4.2) we get

$$\log c(\kappa, t) \geq \log (e^{\varrho_\varepsilon \kappa} - e^\kappa) + (1 + \varepsilon) \log \bar{F}_t(\kappa). \quad (2.4.8)$$

If $\kappa \rightarrow \infty$, since $\log(e^{\varrho_\varepsilon \kappa} - e^\kappa) = \kappa + \log(e^{(\varrho_\varepsilon - 1)\kappa} - 1) \geq \kappa$ eventually, we obtain

$$\begin{aligned} \log c(\kappa, t) &\geq \kappa + (1 + \varepsilon) \log \bar{F}_t(\kappa) = (1 + \varepsilon)(\log \bar{F}_t(\kappa) + \kappa) - \varepsilon\kappa \\ &\geq (1 + 2\varepsilon + \frac{2}{\eta}\varepsilon)(\log \bar{F}_t(\kappa) + \kappa), \end{aligned} \quad (2.4.9)$$

where in the last inequality we have applied (2.4.6). It follows that

$$\limsup \frac{\log c(\kappa, t)}{\log \bar{F}_t(\kappa) + \kappa} \leq 1 + 2\varepsilon + \frac{2}{\eta}\varepsilon, \quad (2.4.10)$$

where the limsup is taken along the given family of values of (κ, t) (note that $\log c(\kappa, t)$ and $\log \bar{F}_t(\kappa) + \kappa$ are negative quantities, cf. (2.4.4), hence the reverse inequality with respect to (2.4.9)). Since $\varepsilon > 0$ is arbitrary and $\eta > 0$ is fixed, we have shown that

$$\limsup \frac{\log c(\kappa, t)}{\log \bar{F}_t(\kappa) + \kappa} \leq 1, \quad (2.4.11)$$

that is we have obtained a sharp bound for (2.1.22).

If $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$, since $\log(e^{\varrho_\varepsilon \kappa} - e^\kappa) \rightarrow \log(e^{\varrho_\varepsilon \bar{\kappa}} - e^{\bar{\kappa}})$ is bounded while $\log \bar{F}_t(\kappa) \rightarrow -\infty$, relation (2.4.8) gives

$$\limsup \frac{\log c(\kappa, t)}{\log \bar{F}_t(\kappa)} \leq 1 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have shown that when $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$

$$\limsup \frac{\log c(\kappa, t)}{\log \bar{F}_t(\kappa)} \leq 1, \quad (2.4.12)$$

obtaining a sharp bound for (2.1.25).

Finally, if $\kappa \rightarrow 0$, since for $\kappa \geq 0$ by convexity $\log(e^{\varrho_\varepsilon \kappa} - e^\kappa) = \kappa + \log(e^{(\varrho_\varepsilon - 1)\kappa} - 1) \geq \kappa + \log((\varrho_\varepsilon - 1)\kappa) = \kappa + \log(\varrho_\varepsilon - 1) + \log \kappa$, relation (2.4.8) yields

$$\log \frac{c(\kappa, t)}{\kappa} = \log c(\kappa, t) - \log \kappa \geq \log(\varrho_\varepsilon - 1) + (1 + \varepsilon) \log \bar{F}_t(\kappa).$$

Again, since $\log(\varrho_\varepsilon - 1)$ is constant and $\log \bar{F}_t(\kappa) \rightarrow -\infty$, and $\varepsilon > 0$ is arbitrary, we get

$$\limsup \frac{\log(c(\kappa, t)/\kappa)}{\log \bar{F}_t(\kappa)} \leq 1, \quad (2.4.13)$$

proving a sharp bound for (2.1.27).

Step 2. Lower bounds on $p(-\kappa, t)$. We are going to prove sharp lower bounds on $p(-\kappa, t)$, that will lead to relations (2.1.28), (2.1.30) and (2.1.33).

Recalling (2.1.1) and (2.4.1), for every $\varepsilon > 0$ we can write

$$p(-\kappa, t) \geq \mathbb{E}[(e^{-\kappa} - e^{X_t})1_{\{X_t \leq -\varrho_\varepsilon \kappa\}}] \geq (e^{-\kappa} - e^{-\varrho_\varepsilon \kappa})F_t(-\varrho_\varepsilon \kappa), \quad (2.4.14)$$

and applying (2.4.2) we obtain

$$\log p(-\kappa, t) \geq \log(e^{-\kappa} - e^{-\varrho_\varepsilon \kappa}) + (1 + \varepsilon) \log F_t(-\kappa). \quad (2.4.15)$$

If $\kappa \rightarrow \infty$, since $\log(e^{-\kappa} - e^{-\varrho_\varepsilon \kappa}) = -\kappa + \log(1 - e^{-(\varrho_\varepsilon - 1)\kappa}) \sim -\kappa$, eventually one has $\log(e^{-\kappa} - e^{-\varrho_\varepsilon \kappa}) \geq -(1 + \varepsilon)\kappa$ and we obtain

$$\log p(-\kappa, t) \geq (1 + \varepsilon)(\log F_t(-\kappa) - \kappa).$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\limsup \frac{\log p(-\kappa, t)}{\log F_t(-\kappa) - \kappa} \leq 1, \quad (2.4.16)$$

which is a sharp bound for (2.1.28).

If $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$, since $\log(e^{-\kappa} - e^{-\varrho_\varepsilon \kappa}) \rightarrow \log(e^{-\bar{\kappa}} - e^{-\varrho_\varepsilon \bar{\kappa}})$ is bounded while $\log F_t(-\kappa) \rightarrow -\infty$, and $\varepsilon > 0$ is arbitrary, relation (2.4.15) gives

$$\limsup \frac{\log p(-\kappa, t)}{\log F_t(-\kappa)} \leq 1, \quad (2.4.17)$$

which is a sharp bound for (2.1.30).

Finally, if $\kappa \rightarrow 0$, since $e^{-\kappa} - e^{-\varrho_\varepsilon \kappa} = e^{-\varrho_\varepsilon \kappa}(e^{(\varrho_\varepsilon - 1)\kappa} - 1) \geq e^{-\varrho_\varepsilon \kappa}(\varrho_\varepsilon - 1)\kappa$ by convexity, since $\kappa \geq 0$, one has eventually

$$\log(e^{-\kappa} - e^{-\varrho_\varepsilon \kappa}) \geq \log \kappa + \log(e^{-\varrho_\varepsilon \kappa}(\varrho_\varepsilon - 1)) \geq \log \kappa + \varepsilon \log F_t(-\kappa),$$

because $\log(e^{-\varrho_\varepsilon \kappa}(\varrho_\varepsilon - 1)) \rightarrow \log(\varrho_\varepsilon - 1) > -\infty$ while $\log F_t(-\kappa) \rightarrow -\infty$. Relation (2.4.15) then yields, eventually,

$$\log \frac{p(-\kappa, t)}{\kappa} = \log p(-\kappa, t) - \log \kappa \geq (1 + 2\varepsilon) \log F_t(-\kappa).$$

Since $\varepsilon > 0$ is arbitrary, we have shown that

$$\limsup \frac{\log(p(-\kappa, t)/\kappa)}{\log F_t(-\kappa)} \leq 1, \quad (2.4.18)$$

obtaining a sharp bound for (2.1.33).

Step 3. Upper bounds on $c(\kappa, t)$. We are going to prove sharp upper bounds on $c(\kappa, t)$, that will complete the proof of relations (2.1.22), (2.1.25) and (2.1.27). We first consider the case when *the moment assumptions* (2.1.17) and (2.1.19) hold for every $\eta > 0$.

Let us look at the regimes $\kappa \rightarrow \infty$ and $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$ (i.e. κ is bounded away from zero), assuming that condition (2.1.17) holds for every $\eta > 0$. By Hölder's inequality,

$$c(\kappa, t) = \mathbb{E}[(e^{X_t} - e^\kappa) \mathbf{1}_{\{X_t > \kappa\}}] \leq \mathbb{E}[e^{X_t} \mathbf{1}_{\{X_t > \kappa\}}] \leq \mathbb{E}[e^{(1+\eta)X_t}]^{\frac{1}{1+\eta}} \bar{F}_t(\kappa)^{\frac{\eta}{1+\eta}}. \quad (2.4.19)$$

Let us fix $\varepsilon > 0$ and choose $\eta = \eta_\varepsilon$ large enough, so that $\frac{\eta}{1+\eta} > 1 - \varepsilon$. By assumption (2.1.17), for some $C \in (0, \infty)$ one has

$$\mathbb{E}[e^{(1+\eta)X_t}]^{\frac{1}{1+\eta}} \leq C,$$

hence eventually, recalling that $\log \bar{F}_t(\kappa) \rightarrow -\infty$, by (2.4.3),

$$\log c(\kappa, t) \leq \log C + (1 - \varepsilon) \log \bar{F}_t(\kappa) \leq (1 - 2\varepsilon) \log \bar{F}_t(\kappa). \quad (2.4.20)$$

Since $\varepsilon > 0$ is arbitrary, this shows that

$$\liminf \frac{\log c(\kappa, t)}{\log \bar{F}_t(\kappa)} \geq 1, \quad (2.4.21)$$

which together with (2.4.11) completes the proof of (2.1.22), if $\kappa \rightarrow \infty$, because $\log \bar{F}_t(\kappa) \sim \log \bar{F}_t(\kappa) - \kappa$ when condition (2.1.17) holds for every $\eta > 0$, by (2.4.5) (cf. also Remark 2.1.9). If $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$, relation (2.4.21) together with (2.4.12) completes the proof of (2.1.25).

We then consider the regime $\kappa \rightarrow 0$, assuming that condition (2.1.19) holds *for every* $\eta > 0$. We modify (2.4.19) as follows: since $(e^{X_t} - e^\kappa) \leq (e^{X_t} - 1) \leq |e^{X_t} - 1|$,

$$c(\kappa, t) \leq \mathbb{E}[|e^{X_t} - 1| \mathbf{1}_{\{X_t > \kappa\}}] \leq \kappa \mathbb{E} \left[\left| \frac{e^{X_t} - 1}{\kappa} \right|^{1+\eta} \right]^{\frac{1}{1+\eta}} \bar{F}_t(\kappa)^{\frac{\eta}{1+\eta}}. \quad (2.4.22)$$

Let us fix $\varepsilon > 0$ and choose $\eta = \eta_\varepsilon$ large enough, so that $\frac{\eta}{1+\eta} > 1 - \varepsilon$. By assumption (2.1.19), for some $C \in (0, \infty)$ one has

$$\mathbb{E} \left[\left| \frac{e^{X_t} - 1}{\kappa} \right|^{1+\eta} \right]^{\frac{1}{1+\eta}} \leq C, \quad (2.4.23)$$

hence relation (2.4.22) yields eventually

$$\log \frac{c(\kappa, t)}{\kappa} \leq \log C + (1 - \varepsilon) \log \bar{F}_t(\kappa) \leq (1 - 2\varepsilon) \log \bar{F}_t(\kappa). \quad (2.4.24)$$

Since $\varepsilon > 0$ is arbitrary, we have proved that

$$\liminf \frac{\log(c(\kappa, t)/\kappa)}{\log \bar{F}_t(\kappa)} \geq 1, \quad (2.4.25)$$

which together with (2.4.13) completes the proof of (2.1.27).

It remains to consider the case when the moment assumptions (2.1.17) and (2.1.19) holds *for some* $\eta > 0$, but in addition conditions (2.1.21) (if $\kappa \rightarrow \infty$ or $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$) or (2.1.24) (if $\kappa \rightarrow 0$) holds. We start with considerations that are valid in any regime of κ .

Defining the constant

$$A := \limsup \left\{ \frac{-\kappa}{\log \bar{F}_t(\kappa) + \kappa} \right\} + 1, \quad (2.4.26)$$

where the limsup is taken along the given family of values of (κ, t) , we claim that $A < \infty$. This follows by (2.4.4) if $\kappa \rightarrow 0$ or if $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$ (in which case, plainly, $A = 1$), while if $\kappa \rightarrow +\infty$ it suffices to apply (2.4.6) to get $A \leq 2/\eta + 1$. It follows by (2.4.26) that eventually

$$\kappa \leq -A(\log \bar{F}_t(\kappa) + \kappa). \quad (2.4.27)$$

Next we show that, for all fixed $\varepsilon > 0$ and $1 < M < \infty$, eventually one has

$$\log \left(\sup_{y \in [1, M]} e^{\kappa y} \bar{F}_t(\kappa y) \right) \leq (1 - \varepsilon)(\log \bar{F}_t(\kappa) + \kappa), \quad (2.4.28)$$

which means that the sup is approximately attained for $y = 1$. This is easy if $\kappa \rightarrow 0$ or if $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$: in fact, since $\kappa \rightarrow \bar{F}_t(\kappa)$ is non-increasing, we can write

$$\begin{aligned} \log \left(\sup_{y \in [1, M]} e^{\kappa y} \bar{F}_t(\kappa y) \right) &\leq \log(e^{\kappa M} \bar{F}_t(\kappa)) = \kappa M + \log \bar{F}_t(\kappa) \\ &= (\log \bar{F}_t(\kappa) + \kappa) + (M - 1)\kappa, \end{aligned}$$

and since $\log \bar{F}_t(\kappa) + \kappa \rightarrow -\infty$ by (2.4.4), while $(M-1)\kappa$ is bounded, (2.4.28) follows.

To prove (2.4.28) in the regime $\kappa \rightarrow \infty$, we are going to exploit the assumption (2.1.21). First we fix $\delta > 0$, to be defined later, and set $\bar{n} := \lceil \frac{M-1}{\delta} \rceil$ and $a_n := 1 + n\delta$ for $n = 0, \dots, \bar{n}$, so that $[1, M] \subseteq \bigcup_{n=1}^{\bar{n}} [a_{n-1}, a_n]$. For all $y \in [a_{n-1}, a_n]$ one has, by (2.1.15),

$$\log \bar{F}_t(\kappa y) \leq \log \bar{F}_t(\kappa a_{n-1}) \sim I_+(a_{n-1}) \log \bar{F}_t(\kappa) \leq a_{n-1} \log \bar{F}_t(\kappa),$$

having used that $I_+(\varrho) \geq \varrho$, by (2.1.21), hence eventually

$$\log \bar{F}_t(\kappa y) \leq (1 - \delta)a_{n-1} \log \bar{F}_t(\kappa), \quad \forall y \in [a_{n-1}, a_n].$$

Recalling that $a_n = a_{n-1} + \delta$, we can write $a_n \leq (1 - \delta)a_{n-1} + \delta(1 + M)$, because $a_{n-1} \leq M$ by construction, and since $e^{\kappa y} \leq e^{\kappa a_n}$ for $y \in [a_{n-1}, a_n]$, it follows that

$$\begin{aligned} \log \left(\sup_{y \in [1, M]} e^{\kappa y} \bar{F}_t(\kappa y) \right) &\leq \max_{n=1, \dots, \bar{n}} (a_n \kappa + (1 - \delta)a_{n-1} \log \bar{F}_t(\kappa)) \\ &= \max_{n=1, \dots, \bar{n}} ((1 - \delta)a_{n-1} (\log \bar{F}_t(\kappa) + \kappa) + \delta(1 + M)\kappa). \end{aligned}$$

Plainly, the max is attained for $n = 1$, for which $a_{n-1} = a_0 = 1$. Recalling (2.4.27), we get

$$\log \left(\sup_{y \in [1, M]} e^{\kappa y} \bar{F}_t(\kappa y) \right) \leq (1 - \delta(1 + A + AM)) (\log \bar{F}_t(\kappa) + \kappa).$$

Choosing $\delta := \varepsilon / (1 + A + AM)$, the claim (2.4.28) is proved.

We are ready to give sharp upper bounds on $c(\kappa, t)$, refining (2.4.19). For fixed $M \in (0, \infty)$, we write

$$c(\kappa, t) = \mathbb{E}[(e^{X_t} - e^\kappa) \mathbf{1}_{\{\kappa < X_t \leq \kappa M\}}] + \mathbb{E}[(e^{X_t} - e^\kappa) \mathbf{1}_{\{X_t > \kappa M\}}], \quad (2.4.29)$$

and we estimate the first term as follows: by Fubini-Tonelli's theorem and (2.4.28),

$$\begin{aligned} \mathbb{E}[(e^{X_t} - e^\kappa) \mathbf{1}_{\{\kappa < X_t \leq \kappa M\}}] &= \mathbb{E} \left[\left(\int_\kappa^\infty e^x \mathbf{1}_{\{x < X_t\}} dx \right) \mathbf{1}_{\{\kappa < X_t \leq \kappa M\}} \right] \\ &= \int_\kappa^{\kappa M} e^x \mathbb{P}(x < X_t \leq \kappa M) dx \leq \int_\kappa^{\kappa M} e^x \bar{F}_t(x) dx \\ &= \kappa \int_1^M e^{\kappa y} \bar{F}_t(\kappa y) dy \leq \kappa (M - 1) e^{(1-\varepsilon)(\log \bar{F}_t(\kappa) + \kappa)}. \end{aligned} \quad (2.4.30)$$

To estimate the second term in (2.4.29), we start with the cases $\kappa \rightarrow \infty$ and $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$, where we assume that (2.1.17) holds for some $\eta > 0$, as well as (2.1.24), hence we can fix $M > 1$ such that $I_+(M) > \frac{1+\eta}{\eta}$. Bounding $(e^{X_t} - e^\kappa) \leq e^{X_t}$, Hölder's inequality yields

$$\mathbb{E}[(e^{X_t} - e^\kappa) \mathbf{1}_{\{X_t > \kappa M\}}] \leq \mathbb{E}[e^{(1+\eta)X_t}]^{\frac{1}{1+\eta}} \bar{F}_t(\kappa M)^{\frac{\eta}{1+\eta}} = C \bar{F}_t(\kappa M)^{\frac{\eta}{1+\eta}},$$

where $C \in (0, \infty)$ is an absolute constant, by (2.1.17). Applying relation (2.1.15) together with $I_+(M) > \frac{1+\eta}{\eta}$ we obtain

$$\frac{\eta}{1+\eta} \log \bar{F}_t(\kappa M) \sim \frac{\eta}{1+\eta} I_+(M) \log \bar{F}_t(\kappa) \leq \log \bar{F}_t(\kappa), \quad (2.4.31)$$

hence eventually

$$\log \mathbb{E}[(e^{X_t} - e^\kappa) \mathbf{1}_{\{X_t > \kappa M\}}] \leq (1 - \varepsilon) \log \bar{F}_t(\kappa) \leq (1 - \varepsilon)(\log \bar{F}_t(\kappa) + \kappa). \quad (2.4.32)$$

Recalling (2.4.6) and (2.4.4), eventually $\kappa(M - 1) \leq e^{-\varepsilon(\log \bar{F}_t(\kappa) + \kappa)}$, hence by (2.4.30)

$$\log \mathbb{E}[(e^{X_t} - e^\kappa) \mathbf{1}_{\{\kappa < X_t \leq \kappa M\}}] \leq (1 - 2\varepsilon)(\log \bar{F}_t(\kappa) + \kappa). \quad (2.4.33)$$

Looking back at (2.4.29), since

$$\log(a + b) \leq \log 2 + \max\{\log a, \log b\}, \quad \forall a, b > 0, \quad (2.4.34)$$

by (2.4.32), (2.4.33) and again (2.4.4) one has eventually

$$\log c(\kappa, t) \leq \log 2 + (1 - 2\varepsilon)(\log \bar{F}_t(\kappa) + \kappa) \leq (1 - 3\varepsilon)(\log \bar{F}_t(\kappa) + \kappa).$$

Since $\varepsilon > 0$ is arbitrary, this shows that

$$\liminf \frac{\log c(\kappa, t)}{\log \bar{F}_t(\kappa) + \kappa} \geq 1, \quad (2.4.35)$$

which together with (2.4.11) completes the proof of (2.1.22), if $\kappa \rightarrow \infty$. Since $\log \bar{F}_t(\kappa) + \kappa \sim \log \bar{F}_t(\kappa)$ if $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$, by (2.4.3), we can rewrite (2.4.35) in this case as

$$\liminf \frac{\log c(\kappa, t)}{\log \bar{F}_t(\kappa)} \geq 1, \quad (2.4.36)$$

which together with (2.4.12) completes the proof of (2.1.25).

It remains to consider the case when $\kappa \rightarrow 0$, where we assume that relation (2.1.19) holds for some $\eta \in (0, \infty)$, together with (2.1.24). As before, we fix $M > 1$ such that $I_+(M) > \frac{1+\eta}{\eta}$. Since

$$\mathbb{E} \left[\left(\frac{e^{X_t} - e^\kappa}{\kappa} \right)^{1+\eta} \mathbf{1}_{\{X_t > \kappa\}} \right] \leq \mathbb{E} \left[\left| \frac{e^{X_t} - 1}{\kappa} \right|^{1+\eta} \right] \leq C, \quad (2.4.37)$$

for some absolute constant $C \in (0, \infty)$, by (2.1.19), the second term in (2.4.29) is bounded by

$$\mathbb{E}[(e^{X_t} - e^\kappa) \mathbf{1}_{\{X_t > \kappa M\}}] \leq \kappa \mathbb{E} \left[\left| \frac{e^{X_t} - e^\kappa}{\kappa} \right|^{1+\eta} \right]^{\frac{1}{1+\eta}} \bar{F}_t(\kappa M)^{\frac{\eta}{1+\eta}} \leq \kappa C \bar{F}_t(\kappa M)^{\frac{\eta}{1+\eta}}. \quad (2.4.38)$$

In complete analogy with (2.4.31)-(2.4.32), we obtain that eventually

$$\log \frac{\mathbb{E}[(e^{X_t} - e^\kappa) \mathbf{1}_{\{X_t > \kappa M\}}]}{\kappa} \leq (1 - \varepsilon) \log \bar{F}_t(\kappa). \quad (2.4.39)$$

By (2.4.4), eventually $(M - 1) \leq e^{-\varepsilon(\log \bar{F}_t(\kappa) + \kappa)}$, hence by (2.4.30)

$$\log \frac{\mathbb{E}[(e^{X_t} - e^\kappa) \mathbf{1}_{\{\kappa < X_t \leq \kappa M\}}]}{\kappa} \leq (1 - 2\varepsilon)(\log \bar{F}_t(\kappa) + \kappa). \quad (2.4.40)$$

Recalling (2.4.29) and (2.4.34), we can finally write

$$\log \frac{c(\kappa, t)}{\kappa} \leq \log 2 + (1 - 2\varepsilon)(\log \bar{F}_t(\kappa) + \kappa) \leq (1 - 3\varepsilon) \log \bar{F}_t(\kappa),$$

because $\kappa \rightarrow 0$ and $\log \bar{F}_t(\kappa) \rightarrow -\infty$. Since $\varepsilon > 0$ is arbitrary, we have proved that

$$\liminf \frac{\log (c(\kappa, t)/\kappa)}{\log \bar{F}_t(\kappa)} \geq 1, \quad (2.4.41)$$

which together with (2.4.13) completes the proof of (2.1.27).

Step 4. Upper bounds on $p(-\kappa, t)$. We are going to prove sharp upper bounds on $p(-\kappa, t)$, that will complete the proof of relations (2.1.28), (2.1.30) and (2.1.33).

By (2.1.1) we can write

$$p(-\kappa, t) = \mathbb{E}[(e^{-\kappa} - e^{X_t})\mathbf{1}_{\{X_t \leq -\kappa\}}] \leq e^{-\kappa} F_t(-\kappa),$$

therefore

$$\frac{\log p(-\kappa, t)}{\log F_t(-\kappa) - \kappa} \geq 1, \quad (2.4.42)$$

which together with (2.4.16) completes the proof of (2.1.33), if $\kappa \rightarrow \infty$. On the other hand, if $\kappa \rightarrow \bar{\kappa} \in (0, \infty)$, since relation (2.4.42) implies (recall that $\kappa \geq 0$)

$$\frac{\log p(-\kappa, t)}{\log F_t(-\kappa)} \geq 1, \quad (2.4.43)$$

in view of (2.4.17), the proof of (2.1.30) is completed.

It remains to consider the case $\kappa \rightarrow 0$. If relation (2.1.19) holds *for every* $\eta \in (0, \infty)$, we argue in complete analogy with (2.4.22)-(2.4.23)-(2.4.24), getting

$$\liminf \frac{\log (p(-\kappa, t)/\kappa)}{\log F_t(-\kappa)} \geq 1, \quad (2.4.44)$$

which together with (2.4.18) completes the proof of (2.1.33). If, on the other hand, relation (2.1.19) holds *only for some* $\eta \in (0, \infty)$, we also assume that condition (2.1.32) holds, hence we can fix $M > 1$ such that $I_-(M) > \frac{1+\eta}{\eta}$. Let us write

$$p(-\kappa, t) = \mathbb{E}[(e^{-\kappa} - e^{X_t})\mathbf{1}_{\{-\kappa M < X_t \leq -\kappa\}}] + \mathbb{E}[(e^{-\kappa} - e^{X_t})\mathbf{1}_{\{X_t \leq -\kappa M\}}]. \quad (2.4.45)$$

In analogy with (2.4.30), for every fixed $\varepsilon > 0$, the first term in the right hand side can be estimated as follows (note that $y \mapsto F_t(-\kappa y)$ is decreasing):

$$\begin{aligned} \mathbb{E}[(e^{-\kappa} - e^{X_t})\mathbf{1}_{\{-\kappa M < X_t \leq -\kappa\}}] &\leq \int_{-\kappa M}^{-\kappa} e^x F_t(x) dx = \kappa \int_1^M e^{-\kappa y} F_t(-\kappa y) dy \\ &\leq \kappa(M-1) F_t(-\kappa) \leq \kappa e^{(1-\varepsilon) \log F_t(-\kappa)}. \end{aligned}$$

The second term in (2.4.45) is estimated in complete analogy with (2.4.37)-(2.4.38)-(2.4.39), yielding

$$\log \frac{\mathbb{E}[(e^{-\kappa} - e^{X_t})\mathbf{1}_{\{X_t \leq -\kappa M\}}]}{\kappa} \leq (1 - \varepsilon) \log F_t(-\kappa).$$

Recalling (2.4.34), we obtain from (2.4.45)

$$\log \frac{p(-\kappa, t)}{\kappa} \leq \log 2 + (1 - \varepsilon) \log F_t(-\kappa) \leq (1 - 2\varepsilon) \log F_t(-\kappa),$$

and since $\varepsilon > 0$ is arbitrary we have proved that relation (2.4.44) still holds, which together with (2.4.18) completes the proof of (2.1.33), and of the whole Theorem 2.1.5. \square

2.4.2 Proof of Theorem 2.1.11

By Skorokhod's representation theorem, we can build a coupling of the random variables $(X_t)_{t \geq 0}$ and Y such that relation (2.1.36) holds a.s.. Since the function $z \mapsto z^+$ is continuous, recalling that $\gamma_t \rightarrow 0$, for $\kappa \sim a\gamma_t$ we have a.s.

$$\frac{(e^{X_t} - e^\kappa)^+}{\gamma_t} = \left(\frac{e^{Y\gamma_t(1+o(1))} - 1}{\gamma_t} - \frac{e^{a\gamma_t(1+o(1))} - 1}{\gamma_t} \right)^+ \xrightarrow[t \downarrow 0]{a.s.} (Y - a)^+, \quad (2.4.46)$$

and analogously for $\kappa \sim -a\gamma_t$

$$\frac{(e^\kappa - e^{X_t})^+}{\gamma_t} \xrightarrow[t \downarrow 0]{a.s.} (-a - Y)^+ = (Y + a)^-. \quad (2.4.47)$$

Taking the expectation of both sides of these relations, one would obtain precisely (2.1.39). To justify the interchanging of limit and expectation, we observe that the left hand sides of (2.4.46) and (2.4.47) are uniformly integrable, being bounded in $L^{1+\eta}$. In fact

$$\frac{|e^{X_t} - e^\kappa|}{\gamma_t} \leq \frac{|e^{X_t} - 1|}{\gamma_t} + \frac{|e^\kappa - 1|}{\gamma_t},$$

and the second term in the right hand side is uniformly bounded (recall that $\kappa \sim a\gamma_t$ by assumption), while the first term is bounded in $L^{1+\eta}$, by (2.1.38). \square

2.5 Miscellanea

2.5.1 About conditions (2.1.3) and (2.1.4)

Recall from §2.1.1 that $(X_t)_{t \geq 0}$ denotes the risk-neutral log-price, and assume that $X_t \rightarrow X_0 := 0$ in distribution as $t \rightarrow 0$ (which is automatically satisfied if X has right-continuous paths). For an arbitrary family of values of (κ, t) , with $t > 0$ and $\kappa \geq 0$, we show that condition (2.1.3) implies (2.1.4).

Assume first that $t \rightarrow 0$ (with no assumption on κ). Since $\kappa \geq 0$, one has $(e^{X_t} - e^\kappa)^+ \rightarrow (1 - e^\kappa)^+ = 0$ in distribution, hence $c(\kappa, t) \rightarrow 0$ by (2.1.1) and Fatou's lemma. With analogous arguments, one has $p(-\kappa, t) \rightarrow 0$, hence (2.1.4) is satisfied.

Next we assume that $\kappa \rightarrow \infty$ and t is bounded, say $t \in (0, T]$ for some fixed $T > 0$. Since $z \mapsto (z - c)^+$ is a convex function and $(e^{X_t})_{t \geq 0}$ is a martingale, the process $((e^{X_t} - e^\kappa)^+)_{t \geq 0}$ is a submartingale and by (2.1.1) we can write

$$0 \leq c(\kappa, t) \leq \mathbb{E}[(e^{X_T} - e^\kappa)^+] = \mathbb{E}[(e^{X_T} - e^\kappa)1_{\{X_T > \kappa\}}] \leq \mathbb{E}[e^{X_T} 1_{\{X_T > \kappa\}}].$$

It follows that, if $\kappa \rightarrow +\infty$, then $c(\kappa, t) \rightarrow 0$. With analogous arguments, one shows that $p(-\kappa, t) \rightarrow 0$, hence condition (2.1.4) holds.

2.5.2 Proof of Proposition 2.3.2

Let us first prove (2.3.11) and (2.3.12). Since $\phi(d_2)e^k = \phi(d_1)$, cf. (2.3.1) and (2.3.8), recalling (2.3.2) we can rewrite the Black&Scholes formula (2.3.7) as follows:

$$\mathbf{C}_{\text{BS}}(\kappa, v) = \phi(d_1)(U(-d_1) - U(-d_2)) = \phi(d_1)(U(-d_1) - U(-d_1 + v)). \quad (2.5.1)$$

If $d_1 \rightarrow -\infty$, applying (2.3.3) we get

$$U(-d_1) - U(-d_1 + v) = - \int_{-d_1}^{-d_1+v} U'(z) dz \sim \int_{-d_1}^{-d_1+v} \frac{1}{z^2} dz = \frac{v}{-d_1(-d_1 + v)},$$

and (2.3.11) is proved. Next we assume that $v \rightarrow 0$. By convexity of $U(\cdot)$ (cf. Lemma 2.3.1),

$$-U'(-d_1 + v) \leq \frac{U(-d_1) - U(-d_1 + v)}{v} \leq -U'(-d_1),$$

hence to prove (2.3.12) it suffices to show that $U'(-d_1 + v) \sim U'(-d_1)$. To this purpose, by a subsequence argument, we may assume that $d_1 \rightarrow \bar{d}_1 \in \mathbb{R} \cup \{\pm\infty\}$. Since $d_1 \leq \frac{v}{2}$ for $\kappa \geq 0$, when $v \rightarrow 0$ necessarily $\bar{d}_1 \in [-\infty, 0]$. If $\bar{d}_1 = -\infty$, i.e. $-d_1 \rightarrow +\infty$, then $-d_1 + v \sim -d_1 \rightarrow +\infty$ and $U'(-d_1 + v) \sim U'(-d_1)$ follows by (2.3.3). On the other hand, if $\bar{d}_1 \in (-\infty, 0]$ then both $U'(-d_1)$ and $U'(-d_1 + v)$ converge to $U'(-\bar{d}_1) \neq 0$, by continuity of U' , hence $U'(-d_1)/U'(-d_1 + v) \rightarrow 1$, i.e. $U'(-d_1 + v) \sim U'(-d_1)$ as requested.

Let us now prove (2.3.10). Assume that $\min\{d_1, \log v\} \rightarrow -\infty$, and note that for every subsequence we can extract a sub-subsequence along which either $d_1 \rightarrow -\infty$ or $v \rightarrow 0$. We can then apply (2.3.11) and (2.3.12) to show that $\mathbf{C}_{\text{BS}}(\kappa, v) \rightarrow 0$:

- if $d_1 \rightarrow -\infty$, the right hand side of (2.3.11) is bounded from above by $\phi(d_1)/(-d_1) \rightarrow 0$;
- If $\kappa \geq 0$ and $v \rightarrow 0$, then $d_1 \leq \frac{v}{2} \rightarrow 0$ and consequently $\phi(d_1)U'(-d_1)$ is uniformly bounded from above, hence the right hand side of (2.3.12) vanishes (since $v \rightarrow 0$).

Finally, we assume that $\min\{d_1, \log v\} \not\rightarrow -\infty$ and show that $\mathbf{C}_{\text{BS}}(\kappa, v) \not\rightarrow 0$. Extracting a subsequence, we have $\min\{d_1, \log v\} \geq -M$ for some fixed $M \in (0, \infty)$, i.e. both $v \geq \varepsilon := e^{-M} > 0$ and $d_1 \geq -M$, and we may assume that $v \rightarrow \bar{v} \in [\varepsilon, +\infty]$ and $d_1 \rightarrow \bar{d}_1 \in [-M, +\infty]$. Consider first the case $\bar{v} = +\infty$, i.e. $v \rightarrow +\infty$: by (2.3.8) one has $-d_1 + v = -d_2 \geq \frac{v}{2} \rightarrow +\infty$, hence $\phi(d_1)U(-d_1 + v) \rightarrow 0$ (because ϕ is bounded), and recalling (2.3.2) relation (2.5.1) yields

$$\mathbf{C}_{\text{BS}}(\kappa, v) = \Phi(d_1) - \phi(d_1)U(-d_1 + v) \rightarrow \Phi(\bar{d}_1) > 0.$$

Next consider the case $\bar{v} < +\infty$: since $d_1 \leq \frac{v}{2}$, we have $\bar{d}_1 \leq \frac{\bar{v}}{2}$ and again by (2.5.1) we obtain $\mathbf{C}_{\text{BS}}(\kappa, v) \rightarrow \phi(\bar{d}_1)(U(-\bar{d}_1) - U(-\bar{d}_1 + \bar{v})) > 0$. In both cases, $\mathbf{C}_{\text{BS}}(\kappa, v) \not\rightarrow 0$. \square

Chapter 3

The asymptotic smile of a multiscaling stochastic volatility model

In this chapter, we focus on a stochastic volatility model for the log-price of a financial asset, recently introduced in [ACDP12], in which the volatility jumps at the jump times of a Poisson process, which represent shocks in the market. Despite the few parameters, this model was shown to capture some relevant stylized facts of financial series, such as the change in the log-return distribution from power-law tails (small time) to a Gaussian behavior (large time), the slow decay in the volatility autocorrelation and the so-called *multiscaling of moments*. We point out that this phenomenon can be observed for a general class of stochastic volatility models, in which the volatility is driven by a Lévy subordinator under a super-linear mean-reversion, cf. [DP14].

In this chapter, we look at this model from the viewpoint of pricing. Applying the results of Chapter 2, we obtain sharp asymptotic formulas for option prices and implied volatility, that are valid both in the limit of small maturity (with arbitrary strike) and of large strike (with bounded maturity). Remarkably, despite the price having continuous paths, the out-of-the-money implied volatility for this model is shown to *diverge* in the small-maturity limit, with an explicit limiting shape displaying a very pronounced smile.

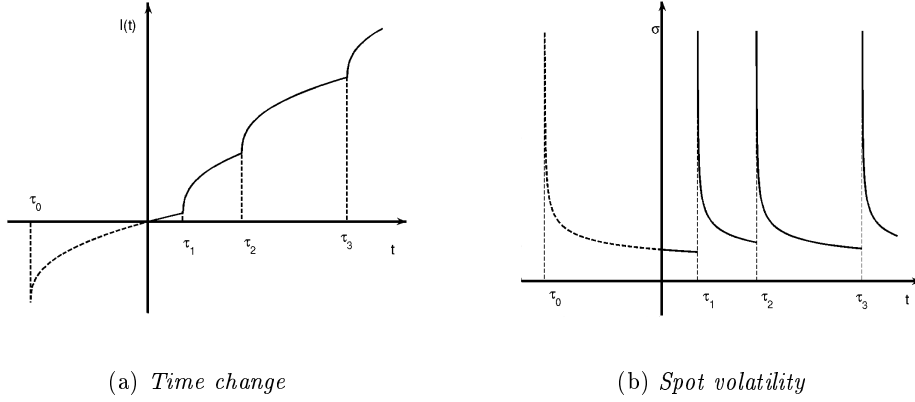
Our approach is based on estimates on the tail decay of the log-return distribution. Even though the moment generating function admits no explicit formula, we can extract asymptotic estimates that are sharp enough to apply *large deviations* techniques, notably the Gärtner-Ellis theorem.

3.1 The model

We recall the definition of the model $(Y_t)_{t \geq 0}$ introduced in [ACDP12] for the detrended log-price under the historical measure, after which we look at the risk-neutral measure.

3.1.1 The historical measure

We have two *independent* sources of randomness:

Figure 3.1: Paths of the *time change* and of the *spot volatility process*

- a standard Brownian motion $(W_t)_{t \geq 0}$;
- a Poisson process $(N_t)_{t \geq 0}$ of intensity $\lambda \in (0, \infty)$, whose jump times are denoted by $0 < \tau_1 < \tau_2 < \dots$

For fixed parameters $V \in (0, \infty)$, $\tau_0 \in (-\infty, 0)$ and $D \in (0, \frac{1}{2})$, we define

$$Y_t := W_{I_t}, \quad (3.1.1)$$

i.e. Y is a *time-changed Brownian motion*, where the clock process $(I_t)_{t \geq 0}$ is defined by

$$I_t := c \left\{ (t - \tau_{N_t})^{2D} - (-\tau_0)^{2D} + \sum_{k=1}^{N_t} (\tau_k - \tau_{k-1})^{2D} \right\}, \quad \text{with} \quad c := \frac{V^2 \lambda^{2D-1}}{\Gamma(2D+1)}, \quad (3.1.2)$$

where $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x} dx$ denotes Euler's gamma function, and with the convention that the sum in (3.1.2) is zero when $N_t = 0$. We refer to Figure 3.1 for a sample path of I_t .

Being a function of $(N_t)_{t \geq 0}$, the process $(I_t)_{t \geq 0}$ is independent of the Brownian motion $(W_t)_{t \geq 0}$. The trajectories $t \mapsto I_t$ are continuous, and also differentiable at every $t \geq 0$ which is not a jump time τ_k of the Poisson process, i.e. for $t \neq \tau_{N_t}$ (note that τ_{N_t} is the last jump time of the Poisson process before t). The derivative of I_t is simply given by

$$I'_t = 2D c (t - \tau_{N_t})^{2D-1}, \quad (3.1.3)$$

where we stress that the exponent $2D - 1$ in (3.1.3) is *negative*, since $D < \frac{1}{2}$.

Observe that in the limiting case $D = \frac{1}{2}$, the model becomes $Y_t = W_{V^2 t}$, i.e. Brownian motion with constant volatility V .

An alternative, equivalent definition of the model $(Y_t)_{t \geq 0}$ is to observe that, by (3.1.1), it is the solution of the following stochastic differential equation:

$$dY_t = \sigma_t dB_t, \quad \text{with} \quad \sigma_t := \sqrt{I'_t}, \quad (3.1.4)$$

where $(B_t)_{t \geq 0}$ denotes another Brownian motion, independent of $(\sigma_t)_{t \geq 0}$ (see [ACDP12]). In other terms, Y_t can be described as a *stochastic volatility model*, where the volatility is

the square root of the time-change process $(I_t)_{t \geq 0}$ and, by (3.1.3), it explodes at each jump time of the Poisson process, after which it decays as an inverse power (see Figure 3.1).

Remark 3.1.1. The four parameters λ, D, V, τ_0 have the following meaning:

- $\lambda \in (0, \infty)$ represents the average frequency of shocks;
- $D \in (0, \frac{1}{2}]$ tunes the non-linear evolution of the volatility after a shock;
- $V \in (0, \infty)$ represents the *large-time volatility*, i.e. $V = \lim_{t \rightarrow \infty} \sqrt{\mathbb{E}[\sigma_t^2]}$;†
- $\tau_0 \in (-\infty, 0)$, which according to (3.1.2) plays the role of the “last jump” before time 0, determines the *initial volatility* σ_0 , cf. (3.1.3) and (3.1.4):

$$\sigma_0 = \frac{\lambda^{D-\frac{1}{2}} V}{\sqrt{\Gamma(2D)}} (-\tau_0)^{D-\frac{1}{2}} = \sqrt{(2D)\mathfrak{c}} (-\tau_0)^{D-\frac{1}{2}}. \quad (3.1.5)$$

Given this correspondence, *one can equivalently use σ_0 as a parameter instead of τ_0* .‡

3.1.2 The risk-neutral measure

Under the natural risk-neutral measure, the price $(S_t)_{t \geq 0}$, say with $S_0 = 1$, evolves according to the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \sigma_t dB_t, \quad (3.1.6)$$

where σ_t is the process defined in (3.1.4)-(3.1.3), namely

$$\sigma_t := \frac{\lambda^{D-\frac{1}{2}} V}{\sqrt{\Gamma(2D)}} (t - \tau_{N_t})^{D-\frac{1}{2}}.$$

As we describe in Chapter 4, there is a one-parameter class of equivalent martingale measures for our model, which allow to modify the value of the parameter $\lambda \in (0, \infty)$ freely. Here we assume to have fixed that parameter, and still call it λ .

The *log-price* process $(X_t)_{t \geq 0}$ is defined by $X_t := \log S_t = \log \frac{S_t}{S_0}$, since $S_0 = 1$. It follows by (3.1.6) that $dX_t = \frac{dS_t}{S_t} - \frac{1}{2} \frac{d(S)_t}{S_t^2} = \sigma_t dB_t - \frac{1}{2} \sigma_t^2 dt$, hence by (3.1.4) and (3.1.1) we have

$$X_t = W_{I_t} - \frac{1}{2} I_t, \quad (3.1.7)$$

where $(W_t)_{t \geq 0}$ is a Brownian motion and $(I_t)_{t \geq 0}$ is an independent process, defined in (3.1.2) As a consequence, the price $(S_t)_{t \geq 0}$, which equals

$$S_t = e^{X_t} = e^{W_{I_t} - \frac{1}{2} I_t}, \quad (3.1.8)$$

is a *time-changed geometric Brownian motion*, with independent time-change process.

Let us stress that *equations (3.1.7) and (3.1.8), together with (3.1.2), can be taken as the definitions of the log-price X_t and price S_t processes.*

†We point out that in [ACDP12] the parameter V was replaced by $\sqrt{\mathfrak{c}}$, appearing in (3.1.2).

‡In [ACDP12] the parameter $-\tau_0$ was chosen randomly, as an $Exp(\lambda)$ random variable (like $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \dots$) independent of $(N_t)_{t \geq 0}$ and $(W_t)_{t \geq 0}$. With this choice, the process $(t - \tau_{N_t})_{t \geq 0}$ is *stationary* (with $Exp(\lambda)$ one-time marginal distributions), hence the volatility $(\sigma_t)_{t \geq 0}$ is a stationary process. In our context, it is more natural to have a fixed value for the initial volatility.

3.1.3 Option price and implied volatility

The price of a European call, with log-strike $\kappa \in \mathbb{R}$ and maturity $t \geq 0$, under our model is

$$c(\kappa, t) := \mathbb{E}[(S_t - e^\kappa)^+] = \mathbb{E}[(e^{X_t} - e^\kappa)^+]. \quad (3.1.9)$$

We recall from Chapter 2, §2.3.2, that the Black&Scholes price of a call option, with volatility $\sigma \in (0, \infty)$, is given by $\mathbf{C}_{\text{BS}}(\kappa, \sigma\sqrt{t})$, where

$$\mathbf{C}_{\text{BS}}(\kappa, v) := \mathbb{E}[(e^{W_{v^2} - \frac{1}{2}v^2} - e^\kappa)^+] = \Phi(d_1) - e^\kappa \Phi(d_2), \quad (3.1.10)$$

where

$$\Phi(x) := \int_{-\infty}^x \frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}} dt, \quad d_1 := -\frac{\kappa}{v} + \frac{v}{2}, \quad d_2 := -\frac{\kappa}{v} - \frac{v}{2}. \quad (3.1.11)$$

For $t > 0$, the *implied volatility* $\sigma_{\text{imp}}(\kappa, t)$ of our model is defined as the unique value of $\sigma \in (0, \infty)$ such that $c(\kappa, t) = \mathbf{C}_{\text{BS}}(\kappa, \sigma\sqrt{t})$, that is

$$c(\kappa, t) = \mathbf{C}_{\text{BS}}(\kappa, \sigma_{\text{imp}}(\kappa, t)\sqrt{t}). \quad (3.1.12)$$

Our goal is to determine sharp asymptotic estimates on $\sigma_{\text{imp}}(\kappa, t)$.

It follows by (3.1.9) and (3.1.8), since $(I_t)_{t \geq 0}$ is independent of $(W_t)_{t \geq 0}$, that the following Hull-White [HW87] formula holds:

$$c(\kappa, t) = \mathbb{E}[\mathbf{C}_{\text{BS}}(\kappa, v)|_{v=\sqrt{I_t}}], \quad (3.1.13)$$

that is the price of a call under our model is obtained by averaging the Black&Scholes price $\mathbf{C}_{\text{BS}}(\kappa, v)$ with a random total volatility $v = \sqrt{I_t}$. In Chapter 4 we obtain a generalized version of this formula, which holds also when there are correlations between the time-change process and the Brownian motion.

By (3.1.10)-(3.1.11) and $\Phi(-x) = 1 - \Phi(x)$, the Black&Scholes call price satisfies

$$\mathbf{C}_{\text{BS}}(-\kappa, v) = 1 - e^{-\kappa} + e^{-\kappa} \mathbf{C}_{\text{BS}}(\kappa, v).$$

Then it follows by (3.1.13) that an analogous relation holds for our model:

$$c(-\kappa, t) = 1 - e^{-\kappa} + e^{-\kappa} c(\kappa, t).$$

Looking at (3.1.12), it follows that *the implied volatility of our model is symmetric in κ* :

$$\sigma_{\text{imp}}(-\kappa, t) = \sigma_{\text{imp}}(\kappa, t).$$

Note that this property holds for any stochastic volatility model in which the volatility process is independent of the price, as first observed in [RT96].

As a consequence, in the sequel we focus on the regime $\kappa \geq 0$.

3.2 Main results (I): implied volatility

In this section we present our main asymptotic results concerning the implied volatility of our model, that hold in the general regime when

$$\text{either } t \rightarrow 0 \text{ with arbitrary } \kappa \geq 0, \quad \text{or } t \rightarrow \bar{t} \in (0, \infty) \text{ and } \kappa \rightarrow \infty. \quad (3.2.1)$$

These results descend from Theorems 2.1.5 and 2.1.6 and are linked to the asymptotic behavior of the call price and tail probability, that will be stated in the next Section 3.3.

Let us define a continuous, increasing function $f : [0, \infty) \rightarrow [1, \infty)$ by[†]

$$f(a) := \min_{m \geq 1} f_m(a) := \min_{m \geq 1} \left(m + \frac{a^2}{2cm^{1-2D}} \right), \quad (3.2.2)$$

where c is the constant defined in (3.1.2). Note that $f(0) = 1$. As $a \rightarrow \infty$, optimizing over m shows that the minimum in (3.2.2) is attained for $m \simeq \left(\frac{(1-2D)a^2}{2c} \right)^{1/(2-2D)}$, hence

$$f(a) \sim \tilde{C} a^{\frac{1}{1-D}}, \quad \text{where} \quad \tilde{C} := \frac{2(1-D)}{(2c)^{\frac{1}{2(1-D)}} (1-2D)^{\frac{1-2D}{2(1-D)}}}. \quad (3.2.3)$$

We also define the two functions

$$\kappa_1(t) := \sqrt{t} \sqrt{\log \frac{1}{t}}, \quad \kappa_2(t) := t^D \sqrt{\log \frac{1}{t}}, \quad (3.2.4)$$

which, as we will see in a moment, act as boundaries for κ , separating different asymptotic regimes for $\sigma_{\text{imp}}(\kappa, t)$ as $t \rightarrow 0$. (Note that $\kappa_1(t) < \kappa_2(t)$, since $D < \frac{1}{2}$.)

We are ready to state our first main result, proved in Section 3.5 (see Figure 3.2 for a plot). We recall that $f \sim g$ means $f/g \rightarrow 1$, while $f \ll g$ means $f/g \rightarrow 0$.

Theorem 3.2.1 (Implied volatility). *Consider a family of values of (κ, t) with $\kappa \geq 0, t > 0$. If $t \rightarrow \bar{t} \in (0, \infty)$ and $\kappa \rightarrow \infty$, the following relation holds:*

$$\sigma_{\text{imp}}(\kappa, t) \sim \left\{ \frac{1}{\sqrt{2\bar{C}}} \right\} \left(\frac{\frac{\kappa}{t}}{\sqrt{\log \frac{\kappa}{t}}} \right)^{\frac{1-2D}{2-2D}}, \quad (3.2.5)$$

for an explicit constant \bar{C} given by

$$\bar{C} := \frac{2(1-D)^{\frac{1}{2(1-D)}}}{(2c)^{\frac{1}{2(1-D)}} (1-2D)^{\frac{1-2D}{2(1-D)}}}. \quad (3.2.6)$$

If $t \rightarrow 0$, we distinguish various regimes. Recall that σ_0 is the constant defined in (3.1.5), while the functions $f(\cdot)$ and $\kappa_1(\cdot), \kappa_2(\cdot)$ are defined in (3.2.2), (3.2.4).

- If $0 \leq \kappa \leq \sqrt{2D+1} \sigma_0 \kappa_1(t)$,

$$\sigma_{\text{imp}}(\kappa, t) \sim \sigma_0; \quad (3.2.7)$$

[†]The function $f(\cdot)$ is continuous because for any $a \in [0, \infty)$ one can restrict the minimum in (3.2.2) over the finite set $m \in \{1, \dots, \lfloor f_1(a) \rfloor\}$, since $f_i(a) \geq i$ for all $i \geq 1$.

- if $\sqrt{2D+1}\sigma_0\kappa_1(t) \leq \kappa \ll \kappa_2(t)$

$$\sigma_{\text{imp}}(\kappa, t) \sim \left\{ \frac{1}{\sqrt{2(D+1 - \frac{\log \kappa}{\log t})}} \right\} \frac{\kappa}{\kappa_1(t)}, \quad (3.2.8)$$

and note that the quantity inside the brackets is of order 1, because $\frac{\log \kappa}{\log t}$ varies smoothly between $\frac{1}{2}$ and D , for κ in the range under consideration;

- if $\kappa \sim a\kappa_2(t)$, for some $a \in (0, \infty)$,

$$\sigma_{\text{imp}}(\kappa, t) \sim \left\{ \frac{1}{\sqrt{2f(a)}} \right\} \frac{\kappa}{\kappa_1(t)}; \quad (3.2.9)$$

- finally, if $\kappa \gg \kappa_2(t)$, the asymptotic relation (3.2.5) holds.

If we fix $t > 0$ small and increase κ , Theorem 3.2.1 shows that the implied volatility $\sigma_{\text{imp}}(\kappa, t)$ of our model is roughly equal to the constant value σ_0 from $\kappa = 0$ until $\kappa \approx \kappa_1(t) \approx \sqrt{t}$, cf. (3.2.7), then it starts growing linearly until $\kappa \approx \kappa_2(t) \approx t^D$, cf. (3.2.8), after which it grows sublinearly as $\approx (\kappa/t)^\gamma$, cf. (3.2.5), where the exponent $\gamma = \frac{1-2D}{2-2D}$ can take any value in $(0, \frac{1}{2})$, depending on D . See Figure 3.2 for a graphical representation.

Remark 3.2.2. For fixed $\kappa > 0$, the implied volatility *diverges* as $t \downarrow 0$, by (3.2.5). This phenomenon, which is typical for models with jumps [AL12], also happens in our model, despite the fact that it has continuous paths, and is linked to the fact that the distribution of the time-change process I_t displays *approximate polynomial tails* as $t \downarrow 0$. Incidentally, this is the same mechanism that produces the *multiscaling of moments* [ACDP12]. We point out that these features are absent in most stochastic volatility models, where the distribution of the stochastic volatility has thin tails as $t \downarrow 0$, such as the Heston model [FJL12].

Remark 3.2.3. The four relations (3.2.7), (3.2.8), (3.2.9) and (3.2.5) match perfectly at the boundaries of the respective intervals of applicability:

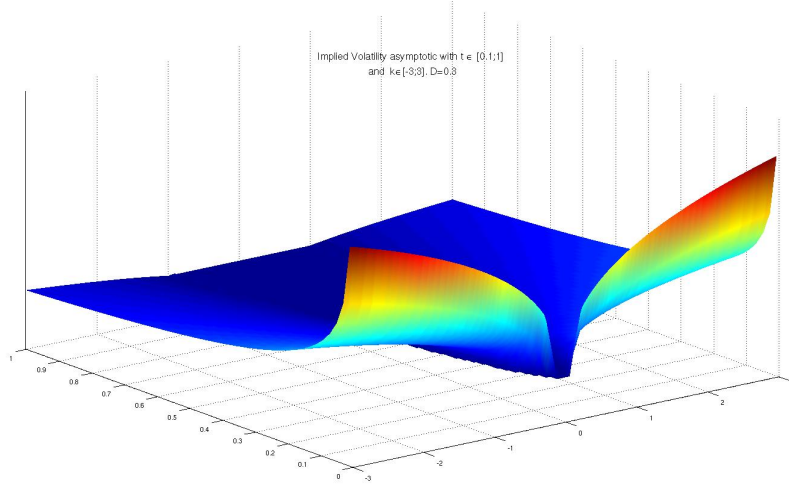
- relations (3.2.7) and (3.2.8) coincide for $\kappa = (\sigma_0\sqrt{2D+1})\kappa_1(t)$;
- since $f(a) \rightarrow 1$, letting $a \downarrow 0$ in (3.2.9) yields relation (3.2.8) with $\kappa \approx \kappa_2(t) \approx t^D$;
- recalling (3.2.3) and noting that $\bar{C} := (1-D)^{\frac{2D-1}{2-2D}}\tilde{C}$, if we let $a \uparrow \infty$ in (3.2.9) we obtain relation (3.2.5) (note that $\log \frac{\kappa}{t} \sim (1-D)\log \frac{1}{t}$ in (3.2.5) when $\kappa \approx \kappa_2(t)$).

Remark 3.2.4. In the limiting case $D = \frac{1}{2}$ one has

$$\sigma_0 = V, \quad \mathbf{c} = V^2, \quad \kappa_1(t) = \kappa_2(t), \quad f(a) = \frac{a^2}{2\mathbf{c}},$$

cf. (3.1.5), (3.1.2) and (3.2.4) and (3.2.2) (where the min should range over $m \geq 0$, but $m = 0$ is automatically ruled out when $D < \frac{1}{2}$). Consequently, relations (3.2.7), (3.2.9) and (3.2.5) reduce to $\sigma_{\text{imp}}(\kappa, t) \sim V$, in perfect agreement with the fact that for $D = \frac{1}{2}$ our model becomes Black&Scholes model with constant volatility V .[†]

[†]Note that relation (3.2.8) does not apply for $D = \frac{1}{2}$, because in this case $\kappa_1(t) = \kappa_2(t)$ and consequently there is no κ for which $(\sigma_0\sqrt{2D+1})\kappa_1(t) \leq \kappa \ll \kappa_2(t)$.

Figure 3.2: Smile asymptotics for $D = 0.3$

3.3 Main results (II): tail probability and option price

We now present explicit estimates for the tail probability $\bar{F}_t(\kappa) = \mathbb{P}(X_t > \kappa)$ of our model, together with estimates on the option price $c(\kappa, t)$, based on Theorems 2.1.5 and 2.1.11 in Chapter 2. These results yield the sharp asymptotic behavior of the implied volatility, described in Theorem 3.2.1.

We first observe that as $t \downarrow 0$ we have the convergence in law

$$\frac{X_t}{\sqrt{t}} \xrightarrow{d} \sigma_0 W_1, \quad (3.3.1)$$

where σ_0 is the constant in (3.1.5). To prove this fact, note that for any $t \geq 0$, by (3.1.7),

$$X_t \stackrel{d}{=} \sqrt{I_t} W_1 - \frac{1}{2} I_t.$$

Since $I_t = I_0 + I_0' t + o(t) = \sigma_0^2 t + o(t)$, cf. (3.1.4)-(3.1.5), one has $I_t/t \rightarrow \sigma_0^2$ a.s. as $t \downarrow 0$, hence $I_t/\sqrt{t} \rightarrow 0$, and (3.3.1) follows.

Relation (3.3.1) shows that $\kappa = O(\sqrt{t})$ is the regime of *typical deviations*, for which we can state the following result, proved in Section 3.6 below.

Theorem 3.3.1 (Tail probability and option price: typical deviations). *Consider a family of values of (κ, t) with $\kappa \geq 0$, $t > 0$ such that*

$$t \rightarrow 0 \quad \text{and} \quad \kappa \sim a\sqrt{t}, \quad \text{for some } a \in [0, \infty).$$

Then

$$\mathbb{P}(X_t > \kappa) \xrightarrow{t \downarrow 0} 1 - \Phi\left(\frac{a}{\sigma_0}\right), \quad c(\kappa, t) \underset{t \downarrow 0}{\sim} a\sqrt{t} D\left(\frac{a}{\sigma_0}\right), \quad (3.3.2)$$

where $D(x) := \frac{1}{x}\phi(x) - \Phi(-x)$ is a smooth decreasing bijection from $(0, \infty)$ to $(0, \infty)$, cf. (2.1.5), and $\phi(\cdot)$ and $\Phi(\cdot)$ are the density and distribution function of a standard Gaussian.

Next we consider the regime of *atypical deviations*, i.e. families of (κ, t) with $\kappa \gg \sqrt{t}$. The following result gives asymptotics for the tail probability and is proved in §3.7 below.

Theorem 3.3.2 (Tail probability: atypical deviations). *Consider a family of values of (κ, t) with $\kappa \geq 0$, $t > 0$. If $t \rightarrow \bar{t} \in (0, \infty)$ and $\kappa \rightarrow \infty$, the following asymptotics holds:*

$$\log \mathbb{P}(X_t > \kappa) \sim -\bar{C} \left(\frac{\kappa}{t^D} \right)^{\frac{1}{1-D}} \left(\log \frac{\kappa}{t} \right)^{\frac{1-2D}{2-2D}}, \quad (3.3.3)$$

where the constants σ_0 and \bar{C} are defined in (3.1.5) and (3.2.6).

If $t \rightarrow 0$ and $\kappa \gg \sqrt{t}$, i.e. $\kappa/\sqrt{t} \rightarrow \infty$, the following asymptotics holds:

$$\log \mathbb{P}(X_t > \kappa) \sim \begin{cases} -\frac{1}{2\sigma_0^2} \left(\frac{\kappa}{\kappa_1(t)} \right)^2 \log \frac{1}{t} & \text{if } \kappa \leq \sqrt{2} \sigma_0 \kappa_1(t), \\ -f \left(\frac{\kappa}{\kappa_2(t)} \right) \log \frac{1}{t} & \text{if } \kappa > \sqrt{2} \sigma_0 \kappa_1(t), \end{cases} \quad (3.3.4)$$

where $f(\cdot)$, $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ are defined in (3.2.2) and (3.2.4). More explicitly:

- if $\sqrt{t} \ll \kappa \ll \kappa_2(t)$, since $f(0) = 1$,

$$\log \mathbb{P}(X_t > \kappa) \sim -\min \left\{ \frac{a^2}{2\sigma_0^2}, 1 \right\} \log \frac{1}{t}, \quad \text{where } a := \frac{\kappa}{\kappa_1(t)}; \quad (3.3.5)$$

- if $\kappa \sim a \kappa_2(t)$, for some $a \in (0, \infty)$,

$$\log \mathbb{P}(X_t > \kappa) \sim -f(a) \log \frac{1}{t}; \quad (3.3.6)$$

- if $\kappa \gg \kappa_2(t)$, the asymptotic relation (3.3.3) holds.

Finally, we give the corresponding asymptotics for the option price. The following Theorem is proved in Section 3.8 below.

Theorem 3.3.3 (Option price: atypical deviations). *Consider a family of values of (κ, t) with $\kappa \geq 0$, $t > 0$. If $t \rightarrow \bar{t} \in (0, \infty)$ and $\kappa \rightarrow \infty$, the following asymptotics holds:*

$$\log c(\kappa, t) \sim \log \mathbb{P}(X_t > \kappa), \quad (3.3.7)$$

and the right hand side can be read from (3.3.3).

If $t \rightarrow 0$ and $\sqrt{t} \ll \kappa \ll \kappa_1(t)$, or $\kappa \gg \kappa_2(t)$, the following relation holds:

$$\log (c(\kappa, t)/\kappa) \sim \log \mathbb{P}(X_t > \kappa), \quad (3.3.8)$$

and the right hand side can be read from (3.3.5) (if $\kappa \ll \kappa_1(t)$) or (3.3.3) (if $\kappa \gg \kappa_1(t)$).

If $t \rightarrow 0$ and $\kappa \sim a \kappa_1(t)$, for some $a \in (0, \infty)$,

$$\log (c(\kappa, t)/\kappa) \sim -\min \left\{ \frac{a^2}{2\sigma_0^2}, D + \frac{1}{2} \right\} \log \frac{1}{t}, \quad (3.3.9)$$

while if $\kappa_1(t) \ll \kappa \leq M \kappa_2(t)$, for some $M \in (0, \infty)$,

$$\log (c(\kappa, t)/\kappa) \sim -\log \frac{1}{t} - \log \frac{\kappa}{t^D}. \quad (3.3.10)$$

3.4 Preliminary results

We start stating a useful upper bound on I_t (which, we recall, is defined in (3.1.2)).

Lemma 3.4.1. *For all $t \geq 0$ the following upper bound holds:*

$$I_t \leq \sigma_0^2 t + c N_t^{1-2D} t^{2D}, \quad (3.4.1)$$

where the constants σ_0 and c are defined in (3.1.5) and (3.1.2).

Proof. Since $(a+b)^{2D} - b^{2D} \leq 2D b^{2D-1} a$ for all $a, b > 0$ by concavity (recall that $D < \frac{1}{2}$), on the event $\{N_t = 0\}$ we can write, recalling (3.1.2) and (3.1.5),

$$I_t = c \{(t - \tau_0)^{2D} - (-\tau_0)^{2D}\} \leq c 2D (-\tau_0)^{2D-1} t = \sigma_0^2 t, \quad (3.4.2)$$

proving (3.4.1). Analogously, on the event $\{N_t \geq 1\} = \{0 \leq \tau_1 \leq t\}$ we have

$$\begin{aligned} I_t &:= c \left\{ (\tau_1 - \tau_0)^{2D} - (-\tau_0)^{2D} + \sum_{k=2}^{N_t} (\tau_k - \tau_{k-1})^{2D} + (t - \tau_{N_t})^{2D} \right\} \\ &\leq c \left\{ 2D (-\tau_0)^{2D-1} t + \sum_{k=2}^{N_t} (\tau_k - \tau_{k-1})^{2D} + (t - \tau_{N_t})^{2D} \right\}. \end{aligned} \quad (3.4.3)$$

For all $\ell \in \mathbb{N}$ and $x_1, \dots, x_\ell \in \mathbb{R}$, it follows by Hölder's inequality with $p := \frac{1}{2D}$ that

$$\sum_{k=1}^{\ell} x_k^{2D} \leq \left(\sum_{k=1}^{\ell} (x_k^{2D})^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\ell} 1 \right)^{1-\frac{1}{p}} = \left(\sum_{k=1}^{\ell} x_k \right)^{2D} \ell^{1-2D}. \quad (3.4.4)$$

Choosing $\ell = N_t$ and $x_1 = \tau_2 - \tau_1$, $x_k = (\tau_{k+1} - \tau_k)$ for $2 \leq k \leq \ell - 1$ and $x_\ell = (t - \tau_{\ell-1})$, since $\sum_{k=1}^{\ell} x_k = t - \tau_1 \leq t$, we get from (3.4.3)

$$I_t \leq c \left(2D (-\tau_0)^{2D-1} t + N_t^{1-2D} t^{2D} \right) = \sigma_0^2 t + c N_t^{1-2D} t^{2D},$$

completing the proof of (3.4.1). \square

Corollary 3.4.2 (No moment explosion). *For every $t \in [0, \infty)$ and $p \in \mathbb{R}$ one has*

$$\mathbb{E}[e^{pX_t}] = \mathbb{E}[e^{\frac{1}{2}p(p-1)I_t}] < \infty. \quad (3.4.5)$$

Proof. Recalling the definition (3.1.7) of X_t , the independence of I and W gives

$$\mathbb{E}[e^{pX_t}] = \mathbb{E}[e^{p(W_{I_t} - \frac{1}{2}I_t)}] = \mathbb{E}[e^{p(\sqrt{I_t}W_1 - \frac{1}{2}I_t)}] = \mathbb{E}[e^{\frac{1}{2}(p\sqrt{I_t})^2 - \frac{1}{2}pI_t}] = \mathbb{E}[e^{\frac{1}{2}p(p-1)I_t}],$$

which proves the equality in (3.4.5). Applying the upper bound (3.4.1) yields

$$\mathbb{E}[e^{\frac{1}{2}p(p-1)I_t}] \leq \mathbb{E}[e^{\frac{1}{2}p(p-1)(\sigma_0^2 t + c N_t^{1-2D} t^{2D})}] = \mathbb{E}[e^{c_1 t + c_2 t^{2D} N_t^{1-2D}}] \leq \mathbb{E}[e^{c_1 t + c_2 t^{2D} N_t}],$$

for suitable $c_1, c_2 \in (0, \infty)$ depending on p and on the parameters of the model. The right hand side is finite because $N_t \sim \text{Pois}(\lambda t)$ has finite exponential moments of all orders. \square

Corollary 3.4.3. *There exists a constant $C \in (0, \infty)$ (depending on the parameters of the model) such that*

$$\mathbb{E}[e^{2X_t}] \leq 1 + Ct, \quad \forall 0 \leq t \leq 1.$$

Proof. By the equality in (3.4.5) and the upper bound (3.4.1), we can write

$$\mathbb{E}[e^{2X_t}] = \mathbb{E}[e^{I_t}] \leq e^{\sigma_0^2 t} \mathbb{E}[e^{ct^{2D} N_t^{1-2D}}].$$

Next observe that, by Hölder's inequality,

$$\begin{aligned} \mathbb{E}[e^{ct^{2D} N_t^{1-2D}}] &= \mathbb{P}(N_t = 0) + e^{ct^{2D}} \mathbb{P}(N_t = 1) + \mathbb{E}[e^{ct^{2D} N_t^{1-2D}} \mathbf{1}_{\{N_t \geq 2\}}] \\ &\leq e^{-\lambda t} + e^{ct^{2D}} \lambda t + \sqrt{\mathbb{E}[e^{2ct^{2D} N_t^{1-2D}}] \mathbb{P}(N_t \geq 2)}. \end{aligned}$$

Note that $\mathbb{P}(N_t \geq 2) = 1 - e^{-\lambda t}(1 + \lambda t) = \frac{1}{2}(\lambda t)^2 + o(t^2)$ as $t \downarrow 0$. For all $0 \leq t \leq 1$ we can write $\mathbb{E}[e^{2ct^{2D} N_t^{1-2D}}] \leq \mathbb{E}[e^{2cN_1^{1-2D}}] =: c_1 < \infty$, and $e^{ct^{2D}} \leq e^c$, hence

$$\mathbb{E}[e^{ct^{2D} N_t^{1-2D}}] \leq 1 + e^c \lambda t + \sqrt{\frac{c_1 \lambda^2}{2}} (t + o(t)) \leq 1 + c_2 t,$$

for some $c_2 < \infty$. Consequently

$$\mathbb{E}[e^{2X_t}] \leq e^{\sigma_0^2 t} (1 + c_2 t) = (1 + \sigma_0^2 t + o(t))(1 + c_2 t) \leq 1 + Ct,$$

for some $C < \infty$. □

3.5 Proof of Theorem 3.2.1

We are going to apply Theorem 2.1.1 in Chapter 2, exploiting the asymptotic behavior of the call price $c(\kappa, t)$ of Theorems 3.3.1 and 3.3.3 (which are proved in Sections 3.6 and 3.8).

3.5.1 Proof of (3.2.5)

Consider a family of values of (κ, t) with $\kappa \geq 0$, $t > 0$. If $t \rightarrow \bar{t} \in (0, \infty)$ and $\kappa \rightarrow \infty$, or alternatively if $t \rightarrow 0$ and $\kappa \rightarrow \bar{\kappa} \in (0, \infty]$ (so that, in particular, $\kappa \gg \kappa_2(t)$), our goal is to prove that relation (3.2.5) holds. Applying relation (3.3.7)[†] and (3.3.3), we get

$$\log c(\kappa, t) \sim -\log \mathbb{P}(X_t > \kappa) \sim \bar{C} \left(\frac{\kappa}{t^D} \right)^{\frac{1}{1-D}} \left(\log \frac{\kappa}{t} \right)^{\frac{1-2D}{2-2D}}, \quad (3.5.1)$$

Since κ is bounded away from zero in these cases, we can apply relation (2.1.7), which reduces to (2.1.10) (because $|\log \mathbb{P}(X_t > \kappa)| \gg |\log \kappa|$ by (3.3.3)), that is

$$\sigma_{\text{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t(-\log c(\kappa, t))}}.$$

Plugging in the asymptotic relation (3.5.1), we obtain our goal (3.2.5).

[†]In case $t \rightarrow 0$ we should apply (3.3.8), i.e. $\log(c(\kappa, t)/\kappa) \sim -\log \mathbb{P}(X_t > \kappa)$. This however is equivalent to (3.3.7), because $|\log \mathbb{P}(X_t > \kappa)| \gg |\log \kappa|$, as it follows by (3.3.3).

3.5.2 Proof of (3.2.7)

Next we consider a family of values of (κ, t) with $t \rightarrow 0$ and $0 \leq \kappa \leq \sqrt{2D+1} \sigma_0 \kappa_1(t)$, and our goal is to prove (3.2.7).

First we consider the case of typical deviations, i.e. when $\kappa = O(\sqrt{t})$, say $\kappa \sim a\sqrt{t}$ for some $a \in [0, \infty)$. Relation (3.3.2) gives

$$c(\kappa, t) \sim a\sqrt{t} D \left(\frac{a}{\sigma_0} \right) \sim \kappa D \left(\frac{\kappa}{\sigma_0 \sqrt{t}} \right),$$

Applying relations (2.1.8), or better its simplified form given by the second line of (2.1.11), yields $\sigma_{\text{imp}}(\kappa, t) \rightarrow \sigma_0$, i.e. our goal (3.2.7).

Next we consider the case of atypical deviations, i.e. when $\kappa \gg \sqrt{t}$. Relations (3.3.8) and (3.3.9), together with (3.3.5), can be rewritten as

$$\log(c(\kappa, t)/\kappa) \sim -\frac{\kappa^2}{2\sigma_0^2 t}.$$

Since $\kappa \rightarrow 0$, we can apply the first line of relation (2.1.11), getting

$$\sigma_{\text{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t(-\log(c(\kappa, t)/\kappa))}} \sim \sigma_0,$$

proving our goal (3.2.7) also in this case.

3.5.3 Proof of (3.2.8)

Next we consider a family of values of (κ, t) with $t \rightarrow 0$ and $\sqrt{2D+1} \sigma_0 \kappa_1(t) \leq \kappa \ll \kappa_2(t)$, and our goal is to prove (3.2.8). In this case relation (3.3.9) becomes

$$\log(c(\kappa, t)/\kappa) \sim -\left(D + \frac{1}{2}\right) \log \frac{1}{t} \sim -\log \frac{1}{t} \left(1 + D - \frac{\log \kappa}{\log t}\right)$$

and also (3.3.10) can be rewritten as

$$\log(c(\kappa, t)/\kappa) \sim -\log \frac{1}{t} \left(1 + D - \frac{\log \kappa}{\log t}\right).$$

Since $\kappa \rightarrow 0$, we can apply the first line of relation (2.1.11), getting

$$\sigma_{\text{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t(-\log(c(\kappa, t)/\kappa))}} \sim \left\{ \frac{1}{\sqrt{2\left(D+1-\frac{\log \kappa}{\log t}\right)}} \right\} \frac{\kappa}{\kappa_1(t)}$$

proving our goal (3.2.8).

3.5.4 Proof of (3.2.9)

Next we consider a family of values of (κ, t) with $t \rightarrow 0$ and $\kappa \sim a\kappa_2(t)$ for some $a \in (0, \infty)$, and our goal is to prove (3.2.9). Relation (3.3.7) together with (3.3.6) can be rewritten as

$$\log(c(\kappa, t)/\kappa) \sim -f(a) \log \frac{1}{t}.$$

Since $\kappa \rightarrow 0$, we can apply the first line of relation (2.1.11), getting

$$\sigma_{\text{imp}}(\kappa, t) \sim \left\{ \frac{1}{\sqrt{2f(a)}} \right\} \frac{\kappa}{\kappa_1(t)}$$

proving our goal (3.2.9).

3.6 Proof of Theorem 3.3.1

The first relation in (3.3.2) follows immediately from (3.3.1).

For the second relation in (3.3.2), we are going to apply Theorem 2.1.11. Note that Hypothesis 2.1.10 of Chapter 2 is satisfied with $\gamma = \sqrt{t}$ and $Y = \sigma_0 W_1$, again by (3.3.1), and assumption (2.1.38) is verified for $\eta = 1$ by Corollary 3.4.3 (cf. (2.1.20)). We can then apply relation (2.1.39) in Theorem 2.1.11, which for $\kappa \sim a\sqrt{t}$ yields

$$\begin{aligned} c(\kappa, t) &\sim \sqrt{t} \sigma_0 \mathbb{E} \left[\left(W_1 - \frac{a}{\sigma_0} \right)^+ \right] = \sqrt{t} \sigma_0 \left[\int_{\frac{a}{\sigma_0}}^{\infty} x \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx - \frac{a}{\sigma_0} \int_{\frac{a}{\sigma_0}}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \right] \\ &= \sqrt{t} \sigma_0 \left(\frac{e^{-\frac{a^2}{2\sigma_0^2}}}{\sqrt{2\pi}} - \frac{a}{\sigma_0} \left(1 - \Phi \left(\frac{a}{\sigma_0} \right) \right) \right) = \sqrt{t} \sigma_0 \left(\phi \left(\frac{a}{\sigma_0} \right) - \frac{a}{\sigma_0} \Phi \left(-\frac{a}{\sigma_0} \right) \right) \\ &= a\sqrt{t} \left(\frac{\phi \left(\frac{a}{\sigma_0} \right)}{\frac{a}{\sigma_0}} - \Phi \left(-\frac{a}{\sigma_0} \right) \right) = a\sqrt{t} D \left(\frac{a}{\sigma_0} \right), \end{aligned} \tag{3.6.1}$$

which is precisely the second relation in (3.3.2). \square

3.7 Proof of Theorem 3.3.2

We split the proof in two parts, focusing first on relation (3.3.3) and then on (3.3.4).

3.7.1 Proof of relation (3.3.3)

Recall the definition of $\kappa_1(t)$ and $\kappa_2(t)$ in (3.2.4). Let us fix a family of (κ, t) with $\kappa > 0$, $t > 0$ such that

$$\text{either } t \rightarrow \bar{t} \in (0, \infty) \text{ and } k \rightarrow \infty, \quad \text{or } t \rightarrow 0 \text{ and } \frac{\kappa}{\kappa_2(t)} \rightarrow \infty. \tag{3.7.1}$$

We are going to prove the following result, which is stronger than (3.3.3).

Theorem 3.7.1 (Large deviations principle). *As (κ, t) run along a family satisfying (3.7.1), the random variables $\frac{X_t}{\kappa}$ satisfy the large deviations principle (LDP) with rate $\alpha_{t,\kappa}$ and good rate function $I(\cdot)$ given by*

$$\alpha_{t,\kappa} = \left(\frac{\kappa}{t^D}\right)^{\frac{1}{1-D}} \left(\log \frac{\kappa}{t}\right)^{\frac{1-2D}{2-2D}}, \quad I(x) = \bar{C} |x|^{\frac{1}{1-D}} \quad (3.7.2)$$

where \bar{C} is defined in (3.2.6). This means that for every Borel set $A \subseteq \mathbb{R}$

$$-\inf_{x \in \overset{\circ}{A}} I(x) \leq \liminf_{\alpha_{t,\kappa}} \frac{1}{\alpha_{t,\kappa}} \log \mathbb{P} \left(\frac{X_t}{\kappa} \in A \right) \leq \limsup_{\alpha_{t,\kappa}} \frac{1}{\alpha_{t,\kappa}} \log \mathbb{P} \left(\frac{X_t}{\kappa} \in A \right) \leq -\inf_{x \in \bar{A}} I(x),$$

where $\overset{\circ}{A}$ and \bar{A} denote respectively the interior part and the closure of A . In particular, choosing $A = (1, \infty)$, relation (3.3.3) in Theorem 3.3.2 holds.

We are going to show that, with $\alpha_{t,\kappa}$ as in (3.7.2), the following limit exists for $\beta \in \mathbb{R}$:

$$\Lambda(\beta) := \lim_{\alpha_{t,\kappa}} \frac{1}{\alpha_{t,\kappa}} \log \mathbb{E}[e^{\beta \alpha_{t,\kappa} \frac{X_t}{\kappa}}], \quad (3.7.3)$$

where $\Lambda : \mathbb{R} \rightarrow \mathbb{R}$ is everywhere finite and differentiable. By the Gärtner-Ellis Theorem [DZ98, Theorem 2.3.6], it follows that $\frac{X_t}{\kappa}$ satisfies a LDP with good rate $\alpha_{t,\kappa}$ and with rate function $I(\cdot)$ given by the Fenchel-Legendre transform of $\Lambda(\cdot)$, i.e.

$$I(x) = \sup_{\beta \in \mathbb{R}} \{\beta x - \Lambda(\beta)\}. \quad (3.7.4)$$

The proof of Theorem 3.7.1 is thus reduced to computing $\Lambda(\beta)$ and then showing that $I(x)$ coincides with the one given in (3.7.2). By (3.4.5), the determination of $\Lambda(\beta)$ in (3.7.3) is reduced to the asymptotic behaviour of exponential moments of I_t . This is possible by the following crucial Proposition, proved below.

Proposition 3.7.2. *For any family of values of (b, t) such that*

$$\text{either } t \rightarrow \bar{t} \in (0, \infty) \text{ and } b \rightarrow \infty, \quad \text{or } t \rightarrow 0 \text{ and } \frac{b}{t^{2D} \log \frac{1}{t}} \rightarrow \infty, \quad (3.7.5)$$

the following asymptotic relation holds:

$$\log \mathbb{E}[e^{bI_t}] \sim \tilde{C} t b^{\frac{1}{2D}} (\log b)^{\frac{2D-1}{2D}}, \quad (3.7.6)$$

where the constant \tilde{C} is given by

$$\tilde{C} = c^{\frac{1}{2D}} (2D)^{\frac{1}{2D}} (1-2D)^{\frac{1-2D}{2D}}. \quad (3.7.7)$$

Proof of Theorem 3.7.1. Let us fix a family of values of (κ, t) satisfying (3.7.1). We want to apply Proposition 3.7.2 with b given by (recall (3.7.2))

$$b = b_{t,\kappa} := \frac{1}{2} \beta \frac{\alpha_{t,\kappa}}{\kappa} \left(\beta \frac{\alpha_{t,\kappa}}{\kappa} - 1 \right) \sim \frac{1}{2} \beta^2 \frac{\alpha_{t,\kappa}^2}{\kappa^2} \sim \frac{\beta^2}{2} \left(\frac{\kappa}{t} \right)^{\frac{2D}{1-D}} \left(\log \frac{\kappa}{t} \right)^{\frac{1-2D}{1-D}}, \quad (3.7.8)$$

where $\beta \in \mathbb{R} \setminus \{0\}$ is a fixed parameter. With this choice, we can write

$$\frac{b}{t^{\frac{1}{2D}} \log \frac{1}{t}} \sim \frac{\beta^2}{2} \left(\frac{\kappa}{t^D \sqrt{\log \frac{1}{t}}} \right)^{\frac{2D}{1-D}} \left(\frac{\log \frac{\kappa}{t}}{\log \frac{1}{t}} \right)^{\frac{1-2D}{1-D}},$$

which diverges to ∞ under assumption (3.7.1) (note that $\log \frac{\kappa}{t} \geq (1-D) \log \frac{1}{t}$, by (3.7.1)). The assumptions of Proposition 3.7.2 are thus verified. By (3.4.5) and (3.7.8), we get

$$\begin{aligned} \log \mathbb{E}[e^{\beta \alpha_{t,\kappa} \frac{X_t}{\kappa}}] &= \log \mathbb{E}[e^{b_{t,k} I_t}] \sim \tilde{C} t b_{t,k}^{\frac{1}{2D}} (\log b_{t,k})^{\frac{2D-1}{2D}} \\ &\sim \tilde{C} t \left(\frac{\beta^2}{2} \right)^{\frac{1}{2D}} \left(\frac{\kappa}{t} \right)^{\frac{1}{1-D}} \left(\log \frac{\kappa}{t} \right)^{\frac{1-2D}{2D(1-D)}} \left(\frac{2D}{1-D} \log \frac{\kappa}{t} \right)^{\frac{2D-1}{2D}} \\ &= c^{\frac{1}{2D}} D \left(\frac{(1-2D)(1-D)}{2} \right)^{\frac{1-2D}{2D}} |\beta|^{\frac{1}{D}} \alpha_{t,\kappa}, \end{aligned}$$

where in the last step we have used the definitions (3.7.2), (3.7.7) of $\alpha_{t,\kappa}$ and \tilde{C} . This shows that the limit (3.7.3) exists with

$$\Lambda(\beta) = \hat{C} |\beta|^{\frac{1}{D}}, \quad \text{and} \quad \hat{C} = c^{\frac{1}{2D}} D \left(\frac{(1-2D)(1-D)}{2} \right)^{\frac{1-2D}{2D}}.$$

To determine the rate function $I(x)$ in (3.7.4) we have to maximize over $\beta \in \mathbb{R}$ the function

$$h(\beta) := \beta x - \Lambda(\beta).$$

Since $h'(\beta) = x - \Lambda'(\beta) = x - \frac{1}{D} \hat{C} \text{sign}(\beta) |\beta|^{\frac{1}{D}-1}$, the only solution to $h'(\bar{\beta}) = 0$ is

$$\bar{\beta} = \bar{\beta}_x = \text{sign}(x) \left(\frac{D|x|}{\hat{C}} \right)^{\frac{D}{1-D}}$$

and consequently

$$I(x) = h(\bar{\beta}_x) = \bar{\beta}_x x - \Lambda(\bar{\beta}_x) = |x|^{\frac{1}{1-D}} \left(\frac{D}{\hat{C}} \right)^{\frac{D}{1-D}} (1-D) = \bar{C} |x|^{\frac{1}{1-D}},$$

where \bar{C} is the constant defined in (3.2.6). Having shown that $I(x)$ coincides with the one given in (3.7.2), the proof of Theorem 3.7.1 is completed. \square

Proof of Proposition 3.7.2. We set

$$B_{t,b} = t b^{\frac{1}{2D}} (\log b)^{\frac{2D-1}{2D}}. \quad (3.7.9)$$

To prove (3.7.6) we start by showing that

$$\limsup \frac{1}{B_{t,b}} \log \mathbb{E}[e^{b I_t}] \leq \tilde{C}. \quad (3.7.10)$$

The upper bound (3.4.1) on I_t yields

$$\mathbb{E}[e^{bI_t}] = \sum_{j=0}^{\infty} \mathbb{E}[e^{bI_t} | N_t = j] \mathbb{P}(N_t = j) \leq e^{\sigma_0^2 tb} \sum_{j=0}^{\infty} e^{ct^{2D} b j^{1-2D}} e^{-\lambda t} \frac{(\lambda t)^j}{j!}.$$

Since $j! \sim j^j e^{-j} \sqrt{2\pi j}$ as $j \uparrow \infty$, there is $c_1 \in (0, \infty)$ such that $j! \geq \frac{1}{c_1} j^j e^{-j}$ for all $j \in \mathbb{N}_0$. Bounding $e^{-\lambda t} \leq 1$, we thus obtain

$$\mathbb{E}[e^{bI_t}] \leq c_1 e^{\sigma_0^2 tb} \sum_{j=0}^{\infty} e^{ct^{2D} b j^{1-2D}} \frac{(\lambda t)^j}{j^j e^{-j}} = c_1 e^{\sigma_0^2 tb} \sum_{j=0}^{\infty} e^{f(j)}, \quad (3.7.11)$$

where for $x \in [0, \infty)$ we set

$$f(x) = f_{t,b}(x) := c (t^{2D} b) x^{1-2D} - x \left(\log \frac{x}{\lambda t} - 1 \right), \quad (3.7.12)$$

with the convention $0 \log 0 = 0$. Note that

$$f'(x) = (1 - 2D)cb \left(\frac{x}{t} \right)^{-2D} - \log \left(\frac{x}{t} \right) + \log \lambda, \quad (3.7.13)$$

hence $f'(x)$ is continuous and strictly decreasing on $(0, \infty)$, with $\lim_{x \downarrow 0} f'(x) = +\infty$ and $\lim_{x \uparrow \infty} f'(x) = -\infty$. As a consequence, there is a unique $\bar{x} = \bar{x}_{t,b} \in (0, \infty)$ with $f'(\bar{x}_{t,b}) = 0$ and the function $f(x)$ attains its global maximum on $[0, \infty)$ at the point $x = \bar{x}_{t,b}$.

Heuristically, the leading contribution to the sum in (3.7.11) is given by a single term $e^{f(j)}$, for $j \approx \bar{x}_{t,b}$. To make this rigorous, we need asymptotic estimates on $\bar{x}_{t,b}$ and $f(\bar{x}_{t,b})$. Since $b \rightarrow \infty$ and assumption (3.7.5) holds, for bounded x , say $0 \leq x \leq M$, one has

$$f'(x) \geq (1 - 2D)cb \left(\frac{M}{t} \right)^{-2D} - \log \left(\frac{M}{t} \right) + \log \lambda \rightarrow \infty.$$

Since $\bar{x}_{t,b}$ is the solution of $f'(x) = 0$, and $f'(\cdot)$ is decreasing, it follows that $\bar{x}_{t,b} > M$ eventually. Since $M \in (0, \infty)$ is arbitrary, we have shown that

$$\bar{x}_{t,b} \rightarrow \infty, \quad (3.7.14)$$

and this implies $\bar{x}_{t,b}/t \rightarrow \infty$, because t is bounded from above by assumption (3.7.5). In particular, by (3.7.13) the equation $f'(\bar{x}_{t,b}) = 0$ yields

$$\frac{\bar{x}_{t,b}}{t} = \left(\frac{(1 - 2D)cb}{\log \frac{\bar{x}_{t,b}}{t} + \log \lambda} \right)^{\frac{1}{2D}} \sim \left(\frac{(1 - 2D)cb}{\log \frac{\bar{x}_{t,b}}{t}} \right)^{\frac{1}{2D}}. \quad (3.7.15)$$

We stress that $b \rightarrow \infty$ under assumption (3.7.5). Rewriting (3.7.15) as

$$\frac{\log \frac{\bar{x}_{t,b}}{t}}{b} \sim (1 - 2D)c \frac{t^{2D}}{\bar{x}_{t,b}^{2D}} \rightarrow 0 \quad (3.7.16)$$

shows that $\log \frac{\bar{x}_{t,b}}{t} = o(b)$. Taking log in (3.7.15), since $b \rightarrow \infty$ by assumption, yields

$$\log \frac{\bar{x}_{t,b}}{t} \sim \frac{1}{2D} \left\{ \log[(1-2D)c] + \log b - \log \left(\log \frac{\bar{x}_{t,b}}{t} \right) \right\} \sim \frac{1}{2D} \log b,$$

which plugged into relation (3.7.15) gives the desired estimate on $\bar{x}_{t,b}$:

$$\bar{x}_{t,b} \sim \left(\frac{2D(1-2D)cb}{\log b} \right)^{\frac{1}{2D}} t. \quad (3.7.17)$$

The estimate on $f(\bar{x}_{t,b})$ then follows by (3.7.12), using (3.7.14) and (3.7.16):

$$\begin{aligned} f(\bar{x}_{t,b}) &\sim c(t^{2D}b) \bar{x}_{t,b}^{1-2D} - \bar{x}_{t,b} \log \frac{\bar{x}_{t,b}}{\lambda t} \\ &\sim c(t^{2D}b) \bar{x}_{t,b}^{1-2D} - \bar{x}_{t,b} \frac{(1-2D)c(t^{2D}b)}{\bar{x}_{t,b}^{2D}} = 2Dc(t^{2D}b) \bar{x}_{t,b}^{1-2D} \\ &= (2D)^{\frac{1}{2D}} (1-2D)^{\frac{1}{2D}-1} c^{\frac{1}{2D}} \frac{t b^{\frac{1}{2D}}}{(\log b)^{\frac{1}{2D}-1}} = \tilde{C} B_{t,b}, \end{aligned} \quad (3.7.18)$$

where we recall that $B_{t,b}$ and \tilde{C} are defined in (3.7.9) and (3.7.7).

We can finally come back to the problem of estimating (3.7.11). Henceforth we set $\bar{x} := \bar{x}_{t,b}$ to lighten notation. We can control $f(x)$ for $x \geq 2\bar{x}$ using Taylor's formula with integral remainder: since $f'(\cdot)$ is strictly decreasing, we get

$$f(x) = f(2\bar{x}) + \int_{2\bar{x}}^x f'(s) ds \leq f(\bar{x}) + f'(2\bar{x})(x - 2\bar{x}),$$

because $f(\bar{x}) = \max_{y \in [0, \infty)} f(y)$. Observe that $f'(2\bar{x}) < 0$, hence

$$\sum_{j \geq 2\bar{x}} e^{f(j)} \leq e^{f(\bar{x})} \sum_{j \geq 2\bar{x}} e^{-|f'(2\bar{x})|(j-2\bar{x})} = \frac{e^{f(\bar{x})}}{1 - e^{-|f'(2\bar{x})|}}. \quad (3.7.19)$$

By (3.7.13), recalling that $f'(\bar{x}) = 0$, we can write

$$f'(2\bar{x}) = f'(2\bar{x}) - 2^{-2D} f'(\bar{x}) = 2^{-2D} \log \left(\frac{\bar{x}}{t} \right) - \log \left(\frac{2\bar{x}}{t} \right) \rightarrow -\infty,$$

because $\bar{x}/t \rightarrow \infty$. In particular, $1 - e^{-|f'(2\bar{x})|} > \frac{1}{2}$ eventually and (3.7.19) yields

$$\sum_{j \geq 2\bar{x}} e^{f(j)} \leq 2e^{f(\bar{x})}. \quad (3.7.20)$$

The initial part of the sum can be simply bounded by

$$\sum_{j < 2\bar{x}} e^{f(j)} \leq (2\bar{x} + 1) e^{f(\bar{x})}. \quad (3.7.21)$$

Looking back at (3.7.11), we can finally write

$$\log \mathbb{E}[e^{bI_t}] \leq \sigma_0^2 b t + f(\bar{x}) + \log(2\bar{x} + 3). \quad (3.7.22)$$

By (3.7.17) and (3.7.18), one has $\bar{x} = O(f(\bar{x})/b) = o(f(\bar{x}))$, since $b \rightarrow \infty$ by assumption, hence $\log(2\bar{x}+3) = o(f(\bar{x}))$. Again by (3.7.18) we have $bt = o(\bar{x}) = o(f(\bar{x}))$, because $D < \frac{1}{2}$. Since $f(\bar{x}) \sim \tilde{C} B_{t,b}$, by relation (3.7.18), we obtain

$$\limsup \frac{1}{B_{t,b}} \log \mathbb{E}[e^{bI_t}] \leq \tilde{C},$$

proving the desired upper bound.

It remains to prove the corresponding lower bound. The strategy is suggested by the proof of the upper bound: Hölder's inequality (3.4.4) becomes an equality when all the terms x_k are equal. We thus introduce the event A_m defined by

$$A_m := \bigcap_{i=1}^m \left\{ \left| \tau_i - (i-1) \frac{t}{m} \right| < \frac{\varepsilon}{m} \right\}, \quad (3.7.23)$$

so that $(1-2\varepsilon)\frac{t}{m} \leq \tau_k - \tau_{k-1} \leq (1+2\varepsilon)\frac{t}{m}$ for all $2 \leq k \leq m$ and $(1-2\varepsilon)\frac{t}{m} \leq t - \tau_m \leq (1+2\varepsilon)\frac{t}{m}$. In particular, recalling the expression (3.1.2) for I_t , on the event A_m we have the lower bound

$$I_t \geq c \sum_{k=1}^m \left((1-2\varepsilon)\frac{t}{m} \right)^{2D} = (1-2\varepsilon)^{2D} c m^{1-2D} t^{2D} =: c_\varepsilon c m^{1-2D} t^{2D}, \quad (3.7.24)$$

Since $(\tau_k - \tau_{k-1})_{k \in \mathbb{N}}$ are i.i.d. $Exp(\lambda)$ random variables, a direct estimate yields

$$\mathbb{P}(A_m) \geq (\lambda e^{-\lambda(1+2\varepsilon)\frac{t}{m}})^m (2\varepsilon \frac{t}{m})^m = e^{-\lambda(1+2\varepsilon)t} (2\varepsilon)^m \frac{(\lambda t)^m}{m^m}. \quad (3.7.25)$$

It follows by (3.7.24) and (3.7.25) that

$$\mathbb{E}[e^{bI_t}] \geq \mathbb{E}[e^{bI_t} \mathbf{1}_{A_m}] \geq e^{(1-2\varepsilon)^{2D} c (t^{2D} b) m^{1-2D}} \mathbb{P}(A_m) \geq e^{\tilde{f}(m)} \quad (3.7.26)$$

where we define $\tilde{f}(x)$, for $x \geq 0$ by

$$\tilde{f}(x) = \tilde{f}_{t,b,\varepsilon}(x) := (1-2\varepsilon)^{2D} c (t^{2D} b) x^{1-2D} - x \log \frac{x}{2\varepsilon \lambda t} - (1+2\varepsilon)\lambda t$$

with the convention $0 \log 0 = 0$.

Since $\tilde{f}(x)$ resembles $f(x)$, defined in (3.7.12), and since the leading contribution to the upper bound was given by $e^{f(\bar{x})}$, where $\bar{x} = \bar{x}_{b,t}$ was the maximizer of $f(\cdot)$, cf. (3.7.17), it is natural to choose $m = \bar{x}$, or more precisely $m = \lfloor \bar{x} \rfloor$, in the lower bound (3.7.26). A computation completely analogous to (3.7.18), recalling (3.7.17) and (3.7.14), gives

$$\tilde{f}(\lfloor \bar{x} \rfloor) \sim \tilde{f}(\bar{x}) \sim \frac{(1-2\varepsilon)^{2D} - 1 + 2D}{2D} f(\bar{x}) \sim \frac{(1-2\varepsilon)^{2D} - 1 + 2D}{2D} \tilde{C} B_{t,b},$$

which coupled to (3.7.26) yields

$$\liminf \frac{1}{B_{t,b}} \log \mathbb{E}[e^{bI_t}] \geq \frac{(1-2\varepsilon)^{2D} - 1 + 2D}{2D} \tilde{C}.$$

Letting $\varepsilon \rightarrow 0$ we complete the proof. \square

3.7.2 Proof of relation (3.3.4)

We focus on family of values of (κ, t) with $t \rightarrow 0$ and $\kappa \geq 0$. Recall that we have already proved that relation (3.3.3) holds for $\kappa \gg \kappa_2(t)$. Consequently, in order to prove (3.3.4) it suffices to prove relations (3.3.5) and (3.3.6). We start with the former, assuming that

$$\sqrt{t} \ll \kappa \ll \kappa_2(t). \quad (3.7.27)$$

Since $N_t \sim \text{Pois}(\lambda t)$, for every $M \in \mathbb{N}_0$

$$\mathbb{P}(N_t \geq M + 1) = \sum_{k=M+1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \leq (\lambda t)^{M+1},$$

hence as $t \rightarrow 0$ we can write

$$\mathbb{P}(X_t > \kappa) = \sum_{m=0}^M \mathbb{P}(X_t > \kappa | N_t = m) e^{-\lambda t} \frac{(\lambda t)^m}{m!} + O(t^{M+1}). \quad (3.7.28)$$

Recall the definition (3.1.2) of the time-change process I_t . On the event $\{N_t = 0\}$ we have

$$I_t = (t - \tau_0)^{2D} - (-\tau_0)^{2D} \underset{t \downarrow 0}{\sim} \sigma_0^2 t,$$

where σ_0^2 is defined in (3.1.5). Consequently, by the definition (3.1.7) of X_t ,

$$\begin{aligned} \mathbb{P}(X_t > \kappa | N_t = 0) &= \mathbb{P}\left(W_1 > \frac{\kappa}{\sqrt{I_t}} + \frac{1}{2}\sqrt{I_t} \mid N_t = 0\right) \\ &= 1 - \Phi\left(\frac{\kappa}{\sigma_0 \sqrt{t}}(1 + o(1))\right) = \exp\left(-\frac{\kappa^2}{2\sigma_0^2 t}(1 + o(1))\right) \\ &= \exp\left(-\frac{1}{2\sigma_0^2} \left(\frac{\kappa}{\kappa_1(t)}\right)^2 \log \frac{1}{t}(1 + o(1))\right), \end{aligned} \quad (3.7.29)$$

where $\Phi(z) = \mathbb{P}(W_1 \leq z)$, we have used the standard estimate $\log(1 - \Phi(z)) \sim -\frac{1}{2}z^2$ as $z \rightarrow \infty$ and the definition (3.2.4) of $\kappa_1(t)$. If we define, as in (3.3.5),

$$a := \frac{\kappa}{\kappa_1(t)} = \frac{\kappa}{\sqrt{t} \sqrt{\log \frac{1}{t}}}, \quad (3.7.30)$$

we can rewrite (3.7.29) as

$$\mathbb{P}(X_t > \kappa | N_t = 0) = e^{-\frac{a^2}{2\sigma_0^2} \log \frac{1}{t}(1+o(1))} = t^{\frac{a^2}{2\sigma_0^2} + o(1)}. \quad (3.7.31)$$

Since $\mathbb{P}(N_t = 0) = 1 - e^{-\lambda t} \rightarrow 1$ as $t \rightarrow 0$, relation (3.7.28) for $M = 0$ gives

$$\mathbb{P}(X_t > \kappa) = t^{\frac{a^2}{2\sigma_0^2} + o(1)} + O(t).$$

In case $a < \sqrt{2\sigma_0}$, the $O(t)$ term can be neglected and we have proved (3.3.5).

Henceforth we assume that $a \geq \sqrt{2\sigma_0}$, cf. (3.7.30), so that by (3.7.31)

$$\mathbb{P}(X_t > \kappa | N_t = 0) \leq t^{1+o(1)}. \quad (3.7.32)$$

Let us look at the other terms in (3.7.28): by (3.4.1), on the event $\{N_t = m\}$ with $m \geq 1$

$$I_t \leq \sigma_0^2 t + cN_t^{1-2D}t^{2D} = \sigma_0^2 t + cm^{1-2D}t^{2D} \underset{t \downarrow 0}{\sim} cm^{1-2D}t^{2D},$$

hence, in analogy with (3.7.29), we get the upper bound

$$\begin{aligned} \mathbb{P}(X_t > \kappa | N_t = m) &\leq 1 - \Phi\left(\frac{\kappa}{\sqrt{ct^D m^{\frac{1}{2}-D}}}(1 + o(1))\right) \\ &= \exp\left(-\frac{1}{2cm^{1-2D}}\left(\frac{\kappa}{\kappa_2(t)}\right)^2 \log \frac{1}{t}(1 + o(1))\right), \end{aligned} \quad (3.7.33)$$

by the definition (3.2.4) of $\kappa_2(t)$. Since $\kappa/\kappa_2(t) \rightarrow 0$ under assumption (3.7.27), relation (3.7.33) for $m = 1$ yields

$$\mathbb{P}(X_t > \kappa | N_t = 1) \leq e^{-o(1) \log \frac{1}{t}} = t^{o(1)}.$$

Since $\mathbb{P}(N_t = 1) = e^{-\lambda t} \lambda t \sim \lambda t$ as $t \downarrow 0$, recalling (3.7.32), relation (3.7.28) for $M = 0$ gives

$$\mathbb{P}(X_t > \kappa) \leq t^{1+o(1)} + \lambda t t^{o(1)} + O(t^2) = t^{1+o(1)},$$

where the $o(1)$ term changes from side to side. We have proved ‘‘half’’ of relation (3.3.5) for $a \geq \sqrt{2\sigma_0}$, namely

$$\limsup \frac{\log \mathbb{P}(X_t > \kappa)}{\log \frac{1}{t}} \leq -1. \quad (3.7.34)$$

To get an analogous lower bound, we argue as we did in the proof of Proposition 3.7.2. For any fixed $\varepsilon > 0$, on the event $A_m \subseteq \{N_t = m\}$ defined in (3.7.23), with $m \geq 1$, one has the lower bound (3.7.24) on I_t and (3.7.25) on $\mathbb{P}(A_m)$, hence

$$\begin{aligned} \mathbb{P}(X_t > \kappa | N_t = m) &\geq \mathbb{P}(X_t > \kappa | A_m) \frac{\mathbb{P}(A_m)}{\mathbb{P}(N_t = m)} \\ &\geq \left(1 - \Phi\left(\frac{\kappa}{\sqrt{c_\varepsilon ct^D m^{\frac{1}{2}-D}}}(1 + o(1))\right)\right) e^{-\lambda(2\varepsilon)t} (2\varepsilon)^m \frac{m!}{m^m} \\ &= \exp\left(-\frac{1}{2c_\varepsilon cm^{1-2D}}\left(\frac{\kappa}{\kappa_2(t)}\right)^2 \log \frac{1}{t}(1 + o(1))\right) e^{-\lambda(2\varepsilon)t} (2\varepsilon)^m \frac{m!}{m^m}, \end{aligned} \quad (3.7.35)$$

with $c_\varepsilon := (1 - 2\varepsilon)^{2D}$. In the special case $m = 1$ this relation yields

$$\mathbb{P}(X_t > \kappa | N_t = 1) \geq e^{-o(1) \log \frac{1}{t}} e^{-\lambda(2\varepsilon)t} (2\varepsilon) \sim t^{o(1)} (2\varepsilon),$$

hence, recalling that $\mathbb{P}(N_t = 1) \sim \lambda t$,

$$\mathbb{P}(X_t > \kappa) \geq \mathbb{P}(X_t > \kappa | N_t = 1) \mathbb{P}(N_t = 1) \geq t^{1+o(1)},$$

which yields

$$\liminf \frac{\log P(X_t > \kappa)}{\log \frac{1}{t}} \geq -1.$$

Together with (3.7.34), this completes the proof of relation (3.3.5) for $a \geq \sqrt{2\sigma_0}$.

It remains to prove relation (3.3.6), hence we assume that

$$\kappa \sim b \kappa_2(t), \quad \text{for some } b \in (0, \infty). \quad (3.7.36)$$

By (3.7.29) we have

$$P(X_t > \kappa | N_t = 0) \leq \exp\left(-\frac{b^2}{2\sigma_0^2} \frac{1}{t^{1-2D}} \log \frac{1}{t} (1 + o(1))\right) = o(t^{M+1}),$$

for any fixed $M \in \mathbb{N}$. As a consequence, relation (3.7.28) together with the upper bounds (3.7.32) and (3.7.33) yields, for every fixed $M \in \mathbb{N}$,

$$\begin{aligned} P(X_t > \kappa) &\leq \sum_{m=1}^M \exp\left(-\frac{b^2}{2cm^{1-2D}} \log \frac{1}{t} (1 + o(1))\right) (\lambda t)^m + O(t^{M+1}) \\ &\leq M \max_{m \in \{1, \dots, M\}} t^{\frac{b^2}{2cm^{1-2D}} + m + o(1)} + O(t^{M+1}) \\ &\leq M t^{f(b) + o(1)} + O(t^{M+1}), \end{aligned}$$

where $f(\cdot)$ is defined in (3.2.2). If we fix M large enough, so that $M + 1 > f(b)$, the term $O(t^{M+1})$ can be neglected and we obtain

$$\limsup \frac{\log P(X_t > \kappa)}{\log \frac{1}{t}} \leq -f(b), \quad (3.7.37)$$

which is ‘‘half’’ of relation (3.3.6).

To prove the corresponding lower bound, let $\bar{m} = \bar{m}_b \in \mathbb{N}$ be the value of $m \in \{1, \dots, M\}$ for which the minimum in the definition (3.2.2) of $f(b)$ is attained, i.e.

$$f(b) = \frac{b^2}{2c\bar{m}^{1-2D}} + \bar{m}. \quad (3.7.38)$$

Recalling (3.7.36), the lower bound (3.7.35) for $m = \bar{m}$ gives

$$\begin{aligned} P(X_t > \kappa | N_t = \bar{m}) &\geq \exp\left(-\frac{b^2}{2c_\varepsilon c \bar{m}^{1-2D}} \log \frac{1}{t} (1 + o(1))\right) e^{-\lambda(2\varepsilon)t} (2\varepsilon)^{\bar{m}} \frac{\bar{m}!}{\bar{m}^{\bar{m}}} \\ &\sim t^{\frac{b^2}{2c_\varepsilon c \bar{m}^{1-2D}} + o(1)} (\text{const.}) \end{aligned}$$

where (const.) is a constant depending on ε and \bar{m} . Since $P(N_t = \bar{m}) \geq (\text{const.}') t^{\bar{m}}$, we get

$$P(X_t > \kappa) \geq P(X_t > \kappa | N_t = \bar{m}) P(N_t = \bar{m}) = t^{\frac{b^2}{2c_\varepsilon c \bar{m}^{1-2D}} + \bar{m} + o(1)},$$

hence

$$\liminf \frac{\log P(X_t > \kappa)}{\log \frac{1}{t}} \geq -\left(\frac{b^2}{2c_\varepsilon c \bar{m}^{1-2D}} + \bar{m}\right).$$

Since $c_\varepsilon := (1 - 2\varepsilon)^{2D}$ and $\varepsilon > 0$ is arbitrary, we can let $\varepsilon \rightarrow 0$ in this relation, so that the right hand side becomes $-f(b)$, by (3.7.38). Recalling (3.7.37), we have completed the proof of relation (3.3.6). \square

3.8 Proof of Theorem 3.3.3

We split the proof of various steps.

3.8.1 Proof of (3.3.7)

Let us fix a family of values of (κ, t) with $\kappa \geq 0$ and $t > 0$, such that $t \rightarrow \bar{t} \in (0, \infty)$ and $\kappa \rightarrow \infty$. Our goal is to prove relation (3.3.7). Let us check the assumptions of Theorem 2.1.5 in Chapter 2. Relation (3.3.3) shows that for all $\varrho \geq 1$

$$I_+(\varrho) := \lim \frac{\log \mathbb{P}(X_t > \varrho \kappa)}{\log \mathbb{P}(X_t > \kappa)} = \varrho^{\frac{1}{1-D}}, \quad (3.8.1)$$

hence Hypothesis 2.1.4 is satisfied, together with the requirement $I_+(\varrho) \geq \varrho$, cf. (2.1.21), since $\frac{1}{1-D} > 1$. Relation (3.4.5) shows that the moment condition (2.1.18) holds for all $\eta, T \in (0, \infty)$, hence (2.1.17) is satisfied. We can thus apply Theorem 2.1.5: observing that $-\log \mathbb{P}(X_t > \kappa)/\kappa \rightarrow \infty$, again by (3.3.3) and $\frac{1}{1-D} > 1$, relation (2.1.25) gives

$$\log c(\kappa, t) \sim \log \mathbb{P}(X_t > \kappa), \quad (3.8.2)$$

which coincides precisely with (3.3.7).

3.8.2 Proof of (3.3.8)

Next we fix a family of values of (κ, t) with $t \rightarrow 0$ and either $\sqrt{t} \ll \kappa \ll \kappa_1(t)$ or $\kappa \gg \kappa_2(t)$, where we recall that $\kappa_1(t)$ and $\kappa_2(t)$ are defined in (3.2.4). Our goal is to prove relation (3.3.8). Again we check the assumptions of Theorem 2.1.5.

- In case $\kappa \gg \kappa_2(t)$, relation (3.3.3) still holds, by the last point in Theorem 3.3.2, hence (3.8.1) applies again.
- In case $\sqrt{t} \ll \kappa \ll \kappa_1(t)$, relation (3.3.5) shows that

$$\log \mathbb{P}(X_t > \kappa) \sim -\frac{1}{2\sigma_0^2} \left(\frac{\kappa}{\kappa_1(t)} \right)^2 \log \frac{1}{t},$$

hence for all $\varrho \geq 1$

$$I_+(\varrho) := \lim \frac{\log \mathbb{P}(X_t > \varrho \kappa)}{\log \mathbb{P}(X_t > \kappa)} = \varrho^2. \quad (3.8.3)$$

In both cases, Hypothesis 2.1.4 and relation (2.1.21) are satisfied.

In case $\sqrt{t} \ll \kappa \ll \kappa_1(t)$ one has, of course, $\kappa \rightarrow 0$, while in case $\kappa \gg \kappa_2(t)$, by extracting a subsequence, we may assume that $\kappa \rightarrow \bar{\kappa} \in [0, \infty]$. Let us consider first the subcase $\bar{\kappa} \in (0, \infty]$. Having already checked the moment condition (2.1.17), we can again apply Theorem 2.1.5: relation (2.1.25) gives (3.8.2), which can be written equivalently as

$$\log (c(\kappa, t)/\kappa) = \log c(\kappa, t) - \log \kappa \sim \log \mathbb{P}(X_t > \kappa),$$

because $|\log \mathbb{P}(X_t > \kappa)| \gg |\log \kappa|$, by (3.3.3). This proves (3.3.8) if $\bar{\kappa} > 0$.

Next we consider the regime $\kappa \rightarrow 0$, for both cases $\sqrt{t} \ll \kappa \ll \kappa_1(t)$ and $\kappa \gg \kappa_2(t)$. In this regime, Theorem 2.1.5 requires to check the moment condition (2.1.19). In the special case $\eta = 1$, this condition reduces to (2.1.20), namely we have to show that

$$\mathbb{E}[e^{2X_t}] \leq 1 + C\kappa^2,$$

for some $C < \infty$. This, however, follows immediately by Corollary 3.4.3, because in both cases under consideration $\kappa \gg \sqrt{t}$. We can thus apply Theorem 2.1.5, and specifically relation (2.1.27), which coincides precisely with our goal (3.3.8).

3.8.3 Proof of (3.3.9) and (3.3.10)

In this last case we can no longer apply Theorem 2.1.5, because proving relations (3.3.9) and (3.3.10) means that equation (3.3.8) fails. We proceed by bare hands estimates.

Let us first consider a family of values of (κ, t) with $\kappa \geq 0$ and $t > 0$, such that

$$t \rightarrow 0 \quad \text{and} \quad \kappa \sim a \kappa_1(t), \quad \text{for some } a \in (0, \sqrt{2D+1}\sigma_0). \quad (3.8.4)$$

(The case $a \geq \sqrt{2D+1}\sigma_0$ will be treated later.) Our goal is to prove (3.3.9), which for $a < \sqrt{2D+1}\sigma_0$ can be rewritten as

$$\log(c(\kappa, t)/\kappa) \sim -\frac{a^2}{2\sigma_0^2} \log \frac{1}{t}. \quad (3.8.5)$$

We prove separately upper and lower bounds for this relation.

Let us set

$$k' := \sqrt{2}\sigma_0 \kappa_1(t), \quad k'' := B \kappa_2(t), \quad (3.8.6)$$

for fixed $B \in (0, \infty)$, chosen later. Noting that $\kappa < k' < k''$, since $D < \frac{1}{2}$, we can write

$$\begin{aligned} c(\kappa, t) &= \mathbb{E}[(e^{X_t} - e^\kappa)1_{\{X_t > \kappa\}}] \\ &= \mathbb{E}[(e^{X_t} - e^\kappa)1_{\{\kappa < X_t \leq k'\}}] + \mathbb{E}[(e^{X_t} - e^\kappa)1_{\{k' < X_t \leq k''\}}] + \mathbb{E}[(e^{X_t} - e^\kappa)1_{\{X_t > k''\}}] \\ &= (1) + (2) + (3). \end{aligned} \quad (3.8.7)$$

By Fubini's theorem, for $\kappa \geq 0$ and $0 \leq a < b$,

$$\begin{aligned} \mathbb{E}[(e^{X_t} - e^\kappa)1_{\{a < X_t \leq b\}}] &= \mathbb{E}\left[\int_\kappa^\infty e^x 1_{\{x < X_t\}} dx 1_{\{a < X_t \leq b\}}\right] \\ &= \int_\kappa^b e^x \mathbb{P}(\max\{a, x\} < X \leq b) dx \\ &\leq (e^b - 1) \mathbb{P}(X_t > \max\{a, \kappa\}), \end{aligned} \quad (3.8.8)$$

hence

$$(1) = \mathbb{E}[(e^{X_t} - e^\kappa)1_{\{\kappa < X_t \leq k'\}}] \leq (e^{k'} - 1) \mathbb{P}(X_t > \kappa) \sim k' \mathbb{P}(X_t > \kappa), \quad (3.8.9)$$

because $k' \rightarrow 0$. Note that, by (3.3.5),

$$\log \mathbb{P}(X_t > \kappa) \sim -\frac{a^2}{2\sigma_0^2} \log \frac{1}{t}, \quad (3.8.10)$$

hence recalling (3.8.4) and (3.8.6) it follows that

$$\log \frac{(1)}{\kappa} \leq -\frac{a^2}{2\sigma_0^2} \log \frac{1}{t} (1 + o(1)).$$

In a similar way, always using (3.8.8), since $\kappa < \kappa'$ and $\kappa'' \rightarrow 0$,

$$(2) = \mathbb{E} [(e^{X_t} - e^\kappa) 1_{\{\kappa' < X_t \leq \kappa''\}}] \leq (e^{\kappa''} - 1) \mathbb{P}(X_t > \kappa') \sim \kappa'' \mathbb{P}(X_t > \kappa'). \quad (3.8.11)$$

Again by (3.8.10) with $a = \sqrt{2}\sigma_0$, noting that $\frac{\kappa}{\kappa''} \sim \frac{a}{B} \left(\frac{1}{t}\right)^{D-\frac{1}{2}}$, we can write

$$\log \frac{(2)}{\kappa} \leq -(1 + o(1)) \log \frac{1}{t} - \log \frac{\kappa}{\kappa''} \leq -\left(D + \frac{1}{2} + o(1)\right) \log \frac{1}{t}.$$

Finally, by Cauchy-Schwarz inequality

$$(3) = \mathbb{E} [(e^{X_t} - e^\kappa) 1_{\{X_t > \kappa''\}}] \leq \kappa \sqrt{\mathbb{E} \left[\left(\frac{e^{X_t} - e^\kappa}{\kappa} \right)^2 \right] \mathbb{P}(X_t > \kappa'')}. \quad (3.8.12)$$

By Corollary 3.4.3 and $\mathbb{E}[e^{X_t}] = 1$ (recall that $(e^{X_t})_{t \geq 0}$ is a martingale) we have

$$\mathbb{E} \left[\left(\frac{e^{X_t} - e^\kappa}{\kappa} \right)^2 \right] = \frac{\mathbb{E}[e^{2X_t}] - 2e^\kappa + e^{2\kappa}}{\kappa^2} \leq \frac{1 + Ct - 2 + e^{2\kappa}}{\kappa^2} = \frac{Ct}{\kappa^2} + \frac{e^{2\kappa} - 1}{\kappa^2} \rightarrow 0,$$

because $\kappa \rightarrow 0$ and $\kappa/\sqrt{t} \rightarrow \infty$, by (3.8.4) and the definition of $\kappa_1(t)$. In particular, for some constant $C' < \infty$ we have

$$(3) \leq \kappa \sqrt{C' \mathbb{P}(X_t > \kappa'')}.$$

Recalling (3.3.6), it follows that

$$\log \frac{(3)}{\kappa} \leq -(1 + o(1)) \frac{1}{2} f(B) \log \frac{1}{t}. \quad (3.8.13)$$

Since $\log(a + b + c) \leq \log 3 + \max\{\log a, \log b, \log c\}$, we obtain by (3.8.7)

$$\log \frac{c(\kappa, t)}{\kappa} \leq -(1 + o(1)) \min \left\{ \frac{a^2}{2\sigma_0^2}, D + \frac{1}{2}, \frac{f(B)}{2} \right\} \log \frac{1}{t}. \quad (3.8.14)$$

We now choose $B > 0$ large enough, so that $\frac{f(B)}{2} > D + \frac{1}{2}$. Since $a < \sqrt{2D+1}\sigma_0$ by assumption, cf. (3.8.4), we have shown that

$$\log \frac{c(\kappa, t)}{\kappa} \leq -(1 + o(1)) \frac{a^2}{2\sigma_0^2} \log \frac{1}{t}, \quad (3.8.15)$$

which is “half” of our goal (3.8.5).

In order to obtain the corresponding lower bound, we observe that for every $\hat{\kappa} > \kappa$

$$\begin{aligned} c(\kappa, t) &= \mathbb{E} [(e^{X_t} - e^\kappa) 1_{\{X_t > \kappa\}}] \geq \mathbb{E} [(e^{X_t} - e^\kappa) 1_{\{X_t > \hat{\kappa}\}}] \geq (e^{\hat{\kappa}} - e^\kappa) \mathbb{P}(X_t > \hat{\kappa}) \\ &\geq (\hat{\kappa} - \kappa) \mathbb{P}(X_t > \hat{\kappa}). \end{aligned} \quad (3.8.16)$$

Always for κ as in (3.8.4), choosing $\hat{\kappa} = (1 + \varepsilon)\kappa$ gives, recalling (3.8.10),

$$\log \frac{c(\kappa, t)}{\kappa} \geq \log \varepsilon + \log \mathbb{P}(X_t > (1 + \varepsilon)\kappa) = -(1 + \varepsilon)^2 \frac{a^2}{2\sigma_0^2} \log \frac{1}{t} (1 + o(1)). \quad (3.8.17)$$

This shows that, along the given family of values of (κ, t) ,

$$\liminf \frac{\frac{c(\kappa, t)}{\kappa}}{\log \frac{1}{t}} \geq -(1 + \varepsilon)^2 \frac{a^2}{2\sigma_0^2}.$$

Since $\varepsilon > 0$ is arbitrary, we have shown that

$$\log \frac{c(\kappa, t)}{\kappa} \geq -(1 + o(1)) \frac{a^2}{2\sigma_0^2} \log \frac{1}{t}. \quad (3.8.18)$$

Together with (3.8.15), this completes the proof of our goal (3.8.5), i.e. of relation (3.3.9) under the assumption (3.8.4).

Finally, we consider a family of values of (κ, t) with $\kappa \geq 0$ and $t > 0$, such that

$$t \rightarrow 0 \quad \text{and} \quad \sqrt{2D+1} \sigma_0 \kappa_1(t) \leq \kappa \leq M \kappa_2(t), \quad \text{for some } M \in (0, \infty). \quad (3.8.19)$$

This includes, in particular, the case when $\kappa \sim a \kappa_1(t)$ with $a \geq \sqrt{2D+1} \sigma_0$, that was left out from (3.8.4). Our goal is to prove (3.3.10), that is

$$\log (c(\kappa, t)/\kappa) \sim -\log \frac{1}{t} - \log \frac{\kappa}{t^D}. \quad (3.8.20)$$

Note that this relation also includes (3.3.9) for $a \geq \sqrt{2D+1} \sigma_0$.

Consider first the subcase of (3.8.19) given by $\kappa \leq \sqrt{2} \sigma_0 \kappa_1(t)$, so assume (without loss of generality, by extracting a subsequence) that $\kappa \sim a \kappa_1(t)$ with $a \in [\sqrt{2D+1} \sigma_0, \sqrt{2} \sigma_0]$. Note that all the steps from (3.8.6) until (3.8.14) can be applied verbatim. However, since $a \geq \sqrt{2D+1} \sigma_0$, one has $\frac{a^2}{2\sigma_0^2} \geq D + \frac{1}{2}$, and instead of relation (3.8.15) we get

$$\log \frac{c(\kappa, t)}{\kappa} \leq -(1 + o(1)) \left(D + \frac{1}{2} \right) \log \frac{1}{t}. \quad (3.8.21)$$

Note that the right hand side of (3.8.21) coincides with the right hand side of our goal (3.8.20) for $\kappa \sim a \kappa_1(t)$, since in this case $\log \frac{\kappa}{t^D} \sim (D - \frac{1}{2}) \log \frac{1}{t}$.

Next we consider the subcase of (3.8.19) when $\kappa > \sqrt{2} \sigma_0 \kappa_1(t)$. Defining $\kappa'' := B \kappa_2(t)$ as in (3.8.6), we modify (3.8.7) as follows:

$$c(\kappa, t) = \mathbb{E} [(e^{X_t} - e^\kappa) 1_{\{\kappa < X_t \leq \kappa''\}}] + \mathbb{E} [(e^{X_t} - e^\kappa) 1_{\{X_t > \kappa''\}}] =: (A) + (B). \quad (3.8.22)$$

Applying (3.8.8), we estimate the first term as follows, since $\kappa'' \rightarrow 0$:

$$(A) = \mathbb{E} [(e^{X_t} - e^\kappa) 1_{\{\kappa < X_t \leq \kappa''\}}] \leq (e^{\kappa''} - 1) \mathbb{P}(X_t > \kappa) \sim \kappa'' \mathbb{P}(X_t > \kappa).$$

Observe that $\log \mathbb{P}(X_t > \kappa) \sim -(1 + o(1)) \log \frac{1}{t}$, by (3.3.5) with $\kappa > \sqrt{2} \sigma_0 \kappa_1(t)$, and moreover $\log(\kappa''/\kappa) \sim \log(t^D/\kappa)$ by definition of $\kappa_2(t)$, hence

$$\log \frac{(A)}{\kappa} \leq \log \frac{\kappa''}{\kappa} + \log \mathbb{P}(X_t > \kappa) \leq -(1 + o(1)) \left(\log \frac{1}{t} + \log \frac{\kappa}{t^D} \right).$$

The term (B) in (3.8.22) coincides with term (3) in (3.8.12), hence by (3.8.13)

$$\begin{aligned} \log \frac{(B)}{\kappa} &\leq -(1+o(1)) \frac{f(B)}{2} \log \frac{1}{t} \leq -(1+o(1)) \frac{f(B)}{2} \left(\log \frac{1}{t} + \log \frac{\kappa}{M\kappa_2(t)} \right) \\ &\sim -(1+o(1)) \frac{f(B)}{2} \left(\log \frac{1}{t} + \log \frac{\kappa}{t^D} \right), \end{aligned}$$

where the second inequality holds just because $\kappa \leq M\kappa_2(t)$ by (3.8.19). If we choose B large enough, so that $f(B) > 2$, the usual estimate $\log(a+b) \leq \log 2 + \log \max\{a, b\}$ yields

$$\log \frac{c(\kappa, t)}{\kappa} \leq -(1+o(1)) \left(\log \frac{1}{t} + \log \frac{\kappa}{t^D} \right). \quad (3.8.23)$$

We have thus proved “half” of our goal (3.8.20).

We finally turn to the lower bound, for which we do not need to distinguish subcases, but we work in the general regime (3.8.19). We are going to apply (3.8.16) with $\hat{\kappa} = \varepsilon\kappa_2(t)$. Recalling that $\log P(X_t > \varepsilon\kappa_2(t)) \sim -f(\varepsilon) \log \frac{1}{t}$ by (3.3.6), and moreover

$$\log \frac{\hat{\kappa} - \kappa}{\kappa} \sim \log \left(\frac{\varepsilon\kappa_2(t)}{\kappa} - 1 \right) \sim \log \frac{t^D}{\kappa},$$

relation (3.8.16) gives

$$\log \frac{c(\kappa, t)}{\kappa} \geq -(1+o(1)) \left(f(\varepsilon) \log \frac{1}{t} + \log \frac{\kappa}{t^D} \right). \quad (3.8.24)$$

Since $\varepsilon > 0$ is arbitrary and $\lim_{\varepsilon \downarrow 0} f(\varepsilon) = f(0) = 1$, cf. (3.2.2), we have shown that

$$\log \frac{c(\kappa, t)}{\kappa} \geq -(1+o(1)) \left(\log \frac{1}{t} + \log \frac{\kappa}{t^D} \right).$$

Together with (3.8.21) and (3.8.23), this completes the proof of our goal (3.8.20). \square

3.9 Numerical results

In this section present some graphical results on the asymptotics of implied volatility.

We have simulated the price of european call using the Monte Carlo algorithm presented in chapter 5 and then compared the implied volatility obtained with the theoretical asymptotics.

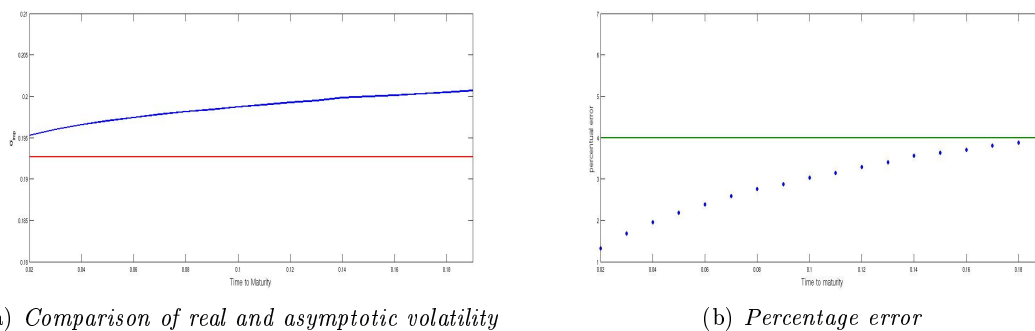


Figure 3.3: At the money regime when $t \rightarrow 0$. Comparison between the implied volatility obtained via simulations (blue) and the asymptotic value σ_0 (red) on the left and percentage error on the right. The error is lower than 4% (green line) already when $t = 0.19$, and it diminishes when t becomes closer to 0 (for $t = 0.02$ it is 1.32%). The parameters used are $D = 0.2, V = 0.2, \lambda = 0.2, \tau_0 = 1.5$.

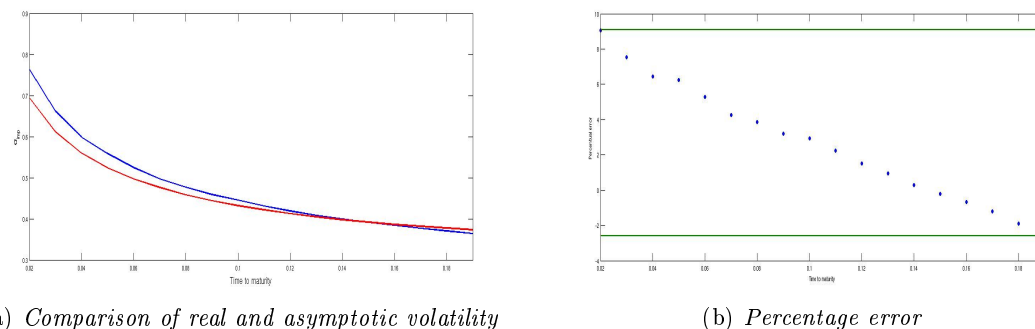


Figure 3.4: Out of the money regime with log-strike fixed $\kappa = 0.5$ when $t \rightarrow 0$. Comparison between the implied volatility obtained via simulations (blue) and the asymptotic value σ (red) on the left and percentage error on the right. The growth in the error as $t \rightarrow 0$ is probably due to the inaccuracy of the Monte Carlo method, anyway it stays under the 10% (upper green line) and above -2% (bottom green line), in absolute term it is around 0.07 when the expected implied volatility is $\sigma_{imp} = 0.7653$. The parameters used are $D = 0.2, V = 0.2, \lambda = 0.2, \tau_0 = 1.5$.

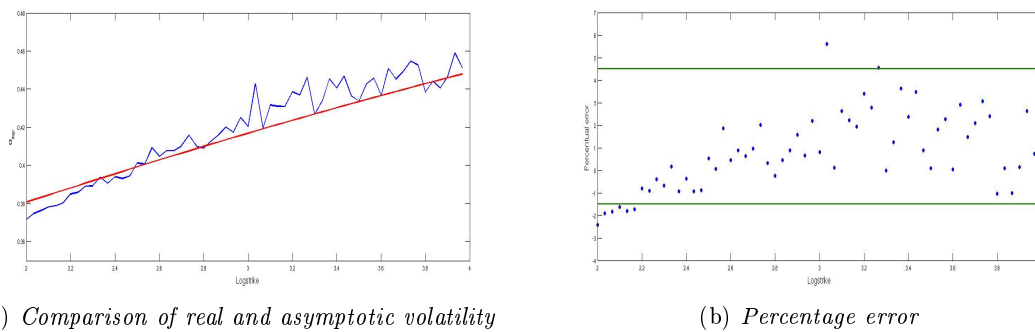


Figure 3.5: Out of the money regime with log-strike fixed $\kappa = 0.5$ when $t \rightarrow 0$. Comparison between the implied volatility obtained via simulations (blue) and the asymptotic value σ (red) on the left and percentage error on the right. The absolute percentage error varies but is most of the time between -1.5% and 4.5% (green lines). The variation is probably due to the Monte Carlo simulations, as it is possible to observe from the fact that the implied volatility from the simulations is not smooth. The parameters used are $D = 0.2, V = 0.2, \lambda = 0.2, \tau_0 = 1.5$.

Chapter 4

Enriching the model and pricing

In this chapter we investigate a possible way of enriching the model introduced in [ACDP12] in order to take into account the so-called *leverage effect*, and we prove some of its properties. We moreover introduce a family of equivalent martingale measures for this enriched model (and, by extension, also for the original model [ACDP12]) under which the price of a European call option can be expressed through a *generalized Hull & White formula*.

4.1 An enriched version of the model

We recall that, given a standard Brownian motion $(W_t)_{t \geq 0}$ and an independent Poisson process $(N_t)_{t \geq 0}$ of rate λ , the original model $(Y_t)_{t \geq 0}$ for the (de-trended) log-price of an asset is

$$Y_t := W_{I_t}, \quad (4.1.1)$$

where the time-change process $(I_t)_{t \geq 0}$ is defined as follows: denoting by $0 < \tau_1 < \tau_2 < \dots$ the jump times of the Poisson process $(N_t)_{t \geq 0}$, and fixing a further parameter $\tau_0 \in (-\infty, 0)$, we set

$$I_t := \mathfrak{c} \left\{ (t - \tau_{N_t})^{2D} - (-\tau_0)^{2D} + \sum_{k=1}^{N_t} (\tau_k - \tau_{k-1})^{2D} \right\}, \quad (4.1.2)$$

where

$$\mathfrak{c} = \frac{V^2 \lambda^{2D-1}}{\Gamma(2D+1)}$$

with the convention that the sum is zero when $N_t = 0$.

Being a function of $(N_t)_{t \geq 0}$ and of the parameters D, λ , the time-change process $(I_t)_{t \geq 0}$ is independent of the Brownian motion $(W_t)_{t \geq 0}$. The trajectories $t \mapsto I_t$ are continuous, and also differentiable at every $t \geq 0$ which is not a jump time, i.e. for $t \neq \tau_{N_t}$, with

$$I'_t = 2D \mathfrak{c} (t - \tau_{N_t})^{2D-1}. \quad (4.1.3)$$

It is standard to show, cf. [ACDP12], that $(Y_t)_{t \geq 0}$ solves the stochastic differential equation

$$dY_t = \sigma_t dB_t, \quad \text{with} \quad \sigma_t := \sqrt{I'_t}, \quad (4.1.4)$$

where $(B_t)_{t \geq 0}$ is a suitable Brownian motion, independent of $(\sigma_t)_{t \geq 0}$.

In order to take into account the so-called *leverage effect* (i.e. the asymmetry of the smile volatility), we enrich the model introducing a jump component in the log-price, using the same Poisson process $(N_t)_{t \geq 0}$ that drives the time change process $(I_t)_{t \geq 0}$, cf. (4.1.2). The intuitive meaning is that shocks in the market, represented by jumps in the Poisson process, determine both an increase in the volatility and a jump in the price.

We thus introduce a further parameter $\varrho \in \mathbb{R}$, which represent the jump size in the log-price at shock-times, and define the new (de-trended) log-price $(Y_t)_{t \geq 0}$ by

$$Y_t := W_{I_t} + \varrho(N(t) - \lambda t), \quad (4.1.5)$$

which reduces to the original model (4.1.1) for $\varrho = 0$.

Remark 4.1.1. *In Chapter 5 of Gatheral [Gat06] it has been pointed out that jumps are necessary in a model in order to take account of the steepness of the skew for very short-dated term structure. We would like to point out that this is not the case for our model: choosing the right parameter we can obtain any possible skew. We have decided to introduce (negative-) correlated jumps in the price in order to generate asymmetry in the distribution (and consequently in the implied volatility).*

We now generalize some of the results proved in [ACDP12] for $\varrho = 0$ to the case $\varrho \neq 0$ (and also to the case when τ_0 is a fixed parameter, that we consider here).

We start with the convergence in distribution of the increments $Y_{t+h} - Y_t$, both when $h \downarrow 0$ and $h \uparrow +\infty$, generalizing Theorem B.2.1 in appendix B. As it is expected, the jumps in the log price do not influence the limiting distribution for small times, since they are rare events, while they influence the limiting distribution for large times.

Theorem 4.1.2 (Diffusive scaling). *The following convergences in distribution hold for any choice of the parameters D, λ, V, ϱ and for every $\tau_0 \in (-\infty, 0)$.*

- *Small-time diffusive scaling:*

$$\frac{(Y_{t+h} - Y_t)}{\sqrt{h}} \xrightarrow[h \downarrow 0]{d} f(x) dx := \text{law of } \frac{V}{\sqrt{\Gamma(2D)}} (S_{\lambda t, \lambda \tau_0})^{D-\frac{1}{2}} W_1, \quad (4.1.6)$$

where for $a < 0 < b$ we set $S_{b,a} := (b-a)1_{\{E>b\}} + E1_{\{E \leq b\}}$, with $E \sim \text{Exp}(1)$, and where $W_1 \sim \mathcal{N}(0, 1)$ is an independent random variable. The density f is thus a mixture of centered Gaussian densities and, when $D < \frac{1}{2}$, has power-law tails.

- *Large-time diffusive scaling:*

$$\frac{(Y_{t+h} - Y_t)}{\sqrt{h}} \xrightarrow[h \uparrow \infty]{d} \frac{e^{-x^2/(2c^2)}}{\sqrt{2\pi c}} dx = \mathcal{N}(0, c^2), \quad \text{with } c^2 = V^2 + \varrho^2 \lambda. \quad (4.1.7)$$

We now look at one of the most interesting features of the original model: the *multiscaling of the moments* for small time (see Theorem B.2.3 in appendix B). It turns out that this property *disappears* with the introduction of jumps.

Proposition 4.1.3. *Let $q > 0$, then the quantity $m_q(h) := \mathbb{E}(|Y_{t+h} - Y_t|^q)$ is finite and has the following asymptotic behavior as $h \downarrow 0$:*

$$m_q(h) = h^{A(q)+o(1)}, \quad (4.1.8)$$

where the exponent $A(q)$ is given for $\varrho \neq 0$ by

$$A(q) = \begin{cases} \frac{q}{2} & \text{if } q < 2 \\ 1 & \text{if } q \geq 2 \end{cases}, \quad (4.1.9)$$

and for $\varrho = 0$ by

$$A(q) = \begin{cases} \frac{q}{2} & \text{if } q < q^* \\ Dq + 1 & \text{if } q > q^* \end{cases}, \quad \text{where } q^* := \frac{1}{(\frac{1}{2} - D)}. \quad (4.1.10)$$

Thus, even if the jumps in the log-price do not affect the limit distribution for small h (note that there is no dependence on ϱ in (4.1.6)), they have a big impact on the moments $E(|Y_{t+h} - Y_t|^q)$, as Proposition 4.1.3 shows: the anomalous scaling exponent $A(q) = Dq + 1$ for $q > q^*$, that is observed when $\varrho = 0$, disappears as soon as $\varrho \neq 0$. Intuitively, the reason for such a dramatic change is that for a small time increment h the effect on the moments $E(|Y_{t+h} - Y_t|^q)$ given by the jumps always overcomes that of the continuous component, because the jump size does not vanish as the time increment $h \downarrow 0$.

In order to see a different behavior, one possibility is to send $\varrho \downarrow 0$ together with $h \downarrow 0$: in this case, we now show that when $\varrho = O(\sqrt{h})$ the same multiscaling (4.1.10) as for $\varrho = 0$ is observed. This has the following interpretation: if ϱ is small, then $E(|Y_{t+h} - Y_t|^q)$ exhibits the non-trivial multiscaling (4.1.10) for h small, but not smaller than ϱ^2 .

Theorem 4.1.4. *For every fixed $q > 0$, if $h \downarrow 0$ and $\varrho \downarrow 0$ simultaneously with $\varrho = O(\sqrt{h})$, the quantity $m_q(h) := E(|Y_{t+h} - Y_t|^q)$ scales as (4.1.8), with $A(q)$ given in (4.1.10).*

It is also possible to generalize Theorem B.2.5 about the correlation decay to the case $\varrho \neq 0$ in the following way

Theorem 4.1.5. *The following relation holds as $h \downarrow 0$, for all $t > s > 0$:*

$$\text{Cov}(|Y_{s+h} - Y_s|, |Y_{t+h} - Y_t|) = \frac{4D}{\pi} c e^{-\lambda|t-s|} (\lambda^{1-2D} \phi_s(\lambda(t-s)) + F(t, s)) h + o(h), \quad (4.1.11)$$

where

$$\phi_y(x) := \text{Cov}(S^{D-1/2}, (S+x)^{D-1/2}) \quad (4.1.12)$$

and $S \sim \text{Exp}(1) \wedge y$ and for every $0 < y < x$

$$F(x, y) = e^{-\lambda y} \left\{ (y - \tau_0)^{D-\frac{1}{2}} \left((x - \tau_0)^{D-\frac{1}{2}} - E[(\lambda(x-y) + S)^{D-\frac{1}{2}}] \lambda^{\frac{1}{2}-D} \right) (1 - e^{-\lambda y}) \right. \\ \left. + \gamma \left(\frac{1}{2} + D, \lambda y \right) \left(E[(\lambda(x-y) + S)^{D-\frac{1}{2}}] \lambda^{1-2D} - (x - \tau_0)^{D-\frac{1}{2}} \lambda^{\frac{1}{2}-D} \right) \right\}. \quad (4.1.13)$$

Remark 4.1.6. *The fact that the correlation between the increments of the log-price is the same in both the cases in which there are or not jumps in it is due to the fact that on each increment the event jumps occur gives contribution of order h , while that of the event no*

jumps is of order \sqrt{h} . The jumps component contribution will become dominant only if we compute

$$\text{Cov}(|Y_{s+h} - Y_s|^q, |Y_{t+h} - Y_t|^q)$$

for $q \geq 2$, exactly for the same reasons that lead to the disappearance of the multiscaling of the moments.

In conclusion, the addition of jumps to the basic model (4.1.1), leading to the enriched model (4.1.5), appears to be an effective way to account for the so-called leverage effect, introducing a skew in the log-return distribution. However, such addition is not completely satisfactory, for what concerns the asymptotic properties of the model for small time increments. Alternative ways of introducing correlations, possibly without jumps, are currently under investigation.

4.2 Pricing under the enriched model

As said at the beginning of the chapter, the (de-trended) log-price $(Y_t)_{t \geq 0}$ is defined as

$$Y_t := W_{I_t} + \varrho(N_t - \lambda t).$$

Observe that a *compensated* Poisson process appears in (4.1.5), so that the (de-trended) log-price is a martingale.

The price process $S_0 e^{Y_t}$ is clearly not a martingale (it is a submartingale, because Y_t is a martingale). We consider a natural class of equivalent measures $(\tilde{\mathbb{P}}_{\tilde{\lambda}})_{\tilde{\lambda} \in (0, \infty)}$, defined by

$$\frac{d\tilde{\mathbb{P}}_{\tilde{\lambda}}}{d\mathbb{P}} := \exp \left(\int_0^T \phi_s dB_s - \frac{1}{2} \int_0^T \phi_s^2 ds + \left(\log \frac{\tilde{\lambda}}{\lambda} \right) N_T - (\tilde{\lambda} - \lambda)T \right), \quad (4.2.1)$$

where $T > 0$ is a fixed time horizon, and the process $(\phi_t)_{t \in [0, T]}$ is defined by

$$\phi_t := \frac{\lambda}{\sigma_t} - \frac{\sigma_t}{2} - \frac{(e^\varrho - 1)\tilde{\lambda}}{\sigma_t} \quad \text{with} \quad \sigma_t = \sqrt{I'_t}. \quad (4.2.2)$$

This is really a martingale measure as we prove in the following Theorem.

Theorem 4.2.1. *Under $\tilde{\mathbb{P}}_{\tilde{\lambda}}$ the price process $(S_0 e^{Y_t})_{t \in [0, T]}$ is a martingale, which is distributed as the following process:*

$$S_t := S_0 e^{W_{I_t} - \frac{1}{2}I_t} e^{\varrho N_t - (e^\varrho - 1)\tilde{\lambda}t}, \quad (4.2.3)$$

where $(I_t)_{t \geq 0}$ denotes the time-change process (4.1.2) in which $(N_t)_{t \geq 0}$ is a Poisson process of rate $\tilde{\lambda}$, and where $(W_t)_{t \geq 0}$ is an independent Brownian motion.

4.2.1 A generalized Hull&White formula

For later convenience, we perform a slight change of notation respect the previous chapters and we set the price of a call option with strike K (instead of log-strike $\kappa = \log \frac{K}{S_0}$)

$$C(S_0, K, t) := \tilde{\mathbb{E}}_{\tilde{\lambda}} [(S_t - K)^+] = \tilde{\mathbb{E}}_{\tilde{\lambda}} [(S_0 e^{Y_t} - K)^+] = \mathbb{E} [(S_0 e^{X_t} - K)^+] \quad (4.2.4)$$

where by (4.2.3)

$$X_t := \log \frac{S_t}{S_0} = W_{I_t} - \frac{1}{2}I_t + \varrho N_t - (e^\varrho - 1)\tilde{\lambda}t.$$

As said in the proof of Theorem 4.2.1 in analogy with (4.1.4), this process solves the SDE

$$dX_t = \sigma_t dB_t - \frac{1}{2}\sigma_t^2 dt + \varrho dN_t - (e^\varrho - 1)\tilde{\lambda} dt, \quad \text{with} \quad \sigma_t := \sqrt{I'_t}, \quad (4.2.5)$$

where $(B_t)_{t \geq 0}$ is a suitable Brownian motion, independent of the Poisson process $(N_t)_{t \geq 0}$.

Note that the log-price $(X_t)_{t \geq 0}$ in (4.2.5) is a stochastic volatility model with two independent driving noises, $(B_t)_{t \geq 0}$ and $(N_t)_{t \geq 0}$, and with a volatility process $(\sigma_t)_{t \geq 0}$ correlated with $(N_t)_{t \geq 0}$. Despite this fact, we can give a representation formula for the call price $C(x, t)$ in the spirit of Hull&White [HW87]. Expressing the Black&Scholes formula in terms of S_0 and K :

$$C_{BS}^\sigma(S_0, K, t) := \mathbb{E}[(S_0 e^{W_{\sigma^2 t} - \frac{1}{2}\sigma^2 t} - K)^+] = S_0 \Phi(d_1) - K \Phi(d_2). \quad (4.2.6)$$

where Φ is the cumulative distribution function of a normal gaussian random variable and

$$d_1 := \frac{\log \frac{S_0}{K} + \frac{1}{2}\sigma^2 t}{\sigma \sqrt{t}} \quad d_2 := d_1 - \sigma \sqrt{t}.$$

Theorem 4.2.2 (Generalized Hull&White formula). *For all $S_0, K > 0$ and $t \geq 0$ the following formula holds:*

$$C(S_0, K, t) = \mathbb{E} \left[C_{BS}^{\tilde{\sigma}}(\tilde{S}_0, K, t) \Big|_{\tilde{\sigma} = \sqrt{\frac{1}{t} I_t}, \tilde{S}_0 = S_0 e^{\varrho N_t - (e^\varrho - 1)\tilde{\lambda}t}} \right]. \quad (4.2.7)$$

In words: the price of a call option in our (enriched) model is obtained by averaging the Black&Scholes formula with respect to a *random volatility* and a *random spot price*. This allows for a *fast Monte Carlo evaluation of option prices*. Note in fact that, for a given maturity t , it suffices to generate a sample of the Poisson process $(N_s)_{s \in [0, t]}$ in the interval $[0, t]$ to get a realization of both $\tilde{\sigma}$ and \tilde{S}_0 ; averaging over a sufficiently large number of samples, one can obtain with a good accuracy the price $C(S_0, K, t)$ (and hence the implied volatility) simultaneously for every K .

This method as been used for implementing a fast and effective simulation program using a *stratification algorithm* that we will discuss in the Chapter 5

Remark 4.2.3. A generalized Hull-White formula like (4.2.7) holds for many correlated stochastic volatility models. E.g., assume that under the pricing measure

$$\begin{cases} dS_t = -\frac{1}{2}Y_t dt + \sqrt{Y_t} dW_t \\ dY_t = f(Y_t) dt + g(Y_t) dZ_t \end{cases}, \quad (4.2.8)$$

for some functions f, g , where $(W_t)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ are (jointly Gaussian) Brownian motions with $\text{Cov}(W_t, Z_t) = \varrho t$. We can write $W_t = \varrho Z_t + \sqrt{1 - \varrho^2} B_t$, where $(B_t)_{t \geq 0}$ is a Brownian motion *independent of* $(Z_t)_{t \geq 0}$, getting

$$S_t = S_0 e^{\int_0^t \sqrt{(1-\varrho^2)Y_s} dB_s + \varrho \int_0^t \sqrt{Y_s} dZ_s - \frac{1}{2} \int_0^t Y_s ds} = \tilde{S}_0 e^{\int_0^t \sqrt{(1-\varrho^2)Y_s} dB_s - \frac{1}{2} \int_0^t (1-\varrho^2)Y_s ds},$$

where we set

$$\tilde{S}_0 := S_0 e^{\varrho \int_0^t \sqrt{Y_s} dZ_s - \frac{1}{2} \int_0^t \varrho^2 Y_s ds}. \quad (4.2.9)$$

If \mathcal{G} denotes the σ -algebra generated by $(Z_t)_{t \geq 0}$, the process $(Y_t)_{t \geq 0}$ is \mathcal{G} -measurable under standard assumptions (ensuring e.g. strong existence and uniqueness for the SDE in (4.2.8)). Since the Brownian motion $(B_t)_{t \geq 0}$ is independent of \mathcal{G} , it follows that *conditionally on \mathcal{G}* the stochastic integral $\int_0^t \sqrt{(1 - \varrho^2)Y_s} dB_s$ is actually a Wiener integral, hence its conditional law is $\mathcal{N}(0, \tilde{\sigma}^2 t)$ with

$$\tilde{\sigma}^2 := \frac{1}{t} \int_0^t (1 - \varrho^2) Y_s ds. \quad (4.2.10)$$

Since \tilde{S}_0 and $\tilde{\sigma}$ are \mathcal{G} -measurable, it follows that

$$\mathbb{E}[(S_t - K)^+ | \mathcal{G}] = \mathbb{E}[(\tilde{S}_0 e^{\mathcal{N}(0, \tilde{\sigma}^2 t) - \frac{1}{2} \tilde{\sigma}^2 t} - K)^+ | \mathcal{G}] = C_{BS}^{\tilde{\sigma}}(\tilde{S}_0, K, t).$$

Taking expectation of both sides, we get the generalized Hull-White formula:

$$C(S_0, K, t) := \mathbb{E}[(S_t - K)^+] = \mathbb{E}[C_{BS}^{\tilde{\sigma}}(\tilde{S}_0, K, t)],$$

where the random variables $\tilde{\sigma}$, \tilde{S}_0 are defined in (4.2.9) and (4.2.10).

4.3 Proof of the properties of the enriched model

4.3.1 Proof of Theorem 4.1.2

By (4.1.5) we can write

$$\begin{aligned} Y_{t+h} - Y_t &= W_{I_{t+h}} - W_{I_t} + \varrho(N_{t+h} - N_t - \lambda h) \\ &\stackrel{d}{=} \sqrt{I_{t+h} - I_t} W_1 + \varrho(N_{t+h} - N_t - \lambda h). \end{aligned} \quad (4.3.1)$$

We know that $\mathbb{P}(N_{t+h} - N_t \geq 1) = 1 - e^{-\lambda h} = O(h) \rightarrow 0$ as $h \downarrow 0$, so we can focus on the event $\{N_{t+h} = N_t\}$, on which we have (recall (4.1.3))

$$\lim_{h \downarrow 0} \frac{I_{t+h} - I_t}{h} = I'_t = \frac{V^2 \lambda^{2D-1}}{\Gamma(2D)} (t - \tau_{N_t})^{2D-1}.$$

Note that $(t - \tau_{N_t}) \mathbf{1}_{\{N(t) \geq 1\}}$ is distributed like $(\lambda^{-1} E) \mathbf{1}_{\{(\lambda^{-1} E) \leq t\}}$, with $E \sim \text{Exp}(1)$, while $(t - \tau_{N_t}) \mathbf{1}_{\{N(t) = 0\}}$ is distributed like $(t - \tau_0) \mathbf{1}_{\{(\lambda^{-1} E) > t\}}$. In conclusion,

$$(t - \tau_{N_t}) \mathbf{1}_{\{N(t) \geq 1\}} \stackrel{d}{=} \lambda^{-1} S_{\lambda t, \lambda \tau_0}, \quad (4.3.2)$$

where we set $S_{b,a} := (b - a) \mathbf{1}_{\{E > b\}} + E \mathbf{1}_{\{E \leq b\}}$, as in the statement of Theorem 4.1.2.

Concerning the second part of the right term of equation (4.3.1), the probability that it is different from $-\lambda h$ is $O(h)$ and $\frac{-\lambda h}{\sqrt{h}} \rightarrow 0$ when $h \rightarrow 0$, so when divided by \sqrt{h} it converges in probability to 0. In the end, by (4.3.2) we get

$$\frac{(X_{t+h} - X_t)}{\sqrt{h}} \xrightarrow[h \downarrow 0]{d} \frac{V \lambda^{D-\frac{1}{2}}}{\sqrt{\Gamma(2D)}} (t - \tau_{N_t})^{D-\frac{1}{2}} W_1 \stackrel{d}{=} \frac{V}{\sqrt{\Gamma(2D)}} (S_{\lambda t, \lambda \tau_0})^{D-\frac{1}{2}} W_1, \quad (4.3.3)$$

where $E \sim \text{Exp}(\lambda)$ and $B \sim \mathcal{B}(1 - e^{-\lambda t})$.

We now focus on the case $h \uparrow \infty$. We have to study the convergence of

$$\frac{Y_{t+h} - Y_t}{\sqrt{h}} = \frac{W_{I_{t+h}} - W_{I_t}}{\sqrt{h}} + \varrho \frac{N(t+h) - N(t) - \lambda h}{\sqrt{h}} \quad (4.3.4)$$

when $h \uparrow \infty$. We have

$$I_{t+h} - I_t = \mathfrak{c} \left((t+h - \tau_{N_{t+h}})^{2D} + (\tau_{N_{t+1}} - \tau_{N_t})^{2D} - (t - \tau_{N_t})^{2D} + \sum_{k=N_t+2}^{N_{t+h}} (\tau_k - \tau_{k-1})^{2D} \right).$$

The random variables $((\tau_k - \tau_{k-1})^{2D})_{k \geq N_t+2}$ are independent and identically distributed with finite mean, hence by the strong law of large numbers (since $\frac{\tau_{N_t+1} - \tau_{N_t}}{n} \rightarrow 0$, $\frac{t - \tau_{N_t}}{n} \rightarrow 0$ and $\frac{t+h - \tau_{N_{t+h}}}{n} \rightarrow 0$ a.s.)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\tau_k - \tau_{k-1})^{2D} = \mathbb{E}[(\tau_1)^{2D}] = \lambda^{-2D} \Gamma(2D+1) \quad \text{a.s.}$$

Plainly, $\lim_{h \rightarrow +\infty} N_h/h = \lambda$ a.s., by the strong law of large numbers applied to the random variables $\{\tau_k\}_{k \geq 1}$. Recalling (4.1.2), it follows easily that

$$\lim_{h \uparrow \infty} \frac{I_{t+h} - I_t}{h} = V^2 \quad \text{a.s.}$$

Since $W_{I_{t+h}} - W_{I_t} \stackrel{d}{=} \sqrt{I_{t+h} - I_t} W_1$ we obtain for the first term in the right hand side of (4.3.4) the convergence in distribution

$$\frac{W_{I_{t+h}} - W_{I_t}}{\sqrt{h}} \xrightarrow{d} V W_1 \quad \text{as } h \uparrow \infty.$$

From the central limit theorem for the Poisson Process, we have the convergence in distribution of the term $\frac{N(t+h) - N(t) - \lambda h}{\sqrt{h}}$ to $\sqrt{\lambda} W'_1$, where $W'_1 \sim \mathcal{N}(0, 1)$. Since $(N_t)_{t \geq 0}$ is independent of W_1 and since $\frac{I_{t+h} - I_t}{h}$ converges to the constant V^2 we have that the two limit random variables W_1 and W'_1 are independent. In the end we obtain

$$\frac{Y_{t+h} - Y_t}{\sqrt{h}} \xrightarrow{d} V W_1 + \varrho \sqrt{\lambda} W'_1 \stackrel{d}{=} \mathcal{N}(0, V^2 + \varrho^2 \lambda) \quad \text{as } h \uparrow \infty,$$

which coincides with (4.1.7). □

4.3.2 Proof of Theorem 4.1.3

We exploit relation (4.3.1) and assume that $\varrho \neq 0$ (the $\varrho = 0$ case was considered in Theorem B.2.3). We already know by Theorem B.2.3 that for every $q > 0$, as $h \downarrow 0$

$$\mathbb{E} [|W_{I_{t+h}} - W_{I_t}|^q] = h^{A(q)+o(1)}, \quad (4.3.5)$$

with $A(q)$ given by (4.1.10). Note that for $c, c', a, a' > 0$ we have, as $h \downarrow 0$,

$$ch^a + c'h^{a'} = h^{\min(a, a') + o(1)}.$$

Then

$$\begin{aligned} \mathbb{E}[|\varrho|^q |N(t+h) - N(t) - \lambda h|^q] &= |\varrho|^q e^{-\lambda h} \sum_{k=0}^{\infty} |k - \lambda h|^q \frac{(\lambda h)^k}{k!} \\ &= |\varrho|^q e^{-\lambda h} ((\lambda h)^q + \lambda h + O(h^2)) = h^{q+o(1)} + h^{1+o(1)} = h^{\min(q, 1) + o(1)}, \end{aligned} \quad (4.3.6)$$

because the term in the sum for $k = 1$ equals $(1 - \lambda h)^q \lambda h = \lambda h + O(h^2)$ and

$$\sum_{k=2}^{\infty} |k - \lambda h|^q \frac{(\lambda h)^k}{k!} = (\lambda h)^2 \sum_{k=2}^{\infty} |k - \lambda h|^q \frac{(\lambda h)^{k-2}}{k!} \leq (\lambda h)^2 \sum_{k=2}^{\infty} \frac{k^q}{k!} = (\text{const.})(\lambda h)^2,$$

where the inequality holds for $\lambda h \leq 1$.

We consider first the case $q \geq 1$ (in particular, $\min(q, 1) = 1$ in (4.3.6)). Computing the q -norm of $Y_{t+h} - Y_t$ we obtain

$$(\mathbb{E}[|Y_{t+h} - Y_t|^q])^{\frac{1}{q}} = \left\| \underbrace{W_{I_{t+h}} - W_{I_t}}_{(1)} + \underbrace{\varrho(N(t+h) - N(t) - \lambda h)}_{(2)} \right\|_q = \|(1) + (2)\|_q. \quad (4.3.7)$$

Using Minkowski inequality we get

$$\begin{aligned} \|Y_{t+h} - Y_t\|_q &\leq \|(1)\|_q + \|(2)\|_q = \left(h^{A(q)+o(1)}\right)^{\frac{1}{q}} + \left(h^{1+o(1)}\right)^{\frac{1}{q}} \\ &= h^{\frac{A(q)}{q} + o(1)} + h^{\frac{1}{q} + o(1)} = h^{\min(\frac{A(q)}{q}, \frac{1}{q}) + o(1)}, \end{aligned}$$

hence

$$\mathbb{E}[|Y_{t+h} - Y_t|^q] = (\|Y_{t+h} - Y_t\|_q)^q \leq h^{\min(A(q), 1) + o(1)}. \quad (4.3.8)$$

Note that $A(q) \leq \frac{q}{2}$ for every q and $A(q) \geq 1$ for $q \geq 2$, by (4.1.10), hence

$$\min(A(q), 1) = \begin{cases} \frac{q}{2} & \text{if } q \leq 2 \\ 1 & \text{if } q \geq 2 \end{cases}. \quad (4.3.9)$$

Using the triangle inequality we also get

$$\|Y_{t+h} - Y_t\|_q \geq \left| \|(1)\|_q - \|(2)\|_q \right| = \left| \left(h^{A(q)+o(1)}\right)^{\frac{1}{q}} - \left(h^{\min(q, 1) + o(1)}\right)^{\frac{1}{q}} \right|.$$

Note that, by (4.3.9), we have $h^{A(q)+o(1)} \gg h^{1+o(1)}$ if $q < 2$, while $h^{1+o(1)} \gg h^{A(q)+o(1)}$ if $q > 2$. Recalling (4.3.8), we have proved that Theorem 4.1.3 (with $\varrho \neq 0$) holds, i.e.

$$\mathbb{E}[|Y_{t+h} - Y_t|^q] = h^{\min(\frac{q}{2}, 1) + o(1)}, \quad (4.3.10)$$

for $q \geq 1$. It remains to consider the case $q < 1$. Since $|a + b|^q \leq |a|^q + |b|^q$ we have

$$\mathbb{E}[|Y_{t+h} - Y_t|^q] \leq \mathbb{E}[|(1)|^q] + \mathbb{E}[|(2)|^q] = h^{A(q)+o(1)} + h^{q+o(1)} = h^{\frac{q}{2} + o(1)},$$

since $A(q) = \frac{q}{2}$ for $q < 1$. In a similar way, writing $a = a + b - b$ we obtain $|a + b|^q \geq ||a|^q - |b|^q|$, hence

$$\mathbb{E}[|Y_{t+h} - Y_t|^q] \geq |\mathbb{E}[|(1)|^q] - \mathbb{E}[|(2)|^q]| = \left| h^{A(q)+o(1)} - h^{q+o(1)} \right| = h^{\frac{q}{2}+o(1)}$$

Combining all the estimates obtained in the previous equations we can conclude that (4.3.10) holds also for $q < 1$, completing the proof. \square

4.3.3 Proof of Theorem 4.1.4

We use the notation of equation (4.3.7). Recall that we now assume that $\varrho = O(h^{\frac{1}{2}})$. Looking at (4.3.6), we get

$$\mathbb{E}[|(2)|^q] = \mathbb{E}[\varrho^q |N(t+h) - N(t) - \lambda h|^q] = h^{\frac{q}{2}+\min(q,1)+o(1)}, \quad (4.3.11)$$

while we recall that by (4.3.5)

$$\mathbb{E}[|(1)|^q] = h^{A(q)+o(1)},$$

with $A(q)$ given by (4.1.10), from which it follows that $A(q) < \frac{q}{2} + \min(q, 1)$ for all $q > 0$. As a consequence, in this case $\mathbb{E}[|(2)|^q] \ll \mathbb{E}[|(1)|^q]$. With the same estimates done in the proof of Proposition 4.1.3, it follows that $\mathbb{E}[|Y_{t+h} - Y_t|^q]$ is of the same order as $\mathbb{E}[|(1)|^q]$, completing the proof.

4.3.4 Proof of Theorem 4.1.5

In the proof we suppose without loss of generality that $s+h < t$ (this is possible since $s < t$ and $h \downarrow 0$).

We indicate with \mathcal{G} the σ -algebra generated by the Poisson process. We can write

$$\begin{aligned} & Cov(|Y_{s+h} - Y_s|, |Y_{t+h} - Y_t|) \\ &= Cov(\mathbb{E}(|Y_{s+h} - Y_s| | \mathcal{G}), \mathbb{E}(|Y_{t+h} - Y_t| | \mathcal{G})) \\ &+ \mathbb{E}[Cov(|Y_{s+h} - Y_s|, |Y_{t+h} - Y_t| | \mathcal{G})] \end{aligned} \quad (4.3.12)$$

We recall that $Y_t = W_t + \varrho(N(t) - \lambda t)$, and the process $(I_t)_{t \geq 0}$ is \mathcal{G} -measurable and independent of the process $(W_t)_{t \geq 0}$. It follows that conditionally on \mathcal{G} the process $(Y_t)_{t \geq 0}$ has independent increments, hence the second term on the right hand side of (4.3.12) vanishes because $Cov(|Y_{s+h} - Y_s|, |Y_{t+h} - Y_t| | \mathcal{G}) = 0$ a.s.. For fixed h , from the equality in law

$$Y_{t+h} - Y_t = W_{I_{t+h}} - W_{I_t} + \varrho(N(t+h) - N(t) - \lambda h) \sim \sqrt{I_{t+h} - I_t} W_1 + \varrho(N(t+h) - N(t) - \lambda h)$$

it follows that $\mathbb{E}[|Y_{t+h} - Y_t| | \mathcal{G}] = c_1 \sqrt{I_{t+h} - I_t} + \varrho(N(t+h) - N(t) - \lambda h)$ where $c_1 = \sqrt{2/\pi}$. Analogously $\mathbb{E}[|Y_{s+h} - Y_s| | \mathcal{G}] = c_1 \sqrt{I_{s+h} - I_s} + \varrho(N(s+h) - N(s) - \lambda h)$ and so (4.3.12) reduces to

$$\begin{aligned} & Cov(|Y_{s+h} - Y_s|, |Y_{t+h} - Y_t|) = \\ & Cov\left(\left|\sqrt{\frac{2}{\pi}} \sqrt{I_{s+h} - I_s} + \varrho(N_{s+h} - N_s - \lambda h)\right|, \left|\sqrt{\frac{2}{\pi}} \sqrt{I_{t+h} - I_t} + \varrho(N_{t+h} - N_t - \lambda h)\right|\right) \\ &= Cov(\star_s, \star_t) \end{aligned} \quad (4.3.13)$$

In analogy with the proof in the case $\varrho = 0$ we can replace \star_t with $\star_t 1_{\{N_t - N_{s+h} = 0\}}$, in fact if a jump occurs at time t' between $s+h$ and t , denoting with \mathcal{G}_I the σ -algebra generated by $(N_t)_{t \in I}$, \star_s would be $\mathcal{G}_{(-\infty, s+h]}$ measurable while \star_t would be $\mathcal{G}_{[t', +\infty)}$ -measurable, and so they would be independent. Therefore we can write

$$\text{Cov}(|Y_{s+h} - Y_s|, |Y_{t+h} - Y_t|) = \text{Cov}(\star_s, \star_t 1_{\{N_t - N_{s+h} = 0\}}). \quad (4.3.14)$$

Now we can decompose the right hand side as follows

$$\begin{aligned} \text{Cov}(\star_s, \star_t 1_{\{N_t - N_{s+h} = 0\}}) &= \mathbb{E} \left((\star_s - \mathbb{E}(\star_s)) \star_t 1_{\{N_t - N_{s+h} = 0\}} \right) \\ &= \frac{2}{\pi} \mathbb{E} \left(\left(\sqrt{I_{s+h} - I_s} - \mathbb{E} \left(\sqrt{I_{s+h} - I_s} \right) \right) \sqrt{I_{t+h} - I_t} 1_{\{N_t - N_{s+h} = 0\}} \right) \\ &\quad + \sqrt{\frac{2}{\pi}} \varrho \mathbb{E} \left((N_{s+h} - N_s - \mathbb{E}(N_{s+h} - N_s)) \sqrt{I_{t+h} - I_t} 1_{\{N_t - N_{s+h} = 0\}} \right) \\ &\quad + \sqrt{\frac{2}{\pi}} \varrho \mathbb{E} \left(\left(\sqrt{I_{s+h} - I_s} - \mathbb{E} \left(\sqrt{I_{s+h} - I_s} \right) \right) (N_{t+h} - N_t - \lambda h) 1_{\{N_t - N_{s+h} = 0\}} \right) \\ &\quad + \varrho^2 \mathbb{E} \left((N_{s+h} - N_s - \mathbb{E}(N_{s+h} - N_s)) (N_{t+h} - N_t - \lambda h) 1_{\{N_t - N_{s+h} = 0\}} \right) \\ &= (1) + (2) + (3) + (4) \end{aligned} \quad (4.3.15)$$

The (1) term is exactly what appears in the case $\varrho = 0$ (see Theorem B.2.5 in appendix B) in particular is equal to

$$(1) = 2D c e^{-\lambda|t-s|} (\lambda^{1-2D} \phi_s(\lambda(t-s)) + F(t,s)) h + o(h), \quad (4.3.16)$$

where ϕ_s and F are defined in (4.1.12) and (4.1.13).

We now study separately the other term showing that they are $o(h)$.

We note that $\mathbb{E}[N_{s+h} - N_s] = \lambda h$ and so term (2) becomes

$$\begin{aligned} \sqrt{\frac{\pi}{2\varrho^2}} (2) &= \mathbb{E} \left((N_{s+h} - N_s) \sqrt{I_{t+h} - I_t} 1_{\{N_t - N_{s+h} = 0\}} \right) \\ &\quad - \lambda h \mathbb{E} \left(\sqrt{I_{t+h} - I_t} 1_{\{N_t - N_{s+h} = 0\}} \right) \end{aligned} \quad (4.3.17)$$

We already know that the expectation in the second term is of order \sqrt{h} when $h \rightarrow 0$ (which implies that the second term is $\mathcal{O}(h^{3/2})$), so we concentrate on the first one which is also negligible since

$$\begin{aligned} \mathbb{E}((N_{t+h} - N_t) \sqrt{I_{t+h} - I_t} 1_{\{N_t - N_{s+h} = 0\}}) &= \\ &= \sum_{i=0}^{+\infty} \mathbb{E} \left(i \sqrt{I_{t+h} - I_t} 1_{\{N_t - N_{s+h} = 0\}} \right) \mathbb{P}(N_{t+h} - N_t = i) \\ &= \mathbb{E} \left(\sqrt{I_{t+h} - I_t} 1_{\{N_t - N_{s+h} = 0\}} \right) \sum_{i=0}^{+\infty} i \mathbb{P}(N_{t+h} - N_t = i) = \mathcal{O}(h^{\frac{3}{2}}). \end{aligned}$$

Term (3) can be treated in the same way of term (2), and so we are left only with term (4)

$$\begin{aligned} \frac{1}{\varrho^2}(4) &= \mathbb{E} \left((N_{s+h} - N_s)(N_{t+h} - N_t) 1_{\{N_t - N_{s+h} = 0\}} \right) \\ &\quad - \lambda h \mathbb{E} \left((N_{s+h} - N_s) 1_{\{N_t - N_{s+h} = 0\}} \right) \\ &\quad - \mathbb{E} \left((N_{s+h} - N_s) \right) \mathbb{E} \left((N_{t+h} - N_t) 1_{\{N_t - N_{s+h} = 0\}} \right) \\ &\quad + \lambda h \mathbb{E} \left((N_{s+h} - N_s) \right) \mathbb{E} \left(1_{\{N_t - N_{s+h} = 0\}} \right). \end{aligned} \quad (4.3.18)$$

Now, since $\mathbb{E}(N_{t+h} - N_t) = \mathbb{E}(N_{s+h} - N_s) = \lambda h$ and $N_{t+h} - N_t$, $N_t - N_{s+h}$ and $N_{s+h} - N_s$ are mutually independent, it becomes

$$\frac{1}{\varrho^2}(4) = \lambda^2 h^2 e^{-\lambda(t-s-h)} - \lambda^2 h^2 e^{-\lambda(t-s-h)} - \lambda^2 h^2 e^{-\lambda(t-s-h)} + \lambda^2 h^2 e^{-\lambda(t-s-h)} = 0. \quad (4.3.19)$$

4.4 Proof of the results on the pricing

4.4.1 Proof of Theorem 4.2.1

In analogy with (4.1.4), the process

$$Y_t = W_{I_t} + \varrho(N_t - \lambda t)$$

solves the SDE

$$dY_t = \sigma_t dB_t + \varrho(dN_t - \lambda dt), \quad \text{with} \quad \sigma_t := \sqrt{I'_t}, \quad (4.4.1)$$

and $(B_t)_{t \geq 0}$ is a suitable Brownian motion independent of $(N_t)_{t \geq 0}$.

Let us denote by \mathcal{G} the σ -algebra generated by the Poisson process $(N_t)_{t \geq 0}$. Note that the process ϕ_t in (4.2.2) is \mathcal{G} -measurable, and it is not difficult to check that its trajectories are in L^2_{loc} . Therefore, *conditionally on \mathcal{G}* , Novikov's condition is satisfied and under $\tilde{\mathbb{P}}_{\tilde{\lambda}}$ the process

$$\tilde{B}_t := B_t - \int_0^t \phi_s ds \quad (4.4.2)$$

is a Brownian motion. Thus the distribution of $(\tilde{B}_t)_{t \in [0, T]}$ conditionally on \mathcal{G} is always the same, i.e. the Wiener measure. This implies that $(\tilde{B}_t)_{t \in [0, T]}$ is independent of \mathcal{G} , i.e. it is *independent of $(N_t)_{t \geq 0}$* , hence of $(I_t)_{t \geq 0}$ and $(\sigma_t)_{t \geq 0}$.

Now observe that the process $(N_t)_{t \in [0, T]}$ under $\tilde{\mathbb{P}}_{\tilde{\lambda}}$ is a Poisson process with rate $\tilde{\lambda}$, as it follows by the explicit form of the Radon-Nikodym density (4.2.1).

Summarizing: under $\tilde{\mathbb{P}}_{\tilde{\lambda}}$ the process $(\tilde{B}_t)_{t \in [0, T]}$ in (4.4.2) is a standard Brownian motion and $(N_t)_{t \in [0, T]}$ is a Poisson process with rate $\tilde{\lambda}$, and these two processes are independent. Rewriting (4.4.1) as

$$dY_t = \sigma_t d\tilde{B}_t - \frac{1}{2}\sigma_t^2 dt + \varrho dN_t - (e^\varrho - 1)\tilde{\lambda} dt,$$

and recalling (4.1.1) and (4.1.4), it follows that under $\tilde{\mathbb{P}}_{\tilde{\lambda}}$ the price process $(S_0 e^{Y_t})_{t \in [0, T]}$ has the same joint distribution as

$$S_t := S_0 e^{W_{I_t} - \frac{1}{2}I_t} e^{\varrho N_t - (e^\varrho - 1)\tilde{\lambda} t}, \quad (4.4.3)$$

where $(W_t)_{t \geq 0}$ is a Brownian motion independent of $(I_t)_{t \geq 0}$. This proves (4.2.3).

4.4.2 Proof of Theorem 4.2.2

The proof follows the arguments described in Remark 4.2.3. By (4.4.3) we can write

$$S_t = \tilde{S}_0 e^{W_{\tilde{\sigma}^2 t} - \frac{1}{2}\tilde{\sigma}^2 t}, \quad \text{where} \quad \tilde{\sigma}^2 := \frac{1}{t}I_t, \quad \tilde{S}_0 := e^{\rho N_t - (e^\rho - 1)\tilde{\lambda}t}.$$

Denoting by \mathcal{G} the σ -algebra generated by the Poisson process $(N_t)_{t \geq 0}$, note that the random variables $\tilde{\sigma}$ and \tilde{S}_0 are \mathcal{G} -measurable, while $(W_t)_{t \geq 0}$ is independent of \mathcal{G} . As a consequence,

$$\mathbb{E}[(S_t - K)^+ | \mathcal{G}] = \mathbb{E}[(\tilde{S}_0 e^{W_{\tilde{\sigma}^2 t} - \frac{1}{2}\tilde{\sigma}^2 t} - K)^+ | \mathcal{G}] = C_{BS}^{\tilde{\sigma}}(\tilde{S}_0, K, t), \quad (4.4.4)$$

where we recall that $C_{BS}^{\tilde{\sigma}}(\tilde{S}_0, K, t)$ was defined in (4.2.6). Taking the expectation of both sides of (4.4.4) we obtain relation (4.2.7), proving Theorem 4.2.2.

Chapter 5

Simulation and numerics

In this chapter we present a Monte Carlo algorithm for the pricing of the call based on the Hull&White formula presented in the previous chapter. In order to speed up the calculation, we use a *stratification method* which enables us to reduce the number of simulations needed to have a precise price.

5.1 The Monte Carlo methods

Since, for generic models, there are not closed formula for computing European option prices, Monte-Carlo methods are extensively used in finance.

Let us describe the principle of the Monte-Carlo methods on an elementary example. Suppose want to compute

$$I := \int_{[0,1]^d} f(x)dx$$

where $f(\cdot)$ is a bounded real valued function, we can represent the integral above as $E(f(U))$, where U is a uniformly distributed random variable on $[0, 1]^d$. By the Strong Law of Large Numbers, if $(U_i)_{i \geq 1}$ is a family of uniformly distributed independent random variables on $[0, 1]^d$ then the average

$$S_N = \frac{1}{N} \sum_{i=1}^N f(U_i)$$

converges to $E(f(U))$ almost surely when N tends to infinity. This suggests a very simple algorithm to approximate I : call a random generator N times and compute the average above. In order to efficiently use the Monte-Carlo method we need to know the rate of convergence, and if it is more efficient than the deterministic algorithms. The Central Limit Theorem shows that the error decays as $\frac{\sigma}{\sqrt{N}}$, where σ^2 is the variance of $g(X)$, which is rather slow, moreover the approximation error is random and may take large values even if N is large (however the probability of such an event tends to 0 when N grows).

5.1.1 Stratification methods

Since the accuracy of a Monte-Carlo method with N simulations is given by the ration σ/N , one always wants to rewrite the quantity to compute as the expectation of a random variable

which has smaller variance: this is the basic idea of the variance reduction techniques.

One way to reduce the variance are the so called *stratification methods*. Assume that we want to compute the expectation

$$I = \mathbf{E}(g(X)) = \int_{\mathbb{R}^d} g(x)f(x)dx,$$

where X is a \mathbb{R}^d valued random variable with density $f(x)$. Let $(D_i, 1 \leq i \leq m)$ be a partition of \mathbb{R}^d . I can be expressed as

$$I = \sum_{i=1}^m \mathbf{E}(1_{\{X \in D_i\}}g(X)) = \sum_{i=1}^m \mathbf{E}(g(X)|X \in D_i)\mathbf{P}(X \in D_i),$$

where

$$\mathbf{E}(g(X)|X \in D_i) = \frac{\mathbf{E}(1_{\{X \in D_i\}}g(X))}{\mathbf{P}(X \in D_i)}.$$

Note that $\mathbf{E}(g(X)|X \in D_i)$ can be interpreted as $\mathbf{E}(g(X^i))$, where X^i is a random variable whose law is the law of X conditioned by X belongs to D_i , whose density is

$$\frac{1}{\int_{D_i} f(y)dy} 1_{x \in D_i} f(x)dx.$$

When the numbers $p_i = \mathbf{P}(X \in D_i)$ can be explicitly computed one can use a Monte-Carlo method to approximate each conditional expectation $I_i = \mathbf{E}(g(X)|X \in D_i)$ by

$$\tilde{I}_i = \frac{1}{n_i} (g(X_1^i) + \dots + g(X_{n_i}^i)),$$

where $(X_1^i, \dots, X_{n_i}^i)$ are independent copies of X^i . An estimator \tilde{I} of I is then

$$\tilde{I} = \sum_{i=1}^m p_i \tilde{I}_i.$$

Of course, since the samples used to compute \tilde{I}_i are supposed to be independent, the variance of \tilde{I} is

$$\sum_{i=1}^m p_i^2 \frac{\sigma_i^2}{n_i}$$

where σ_i^2 is the variance of $g(X^i)$.

Fix the total number of simulations $\sum_{i=1}^m n_i = N$. In order to minimize the variance above one must choose

$$n_i = N \frac{p_i \sigma_i}{\sum_{i=1}^m p_i \sigma_i}.$$

For this value of n_i , the variance of \tilde{I} is given by

$$\frac{1}{N} \left(\sum_{i=1}^m p_i \sigma_i \right)^2.$$

Indeed the variance obtained is smaller than the one obtained without stratification since

$$\begin{aligned} \text{Var}(g(X)) &= \mathbb{E}(g(X)^2) - \mathbb{E}(g(X))^2 \\ &= \sum_{i=1}^m p_i \mathbb{E}(g(X)^2 | X \in D_i) - \left(\sum_{i=1}^m p_i \mathbb{E}(g(X) | X \in D_i) \right)^2 \\ &= \sum_{i=1}^m p_i \text{Var}(g(X)^2 | X \in D_i) + \sum_{i=1}^m p_i \mathbb{E}(g(X) | X \in D_i)^2 \\ &\quad - \left(\sum_{i=1}^m p_i \mathbb{E}(g(X) | X \in D_i) \right)^2. \end{aligned}$$

Using the convexity inequality for x^2 we obtain, since $\sum_{i=1}^m p_i = 1$,

$$\text{Var}(g(X)) \geq \sum_{i=1}^m p_i \text{Var}(g(X)^2 | X \in D_i) \geq \left(\sum_{i=1}^m p_i \sigma_i \right)^2.$$

Remark 5.1.1. The optimal stratification involves the knowledge of the σ_i s which are seldom explicitly known, so one need to estimate their values by Monte-Carlo simulations.

A common way to bypass this problem is to choose $n_i = Np_i$. The corresponding variance is

$$\frac{1}{N} \sum_{i=1}^m p_i \sigma_i^2,$$

which is always smaller than the original one. This choice is often made when the probabilities p_i can be computed.

5.2 The stratification method applied to our model

The idea is to apply the stratification method to the pricing formula obtained in the chapter before from the Hull & white formula:

$$C(S_0, K, t) = \mathbb{E} \left[C_{BS}^{\tilde{\sigma}}(\tilde{S}_0, K, t) \Big|_{\tilde{\sigma}=\sqrt{\frac{1}{t}I_t}, \tilde{S}_0=S_0 e^{\varrho N_t - (e^{\varrho}-1)\tilde{\lambda}t}} \right]. \quad (5.2.1)$$

In this case we partition on the number of jumps of the Poisson process N_t driving the time change

$$I_t := \mathfrak{c} \left\{ (t - \tau_{N_t})^{2D} - (-\tau_0)^{2D} + \sum_{k=1}^{N_t} (\tau_k - \tau_{k-1})^{2D} \right\}, \quad (5.2.2)$$

and the jumps in the log-price:

$$Y_t := W_{I_t} + \varrho(N(t) - \lambda t).$$

In fact, once the number of jumps N_t becomes fixed $\tilde{S}_0 = S_0 e^{\varrho N_t - (e^{\varrho}-1)\tilde{\lambda}t}$ becomes deterministic, and the simulation of I_t is much faster. In particular, when no jumps occur I_t is deterministic and equal to

$$I_t = \mathfrak{c} [(t - \tau_0)^{2D} - (-\tau_0)^{2D}].$$

Since this event when $t < \frac{1}{\lambda}$ is the most likely and so the one with the greatest n_i , the stratification method permits to highly reduce the number of computation, because it is enough to compute just one time the $N_t = 0$ layer and then multiply it for its probability in the final average.

Even on the event $N_t = n > 0$ once that the number of jumps is known the computation for obtaining the value of I_t are reduced since the jumps have a uniform distribution of parameters $(n, [0, t])$ which is very easy to simulate.

Once we have both N_t and I_t , computing the price via the Hull & White formula becomes very easy: for every occurrence of the jumps we compute the single price and then we just average on the total.

Observe that this method not only reduces the number of computation thanks to the 0-jumps layer but also, thanks to the stratification give a more precise result, since the distribution of the number of jumps is less randomized and optimized.

5.2.1 The algorithm

In this section we present the algorithm on which the pricing program works, the code both in C and MATLAB[®] languages can be found in appendix A.

The core of the code, as already said, is a stratified Monte-Carlo method. We use a 4 layers partition: the Poisson process in $[0, t]$ realizes 0 jumps, 1 jump, 2 jumps and 3 or more jumps. We have two different choices for the number of simulations in each partition, or we go for the fastest choice, which is just choosing $n_i = N * p_i$ or we can “sacrifice” part of the simulation in order to find the optimal σ_i for the stratification and choose $n_i = N \frac{p_i \sigma_i}{p_0 \sigma_0 + p_1 \sigma_1 + p_2 \sigma_2 + p_3 \sigma_3} \vee 1$ [†].

The first method result much faster than the second, even if the precision obtained is similar since the number of simulation for the 0-jumps layers is directly reduced from n_0 to 1, and so the total number of simulations is strictly lesser than N , and since in general the first layer is the more important one this reduces a lot the number of computations. The second method, on the other hand, performs N simulation since even if the 0-jumps layer needs only one simulation, the other $N - 1$ are distributed on the other layers.

Once the number of simulation for each layer is fixed we can compute the price using the Hull&white formula given in Theorem 4.2.2: every time we just simulate the value of I_t (knowing that i jumps occur), and then we use the Black&Scholes formula, with volatility $\sigma = \sqrt{\frac{I_t}{t}}$ and value of the underline at time 0 $\tilde{S}_0 = S_0 \exp(\rho i - \lambda t(e^\rho - 1))$. Once we have finished the simulations we just average obtaining the real price.

The different layers deals in different ways with the simulations of the value I_t . In the zero jumps layer we just put the deterministic value $I_t = c[(t - \tau_0)^{2D} - (-\tau_0)^{2D}]$.

In the case of one or two jumps the simulation is based on the fact that, give that there are i jumps in $[0, t]$, these are uniformly distributed and so can be easily generated.

[†]The $\vee 1$ is due to the fact that the 0-jumps layer has 0 variance because the price of the call is simply given by the Black & Scholes formula with

$$\sigma = \sqrt{\frac{c[(t - \tau_0)^{2D} - (-\tau_0)^{2D}]}{t}} \quad \tilde{S}_0 = S_0 \exp(-\lambda t(e^\rho - 1))$$

Finally, for the fourth layer, the one with more than 2 jumps, we firstly simulate the position of the third jump τ_3 , we then create the first and the second jumps using a 2-valued uniform distribution in $[0, \tau_3]$ and then we generate the next jumps using independent exponential distribution, since $\tau_j - \tau_{j-1} \sim \text{Exp}(\lambda)$, until $\tau_j > t$, in which case we obtain exactly $j - 1$ jumps.

In order to reduce once more the machine-time we observed that if we have multiple strikes for a single maturity, we can compute at once all the prices since both the C and MATLAB[®] language have a command which permits the computation of a vector of European call option with the same maturity but different strikes. This reduction is due to the fact that we do not have to perform $N \# \text{strikes}$ simulations, but only N .

5.3 Calibration

We used the algorithm above to perform a calibration (still in progress) on implied volatility obtained from real data from the DAX index (we thank Martino Grasselli for the data).

We have fitted the data with the parameters that minimize the root square error with the real volatility at different maturities obtaining the following results

Maturity (days)	D	V	λ	ϱ	τ_0 (years)
50	0.4	0.175	0.7	-0.15	-2.0
85	0.45	0.15	0.7	-0.2	-2.0
113	0.45	0.15	0.6	-0.225	-5.0
204	0.3	0.175	0.7	-0.25	-3.0
Total	0.375	0.175	0.8	-0.15	-2.0

Table 5.1: Parameters of the model optimizing the fitting, for the single maturities and for all of them at the same time.

The preliminary results show good agreement for fixed maturity (see 5.1), by the way it is not a stable method: when there is more than one maturity the error begins to grow (see Figure 5.2). We would like to remark that this is not a feature of our model only, for example Gatheral [Gat06] points out how the Heston model (and more in general stochastic volatility models) performs poorly for small maturities, underestimating the skew of the implied volatility.

We observe that the parameters found are very different from the ones found in [ACDP12], in fact the main weight on the price is given by the jump component, while the time change gives a smaller contribute (the exponent D is close to 0.5 and so the smile flattens).

This suggests once more that even if the introduction of jumps in the log price allows to reproduce the *leverage effect* and the *asymmetry* of the smile, it is possibly not the most natural way to enrich our model, as also the theoretical results of the disappearance of the multiscaling of the moments (cf. Theorem 4.1.3) hinted.

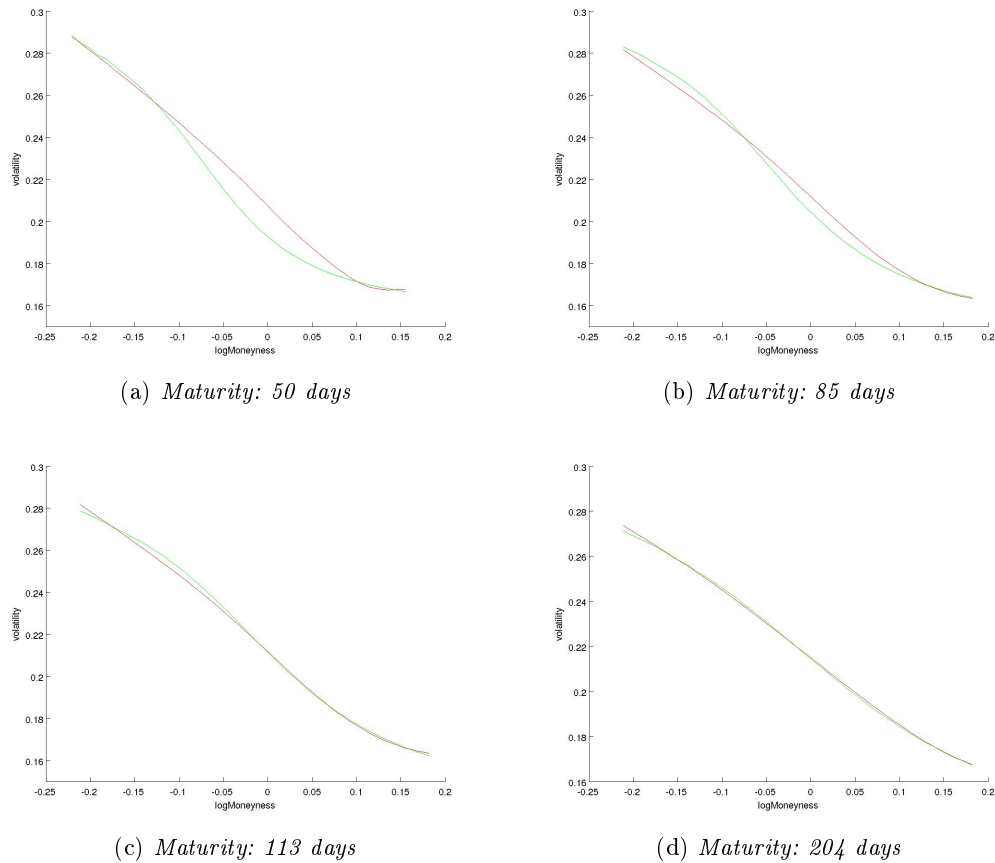


Figure 5.1: This figure contains the plot (in red) of the implied volatility of the European Call options on the DAX index in date 28/08/2008 with maturities 50, 85, 113 and 204 days, and the plot (in green) of the implied volatility of our model for call options with the same maturities, where we have used the fitting parameters contained in the first four rows of Table 5.1 (every maturity has its best fitting parameters)

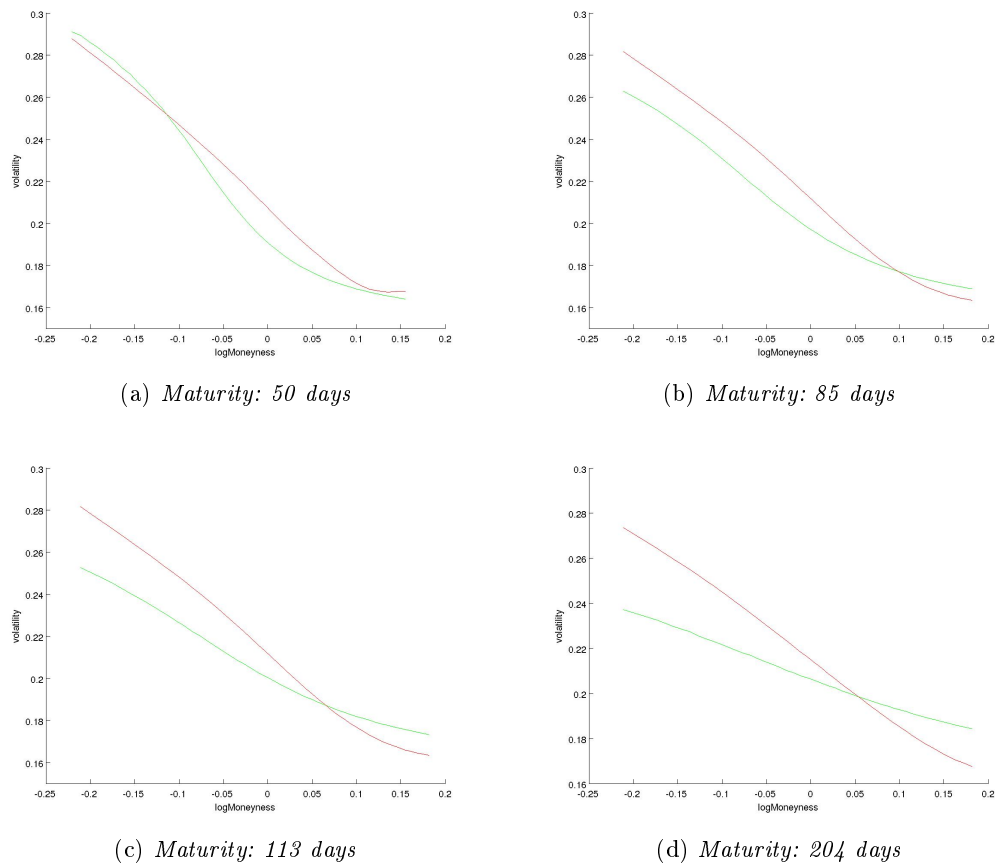


Figure 5.2: This figure contains the plot (in red) of the implied volatility of the European Call options on the DAX index in date 28/08/2008 with maturities 50, 85, 113 and 204 days, and the plot (in green) of the implied volatility of our model for call options with the same maturities, where we have used the fitting parameters contained in the last row of Table 5.1 (best fitting parameters of all the maturities at the same times)

Appendix A

The code

In this appendix we present the C and the MATLAB[®] codes used for pricing the european call in the model presented in chapter 4.

For the C code in particular we have used some tools present in the `Premia` library, which can be downloaded at <https://www.rocq.inria.fr/mathfi/Premia/index.html>.

A.1 The C code

We now present separately the main routines for the C code, there will be one for pricing, two for generating the jumps and one for generating the time change. We will also introduce a function that permits to find the third jump.

A.1.1 The jump makers

We start with the jumps generators: we have two different routines, one when the number of jumps is fixed (and equal to one or two), and the second when there are more than two jumps.

The following generates the jumps when their number is fixed. Observe that in principle this works also when the number of jumps is greater than two but still fixed, which could be useful if we consider call options with large maturities, in which case we could easily change the code in order to have a different stratification with $k + 1$ layers: k with fixed jumps and one for the case "more than $k - 1$ jumps". Moreover both this routine and the following generate a random τ_0 . This is done because it permits to work also with the original model, as presented in [ACDP12], where it was not fixed. We put the deterministic value in the part of the code used for the pricing.

```
void jumpGenerator(double lambda, double sigma,
                  double D, double T, int k, PnlVect* Tau){
    int i=0;
    PnlVect* Valori;
    Valori= pnl_vect_create(k+2);
    double fattcom;
    double prodpar;
    for(i=0;i<=k+1;i++){
```

```

    pnl_vect_set(Valori,i,unif());
}
pnl_vect_set(Tau,0,log(pnl_vect_get(Valori,0))/lambda);
if(k>0){
    fattcom=1.0;
    prodpar=1.0;
    for(i=1;i<=k+1;i++){
        fattcom=fattcom*pnl_vect_get(Valori,i);
    }
    for(i=1;i<=k;i++){
        prodpar=prodpar*pnl_vect_get(Valori,i);
        pnl_vect_set(Tau,i,T*log(prodpar)/log(fattcom));
    }
}
pnl_vect_set(Tau,k+1,T);
pnl_vect_free(&Valori);
}

```

In order to simulate the k jumps, we simulate $k + 1$ uniform random variable (using the function `unif()`), then we make their product and we obtain the jumps using the logarithm.

In case we have more than two jumps (and we don't know how many) we use the routine `jumpGenerator3`. In this routine we firstly find the third jumping time τ_3 (conditioning it is less than t) then we add exponential of parameter λ until we surpass the threshold t ; as before τ_1 and τ_2 are found using a 2 valued uniform in $[0, \tau_3]$. In order to find the value τ_3 we have to compute $P(\tau_3 < t) = P(Ga < t)$, where Ga follows a $\Gamma(3, \frac{1}{\lambda})$ we then simulate a uniform random variable $u \sim U([0, 1])$ and we solve

$$\frac{P(Ga < x)}{P(Ga < t)} = u, \quad (\text{A.1.1})$$

which we create as the function `terzoSalto`, using the function `pnl_root_brent` from the `Premia` library.

```

void jumpGenerator3(double lambda, double sigma,
    double D, double T, PnlVect* Tau){
    int i=0;
    int crazy;
    double x;
    double tmp[3];
    double u;
    double t;
    double tol;
    double fattcom;
    double prodpar;

    PnlVect* Valori;
    Valori= pnl_vect_create(3);

```



```

tol=T/1000;
crazy=pnl_vect_resize (Tau, 1);
x=unif();
pnl_vect_set(Tau, 0, log(x)/lambda);
u=unif();
tmp[0]= lambda;
tmp[1]=T;
tmp[2]=u;
PnlFunc func;
func.function = terzoSalto;
func.params = (void*) tmp;
t = pnl_root_brent(&func, 0.0, T ,&tol);
crazy=pnl_vect_resize(Tau,4);
pnl_vect_set(Tau, 3, t);

for(i=0;i<=2;i++){
    pnl_vect_set(Valori,i,unif());
}
fattcom=1.0;
prodpar=1.0;
for(i=0;i<=2;i++){
    fattcom=fattcom*pnl_vect_get(Valori,i);
}
for(i=0;i<2;i++){
    prodpar=prodpar*pnl_vect_get(Valori,i);
    pnl_vect_set(Tau,i+1, t*(log(prodpar)/log(fattcom)));
}
i=4;
do{
    x=unif();
    t=t-log(x)/lambda;
    if(t<T){
        crazy=pnl_vect_resize(Tau,i+1);
        pnl_vect_set(Tau, i, t);
        i=i+1;
    }
}while(t<T);
crazy=pnl_vect_resize(Tau,i+1);
pnl_vect_set(Tau, i, T);
pnl_vect_free(&Valori);
}

```

The equation (A.1.1) is defined in the following way, using the object `func` of the library

```

static double terzoSalto(double x, void *v)
{
    double *vi = (double*)v;

```

```

double p,q;
p=1-(exp(-vi[0]*x)*(1+vi[0]*x+0.5*x*x*vi[0]*vi[0]));
q=1-(exp(-vi[0]*vi[1])*(1+vi[0]*vi[1]+0.5*vi[1]*vi[1]*vi[0]*vi[0]));
return ((p/q)-vi[2]);
}

```

A.1.2 The clock-time generators

The routine that gives the time change I_t , taking as an input the jumping time is simply an application of the formula I_t :

$$I_t := c \left\{ (t - \tau_{N_t})^{2D} - (-\tau_0)^{2D} + \sum_{k=1}^{N_t} (\tau_k - \tau_{k-1})^{2D} \right\},$$

obtained in the following way

```

double changedTimeGeneratorVec(double sigma,double D,
                               int k, PnlVect* Jumps){
    int i;
    double finale;

    i=0;
    finale=-pow(- pnl_vect_get(Jumps,0),2*D);
    if(k>0){
        for(i=1;i<=k;i++){
            finale=finale+pow((pnl_vect_get(Jumps,i)
                               -pnl_vect_get(Jumps,i-1)),2*D);
        }
    }
    finale=SQR(sigma)*(finale+pow((pnl_vect_get(Jumps,k+1)
                                   -pnl_vect_get(Jumps,k)),2*D));

    return finale;
}

```

We have also a second routine which generates the time-change taking as an input simply the number of jumps that occurs

```

double changedTimeGenerator(double lambda, double sigma,
                            double D, double tau0,double T, int k){
    int i=0;
    double finale;
    PnlVect* Jumps;

    Jumps=pnl_vect_create(k+2);

    jumpGenerator(lambda,sigma,D,T,k,Jumps);
    pnl_vect_set(Jumps,0, tau0);
}

```

```

finale=-pow(-pnl_vect_get(Jumps,0),2*D);
if(k>0){
  for(i=1;i<=k;i++){
    finale=finale+pow((pnl_vect_get(Jumps,i)
                      -pnl_vect_get(Jumps,i-1)),2*D);
  }
}
finale=SQR(sigma)*(finale+pow((pnl_vect_get(Jumps,k+1)
                              -pnl_vect_get(Jumps,k)),2*D));

pnl_vect_free(&Jumps);

return finale;
}

```

A.1.3 The price maker

The actual pricing is done by the following function, based on the stratification algorithm explained in chapter 5. This permits also to price the put option, thanks to the control variable `isCall` which determines which kind of option we are pricing.

For the choice of the number of simulations in each layer we have chosen the fastest method, that is $n_i = Np_i$, because its degree of precision is very similar to the optimal one, even if it is much faster.

```

double pnl_cf_ACDP_call_put(int isCall,double lambda,
                           double v, double D , double T, double strike,
                           double r, int N, double s0, double tau0,double rho){

if(tau0<0){
  printf("error! tau0 must be positive !\n");
  return 0;
}
else{
  int kMax=3;
  double sigma;
  int j;
  PnlVect* Taun;
  double x;
  double time;
  int k;
  double prezzoPar=0.0;
  double prezzo;
  PnlVect* p;
  p= pnl_vect_create(4);
  PnlVect* Tenta;

```

```

sigma=v*sqrt(pow(lambda,2*D-1)/pnl_tgamma(2*D+1));
x=0;
for(k=0;k<kMax;k++){
    x=pnl_sf_fact(k);
    pnl_vect_set(p,k,exp(-lambda*T)*(pow(lambda*T,k))/x);
}
k=3;
pnl_vect_set(p,k,1.0-pnl_vect_get(p,0)-pnl_vect_get(p,1)-pnl_vect_get(p,2));

if(D==0.5){
    if(rho==0){
        return(pnl_bs_call_put(isCall,s0, strike, T, r,0, sigma));
        //if D=0.5 and rho=0 we have the Black & Scholes model
    }else{
        for(j=1;j<=N;j++){
            Taun=pnl_vect_new ();
            jumpGenerator3(lambda, sigma, D,T, Taun);
            prezzoPar=pnl_bs_call_put(isCall,s0*exp(rho*(Taun->size-2)
                -lambda*T*(exp(rho)-1)), strike, T, r,0,sigma);
            pnl_vect_free(&Taun);
        }
        prezzoPar=pnl_vect_get(p,3)*prezzoPar/N;
        for(k=0;k<3;k++){
            prezzoPar=prezzoPar+(pnl_vect_get(p,k)*pnl_bs_call_put(isCall,
                s0*exp(rho*k-lambda*T*(exp(rho)-1)), strike, T, r,0, sigma));
        }
        return (prezzoPar);
        //in case D=0.5 the only element on which we average in the Hull & White
        // formula is the initial value s0, which depend only on the number of jumps.
        //as a consequence we put all the simulations in the layer more then 2
        //jumps because it is the only one with randomness.
    }
}
else{

    PnlVect* prezzi;

    prezzo=0.0;
    prezzoPar=0.0;

    Tenta= pnl_vect_create(4);
    prezzi= pnl_vect_create(4);

    for(k=0;k<=kMax;k++){

```

```

    pnl_vect_set(Tenta,k,(int)MAX(N*pnl_vect_get(p,k),1));
    x=x+pnl_vect_get(Tenta,k);
}

//first layer
k=0;
time=pow(sigma,2)*(pow(T+tau0,2*D)-pow(tau0,2*D));
prezzoPar=pnl_bs_call_put(isCall,s0*exp(rho*k-lambda*T*(exp(rho)-1)),
                          strike, T, r,0, sqrt(time/T));
pnl_vect_set(prezzi,k,(pnl_vect_get(Tenta,0)*prezzoPar));

//second and third layer
for(k=1; k<kMax ;k++){
    if(pnl_vect_get(Tenta,k)!=0){
        for(j=1;j<=pnl_vect_get(Tenta,k);j++){
            Taun=pnl_vect_create(k+2);
            jumpGenerator(lambda,sigma,D,T,k,Taun);
            pnl_vect_set(Taun,0,-tau0);
            time=changedTimeGeneratorVec(sigma, D, k, Taun);
            prezzoPar=pnl_bs_call_put(isCall,s0*exp(rho*k-lambda*T*(exp(rho)-1)),
                                      strike, T, r,0, sqrt(time/T));
            pnl_vect_set(prezzi,k,pnl_vect_get(prezzi,k)+prezzoPar);
            pnl_vect_free(&Taun);
        }
    }
}

//fourth layer
k=3;
if(pnl_vect_get(Tenta,k)!=0){
    for(j=1;j<=pnl_vect_get(Tenta,k);j++){
        Taun=pnl_vect_new ();
        jumpGenerator3(lambda, sigma, D,T, Taun);
        pnl_vect_set(Taun,0,-tau0);
        time=changedTimeGeneratorVec(sigma,D,(Taun->size-2), Taun);
        prezzoPar=pnl_bs_call_put(isCall,s0*exp(rho*(Taun->size-2)
                                                -lambda*T*(exp(rho)-1)), strike, T, r,0, sqrt(time/T));
        pnl_vect_set(prezzi,k,pnl_vect_get(prezzi,k)+prezzoPar);
        pnl_vect_free(&Taun);
    }
}
for(k=0;k<=3;k++){
    prezzo=pnl_vect_get(prezzi,k)+prezzo;
}
//averaging the price

```

```

    prezzo=prezzo/x;

    return prezzo;

    pnl_vect_free(&Tenta);
    pnl_vect_free(&p);
    pnl_vect_free(&prezzi);

}
}
}

```

In the case we have a single maturity but more then one strike we have the following procedure

```

void pnl_cf_ACDP_call_put_vect(PnlVectInt* isCall, double lambda,
                               double v, double D , double T, PnlVect* strike,
                               double r, int N, double s0, double tau0,double rho,
                               PnlVect* Prezzi){

int l;
if(tau0<0){
    printf("error! tau0 must be positive !\n");
}
else{
    double sigma;
    sigma=v*sqrt(pow(lambda,2*D-1)/pnl_tgamma(2*D+1));
    int kMax=3;
    PnlVect* Taun;
    int j;
    double x;
    double time;
    int k;
    PnlVect* Tenta;
    PnlVect* p;

    double prezzoPar;
    double prezzo;

    prezzo=0.0;
    prezzoPar=0.0;

    Tenta= pnl_vect_create(4);

```

```

p= pnl_vect_create(4);
prezzi= pnl_vect_create(4);

x=0;
for(k=0;k<kMax;k++){
    x=pnl_sf_fact(k);
    pnl_vect_set(p,k,exp(-lambda*T)*(pow(lambda*T,k))/x);
}
k=3;
pnl_vect_set(p,k,1.0-pnl_vect_get(p,0)-pnl_vect_get(p,1)-pnl_vect_get(p,2));
for(k=0;k<=kMax;k++){
    pnl_vect_set(Tenta,k,(int)MAX(N*pnl_vect_get(p,k),1));
    x=x+pnl_vect_get(Tenta,k);
}

//first layer
k=0;
time=pow(sigma,2)*(pow(T+tau0,2*D)-pow(tau0,2*D));
for(l=0,l<(strike->size),l++){
    prezzoPar=pnl_bs_call_put(GET_INT(isCall,L),
        s0*exp(rho*k-lambda*T*(exp(rho)-1),pnl_vect_get(strike,l),
        T, r,0, sqrt(time/T));
    pnl_vect_set(Prezzi,l,(pnl_vect_get(Tenta,0)*prezzoPar));
}

//second and third layer
for(k=1; k<kMax ;k++){
    if(pnl_vect_get(Tenta,k)!=0){
        for(j=1;j<=pnl_vect_get(Tenta,k);j++){
            Taun=pnl_vect_create(k+2);
            jumpGenerator(lambda,sigma,D,T,k,Taun);
            pnl_vect_set(Taun,0,-tau0);
            time=changedTimeGeneratorVec(sigma, D, k, Taun);
            for(l=0,l<(strike->size),l++){
                prezzoPar=pnl_bs_call_put(GET_INT(isCall,L),
                    s0*exp(rho*k-lambda*T*(exp(rho)-1),
                    pnl_vect_get(strike,l),T, r,0, sqrt(time/T));
                pnl_vect_set(Prezzi,l,pnl_vect_get(Prezzi,l)+prezzoPar));
            }
            pnl_vect_free(&Taun);
        }
    }
}

//fourth layer

```

```

k=3;
if(pnl_vect_get(Tenta,k)!=0){
  for(j=1;j<=pnl_vect_get(Tenta,k);j++){
    Taun=pnl_vect_new ();
    jumpGenerator3(lambda, sigma, D,T, Taun);
    pnl_vect_set(Taun,0,-tau0);
    time=changedTimeGeneratorVec(sigma,D,(Taun->size-2), Taun);
    for(l=0,l<(strike->size),l++){
      prezzoPar=pnl_bs_call_put(GET_INT(isCall,L),
                                s0*exp(rho*(Taun->size-2)-lambda*T*(exp(rho)-1)),
                                pnl_vect_get(strike,l),T, r,0, sqrt(time/T));
      pnl_vect_set(Prezzi,l,pnl_vect_get(Prezzi,l)+prezzoPar);
    }
    pnl_vect_free(&Taun);
  }
}
//averaging the price
for(l=0,l<(strike->size),l++){
  Pnl_vect_set(Prezzi,l,pnl_vect_get(Prezzi,l)/x);
}

pnl_vect_free(&Tenta);
pnl_vect_free(&p);

}
}
}

```

A.2 The MATLAB[®] code

The MATLAB[®] code is organized in the same way as the C code but, thanks to the higher level of the language is more compact; the drawback of such a high level language is that it is less performing, causing a slower pricing.

A.2.1 The time changers and jump generators

In this case, in the spirit of the function `changedTimeGenerator` above we have group together the time changers and the jump generators, creating only 2 functions, one for the case with a fixed number of jumps and one for the case in which we have a random number of jumps greater or equal than 3.

The function for the fixed number of jumps is the following

```
function[I]=cambiatempo(k,tau0,T,D, lambda,V)
```



```

tau(1)=-tau0;
x=sort(rand(1,k));
for i=2:k+1
    tau(i)=T*x(i-1);
end
tau(k+2)=T;
I=-(tau0)^(2*D);
for i=2:k+2;
    I=I+(tau(i)-tau(i-1))^(2*D);
end
I=I*V^2*lambda^(2*D-1)/gamma(2*D+1);

```

while the one for the random number of jumps is

```

function[I,i]=cambiatempo3(T,tau0,lambda,V,D)
q=1-(exp(-lambda*T)*(1+lambda*T+0.5*lambda^2*T^2));
u=rand(1);
f=@(x) ((1-(exp(-lambda*x)*(1+lambda*x+0.5*lambda^2*x.^2)))/q)-u;
X=fzero(@(x) f(x), 0.0);
tau(4)=X;
tau(1)=-tau0;
u=sort(rand(1,2));
for i=2:3
    tau(i)=X*u(i-1);
end
i=4;
while(tau(i)<T)
    i=i+1;
    u=rand(1);
    tau(i)=tau(i-1)+abs(log(u)/lambda);
end
tau(i)=T;
I=-(tau0)^(2*D);
for j=2:i;
    I=I+(tau(j)-tau(j-1))^(2*D);
end
I=I*V^2*lambda^(2*D-1)/gamma(2*D+1);
i=i-2;

```

A.2.2 The price maker

The function used for getting the price is exactly the same as the one used in the C code, the difference is that in this case we compute at once both the call and the put value, and also the implied volatility. Moreover, we have given the version of the code which give only one price at a time, we can use this function also for computing the prices of options with different strikes at the same time.

```
function [call,put, impVol]=prezzi(S0, strike, r,
```

```

T,lambda, D, V,rho,tau0,N)

%finding the probability of each layer
proba(1)=exp(-lambda*T);
proba(2)=exp(-lambda*T)*lambda*T;
proba(3)=exp(-lambda*T)*(lambda*T)^2;
proba(4)=1-proba(1)-proba(2)-proba(3);

%number of simulation for layer
tenta(1)=max(N*proba(1),1);
tenta(2)=max(N*proba(2),1);
tenta(3)=max(N*proba(3),1);
tenta(4)=max(N*proba(4),1);

%beginning of the stratification

%first layer (0 jumps)
Volatility=sqrt(V^2*lambda^(2*D-1)/gamma(2*D+1)*
                ((T+tau0)^(2*D)-(tau0)^(2*D))/T);
Price=S0*exp(-lambda*T*(exp(rho)-1));

[callT, putT]=blsprice(Price, strike, r, T, Volatility);
call=tenta(1)*callT;
put=tenta(1)*putT;

%second layer (1 jump)
for j=1:tenta(2)
    IT=cambiotempo(1,tau0,T,D, lambda,V);
    Volatility=sqrt(IT/T);
    Price=S0*exp(rho-lambda*T*(exp(rho)-1));
    [callT, putT]=blsprice(Price, strike, r, T, Volatility);
    call=call+callT;
    put=put+putT;
end

%third layer (2 jumps)
for j=1:tenta(3)
    IT=cambiotempo(2,tau0,T,D, lambda,V);
    Volatility=sqrt(IT/T);
    Price=S0*exp(2*rho-lambda*T*(exp(rho)-1));
    [callT, putT]=blsprice(Price, strike, r, T, Volatility);
    call=call+callT;

```

```
    put=put+putT;
end

%fourth layer (>2 jumps)
for j=1:tenta(2)
    [IT,Nsalti]=cambiatempo3(T,tau0,lambda,V,D);
    Volatility=sqrt(IT/T);
    Price=S0*exp(Nsalti*rho-lambda*T*(exp(rho)-1));
    [callT, putT]=blsprice(Price, strike, r, T, Volatility);
    call=call+callT;
    put=put+putT;
end

% final average of the prices
call=call/(tenta(1)+tenta(2)+tenta(3)+tenta(4));
put=put/(tenta(1)+tenta(2)+tenta(3)+tenta(4));

% implied volatility
impVol=zeros(taglia);
impVol=blsimpv(S0,strike,r,T,call,[],0,[],{'call'});
```


Appendix B

Known properties of the model

In this appendix we recall the proofs of the main properties of the original model, as done in [ACDP12]. We point out that in the original paper the authors considered $-\tau_0 \sim \text{Exp}(\lambda)$ in order to have a process with stationary increment. We then generalize them to the case in which τ_0 is deterministic.

We recall that the model was defined in the following way: given a standard Brownian motion $(W_t)_{t \geq 0}$ and an independent Poisson process $(N_t)_{t \geq 0}$ of rate λ , the (de-trended) log-price $(Y_t)_{t \geq 0}$ of an asset is

$$Y_t := W_{I_t}, \quad (\text{B.0.1})$$

where the time-change process $(I_t)_{t \geq 0}$ is defined as follows: denoting by $0 < \tau_1 < \tau_2 < \dots$ the jump times of the Poisson process $(N_t)_{t \geq 0}$, and for $\tau_0 < 0$

$$I_t := \mathfrak{c} \left\{ (t - \tau_{N_t})^{2D} - (-\tau_0)^{2D} + \sum_{k=1}^{N_t} (\tau_k - \tau_{k-1})^{2D} \right\}, \quad (\text{B.0.2})$$

where

$$\mathfrak{c} = \frac{V^2 \lambda^{2D-1}}{\Gamma(2D+1)}$$

with the convention that the sum is zero when $N_t = 0$.

The parameter $\tau_0 \in (-\infty, 0)$ in the definition (B.0.2) of I_t plays the role of the “last jump time” before 0. It is clear that τ_0 determines the *initial volatility* σ_0 :

$$\sigma_0 = \frac{V \lambda^{D-\frac{1}{2}}}{\sqrt{\Gamma(2D)}} (-\tau_0)^{D-\frac{1}{2}}. \quad (\text{B.0.3})$$

In the original model, presented in [ACDP12], the choice was made of taking $-\tau_0$ as an $\text{Exp}(\lambda)$ random variable (just like $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \dots$) independent of $(N_t)_{t \geq 0}$ and $(W_t)_{t \geq 0}$. Since $\tau_0 \in (-\infty, 0)$ is in one-to-one correspondence with σ_0 , we consider τ_0 as a further parameter, as we have done in chapter 3, 4 and 5, which tunes the initial value σ_0 of the volatility process σ_t (or equivalently, the slope at the origin I'_0 of the time-change process I_t).

It is worth stressing that if $-\tau_0$ is chosen as an independent $\text{Exp}(\lambda)$ random variable, as in [ACDP12], the process $(t - \tau_{N_t})_{t \geq 0}$ is *stationary* (with $\text{Exp}(\lambda)$ one-time marginal distributions), hence also $(I'_t)_{t \geq 0}$ is stationary which means that the time-change process

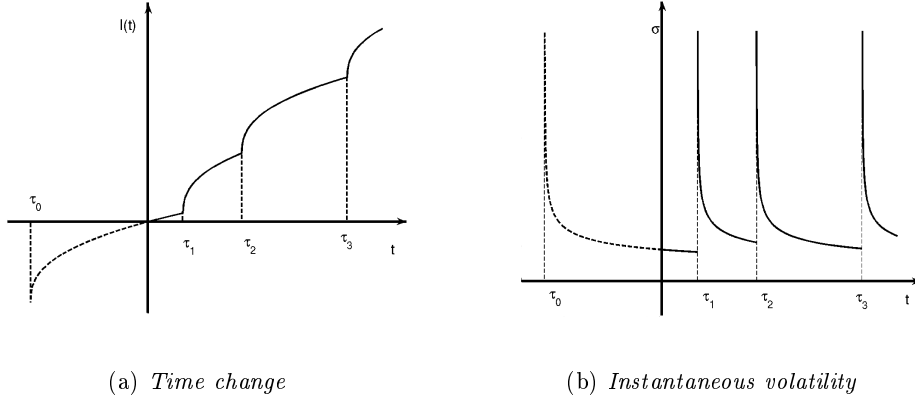


Figure B.1: Trajectories of the *time change process* and of the *instantaneous volatility process*

$(I_t)_{t \geq 0}$ has *stationary increments*. This property breaks down when τ_0 is chosen as a fixed parameter, as we do here, but it still holds asymptotically: for any fixed $\tau_0 \in (-\infty, 0)$, the process $(I_{T+t} - I_T)_{t \geq 0}$ converges in distribution as $T \rightarrow \infty$ toward the process $(I_t)_{t \geq 0}$ of [ACDP12], i.e. in which $-\tau_0$ is chosen as an independent $Exp(\lambda)$ random variable.

B.1 Main properties of the model in the stationary setting

As said in Chapter 1 the main properties of the model are the *crossover of distribution*, the *multiscaling of the moments*, and the *clustering of volatility*. We will prove each result at the end of the appendix.

Theorem B.1.1 (Diffusive scaling: crossover of distributions). *The following convergences in distribution hold for any choice of the parameters D, λ , and V .*

- *Small-time diffusive scaling:*

$$\frac{(Y_{t+h} - Y_t)}{\sqrt{h}} \xrightarrow[h \downarrow 0]{d} f(x) dx := \text{law of } \frac{V}{\sqrt{\Gamma(2D)}} S^{D-\frac{1}{2}} W_1, \quad (\text{B.1.1})$$

where $S \sim Exp(1)$ and $W_1 \sim \mathcal{N}(0, 1)$ are independent random variables. The density f is thus a mixture of centered Gaussian densities and, when $D < \frac{1}{2}$, has power-law tails, more precisely

$$\int |x|^q f(x) dx < +\infty \iff q < q^* := \frac{1}{(\frac{1}{2} - D)}. \quad (\text{B.1.2})$$

- *Large-time diffusive scaling:*

$$\frac{(Y_{t+h} - Y_t)}{\sqrt{h}} \xrightarrow[h \uparrow \infty]{d} \frac{e^{-x^2/(2V^2)}}{\sqrt{2\pi}V} dx = \mathcal{N}(0, V^2). \quad (\text{B.1.3})$$

We now present the main feature of the model, the multiscaling of the moments

Theorem B.1.2. *Let $q > 0$, then the quantity $m_q(h) := \mathbb{E}(|Y_{t+h} - Y_t|^q)$ is finite and has the following asymptotic behavior as $h \downarrow 0$:*

$$m_q(h) \sim \begin{cases} C_q h^{\frac{q}{2}} & \text{if } q < q^* \\ C_q h^{\frac{q}{2}} \log(\frac{1}{h}) & \text{if } q = q^* \\ C_q h^{Dq+1} & \text{if } q > q^* \end{cases}, \quad \text{where } q^* := \frac{1}{(\frac{1}{2} - D)}.$$

The constant $C_q \in (0, \infty)$ is given by

$$C_q := \begin{cases} E(|W_1|^q) c^{\frac{q}{2}} \lambda^{q/q^*} (2D)^{q/2} \Gamma(1 - q/q^*) & \text{if } q < q^* \\ E(|W_1|^q) c^{\frac{q}{2}} \lambda (2D)^{q/2} & \text{if } q = q^* \\ E(|W_1|^q) c^{\frac{q}{2}} \lambda \left[\int_0^\infty ((1+x)^{2D} - x^{2D})^{\frac{q}{2}} dx + \frac{1}{Dq+1} \right] & \text{if } q > q^* \end{cases}, \quad (\text{B.1.4})$$

where $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x} dx$ denotes Euler's Gamma function. This implies

$$A(q) := \lim_{h \downarrow 0} \frac{\log m_q(h)}{\log h} = \begin{cases} \frac{q}{2} & \text{if } q \leq q^* \\ Dq + 1 & \text{if } q \geq q^* \end{cases}, \quad \text{where } q^* := \frac{1}{(\frac{1}{2} - D)}. \quad (\text{B.1.5})$$

The third property we are interested in is the volatility autocorrelation

Theorem B.1.3 (Volatility autocorrelation). *The following relation holds as $h \downarrow 0$, for all $s, t > 0$:*

$$\text{Cov}(|Y_{s+h} - Y_s|, |Y_{t+h} - Y_t|) = \frac{4D}{\pi} c \lambda^{1-2D} e^{-\lambda|t-s|} (\phi(\lambda|t-s|) h + o(h)), \quad (\text{B.1.6})$$

where

$$\phi(x) := \text{Cov}(S^{D-1/2}, (S+x)^{D-1/2}) \quad (\text{B.1.7})$$

and $S \sim \text{Exp}(1)$. Therefore the correlation is given by

$$\begin{aligned} & \lim_{h \downarrow 0} \varrho(|Y_{s+h} - Y_s|, |Y_{t+h} - Y_t|) \\ &= \varrho(t-s) := \frac{2}{\pi \text{Var}(|W_1| S^{D-1/2})} e^{-\lambda|t-s|} \phi(\lambda|t-s|), \end{aligned} \quad (\text{B.1.8})$$

B.2 Deterministic τ_0

In this section we state the corresponding Theorems in the case in which the ‘‘last jump time’’ is deterministic and not distributed as an exponential random variable. The results are very similar to the ones of the previous section, but they also depend on the time t on which we are studying the properties of $|W_{t+h} - W_t|$; in particular, when $t \uparrow +\infty$ we obtain exactly the same results as before.

Theorem B.2.1. *[Diffusive scaling] The following convergences in distribution hold for any choice of the parameters D, λ, V, ϱ and for every $\tau_0 \in (-\infty, 0)$.*

- *Small-time diffusive scaling:*

$$\frac{(Y_{t+h} - Y_t)}{\sqrt{h}} \xrightarrow[h \downarrow 0]{d} f(x) dx := \text{law of } \frac{V}{\sqrt{\Gamma(2D)}} (S_{\lambda t, \lambda \tau_0})^{D-\frac{1}{2}} W_1, \quad (\text{B.2.1})$$

where for $a < 0 < b$ we set $S_{b,a} := (b-a)1_{\{E>b\}} + E1_{\{E \leq b\}}$, with $E \sim \text{Exp}(1)$, and where $W_1 \sim \mathcal{N}(0,1)$ is an independent random variable. The density f is thus a mixture of centered Gaussian densities and, when $D < \frac{1}{2}$, has power-law tails.

- *Large-time diffusive scaling:*

$$\frac{(Y_{t+h} - Y_t)}{\sqrt{h}} \xrightarrow[h \uparrow \infty]{d} \frac{e^{-x^2/(2V^2)}}{\sqrt{2\pi}V} dx = \mathcal{N}(0, V^2), \quad (\text{B.2.2})$$

Remark B.2.2. Note that the choice of a deterministic τ_0 influences only the small time diffusive scaling, in particular, if $t \downarrow 0$ we have

$$\frac{(Y_{t+h} - Y_t)}{\sqrt{h}} \xrightarrow[h \downarrow 0]{d} \sigma_0 W_1$$

while when $t \uparrow \infty$ we converge to (B.1.1). On the other hand if we look at the large time diffusive scaling we obtain exactly the results in (B.1.3).

For the *multiscaling of the moments* we obtain the same results as before with only a slight change in the constants but only in the case $q < q^*$. In fact we have the following

Theorem B.2.3. Let $q > 0$, then the quantity $m_q(h) := \mathbb{E}(|Y_{t+h} - Y_t|^q)$ is finite and has the following asymptotic behavior as $h \downarrow 0$:

$$m_q(h) \sim \begin{cases} C_q h^{\frac{q}{2}} & \text{if } q < q^* \\ C_q h^{\frac{q}{2}} \log(\frac{1}{h}) & \text{if } q = q^* \\ C_q h^{Dq+1} & \text{if } q > q^* \end{cases}, \quad \text{where } q^* := \frac{1}{(\frac{1}{2} - D)}.$$

The constant $C_q \in (0, \infty)$ is given by

$$C_q := \begin{cases} E(|W_1|^q) c^{\frac{q}{2}} (2D)^{\frac{q}{2}} \left(e^{-\lambda t} (t - \tau_0)^{q(D-\frac{1}{2})} + \lambda^{\frac{q}{q^*}} \gamma\left(1 - \frac{q}{q^*}, \lambda t\right) \right) & \text{if } q < q^* \\ E(|W_1|^q) c^{\frac{q}{2}} \lambda (2D)^{q/2} & \text{if } q = q^* \\ E(|W_1|^q) c^{\frac{q}{2}} \lambda \left[\int_0^{+\infty} ((1+x)^{2D} - x^{2D})^{\frac{q}{2}} dx + \frac{1}{Dq+1} \right] & \text{if } q > q^* \end{cases} \quad (\text{B.2.3})$$

where $\gamma(\alpha, t) := \int_0^t x^{\alpha-1} e^{-x} dx$ denotes Euler's lower incomplete Gamma function. This implies

$$A(q) := \lim_{h \downarrow 0} \frac{\log m_q(h)}{\log h} = \begin{cases} \frac{q}{2} & \text{if } q \leq q^* \\ Dq + 1 & \text{if } q \geq q^* \end{cases}, \quad \text{where } q^* := \frac{1}{(\frac{1}{2} - D)}. \quad (\text{B.2.4})$$

Remark B.2.4. We remark that the multiscaling is preserved also in the case in which τ_0 is deterministic; such a choice influences only the behavior of the multiplicative constants and not the exponent and only in the case $q < q^*$.

Finally we prove the correspondent of Theorem B.1.3, i.e. the clustering of volatility.

Theorem B.2.5 (Clustering of volatility). *The following relation holds as $h \downarrow 0$, for all $t > s > 0$:*

$$\text{Cov}(|Y_{s+h} - Y_s|, |Y_{t+h} - Y_t|) = \frac{4D}{\pi} c e^{-\lambda|t-s|} (\lambda^{1-2D} \phi_{\lambda s}(\lambda(t-s)) + F(t, s)) h + o(h), \quad (\text{B.2.5})$$

where

$$\phi_y(x) := \text{Cov}(S^{D-1/2}, (S+x)^{D-1/2}) \quad (\text{B.2.6})$$

and $S \sim \text{Exp}(1) \wedge y$ and for every $0 < y < x$

$$F(x, y) = e^{-\lambda y} \left\{ (y - \tau_0)^{D-\frac{1}{2}} \left((x - \tau_0)^{D-\frac{1}{2}} - \mathbb{E}[(\lambda(x-y) + S)^{D-\frac{1}{2}}] \lambda^{\frac{1}{2}-D} \right) (1 - e^{-\lambda y}) \right. \\ \left. + \gamma \left(\frac{1}{2} + D, \lambda y \right) \left(\mathbb{E}[(\lambda(x-y) + S)^{D-\frac{1}{2}}] \lambda^{1-2D} - (x - \tau_0)^{D-\frac{1}{2}} \lambda^{\frac{1}{2}-D} \right) \right\} \quad (\text{B.2.7})$$

B.3 Proof in the stationary setting

B.3.1 Diffusive scaling: proof of Theorem B.1.1

Since $P(N_h \geq 1) = 1 - e^{-\lambda h} \mathcal{O}(h)$ as $h \downarrow 0$, we may focus on the event $\{N_h = 0\} = \{\mathcal{T} \cap (0, h] = \emptyset\}$, on which we have $I_h = c((h - \tau_0)^{2D} - (-\tau_0)^{2D})$, with $-\tau_0 \sim \text{Exp}(\lambda)$. In particular,

$$\lim_{h \downarrow 0} \frac{I_h}{h} = I'(0) = (2D) \frac{V^2 \lambda^{2D-1}}{\Gamma(2D+1)} (-\tau_0)^{2D-1} \quad \text{a.s.}$$

Since $X_{t+h} - X_t \sim \sqrt{I_h} W_1$, the convergence in distribution (B.1.1) follows:

$$\frac{X_{t+h} - X_t}{\sqrt{h}} \xrightarrow{d} \frac{V \lambda^{D-\frac{1}{2}}}{\sqrt{\Gamma(2D)}} (-\tau_0)^{D-1/2} W_1 \quad \text{as } h \downarrow 0.$$

Since $\tau_0 \sim \text{Exp}(\lambda)$ we obtain the thesis.

Next we focus on the case $h \uparrow \infty$. Under the assumption $E(\sigma^2) < \infty$, the random variables $\{\sigma_{k-1}^2 (\tau_k - \tau_{k-1})^{2D}\}_{k \geq 1}$ are independent and identically distributed with finite mean, hence by the strong law of large numbers

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\tau_k - \tau_{k-1})^{2D} = \mathbb{E}[(\tau_1)^{2D}] = \lambda^{-2D} \Gamma(2D+1) \quad \text{a.s.}$$

Plainly, $\lim_{h \rightarrow +\infty} N_h/h = \lambda$ a.s., by the strong law of large numbers applied to the random variables $\{\tau_k\}_{k \geq 1}$. It follows easily that

$$\lim_{h \uparrow \infty} \frac{I(h)}{h} = V^2 \quad \text{a.s.}$$

Since $X_{t+h} - X_t \sim \sqrt{I_h} W_1$, we obtain the convergence in distribution

$$\frac{X_{t+h} - X_t}{\sqrt{h}} \xrightarrow{d} V W_1 \quad \text{as } h \uparrow \infty,$$

which coincides with (B.1.3). □

B.3.2 Multiscaling: proof of Theorem B.1.2

Since we have stationary increments, i.e. $X_{t+h} - X_t \sim \sqrt{I_h} W_1$, we can write

$$\mathbb{E}(|X_{t+h} - X_t|^q) = \mathbb{E}(|I_h|^{q/2} |W_1|^q) = \mathbb{E}(|W_1|^q) \mathbb{E}(|I_h|^{q/2}) = c_q \mathbb{E}(|I_h|^{q/2}), \quad (\text{B.3.1})$$

where we set $c_q := \mathbb{E}(|W_1|^q)$. We therefore focus on $\mathbb{E}(|I_h|^{q/2})$, that we write as the sum of three terms, that will be analyzed separately:

$$\mathbb{E}(|I_h|^{q/2}) = \mathbb{E}(|I_h|^{q/2} \mathbf{1}_{\{N_h=0\}}) + \mathbb{E}(|I_h|^{q/2} \mathbf{1}_{\{N_h=1\}}) + \mathbb{E}(|I_h|^{q/2} \mathbf{1}_{\{N_h \geq 2\}}). \quad (\text{B.3.2})$$

For the first term in the right hand side of (B.3.2), we note that $P(N_h = 0) = e^{-\lambda h} \rightarrow 1$ as $h \downarrow 0$ and that $I_h = c((h - \tau_0)^{2D} - (-\tau_0)^{2D})$ on the event $\{N_h = 0\}$. Setting $-\tau_0 =: \lambda^{-1}S$ with $S \sim \text{Exp}(1)$, we obtain as $h \downarrow 0$

$$\mathbb{E}(|I_h|^{q/2} \mathbf{1}_{\{N_h=0\}}) = c^{q/2} \lambda^{-Dq} \mathbb{E}(((S + \lambda h)^{2D} - S^{2D})^{q/2}) (1 + o(1)). \quad (\text{B.3.3})$$

Recalling that $q^* := (\frac{1}{2} - D)^{-1}$, we have

$$q \underset{\geq}{\leq} q^* \iff \frac{q}{2} \underset{\geq}{\leq} Dq + 1 \iff -1 \underset{\geq}{\leq} \left(D - \frac{1}{2}\right) q.$$

As $\delta \downarrow 0$ we have $\delta^{-1}((S + \delta)^{2D} - S^{2D}) \uparrow 2D S^{2D-1}$ and note that $\mathbb{E}(S^{(D-\frac{1}{2})q}) = \Gamma(1 - q/q^*)$ is finite if and only if $(D - \frac{1}{2})q > -1$, that is $q < q^*$. Therefore the monotone convergence theorem yields

$$\text{for } q < q^* : \lim_{h \downarrow 0} \frac{\mathbb{E}(((S + \lambda h)^{2D} - S^{2D})^{q/2})}{\lambda^{q/2} h^{q/2}} = (2D)^{q/2} \Gamma(1 - q/q^*) \in (0, \infty). \quad (\text{B.3.4})$$

Next observe that, by the change of variables $s = (\lambda h)x$, we can write

$$\begin{aligned} \mathbb{E}(((S + \lambda h)^{2D} - S^{2D})^{q/2}) &= \int_0^\infty ((s + \lambda h)^{2D} - s^{2D})^{q/2} e^{-s} ds \\ &= (\lambda h)^{Dq+1} \int_0^\infty ((1+x)^{2D} - x^{2D})^{q/2} e^{-\lambda h x} dx. \end{aligned} \quad (\text{B.3.5})$$

Note that $((1+x)^{2D} - x^{2D})^{q/2} \sim (2D)^{q/2} x^{(D-\frac{1}{2})q}$ as $x \rightarrow +\infty$ and that $(D - \frac{1}{2})q < -1$ if and only if $q > q^*$. Therefore, again by the monotone convergence theorem, we obtain

$$\text{for } q > q^* : \lim_{h \downarrow 0} \frac{\mathbb{E}(((S + \lambda h)^{2D} - S^{2D})^{q/2})}{\lambda^{Dq+1} h^{Dq+1}} = \int_0^\infty ((1+x)^{2D} - x^{2D})^{q/2} dx \in (0, \infty). \quad (\text{B.3.6})$$

Finally, in the case $q = q^*$ we have $((1+x)^{2D} - x^{2D})^{q^*/2} \sim (2D)^{q^*/2} x^{-1}$ as $x \rightarrow +\infty$ and we want to study the integral in the second line of (B.3.5). Fix an arbitrary (large) $M > 0$ and note that, integrating by parts and performing a change of variables, as $h \downarrow 0$ we have

$$\begin{aligned} \int_M^\infty \frac{e^{-\lambda h x}}{x} dx &= -\log M e^{-\lambda h M} + \lambda h \int_M^\infty (\log x) e^{-\lambda h x} dx = O(1) + \int_{\lambda h M}^\infty \log\left(\frac{y}{\lambda h}\right) e^{-y} dy \\ &= O(1) + \int_{\lambda h M}^\infty \log\left(\frac{y}{\lambda}\right) e^{-y} dy + \log\left(\frac{1}{h}\right) \int_{\lambda h M}^\infty e^{-y} dy = \log\left(\frac{1}{h}\right) (1 + o(1)). \end{aligned}$$

From this it is easy to see that as $h \downarrow 0$

$$\int_0^\infty ((1+x)^{2D} - x^{2D})^{\frac{q^*}{2}} e^{-\lambda h x} dx \sim (2D)^{\frac{q^*}{2}} \log\left(\frac{1}{h}\right).$$

Coming back to (B.3.5), noting that $Dq + 1 = \frac{q}{2}$ for $q = q^*$, it follows that

$$\lim_{h \downarrow 0} \frac{\mathbb{E}(((S+h)^{2D} - S^{2D})^{\frac{q^*}{2}})}{\lambda^{Dq^*+1} h^{\frac{q^*}{2}} \log(\frac{1}{h})} = (2D)^{\frac{q^*}{2}}. \quad (\text{B.3.7})$$

Recalling (B.3.1) and (B.3.3), the relations (B.3.4), (B.3.6) and (B.3.7) show that the first term in the right hand side of (B.3.2) has the same asymptotic behavior as in the statement of the theorem, except for the regime $q > q^*$ where the constant does not match (the missing contribution will be obtained in a moment).

We now focus on the second term in the right hand side of (B.3.2). Note that, conditionally on the event $\{N_h = 1\} = \{\tau_1 \leq h, \tau_2 > h\}$, we have

$$I_h = \mathbf{c} \left((h - \tau_1)^{2D} + ((\tau_1 - \tau_0)^{2D} - (-\tau_0)^{2D}) \right) \sim \mathbf{c} \left((h - hU)^{2D} + \left(\left(hU + \frac{S}{\lambda} \right)^{2D} - \left(\frac{S}{\lambda} \right)^{2D} \right) \right),$$

where $S \sim \text{Exp}(1)$ and $U \sim U(0,1)$ (uniformly distributed on the interval $(0,1)$) are independent. Since $\mathbb{P}(N_h = 1) = \lambda h + o(h)$ as $h \downarrow 0$, we obtain

$$\mathbb{E}(|I_h|^{\frac{q}{2}} \mathbf{1}_{\{N_h=1\}}) = \lambda h^{Dq+1} \mathbf{c} \mathbb{E} \left[\left((1-U)^{2D} + \left(\left(U + \frac{S}{\lambda h} \right)^{2D} - \left(\frac{S}{\lambda h} \right)^{2D} \right)^{\frac{q}{2}} \right) \right]. \quad (\text{B.3.8})$$

Since $(u+x)^{2D} - x^{2D} \rightarrow 0$ as $x \rightarrow \infty$, for every $u \geq 0$, by the dominated convergence theorem we have (for every $q \in (0, \infty)$)

$$\lim_{h \downarrow 0} \frac{\mathbb{E}(|I_h|^{\frac{q}{2}} \mathbf{1}_{\{N_h=1\}})}{h^{Dq+1}} = \lambda \mathbf{c}^{\frac{q}{2}} \mathbb{E}((1-U)^{Dq}) = \lambda \mathbf{c}^{\frac{q}{2}} \frac{1}{Dq+1}. \quad (\text{B.3.9})$$

This shows that the second term in the right hand side of (B.3.2) gives a contribution of the order h^{Dq+1} as $h \downarrow 0$. This is relevant only for $q > q^*$, because for $q \leq q^*$ the first term gives a much bigger contribution of the order $h^{q/2}$ (see (B.3.4) and (B.3.7)). Recalling (B.3.1), it follows from (B.3.9) and (B.3.6) that the contribution of the first and the second term in the right hand side of (B.3.2) matches the statement of the theorem (including the constant).

It only remains to show that the third term in the right hand side of (B.3.2) gives a negligible contribution. We begin by deriving a simple upper bound for I_h . Since $(a+b)^{2D} - b^{2D} \leq a^{2D}$ for all $a, b \geq 0$ (we recall that $2D \leq 1$), when $N_h \geq 1$, i.e. $\tau_1 \leq h$, we can write

$$\begin{aligned} I_h &= \mathbf{c} \left((h - \tau_{N_h})^{2D} + \sum_{k=2}^{N_h} (\tau_k - \tau_{k-1})^{2D} + (\tau_1 - \tau_0)^{2D} - (-\tau_0)^{2D} \right) \\ &\leq \mathbf{c} \left[(h - \tau_{N_h})^{2D} + \sum_{k=2}^{N_h} (\tau_k - \tau_{k-1})^{2D} + \tau_1^{2D} \right], \end{aligned} \quad (\text{B.3.10})$$

where we agree that the sum over k is zero if $N_h = 1$. Since $\tau_k \leq h$ for all $k \leq N_h$, by the definition (B.0.2) of N_h , relation (B.3.10) yields the bound $I_h \leq h^{2D} \mathbf{c}(N_h + 1)$, which holds clearly also when $N_h = 0$. In conclusion, we have shown that for all $h, q > 0$

$$|I_h|^{q/2} \leq h^{Dq} \mathbf{c}^{\frac{q}{2}} (N_h + 1)^{q/2}. \quad (\text{B.3.11})$$

Consider first the case $q > 2$: we obtain

$$\mathbf{E}(|I_h|^{q/2} \mathbf{1}_{\{N_h \geq 2\}}) \leq h^{Dq} \mathbf{c}^{\frac{q}{2}} \mathbf{E}((N_h + 1)^{q/2} \mathbf{1}_{\{N_h \geq 2\}}). \quad (\text{B.3.12})$$

A corresponding inequality for $q \leq 2$ is derived from (B.3.11) and the inequality $(N_t + 1)^{q/2} \leq N_t + 1$:

$$\mathbf{E}(|I_h|^{q/2} \mathbf{1}_{\{N_h \geq 2\}}) \leq h^{Dq} \mathbf{c}^{\frac{q}{2}} \mathbf{E}((N_h + 1) \mathbf{1}_{\{N_h \geq 2\}}). \quad (\text{B.3.13})$$

For any fixed $a > 0$, by the Hölder inequality with $p = 3$ and $p' = 3/2$ we can write for $h \leq 1$

$$\begin{aligned} \mathbf{E}((N_h + 1)^a \mathbf{1}_{\{N_h \geq 2\}}) &\leq \mathbf{E}((N_h + 1)^{3a})^{1/3} P(N_h \geq 2)^{2/3} \\ &\leq E((N_1 + 1)^{3a})^{1/3} (1 - e^{-\lambda h} - e^{-\lambda h} \lambda h)^{2/3} \leq (\text{const.}) h^{4/3}, \end{aligned} \quad (\text{B.3.14})$$

because $\mathbf{E}((N_1 + 1)^{3a}) < \infty$ (recall that $N_h \sim Po(\lambda)$) and $(1 - e^{-\lambda h} - e^{-\lambda h} \lambda h) \sim \frac{1}{2}(\lambda h)^2$ as $h \downarrow 0$. Then it follows from (B.3.12) and (B.3.13) and (B.3.14) that

$$\mathbf{E}(|I_h|^{q/2} \mathbf{1}_{\{N_h \geq 2\}}) \leq (\text{const.}') h^{Dq+4/3}.$$

This shows that the contribution of the third term in the right hand side of (B.3.2) is always negligible with respect to the contribution of the second term (recall (B.3.9)). \square

B.3.3 Decay of correlation: proof of Theorem B.1.3

Given a Borel set $I \subseteq \mathbb{R}$, we let \mathcal{G}_I denote the σ -algebra generated by the family of random variables $(\tau_k \mathbf{1}_{\{\tau_k \in I\}})_{k \geq 0}$. Informally, \mathcal{G}_I may be viewed as the σ -algebra generated by the variables τ_k for the values of k such that $\tau_k \in I$. From the basic property of the Poisson process, it follows that for disjoint Borel sets I, I' the σ -algebras $\mathcal{G}_I, \mathcal{G}_{I'}$ are independent. We set for short $\mathcal{G} := \mathcal{G}_{\mathbb{R}}$, which is by definition the σ -algebra generated by all the variables $(\tau_k)_{k \geq 0}$, which coincides with the σ -algebra generated by the process $(I_t)_{t \geq 0}$.

We have to prove (B.1.6). Plainly, by translation invariance we can set $s = 0$ without loss of generality. We also assume that $h < t$. We start writing

$$\begin{aligned} &Cov(|X_h|, |X_{t+h} - X_t|) \\ &= Cov(\mathbf{E}(|X_h| | \mathcal{G}), \mathbf{E}(|X_{t+h} - X_t| | \mathcal{G})) + \mathbf{E}(Cov(|X_h|, |X_{t+h} - X_t| | \mathcal{G})). \end{aligned} \quad (\text{B.3.15})$$

We recall that $X_t = W_{I_t}$ and the process $(I_t)_{t \geq 0}$ is \mathcal{G} -measurable and independent of the process $(W_t)_{t \geq 0}$. It follows that, conditionally on $(I_t)_{t \geq 0}$, the process $(X_t)_{t \geq 0}$ has independent increments, hence the second term in the right hand side of (B.3.15) vanishes, because $Cov(|X_h|, |X_{t+h} - X_t| | \mathcal{G}) = 0$ a.s.. For fixed h , from the equality in law $X_h = W_{I_h} \sim \sqrt{I_h} W_1$ it follows that $\mathbf{E}(|X_h| | \mathcal{G}) = c_1 \sqrt{I_h}$, where $c_1 = \mathbf{E}(|W_1|) = \sqrt{2/\pi}$. Analogously $\mathbf{E}(|X_{t+h} - X_t| | \mathcal{G}) = \sqrt{2/\pi} \sqrt{I_{t+h} - I_t}$ and (B.3.15) reduces to

$$Cov(|X_h|, |X_{t+h} - X_t|) = \frac{2}{\pi} Cov(\sqrt{I_h}, \sqrt{I_{t+h} - I_t}). \quad (\text{B.3.16})$$

Recall the definition (B.0.2) of the variable I_t . We now claim that we can replace $\sqrt{I_{t+h} - I_t}$ by $\sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (h, t] = \emptyset\}}$ in (B.3.16). In fact we can write

$$I_{t+h} - I_t = c \left((t+h - \tau_{i(t+h)})^{2D} + \sum_{k=N_t+1}^{i(t+h)} (\tau_k - \tau_{k-1})^{2D} - (t - \tau_{N_t})^{2D} \right),$$

where we agree that the sum in the right hand side is zero if $N_{t+h} = N_t$. This shows that $(I_{t+h} - I_t)$ is a function of the variables τ_k with index $N_t \leq k \leq N_{t+h}$. Since $\{\mathcal{T} \cap (h, t] \neq \emptyset\} = \{\tau_{N_t} > h\}$, this means that $\sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (h, t] \neq \emptyset\}}$ is $\mathcal{G}_{(h, t+h]}$ -measurable, hence independent of $\sqrt{I_h}$, which is clearly $\mathcal{G}_{(-\infty, h]}$ -measurable. This shows that $Cov(\sqrt{I_h}, \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (h, t] \neq \emptyset\}}) = 0$, therefore from (B.3.16) we can write

$$Cov(|X_h|, |X_{t+h} - X_t|) = \frac{2}{\pi} Cov(\sqrt{I_h}, \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (h, t] = \emptyset\}}). \quad (\text{B.3.17})$$

Now we decompose this last covariance as follows:

$$\begin{aligned} Cov(\sqrt{I_h}, \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (h, t] = \emptyset\}}) &= \mathbb{E} \left[\left(\sqrt{I_h} - \mathbb{E}(\sqrt{I_h}) \right) \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (h, t] = \emptyset\}} \right] \\ &= \mathbb{E} \left[\left(\sqrt{I_h} - \mathbb{E}(\sqrt{I_h}) \right) \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (0, t+h] = \emptyset\}} \right] \\ &\quad + \mathbb{E} \left[\left(\sqrt{I_h} - \mathbb{E}(\sqrt{I_h}) \right) \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (h, t] = \emptyset\}} \mathbf{1}_{\{\mathcal{T} \cap ([0, h] \cup (t, t+h]) \neq \emptyset\}} \right] \end{aligned} \quad (\text{B.3.18})$$

We deal separately with the two terms in the r.h.s. of (B.3.18). The first gives the dominant contribution. To see this, observe that, on $\{\mathcal{T} \cap (0, t+h] = \emptyset\}$

$$I_h = c [(h - \tau_0)^{2D} - (-\tau_0)^{2D}]$$

and

$$I_{t+h} - I_t = c [(t+h - \tau_0)^{2D} - (t - \tau_0)^{2D}].$$

Since both $c [(h - \tau_0)^{2D} - (-\tau_0)^{2D}]$ and $c [(t+h - \tau_0)^{2D} - (t - \tau_0)^{2D}]$ are independent of $\{\mathcal{T} \cap (0, t+h] = \emptyset\}$, we have

$$\begin{aligned} &\mathbb{E} \left[\left(\sqrt{I_h} - \mathbb{E}(\sqrt{I_h}) \right) \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (0, t+h] = \emptyset\}} \right] \\ &= \mathbb{E} \left[\left(\sqrt{c} \sqrt{(h - \tau_0)^{2D} - (-\tau_0)^{2D}} - \mathbb{E}(\sqrt{I_h}) \right) \sqrt{c} \sqrt{(t+h - \tau_0)^{2D} - (t - \tau_0)^{2D}} \mathbf{1}_{\{\mathcal{T} \cap (0, t+h] = \emptyset\}} \right] \\ &= c e^{-\lambda(t+h)} \mathbb{E} \left[\left(\sqrt{(h - \tau_0)^{2D} - (-\tau_0)^{2D}} - \mathbb{E}(\sqrt{I_h}) \right) \sqrt{(t+h - \tau_0)^{2D} - (t - \tau_0)^{2D}} \right] \\ &= c e^{-\lambda(t+h)} \left\{ Cov \left(\sqrt{(h - \tau_0)^{2D} - (-\tau_0)^{2D}}, \sqrt{(t+h - \tau_0)^{2D} - (t - \tau_0)^{2D}} \right) \right. \\ &\quad \left. + \left[\mathbb{E} \left(\sqrt{(h - \tau_0)^{2D} - (-\tau_0)^{2D}} \right) - \mathbb{E}(\sqrt{I_h}) \right] \mathbb{E} \left(\sqrt{(t+h - \tau_0)^{2D} - (t - \tau_0)^{2D}} \right) \right\}. \end{aligned} \quad (\text{B.3.19})$$

Since $\delta^{-1}((\delta + x)^{2D} - x^{2D}) \uparrow 2Dx^{2D-1}$ as $\delta \downarrow 0$, by monotone convergence we obtain

$$\begin{aligned} & \lim_{h \downarrow 0} \frac{1}{h} \text{Cov} \left(\sqrt{c} \sqrt{(h - \tau_0)^{2D} - (-\tau_0)^{2D}}, \sqrt{c} \sqrt{(t + h - \tau_0)^{2D} - (t - \tau_0)^{2D}} \right) \\ &= 2Dc \text{Cov} \left((-\tau_0)^{D-1/2}, (t - \tau_0)^{D-1/2} \right) \\ &= 2D\lambda^{1-2D} c \text{Cov} \left(S^{D-1/2}, (\lambda t + S)^{D-1/2} \right) = 2D\lambda^{1-2D} c \phi(\lambda t), \end{aligned} \quad (\text{B.3.20})$$

with $S := \lambda(-\tau_0) \sim \text{Exp}(1)$ and ϕ is defined in (B.1.7). Similarly

$$\lim_{h \downarrow 0} \frac{1}{\sqrt{h}} \mathbb{E} \left(\sqrt{c} \sqrt{(t + h - \tau_0)^{2D} - (t - \tau_0)^{2D}} \right) = \sqrt{2D} \mathbb{E} \left(\sqrt{c} (t - \tau_0)^{D-1/2} \right) < +\infty. \quad (\text{B.3.21})$$

Therefore, if we show that

$$\lim_{h \downarrow 0} \mathbb{E} \left(\sqrt{\frac{I_h}{h}} \right) = \sqrt{2D} \mathbb{E} \left(\sqrt{c} (-\tau_0)^{D-1/2} \right) \quad (\text{B.3.22})$$

using (B.3.19), (B.3.20), (B.3.21), we have

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E} \left[\left(\sqrt{I_h} - \mathbb{E} \sqrt{I_h} \right) \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (0, t+h] = \emptyset\}} \right] = 2D\lambda^{1-2D} \phi(\lambda t) \quad (\text{B.3.23})$$

To complete the proof of (B.3.23), we are left to show (B.3.22). But this is a nearly immediate consequence of Theorem B.1.2: indeed, using the fact that $q^* > 1$,

$$\mathbb{E} \sqrt{I_h} = \frac{1}{\mathbb{E} \sqrt{|W_1|}} \mathbb{E} (|X_h|) = \frac{C_1}{\mathbb{E} |W_1|} \sqrt{h} + o(\sqrt{h}) = \sqrt{2D} \mathbb{E} \left(\sqrt{c} (-\tau_0)^{D-1/2} \right) \sqrt{h} + o(\sqrt{h}).$$

The proof is now completed if we show that the second term in (B.3.18) is negligible, i.e. it is $o(h)$. By Cauchy-Schwarz inequality and the simple fact that $(\sqrt{I_h} - \mathbb{E} \sqrt{I_h})^2 \leq I_h + \mathbb{E}(I_h)$

$$\begin{aligned} & \mathbb{E} \left[\left(\sqrt{I_h} - \mathbb{E} \sqrt{I_h} \right) \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (h, t] = \emptyset\}} \mathbf{1}_{\{\mathcal{T} \cap ((0, h] \cup (t, t+h]) \neq \emptyset\}} \right] \\ & \leq \left(\mathbb{E} \left[\left(\sqrt{I_h} - \mathbb{E} \sqrt{I_h} \right)^2 (I_{t+h} - I_t) \right] \mathbb{P}(\mathcal{T} \cap ((0, h] \cup (t, t+h]) \neq \emptyset) \right)^{1/2} \\ & \leq (\mathbb{E} [(I_h + \mathbb{E}(I_h)) (I_{t+h} - I_t)] \mathbb{P}(\mathcal{T} \cap ((0, h] \cup (t, t+h]) \neq \emptyset))^{1/2} \\ & \leq (2 \mathbb{E} [I_h^2] \mathbb{P}(\mathcal{T} \cap ((0, h] \cup (t, t+h]) \neq \emptyset))^{1/2} = (2 \mathbb{E} [I_h^2])^{1/2} \sqrt{2\lambda h}. \end{aligned} \quad (\text{B.3.24})$$

By Theorem B.1.2, $\mathbb{E} [I_h^2]$ is of order h^2 if $4 < q^*$, and of order h^{4D+1} if $4 > q^*$, with a logarithmic correction for $q^* = 4$. In both cases $(\mathbb{E} [I_h^2])^{1/2} \sqrt{2\lambda h} = o(h)$, and the proof is completed. \square

B.4 Proof in the non-stationary setting

B.4.1 Diffusive scaling: proof of Theorem B.2.1

By (B.0.1) we can write

$$Y_{t+h} - Y_t = W_{I_{t+h}} - W_{I_t} \stackrel{d}{=} \sqrt{I_{t+h} - I_t} W_1 \quad (\text{B.4.1})$$

We know that $P(N_{t+h} - N_t \geq 1) = 1 - e^{-\lambda h} = O(h) \rightarrow 0$ as $h \downarrow 0$, so we can focus on the event $\{N_{t+h} = N_t\}$, on which we have

$$\lim_{h \downarrow 0} \frac{I_{t+h} - I_t}{h} = I'_t = \frac{V^2 \lambda^{2D-1}}{\Gamma(2D)} (t - \tau_{N_t})^{2D-1}.$$

Note that $(t - \tau_{N_t}) \mathbf{1}_{\{N(t) \geq 1\}}$ is distributed like $(\lambda^{-1} E) \mathbf{1}_{\{(\lambda^{-1} E) \leq t\}}$, with $E \sim \text{Exp}(1)$, while $(t - \tau_{N_t}) \mathbf{1}_{\{N(t)=0\}}$ is distributed like $(t - \tau_0) \mathbf{1}_{\{(\lambda^{-1} E) > t\}}$. In conclusion,

$$(t - \tau_{N_t}) \mathbf{1}_{\{N_t \geq 1\}} \stackrel{d}{=} \lambda^{-1} S_{\lambda t, \lambda \tau_0}, \quad (\text{B.4.2})$$

where we set $S_{b,a} := (b - a) \mathbf{1}_{\{E > b\}} + E \mathbf{1}_{\{E \leq b\}}$, as in the statement of Theorem B.2.1.

We now focus on the case $h \uparrow \infty$. We have to study the convergence of

$$\frac{Y_{t+h} - Y_t}{\sqrt{h}} = \frac{W_{I_{t+h}} - W_{I_t}}{\sqrt{h}} \quad (\text{B.4.3})$$

when $h \uparrow \infty$. We have

$$I_{t+h} - I_t = c \left((t + h - \tau_{N_{t+h}})^{2D} + (\tau_{N_{t+1}} - \tau_{N_t})^{2D} - (t - \tau_{N_t})^{2D} + \sum_{k=N_t+2}^{N_{t+h}} (\tau_k - \tau_{k-1})^{2D} \right).$$

The random variables $((\tau_k - \tau_{k-1})^{2D})_{k \geq N_t+2}$ are independent and identically distributed with finite mean, hence by the strong law of large numbers (since $\frac{\tau_{N_t+1} - \tau_{N_t}}{n} \rightarrow 0$, $\frac{t - \tau_{N_t}}{n} \rightarrow 0$ and $\frac{t+h - \tau_{N_{t+h}}}{n} \rightarrow 0$ a.s.)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\tau_k - \tau_{k-1})^{2D} = E[(\tau_1)^{2D}] = \lambda^{-2D} \Gamma(2D + 1) \quad \text{a.s.}$$

Plainly, $\lim_{h \rightarrow +\infty} N_h/h = \lambda$ a.s., by the strong law of large numbers applied to the random variables $\{\tau_k\}_{k \geq 1}$. Recalling (B.0.2), it follows easily that

$$\lim_{h \uparrow \infty} \frac{I_{t+h} - I_t}{h} = V^2 \quad \text{a.s.}$$

Since $W_{I_{t+h}} - W_{I_t} \stackrel{d}{=} \sqrt{I_{t+h} - I_t} W_1$ n

$$\frac{W_{I_{t+h}} - W_{I_t}}{\sqrt{h}} \xrightarrow{d} V W_1 \quad \text{as } h \uparrow \infty.$$

□

B.4.2 Multiscaling: proof of Theorem B.2.3

As in the proof of the stationary case we can write

$$E(|X_{t+h} - X_t|^q) = E(|I_{t+h} - I_t|^{q/2} |W_1|^q) = E(|W_1|^q) E(|I_{t+h} - I_t|^{q/2}) = c_q E(|I_{t+h} - I_t|^{q/2}), \quad (\text{B.4.4})$$

where we set $c_q := \mathbb{E}(|W_1|^q)$. We therefore focus on $\mathbb{E}(|I_{t+h} - I_t|^{\frac{q}{2}})$, that we write as the sum of three terms, that will be analyzed separately:

$$\begin{aligned} \mathbb{E}(|I_{t+h} - I_t|^{\frac{q}{2}}) &= \mathbb{E}(|I_{t+h} - I_t|^{\frac{q}{2}} \mathbf{1}_{\{N_t=N_{t+h}\}}) \\ &\quad + \mathbb{E}(|I_{t+h} - I_t|^{\frac{q}{2}} \mathbf{1}_{\{N_{t+h}=N_t+1\}}) + \mathbb{E}(|I_{t+h} - I_t|^{\frac{q}{2}} \mathbf{1}_{\{N_{t+h}-N_t \geq 2\}}). \end{aligned} \quad (\text{B.4.5})$$

The analysis of the second and third term is exactly the same as the one in the stationary setting, in particular the second term gives contribution $\frac{1}{Dq+1} h^{A(q)}$ only when $q > q^*$ (while is negligible if $q \leq q^*$), and the third is always negligible.

We can focus on the first term and we can rewrite it in the following way

$$\mathbb{E}(|I_{t+h} - I_t|^{\frac{q}{2}} \mathbf{1}_{\{N_{t+h}=N_t\}}) = \mathbb{E}(|I_{t+h} - I_t|^{\frac{q}{2}} \mathbf{1}_{\{N_{t+h}=0\}}) + \mathbb{E}(|I_{t+h} - I_t|^{\frac{q}{2}} \mathbf{1}_{\{N_{t+h}=N_t \geq 1\}}) \quad (\text{B.4.6})$$

The first term is

$$\mathbb{E}(|I_{t+h} - I_t|^{\frac{q}{2}} \mathbf{1}_{\{N_{t+h}=0\}}) = c^{\frac{q}{2}} ((t+h-\tau_0)^{2D} - (t-\tau_0)^{2D})^{\frac{q}{2}} \mathbb{P}(N_{t+h}=0) \quad (\text{B.4.7})$$

and so is asymptotic to

$$\mathbb{E}(|I_{t+h} - I_t|^{\frac{q}{2}} \mathbf{1}_{\{N_{t+h}=0\}}) \sim c^{\frac{q}{2}} (t-\tau_0)^{q(D-\frac{1}{2})} (2D)^{\frac{q}{2}} h^{\frac{q}{2}} e^{-\lambda t}, \quad (\text{B.4.8})$$

independently on the value of q . In particular it will be relevant only if $q < q^*$.

On the other hand, the second term is, since $t - \tau_{N_t} \sim \text{Exp}(\lambda) \wedge t$

$$\mathbb{E}(|I_{t+h} - I_t|^{\frac{q}{2}} \mathbf{1}_{\{N_{t+h}=N_t \geq 1\}}) = c^{\frac{q}{2}} \lambda^{-Dq} \mathbb{E}(((S+\lambda h)^{2D} - S^{2D})^{\frac{q}{2}}) (1-e^{-\lambda t}) (1+o(1)). \quad (\text{B.4.9})$$

where $S = \text{Exp}(1) \wedge \lambda t$ (and in particular its distribution function is $f(x) = \frac{e^{-x}}{1-e^{-\lambda t}}$). Observe that if $q < q^*$ using the same arguments as in the stationary case we obtain

$$\text{for } q < q^* : \quad \lim_{h \downarrow 0} \frac{\mathbb{E}(((S+\lambda h)^{2D} - S^{2D})^{\frac{q}{2}})}{\lambda^{\frac{q}{2}} h^{\frac{q}{2}}} = (2D)^{q/2} \frac{1}{1-e^{-\lambda t}} \gamma(1-q/q^*, \lambda t) \in (0, \infty). \quad (\text{B.4.10})$$

which together with (B.4.6) gives the thesis.

On the other hand observe that, by the change of variables $s = (\lambda h)x$, we can write

$$\begin{aligned} (1-e^{-\lambda t}) \mathbb{E}(((S+\lambda h)^{2D} - S^{2D})^{\frac{q}{2}}) &= \int_0^{\lambda t} ((s+\lambda h)^{2D} - s^{2D})^{\frac{q}{2}} e^{-s} ds \\ &= (\lambda h)^{Dq+1} \int_0^{\frac{t}{h}} ((1+x)^{2D} - x^{2D})^{\frac{q}{2}} e^{-\lambda h x} dx \\ &= (\lambda h)^{Dq+1} \int_0^{\infty} ((1+x)^{2D} - x^{2D})^{\frac{q}{2}} e^{-\lambda h x} \mathbf{1}_{\{x < \frac{t}{h}\}} dx. \end{aligned} \quad (\text{B.4.11})$$

Note that $((1+x)^{2D} - x^{2D})^{\frac{q}{2}} \sim (2D)^{\frac{q}{2}} x^{(D-\frac{1}{2})q}$ as $x \rightarrow +\infty$ and that $(D-\frac{1}{2})q < -1$ if and only if $q > q^*$. Therefore, by the monotone convergence theorem, we obtain

$$\text{for } q > q^* : \quad \lim_{h \downarrow 0} \frac{\mathbb{E}(((S+\lambda h)^{2D} - S^{2D})^{\frac{q}{2}})}{\lambda^{Dq+1} h^{Dq+1}} = \int_0^{\infty} ((1+x)^{2D} - x^{2D})^{\frac{q}{2}} dx \in (0, \infty). \quad (\text{B.4.12})$$

Finally, in the case $q = q^*$ we have $((1+x)^{2D} - x^{2D})^{q^*/2} \sim (2D)^{q^*/2} x^{-1}$ as $x \rightarrow +\infty$ and we want to study the integral in the second line of (B.4.11). Fix an arbitrary (large) $M > 0$ and note that as $h \downarrow 0$ $M < \frac{t}{h}$, so we can write

$$\int_M^{\frac{t}{h}} \frac{e^{-\lambda hx}}{x} dx = \int_M^\infty \frac{e^{-\lambda hx}}{x} dx - \int_{\frac{t}{h}}^\infty \frac{e^{-\lambda hx}}{x} dx$$

Now, integrating by parts and performing a change of variables, as $h \downarrow 0$ the first integral becomes

$$\begin{aligned} \int_M^\infty \frac{e^{-\lambda hx}}{x} dx &= -\log M e^{-\lambda h M} + \lambda h \int_M^\infty (\log x) e^{-\lambda hx} dx = O(1) + \int_{\lambda h M}^\infty \log\left(\frac{y}{\lambda h}\right) e^{-y} dy \\ &= O(1) + \int_{\lambda h M}^\infty \log\left(\frac{y}{\lambda}\right) e^{-y} dy + \log\left(\frac{1}{h}\right) \int_{\lambda h M}^\infty e^{-y} dy = \log\left(\frac{1}{h}\right) (1 + o(1)). \end{aligned}$$

and in the same way the second becomes

$$\begin{aligned} \int_{t/h}^\infty \frac{e^{-\lambda hx}}{x} dx &= -\log \frac{t}{h} e^{-\lambda t} + \lambda h \int_{t/h}^\infty (\log x) e^{-\lambda hx} dx \\ &= \int_{\lambda t}^\infty \log\left(\frac{y}{\lambda h}\right) e^{-y} dy - \log \frac{1}{h} e^{-\lambda t} - \log t e^{-\lambda t} \\ &= + \int_{\lambda t}^\infty \log\left(\frac{y}{\lambda}\right) e^{-y} dy + \log\left(\frac{1}{h}\right) \int_{\lambda t}^\infty e^{-y} dy \log \frac{1}{h} e^{-\lambda t} - \log t e^{-\lambda t} \\ &= \log\left(\frac{1}{h}\right) e^{-\lambda t} + O(1) - \log \frac{1}{h} e^{\lambda t} + O(1) = O(1). \end{aligned}$$

From this it is easy to see that as $h \downarrow 0$

$$\int_0^\infty ((1+x)^{2D} - x^{2D})^{\frac{q^*}{2}} e^{-\lambda hx} \mathbf{1}_{\{x < \frac{t}{h}\}} dx \sim (2D)^{\frac{q^*}{2}} \log\left(\frac{1}{h}\right).$$

and so the thesis follows.

B.4.3 Decay of correlations: proof of Theorem B.2.5

The proof of the decay of correlation is also similar to the one in the stationary case: we can use the same arguments, with just the change $I_h \rightsquigarrow I_{s+h} - I_s$, $\mathcal{T} \cap (h, t] \rightsquigarrow \mathcal{T} \cap (s+h, t]$, $\mathcal{T} \cap (0, t] \rightsquigarrow \mathcal{T} \cap (s, t]$ and $\mathcal{T} \cap ((0, h] \cup (t, t+h]) \rightsquigarrow \mathcal{T} \cap ((s, s+h] \cup (t, t+h])$ until equation (B.3.18), where again we have two terms, with whom we will deal separately.

$$\begin{aligned} & Cov(\sqrt{I_{s+h} - I_s}, \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (s+h, t] = \emptyset\}}) \\ &= \mathbb{E} \left[\left(\sqrt{I_{s+h} - I_s} - \mathbb{E}(\sqrt{I_h}) \right) \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (s+h, t] = \emptyset\}} \right] \\ &= \mathbb{E} \left[\left(\sqrt{I_{s+h} - I_s} - \mathbb{E}(\sqrt{I_{s+h} - I_s}) \right) \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (s, t+h] = \emptyset\}} \right] \\ &+ \mathbb{E} \left[\left(\sqrt{I_{s+h} - I_s} - \mathbb{E}(\sqrt{I_{s+h} - I_s}) \right) \sqrt{I_{t+h} - I_t} \mathbf{1}_{\{\mathcal{T} \cap (s+h, t] = \emptyset\}} \mathbf{1}_{\{\mathcal{T} \cap ((s, h] \cup (t, t+h]) \neq \emptyset\}} \right] \end{aligned} \tag{B.4.13}$$

Again, using the same arguments as in the original proof, we can prove that the second term in the r.h.s of (B.4.13) is negligible. On the other hand we observe that on $\{\mathcal{T} \cap (s+h, t] = \emptyset\}$

$$I_{s+h} - I_s = c[(s+h - \tau_{N_s})^{2D} - (s - \tau_{N_s})^{2D}] = \star_s$$

and

$$I_{t+h} - I_t = c[(t+h - \tau_{N_s})^{2D} - (t - \tau_{N_s})^{2D}] = \star_t$$

and since both \star_s and \star_t are independent of $\{\mathcal{T} \cap (s+h, t] = \emptyset\}$, we have

$$\begin{aligned} & \mathbb{E} \left[(\sqrt{\star_s} - \mathbb{E}(\sqrt{I_{s+h} - I_s}) \sqrt{\star_t} \mathbf{1}_{\{\mathcal{T} \cap (s+h, t] = \emptyset\}}) \right] \\ &= e^{-\lambda(t+h-s)} \mathbb{E} \left[(\sqrt{\star_s} - \mathbb{E}(\sqrt{I_{s+h} - I_s}) \sqrt{\star_t}) \right] \\ &= e^{-\lambda(t+h-s)} \mathbb{E} \left[(\sqrt{\star_s} - \mathbb{E}(\sqrt{I_{s+h} - I_s}) \sqrt{\star_t} (\mathbf{1}_{N_s=0} + \mathbf{1}_{N_s>0})) \right] \end{aligned} \quad (\text{B.4.14})$$

We study the two cases separately: if $N_t = 0$ we have

$$\begin{aligned} & \mathbb{E} \left[(\sqrt{\star_s} - \mathbb{E}(\sqrt{I_{s+h} - I_s}) \sqrt{\star_t} \mathbf{1}_{N_s=0}) \right] \\ &= e^{-\lambda s} c \left[\sqrt{(s+h - \tau_0)^{2D} - (s - \tau_0)^{2D}} \sqrt{(t+h - \tau_0)^{2D} - (t - \tau_0)^{2D}} \right] \\ & \quad - \sqrt{c(t+h - \tau_0)^{2D} - (t - \tau_0)^{2D}} \mathbb{E}[\sqrt{I_{s+h} - I_s}]. \end{aligned} \quad (\text{B.4.15})$$

Since $\delta^{-1}((\delta + x)^{2D} - x^{2D}) \uparrow 2Dx^{2D-1}$ as $\delta \downarrow 0$, by monotone convergence and using the results in (B.2.3) we obtain

$$\lim_{h \rightarrow 0} \frac{1}{h} (\text{B.4.15}) = hc 2D e^{-\lambda s} (t - \tau_0)^{D-\frac{1}{2}} \left((1 - e^{-\lambda s}) (s - \tau_0)^{D-\frac{1}{2}} - \lambda^{\frac{1}{2}-D} \gamma \left(\frac{1}{2} + D, \lambda s, \right) \right) \quad (\text{B.4.16})$$

where as before γ denotes the lower incomplete Gamma function.

On the other hand if $N_s > 0$ we have

$$\begin{aligned} & \mathbb{E} \left[(\sqrt{\star_s} - \mathbb{E}(\sqrt{I_{s+h} - I_s}) \sqrt{\star_t} \mathbf{1}_{N_s>0}) \right] \\ &= (1 - e^{-\lambda s}) \left[\text{Cov} \left(\sqrt{\star_s}, \sqrt{\star_t} \right) + \mathbb{E} \left[\sqrt{\star_s} - \mathbb{E}(\sqrt{I_{s+h} - I_s}) \right] \mathbb{E}(\sqrt{\star_t}) \right] \end{aligned} \quad (\text{B.4.17})$$

where in this case $s - \tau_{N_s} \sim \text{Exp}(\lambda) \wedge s$. Using the usual asymptotics $(\delta + x)^{2D} - x^{2D} \uparrow 2Dx^{2D-1}$ as $\delta \downarrow 0$, we obtain

$$\begin{aligned} & c \mathbb{E} \left[\sqrt{(s+h - \tau_{N_s})^{2D} - (s - \tau_{N_s})^{2D}} \right] \mathbb{E} \left(\sqrt{(t+h - \tau_{N_s})^{2D} - (t - \tau_{N_s})^{2D}} \right) \\ & \sim c(2D) h \mathbb{E}[(s - \tau_{N_s})^{D-\frac{1}{2}}] \mathbb{E}[(t - \tau_{N_s})^{D-\frac{1}{2}}] \\ & = c(2D) h \mathbb{E}[(S)^{D-\frac{1}{2}}] \mathbb{E}[(\lambda(t-s) + S)^{D-\frac{1}{2}}] \lambda^{1-2D} \end{aligned} \quad (\text{B.4.18})$$

where $S \sim Exp(1) \wedge \lambda s$ (and so in particular $E[(S)^{D-\frac{1}{2}}] = \frac{1}{1-e^{-\lambda s}} \gamma\left(\frac{1}{2} + D, \lambda s\right)$). Moreover we have

$$\begin{aligned} & E(\sqrt{I_{s+h} - I_s}) E(\sqrt{c((t+h - \tau_{N_s})^{2D} - (t - \tau_{N_s})^{2D})}) \\ & \sim c(2D)h E[(\lambda(t-s) + S)^{D-\frac{1}{2}}] \lambda^{\frac{1}{2}-D} \left(e^{-\lambda s} (s - \tau_0)^{D-\frac{1}{2}} + \lambda^{\frac{1}{2}-D} \gamma\left(\frac{1}{2} + D, \lambda s\right) \right), \end{aligned} \quad (\text{B.4.19})$$

which combined with (B.4.18) gives

$$\begin{aligned} & E \left[\sqrt{c((s+h - \tau_{N_s})^{2D} - (s - \tau_{N_s})^{2D})} - E(\sqrt{I_{s+h} - I_s}) \right] E(\sqrt{c((t+h - \tau_{N_s})^{2D} - (t - \tau_{N_s})^{2D})}) \\ & \sim c 2D h E[(\lambda(t-s) + S)^{D-\frac{1}{2}}] \lambda^{1-2D} e^{-\lambda s} \left(\frac{1}{1-e^{-\lambda s}} \gamma\left(\frac{1}{2} + D, \lambda s\right) - (s - \tau_0)^{D-\frac{1}{2}} \lambda^{D-\frac{1}{2}} \right). \end{aligned} \quad (\text{B.4.20})$$

Finally the covariance term gives

$$\begin{aligned} & \lim_{h \downarrow 0} \frac{1}{h} Cov \left(\sqrt{c} \sqrt{(s+h - \tau_{N_s})^{2D} - (s - \tau_{N_s})^{2D}}, \sqrt{c} \sqrt{(t+h - \tau_{N_s})^{2D} - (t - \tau_{N_s})^{2D}} \right) \\ & = c 2D Cov \left((s - \tau_{N_s})^{D-1/2}, (t - \tau_{N_s})^{D-1/2} \right) \\ & = c 2D \lambda^{1-2D} Cov \left(S^{D-1/2}, (\lambda(t-s) + S)^{D-1/2} \right) \end{aligned} \quad (\text{B.4.21})$$

where, as usual $S \sim Exp(1) \wedge \lambda s$.

If now we combine everything together we obtain

$$\begin{aligned} (\text{B.4.14}) & \sim h e^{-\lambda(t-s)} 2D c \left\{ e^{-\lambda s} (t - \tau_0)^{D-\frac{1}{2}} (s - \tau_0)^{D-\frac{1}{2}} (1 - e^{-\lambda s}) \right. \\ & \quad - e^{-\lambda s} (t - \tau_0)^{D-\frac{1}{2}} \lambda^{\frac{1}{2}-D} \gamma\left(\frac{1}{2} + D, \lambda s\right) \\ & \quad - e^{-\lambda s} E[(\lambda(t-s) + S)^{D-\frac{1}{2}}] \lambda^{\frac{1}{2}-D} (s - \tau_0)^{D-\frac{1}{2}} \\ & \quad + e^{-\lambda s} \lambda^{1-2D} E[(\lambda(t-s) + S)^{D-\frac{1}{2}}] \gamma\left(\frac{1}{2} + D, \lambda s\right) \\ & \quad \left. + (1 - e^{-\lambda s}) \lambda^{1-2D} Cov \left(S^{D-1/2}, (\lambda(t-s) + S)^{D-1/2} \right) \right\} \\ & = h e^{-\lambda(t-s)} 2D c e^{-\lambda s} \left\{ (s - \tau_0)^{D-\frac{1}{2}} \left((t - \tau_0)^{D-\frac{1}{2}} - E[(\lambda(t-s) + S)^{D-\frac{1}{2}}] \lambda^{\frac{1}{2}-D} \right) (1 - e^{-\lambda s}) \right. \\ & \quad \left. + \gamma\left(\frac{1}{2} + D, \lambda s\right) \left(E[(\lambda(t-s) + S)^{D-\frac{1}{2}}] \lambda^{1-2D} - (t - \tau_0)^{D-\frac{1}{2}} \lambda^{\frac{1}{2}-D} \right) \right\} \\ & \quad + h e^{-\lambda(t-s)} 2D c \lambda^{1-2D} Cov \left(S^{D-1/2}, (\lambda(t-s) + S)^{D-1/2} \right) (1 - e^{-\lambda s}) \\ & = h e^{-\lambda(t-s)} 2D c e^{-\lambda s} \{(1) + (2)\} + h e^{-\lambda(t-s)} 2D c (3) \end{aligned} \quad (\text{B.4.22})$$

Observe that

$$e^{-\lambda s} \{(1+)(2)\} = F(t, s)$$

and

$$(3) = \phi_{\lambda s}(\lambda(t - s))$$

where ϕ and F were defined respectively in (B.2.6) and (B.2.7).

The only difference between (B.4.22) and (B.2.5) is a $\frac{2}{\pi}$ factor which is due to the fact that

$$\text{Cov}(|Y_{s+h} - Y_s|, |Y_{t+h} - Y_t|) = \frac{2}{\pi} \text{Cov}(\sqrt{I_{s+h} - I_s}, \sqrt{I_{t+h} - I_t}),$$

which concludes the proof. \square

Remark B.4.1. From equation (B.4.22) we notice two facts:

- if $s \downarrow 0$ then $\frac{(B.4.22)}{h} \rightarrow 0$, since (1) and (3) go to 0 because appears a $(1 - e^{-\lambda s})$ factor in both of them, while (2) goes also to 0 because $\gamma(\alpha, \delta) \rightarrow 0$ when $\delta \downarrow 0$;
- if $s \uparrow \infty$ we get the same result as in the stationary case: in fact F vanishes thanks to the $e^{-\lambda s}$ factor, while the random variable $S \sim \text{Exp}(1) \wedge \lambda t$ converges to $S' \sim \text{Exp}(1)$ and, by dominate convergence

$$\text{Cov}(S^{D-\frac{1}{2}}, (\lambda(t - s) + S)^{D-\frac{1}{2}}) \rightarrow \text{Cov}((S')^{D-\frac{1}{2}}, (\lambda(t - s) + S')^{D-\frac{1}{2}})$$

Bibliography

- [AL12] L. Andersen and A. Lipton, *Asymptotics for exponential Lévy processes and their volatility smile: Survey and new results.*, preprint (2012), arXiv.org:1206.6787v1.
- [ACDP12] A. Andreoli, F. Caravenna, P. Dai Pra, G. Posta. *Scaling and multiscaling in financial series: a simple model.* Adv. in Appl. Probab. **44** (2012), 1018-1051.
- [AP07] L. B. G. Andersen and V. V. Piterbarg, *Moment explosions in stochastic volatility models*, Finance Stoch. (2007), 29–50.
- [AFZ13] J. Armstrong, M. Forde and H. Zhang, *Small-time asymptotics for a general local-stochastic volatility model with a jump-to-default: curvature and the heat kernel expansion*, preprint (2013), arXiv:1312.2281
- [BKS82] O. Barndorff-Nielsen, J. Kent and M. Sørensen, *Normal variance-mean mixtures and z distributions*, International Statistical Review/Revue Internationale de Statistique (1982): 145-159.
- [Bar97] O.E. Barndorff-Nielsen, *Processes of normal inverse Gaussian type*, Finance and stochastics 2.1 (1997): 41-68.
- [BF08] S. Benaim and P. Friz. *Smile asymptotics II: models with known moment generating function*, J. Appl. Probab. **45** (2008), 16–32.
- [BF09] S. Benaim and P. Friz, *Regular variation and smile asymptotics* Mathematical Finance, 19 (2009): 1–12.
- [BFL08] S. Benaim, P. Friz, and R. Lee, *On Black-Scholes implied volatility at extreme strikes*, Frontiers in Quantitative Finance: Volatility and Credit Risk Modeling (2008): 19-45.
- [BBF02] H. Berestycki, J. Busca, and I. Florent, *Asymptotics and calibration of local volatility models*, Quantitative finance 2.1 (2002): 61-69.
- [BBF04] H. Berestycki, J. Busca, and I. Florent, *Computing the implied volatility in stochastic volatility models*, Communications on Pure and Applied Mathematics 57.10 (2004): 1352-1373.
- [BGT89] N. H. Bingham, C. H. Goldie and J. L. Teugels, *Regular variation*, Cambridge University Press (1989).

- [CW04] P. Carr and L. Wu, *The Finite Moment Log Stable Process and Option Pricing*, J. Finance 58 (2004), 753–778.
- [DG11] J. Da Fonseca and M. Grasselli, *Riding on the smiles*, Quantitative Finance 11.11 (2011): 1609-1632.
- [DP14] P. Dai Pra and P. Pigato, *Multi-scaling of moments in stochastic volatility models*, preprint (2014), arXiv.org:1403.7387.
- [DHJ13] S. De Marco, C. Hillairet and A. Jacquier, *Shapes of implied volatility with positive mass at zero*, preprint(2013), arXiv:1310.1020 .
- [DZ98] A. Dembo and O. Zeitouni, *Large deviations techniques and applications*, Second Edition, Springer–Verlag, New York, 1998.
- [DFJV14] J.D. Deuschel, P.K. Friz, A. Jacquier and S. Violante, *Marginal density expansions for diffusions and stochastic volatility, Part II: Applications*, Communications on Pure and Applied Mathematics, 67(2) (2014), 321–350
- [EM85] P. Embrechts et al. *Approximations for compound Poisson and Pólya processes*, Advances in Applied Probability (1985): 623-637.
- [FJ09] M. Forde and A. Jacquier, *Small-time asymptotics for implied volatility under the Heston model*, Int J. Theor and Appl. Finance **12** (2009), 861–876.
- [FJ11] M. Forde and A. Jacquier, *The large-maturity smile for the Heston model*, Finance & Stochastics, 15 (4): 755-780, 2011.
- [FJ11b] M. Forde and A. Jacquier, *Small-time asymptotics for an uncorrelated local-stochastic volatility model*, Applied Mathematical Finance 18.6 (2011): 517-535.
- [FJL12] M. Forde, A. Jacquier and R. Lee, *The small-time smile and term structure of implied volatility under the Heston model* SIAM Journal on Financial Mathematics, 3 (1): 690-708, 2012.
- [FJM10] M. Forde, A. Jacquier and A. Mijatović *Asymptotic formulae for implied volatility in the Heston model*, Proceedings of the Royal Society A, 466 (2124): 3593-3620, 2010.
- [FGGS11] P. Friz, S. Gerhold, A. Gulisashvili and S. Sturm, *Refined implied volatility expansions in the Heston model*, Quantitative Finance, 11 (8): 1151-1164, 2011.
- [FGY14] P. Friz, S. Gerhold and M. Yor (2014), *How to make Dupire’s local volatility work with jumps* Quantitative Finance, DOI: 10.1080/14697688.2013.874622
- [FK09] P. Friz and M. Keller-Ressel, *Moment explosions in stochastic volatility models*, Encyclopedia of Quantitative Finance (2009).
- [GL14] K. Gao, R. Lee. *Asymptotics of implied volatility to arbitrary order*, Finance Stoch. **18** (2014), 349–392.
- [Gat06] J. Gatheral. *The Volatility Surface: A Practitioner’s Guide*. Wiley (2006).

- [GHLOW12] J. Gatheral, E.P. Hsu, P. Laurence, C. Ouyang and T-H. Wong, *Asymptotics of implied volatility in local volatility models*, Mathematical Finance, 22 591-620, 2012
- [GG13] S. Gerhold and I.C. Gulum, *The small maturity implied volatility slope for Lévy models*, preprint (2013) arXiv:1310.3061
- [GMZ14] S. Gerhold, J. F. Morgenbesser and A. Zrunek (2014), *Refined wing asymptotics for the Merton and Kou jump diffusion models*, to appear on Banach Center Publications, arXiv.org: 1401.1954.
- [Gu10] A. Gulisashvili, *Asymptotic formulas with error estimates for call pricing functions and the implied volatility at extreme strikes*, SIAM Journal on Financial Mathematics 1.1 (2010): 609-641.
- [Gu12] A. Gulisashvili, *Asymptotic equivalence in Lee's moment formulas for the implied volatility and Piterbarg's conjecture*, International Journal of Theoretical and Applied Finance, 15, 1250020 (2012)
- [HW87] J. Hull, A. White. *The Pricing of Options on Assets with Stochastic Volatilities*. J. Finance **XLII**, no. 2, June 1987.
- [Hes93] S. L. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of financial studies 6.2 (1993): 327-343.
- [JKM13] A. Jacquier, M. Keller-Ressel and A. Mijatović *Large deviations and stochastic volatility with jumps: asymptotic implied volatility for affine models*, Stochastics An International Journal of Probability and Stochastic Processes 85.2 (2013): 321-345.
- [JR13] A. Jacquier and P. Roome, *The small-maturity Heston forward smile* SIAM Journal on Financial Mathematics 4.1 (2013): 831-856.
- [JO13] P. Junod and R. Oeuvray, *On time scaling of semivariance in a jump-diffusion process*, arXiv preprint arXiv:1311.1122 (2013).
- [Ke11] M. Keller-Ressel, *Moment Explosions and Long-Term Behavior of Affine Stochastic Volatility Models*, Mathematical Finance, 21 (1): 73-98, 2011.
- [Le04] R. W. Lee, *The moment formula for implied volatility at extreme strikes*, Mathematical Finance 14.3 (2004): 469-480.
- [Le05] R. W. Lee, *Implied volatility: Statics, dynamics, and probabilistic interpretation*, Recent advances in applied probability. Springer US, 2005. 241-268.
- [Ma04] K. Matsuda, *Introduction to merton jump diffusion model*, Department of Economics. The Graduate Center, The City University of New York (2004).
- [M76] R. C. Merton (1976), *Option pricing when underlying stock returns are discontinuous*, J. Financial Economics **3**, 125-144.

- [Mi11] S. Mitra, *A review of volatility and option pricing*, International Journal of Financial Markets and Derivatives 2.3 (2011): 149-179.
- [P01] I. Pinelis, *Monotonicity Properties of the Relative Error of a Padé Approximation for Mills' Ratio*, J. Inequal. Pure Appl. Math. **3**(2) (2002), article 20.
- [RR09] M. Roper and M. Rutkowski (2009), *On the relationship between the call price surface and the implied volatility surface close to expiry*, Int. J. Theor. Appl. Finance **12**, 427-441.
- [RT96] E. Renault and N. Touzi, *Option hedging and implied volatilities in a stochastic volatility model*, Math. Finance **6** (1996), 279-302.
- [S54] L.R. Shenton, *Inequalities for the normal integral including a new continued fraction*, Biometrika **41** (1954), 177-189.
- [Ta10] P. Tankov, *Pricing and hedging in exponential Lévy models: review of recent results*, Paris-Princeton Lecture Notes in Mathematical Finance, Springer, 2010.
- [Te09] M. Tehranchi, *Asymptotics of implied volatility far from maturity*, J. Appl. Probab. **46** (2009), 629-650.