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Parabolic operators with unbounded coefficients with applications to stochastic optimal control and differential games

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“But beauty, real beauty, begins where an intellectual expression ends”

The picture of Dorian Gray, Oscar Wilde

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Contents

Chapter 1

Introduction

The aim of this thesis consists in providing new results for parabolic Cauchy problems with possible unbounded coefficients, and in solving stochastic problems throughout analytic techniques.

The Chapters 2 and 3 are brief overviews of the main recent results on the theory of partial differential equations and systems, and on the theory of stochastic optimal control problems.

Our investigation begins from Chapter 4. Here, we consider the stochastic control problem

$$\begin{cases} d_\tau X_\tau^u = B(X_\tau^u)d\tau + G(X_\tau^u)r(X_\tau^u, u_\tau)d\tau + G(X_\tau^u)dW_\tau, \quad \tau \in [t,T], \\ \\ X_t^u = x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where B, G are possibly unbounded measurable functions, G is a uniformly positive definite $d \times d-$matrix, r is a measurable bounded function and W is an $\mathbb{R}^d-$valued Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The random process u is called *control*, and for any fixed u, X^u denotes the solution to (1.1) under the control u. For this class of problems we look for a control u such that the cost functional

$$J(x,u) := \mathbb{E}\int_0^T l(X_t^u, u_t)dt + \mathbb{E}\varphi(X_T^u), \tag{1.2}$$

where l and φ are measurable bounded functions, attains its lower value. The control u which realizes this minimizing condition is called *optimal control*. We consider the weak formulation of the problem, which means that the solution consists not only of the process u, but also of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, of the Brownian motion which is defined in such space, and of the process X^u.

We deal with the special case where the diffusion term G does not depend on the control u, and the drift term has the form $B(x) + G(x)r(x,u)$. Following the approach of [41], we are able to link the stochastic control problem with the semilinear Cauchy problem

$$\begin{cases} D_t v(t,x) + Av(t,x) = \psi(x, G(x)\nabla v(t,x)), & t \in [0,T), \quad x \in \mathbb{R}^N, \\ v(T,x) = \varphi(x), & x \in \mathbb{R}^N. \end{cases} \tag{BPDE}$$

which is known as the Hamilton-Jacobi-Bellman equation.

Here, A is the uniformly elliptic operator defined on smooth functions f by

$$Af(x) = \frac{1}{2}\mathrm{Tr}(G(x)G(x)D_x^2 f(x)) + \langle B(x), \nabla f(x)\rangle,$$

where the coefficients may be unbounded and ψ is a continuous function which satisfies

$$\begin{aligned} |\psi(x_1,x_2) - \psi(y_1,y_2)| &\le L_\psi |x_2 - y_2| + L_\psi |x_1 - y_1| \left(1 + |x_2| + |y_2|\right), \\ |\psi(x,0)| &\le L_\psi, \end{aligned} \tag{1.3}$$

for some positive constant L_ψ. The Cauchy problem

$$\begin{cases} D_t w(t,x) = Aw(t,x) + C(x)w(t,x), & t \in (0,+\infty), \quad x \in \mathbb{R}^N, \\ w(0,x) = \varphi(x), & x \in \mathbb{R}^N, \end{cases} \tag{PDE}$$

with possible unbounded coefficients, has been widely studied in recent years. In the paper [79], the authors provide sufficient conditions in order to get existence and uniqueness of a classical solution w to (PDE), for any $\varphi \in C_b(\mathbb{R}^N)$. Moreover, throughout w it is possible to introduce a semigroup of bounded linear operators $\{S(t)\}_{t\ge 0}$ on $C_b(\mathbb{R}^d)$ setting $S(t,\cdot)f := w(t,\cdot)$.

Our aim consists in writing an optimal control for (1.1) in terms of the mild solution of (BPDE), which is defined by means of the variation of constants formula

$$v(t,x) = S(T-t)\varphi(x) - \int_t^T (S(r-t)\psi(\cdot, Q^{1/2}(\cdot)\nabla_x v(r,\cdot)))(x)dr. \tag{1.4}$$

Moreover, the smoothness of v allows us to identify the optimal feedback law for the problem (1.1).

Hence, at first we ask if it is possible to apply $S(r-t)$ to the function $\psi(\cdot, Q^{1/2}(\cdot)\nabla v(s,\cdot))$ which, in general, is unbounded. The answer is positive, and it is a byproduct of the weighted gradient estimate

$$t^{1/2}\|Q^{1/2}\nabla(S(t)\varphi)\|_\infty \le C_T\|\varphi\|_\infty, \quad t \in (0,T], \tag{1.5}$$

which holds for any $\varphi \in C_b(\mathbb{R}^N)$, where C_T is a positive constant, under the following assumptions:

(i) the coefficients Q_{ij} belong to $C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\alpha \in (0,1)$ and any $i,j = 1,\dots,d$;

(ii) the coefficients of the vector B belong to $C^{1+\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$; further, $\langle B(x), x\rangle \le B_0(x)|x|$ for any $x \in \mathbb{R}^d$ and some negative function B_0;

(iii) there exist a positive constant K_0 and positive functions $\gamma_i : \mathbb{R}^d \to \mathbb{R}$, $i = 1, 2$, such that

$$|\langle Q(x), x\rangle| \le K_0(1 + |x|^2)\nu(x), \qquad x \in \mathbb{R}^d,$$
$$|\nabla_x Q^{1/2}(x)Q^{-1/2}(x)| \le \gamma_1(x), \qquad |Q(x)| \le \gamma_2(x), \quad x \in \mathbb{R}^d;$$

(iv) the functions γ_1 and γ_2 satisfy the following conditions:

$$\lim_{|x|\to+\infty} \frac{(\nu(x))^{-1}(\gamma_1(x))^2(\gamma_2(x))^2}{\omega(x)} = 0,$$

where the function $\omega : \mathbb{R}^d \to \mathbb{R}$ is a negative function which bounds from above the quadratic form associated with the matrix

$$\mathcal{M} := Q^{1/2}(J_x B)^T Q^{-1/2} - \sum_{j=1}^{d} B_j(D_j Q^{1/2})Q^{-1/2} - \sum_{i,j=1}^{d} q_{ij}(D_{ij}Q^{1/2})Q^{-1/2}.$$

Moreover,

$$\liminf_{|x|\to+\infty} \frac{(\nu(x))^2}{\omega(x)} > -\infty,$$
$$\lim_{|x|\to+\infty} \frac{|x|\nu(x)\gamma_1(x)}{B_0(x)} = 0,$$
$$\liminf_{|x|\to+\infty} \frac{|x|(\nu(x))^2}{B_0(x)} > -\infty;$$

(v) there exist $\lambda > 0$ and a function $f \in C^2(\mathbb{R}^d)$ such that

$$\lim_{|x|\to+\infty} f(x) = \infty, \quad \sup_{x\in\mathbb{R}^d} (Af(x) - \lambda f(x)) < \infty. \tag{1.6}$$

The proof of these estimates is based on an application of the Bernstein method and the maximum principle for parabolic differential equation with bounded coefficients.

The existence and uniqueness of a function v in $\mathcal{K}_\delta$ which satisfies (1.4) follows from the Banach's fixed point theorem, where

$$\mathcal{K}_\delta = \left\{ \begin{array}{l} h \in C_b\left([T-\delta, T] \times \mathbb{R}^N\right) \cap C^{0,1}\left([T-\delta, T) \times \mathbb{R}^N\right) : \\ \sup\limits_{\substack{t\in[T-\delta,T) \\ x\in\mathbb{R}^N}} (T-t)^{1/2}|G(x)\nabla h(t,x)| < \infty \end{array} \right\},$$

endowed with the norm

$$\|h\|_{\mathcal{K}_\delta} = \|h\|_\infty + [h]_{\mathcal{K}_\delta}, \tag{1.7}$$

$$[h]_{\mathcal{K}_\delta} := \sup_{t\in[T-\delta,T)} (T-t)^{1/2}\|G\nabla h(t,\cdot)\|_\infty,$$

and δ is a suitable positive constant which belongs to $(0,T]$.

Finally, since ψ is uniformly Lipschitz continuous, we can deduce that the mild solution v is defined in the whole $[0,T]$, for any $\varphi \in C_b(\mathbb{R}^d)$.

The technical tool to connect (1.1) and (BPDE) is a Forward Backward Stochastic Differential Equation (FBSDE), which plays the role of Itô formula when one deals with classical solutions of (BPDE) (see [101, Chp. 4,5]). Indeed, if we consider the system

$$\begin{cases} dY_\tau = \psi(X_\tau, Z_\tau)d\tau + Z_\tau dW_\tau, & \tau \in [t,T], \\ dX_\tau = B(X_\tau)d\tau + G(X_\tau)dW_\tau, & \tau \in [t,T], \\ Y_T = \varphi(X_T), \\ X_t = x, & x \in \mathbb{R}^N, \end{cases} \tag{FBSDE}$$

where ψ is the same function as in (BPDE), then it is possible to prove that the identification formulae

$$Y(s,t,x) := v(s, X(s,t,x)), \quad Z(s,t,x) := G(X(s,t,x))\nabla v(s, X(s,t,x)) \tag{1.8}$$

hold true. Formulae (1.8) have been proved for coefficients which satisfy some growth and smoothness conditions (see [41], [89]). We show that (1.8) hold under our weaker assumptions.

It remains to prove that the cost functional J and an optimal control u for (1.1) are related to the random processes Y and Z. We write the process

$$X^{\mathbb{U}}_\tau = x + \int_t^\tau B(X^{\mathbb{U}}_\sigma)d\sigma + \int_t^\tau G(X^{\mathbb{U}}_\sigma)r(X^{\mathbb{U}}_\sigma, u_\sigma)d\sigma + \int_t^\tau G(X^{\mathbb{U}}_\sigma)dW_\sigma, \quad \tau \in [t,T],$$

with respect to another process $\widetilde{W}$, i.e.,

$$X^{\mathbb{U}}_\tau = x + \int_t^\tau B(X^{\mathbb{U}}_\sigma)d\sigma + \int_t^\tau G(X^{\mathbb{U}}_\sigma)d\widetilde{W}_\sigma, \quad \tau \in [t,T],$$

defined by

$$\widetilde{W}_\tau := W_\tau + \int_t^{t\wedge\tau} r(X^{\mathbb{U}}_\sigma, u_\sigma)d\sigma.$$

The Girsanov theorem guarantees that there exists a probability measure $\tilde{\mathbb{P}}$ on Ω such that $\widetilde{W}$ is a Brownian motion with respect to $\tilde{\mathbb{P}}$. The reason why we consider the problem in weak formulation is clarified by the change of probability, which makes no sense if we consider fixed the probability setting we deal with. In the next step we write the (FBSDE)

$$\widetilde{Y}_\tau + \int_t^\tau \widetilde{Z}_\sigma d\widetilde{W}_\sigma = \varphi(X^{\mathbb{U}}_T) + \int_t^\tau \psi(X^{\mathbb{U}}_\sigma, \widetilde{Z}_\sigma)d\sigma,$$

where $\psi := \inf_u\{l(u,x) + zr(u,x)\}$, which is the Hamiltonian function of the system. Easy computations yield to the inequality

$$v(t,x) \le J(t,x,u),$$

for any control u, and we show that there exists a control $\tilde{u}$ which realizes the equality. Hence, $\tilde{u}$ is an optimal control. Finally, the identification of the feddback law for (1.1) by means of the gradient of v follows from the assumption that the set

$$\Gamma := \{u \in U : \psi(x,z) = l(u,x) + zr(u,x)\}$$

is not empty.

In the second part of this thesis we deal with systems of parabolic differential equations

$$D_t\mathbf{u}(t,x) = (\mathbf{A}(t)\mathbf{u})(t,x), \quad t > s,\ x \in \mathbb{R}^d, \tag{1.9}$$

where $\mathbf{A}(t)$ is the elliptic operator defined on smooth vector-valued functions $\mathbf{v}$

$$(\mathbf{A}(t)\mathbf{v})(x) = \sum_{i,j=1}^{d} q_{ij}(t,x)D^2_{ij}\mathbf{v}(x) + \sum_{j=1}^{d} B_j(t,x)D_j\mathbf{v}(x) + C(t,x)\mathbf{v}(x), \tag{1.10}$$

for any $(t,x) \in I \times \mathbb{R}^d$, with possible unbounded coefficients. Here, I is a right halfline, possibly $I = \mathbb{R}$. Note that the equations are coupled both at zero and first order.

We start from the results of [31], where the terms are coupled only at order zero, to prove the uniqueness and existence of a classical solution $\mathbf{u} \in C_b([s,\infty) \times \mathbb{R}^d; \mathbb{R}^m) \cap C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,\infty) \times \mathbb{R}^d; \mathbb{R}^m)$ to the Cauchy problem

$$\begin{cases} D_t\mathbf{u}(t,x) = (\mathbf{A}(t)\mathbf{u})(t,x), & t > s, \quad x \in \mathbb{R}^d, \\ \mathbf{u}(s,x) = \mathbf{f}(x), & \qquad\quad x \in \mathbb{R}^d, \end{cases} \tag{1.11}$$

which is bounded in all the strips $[s,T] \times \mathbb{R}^d$, for any $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$. However, here we use different techniques from [31], since we have not pointwise estimates of the classical solution to (1.11) in terms of a suitable scalar evolution operator (see [31, Prop 2.4]).

The uniqueness follows from an extension of the methods in [55], related to the case of smooth coefficients in bounded domains, combined with a Lyapunov condition. In particular, we require that

(*i*) there exist $\varepsilon > 0$ and a function $\kappa : I \times \mathbb{R}^d \to \mathbb{R}$, bounded from above by a constant κ_0, such that $\mathcal{K}_\varepsilon(t,x,\eta) \ge 0$ for any $(t,x) \in I \times \mathbb{R}^d$ and any $\eta \in \partial B(1)$, where

$$\begin{aligned} \mathcal{K}_\varepsilon(t,x,\eta) = &\sum_{i,j=1}^{d} a_{ij}(t,x)[\langle B_i(t,x)\eta,\eta\rangle\langle B_j(t,x)\eta,\eta\rangle - \langle B_i(t,x)^*\eta, B_j(t,x)^*\eta\rangle] \\ &- 4\langle C(t,x)\eta,\eta\rangle + 4\varepsilon\kappa(t,x), \end{aligned} \tag{1.12}$$

and $Q(t,x)^{-1} = [a_{ij}(t,x)]$;

(*ii*) for any bounded interval $J \subset I$ there exist a constant λ_J and a positive function $\varphi_J \in C^2(\mathbb{R}^d)$, blowing up as $|x| \to +\infty$, such that

$$\sup_{\eta \in \partial B(1)} \sup_{(t,x) \in J \times \mathbb{R}^d} (\tilde{A}_\eta(t)\varphi_J)(x) - \lambda \varphi_J(x)) < +\infty, \tag{1.13}$$

where $\tilde{A}_\eta = \mathrm{Tr}(QD^2) + \sum_{j=1}^d b_{\eta,j} D_j + 2\varepsilon\kappa$ and $b_{\eta,j} = \langle B_j \eta, \eta \rangle$.

In [55], a condition similar to (*i*) appears; in particular, it is shown that a maximum modulus principle holds if the function $\mathcal{K}_0(t,x,\eta)$ is nonnegative. Here, we provide some examples of the function $\mathcal{K}_\varepsilon$ in special cases. Let us assume that, in (1.10), $B_j = b_j I_m$, for some scalar functions b_j, $j = 1, \ldots, d$; this is the situation in [31]. Condition (*i*) reduces to $\langle C\eta, \eta \rangle \le \varepsilon\kappa |\eta|^2$, for any $\eta \in \mathbb{R}^m$, while in (*ii*) we have $b_{\eta,j} = b_j$, for any $j = 1, \ldots, d$. It easy to check that (*i*) and (*ii*) coincide with Hypotheses 2.1(*iii*) – (*iv*) in [31].

Moreover, if $m = 1$, i.e., in the scalar case where the elliptic operator in (1.10) is $\mathcal{A} = \mathrm{Tr}(QD^2) + \langle b\nabla \rangle + c$, then (*i*) is satisfied with the choices $\varepsilon = 1$ and $\kappa = c$. Further, in the scalar case, the uniqueness of a classical solution to (1.11) follows from the fact that c is bounded from above, and that there exist $\lambda \in \mathbb{R}$ and a positive function $\varphi \in C^2(\mathbb{R}^d)$, blowing up as $|x| \to +\infty$, such that $\mathcal{A}\varphi - \lambda\varphi \le 0$. When this is the case, with the previous choice of ε and κ, also condition (*ii*) is satisfied.

The classical solution $\mathbf{u}$ is defined as the limit of solutions $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ of the Dirichlet-Cauchy problem in $B(n)$

$$\begin{cases} D_t \mathbf{u}_n(t,x) = (\mathbf{A}\mathbf{u}_n)(t,x), & t \in (s,+\infty), \quad x \in B(n), \\ \mathbf{u}_n(t,x) = 0 & t \in (s,+\infty), \quad x \in \partial B(n), \\ \mathbf{u}(s,x) = \mathbf{f}(x), & \qquad\qquad\qquad x \in B(n). \end{cases}$$

The results in [63] and interior estimates for systems of equations allow us to prove that $\mathbf{u}$ has the required smoothness, and that it solves $D_t\mathbf{u}(t,x) = (\mathbf{A}(t)\mathbf{u})(t,x)$, for any $t > s$ and $x \in \mathbb{R}^d$. The continuity up to s is proved using a localization argument and the variation-of-constants formula for solutions to Dirichlet Cauchy problems in bounded and smooth domains. Indeed, fix $M \in \mathbb{N}$ and let ϑ be any smooth function such that $\chi_{B(M-1)} \le \vartheta \le \chi_{B(M)}$. For any $n_k > M$ the function $\mathbf{v}_k := \vartheta \mathbf{u}_{n_k}$ belongs to $C([s,T] \times \overline{B(M)}; \mathbb{R}^m) \cap C^{1,2}((s,T] \times B(M); \mathbb{R}^m)$ and solves the Dirichlet-Cauchy problem

$$\begin{cases} D_t \mathbf{v}_k(t,x) = (\mathbf{A}\mathbf{v}_k)(t,x) + \mathbf{g}_k(t,x), & t \in (s,T], \quad x \in B(M), \\ \mathbf{v}_k(t,x) = 0 & t \in (s,T], \quad x \in \partial B(M), \\ \mathbf{v}_k(s,x) = (\vartheta \mathbf{f})(x), & \qquad\qquad x \in \overline{B(M)}, \end{cases}$$

where $\mathbf{g}_k = -\mathrm{Tr}(QD^2\vartheta)\mathbf{u}_{n_k} - 2(J_x \mathbf{u}_{n_k})Q\nabla\vartheta - \sum_{j=1}^d (B_j \mathbf{u}_{n_k}) D_j \vartheta$, for any $n_k > M$. We thus represent

$$\mathbf{v}_k(t,x) = (\mathbf{G}_M(s,t)(\vartheta\mathbf{f}))(x) + \int_s^t (\mathbf{G}_M(t,r)\mathbf{g}_k(r,\cdot))(x)dr,$$

where $\mathbf{G}_M(s,t)$ is the evolution family associated to the realization of $\mathbf{A}(\cdot)$ in $C_b(\overline{B(M)};\mathbb{R}^m)$ with homogeneous Dirichlet boundary conditions. From a priori estimates, and letting $k \to +\infty$, we get

$$\begin{aligned}|\mathbf{u}(t,x) - \mathbf{f}(x)| \leq & \|\mathbf{G}_M(s,t)(\vartheta\mathbf{f}) - \vartheta\mathbf{f}\|_{L^\infty(B(M-1))} \\ & + K'_M\|\mathbf{f}\|_\infty \int_s^t (1+(t-r)^{-1/2})dr,\end{aligned}$$

which shows that $\mathbf{u}$ is continuous at $t = s$ for any $x \in B(M-1)$, From the arbitrariness of M, we conclude that $\mathbf{u} \in C([s,T] \times \mathbb{R}^d;\mathbb{R}^m)$ and $\mathbf{u}(s,\cdot) = \mathbf{f}$.

At this stage we introduce the evolution operator $\{\mathbf{G}(t,s)\}_{t\geq s\in I}$ on $C_b(\mathbb{R}^d;\mathbb{R}^m)$, defined by $\mathbf{G}(t,s)\mathbf{f} = \mathbf{u}(t,\cdot)$, for any $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$, where $\mathbf{u}$ is the unique classical solution to (1.11). $\{\mathbf{G}(t,s)\}_{t\geq s\in I}$ satisfies remarkable continuity properties and an integral representation holds, i.e., there exists a family of finite Borel signed measures $\{p_{ij}(t,s,x,dy) : t > s \in I,\ x \in \mathbb{R}^d\ \ i,j = 1,\ldots,m\}$ such that

$$((\mathbf{G}(t,s)\mathbf{f})(x))_i = \sum_{j=1}^m \int_{\mathbb{R}^d} p_{ij}(t,s,x,dy)f_j(y), \tag{1.14}$$

where f_j denotes the $j-$th component of $\mathbf{f}$. Moreover, any signed measure $p_{ij}(t,s,x,dy)$, with $t > s$, $x \in \mathbb{R}^d$ and $i,j = 1,\ldots,m$, is absolutely continuous with respect to the Lebesgue measure. As a byproduct, we deduce that $\{\mathbf{G}(t,s)\}_{t\geq s\in I}$ is strong Feller, i.e., it transforms bounded Borel functions in continuous functions, and that it is possible to extend the continuity properties mentioned above and the representation formula (1.14), to any $\mathbf{f} \in B_b(\mathbb{R}^d;\mathbb{R}^m)$.

The absolute continuity of the signed measures has been obtained in a completely different way with respect to the scalar case, where the kernel theory and monotone arguments are used (see e.g. [3, Prop. 3.1]). Here, the key tool is the construction of the set of Borel functions $B_b(\mathbb{R}^d)$ by means of transfinite induction, starting from continuous functions. We briefly present this construction (refer to [61] and [99] for further details). We define

$$B^1 := \{f : f(x) = \lim_{n\to\infty} f_n(x),\ \{f_n\}_{n\in\mathbb{N}} \subset C(\mathbb{R}^d)\},$$

i.e., B^1 is the set of functions which are pointwise limit of continuous functions. B^1 is not closed under pointwise convergence: for instance, the characteristic function of $\mathbb{Q}^d$ can not be obtained as pointwise limit of continuous functions. Hence it is possible to introduce B^2 in an analogous way:

$$B^2 := \{f : f(x) = \lim_{n\to\infty} f_n(x),\ \{f_n\}_{n\in\mathbb{N}} \subset B^1\}.$$

Iterating this argument, we obtain B^n, for any $n \in \mathbb{N}$, and put

$$B^\omega := \bigcup_{n\in\mathbb{N}} B^n,$$

where ω is the first transfinite ordinal number. Again, B^ω is not closed under pointwise convergence, and therefore we define $B^{\omega+1}$ setting

$$B^{\omega+1} := \{f : f(x) = \lim_{n\to\infty} f_n(x),\ \{f_n\}_{n\in\mathbb{N}} \subset B^\omega\}.$$

The above reasoning can be repeated, and finally we get $B(\mathbb{R}^d) = B^{\omega_1}$, where ω_1 is the first non countable transfinite ordinal number.

We further study the compactness properties of $\mathbf{G}(t,s)_{t>s\in J}$, $J \Subset I$, and we relate them to the compactness of the scalar evolution operator $\{G(t,s)\}_{t>s\in J}$ associated to a suitable elliptic operator $\mathcal{A}$, following the idea in [31]. In particular, we provide sufficient conditions such that the compactness of $G(t,s)_{t>s\in J}$ implies the compactness of $\{\mathbf{G}(t,s)\}_{t>s\in J}$, and viceversa.

As first step, we obtain pointwise estimates which relate the vector-valued evolution operator to the scalar one. In such a way we deduce that the compactness of $\{G(t,s)\}_{t>s\in J}$ implies the compactness of $\{\mathbf{G}(t,s)\}_{t>s\in J}$.

Proving that the compactness of the vector-valued evolution operator implies the compactness of the vector-valued one is much more complicated, and we obtain it under some additional conditions related to the growth of the coefficients of operator $\mathbf{A}$. The critical point consists in proving that it is possible to write

$$\begin{aligned}(\mathbf{G}(t,s)\mathbf{f})_j(x) = (G(t,s)f_j)(x) + \int_s^t \left(G(t,r)\sum_i \langle(\tilde{B}_i)_{j\cdot}, D_i(\mathbf{G}(r,s)\mathbf{f})\rangle\right)(x)dr \\ + \int_s^t (G(t,r)\langle C_{j\cdot}, \mathbf{G}(r,s)\mathbf{f}\rangle)(x)dr, \qquad (1.15)\end{aligned}$$

for some $j \in \{1,\ldots,m\}$. Then, we conclude adapting the procedure in [31, Thm. 3.6] to our situation.

We observe that, if the vector $C_{j\cdot} = (C_{j1},\ldots,C_{jm})$ is bounded, for some j, then the last integral makes sense. The first one is more difficult to treat, since, in general, the function under the integral sign is not bounded.

To overcome this problem we prove the following weighted gradient estimates

$$(t-s)\sum_{j=1}^m \|Q^{1/2}(t,\cdot)\nabla_x(\mathbf{G}(t,s)\mathbf{f})_j\|_\infty^2 \le C\|\mathbf{f}\|_\infty^2,$$

for $\mathbf{G}(t,s)\mathbf{f}$, which are obtained with techniques similar to those used to prove (1.5). Hence, from an approximation argument we get (1.15).

The third and last part of the thesis is devoted to study the controlled equation

$$\begin{cases} dX_\tau^{(u)} = b(X_\tau^{(u)})d\tau + G(X_\tau^{(u)})r(X_\tau^{(u)}, u_\tau) + G(X_\tau^{(u)})dW_\tau^{(u)}, & \tau \in [t,T], \\ X_t^{(u)} = x, & x \in \mathbb{R}^d, \end{cases} \qquad (1.16)$$

where

$$b : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad G : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d},$$

are Borel measurable functions, and $r : \mathbb{R}^d \times U \longrightarrow \mathbb{R}^d$ is a measurable and bounded function. We consider m players, and $u = (u^1, \dots, u^m)$ is an $\mathbb{R}^m-$valued random process, where any component u^i represents the strategy of the player i, for any $i = 1, \dots, m$. To any player i, $i = 1, \dots, m$, we associate a cost functional

$$J^i(u) = \mathbb{E}^{(u)} \left[\int_0^T h^i(X_s, u_s) ds + g^i(X_T) \right], \tag{1.17}$$

and we notice that the value of the cost functional for any player i depends on the strategy of all the other ones.

We look for a Nash equilibrium, i.e., a strategy $\tilde{u} = \left(\tilde{u}^1, \dots, \tilde{u}^m\right)$, such that, for any $i = 1, \dots, m$, any $u^i \in U^i$, we have

$$J^i(\tilde{u}) \leq J^i \left(\tilde{u}^1, \dots, \tilde{u}^{i-1}, u^i, \tilde{u}^{i+1}, \dots, u^m\right). \tag{1.18}$$

The above definition of Nash equilibrium implies that, for any $i = 1, \dots, m$, if any player j, with $j \neq i$, chooses the strategy $\tilde{u}^j$, then the best strategy for i is $\tilde{u}^i$.

As in the first part, the solvability of a particular Forward Backward Stochastic Differential System gives information about the existence of a Nash equilibrium for the above game (see [39]). We want to prove that, if $(X, \mathbf{Y}, \mathbf{Z})$ is a solution to

$$\begin{cases} d\mathbf{Y}_\tau = \mathbf{H}(X_\tau, \mathbf{Z}_\tau) d\tau + \mathbf{Z}_\tau dW_\tau, & \tau \in [t, T], \\ dX_\tau = b(X_\tau) d\tau + G(X_\tau) dW_\tau, & \tau \in [t, T], \\ \mathbf{Y}_T = \mathbf{g}(X_T), & \\ X_t = x, & x \in \mathbb{R}^d, \end{cases} \tag{1.19}$$

then $\mathbf{Y}$ and $\mathbf{Z}$ can be represented throughout the identification formulae

$$\mathbf{Y}(s,t,x) := \mathbf{v}(s, X(s,t,x)), \quad \mathbf{Z}(s,t,x) := G(X(s,t,x)) \nabla \mathbf{v}(s, X(s,t,x)). \tag{1.20}$$

Here, $\mathbf{v}$ is a mild solution to

$$\begin{cases} D_t \mathbf{v}(t,x) + \mathbf{A}\mathbf{v}(t,x) = \psi(x, Q^{1/2}(x) \nabla \mathbf{v}(t,x)), & t \in [0,T), \quad x \in \mathbb{R}^d, \\ \mathbf{v}(T,x) = \mathbf{f}(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.21}$$

where $\mathbf{A}$ is the elliptic operator defined on $\mathbb{R}^m-$valued smooth functions ϕ by

$$(\mathbf{A}\phi)_j(x) = Tr[Q(x) D^2 \phi_j(x)] + \sum_{k=1}^m \langle (B)_{jk}(x), \nabla \phi_k \rangle, \quad j = 1, \dots, m,$$

and ψ is involved in the definition of $\mathbf{H}$. Problem (1.21) is similar to problem (BPDE), but now, in view of applications, it does not make sense to assume the function ψ to be Lipschitz; hence, we assume the following weaker hypotheses on ψ:

$$\begin{aligned} |\psi(x, z_1) - \psi(x, z_2)| &\leq C(1 + |z_1| \vee |z_2|) |z_1 - z_2|^\alpha, \\ |\psi(x, z)| &\leq C(1 + |z|), \end{aligned} \tag{1.22}$$

for any $x \in \mathbb{R}^d$ and $z, z_1, z_2 \in \mathbb{R}^{d\times m}$, some positive constant C and $\alpha \in (0,1)$.

Since ψ is not Lipschitz continuous, we can not directly use the Banach fixed point theorem; however, throughout an approximation argument we obtain the desired result. At first, we consider the semilinear Cauchy problem when the function ψ is uniformly Lipschitz continuous with respect the second variable, and follow the same procedure as in Chapter 4, i.e., we prove that there exists a function $\mathbf{v} \in \mathbf{K}_T$ which satisfies

$$\mathbf{v}(t,x) = \mathbf{T}(T-t)\mathbf{f}(x) - \int_t^T (\mathbf{T}(s-t)F(s,\mathbf{v}))(x)ds,$$

for any $t \in [0,T]$ and $x \in \mathbb{R}^d$, where $F(s,\mathbf{w})(x) := \psi(s, Q^{1/2}(x)\nabla_x \mathbf{w}(x))$ for any $(s,\mathbf{w}) \in [0,T) \times \mathbf{K}_T$, and

$$\mathbf{K}_T := \left\{\mathbf{h} \in C_b\left([0,T]\times\mathbb{R}^d;\mathbb{R}^m\right) \cap C^{0,1}\left([0,T)\times\mathbb{R}^d;\mathbb{R}^m\right) : \|\mathbf{h}\|_{\mathbf{K}_T} < \infty\right\},$$

$$\|\mathbf{h}\|_{\mathbf{K}_T} := \|\mathbf{h}\|_\infty + [\mathbf{h}]_{\mathbf{K}_T},\ [\mathbf{h}]_{\mathbf{K}_T} := \sup_{t\in[0,T)} (T-t)^{1/2} \sum_{j=1}^m \|Q^{1/2}(\cdot)\nabla_x \mathbf{h}_j(t,\cdot)\|_\infty.$$

The first step consists in proving that the integral term in the above formula makes sense. This result is a byproduct of the weighted estimates in Proposition 5.21, where we have proved that

$$t\sum_{j=1}^m \|Q^{1/2}\nabla_x(\mathbf{u}(t,\cdot))_j\|_\infty^2 \le C\|\mathbf{f}\|_\infty^2,$$

and $\mathbf{u}$ is the unique classical solution to

$$\begin{cases} D_t\mathbf{u}(t,x) = \mathbf{A}\mathbf{u}(t,x), & t\in(0,T], \quad x\in\mathbb{R}^d, \\ \mathbf{u}(0,x) = \mathbf{f}(x), & \qquad\qquad\quad x\in\mathbb{R}^d. \end{cases}$$

Then, the Banach fixed point Theorem and the linear growth of ψ allow us to conclude.

The main effort now consists in proving that a suitable sequence of mild solutions with Lipschitz data converges, up to a subsequence, to a function which satisfies our requires.

We build the sequence of mild solutions as follows. At first, we approximate ψ by $\psi_n := \vartheta_n(\rho_n \star_z \psi)$, which, for any $n \in \mathbb{N}$, are defined by the convolution only with respect the variable z with a standard sequence of mollifiers $\{\rho_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{m\times d}$, and $\vartheta_n \in C_c^\infty(\mathbb{R}^{m\times d})$ are cut-off functions which satisfy $\chi_{B(n)} \le \vartheta_n \le \chi_{B(n+1)}$.

Since we have already proved the existence and uniqueness of mild solution with Lipschitz data, we consider the sequence $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$, where

$$\mathbf{v}_n(t,x) = \mathbf{T}(T-t)\mathbf{f}(x) - \int_t^T (\mathbf{T}(s-t)F_n(s,\mathbf{v}_n))(x)ds, \tag{1.23}$$

and $F_n(s,\mathbf{w})(x) := \psi_n(s, Q^{1/2}(x)\nabla_x\mathbf{w}(x))$. If we prove that, as $n \to +\infty$, $\mathbf{v}_n$ converges to a function $\mathbf{v}$ in a suitable way, we conclude taking the limit in the left-hand side and in the right-hand side of (1.23).

The proof of the convergence of the sequence $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ relies on two results: the first one is the uniform boundedness of $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ in $\mathbf{K}_T$, i.e., there exists a positive constant K such that $\sup_n \|\mathbf{v}_n\|_{\mathbf{K}_T} \le K$. Further, we introduce and the family of operators on $\mathbf{K}_T$

$$\Phi_k^n(\mathbf{u})(t,x) := (\mathbf{T}(T-t)\mathbf{f})(x) - \int_{t+1/\hat{n}}^T (\mathbf{T}(s-t)F_k(s,\mathbf{u}))(x)ds,$$

for any $k, n \in \mathbb{N}$ and $(t,x) \in [0,T) \times \mathbb{R}^d$, where $\hat{n} := [1/T] + n$.

The second step consists in showing that Φ_k^n is compact from $\mathbf{K}_T$ is $C^{0,1}([0,T-1/\hat{l}] \times B(\hat{l}); \mathbb{R}^m)$ by means of Ascoli-Arzelà theorem, and the proof is a byproduct of the interior estimates for systems of equations in Subsection 5.2.1.

To conclude, we argue as follows: we show that there exists a subsequence $\{\mathbf{w}_n\} \subset \{\mathbf{v}_n\}$ such that $\{\Phi_n^n(\mathbf{w}_n)\}_{n\in\mathbb{N}}$ converges to a function $\mathbf{v}$ in $C^{0,1}([0,T-1/\hat{l}] \times B(\hat{l}); \mathbb{R}^m)$, for any $l \in \mathbb{N}$. Then, we prove that also $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ converges to $\mathbf{v}$ in $C^{0,1}([0,T-1/\hat{l}] \times B(\hat{l}); \mathbb{R}^m)$, as $n \to +\infty$. Finally, since

$$\mathbf{w}_n(t,x) = \mathbf{T}(T-t)\mathbf{f}(x) - \int_t^T (\mathbf{T}(s-t)F_{(n,n)_n}(s,\mathbf{w}_n))(x)ds,$$

where $\{(n,n)_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ is a suitable sequence, we get that $\mathbf{v}$ satisfies

$$\mathbf{v}(t,x) = \mathbf{T}(T-t)\mathbf{f}(x) - \int_t^T (\mathbf{T}(s-t)F(s,\mathbf{v}))(x)ds,$$

for any $t \in [0,T)$ and $x \in \mathbb{R}^d$, and it belongs to $\mathbf{K}_T$.

Then, we show how to obtain a solution of the (S-FBSDE) throughout $\mathbf{v}$. We approximate $\mathbf{H}$ by $\mathbf{H}_n$, defined similarly to ψ_n, even if in this case the convolution also involves the spacial variable. From [41], for any $n \in \mathbb{N}$ we have

$$\mathbf{Y}^n(s,t,x) := \mathbf{v}_n(s, X(s,t,x)), \quad \mathbf{Z}^n(s,t,x) := G(X(s,t,x))\nabla_x \mathbf{v}_n(s, X(s,t,x)),$$

where $(X; \mathbf{Y}^n, \mathbf{Z}^n)$ is the unique predictable solution to the approximate System of Forward Backward Stochastic Differential Equations

$$\begin{cases} d\mathbf{Y}_\tau^n = \mathbf{H}_n(X_\tau, \mathbf{Z}_\tau^n)d\tau + \mathbf{Z}_\tau^n dW_\tau, & \tau \in [t,T], \\ dX_\tau = b(X_\tau)d\tau + G(X_\tau)dW_\tau, & \tau \in [t,T], \\ \mathbf{Y}_T^n = \mathbf{g}(X_T), & \\ X_t = x, & x \in \mathbb{R}^d. \end{cases}$$

Both $\mathbf{v}_n$ and $Q^{1/2}\nabla_x \mathbf{v}_n$ converge to $\mathbf{v}$ and $Q^{1/2}\nabla_x \mathbf{v}$, respectively, and we prove these convergences following the procedure of Subsection 6.2.2. Moreover, we prove that also

$\mathbf{Y}_n$ and $\mathbf{Z}_n$ converge to random variables $\mathbf{Y}$ and $\mathbf{Z}$, and we can conclude that $(X, \mathbf{Y}, \mathbf{Z})$ is a solution to

$$\begin{cases} d\mathbf{Y}_\tau = \mathbf{H}(X_\tau, \mathbf{Z}_\tau)d\tau + \mathbf{Z}_\tau dW_\tau, & \tau \in [t, T], \\ dX_\tau = b(X_\tau)d\tau + G(X_\tau)dW_\tau, & \tau \in [t, T], \\ \mathbf{Y}_T = \mathbf{g}(X_T), & \\ X_t = x, & x \in \mathbb{R}^d, \end{cases}$$

with

$$\mathbf{Y}(s, t, x) := \mathbf{v}(s, X(s, t, x)), \quad \mathbf{Z}(s, t, x) := G(X(s, t, x))\nabla_x \mathbf{v}(s, X(s, t, x)).$$

Chapter 2

Parabolic Cauchy Problems

The starting point of our analysis is the parabolic Cauchy problem

$$\begin{cases} D_t u(t,x) = Au(t,x), & x \in \mathbb{R}^d, \quad t > 0, \\ u(0,\cdot) = f, & x \in \mathbb{R}^d, \end{cases} \tag{PCP}$$

A is the elliptic operator defined on the smooth functions g by

$$Ag(x) = \sum_{i,j=1}^{d} q_{ij}(x) \frac{\partial^2 g}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial g}{\partial x_i}(x) + c(x)g(x), \tag{2.1}$$

where the coefficients of operator A are possibly unbounded, and $f \in C_b(\mathbb{R}^d)$.

In recent years much attention has been paid to the uniformly elliptic operator A, with unbounded coefficients in $\mathbb{R}^N$, since they naturally appear in the theory of Markov processes (for a systematic treatment of this argument see [15]). Moreover, the interest has also been extended to elliptic nonautonomous second order differential operators ([3], [65], [67]).

If its coefficients are bounded and smooth enough, A is a sectorial operator which generates an analytic semigroup (see [70]). Hence, it is possible to study (PCP) using the classical theory of analytic semigroups in order to get existence, uniqueness and smooth properties of the solution u.

The situation is completely different if we consider elliptic operators with unbounded coefficients. Indeed, let us consider the Ornstein-Uhlenbeck operator, defined on the smooth functions g by

$$Ag(x) = \frac{1}{2} \sum_{i,j=1}^{d} q_{ij} \frac{\partial^2 g}{\partial x_i \partial x_j}(x) + \sum_{i,j=1}^{d} b_{ij} x_j \frac{\partial g}{\partial x_i}(x), \tag{2.2}$$

where $[q_{ij}]$ is a constant, symmetric and positive definite matrix, and $[b_{ij}]$ is a constant matrix, whose eigenvalues have non-positive real part. This is the most famous example of second-order elliptic operator with unbounded coefficients, and the semigroup defined by the solution to (PCP) is neither strongly continuous nor analytic in $C_b(\mathbb{R}^d)$ (see [25]). Moreover, the spectrum of the Ornstein-Uhlenbeck operator A in $L^p(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$ contains suitable unbounded subgroups of vertical lines (see [77]). The lack of analyticity has been also shown in [28] and [81] for more general operators of the form $\Delta + \langle F, \nabla\rangle$, with suitable choices of F. This means that we can not use classical techniques to study operators with unbounded coefficients and their associated Cauchy problem in $C_b(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$. However, in recent years important developments has been done in the study of such operators, starting from the problem of existence and uniqueness of the solution to the parabolic Cauchy problem (PCP), solved in the $60'$s ([52], [56], [57], [58], [59]). Remarkable properties of the semigroups associated with operators with unbounded coefficients, such as compactness, estimates of the spacial derivatives of the function $T(t)f$, $t \geq 0$, and the study of invariant measures in $L^p(\mathbb{R}^d)$ (see the monograph [15] and the paper [79]), have been established in the last years.

Also nonautonomous operators are of interest for us, and the first works in this direction deal with the particular case of the Ornstein-Uhlenbeck operator (see [24], [43], [44]). The pioneer paper which set the basis for a general theory of nonautonomous operators is [60], in which the authors consider the parabolic Cauchy problem

$$\begin{cases} D_t u(t,x) = A(t)u(t,x), & t > s, \quad x \in \mathbb{R}^d, \\ u(s,x) = f(x), & \qquad\qquad x \in \mathbb{R}^d. \end{cases} \tag{2.3}$$

The operator $A(t)$ is defined on the smooth functions φ by

$$\begin{aligned} (A(t)\varphi)(x) &= \sum_{i,j=1}^{d} q_{ij}(t,x)\frac{\partial^2\varphi}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(t,x)\frac{\partial\varphi}{\partial x_i}(x) \\ &= \mathrm{Tr}(Q(t,x)D^2\varphi(x)) + \langle B(t,x), \nabla\varphi(x)\rangle, \end{aligned} \tag{2.4}$$

where $x \in \mathbb{R}^d$ and $t \in I$, which is either $\mathbb{R}$ or a right half-line. The Cauchy problem (2.3) is strictly linked to the stochastic differential problem

$$\begin{cases} dX_t = \mu(t,X_t)dt + \sigma(t,X_t)DW_t, & t > s, \\ X_s = x \in \mathbb{R}^d, \end{cases} \tag{2.5}$$

where W_t is a standard $d-$dimensional Brownian motion and μ and σ are respectively regular $\mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R}^d-$ valued functions. If problem (2.5) has a solution $X_t = X(t,s,x)$, then Itô formula implies that, for any $f \in C_b^2(\mathbb{R}^d)$ and $t > s \in I$, the function

$$u(t,s,x) := \mathbb{E}[f(X(t,s,x))] \tag{2.6}$$

solves the differential equation

$$\begin{cases} D_s u(s,x) = \frac{1}{2}\mathrm{Tr}(\sigma(s,x)\sigma^*(s,x)D^2u(s,x)) + \langle b(s,x), \nabla u(s,x)\rangle, & s < t, \quad x \in \mathbb{R}^d, \\ u(t,x) = f(x), & \qquad\qquad x \in \mathbb{R}^d. \end{cases} \tag{2.7}$$

This is a backward Cauchy problem, but reverting time the solutions to (2.7) are transformed into solutions to (2.3).

However, in the paper mentioned above, the Cauchy problem (2.3) is studied in a purely analytic way. Authors find sufficient conditions such that for any $f \in C_b(\mathbb{R}^d)$ the Cauchy problem is solvable by a unique classical solution $u \in C([s,\infty) \times \mathbb{R}^d) \cap C^{1,2}((s,\infty) \times \mathbb{R}^d)$. Then they define a family of evolution operators $\{G(t,s)\}_{t \geq s \in I}$ setting $G(t,s)f(x) = u(t,x)$, and establish several important properties of these operators. These properties concern with the continuity and the smoothing effects of $\{G(t,s)\}_{t \geq s}$, and the key tool to show them is an integral representation of $G(t,s)$. This means that there exists a family of finite Borel measures on $\mathbb{R}^d$ $\{p_{t,s}(x,dy) : x \in \mathbb{R}^d\}$, $t > s \in I$, such that

$$G(t,s)f(x) = \int_{\mathbb{R}^d} f(y) p_{t,s}(x,dy), \tag{2.8}$$

for any $f \in C_b(\mathbb{R}^d), x \in \mathbb{R}^d$ and $t > s \in I$. Moreover, several estimates on the spacial derivatives of $G(t,s)f$ are proved, with $f \in C_b(\mathbb{R}^d)$.

These estimates have been used both to study the asymptotic behavior of $G(t,s)$ and to prove optimal Schauder estimates for non-homogenous Cauchy problems.

Finally, the existence of a system of invariant measures $\{\mu_t\}_{t \in I}$ is proved, i.e. a family of finite Borel measures μ_t such that

$$\int_{\mathbb{R}^d} G(t,s)f(x) d\mu_t(x) = \int_{\mathbb{R}^d} f(x) d\mu_s(x), \tag{2.9}$$

for any $t > s \in I$ and $f \in C_b(\mathbb{R}^d)$.

Nonautonomous elliptic operators with unbounded coefficients with a potential term has been studied in [3]. More precisely, in [3] the operator

$$\begin{aligned}(A(t)\varphi)(x) = \sum_{i,j=1}^{d} q_{ij}(t,x) \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(t,x) \frac{\partial \varphi}{\partial x_i}(x) \\ - c(t,x)\varphi(x)\end{aligned} \tag{2.10}$$

has been considered. The proof of the existence and uniqueness of a solution to

$$\begin{cases} D_t u(t,x) = A(t)u(t,x), & t > s, \quad x \in \mathbb{R}^d, \\ u(s,x) = f(x), & \qquad\quad x \in \mathbb{R}^d, \end{cases}$$

follows the ideas in [60], as the definition of $G(t,s)$. The rest of the paper is devoted to prove compactness properties of $\{G(t,s)\}_{t > s \in J}$ in $C_b(\mathbb{R}^d)$, for any bounded interval $J \subset I$, and the invariance of $L^p(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$ under its action. As in the autonomous case, the compactness of $\{G(t,s)\}_{t > s \in J}$ is equivalent to the tightness of the family of transition measures $\{p_{t,s}(x,dy) : x \in \mathbb{R}^d\}$, and sufficient conditions for this family to be tight are provided.

However, under these conditions $G(t,s)$ is compact but it does not preserve $C_0(\mathbb{R}^d)$; hence finding assumptions which guarantee that $C_0(\mathbb{R}^d)$ is invariant under the action of $G(t,s)$ is not trivial, and under these assumptions the authors of [3] show that $\{G(t,s)\}_{t\geq s\in I}$, restricted to $C_0(\mathbb{R}^d)$, is a strongly continuous family of evolution operators.

As noticed, elliptic operators with unbounded coefficients have been deeply investigated in these last years. Different is the case of systems of elliptic operators with unbounded coefficients. The extension of the classical theory of elliptic systems has been mainly devoted to the case of bounded but not smooth coefficients (see [1], [32], [49], [85], [86], [87], [95], [98]), while the direction of unbounded coefficients has been little beaten. In the papers [50] and [94], a class of weakly coupled systems is considered in the L^p-setting, and the paper [31] keeps on the analysis both in vector values spaces $L^p(\mathbb{R}^N;\mathbb{R}^m)$ and $C_b(\mathbb{R}^N;\mathbb{R}^m)$. Here, some results of previous papers are extended to more general situations, and the operator $\mathcal{A}$ is studied in the space $C_b(\mathbb{R}^d;\mathbb{R}^M)$ of all bounded and continuous vector-valued functions. The operator $\mathcal{A}$ is defined on smooth vector-valued functions $\varphi=(\varphi_1,\dots,\varphi_M)$ by

$$(\mathcal{A}\varphi(x))_k=\sum_{i,j=1}^d q_{ij}(x)D^2_{ij}\varphi_k(x)+\sum_{i=1}^d b_i(x)D_i\varphi_k(x)+\sum_{h=1}^d V_{kh}(x)\varphi_h(x),$$

for $x\in\mathbb{R}^d$ and $k=1,\dots,M$. As a first step the authors of [31] proved that it is possible to associate a semigroup of bounded linear operators on $C_b(\mathbb{R}^d;\mathbb{R}^M)$ to $\mathcal{A}$ under minimal assumptions on the coefficients, which are comparable with those of the case of single equation. As usual, the semigroup is defined by means of the unique classical solution $\mathbf{u}$ to the parabolic Cauchy problem

$$\begin{cases} D_t\mathbf{u}(t,x)=\mathcal{A}\mathbf{u}(t,x), & x\in\mathbb{R}^d,\quad t>0,\\ \mathbf{u}(0,\cdot)=\mathbf{f}, & x\in\mathbb{R}^d,\end{cases}$$

For any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^M)$ we set $\mathbf{T}(t)\mathbf{f}(x):=\mathbf{u}(t,x)$. Though the semigroup may fail to be strongly continuous or analytic in $C_b(\mathbb{R}^d;\mathbb{R}^M)$, a "weak" generator can be associated with in, as in the scalar case. The authors of [31] have also provided sufficient conditions in order to get compactness of $\mathbf{T}(t)$ on $C_b(\mathbb{R}^d;\mathbb{R}^M)$ and estimates of spatial derivatives of the function $T(t)\mathbf{f}$, with $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^M)$. These estimates have been used to prove the optimal Hölder regularity of solutions to non-homogenous parabolic and elliptic Cauchy problem. Moreover, generation of strongly continuous and analytic semigroups $\{\mathbf{T}_p(t)\}_{t\geq 0}$ in $L^p(\mathbb{R}^d;\mathbb{R}^M)$ has been investigated, and conditions which imply that $\mathbf{T}_p(t)$ maps $L^p(\mathbb{R}^d;\mathbb{R}^M)$ into $L^q(\mathbb{R}^d;\mathbb{R}^M)$ for any $1\leq p\leq q<\infty$ are provided.

2.1 Autonomous Cauchy Problems

2.1.1 The Parabolic Equation and the Semigroup

Let A be the differential operator defined on smooth functions g by

$$Ag(x) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 g}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial g}{\partial x_i}(x) + c(x)g(x). \tag{2.11}$$

We assume the following hypotheses on the coefficients of the operator A.

Hypotheses 2.1. **(i)** *$Q = [q_{ij}]$ is a symmetric matrix, i.e. $q_{ij} = q_{ji}$ for any $i, j = 1, \dots, d$, and there exist a positive function ν and a constant $\nu_0 > 0$ such that $\nu(x) \ge \nu_0$ for any $x \in \mathbb{R}^d$ and*

$$\langle Q(x)\xi, \xi\rangle \ge \nu(x)|\xi|^2, \quad \xi, x \in \mathbb{R}^d; \tag{2.12}$$

(ii) *q_{ij}, b_i and c belong to $C^\alpha_{loc}(\mathbb{R}^d)$, for any $i, j = 1, \dots, d$, and some $\alpha \in (0, 1)$;*

(iii) *$c \le k_0$, for some $k_0 \in \mathbb{R}$;*

(iv) *there exist a positive function $\varphi \in C^2(\mathbb{R}^d)$ and a constant $\lambda > 0$ such that $\lim_{|x|\to\infty} \varphi(x) = +\infty$ and*

$$A\varphi(x) - \lambda\varphi(x) \le 0, \quad x \in \mathbb{R}^d. \tag{2.13}$$

Remark 2.2. *The function φ which appears in Hypothesis 2.1(iv) is called a Lyapunov function associated to the operator A.*

Under these assumptions, the following theorem holds.

Theorem 2.3. *If Hypotheses 2.1 are satisfied then, for any $f \in C_b(\mathbb{R}^d)$, there exists a unique classical solution to the Cauchy problem*

$$\begin{cases} D_t u(t,x) - Au(t,x) = 0, & t > 0, \quad x \in \mathbb{R}^d, \\ u(0,x) = f(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.14}$$

i.e., there exists a unique function $u \in C([0,\infty) \times \mathbb{R}^d) \cap C^{1,2}((0,\infty) \times \mathbb{R}^d)$ which satisfies (2.14). Moreover. $u \in C^{1+\alpha/2, 2+\alpha}_{loc}((0,\infty) \times \mathbb{R}^d)$.

The uniqueness is a byproduct of Hypothesis 2.1(*iv*), while the existence is based on an approximation argument and Schauder interior estimates. Let us explain this arguments. For any $n \in \mathbb{N}$ we consider the Dirichlet-Cauchy problem

$$\begin{cases} D_t u_n(t,x) = Au_n(t,x), & x \in B(n), \quad t > 0, \\ u_n(t,x) = 0, & x \in \partial B(n), \quad t > 0, \\ u_n(0,\cdot) = f, & x \in B(n). \end{cases} \tag{PDCP}$$

By classical results for parabolic Cauchy problem in bounded domains (see [63]) we know that there exists a unique solution $u_n \in C([0,\infty) \times B(n) \setminus (\{0\} \times \partial B(n)) \cap C^{1+\alpha/2,2+\alpha}_{loc}((0,\infty) \times B(n))$ to (PDCP). Using classical Schauder estimates and Ascoli-Arzelà Theorem it is possible to show that $\{u_n\}_{n\in\mathbb{N}}$ converges to a function $u \in C((0,\infty)\times\mathbb{R}^d) \cap C^{1+\alpha/2,2+\alpha}_{loc}((0,\infty)\times\mathbb{R}^d)$ in $C^{1+\beta/2,2+\beta}_{loc}((0,\infty)\times\mathbb{R}^d)$, for any $\beta \in (0,\alpha)$ and for any compact set $K \subset (0,\infty)\times\mathbb{R}^d$. Finally, one proves that u can be extended by continuity up to 0 where it equals f. This is obtained using a localization argument: we set $v_n := \vartheta u_n$, where θ is a smooth function which is equal to 1 in $B(M-1)$, and equal to 0 in $B(M)^c$. Hence v_n satisfies the non-homogenous Cauchy problem

$$\begin{cases} D_t v_n(t,x) = Av_n(t,x) + g_n(t,x), & x \in B(M), \quad t>0, \\ u_n(t,x) = 0, & x \in \partial B(M), \quad t > 0, \\ u_n(0,\cdot) = \vartheta f, & x \in B(M), \end{cases}$$

where $g_n = -\mathrm{Tr}(QD^2\vartheta)v_n - 2\sum_{i,j=1}^d q_{ij} D_i\vartheta D_j vn - \sum_{j=1}^d (B_j v_n) D_j\vartheta$.

By means of the variation of constants formula we get

$$v_n(t,x) = T_M(t)\vartheta f(x) + \int_0^t T_M(t-s) g_n(s,\cdot)(x)ds,$$

and

$$|u_n(t,x) - f(x)| \le |T_M(t)\vartheta f(x) - f(x)| + \left| \int_0^t T_M(t-s) g_n(s,\cdot)(x)ds \right|,$$

for any $x \in B(M-1)$. Letting n to $+\infty$ and later t to 0^+ we get that $u(0,x) = f(x)$, for any $x \in B(M-1)$. Since M is arbitrary, we can conclude that $u(0,\cdot) \equiv f$ in $\mathbb{R}^d$. Further, u satisfies

$$|u(t,x)| \le \exp(k_0 t)\|f\|_\infty, \tag{2.15}$$

for any $t > 0$ and $x \in \mathbb{R}^d$, where k_0 is given by Hypothesis 2.1(iii).

Remark 2.4. *If $f \ge 0$ the classical maximum principle implies that u_n is positive and increasing, and therefore u is positive.*

Remark 2.5. *If Hypothesis* 2.1(iv) *fails to hold, the approximation argument explained above still works and shows that problem* (2.14) *admits a classical solution u with the smoothing properties in Theorem 2.3. In general, as we will see in Example 2.6, this solution is not unique. Anyway, using the classical maximum principle, we can show that, for any $f \ge 0$, u is the minimal solution to* (2.14)*, i.e. if v is another classical solution to* (2.14) *then $u \le v$ in $[0,\infty)\times\mathbb{R}^d$.*

Example 2.6. *Let $d = 1$ and consider the operators*

$$A_1 g(x) = g''(x) - x^3 g'(x) \tag{2.16}$$

and

$$A_2 g(x) = g''(x) + x^3 g'(x). \tag{2.17}$$

Then, for any $f \in C_b(\mathbb{R})$ the Cauchy problem associated to A_1 admits a unique classical solution. On the other hand, the Cauchy problem associated to A_2 admits more than a classical solution.

Remark 2.7. *Example above shows that the existence of a Lyapunov function is not only connected to the growth of the coefficients, as it could appear. Indeed the drift terms of A_1 and A_2 differ only in the sign, but their behavior is quite different.*

We recall that, for any $n \in \mathbb{N}$, u_n denotes the unique classical solution to the parabolic Dirichlet-Cauchy problem in $B(n)$ with initial datum f, then

$$u_n(t,x) = \int_{\mathbb{R}^d} G_n(t,x,y) f(y)dy, \tag{2.18}$$

where $G_n \in C((0,\infty) \times B(n) \times B(n))$ is the fundamental solution to (PDCP). By the classical maximum principle one can easily show that $\{G_n\}_{n\in\mathbb{N}}$ is increasing, and so we can define

$$G(t,x,y) := \lim_{n\to\infty} G_n(t,x,y), \quad t>0, \quad x,y \in \mathbb{R}^d. \tag{2.19}$$

Function G is positive and almost everywhere finite, since if $f = \mathbb{1}$ we have

$$\int_{\mathbb{R}^d} G_n(t,x,y)dy \le \exp(k_0 t), \quad t>0, \quad x \in \mathbb{R}^d. \tag{2.20}$$

Now we can get an integral representation of $T(t)$; indeed, defining

$$p(t,x,dy) := G(t,x,y)dy, \quad t>0, \quad x \in \mathbb{R}^d, \tag{2.21}$$

by the monotone convergence we have

$$T(t)f(x) = \int_{\mathbb{R}^d} p(t,x,dy) f(y), \tag{2.22}$$

for any nonnegative $f \in C_b(\mathbb{R}^d)$. Splitting f into positive and negative part, we can extend (2.22) to any $f \in C_b(\mathbb{R}^d)$.

As it has been stressed, in general the convergence is not uniform in $\mathbb{R}^d$. This is the case if f vanishes at infinity, as the next Proposition shows.

Proposition 2.8. *If $f \in C_0(\mathbb{R}^d)$ then $T(t)f$ converges to f uniformly in $C_b(\mathbb{R}^d)$.*

Remark 2.9. *One can ask if, as a byproduct of Proposition 2.8, $\{T(t)\}_{t\ge0}$ is strongly continuous on $C_0(\mathbb{R}^d)$. The answer is negative in general since it is not guaranteed that the semigroup preserves $C_0(\mathbb{R}^d)$, as we will see below.*

Some important properties of $\{T(t)\}_{t\ge0}$, among which the strong Feller property, are consequences of the integral representation (2.22).

Proposition 2.10. *If the sequence* $\{f_n\}_{n\in\mathbb{N}} \subset C_b(\mathbb{R}^d)$ *is uniformly bounded and converges pointwise to* $f \in C_b(\mathbb{R}^d)$ *as* $n \to \infty$*, then* $T(t)f_n$ *converges to* $T(t)f$ *on* K*, for any compact set* $K \subset (0,\infty)\times\mathbb{R}^d$*. Moreover, if* $\{f_n\}_{n\in\mathbb{N}}$ *converges to* f *locally uniformly in* $\mathbb{R}^d$*, then* $T(t)f_n$ *converges locally uniformly in* $[0,\infty)\times\mathbb{R}^d$ *as* $n\to\infty$*.*

Remark 2.11. *As a byproduct of Proposition 2.10 we can extend* $T(t)$ *to the set* $B_b(\mathbb{R}^d)$ *of all the Borel measurable functions on* $\mathbb{R}^d$*.*

Finally, we recall the following definitions.

Definition 2.12. *A semigroup* $\{S(t)\}_{t\geq 0}$ *on* $B_b(\mathbb{R}^d)$ *is said to be irreducible if, for any nonempty open set* $B \in \mathbb{R}^d$*, it holds that* $S(t)\chi_B > 0$*.*

$\{S(t)\}_{t\geq 0}$ *has the strong Feller property if* $S(t)f \in C_b(\mathbb{R}^d)$*, for any* $f \in B_b(\mathbb{R}^d)$*.*

Proposition 2.13. $\{T(t)\}_{t\geq 0}$ *is irreducible and has the strong Feller property.*

The last concept that we introduce in this section is the *weak generator* of the semigroup $\{T(t)\}_{t\geq 0}$. Even if in general $\{T(t)\}_{t\geq 0}$ in neither strongly continuous nor analytic, and hence it is not possible to define the infinitesimal generator in the usual sense, however we can still associate a generator to $\{T(t)\}_{t\geq 0}$, which has properties similar to those of the infinitesimal generator.

We will provide two equivalent definitions of the weak generator.

For any $f \in C_b(\mathbb{R}^d)$ and any $\lambda > k_0$ the function $t \to e^{-\lambda t}T(t)f(x)$ is continuous and integrable in $(0,+\infty)$. Hence, the operators

$$R(\lambda)f(x) := \int_0^\infty e^{-\lambda t}T(t)f(x)dt \tag{2.23}$$

are bounded on $C_b(\mathbb{R}^d)$, and straightforward computations show that the family $\{R(\lambda) : \lambda > k_0\}$ satisfies the resolvent identity. Moreover, $R(\lambda)$ is injective for any $\lambda > k_0$; a result of functional analysis guarantees that there exists a unique closed operator $A_1 : D(A_1) \subset C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ such that

$$R(\lambda) = \mathbb{R}(\lambda, A_1), \quad Im(R(\lambda)) = D(A_1), \quad \lambda > k_0. \tag{2.24}$$

Definition 2.14 (see [19], [53])**.** A_1 *is the Weak Generator of* $\{T(t)\}_{t\geq 0}$*.*

The second definition of the weak generator is based on bounded pointwise convergence: a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C_b(\mathbb{R}^d)$ is said to be boundedly and pointwise convergent to $f \in C_b(\mathbb{R}^d)$ if there exists a positive constant C such that $\|f_n\|_\infty \leq C$, for any $n \in \mathbb{N}$, and $f_n(x) \to f(x)$ as $n \to +\infty$, for any $x \in \mathbb{R}^d$.

Definition 2.15 (see [91], [92])**.** *We call weak generator of the semigroup* $\{T(t)\}_{t\geq 0}$ *the operator* A_2 *defined as follows:*

$$\begin{cases} D(A_2) := \left\{ f \in C_b(\mathbb{R}^d) : \begin{array}{l} \displaystyle\sup_{t\in(0,1)} \frac{\|T(t)f - f\|_\infty}{t} < \infty; \\ \displaystyle\exists g \in C_b(\mathbb{R}^d) : \lim_{t\to 0^+} \frac{T(t)f(x) - f(x)}{t} = g(x), \ x \in \mathbb{R}^d \end{array} \right\} \\ \\ A_2 f(x) = g(x), \quad f \in D(A_2), \quad x \in \mathbb{R}^d. \end{cases} \tag{2.25}$$

Proposition 2.16. *The operators $(A_1, D(A_1))$ and $(A_2, D(A_2))$ coincide.*

Definition 2.17. *The operator $(\hat{A}, D(\hat{A})) := (A_1, D(A_1)) = (A_2, D(A_2))$ is called the weak generator of $\{T(t)\}_{t\geq 0}$.*

The weak generator fulfills the following properties.

Proposition 2.18. *For any $f \in D(\hat{A})$ and any fixed $x \in \mathbb{R}^d$, the function $T(\cdot)f(x)$ is continuously differentiable in $[0, \infty)$ and*

$$\frac{d}{dt}T(t)f(x) = (T(t)\hat{A}f)(x). \tag{2.26}$$

For any sequence $\{f_n\}_{n\in\mathbb{N}} \subset D(\hat{A})$ such that f_n and $\hat{A}f_n$ converge boundedly in a dominated way to $f, g \in C_b(\mathbb{R}^d)$, respectively, then $f \in D(\hat{A})$ and $\hat{A}f = g$.

Proposition 2.19. *We have*

$$D_{max}(A) \cap C_0(\mathbb{R}^d) \subset D(\hat{A}) \subset D_{max}(A), \tag{2.27}$$

where

$$D_{max}(A) := \left\{ f \in C_b(\mathbb{R}^d) : \bigcap_{p>1} W^{2,p}_{loc}(\mathbb{R}^d) : Af \in C_b(\mathbb{R}^d) \right\}. \tag{2.28}$$

Moreover, the following conditions are equivalent:

(i) *$\lambda \in \rho(A)$, for some $\lambda > k_0$;*

(ii) *$(k_0, \infty) \subset \rho(A)$;*

(iii) *$D(\hat{A}) = D_{max}(A)$.*

2.1.2 Compactness of $\{T(t)\}$ in $C_b(\mathbb{R}^d)$

In this section we deal with some properties of the semigroup $\{T(t)\}_{t\geq 0}$ generated by (PCP). In particular, we study the compactness of $\{T(t)\}_{t>0}$ in $C_b(\mathbb{R}^d)$ and the invariance of $C_0(\mathbb{R}^d)$ with respect to $\{T(t)\}_{t\geq 0}$.

We say that $\{T(t)\}_{t\geq 0}$ is conservative if $T(t)\mathbb{1} \equiv \mathbb{1}$ for any $t \geq 0$; let us observe that if $\{T(t)\}_{t\geq 0}$ is conservative then necessarily $c \equiv 0$.

The compactness of $\{T(t)\}_{t>0}$ is strictly connected with the tightness of the family of Borel measures $\{p(t, x, dy) : t > 0, x \in \mathbb{R}^d\}$, where $p(t, x, dy)$ has been defined in (2.21). Here, we recall the definition of tightness fora family of bounded measures.

Definition 2.20. *A family of Borel bounded measures $\{\mu_\alpha\}_{\alpha\in I}$ is said to be tight if for any $\epsilon > 0$ there exists a compact set $K_\epsilon \subset \mathbb{R}^d$ such that*

$$\mu_\alpha(\mathbb{R}^d \setminus K_\epsilon) \leq \epsilon, \quad \alpha \in I. \tag{2.29}$$

Proposition 2.21. *Suppose that* $\{T(t)\}_{t>0}$ *is conservative. Then* $\{T(t)\}_{t>0}$ *is compact in* $C_b(\mathbb{R}^d)$ *if and only if the system of Borel measures* $\{p(t,x,dy) : x \in \mathbb{R}^d\}$ *is tight, for any* $t > 0$.

A sufficient condition for the tightness of $\{p(t,x,dy) : x \in \mathbb{R}^d\}$, for any $t > 0$, is given by the proposition below.

Proposition 2.22. *Suppose that* $\{T(t)\}_{t>0}$ *is conservative. If there exist a strictly positive function* $\varphi \in C^2(\mathbb{R}^d)$ *and a convex and positive function* g *such that*

$$\lim_{|x|\to\infty} \varphi(x) = +\infty, \quad \frac{1}{g} \text{ is integrable at } +\infty, \quad A\varphi(x) \le -g(\varphi(x)), \quad x \in \mathbb{R}^d, \tag{2.30}$$

then $\{p(t,x,dy) : x \in \mathbb{R}^d\}$ *is tight for any* $t > 0$ *and, consequently,* $T(t)$ *is compact on* $C_b(\mathbb{R}^d)$ *for any* $t > 0$.

Example 2.23. *If we consider the operator* A *defined by*

$$Af(x) = \nabla f(x) + \langle b(x), Df(x)\rangle,$$

$x \in \mathbb{R}$, *on smooth functions* f, *with the drift* b *satisfying*

$$\langle b(x), x\rangle \le C - M|x|^{2+\varepsilon},$$

for some $C, M, \varepsilon > 0$, *then the associated semigroup* $\{T(t)\}_{t>0}$ *is compact. Indeed, if*

$$\varphi(x) = |x|^2, \quad g(s) = -(2N + C) + Ms^{1+\varepsilon/2},$$

for any $x \in \mathbb{R}$ *and* $s > 0$, *then we can apply Proposition 2.22.*

Now we assume that $\{\mathbf{T}(t)\}_{t\ge 0}$ is not conservative, so in particular c is not identically zero in $\mathbb{R}^d$. In this case it is possible to relate the compactness of $T(t)$ with the behavior at infinity of $T(t)\mathbb{1}$.

Theorem 2.24. *Fix* $t > 0$. *Then* $T(t)\mathbb{1} \in C_0(\mathbb{R}^d)$ *if and only if* $T(t)$ *is compact and it preserves* $C_0(\mathbb{R}^d)$.

Hence it is sufficient to give some conditions such that $T(t)\mathbb{1}$ belongs to $C_0(\mathbb{R}^d)$ for any $t > 0$ in order to get the compactness of the semigroup. The proposition below gives a sufficient condition, which guarantees that $T(t)\mathbb{1}$ vanishes at infinity, in terms of a suitable Lyapunov function.

Proposition 2.25. *Suppose that there exist* $\lambda_0 > k_0$, *a compact set* $K \subset \mathbb{R}^d$ *and a function* $\varphi \in C^2(\mathbb{R}^d \setminus K) \cap C_0(\mathbb{R}^d \setminus K)$ *such that*

$$\varphi(x) > 0, \ x \in \mathbb{R}^d \setminus K, \qquad \inf_{x\in\mathbb{R}^d\setminus K} A\varphi(x) - \lambda_0\varphi(x) := a > 0. \tag{2.31}$$

Then $T(t)\mathbb{1} \in C_0(\mathbb{R}^d)$ *for any* $t > 0$.

2.1.3 Uniform Estimates and Consequences

In this section we state uniform estimates of the spacial derivatives of $\{T(t)\}_{t>0}$, and some of their consequences. This problem has been widely studied in literature with both analytic and probabilistic techniques. Here, we consider the conservative case.

Since in the case of unbounded coefficients the semigroup $\{T(t)\}_{t\geq 0}$ is not analytic, it is not possible to use the well-known theory of analytic semigroups. However, with additional hypotheses on the coefficients of A it is possible to prove optimal results on the behavior of the semigroup and of its spacial derivatives near zero. The optimality relays on the fact that, near 0, we obtain the same singularity which we get for elliptic operators with bounded and smooth coefficients.

Here we only discuss uniform estimates, which have been studied in [11], [72] and [14], even if pointwise estimates have also been established. These estimates show that that for any $\omega > 0$ and any $0 \leq k \leq l \leq 3$ there exists a positive constant $C_{l,k} = C(l, k, \omega)$ such that

$$\|T(t)f\|_{C_b^l(\mathbb{R}^d)} \leq C_{l,k} t^{-(l-k)/2} e^{\omega t} \|f\|_{C_b^k(\mathbb{R}^d)}, \quad f \in C_b^k(\mathbb{R}^d). \tag{2.32}$$

Estimates (2.32) has been proved using the Bernstein method (see [10]). It consists in introducing the function

$$\begin{aligned} v_R(t,x) &= u_R(t,x)^2 + at\vartheta_R^2|Du_R(t,x)|^2 + a^2t^2\vartheta_R^4|D^2u_R(t,x)|^2 \\ &\quad + a^3t^3\vartheta_R^6|D^3u_R(t,x)|^2, \quad t \in [0,T], \quad x \in \overline{B(R)}, \end{aligned}$$

where, for any $R > 0$, u_R is the solution to the Dirichlet-Cauchy problem in $B(R)$ with initial datum $(\vartheta f)^2$, and ϑ is a suitable smooth function.

It is possible to apply the classical maximum principle to v_R, i.e., there exists $K > 0$ such that $|v_R(t,x)| \leq K\|f\|_\infty$. Since K does not depend on R, letting R to $+\infty$ we get the thesis.

Finally by interpolation it is possible to extend (2.32) to $k, l \in \mathbb{R}_+$, $0 \leq k \leq l \leq 3$.

Below we give the additional hypotheses we need to get inequalities (2.32).

Hypotheses 2.26. *Suppose that Hypothesis* 2.1 *holds. Moreover, assume the following conditions:*

(i) *there exists a constant* $C > 0$ *such that*

$$\left|\sum_{i,j=1}^d q_{ij}(x)x_j\right| \leq C\nu(x)(1+|x|^2), \tag{2.33}$$

$$\mathrm{Tr}(Q(x)) \leq C\nu(x)(1+|x|^2), \tag{2.34}$$

$$\left|\sum_{i=1}^d b_i(x)x_i\right| \leq C\nu(x)(1+|x|^2), \tag{2.35}$$

where $\nu(x)$ *has been defined in Hypothesis* 2.1(*i*).

Moreover, in the next theorem we need one of the following conditions:

(ii-1) $q_{ij}, b_i \in C^{1+\delta}_{loc}(\mathbb{R}^d)$ *for some* $\delta \in (0,1)$ *and any* $i, j = 1, \ldots, d$*. Further, there exist a positive constant* C *and a function* $d : \mathbb{R}^d \longrightarrow \mathbb{R}$ *with*

$$L_1 := \sup_{x \in \mathbb{R}^d} \frac{d(x)}{\nu(x)} < \infty \tag{2.36}$$

such that $|D_k q_{ij}(x)| \le C\nu(x)$ *for any* $i, j, k = 1, \ldots, d$*, and*

$$\sum_{i,j=1}^{d} D_i b_j(x) \xi_i \xi_j \le C d(x), \quad x, \xi \in \mathbb{R}^d; \tag{2.37}$$

(ii-2) $q_{ij}, b_i \in C^{2+\delta}_{loc}(\mathbb{R}^d)$ *for some* $\delta \in (0,1)$ *and any* $i, j = 1, \ldots, d$*, Hypothesis* 2.26*(ii−1) holds true and there exist a positive function* $r : \mathbb{R}^d \longrightarrow \mathbb{R}$ *and three constants* $K_1 \in \mathbb{R}$ *and* $L_2, L_3 > 0$ *such that*

$$\begin{aligned} &|D^\beta b_i(x)| \le r(x), \quad x \in \mathbb{R}^d, \; i = 1, \ldots, d, \; |\beta| = 2, \\ &d(x) + L_2 r(x) \le L_3 \nu(x), \quad x \in \mathbb{R}^d, \\ &\sum_{i,j,h,k=1}^{d} D_{hk} q_{ij}(x) m_{ij} m_{hk} \le K_1 \nu(x) \sum_{h,k=1}^{d} m_{hk}^2, \quad x \in \mathbb{R}^d, \end{aligned} \tag{2.38}$$

for any symmetric matrix $M = [m_{hk}]$*.*

(ii-3) $q_{ij}, b_i \in C^{3+\delta}_{loc}(\mathbb{R}^d)$ *for some* $\delta \in (0,1)$ *and any* $i, j = 1, \ldots, d$*, Hypothesis* 2.26*(ii−2) holds true and there exists* $C > 0$ *such that* $|D^\beta b_i(x)| \le C r(x)$ *and* $|D^\beta q_{ij}(x)| \le C\nu(x)$ *for any* $i, j = 1, \ldots, d$*, any* $|\beta| = 3|$ *and any* $x \in \mathbb{R}^d$*.*

Now we can state the main theorem of this section.

Theorem 2.27. *Let Hypotheses* 2.1*,* 2.26*(i) and* 2.26*(ii−l) hold true for some* $l \in \{1, 2, 3\}$*. Then for any* $\omega > 0$ *and any* $k = 0, 1, \ldots, l$ *there exist constants* $C_{l,k} = C_{l,k}(\omega) > 0$ *such that*

$$\|T(t)f\|_{C^l_b(\mathbb{R}^d)} \le C_{l,k} t^{-(l-k)/2} e^{\omega t} \|f\|_{C^k_b(\mathbb{R}^d)}, \quad f \in C^k_b(\mathbb{R}^d). \tag{2.39}$$

In particular, if $k = l$ *we can take* $\omega = 0$ *in* (2.39)*.*

As we said, the proof is based on an approximating argument. At first, we consider the case $l = 3$ and $k = 0$, since the others can be obtained in the analogous way. We define a smooth function $\vartheta_n : \mathbb{R}^d \longrightarrow \mathbb{R}$ by $\vartheta_n(x) = \vartheta(|x|/n)$, where $\vartheta \in C^\infty_c(\mathbb{R})$ and $\chi_{(-1,1)} \le \vartheta \le \chi_{(-2,2)}$.

For any $f \in C_b(\mathbb{R}^d)$ let u_n be the unique classical solution to

$$\begin{cases} D_t u_n(t,x) = A u_n(t,x), & x \in B(n), \quad t > 0, \\ u_n(t,x) = 0, & x \in \partial B(n), \quad t > 0, \\ u_n(0,\cdot) = \vartheta_n f, & x \in B(n). \end{cases}$$

From the interior Schauder estimates (see [63]) it follows that

$$\|u_n(t,\cdot)-u(t,\cdot)\|_{C^3(B(n))}\to 0,\quad n\to\infty, \tag{2.40}$$

for any $t>0$. For any $x\in B(n)$ and $t\in[0,t_0]$, where t_0 will be chosen in a suitable way, we set

$$\begin{aligned} v_{0,3,n}(t,x) :=& |u_n(t,x)|^2 + at\vartheta_n^2(x)|Du_n(t,x)|^2 + a^2t^2\vartheta_n^4(x)|D^2u_n(t,x)|^2 \\ &+ a^3t^3\vartheta_n^6(x)|D^3u_n(t,x)|^2. \end{aligned} \tag{2.41}$$

This function is smooth and satisfies the Dirichlet-Cauchy problem

$$\begin{cases} D_tv_{0,3,n}(t,x) = Av_{0,3,n}(t,x)+g(t,x), & x\in B(n), \quad t\in(0,t_0], \\ v_{0,3,n}(t,x)=0, & x\in\partial B(n), \quad t\in(0,t_0], \\ v_{0,3,n}(0,\cdot)=\vartheta_n f, & x\in B(n). \end{cases} \tag{2.42}$$

Long computations show that, with a suitable choice of a and t_0, we get $|g(t,x)|\leq 0$ for any $t\in(0,t_0]$ and $x\in B(n)$. The classical maximum principle implies that $v_{0,3,n}(t,x)\leq|\vartheta_n f(x)|^2$ and the uniform local convergence of u_n and ϑ_n respectively to u and $\mathbb{1}$ leads to the desired estimate.

We can now state the optimal regularity result for Cauchy problem

$$\begin{cases} D_tu(t,x)=Au(t,x)+g(t,x), & x\in\mathbb{R}^d, \quad t\in(0,T], \\ u(0,x)=f(x), & x\in\mathbb{R}^d, \end{cases} \tag{2.43}$$

Theorem 2.28. *Suppose that $f\in C_b^{2+\theta}(\mathbb{R}^d)$ for some $\theta\in(0,1)$, and $g\in C([0,T]\times\mathbb{R}^d)$ satisfies $g(t,\cdot)\in C_b^\theta(\mathbb{R}^d)$ for all $t\in[0,T]$ and*

$$[g]_\theta := \sup_{t\in(0,T)}\|g(t,\cdot)\|_{C_b^\theta(\mathbb{R}^d)}<\infty.$$

Then, problem (2.43) admits a unique bounded classical solution u, which is given by the variation of constants formula

$$u(t,x)=T(t)f(x)+\int_0^t (T(t-s)g(s,\cdot))(x)ds,\quad t\in[0,T],\ x\in\mathbb{R}^d. \tag{2.44}$$

Moreover, there exists a positive constant C, independent of u and the data, such that

$$\sup_{t\in(0,T)}\|u(t,\cdot)\|_{C_b^{2+\theta}(\mathbb{R}^d)}\leq C\left(\|f\|_{C_b^{2+\theta}(\mathbb{R}^d)}+[g]_\theta\right). \tag{2.45}$$

2.2 Nonautonomous Elliptic Operators

In this section we present the results [3]. The first paper which contains a systematic treatment of nonautonomous operators with null potential term is [60], after that in [25], [43] and [44] the nonautonomous Ornstein-Uhlenbeck operator has been studied. [3] extends some results of [60] and [71] in the case of non-zero potential and provides sufficient conditions for the compactness of the family of evolution operators associated to A, adapting the technique of [80].

2.2.1 Existence, Uniqueness and Main Properties of the Solution to the Cauchy Problem

Let $I = \mathbb{R}$ or be a right half-line, $\Lambda_I := \{(t, s) \in I \times I : t > s\}$, and $A(t)$ be the differential operator defined on smooth functions φ by

$$(A(t)\varphi)(x) = \sum_{i,j=1}^{d} q_{ij}(t, x)\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(t, x)\frac{\partial \varphi}{\partial x_i}(x) - c(t, x)\varphi(x), \tag{2.46}$$

for any $(t, x) \in I \times \mathbb{R}^d$, under the following hypotheses:

Hypotheses 2.29.

(i) *q_{ij}, b_i and c belong to $C^{\alpha/2,\alpha}_{loc}(I \times \mathbb{R}^d)$, for any $i, j = 1, \ldots, d$;*

(ii) *$c_0 := \inf_{(t,x)\in I\times\mathbb{R}^d} c(t, x) > -\infty$;*

(iii) *$Q(t, x) = [q_{ij}(t, x)]$ is a symmetric matrix for any $(t, x) \in I \times \mathbb{R}^d$; moreover there exist a function $\eta : I \times \mathbb{R}^d \longrightarrow \mathbb{R}$ such that $0 < \eta_0 := \inf_{(t,x)\in I\times\mathbb{R}^d} \eta(t, x)$ and*

$$\langle Q(t, x)\xi, \xi\rangle \geq \eta(t, x)|\xi|^2, \quad t \in I, \quad x, \xi \in \mathbb{R}^d; \tag{2.47}$$

(iv) *there exist a positive function $\varphi \in C^2(\mathbb{R}^d)$ and a constant $\lambda > 0$ such that $\lim_{|x|\to\infty} = +\infty$ and*

$$A(t)\varphi(x) - \lambda\varphi(x) < 0, \quad (t, x) \in I \times \mathbb{R}^d. \tag{2.48}$$

Remark 2.30. *As in the autonomous case, φ is called Lyapunov function.*

Hypothesis 2.29 guarantees that for any $f \in C_b(\mathbb{R}^d)$ the nonautonomous parabolic Cauchy problem

$$\begin{cases} D_t u(t, x) = A(t)u(t, x), & t > s, \quad x \in \mathbb{R}^d, \\ u(s, x) = f(x), & x \in \mathbb{R}^d. \end{cases} \tag{2.49}$$

admits a unique classical solution u which belongs to $C_{loc}^{1+\alpha/2,2+\alpha}((s,\infty)\times\mathbb{R}^d)$. Moreover,

$$|u(t,x)| \le e^{-c_0(t-s)}\|f\|_\infty, \quad s<t,\ x\in\mathbb{R}^d. \tag{2.50}$$

The proof is analogous to that in the autonomous case: the Lypaunov function is the key tool to prove uniqueness, while the existence follows from an approximation argument with solutions to Cauchy problems in bounded domains with Dirichlet or Neumann boundary conditions.

By means of the solution to (2.49) it is possible to define a family of bounded linear operators on $C_b(\mathbb{R}^d)$, setting $G(t,s): C_b(\mathbb{R}^d) \longrightarrow C_b(\mathbb{R}^d)$, $f \mapsto G(t,s)f := u(t,\cdot)$. The following proposition shows the main properties of these operators.

Proposition 2.31.

(i) *The family of operators $\{G(t,s)\}_{t\ge s\in I}$ defines an evolution operator on $C_b(\mathbb{R}^d)$, i.e. $G(t,s)$ is a bounded linear operator on $C_b(\mathbb{R}^d)$, for any $t>s\in I$, $G(s,s)=Id_{C_b(\mathbb{R}^d)}$ and $G(t,r)G(r,s)=G(t,s)$ for any $t>r>s\in I$;*

(ii) *the evolution operator $G(t,s)$ can be represented in the form*

$$G(t,s)f(x) = \int_{\mathbb{R}^d} g(t,s,x,y)f(y)dy, \quad s<t,\ x\in\mathbb{R}^d, \tag{2.51}$$

for any $f\in C_b(\mathbb{R}^d)$, where $g:\Lambda_I\times\mathbb{R}^d\times\mathbb{R}^d\longrightarrow\mathbb{R}$ is a positive function. For any $s\in I$ and almost any $y\in\mathbb{R}^d$ the function $g(\cdot,s,\cdot,y)\in C_{loc}^{1+\alpha/2,2+\alpha}((s,\infty)\times\mathbb{R}^d)$ and it solves the equation $D_tg-Ag=0$ in $(s,\infty)\times\mathbb{R}^d$. Moreover,

$$\|g(t,s,x,\cdot)\|_{L^1(\mathbb{R}^d)} \le e^{-c_0(t-s)}, \quad s<t,\ x\in\mathbb{R}^d. \tag{2.52}$$

g is called the Green function of $D_tu-Au=0$ in $(s,\infty)\times\mathbb{R}^d$;

(iii) *$G(t,s)$ can be extended to $B_b(\mathbb{R}^d)$ by the formula (2.51). Each operator $G(t,s)$ is irreducible and has the strong Feller property.*

The Green function g defined in (2.51) is the key tool to prove the compactness of evolution operator in $C_b(\mathbb{R}^d)$. Indeed it is possible to introduce a family of Borel measures which are equivalent to the Lebesgue measure, and, as in autonomous case, one can prove that the tightness of this family is equivalent to the compactness of evolution operator.

At first, we define this family of Borel measures and present its basic properties which immediately follow from Proposition 2.31.

Corollary 2.32. *For any $(t,s)\in\Lambda_I$ and any $x\in\mathbb{R}^d$ we define the measure $g_{t,s}(x,dy)$ by setting $g_{s,s}(x,dy)=\delta_x$ and*

$$g_{t,s}(x,A) := \int_A g(t,s,x,y)dy, \tag{2.53}$$

for any Borel set $A \subset \mathbb{R}^d$. Then each measure $g_{t,s}(x,dy)$ is equivalent to the Lebesgue measure and for any $t > r > s \in I$ we have

$$g_{t,s}(x,A) = \int_{\mathbb{R}^d} g_{r,s}(y,A) g_{t,r}(x,dy), \quad A \in B(\mathbb{R}^d). \tag{2.54}$$

To prove the compactness of $G(t,s)$ we need some stronger hypothesis on the Lyapunov function.

Hypothesis 2.33. *For any bounded interval $J \subset I$ there exist a positive function $\varphi = \varphi_J \in C^2(\mathbb{R}^d)$ and a positive constant $\lambda = \lambda_J$ such that $\lim_{|x|\to\infty} \varphi(x) = +\infty$ and*

$$(A(t) + c(t,x))\varphi(x) - \lambda\varphi(x) < 0, \quad t \in J,\ x \in \mathbb{R}^d. \tag{2.55}$$

2.2.2 Compactness

The aim of this part consists in giving sufficient conditions which ensure that the operator $G(t,s)$ is compact. In the conservative case ($c \equiv 0$) a sufficient condition for $G(t,s)$ to be compact in $C_b(\mathbb{R}^d)$ has been established in [71].

For any $J \subset I$ we set

$$\tilde{\Lambda}_J := \{(t,s) \in J \times J : t > s\}. \tag{2.56}$$

As in autonomous case, the tightness of the family of measures $\{g_{t,s}(x,dy) : s,t \in \tilde{\Lambda}_J, s < t, x \in \mathbb{R}^d\}$ and the compactness of the evolution operator $\{G(t,s)\}_{s<t}$, $s,t \in J$, are strictly connected.

Proposition 2.34. *Let $J \subset I$ be an interval. The following are equivalent:*

(i) *for any $(t,s) \in \tilde{\Lambda}_J$, $G(t,s)$ is compact on $C_b(\mathbb{R}^d)$;*

(ii) *the family of measures $\{g_{t,s}(x,dy) : x \in \mathbb{R}^d\}$ is tight for any $(t,s) \in \tilde{\Lambda}_J$.*

Hence, to prove the compactness of evolution operator we need to study the tightness of this family of measures.

To conclude, we present a sufficient condition ensuring the compactness of the evolution operator.

Theorem 2.35. *Assume that Hypothesis* 2.33 *is satisfied and there exist $K > 0, d_1, d_2 \in I$ with $d_1 < d_2$, a positive function $\eta \in C^2(\mathbb{R}^d)$ blowing up as $|x| \to \infty$, and a convex function $h : [0,\infty) \longrightarrow \mathbb{R}$ such that $1/h \in L^1(a,\infty)$, for large a, and*

$$((A(s)\eta)(x) \le -h(\eta(x)), \quad s \in [d_1,d_2],\ |x| \ge K. \tag{2.57}$$

Moreover, assume that, for the interval $J = [d_1,d_2] \subset I$there exist $\mu \in \mathbb{R}, R > 0$ and a positive and bounded function $W \in C^2(\mathbb{R}^d \setminus B(0,R))$ such that $\inf_{x\in\mathbb{R}^d\setminus B(0,R)} W(x) > 0$ and

$$A(t)W(x) - \mu W(x) \le 0, \quad (t,x) \in J \times \mathbb{R}^d \setminus B(0,R). \tag{2.58}$$

Then $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$ for any $(t,s) \in \Lambda_I$ such that $s \le d_2$ and $t \ge d_1$, and $t \ne s$.

2.3 Weakly-Coupled Systems

While parabolic PDE's with unbounded coefficients have been widely studied in the last decades, the same has not been done for systems of parabolic PDE's with unbounded coefficients.

The first results have been obtained in L^p spaces, assuming that the diffusion coefficients are bounded in $\mathbb{R}^d$. Here, we follow [31], and we consider the vector-valued elliptic operator $\mathcal{A}$ defined on the smooth function φ by

$$(\mathcal{A}\varphi(x))_k = \sum_{i,j=1}^{d} q_{ij}(x)D^2_{ij}\varphi_k(x) + \sum_{i=1}^{d} b_i(x)D_i\varphi_k(x) + \sum_{h=1}^{d} V_{kh}(x)\varphi_h(x), \tag{2.59}$$

for $x \in \mathbb{R}^d$ and $k = 1, \dots, m$. Let us notice that only the terms of order zero are coupled.

The aim is to define a semigroup $\{\mathbf{T}(t)\}_{t\geq 0}$ in terms of the solution to

$$\begin{cases} D_t\mathbf{u}(t,x) = \mathcal{A}\mathbf{u}(t,x), & x \in \mathbb{R}^d, \quad t > 0, \\ \mathbf{u}(0,\cdot) = \mathbf{f}, & x \in \mathbb{R}^d, \end{cases} \tag{2.60}$$

with $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and to analyze the main properties properties of this semigroup.

2.3.1 The Cauchy Problem and the Definition of the Semigroup

We assume the following hypotheses on the coefficients of operator $\mathcal{A}$ defined in (2.60).

Hypotheses 2.36. *The coefficients of operator $\mathcal{A}$ satisfy the following conditions:*

(i) $a_{ij}, b_i \in C^\alpha_{loc}(\mathbb{R}^d)$ *for any* $i, j = 1, \dots, d$;

(ii) *there exists a continuous function* $\nu : \mathbb{R}^d \longrightarrow (0,\infty)$ *and a positive constant* ν_0 *such that* $\nu(x) \geq \nu_0$, *for any* $x \in \mathbb{R}^d$, *and*

$$\langle Q(x)\xi, \xi\rangle \geq \nu(x)|\xi|^2, \quad x, \xi \in \mathbb{R}^d, \tag{2.61}$$

where $Q(x) = [q_{ij}(x)]$ *is a symmetric matrix;*

(iii) $V_{hk} \in C^\alpha_{loc}(\mathbb{R}^d)$ *for any* $h, k = 1, \dots, m$ *and there exists a function* $k \in C^\alpha_{loc}(\mathbb{R}^d)$ *with* $k_0 := \sup_{x\in\mathbb{R}^d} k(x) < +\infty$ *such that*

$$\langle V(x)\xi, \xi\rangle \leq k(x)|\xi|^2, \quad x, \xi \in \mathbb{R}^d; \tag{2.62}$$

(iv) *there exist a positive function* $\varphi \in C^2(\mathbb{R}^d)$ *which blows up as* $|x| \to +\infty$ *and a constant* $\lambda_0 > 0$ *such that*

$$\lambda_0\varphi(x) - \sum_{i,j=1}^{d} q_{ij}(x)D^2_{ij}\varphi(x) - \sum_{i=1}^{d} b_i(x)D_i\varphi(x) - 2k(x)\varphi(x) > 0, \tag{2.63}$$

for any $x \in \mathbb{R}^d$.

To enlighten the notations, we set

$$\mathcal{A}_{2k} := \sum_{i,j=1}^{d} q_{ij}(x)D^2_{ij} + \sum_{i=1}^{d} b_i(x)D_i + 2k(x),$$

and denote by $\{S_{2k}(t)\}_{t\geq 0}$ the semigroup of bounded linear operators associated to the classical solution to

$$\begin{cases} D_t\mathbf{u}(t,x) = \mathcal{A}_{2k}\mathbf{u}(t,x), & x \in \mathbb{R}^d, \quad t > 0, \\ \mathbf{u}(0,\cdot) = \mathbf{f}, & x \in \mathbb{R}^d, \end{cases}$$

with $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$.

Fixed $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$, we look for a classical solution $\mathbf{u}$ to (2.60), that is a function $\mathbf{u} \in C([0,\infty)\times\mathbb{R}^d;\mathbb{R}^m)\cap C^{1,2}((0,\infty)\times\mathbb{R}^d;\mathbb{R}^m)$ which is bounded in the strip $[0,T]\times\mathbb{R}^d$, for any $T > 0$. Uniqueness follows from Hypotheses 2.36$(iii)-(iv)$, while existence is established by means of limit of solutions in bounded domains with Dirichlet boundary conditions.

Proposition 2.37. *Let $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$, and let $\mathbf{u}$ be a classical solution to* (2.60)*, which is bounded in $[0,T]\times\mathbb{R}^d$ for any $T > 0$. Then,*

$$|\mathbf{u}(t,\cdot)|^2 \leq S_{2k}(t)(|\mathbf{f}|^2), \quad t \geq 0. \tag{2.64}$$

In particular,

$$\|\mathbf{u}(t,\cdot)\|_\infty \leq \sqrt{M}e^{k_0 t}\|\mathbf{f}\|_\infty, \quad t \geq 0, \tag{2.65}$$

where k_0 is the constant in Hypothesis 2.36(iii).

Theorem 2.38. *For any $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ the Cauchy problem* (2.60) *admits a unique classical solution $\mathbf{u}$ which is bounded in $[0,T]\times\mathbb{R}^d$, for any $T > 0$. $\mathbf{u}$ belongs to $C^{1+\alpha/2,2+\alpha}_{loc}((0,\infty)\times\mathbb{R}^d;\mathbb{R}^m)$ and*

$$\|\mathbf{u}(t,\cdot)\|_\infty \leq \sqrt{M}e^{k_0 t}\|\mathbf{f}\|_\infty, \quad t \geq 0, \tag{2.66}$$

where k_0 and α are defined in Hypotheses 2.36$(i),(iii)$.

By Proposition 2.37 and Theorem 2.38 the family of operators $\{\mathbf{T}(t)\}_{t\geq 0}$ defined by $\mathbf{T}(t)\mathbf{f} := \mathbf{u}(t,\cdot)$, where $\mathbf{u}$ is the unique classical solution to (2.60) with initial datum $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$, is a semigroup of bounded linear operators which satisfies

$$\|\mathbf{T}(t)\|_{\mathcal{L}(C_b(\mathbb{R}^d;\mathbb{R}^m))} \leq \sqrt{M}e^{k_0 t}, \quad t > 0. \tag{2.67}$$

According to Definitions 2.14 and 2.15, it is possible to define the weak generator of $\{\mathbf{T}(t)\}_{t\geq 0}$. It suffices to replace $C_b(\mathbb{R}^d)$ with $C_b(\mathbb{R}^d;\mathbb{R}^m)$. We denote by $\hat{A}_1$ the operator defined as in Definition 2.14, and $\hat{A}_2$ the operator defined as in Definition 2.15 . As in the scalar case, these operators coincide and we denote by $A := \hat{A}_1 = \hat{A}_2$ the weak generator of $\{\mathbf{T}(t)\}_{t\geq 0}$. The following proposition gives a useful characterization of the weak generator A.

Proposition 2.39. *Let $D_{max}(A)$ be the maximal domain of the realization of A in $C_b(\mathbb{R}^d;\mathbb{R}^m)$, i.e.*

$$D_{max}(A) := \{\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m) \bigcap_{p>1} W^{2,p}_{loc}(\mathbb{R}^d;\mathbb{R}^m) : \mathcal{A}\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)\}. \tag{2.68}$$

Then $D_{max}(\mathcal{A}) = D(A)$ and $\mathcal{A}\mathbf{f} = A\mathbf{f}$, for any $\mathbf{f} \in D(A)$, where $\mathcal{A}\mathbf{f}$ is understood in the sense of distributions.

The last result of this subsection is the next proposition, which shows that $\mathbf{T}(t)$ and $\mathcal{A}$ commute on $D_{max}(\mathcal{A})$.

Proposition 2.40. *For any $\mathbf{f} \in D_{max}(\mathcal{A})$ and any $t > 0$ $\mathbf{T}(t)\mathbf{f} \in D_{max}(\mathcal{A})$. Moreover, $\mathcal{A}\mathbf{T}(t)\mathbf{f} = \mathbf{T}(t)\mathcal{A}\mathbf{f}$.*

2.3.2 Compactness Properties of $\{\mathbf{T}(t)\}_{t\geq 0}$

Compactness properties of $\{\mathbf{T}(t)\}_{t>0}$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ are linked to those of the scalar semigroups $\{S_0(t)\}_{t>0}$ and $\{S_{2k}(t)\}_{t>0}$ in $C_b(\mathbb{R}^d)$. In particular, it has been proved that the compactness of the scalar semigroup $\{S_{2k}(t)\}_{t>0}$, related to the elliptic operator $\mathcal{A}_{2k}$, implies the compactness of the vector-valued semigroup, and vice versa.

Theorem 2.41. *If $\{S_{2k}(t)\}_{t>0}$ is compact in $C_b(\mathbb{R}^d)$, then also $\{\mathbf{T}(t)\}_{t>0}$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$.*

This result is a consequence of an integral representation of $S_{2k}(t)$ and estimate (2.64). Hence, to prove the compactness of $\{\mathbf{T}(t)\}_{t>0}$ it is sufficient to find conditions which ensure the compactness of $\{S_{2k}(t)\}_{t>0}$. This is the content of the following corollary.

Corollary 2.42. *Suppose that one of the following conditions holds:*

(i) *there exist a positive function $\tilde{\varphi} \in C^2(\mathbb{R}^d)$ blowing up as $|x| \to +\infty$ and a convex function $g \in L^1(a,\infty)$ for large a such that $\mathcal{A}_0\tilde{\varphi}(x) + g(\tilde{\varphi}(x)) \leq 0$ for any $x \in \mathbb{R}^d$;*

(ii) *there exist $\lambda, R > 0, \mu \in \mathbb{R}$, a function $\psi \in C^2(\mathbb{R}^d \setminus B(R))$ with positive infimum, two positive functions $\tilde{\varphi}_1, \tilde{\varphi}_2 \in C^2(\mathbb{R}^d)$ blowing up as $|x| \to +\infty$ and a convex function g as in (i) such that $\mathcal{A}_{2k}\psi - \mu\psi \geq 0$ in $\mathbb{R}^d \setminus B(R)$, $\mathcal{A}_0\tilde{\varphi}_1 - \lambda\tilde{\varphi}_1 < 0$ in $\mathbb{R}^d$ and $\mathcal{A}_{2k}\tilde{\varphi}_2 + g(\tilde{\varphi}_2) \leq 0$ in $\mathbb{R}^d$.*

Then, the semigroup $\{\mathbf{T}(t)\}_{t>0}$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$.

Now we prove the other implication, i.e., we provide sufficient conditions which guarantee that, if $\{\mathbf{T}(t)\}_{t>0}$ is compact, then $\{S_{2k}(t)\}_{t>0}$ is compact as well. We need to consider the operator $\mathcal{A}_0 := A_{2k} - 2k$ and the semigroup $\{S_0(t)\}_{t\geq 0}$ associated to it, but before entering into details, we stress that under Hypotheses 2.36 it is possible that Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}_0 u(t,x), & x \in \mathbb{R}^d, \quad t > 0, \\ \mathbf{u}(0,\cdot) = f, & x \in \mathbb{R}^d, \end{cases} \tag{2.69}$$

admits more then one classical solution. Nevertheless, we can associate a semigroup to the operator $\mathcal{A}_0$ by setting $u(t,x) := \lim_{n\to\infty} u_n(t,x)$, where u_n is the unique classical solution to the Dirichlet-Cauchy problem

$$\begin{cases} D_t u_n(t,x) = \mathcal{A}_0 u_n(t,x), & x \in B(n), \quad t > 0, \\ u_n(t,x) = 0, & x \in \partial B(n), \quad t > 0, \\ u_n(0,\cdot) = f, & x \in B(n), \end{cases}$$

and u is the minimal solution to (2.69) (see Remark 2.5). If we define $S_0(t)f(x) := u(t,x)$, we get that $\{S_0(t)\}_{t\ge 0}$ is a semigroup of bounded operators on $C_b(\mathbb{R}^d)$.

The first step is the proposition below, which links the scalar semigroup with the vector one.

Proposition 2.43. *Assume that $V_{\bar{k}h}$ is bounded in $\mathbb{R}^d$ for any $h = 1,\dots,m$ and some $\bar{k} \in \{1,\dots,m\}$. Then, for any $t > 0$, any $x \in \mathbb{R}^d$ and any $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ we have*

$$(\mathbf{T}(t)\mathbf{f})_{\bar{k}}(x) = S_0(t)\mathbf{f}_{\bar{k}}(x) + \int_0^t (S_0(t-s)(V\mathbf{T}(s)\mathbf{f})_{\bar{k}})(x)ds. \tag{2.70}$$

Now we can state the following result.

Theorem 2.44. *Suppose that $\{\mathbf{T}(t)\}_{t>0}$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$, and that there exists $\bar{k} \in \{1,\dots,m\}$ such that $V_{\bar{k}h}$ is bounded in $\mathbb{R}^d$ for any $h = 1,\dots,m$. Then both $\{S_0(t)\}_{t>0}$ and $\{S_{2k}(t)\}_{t>0}$ are compact in $C_b(\mathbb{R}^d)$.*

2.3.3 Uniform Estimates and Consequences

Here, we state estimates analogous to (2.39) for the vector-valued semigroup $\{\mathbf{T}(t)\}_{t>0}$. In particular, we show that

$$\|\mathbf{T}(t)\psi\|_{C_b^\theta(\mathbb{R}^d;\mathbb{R}^m)} \le C_{\theta,\beta} t^{-(\theta-\beta)/2} e^{\omega_{\theta,\beta}t} \|\psi\|_{C_b^\beta(\mathbb{R}^d;\mathbb{R}^m)}, \quad t > 0, \quad \psi \in C_b^\beta(\mathbb{R}^d;\mathbb{R}^m), \tag{2.71}$$

for any $0 \le \beta \le \theta \le 3$ and some positive constants $C_{\theta,\beta}$ and $\omega_{\theta,\beta}$. Besides Hypotheses 2.36 we also require the following conditions to be satisfy.

Hypotheses 2.45. (i) *q_{ij} and b_i belong to $C_{loc}^{3+\alpha}(\mathbb{R}^d)$, for any $i,j = 1,\dots,d$;*

(ii) *the function ν in (2.61) is bounded from below by a positive constant ν_0;*

(iii) *there exist a positive constant $C > 0$ such that*

$$|Q(x)x| + \mathrm{Tr}(Q(x)) \le C(1+|x|^2)\left(\nu(x)\sqrt{\nu(x)k(x)}\right), \tag{2.72}$$

$$\langle b(x), x\rangle \le C(1+|x|^2)\left(\nu(x)\sqrt{\nu(x)k(x)}\right), \tag{2.73}$$

for any $x \in \mathbb{R}^d$;

(iv) *there exist functions* $h, r : \mathbb{R}^d \longrightarrow \mathbb{R}$ *and positive constants* M, l, p, q *such that* $6l + (5p + 6q)d < 2$, $h + Mr \le C\nu + l|k|$ *in* $\mathbb{R}^d$ *and*

$$|D^\alpha q_{ij}(x)| \le C\nu(x) + g_{|\alpha|}(x), \tag{2.74}$$

$$\langle Db(x)\xi, \xi\rangle \le h(x)|\xi|^2, \tag{2.75}$$

$$|D^\beta b(x)| \le r(x), \tag{2.76}$$

for any $i, j = 1, \ldots, d$, *any* $x, \xi \in \mathbb{R}^d$ *and any multi-index* α *and* β *with* $|\alpha| = 1, 2, 3$ *and* $|\beta| = 2, 3$, *where* $g_1(x) = p\sqrt{\nu|k|}, g_2(x) = q|k|, g_3(x) = C|k|$;

(v) *for any multi-index* β *with* $|\beta| = 1, 2, 3$, *it holds that*

$$|D^\beta V(x)| \le C\left(\sqrt{\nu(x)|k(x)|} + |k(x)| + 1\right), \quad x \in \mathbb{R}^d. \tag{2.77}$$

Theorem 2.46. *Under Hypotheses* 2.36 *and* 2.45 *estimates* (2.71) *hold true.*

As a consequence of Theorem (2.46), we get an optimal regularity result for Cauchy problem

$$\begin{cases} D_t\mathbf{u}(t,x) = \mathcal{A}\mathbf{u}(t,x) + \mathbf{g}(t,x), & x \in \mathbb{R}^d, \quad t \in (0,T], \\ \\ \mathbf{u}(0,x) = \mathbf{f}(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.78}$$

Theorem 2.47. *Suppose that* $\mathbf{f} \in C_b^{2+\theta}(\mathbb{R}^d;\mathbb{R}^m)$, *for some* $\theta \in (0,1)$, *and* $\mathbf{g} \in C([0,T] \times \mathbb{R}^d;\mathbb{R}^m)$ *satisfies* $\mathbf{g}(t,\cdot) \in C_b^\theta(\mathbb{R}^d;\mathbb{R}^m)$ *for all* $t \in [0,T]$ *and*

$$[\mathbf{g}]_\theta := \sup_{t\in(0,T)} \|\mathbf{g}(t,\cdot)\|_{C_b^\theta(\mathbb{R}^d;\mathbb{R}^m)} < \infty.$$

Then problem (2.78) *admits a unique bounded classical solution* $\mathbf{u}$, *which is given by the variation of constants formula*

$$\mathbf{u}(t,x) = \mathbf{T}(t)\mathbf{f}(x) + \int_0^t (\mathbf{T}(t-s)\mathbf{g}(s,\cdot))(x)ds, \quad t \in [0,T],\ x \in \mathbb{R}^d. \tag{2.79}$$

Moreover, there exists a positive constant C, *independent of* $\mathbf{u}$ *and the data, such that*

$$\sup_{t\in(0,T)} \|\mathbf{u}(t,\cdot)\|_{C_b^{2+\theta}(\mathbb{R}^d;\mathbb{R}^m)} \le C\left(\|\mathbf{f}\|_{C_b^{2+\theta}(\mathbb{R}^d;\mathbb{R}^m)} + [\mathbf{g}]_\theta\right). \tag{2.80}$$

Chapter 3

Optimal Control Problems

The *optimal control theory* appeared for the first time only at the half of the past century, even if its interest is related to problems whose nature has a long history. The deterministic case consists in the treatment of a dynamical system

$$\begin{cases} \dot{x}(t) = b(t, x(t), u(t)), \\ x(0) = x_0 \in \mathbb{R}^d, \end{cases} \tag{3.1}$$

associated with a cost functional

$$J(u) := \int_0^T f(t, x(t), u(t))dt + h(x(T)), \tag{3.2}$$

and the aim is to minimize (or maximize) J over all u belonging to a suitable space.

Randomness was considered in the early stages of the development of this theory, and the first paper in which the expression "stochastic control" appeared was that of Bellman ([9]). However, in that paper the Itô type differential equation was not involved. The first connection between control theory and stochastic differential equations arose in [36], where Bellman's dynamic programming (see [8]) was used to derive a partial differential equation associated with a continuous-time controlled Markov process. Given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in which a $d-$dimensional Brownian motion $W(\cdot)$ is defined, the stochastic control problem has the following form. We deal with a state equation as

$$\begin{cases} dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \\ x(0) = x_0 \in \mathbb{R}^d, \end{cases} \tag{3.3}$$

where β and σ are Borel mesurable functions which take values in $\mathbb{R}^d$, $\mathbb{R}^{m \times d}$ respectively, and a cost functional as

$$J(u) := \mathbb{E}\left[\int_0^T f(t, x(t), u(t))dt\right] + \mathbb{E}[h(x(T))], \tag{3.4}$$

where x and u are stochastic processes which have to satisfy some adaptability conditions, and the goal is always to find the optimal value of J and some pair $(\bar{x}, \bar{u})$ for which this value is reached.

The main steps of the classical dynamic programming approach are the following. At first, we let the initial time and state vary and define the value function V. Then, we establish the Bellman's Optimality Principle, together with some continuity and local boundedness of the value function, and we prove that the value function is the unique solution to a certain partial differential equation. Finally, under suitable regularity assumptions, the application of some verification theorem concludes the procedure.

This means that, if $T > 0$ is fixed, for any $(s, y) \in [0, T) \times \mathbb{R}^d$ we consider the state equation (3.3) and the cost functional (3.4).

Hence, we define the value function

$$\begin{cases} V(s,y) = \inf_{u \in \mathcal{U}^w[s,T]} J(s,y,u), & (s,y) \in [0,T) \times \mathbb{R}^d, \\ V(T,y) = h(y), & y \in \mathbb{R}^d, \end{cases}$$

where

$$\mathcal{U}^w[0,T] := \{u : [0,T] \times \Omega \longrightarrow U | u \text{ is } \{\mathcal{F}_t\}_{t \geq 0}\text{−adapted}\},$$

and $U \subset \mathbb{R}^m$.

if $V \in C^{1,2}([0,T] \times \mathbb{R}^d)$, then it satisfies the backward partial differential equation

$$\begin{cases} -v_t + \sup_{u \in U} G(t,x,u,-v_x,-v_{xx}) = 0, & (t,x) \in [0,T) \times \mathbb{R}^d, \\ v(T,x) = h(x), & x \in \mathbb{R}^d, \end{cases}$$

which is called the *Hamilton-Jacobi-Bellman equation*, HJB for short, where

$$G(t,x,u,p,q) := \mathrm{Tr}(\sigma(t,x,u)\sigma^*(t,x,u)^t q) + \langle b(t,x,u), p \rangle - f(t,x,u),$$

for any $(t,x) \in [0,T) \times \mathbb{R}^d$, $u \in U$, $p \in \mathbb{R}^d$ and $\in \mathbb{R}^{d \times d}$. Here, $\mathcal{U}^w[s,T]$ is the space of adapted controls with respect a suitable filtration.

The tools to connect the value function, defined starting from the stochastic optimal control problem, and the above PDE are the *Bellman's Optimality Principle* and the Itô formula. The first gives a representation formula of $V(s,y)$ in terms of the functional cost, while the second is the classical bridge between SDE's and PDE's. However a problem arises, and it is linked with the regularity required to V in order to apply the Itô formula; indeed in general the solution of the HJB equation is not smooth enough, and so it is not possible to use these techniques.

Since, in general, the smoothness of the solution V to HJB equation fails, different approaches to the stochastic control problem has been considered, which give rise to different notions of solution.

A possible alternative is the notion of viscosity solutions. It was introduced in [22] and [23] for first order Hamilton-Jacobi equations, which are related to deterministic optimal control problems, and in [21] for second order partial differential equations, connected with stochastic optimal control problems. Based on the optimality principle, it is possible to prove in a natural way that the value function is a viscosity solution to the corresponding HJB equation. Finally, uniqueness of viscosity solutions was proved with different methods (see [54] and [51]).

As mentioned above, we follow a different method. which has as starting point the backward stochastic differential equations (BSDE for short), i.e., Itô equations with final conditions. Firstly, Bismut (see [16], [17]) introduced a linear BSDE with adapted solutions when he was studying adjoint equations of the stochastic optimal control problem, but the first systematic treatment of the nonlinear BSDE

$$\begin{cases} dY(t) = f(t, Y(t), Z(t))dt + Z(t)dW(t), \quad t \in [0, T], \\ \\ Y(T) = \xi, \end{cases}$$

is due to Pardoux and Peng ([89]).

Another interesting situation is when f and ξ depend on a given process X, which is aolution to a forward equation. In this case we talk about the forward-backward stochastic differential equation (FBSDE for short)

$$\begin{cases} dY(t) = \psi(t, X(t), Y(t), Z(t))dt + Z(t)dW(t), \quad t \in [0, T], \\ \\ Y(T) = \varphi(X(T)), \end{cases} \tag{3.5}$$

initially studied in [5].

If X is the solution to a SDE of Itô type, then it is possible to connect FBSDE with the semi-linear parabolic PDE

$$\begin{cases} D_t u(t, x) + Au(t, x) = \psi(t, x, u(t, x), \nabla u(t, x)G(x)), & x \in \mathbb{R}^d, \ t \in [0, T), \\ \\ u(T, x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.6}$$

where A is an uniformly elliptic operator defined by means of the SDE (see [89]). In these directions a lot of developments were done, see e.g. [33], [34], [75], [88], [90], [30].

Finally, following the approach of [41], which holds both in finite and infinite dimension, we consider an application of FBSDE to a particular stochastic optimal control problem in weak formulation. Indeed, the authors consider the problem (3.3) in the following form,

$$\begin{cases} dx(t) = b(x(t))dt + \sigma(x(t))r(x(t), u(t))dt + \sigma(x(t))dW(t), \\ \\ x(0) = x_0 \in \mathbb{R}^d, \end{cases} \tag{3.7}$$

where $\sigma(x)$ is a $d-$dimensional square matrix, and the cost functional is given by

$$J(u) := \mathbb{E}\left[\int_0^T l(x(t), u(t))dt\right] + \mathbb{E}[\varphi(x(T))]. \tag{3.8}$$

Then, setting

$$\psi(x,z) := \inf_u [l(x,u) + zr(x,u)],$$

they show that the existence of an optimal control u is related with the solvability (3.5).

We also consider the state equation (3.7) and the cost functional (3.8)

Our aim is to prove the existence of an optimal control for the above problem through the existence and uniqueness of a mild solution to the semilinear Cauchy problem

$$\begin{cases} D_t u(t,x) + Au(t,x) = \psi(t,x,u(t,x),\nabla u(t,x)\sigma(x)), & x \in \mathbb{R}^n,\ t \in [0,T), \\ u(T,x) = \varphi(x), & x \in \mathbb{R}^n, \end{cases}$$

where A is the uniformly elliptic operator defined on smooth function g by

$$Ag(x) = \frac{1}{2}\mathrm{Tr}(\sigma(x)\sigma^*(x)D^2 g(x)) + \langle b(x), \nabla g(x)\rangle,$$

for any $x \in \mathbb{R}^d$. This approach is based on the study of a forward backward stochastic differential system, following the idea of [41].

3.1 A formulation of Stochastic Control Problem

There are two possible formulations of a stochastic optimal control problem. Exactly as for stochastic differential equations, the difference arises form the concept of solution; indeed, if we consider the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ being fixed and we look for a solution (u,x) of the stochastic controlled equation, we are speaking about strong formulation. Otherwise, if also the probability space is part of the solution, the setting is that of weak formulation. The formal differences of the two definitions are explained below.

3.1.1 Strong Formulation

Fixed a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$ satisfying the usual conditions on which an $m-$dimensional standard Brownian motion W is defined, consider the following controlled stochastic differential equation

$$\begin{cases} dx(t) = b(t,x(t),u(t))dt + \sigma(t,x(t),u(t))dW(t), \\ x(0) = x_0 \in \mathbb{R}^d, \end{cases} \tag{3.9}$$

where $b : [0,T]\times\mathbb{R}^d\times U \longrightarrow \mathbb{R}^d$, $\sigma : b : [0,T]\times\mathbb{R}^d\times U \longrightarrow \mathbb{R}^{d\times m}$, with U being a separable metric space, and $T \in (0,\infty)$ being fixed. The function u is called *control* and represents the action or decision or policy of the decision-maker (*controller*). At time $t \in (0,T)$ the controller is knowledgeable of some information about what happened till that moment, but is not able to foretell what is going to happen. This non-anticipative restriction in mathematical terms can be represented by the condition that u is $\{\mathcal{F}_t\}_{t\geq 0}-$adapted. Hence we define

$$\mathcal{U}^s[0,T] := \{u : [0,T] \times \Omega \longrightarrow U | u \text{ is } \{\mathcal{F}_t\}_{t\geq 0}-\text{adapted}\}, \tag{3.10}$$

and we consider the cost functional (3.4).

Definition 3.1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be probability space satisfying the usual conditions on which an $m-$dimensional standard Brownian Motion W is defined. A control u is called an $s-$admissible control ($s-$a.c for short), and (u, x) an $s-$admissible pair ($s-$a.p. for short) if*

(i) $u \in \mathcal{U}^s[0,T]$;

(ii) *x is the unique strong solution to* (3.9) *(see Section A.3);*

(iii) $f(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}(0,T;\mathbb{R})$, $h(x(T)) \in L^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$ *and* $\sigma(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}(0,T;\mathbb{R})$.

Here,

$$L^i_{\mathcal{F}}(0,T;\mathbb{R}) := \{f \in L^i(\Omega \times (0,T);\mathbb{R}) : f \text{ is } \mathcal{F} - \text{measuable}\},$$
$$L^1_{\mathcal{F}}(\Omega;\mathbb{R}) := \{f \in L^1(\Omega;\mathbb{R}) : f \text{ is } \mathcal{F} - \text{measuable}\},$$

with $i = 1, 2$.

The set of all the $s-$a.c. is denoted by $\mathcal{U}^s_{ad}[0,T]$. The stochastic optimal control under strong formulation can be formulated as follows:

Problem 3.2. *Minimize* (**??**) *over* $\mathcal{U}^s_{ad}[0,T]$.

Our goal is to find $\bar{u} \in \mathcal{U}^s_{ad}[0,T]$, if it exists, such that

$$J(\bar{u}) = \inf_{u \in \mathcal{U}_{ad}[0,T]} J(u). \tag{3.11}$$

Problem 3.2 is said to be $s-$finite if the right-hand side of (3.11) is finite, and is said to be (unique) solvable if there exists a (unique) $s-$a.c. $\bar{u}$ which satisfies (3.11). Such $\bar{u}$ is called $s-$optimal control, and the corresponding state process $\bar{x}$ and the pair $(\bar{u}, \bar{x})$ are called $s-$optimal state process and $s-$optimal pair, respectively.

3.1.2 Weak Formulation

As we said above, in the weak context the probability space is not fixed, but it is contained into the solution. Hence also the definition of admissible control changes, as it is shown below.

Definition 3.3. *We say that a $6-$tuple $\pi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W, u)$ is a $w-$admissible control ($w-$a.c. for short), and (u, x) a $w-$admissible pair ($w-$a.p. for short), if*

(i) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *is a filtered probability space which satisfies the usual conditions;*

(ii) W *is an* $m-$*dimensional standard Brownian motion on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$*;*

(iii) u *is an* $\{\mathcal{F}_t\}_{t\geq 0}-$*adapted process on* $(\Omega, \mathcal{F}, \mathbb{P})$ *taking values on* U*;*

(iv) x *is the unique solution of equation* (3.9) *on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$*;*

(v) $f(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$, $h(x(T)) \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$ *and* $\sigma(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$, *where* $L^1_{\mathcal{F}}(0, T; \mathbb{R})$, $L^2_{\mathcal{F}}(0, T; \mathbb{R})$ *and* $L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$ *are defined on the given filtered probability space associated with the* $6-$*tuple introduced above.*

The set of all $w-$a.c. is denoted by $\mathcal{U}^w_{ad}[0, T]$, and we write u instead of $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W, u)$ when there is no possibility of confusion. The stochastic optimal control problem in the setting of weak formulation is the following:

Problem 3.4. *minimize* (**??**) *over* $\mathcal{U}^w_{ad}[0, T]$.

As for the strong formulation, we say that Problem 3.4 is $w-$finite if the right-hand side of (3.11) is finite, and it is (unique) solvable if there exists a (unique) $w-$a.c. $\bar{u}$ which satisfies (3.11). Such $\bar{u}$ is called $w-$optimal control, and the corresponding state process $\bar{x}$ and the pair $(\bar{u}, \bar{x})$ are called $w-$optimal state process and $w-$optimal pair, respectively.

3.2 Dynamic Programming and Hamilton-Jacobi-Bellman Equation

3.2.1 Stochastic Dynamic Programming

As above, we have a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual condition, on which a $m-$dimensional standard Brownian motion W is defined. Moreover, we consider the state equation (3.3) and the associated cost functional (3.4).

We now set up the framework. Let $T > 0$ fixed, and let U be a metric space. For any $(s, y) \in [0, T) \times \mathbb{R}^d$ we consider the state equation

$$\begin{cases} dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), & t \in [s, T] \\ x(s) = y \in \mathbb{R}^d, \end{cases} \tag{3.12}$$

along with the cost functional

$$J(s, y, u) := \mathbb{E}\left[\int_s^T f(t, x(t), u(t))dt\right] + \mathbb{E}[h(x(T))]. \tag{3.13}$$

Fixing $s \in [0, T)$ we denote by $U^w[s, T]$ the set of all $5-$tuple which satisfy the following conditions:

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space;

(ii) $\{W(t)\}_{t \geq s}$ is an $m-$dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ over $[s, T]$ (with $W(s) = 0$ almost surely) and $\{\mathcal{F}_t^s\} := \sigma\{W(\sigma) : s \leq \sigma \leq t\}$;

(iii) $u : [s, T] \times \Omega \longrightarrow U$ is an $\{\mathcal{F}_t^s\}-$adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$;

(iv) for any $\{\mathcal{F}_t^s\}-$adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$ u and any $y \in \mathbb{R}^d$, the equation (3.12) admits a unique solution x on $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}, \mathbb{P})$;

(v) $f(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$ and $h(x(T)) \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$, where the spaces $L^1_{\mathcal{F}}(0, T; \mathbb{R})$ and $L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$ are defined on the given filtered probability space associated with the 5−tuple introduced above.

In general, in the weak formulation the filtration is also part of the solution, while here we are considering the filtration $\{\mathcal{F}_t^s\}$ generated by the Brownian motion. We can formulate the following optimal control problem:

Problem 3.5 ((S_{sy}))**.** *For any $(s, y) \in [0, T) \times \mathbb{R}^d$ we want to minimize* (3.13) *among all $u \in \mathcal{U}^w[s, T]$.*

We need the following assumptions:

Hypotheses 3.6. (i) *(U, d) is a polish space (i.e. U is a sparable completely metrizable space) and $T > 0$;*

(ii) *functions b, σ, f, h are uniformly continuous in their domains and there exists $L > 0$ such that, if $\varphi(t, x, u)$ denotes $b(t, x, u), \sigma(t, x, u), f(t, x, u), h(x)$, we have*

$$\begin{cases} |\varphi(t, x, u) - \varphi(t, \hat{x}, u)| \leq L|x - \hat{x}|, \quad t \in [0, T], \; x, \hat{x} \in \mathbb{R}^d, \; u \in U, \\ \\ |\varphi(t, 0, u)| \leq L, \quad (t, u) \in [0, T] \times U. \end{cases} \tag{3.14}$$

Note that under Hypotheses 3.6 problem (3.12) admits a unique solution x and the cost functional is well defined. Hence, we can introduce the following function.

Definition 3.7. *The function*

$$\begin{cases} V(s, y) = \inf_{u \in \mathcal{U}^w[s,T]} J(s, y, u), & (s, y) \in [0, T) \times \mathbb{R}^d, \\ \\ V(T, y) = h(y), & y \in \mathbb{R}^d, \end{cases} \tag{3.15}$$

is called value function.

The value function has the following regularity properties.

Proposition 3.8. *Let Hypotheses* 3.6 *hold. Then the value function* V *satisfies the following conditions:*

$$|V(s,y)| \leq \quad K(1+|y|), \quad (s,y) \in [0,T] \times \mathbb{R}^d, \tag{3.16}$$

$$|V(s,y) - V(\hat{s},\hat{y})| \leq \quad K(|y-\hat{y}| + (1+|y| \vee |\hat{y}||s-\hat{s}|^{1/2}), \tag{3.17}$$

$$s, \hat{s} \in [0,T], \;\; y, \hat{y} \in \mathbb{R}^d.$$

3.2.2 Principle of Optimality and HJB Equation

Now we are ready to present a crucial result. It is the *Bellman Principle of Optimality* in the stochastic version, which allows us of writing the value function in terms of the function f of the cost functional J.

Theorem 3.9. *Suppose that Hypotheses* 3.6 *hold. Then, for any* $(s,y) \in [0,T) \times \mathbb{R}^d$

$$V(s,y) = \inf_{u \in \mathcal{U}^w[s,T]} \mathbb{E}\left\{ \int_s^{\hat{s}} f(t,x(t),u(t))dt + V(\hat{s}, x(\hat{s})) \right\}, \tag{3.18}$$

for any $0 \leq s \leq \hat{s} \leq T$*, where* $x(t) = x(t,s,y,u(t))$*.*

Equation (3.18) is called *dynamic programming equation.* Its importance is the following: if $(\bar{x}, \bar{u})$ is optimal for the problem (S), then it has to satisfy a certain relationship with the value function, as shown in the theorem below.

Theorem 3.10. *If Hypotheses 3.6 hold and* $(\bar{x}, \bar{u})$ *is optimal for* (S)*, then*

$$V(t,\bar{x}(t)) = \mathbb{E}\left\{ \int_t^T f(r,\bar{x}(r),\bar{u}(r))dr + h(\bar{x}(T)) | \mathcal{F}_t^s \right\}, \quad \mathbb{P}-a.s \; t \in [s,T]. \tag{3.19}$$

The dynamic programming equation is complicated and difficult to solve directly; hence it is useful to find other ways in order to use this result. This is the case when the value function V is smooth enough: indeed in such a situation V satisfies a partial differential equation which is derived from (3.18). This fact is clarified in the next proposition.

Proposition 3.11. *Suppose that Hypotheses 3.6 hold, and that* $V \in C^{1,2}([0,T] \times \mathbb{R}^d)$*. Then,* V *is a solution of the following terminal value problem of a (possible degenerate) second order partial differential equation*

$$\begin{cases} -v_t + \sup_{u \in U} G(t,x,u,-v_x,-v_{xx}) = 0, & (t,x) \in [0,T) \times \mathbb{R}^d, \\ v(T,x) = h(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.20}$$

where

$$G(t,x,u,p,q) := \mathrm{Tr}(\sigma(t,x,u)\sigma^*(t,x,u)q) + \langle b(t,x,u),p\rangle - f(t,x,u), \tag{3.21}$$

for any $(t,x) \in [0,T) \times \mathbb{R}^d$, $u \in U$, $p \in \mathbb{R}^d$ *and* $\in \mathbb{R}^{d\times d}$.

Equation (3.20) is called *Hamilton-Jacobi-Bellman equation.* The function G is called Hamiltonian, and if $\sigma\sigma^t$ is uniformly positive definite it is an elliptic operator. Moreover, if its coefficients satisfy suitable conditions, the HJB equation admits a solution in $C^{1,2}([0,T]\times\mathbb{R}^d)$ (see [35]).

3.2.3 Stochastic Verification Theorem and Optimal Feedback Control

Solving an optimal control problem means finding an optimal control and the corresponding state function. The introduction of dynamic programming is motivated by the fact that one might be able to construct an optimal feedback control via the value function. The following result, called *classical stochastic verification theorem*, gives a way of testing if a pair is optimal and suggests how to construct an optimal feedback control. We refer to [101] for the following results.

Theorem 3.12. *Let Hypotheses* 3.6 *hold, and let* $v \in C^{1,2}([0,T]\times\mathbb{R}^d)$ *be a solution of the HJB equation* (3.20)*. Then*

$$v(s,y) \le J(s,y,u), \quad u \in \mathcal{U}^w[s,T],\ (s,y) \in [0,T) \times \mathbb{R}^d. \tag{3.22}$$

Moreover, an admissible pair $(\bar{x},\bar{u})$ *is optimal for problem* (S_{sy}) *if and only if*

$$\begin{aligned} v_t(t,\bar{x}(t)) &= \max_{u\in U} G(t,\bar{x}(t),u,-v_x(t,\bar{x}(t)),-v_{xx}(t,\bar{x}(t))) \\ &= G(t,\bar{x}(t),\bar{u}(t),-v_x(t,\bar{x}(t)),-v_{xx}(t,\bar{x}(t))), \end{aligned} \tag{3.23}$$

a.e. $t \in [s,T]$, $\mathbb{P}-$*a.s.*

Now we can show how to construct an optimal feedback control. At first, we need of the following definition:

Definition 3.13. *A measurable function* $u : [0,T]\times\mathbb{R}^d \longrightarrow U$ *is called admissible feedback control if for any* $(s,y) \in [0,T)\times\mathbb{R}^d$ *there is a weak solution of equation*

$$\begin{cases} dx(t) = b(t,x(t),u(t,x(t)))dt + \sigma(t,x(t),u(t,x(t)))dW(t), \quad t\in[s,T] \\ x(s) = \xi \in \mathbb{R}^d. \end{cases} \tag{3.24}$$

An admissible feedback control is said to be optimal if for any $(s,y) \in [0,T)\times\mathbb{R}^d$ *the pair* $(\bar{x},\bar{u}(\cdot,\bar{x}))$ *is optimal for the problem* (S_{sy})*, where* $\bar{x}$ *is the solution associated to* $\bar{u}$*. .*

The following theorem gives a sufficient condition to obtain an optimal feedback control.

Theorem 3.14. *Suppose that Hypotheses* 3.6 *hold, and let* $v \in C^{1,2}([0,T] \times \mathbb{R}^d)$ *be a solution to the HJB equation. If* $\bar{u}$ *is an admissible feedback control which satisfies*

$$G(t,x,\bar{u}(t,x),-v_x(t,x),-v_{xx}(t,x)) = \max_{u\in U} G(t,x,u,-v_x(t,x),-v_{xx}(t,x)), \tag{3.25}$$

for any $(t,x) \in [0,T] \times \mathbb{R}^d$*, then* $u(t) := \bar{u}(t,\bar{x}(t))$ *is an optimal control, where* $\bar{x}$ *is the solution to* (3.24) *related to* $\bar{u}$*.*

3.3 Backward Stochastic Differential Equations

3.3.1 Nonlinear Backward Stochastic Differential Equations

We consider a complete probability space $\pi = (\Omega, \mathcal{F}, \mathbb{P})$, an $d-$dimensional Brownian motion on π W and $\{\mathcal{F}_t\}_{t\geq 0}$ the filtration generated by W, augmented by all the $\mathbb{P}-$null set of Ω.

In the following we deal with the nonlinear BSDE

$$\begin{cases} dY(t) = f(t,Y(t),Z(t))dt + Z(t)dW(t), \quad t \in [0,T], \\ Y(T) = \xi, \end{cases} \tag{3.26}$$

where $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^m$ and $\xi : \Omega \longrightarrow \mathbb{R}^m$ are given functions, and we look for unknown adapted processes Y and Z. Let us observe that f could be stochastic, but we obmits the possible dependence on ω.

As in the linear case, we say that (Y,Z) is a solution to (3.26) if it is adapted, satisfies

$$Y(t) = \xi - \int_t^T f(s,Y(s),Z(s))ds - \int_t^T Z(s)dW(s), \tag{3.27}$$

and $(Y,Z) \in \mathcal{M}[0,T]$, where

$$\mathcal{M}[0,T] := L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^m)) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m\times d}),$$

equipped with the norm

$$\|(Y,Z)\|_{\mathcal{M}[0,T]} := \left\{ \mathbb{E}\left(\sup_{t\in[0,T]} |Y(t)|^2 \right) + \mathbb{E}\int_0^T |Z(s)|^2 ds \right\}^{1/2}. \tag{3.28}$$

Let us observe that equation (3.27) makes sense if $f(\cdot,Y(\cdot),Z(\cdot) \in M^1_{\text{loc}}[0,T]$

We assume the following hypotheses.

Hypotheses 3.15. **(i)** $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^m)$*;*

(ii) f *is* $\{\mathcal{F}_t\}_{t\geq 0}-$*adapted and there exists* $K \geq 0$ *such that, for any* $t \in [0,T]$*, any* $y,y' \in \mathbb{R}^m$ *and any* $z,z' \in \mathbb{R}^{m\times d}$*, we have*

$$|f(t,y,z) - f(t,y',z')| \leq K(|y-y'| + \|z-z'\|), \quad \mathbb{P}-a.s.; \tag{3.29}$$

(iii) $f(\cdot, 0, 0) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$.

Under the above conditions, the following theorem holds.

Theorem 3.16. *Let Hypotheses* 3.15 *hold. Then for any given* $\xi \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ *equation* (3.26) *admits a unique solution in* $\mathcal{M}[0, T]$.

3.3.2 Forward-Backward Stochastic Differential Equations and Connection with PDEs

In this subsection we consider a particular BSDE in which both ξ and f depend on another stochastic process X, that we assume given in advance, and the case $m = 1$. Then, we will show the connection of BSDE with a (possible degenerate) quasi-linear parabolic differential equation. Here we follow closely [89].

Suppose that X is a process with values in $\mathbb{R}^n$ and that it belongs to $L^p_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^n))$, for any $p \in [1, \infty)$. This means that for any $p \in [1, \infty)$ we have $\mathbb{E} \sup_{t \in [0,T]} |X(t)|^p < \infty$.

We consider the BSDE

$$\begin{cases} dY(t) = \psi(t, X(t), Y(t), Z(t))dt + Z(t)dW(t), \quad t \in [0, T], \\ Y(T) = \varphi(X(T)), \end{cases} \tag{3.30}$$

where $\psi : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times (\mathbb{R}^d)^* \longrightarrow \mathbb{R}$ and $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ are given Borel functions. As before, we look for adapted processes Y and Z which satisfy (3.30).

Now, suitable assumptions on ψ and φ allow us of connecting with Hypotheses 3.15.

Hypotheses 3.17. *For all* $t \in [0, T]$, $x \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $z, z' \in (\mathbb{R}^d)^*$, *we have, for some* $k, K \geq 0$,

(i) $|\psi(t, x, y, z) - \psi(t, x, y', z')| \leq K\left(|y - y'| + \|z - z'\|\right)$;

(ii) $|\varphi(x)| + |\psi(t, x, 0, 0)| \leq K(1 + |x|^k)$.

Setting $\xi = \varphi(X(T))$ and $f(t, y, z) = \psi(t, X(t), y, z)$ we can easily check that ξ and f satisfy Hypotheses 3.15. Thus there exists a unique solution $(Y, Z) \in \mathcal{M}[0, T]$.

Now we consider the special case when X is given as a solution of another standard stochastic differential equation (*forward*). Moreover, we assume that $d = 1$. So, for any interval $[t, T] \subset [0, T]$ we are concerned with

$$\begin{cases} dX(s) = F(X(s))dt + G(X(s))dW(s), \quad s \in [t, T], \\ X(t) = x \in \mathbb{R}^n, \end{cases} \tag{3.31}$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $G : \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times d}$ are Borel functions and satisfy:

Hypothesis 3.18. *For all* $x, x' \in \mathbb{R}^d$ *and for some constant* $K \geq 0$ *we have*

$$|F(x) - F(x')| + \|G(x) - G(x')\| \leq K|x - x'|. \tag{3.32}$$

Under the above hypotheses, there exists a unique continuous and adapted process $\{X(s) : s \in [t,T]\}$, solution of (3.31), and for convenience we set $X(r) = x$, for any $r \in [0,t]$. We denote by $\{X(s,t,x) : s \in [0,T]\}$ the solution process, to stress the dependence on the parameters t and x. Then, for every $p \in [1,\infty)$ we have

$$\begin{aligned} \|X\|_p &:= \mathbb{E} \sup_{t\in[0,T]} |X(t)|^p \leq c(1+|x|^p), \\ \|X(\cdot,t,x) - X(\cdot,t',x')\|_p &\leq c|x-x|^p, \end{aligned} \tag{3.33}$$

for some positive constant c independent of x.

By the previous results, choosing $X = X(\cdot,t,x)$ in (3.30), such a backward equation admits a unique adapted solution, i.e., the system

$$\begin{cases} dY(s) = \psi(s,X(s),Y(s),Z(s))ds + Z(s)dW(s), & s \in [t,T], \\ dX(s) = B(X(s))ds + G(X(s))dW(s), & s \in [t,T], \\ Y(T) = \varphi(X(T)), \\ X(t) = x, \end{cases} \tag{3.34}$$

which we call *forward-backward stochastic differential equations*, admits a unique solution $(X,Y,Z) \in L^p_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times \mathcal{M}[0,T]$. Sometimes we will write $\{X(s,t,x), Y(s,t,x), Z(s,t,x) : s \in [t,T]\}$ to point out the dependence on t and x.

Lemma 3.19. *$Y(t,t,x)$ is deterministic.*

This result follows from the fact that $Y(t,t,x)$ is measurable both with respect to $\mathcal{F}_t$ and with respect to $\mathcal{F}_{t,T}$, where $\mathcal{F}_{t,T}$ denotes the $\sigma-$field generated by the random variables $W_\tau - W_t$, $\tau \in [t,T]$, augmented with $\mathbb{P}-$null sets.

Let us introduce the differential operator A, defined on smooth functions f by

$$Af(x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x) D^2_{ij} f(x) + \sum_{i=1}^{n} F_i(x) D_i f(x), \quad x \in \mathbb{R}^n, \tag{3.35}$$

where F_i, for $i = 1,\dots,n$, are the components of F and, denoting G_{ik} the components of G, we have

$$a_{ij}(x) = \sum_{k=1}^{d} G_{ik}(x) G_{jk}(x). \tag{3.36}$$

A is an elliptic second order differential operator, but in general not uniformly elliptic. We consider the backward Cauchy problem associated to A:

$$\begin{cases} D_t u(t,x) + Au(t,x) = \psi(t,x,u(t,x),\nabla u(t,x) G(x)), & x \in \mathbb{R}^n,\ t \in [0,T), \\ u(T,x) = \varphi(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.37}$$

This equation is not easy to treat by analytic methods, since the diffusion term a is not, in general, uniformly elliptic. Hence the probabilistic approach, based on FBSDEs, is a useful tool to overcome this difficulty.

Indeed, a first result is due to a straightforward application of Itô formula.

Theorem 3.20. *Assume that Hypotheses* 3.17 *and* 3.18 *hold true. If* $u \in C^{1,2}([0,T]\times\mathbb{R}^n)$ *is a classical solution to* (3.37)*, then* $u(t,x) = Y(t,t,x)$*, where* (X,Y,Z) *denotes the solution of the FBSDE* (3.34)*.*

The inverse result is harder to reach, and need of additional hypotheses. The following conditions have been introduced by Pardoux and Peng in [89], and guarantee the existence and uniqueness of a classical solution to (3.37).

Hypotheses 3.21. **(i)** *F and G are of class C^3, and their derivatives of order* $1,2,3$ *are bounded;*

(ii) *φ is of class C^3, and it has polynomial growth together with its derivatives of order* $1,2,3$*;*

(iii) *$\psi(t,\cdot,\cdot,\cdot)$ is of class C^3, for all $t \in [0,T]$;*

(iv) *$\psi(t,\cdot,y,z)$ has polynomial growth with its derivatives of order* $1,2,3$*, for any $t \in [0,T]$, $y \in \mathbb{R}$, $z \in (\mathbb{R}^d)^*$; moreover, $|\nabla_x\psi(t,x,y,z)| \le K(1+|z|)(1+|x|+|y|)^{\mu}$ for suitable constants $K,\mu \ge 0$;*

(v) *$\nabla_x\psi$ and $\nabla_z\psi$ are bounded together with their derivatives of order* $1,2$ *with respect to y and z.*

Under these assumptions, we obtain the following result.

Theorem 3.22. *Let us assume that Hypotheses* 3.21 *hold. Then setting $u(t,x) = Y(t,t,x)$, we have that $u \in C^{1,2}([0,T]\times\mathbb{R}^n)$ and it is the unique solution to* (3.37)*.*

Uniqueness is a byproduct of Theorem 3.20. The regularity of u and the fact that it is a solution is a consequence of the following propositions.

Proposition 3.23. *For every $(t,x) \in [0,T]\times\mathbb{R}^n$, $\mathbb{P}-a.s.$ we have*

$$Y(s,t,x) = u(s,X(s,t,x)), \quad s \in [t,T]. \tag{3.38}$$

Proposition 3.24. *For every $(t,x) \in [0,T]\times\mathbb{R}^n$, $\mathbb{P}-a.s.$ we have*

$$Z(s,t,x) = \nabla_x u(s,X(s,t,x))G(X(s,t,x)), \quad \mathbb{P}-a.s. s \in [t,T]. \tag{3.39}$$

However, sometimes in the applications to stochastic optimal control it is not necessary having a classical solution to (3.37), but it is sufficient that the function $u \in C^{0,1}([0,T] \times \mathbb{R}^n)$.

To this aim we introduce the concept of mild solution. We define the transition semigroup $P_{t,\tau}$, for any function ϕ with polynomial growth, by

$$P_{t,\tau}[\phi](x) := \mathbb{E}\Phi(X(\tau,t,x)),$$

for any $x \in \mathbb{R}^n$ and $0 \le t \le \tau \le T$. $P_{t,\tau}$ is well defined, as estimate $\mathbb{E}\sup_{\tau\in[t,T]} |X_\tau|^p \le C(1+|x|^p)$ shows, and also $P_{t,\tau}[\phi](x)$ has polynomial growth.

The operator A introduced in (3.35) is called the generator of $P_{t,\tau}$, and it is well defined on smooth functions. By means of variation of constants formula for the Cauchy problem (3.37), we define the function

$$u(t,x) := P_{t,T}[\phi](x) - \int_t^T P_{t,s}[\psi(\cdot, G(\cdot)\nabla u(r,\cdot))](x), \tag{3.40}$$

for any $x \in \mathbb{R}^n$ and $0 \le t \le T$. Moreover, the above formula is meaningful if ψ and ϕ has polynomial growth.

Now we provide the definition of mild solution to the Cauchy problem (3.37).

Definition 3.25. *We say that a function u is a mild solution to* (3.37) *if it satisfies the following conditions:*

(i) $u \in C^{0,1}([0,T] \times \mathbb{R}^n)$*;*

(ii) *there exists a positive constant C such that $|\nabla_x u(t,x)| \le C(1+|x|^k)$, for any $t \in [0,T]$;*

(iii) *equality* (3.40) *holds.*

Under the following assumptions, we have the existence and uniqueness of a mild solution to (3.37) (see [41]).

Hypotheses 3.26. (i) *F and G are of class C^1 and their derivatives of order* 1 *are bounded;*

(ii) *φ is of class C^1 and it has polynomial growth together with its derivatives of order* 1*;*

(iii) *$\psi(t,\cdot,\cdot,\cdot)$ is of class C^1, for all $t \in [0,T]$;*

(iv) $|\nabla_x \psi(t,x,y,z)| \le K(1+|z|)(1+|x|+|y|)^\mu$ *for suitable constants $K, \mu \ge 0$;*

(v) *$\nabla_x \psi$ and $\nabla_z \psi$ are bounded with respect to y and z.*

Theorem 3.27. *If Hypotheses* 3.26 *hold true, then there exists a unique mild solution u to* (3.37)*. Moreover, u is given by the formula*

$$u(t,x) = Y(t,t,x),$$

where (X,Y,Z) is the solution to (3.34)*.*

3.3.3 Applications to Optimal Control

In this subsection we consider the controlled stochastic differential equation

$$\begin{cases} dX(s) = F(s, X(s))ds + G(s, X(s))r(s, X(s), u(s))ds + G(s, X(s))dW(s), \\ \quad s \in [t, T] \\ X(t) = x_0 \in \mathbb{R}^n, \end{cases} \tag{3.41}$$

where u is an adapted stochastic process with values in some specified set $\mathcal{U} \in \mathbb{R}^m$ and W is an $d-$dimensional standard Brownian motion. The particular equation that we consider allows us to prove the existence of an optimal control to the stochastic control problem, introducing a suitable FBSDE.

We work in weak formulation, hence our purpose is to minimize the cost functional

$$J(u) = \mathbb{E}\left[\int_0^T l(s, X(s), u(s))ds\right] + \mathbb{E}[\varphi(X(T))], \tag{3.42}$$

over all $w-$admissible controls. We require the following hypotheses on the coefficients of (3.41), which are comparable with Hypotheses 3.15 and 3.17.

Hypothesis 3.28. *$\mathcal{U}$ is a Borel set of $\mathbb{R}^m$, the functions F, G, r, φ, l are Borel measurable, the function $x \mapsto F(t, x)$ is continuous on $\mathbb{R}^n$ for any $t \in [0, T]$ and there exists a constant $C > 0$ such that*

$$\begin{aligned} |\varphi(x) - \varphi(x')| + |F(t, x) - F(t, x')| + |G(t, x - G(t, x')| &\le C|x - x'|, \\ |r(t, x, u) - r(t, x', u')| + l(t, x, u) - l(t, x', u')| &\le C(|x - x'| + |u - u'|), \\ |G(t, x)| + |F(t, 0)| + |r(t, x, u)| + |l(t, 0, u)| &\le C, \end{aligned} \tag{3.43}$$

for any $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $u, u' \in \mathcal{U}$.

As said above, we work under weak formulation, hence the solution we are looking for is a $6-$tuple $\mathbb{U} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P}, W, u)$ and has to satisfies Definition 3.3. If for any $w-$a.c. $\mathbb{U}$ we consider the process $X^{\mathbb{U}}$ solution to the Itô stochastic equation

$$\begin{aligned} X^{\mathbb{U}}(s) = x &+ \int_t^s F(\sigma, X^{\mathbb{U}}(\sigma))d\sigma + \int_t^s G(\sigma, X^{\mathbb{U}}(\sigma))r(\sigma, X^{\mathbb{U}}(\sigma), u(\sigma))d\sigma \\ &+ \int_s^t G(\sigma, X^{\mathbb{U}}(\sigma))dW(\sigma), \quad s \in [t, T], \end{aligned} \tag{3.44}$$

$\mathbb{P}-$a.s. For any $w-$a.c. this equation has a continuous and $\{\mathcal{F}_t\}_{t\ge0}-$adapted solution, unique up to indistinguishability. In this setting the cost functional is

$$J(t, x, \mathbb{U}) = \mathbb{E}\left[\int_t^T l(s, X^{\mathbb{U}}(s), u(s))ds\right] + [\varphi(X^{\mathbb{U}}(T))], \tag{3.45}$$

for any $x \in \mathbb{R}^n$, any $t \in [0,T]$, $\mathbb{U}$ $w-$a.c. Finally, we recall that the value function is $V : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined as

$$V_w(t,x) = \inf_{\mathbb{U}} J(t,x,\mathbb{U}), \quad t \in [0,T],\ x \in \mathbb{R}^n. \tag{3.46}$$

In this case, the hamiltonian function $\psi : [0,T] \times \mathbb{R}^n \times (\mathbb{R}^d)^* \longrightarrow \mathbb{R}$ is defined as

$$\psi(t,x,z) := \inf_{u \in \mathcal{U}} \{l(t,x,u) + zr(t,x,u)\}, \quad t \in [0,T],\ x \in \mathbb{R}^n,\ z \in (\mathbb{R}^d)^*, \tag{3.47}$$

and we can introduce the following, possible empty, set:

$$\Gamma((t,x,z) := \{u \in \mathcal{U} : \psi(t,x,z) = l(t,x,u) + zr(t,x,u)\}, \tag{3.48}$$

for any $t \in [0,T]$, $x \in \mathbb{R}^n$ and $z \in (\mathbb{R}^d)^*$.

The hamiltonian enjoys the properties stated in the lemma below.

Lemma 3.29. *Assume that Hypothesis 3.28 holds. Then there exists a constant $c > 0$ such that*

$$|\psi(t,0,0)| \le c, \quad |\psi(t,x,z) - \varphi(t,x',z')| \le c|z - z'| + c|x - x'|(1 + |z| + |z'|), \tag{3.49}$$

for any $t \in [0,T],\ x \in \mathbb{R}^n,\ z \in (\mathbb{R}^d)^*$.

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a complete probability space, on which is defined an $d-$dimensional standard Brownian motion $\tilde{W}$ is defined. As usual, we consider the $\sigma-$field $\tilde{\mathcal{F}}_{[t,s]}$ generated by $\tilde{W}(s) - \tilde{W}(t)$, augmented by the $\tilde{\mathbb{P}}-$null sets of $\tilde{\mathcal{F}}$. For any fixed $t \in [0,T]$ we consider the equation

$$\tilde{X}(s) = x + \int_t^s F(\sigma, \tilde{X}(\sigma))d\sigma + \int_s^t G(\sigma, \tilde{X}(\sigma))d\tilde{W}(\sigma), \quad s \in [t,T]. \tag{3.50}$$

The solution $\{\tilde{X}(s) : s \in [t,T]\}$ is adapted to the filtration $\{\tilde{\mathcal{F}}_{[t,s]}\}_{s \in [t,T]}$ and the law of $(\tilde{X}, \tilde{W})$ is uniquely determined by x, F, G. If we take the BSDE

$$\tilde{Y}(\tau) + \int_\tau^T \tilde{Z}(s)d\tilde{W}(s) = \varphi(\tilde{X}(T)) + \int_\tau^T \psi(s, \tilde{X}(s), \tilde{Y}(s), \tilde{Z}(s))ds, \quad \tau \in [t,T], \tag{3.51}$$

by Theorem 3.16 there exists a unique solution $(\tilde{Y}, \tilde{Z})$ on interval $[t,T]$, and Lemma 3.19 implies that $\tilde{Y}(t)$ is deterministic. Moreover, the law of $\tilde{Y}$ depends only on $(\tilde{X}, \tilde{W})$ and ψ, φ. We set

$$J^{\#}(t,x) := \tilde{Y}(t,x), \quad t \in [0,T],\ x \in \mathbb{R}^d. \tag{3.52}$$

The following result is obtained by means of equality

$$\begin{aligned} J^{\#}(t,x) &= J(t,x,\mathbb{U}) \\ &+ \int_t^T \Big(\psi(s, X^{\mathbb{U}}(s), Z^{\mathbb{U}}(s)) - l(s, X^{\mathbb{U}}(s), u(s)) - Z^{\mathbb{U}}(s) r(s, X^{\mathbb{U}}(s), u(s))\Big)\, ds, \end{aligned} \tag{3.53}$$

which holds for any $\mathbb{U}$.

Proposition 3.30. *Assume that Hypothesis 3.28 holds. Then for any* $t \in [0, T]$*, any* $x \in \mathbb{R}^d$ *and any* $\mathbb{U}$ *w−a.c., we have that* $J^{\#}(t, x) \leq J(t, x, \mathbb{U})$*.*

Indeed, from (3.53) and from the definition of the hamiltonian ψ we easily deduce that $J^{\#}(t, x) \leq J(t, x, \mathbb{U})$, and the equality holds if and only if $u(s) \in \Gamma(s, X^{\mathbb{U}}(s), Z^{\mathbb{U}}(s))$ for a.e. $s \in [t, T]$. Hence, the existence of an optimal control is strictly connected with the achievement of the minimum in (3.47). We introduce the following additional hypothesis.

Hypothesis 3.31. *The set* Γ *is non-empty.*

From Hypothesis 3.31 it follows that (see [6, Thm 8.2.10]) there exists a measurable map $\gamma : [0, T] \times \mathbb{R}^n \times (\mathbb{R}^d)^* \longrightarrow U$ such that

$$\psi(t, x, z) = l(t, x, \gamma(t, x, z)) + zr(t, x, \gamma(t, x, z)), \tag{3.54}$$

with $(t, x, z) \in [0, T] \times \mathbb{R}^n \times (\mathbb{R}^d)^*$.

Proposition 3.32. *Under Hypotheses 3.28 and 3.31 for every* $t \in [0, T]$ *and* $x \in \mathbb{R}^n$ *there exists a w−a.c.* $\mathbb{U}$ *verifying* $J(t, x, \mathbb{U}) = J^{\#}(t, x)$*. Consequently,* $J^{\#}(t, x) = V_w(t, x)$*.*

Finally, we talk about the *optimal feedback law*. Suppose that both Hypotheses 3.28 and 3.31 hold, and we refer to $\{\tilde{X}(s, t, x), \tilde{Y}(s, t, x), \tilde{Z}(s, t, x) : s \in [t, T], x \in \mathbb{R}^n\}$ as the solution to (3.50) and (3.51) for given $t \in [0, T]$ and $x \in \mathbb{R}^n$.

Lemma 3.33. *There exists a Borel measurable function* $\zeta : [0, T] \times \mathbb{R}^n \longrightarrow (\mathbb{R}^d)^*$ *such that, for any* $t \in [0, T]$ *and* $x \in \mathbb{R}^n$*,* $\mathbb{P}$*−a.s.*

$$\zeta(s, \tilde{X}(s, t, x)) = \tilde{Z}(s, t, x), \quad a.e. s \in [t, T]. \tag{3.55}$$

Moreover, ζ *depends only on* F, G, φ, ψ *and not on the probability space.*

For the proof of Lemma 3.33 see [7].

Corollary 3.34. *If* $\{(X, Y, Z)\}$ *denotes the unique solution to*

$$\begin{cases} X(s) = \displaystyle\int_t^s F(\sigma, X(\sigma))d\sigma + \int_t^s G(\sigma, X(\sigma))r(\sigma, X(\sigma), \gamma(\sigma, X(\sigma), Z(\sigma)))d\sigma \\ \quad + \displaystyle\int_t^s G(\sigma, X(\sigma))dW(\sigma), \\ Y(s) + \displaystyle\int_s^T Z(\sigma)dW(\sigma) = \varphi(X(T)) + \int_s^T l(\sigma, X(\sigma), \gamma(, \sigma, X(\sigma), Z(\sigma)))d\sigma, \\ \quad s \in [t, T], \end{cases} \tag{3.56}$$

then $\mathbb{P}$*−a.s.* $\zeta(s, X(s)) = Z(s)$*, for a.e.*$s \in [t, T]$*.*

To prove Corollary 3.34, we set

$$W(t) := \tilde{W}(t) - \int_0^t r(\sigma, \tilde{X}(\sigma), \gamma(\sigma, \tilde{X}(\sigma), \tilde{Z}(\sigma)))ds,$$

and we recall that $(\tilde{X}, \tilde{Y}, \tilde{Z})$ solves (3.50) and (3.51), and (3.55) and (3.54) hold. Hence,

$$\begin{aligned}\tilde{X}(s) &= \int_t^s F(\sigma, \tilde{X}(\sigma))d\sigma + \int_t^s G(\sigma, \tilde{X}(\sigma))r(\sigma, \tilde{X}(\sigma), \gamma(\sigma, \tilde{X}(\sigma), \tilde{Z}(\sigma)))d\sigma \\ &+ \int_t^s G(\sigma, \tilde{X}(\sigma))dW(\sigma)\end{aligned}$$

and

$$\begin{aligned}\tilde{Y}(s) &= \varphi(\tilde{X}(T)) + \int_s^T \psi(\sigma, \tilde{X}(\sigma), \gamma(, \sigma, \tilde{X}(\sigma), \tilde{Z}(\sigma)))d\sigma - \int_s^T \tilde{Z}(\sigma)d\tilde{W}(\sigma) \\ &= \varphi(\tilde{X}(T)) + \int_s^T l(\sigma, \tilde{X}(\sigma), \gamma(, \sigma, \tilde{X}(\sigma), \tilde{Z}(\sigma)))d\sigma \\ &+ \int_s^T \tilde{Z}(\sigma)r(\sigma, \tilde{X}(\sigma), \gamma(\sigma, \tilde{X}(\sigma), \tilde{Z}(\sigma)))d\sigma - \int_s^T \tilde{Z}(\sigma)d\tilde{W}(\sigma) \\ &= \varphi(\tilde{X}(T)) + \int_s^T l(\sigma, \tilde{X}(\sigma), \gamma(, \sigma, \tilde{X}(\sigma), \tilde{Z}(\sigma)))d\sigma - \int_s^T \tilde{Z}(\sigma)dW(\sigma)\end{aligned}$$

Through ζ we can define a Borel measurable function $\underline{u} : [0, T] \times \mathbb{R}^n \longrightarrow U$ by

$$\underline{u}(t, x) = \gamma(t, x, \zeta(t, x)), \quad t \in [0, T],\ x \in \mathbb{R}^n, \tag{3.57}$$

and we introduce the so called *closed-loop equation*

$$\begin{aligned}\bar{X}(s) &= + \int_t^s F(\sigma, \bar{X}(\sigma))d\sigma + \int_t^s G(\sigma, \bar{X}(\sigma))r(\sigma, \bar{X}(\sigma), \underline{u}(\sigma, \bar{X}(\sigma)))d\sigma \\ &+ \int_t^s G(\sigma, \bar{X}(\sigma))dW(\sigma), \quad s \in [t, T].\end{aligned} \tag{3.58}$$

Since r is bounded, it makes sense to look for an $\{\mathcal{F}_t\}_{t \geq 0}$−adapted solution of this equation. Indeed, if we set

$$\bar{W}(\tau) = W(\tau) + \int_t^{t \wedge \tau} r(\sigma, \bar{X}(\sigma), \underline{u}(\sigma, \bar{X}(\sigma)))d\sigma,$$

the Girsanov theorem implies that there exists a probability measure $\bar{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ such that $\bar{W}$ is a Brownian motion under $\bar{\mathbb{P}}$.

We have the following result:

Theorem 3.35. *If Hypothesis* 3.28 *holds, then* $\mathbb{U}$ *is optimal for the control problem starting from* x *at time* t *if and only if* $u(s) \in \Gamma(s, X^{\mathbb{U}}(s), \zeta(s, X^{\mathbb{U}}(s)))$ $\mathbb{P}-$*a.s., a.e.* $s \in [t, T]$.

Moreover, if Hypothesis 3.31 *is satisfied and* $u(t) = \underline{u}(t, X^{\mathbb{U}}(s))$, *then* $\mathbb{U}$ *is optimal for the control problem starting from* x *at time* t.

Finally, if Hypotheses 3.21 *hold, then*

$$\zeta(t, x) = \nabla_x u(t, x) G(t, x), \tag{3.59}$$

where u *is the unique classical solution to the semi-linear Cauchy problem* (3.37) *(that in this context is the HJB equation). Hence the feedback law has the form*

$$\underline{u}(t, x) = \gamma(t, x, \nabla_x u(t, x) G(t, x)), \tag{3.60}$$

and $\mathbb{U}$ *is optimal for the control problem starting from* x *at time* t *if and only if*

$$u(s) \in \Gamma(s, X^{\mathbb{U}}(s), \nabla_x u(s, X^{\mathbb{U}}(s)) G(s, X^{\mathbb{U}}(s))), \quad \mathbb{P} - a.s., \ a.e.\ s \in [t, T]. \tag{3.61}$$

Remark 3.36. *The above formulae hold true also if* u *is a mild solution to* (3.37). *Hence, if Hypotheses 3.26 are satisfied, then* (3.59), (3.60) *and* (3.61) *are still true, where* u *is the unique mild solution to* (3.37).

Chapter 4

Hamilton-Jacobi-Bellman Equation and Applications to Optimal Control

4.1 Introduction

The aim of this chapter is the study of the backward parabolic Cauchy problem (BPDE) of HJB type

$$\begin{cases} D_t v(t,x) + Av(t,x) = \psi(x, G(x)\nabla_x v(t,x)), & t \in [0,T), \quad x \in \mathbb{R}^d, \\ v(T,x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{BPDE}$$

by analytic methods, and show some of its applications to a particular case of stochastic optimal control problems. Here, A is the uniformly elliptic differential operator defined on smooth functions f by

$$Af(x) = \frac{1}{2}\mathrm{Tr}(G(x)G(x)D_x^2 f(x)) + \langle B(x), \nabla f(x)\rangle,$$

where $G : \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^d$, $B : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, ψ is a continuous function which satisfies some additional conditions and $\varphi \in C_b(\mathbb{R}^d)$.

Under suitable assumptions on the coefficients of the operator A, we prove the existence and uniqueness of a mild solution to problem (BPDE). More precisely, let $\{S(t)\}_{t\geq 0}$ be the bounded linear semigroup on $C_b(\mathbb{R}^d)$ associated to the Cauchy problem

$$\begin{cases} D_t v(t,x) = Av(t,x), & t \in (0,+\infty), \quad x \in \mathbb{R}^d, \\ v(0,x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases}$$

and let F be the functional defined by

$$F(t,u)(x) = \psi(x, G(x)\nabla u(t,x)), \quad t \in [0,T),\ x \in \mathbb{R}^d,$$

for suitable functions u. We show that the functional

$$(\Gamma v)(t,x) := S(T-t)\varphi(x) - \int_t^T S(r-t)F(r,v)(x)dr,$$

admits a unique fixed point $v \in C_b([0,T]\times\mathbb{R}^d)\cap C^{0,1}([0,T)\times\mathbb{R}^d)$, i.e., there exists a unique function $v \in C_b([0,T]\times\mathbb{R}^d)\cap C^{0,1}([0,T)\times\mathbb{R}^d)$ which satisfies

$$v(t,x) = S(T-t)\varphi(x) - \int_t^T S(r-t)F(r,v)(x)dr.$$

The proof is based on weighted gradient estimates, which guarantees that, for any $T>0$, there exists a positive constant C_T such that

$$\|G\nabla_x(S(t)\varphi)\|_\infty \le \frac{C_T}{t^{1/2}}\|\varphi\|_\infty, \quad t\in[0,T),$$

for any $\varphi \in C_b(\mathbb{R}^d)$. This estimate allows us to apply the Banach fixed point Theorem to an appropriate space of functions, and therefore to deduce the existence of the mild solution v.

As it is well known equation (BPDE) is the Hamilton Jacobi Bellman (HJB) equation corresponding to an optimal stochastic control problem (see Chapter 3, Section 3.3). Moreover, if $\varphi \in BUC(\mathbb{R}^d)$, then the regularity of the mild solution v allows us to show that it is the Value Function (see [101, Chp. 4]) associated to the control problem given by the state equation

$$\begin{cases} d_\tau X^u_\tau = B(X^u_\tau)d\tau + G(X^u_\tau)r(X^u_\tau,u_\tau)d\tau + G(X^u_\tau)dW_\tau, & \tau\in[t,T],\\ X^u_t = x\in\mathbb{R}^d, \end{cases} \tag{4.1}$$

and the cost functional is

$$\mathbb{E}\int_0^T l(X^u_t,u_t)dt + \mathbb{E}\varphi(X^u_T), \tag{4.2}$$

where l and φ are measurable functions, and B and G has been introduced above. Moreover, the existence of $\nabla_x v$ and the estimate on its growth allow us to identify the optimal feedback law for the control problems.

The key tool to link the HJB equation and the optimal control problem is the backward stochastic differential equation. The interaction between backward stochastic differential equations and backward partial differential equations was proved in [89] and [90] for the finite dimensional case and for classical solutions of the parabolic Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),(\nabla_x u\, G)(t,x)) = 0, & t\in[0,T], \quad x\in\mathbb{R}^d,\\ u(T,x) = g(x), & x\in\mathbb{R}^d, \end{cases}$$

where

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{N}(GG^*)_{ij}(t,x)\frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^{N} b_i(t,x)\frac{\partial}{\partial x_i},$$

$G(x)$ is a $(N \times d)-$matrix valued function and b_i are scalar functions, for $i = 1, \dots, N$.

Forward backward stochastic differential system

$$\begin{cases} dY_\tau = \psi(X_\tau, Z_\tau)d\tau + Z_\tau dW_\tau, & \tau \in [t,T], \\ dX_\tau = B(X_\tau)d\tau + G(X_\tau)dW_\tau, & \tau \in [t,T], \\ Y_T = \varphi(X_T), & \\ X_t = x, & x \in \mathbb{R}^d, \end{cases} \tag{FBSDE}$$

admits a solution (X, Y, Z) with X, Y, Z belonging to some suitable spaces, and under opportune regularity and growth assumptions on ψ, B, G, φ the processes Y and Z can indeed be represented by $v(t, X_t)$ and $G(X_t)\nabla_x v(t, X_t)$, respectively (see [89]). Our analytic results allow us to obtain these identifications relaxing the hypotheses on the terms of the Cauchy problem, and so to study the control problem in a more general situation.

The chapter is organized as follows. In Section 4.2 we prove the existence and uniqueness of a mild solution to (BPDE), and study some of its regularity properties. In the first subsection, we show that the estimate

$$\|G\nabla_x S(t)\varphi\|_\infty \leq \frac{C}{t^{1/2}}\|\varphi\|_\infty, \quad t \in (0,T],$$

holds for any $\varphi \in C_b(\mathbb{R}^d)$, any $T > 0$ and some positive constant $C = C(T)$.

In the second subsection, we show that the integral term of the mild solution has the required smoothness. Moreover, a classical fixed point argument shows the existence and uniqueness of a local solution to the Cauchy problem (BPDE), solution which can be extended to the whole interval $[0, T]$.

Section 4.3 is devoted to the study of the (FBSDE) which, as we stressed above, is the key tool to prove that v is indeed the Value Function associated to problem (4.1).

Finally, in Section 4.4 we introduce the stochastic controlled equation. The regularity of v and the solvability of (FBSDE) enable us to prove that v is the value function and that, under suitable assumptions, the feedback law is verified.

4.2 The Semi-Linear PDE

Let us consider the backward Cauchy problem

$$\begin{cases} D_t u(t,x) + Au(t,x) = \psi(x, Q^{1/2}(x)\nabla_x u(t,x)), & t \in [0,T), \quad x \in \mathbb{R}^d, \\ u(T,x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{4.3}$$

where A is the second order elliptic operator, defined on smooth functions f by

$$Af(x) = \frac{1}{2}\mathrm{Tr}(Q(x)D_x^2 f(x)) + \langle B(x), \nabla_x f(x)\rangle, \tag{4.4}$$

$Q(x) = [Q_{ij}(x)]$ is a positive defined matrix for any $x \in \mathbb{R}^d$, $Q^{1/2}$ denotes its square root, $\varphi \in C_b(\mathbb{R}^d)$, and ψ is a continuous function, which satisfies the following conditions:

Hypotheses 4.1. *There exists a positive constant L_ψ such that*

$$\begin{aligned} |\psi(x_1, x_2) - \psi(y_1, y_2)| &\leq L_\psi |x_2 - y_2| + L_\psi |x_1 - y_1| \left(1 + |x_2| + |y_2|\right), \\ |\psi(x, 0)| &\leq L_\psi, \end{aligned} \tag{4.5}$$

for any $x, x_1, x_2, y_1, y_2 \in \mathbb{R}^d$.

Hypotheses 4.2. (i) *The coefficients Q_{ij} belong to $C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\alpha \in (0, 1)$ and any $i, j = 1, \ldots, d$;*

(ii) *the coefficients of the vector B belong to $C^{1+\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$; further, $\langle B(x), x\rangle \leq B_0(x)|x|$ for any $x \in \mathbb{R}^d$ and some negative function B_0;*

(iii) *there exist a positive constant K_0 and positive functions $\gamma_i : \mathbb{R}^d \to \mathbb{R}$, $i = 1, 2$, such that*

$$|\langle Q(x), x\rangle| \leq K_0(1 + |x|^2)\nu(x), \qquad x \in \mathbb{R}^d, \tag{4.6}$$

$$|\nabla_x Q^{1/2}(x)||Q^{-1/2}(x)| \leq \gamma_1, \qquad |Q(x)| \leq \gamma_2, \quad \forall x \in \mathbb{R}^d; \tag{4.7}$$

(iv) *the functions γ_1 and γ_2 satisfy the following conditions:*

$$\lim_{|x|\to+\infty} \frac{(\nu(x))^{-1}(\gamma_1(x))^2(\gamma_2(x))^2}{\omega(x)} = 0, \tag{4.8}$$

where the function $\omega : \mathbb{R}^d \to \mathbb{R}$ is a negative function which bounds from above the quadratic form associated with the matrix

$$\mathcal{M} := Q^{1/2}(J_x B)^T Q^{-1/2} - \sum_{j=1}^{d} B_j (D_j Q^{1/2}) Q^{-1/2} - \sum_{i,j=1}^{d} q_{ij}(D_{ij} Q^{1/2}) Q^{-1/2}.$$

Moreover,

$$\liminf_{|x|\to+\infty} \frac{(\nu(x))^2}{\omega(x)} > -\infty, \tag{4.9}$$

$$\lim_{|x|\to+\infty} \frac{|x|\nu(x)\gamma_1(x)}{B_0(x)} = 0, \tag{4.10}$$

$$\liminf_{|x|\to+\infty} \frac{|x|(\nu(x))^2}{B_0(x)} > -\infty; \tag{4.11}$$

(v) *there exist* $\lambda > 0$ *and a function* $f \in C^2(\mathbb{R}^d)$ *such that*

$$\lim_{|x|\to+\infty} f(x) = \infty, \quad \sup_{x\in\mathbb{R}^d} (Af(x) - \lambda f(x)) < \infty.$$

Here, we provide an example of operator A whose coefficients satisfy Hypotheses 4.2.

Example 4.3. *We consider the operator* A *with coefficents*

$$Q_{ij}(x) = Q_{ij}(1 + |x|^2)^l, \quad B_i(x) = -bx_i(1 + |x|^2)^p, \quad i, j = 1, \ldots, d,$$

where $[Q_{ij}]$ *is a positive definite matrix and* $b, l, p > 0$*. Hence, in Hypotheses* $4.2(ii)-(iii)$ *we can choose* $K_0 = 1$*,*

$$B_0(x) = |x|(1 + |x|^2)^p, \qquad \gamma_1(x) = 2l \max_{i,j=1,\ldots,d} |Q_{ij}^{1/2}|(1 + |x|^2)^{-1/2},$$
$$\gamma_2(x) = |Q|(1 + |x|^2)^l,$$

and

$$\omega(x) \leq -b\sum_{i=1}^{d}(1 + |x|^2)^p - 2l\nu_0(1 + |x|^2)^{l-2}(d + (2l + d - 1)|x|^2)$$
$$+ lb|x|^2(1 + |x|^2)^{p-1}.$$

Finally, the growth conditions in Hypotheses $4.2(iv)-(v)$ *hold if*

$$l \geq 1, \quad p \geq 2l.$$

Under Hypotheses $4.2(i),(v)$, the Cauchy problem

$$\begin{cases} D_t u(t, x) = Au(t, x), & t > 0, \quad x \in \mathbb{R}^d, \\ u(0, x) = \varphi(x), & \qquad\quad\; x \in \mathbb{R}^d, \end{cases}$$

admits a unique classical solution

$$u \in C([0, \infty) \times \mathbb{R}^d) \cap C_{loc}^{1+\alpha/2, 2+\alpha}((0, \infty) \times \mathbb{R}^d),$$

for any $\alpha \in (0, 1)$, satisfying

$$|u(t, x)| \leq \|\varphi\|_\infty, \quad t > 0, \ x \in \mathbb{R}^d.$$

(see 2).

We define a family of linear bounded operators $\{S(t)\}_{t\geq0}$ by $S(t)f(x) = u(t, x)$, for any $t \geq 0$, $x \in \mathbb{R}^d$, where $\{S(t)\}_{t\geq0}$ is the contractive semigroup of linear operators associated to the operator A.

Now, we introduce a class of function spaces, which is a natural environment where to set the Cauchy problem (4.3):

Definition 4.4. *For any $a > 0$, let us consider the space*

$$\mathcal{K}_a = \left\{ \begin{array}{l} h \in C_b\left([T-a,T] \times \mathbb{R}^d\right) \cap C^{0,1}\left([T-a,T) \times \mathbb{R}^d\right) : \\ \\ \sup\limits_{\substack{t\in[T-a,T)\\ x\in\mathbb{R}^d}} (T-t)^{1/2}|Q^{1/2}(x)\nabla_x h(t,x)| < \infty \end{array} \right\},$$

endowed with the norm

$$\|h\|_{\mathcal{K}_a} = \|h\|_\infty + [h]_{\mathcal{K}_a}, \tag{4.12}$$

where

$$[h]_{\mathcal{K}_a} := \sup_{t\in[T-a,T)} (T-t)^{1/2}\|Q^{1/2}\nabla_x h(t,\cdot)\|_\infty.$$

For any $a > 0$ we define the function $F_a : [T-a,T) \times \mathcal{K}_a \longrightarrow C(\mathbb{R}^d)$ by

$$F(t,u)(x) = \psi(x, G(x)\nabla_x u(t,x)). \tag{4.13}$$

At this stage formula

$$v(t,x) = S(T-t)\varphi(x) - \int_t^T S(r-t)F(r,v)(x)dr, \tag{4.14}$$

is just formal. Since ψ and $Q^{1/2}$ may be unbounded, to justify this formula we need first to show that the semigroup $\{S(t)\}_{t\geq 0}$ can actually be applied to F.

4.2.1 Weighted gradient estimates

Our purpose is to prove that, for any $\varphi \in C_b(\mathbb{R}^d)$ and any $t > 0$, the function $x \mapsto Q^{1/2}(x)S(t)\varphi(x)$ is bounded in $\mathbb{R}^d$ and that, for any $T > 0$, there exists a positive constant C_T such that

$$\|Q^{1/2}\nabla_x S(t)\varphi\|_\infty \leq \frac{C_T}{t^{1/2}}\|\varphi\|_\infty, \quad t \in (0,T].$$

To this aim, for any $R \geq 1$, we introduce the function η_R defined by $\eta_R(x) = \eta(|x|/R)$ for any $x \in \mathbb{R}^d$, where

$$\eta(t) = \begin{cases} 1, & t \in [0,1/2], \\ \exp\left(1 - \frac{1}{1-(4t-2)^3}\right), & t \in (1/2,3/4), \\ 0 & t \geq 3/4. \end{cases}$$

Clearly, $\eta_R \in C_c^2(\mathbb{R}^d)$, $0 \leq \eta_R \leq 1$ in $\mathbb{R}^d$, $\eta_R \equiv 1$ in $B(R/2)$, and $\eta_R \equiv 0$ outside the ball $B(R)$. Moreover, we have $D_i\eta_R(x) = -x_i\eta_R\mathcal{K}_R(x)$, where

$$\mathcal{K}_R(x) := \chi_{[1/2,3/4)}(|x|/R)\frac{12(4|x|/R-2)^2}{|x|R\left(1-(4|x|/R-2)^3\right)^2},.$$

Computing the first and second orders derivatives of η_R, we easily obtain

$$(i)\ |Q\nabla\eta_R| \leq cR^2\mathcal{K}_R\nu\eta_R, \quad (ii)\ |\mathrm{Tr}(QD^2\eta_R)| \leq c\nu, \tag{4.15}$$

where c is a positive constant independent of R.

Theorem 4.5. *Let Hypotheses 4.1 and 4.2 be fulfilled, and let* $\varphi \in C_b(\mathbb{R}^d)$. *If* u *is a classical solution to the homogenous Cauchy problem*

$$\begin{cases} D_t u(t,x) = Au(t,x), & t > 0, \quad x \in \mathbb{R}^d, \\ u(0,x) = \varphi(x), & \qquad\quad x \in \mathbb{R}^d, \end{cases}$$

i.e., $u \in C_b\left([0,\infty) \times \mathbb{R}^d\right) \cap C^{1,2}\left((0,\infty) \times \mathbb{R}^d\right)$ *and it satisfies the above equation and the initial condition, then the function*

$$(t,x) \mapsto Q^{1/2}(x)\nabla_x u(t,x)$$

is bounded in $[\varepsilon, T] \times \mathbb{R}^d$, *for any* $0 < \varepsilon < T$. *Moreover, there exists a positive constant* C_T *such that*

$$t^{1/2}\|Q^{1/2}\nabla_x u(t,\cdot)\|_\infty \le C_T\|\varphi\|_\infty, \quad t \in (0,T]. \tag{4.16}$$

Proof. Fix $R \ge 1$, $T > 0$ and let $u_R \in C_b\left([0,\infty) \times \overline{B(R)}\right) \cap C^{1,2}\left((0,\infty) \times \overline{B(R)}\right)$ be the solution to the Cauchy Dirichlet problem

$$\begin{cases} D_t u_R(t,x) = Au_R(t,x), & t > 0, \quad x \in B(R), \\ u_R(t,x) = 0, & t > 0, \quad x \in \partial B(R), \\ u_R(0,x) = \eta_R(x)\varphi(x), & \qquad\quad x \in \overline{B(R)}. \end{cases} \tag{4.17}$$

We set

$$v_R(t,x) = u_R(t,x)^2 + at\eta_R^2|Q^{1/2}(x)\nabla_x u_R(t,x)|^2, \quad t \in [0,T], \quad x \in \overline{B(R)}.$$

Function v_R is continuous in its domain, and it solves the Cauchy problem

$$\begin{cases} D_t v_R(t,x) - Av_R(t,x) = g_R(t,x), & t \in [0,T], \quad x \in B(R), \\ v_R(t,x) = 0, & t \in [0,T], \quad x \in \partial B(R), \\ v_R(0,x) = (\eta_R\varphi)^2(x), & \qquad\qquad x \in \overline{B(R)}, \end{cases} \tag{4.18}$$

where $g_R(t,x) = t\sum_{i=1}^{5} g_{i,R}(t,x)$, $\mathcal{G} := |Q^{1/2}\nabla_x u_R|^2$, $\mathcal{H} := \sum_{i=1}^d |Q^{1/2}\nabla_x(D_i u_R)|^2$, and

$$g_{1,R} = -2t^{-1}\mathcal{G} - 2a\eta_R\sum_{i=1}^d B_i D_i\eta_R\mathcal{G} - 2a\eta_R^2\sum_{i,j=1}^d Q_{ij}\langle Q^{1/2}\nabla_x(D_i u_R), Q^{1/2}\nabla(D_j u_R)\rangle,$$

$$g_{2,R} = 2a\eta_R^2\langle Q^{1/2}(DB)\nabla_x u_R, Q^{1/2}\nabla_x u_R\rangle - 2a\eta_R^2\sum_{i,j=1}^d Q_{ij}\langle (D_{ij}Q^{1/2})\nabla_x u_R, Q^{1/2}\nabla_x u_R\rangle$$

$$
\begin{aligned}
&- 2a\eta_R^2\sum_{j=1}^d B_j\langle (D_jQ^{1/2})\nabla_x u_R, Q^{1/2}\nabla_x u_R\rangle,\\
g_{3,R} &= -2a|Q^{1/2}\nabla_x u_R|^2|Q^{1/2}\nabla_x\eta_R|^2 - 2a\eta_R^2\sum_{i,j=1}^d Q_{ij}\langle (D_jQ^{1/2})\nabla_x u_R, (D_iQ^{1/2})\nabla_x u_R\rangle,\\
g_{4,R} &= -2a\eta_R \mathrm{Tr}(Q(D^2\eta_R))\mathcal{G} - 8a\eta_R\sum_{i,j=1}^d Q_{ij}(D_i\eta_R)\langle (D_jQ^{1/2})\nabla_x u_R, G\nabla_x u_R\rangle\\
&- 8a\eta_R\sum_{i,j=1}^d Q_{ij}(D_i\eta_R)\langle Q^{1/2}\nabla_x(D_ju_R), Q^{1/2}\nabla_x u_R\rangle,\\
g_{5,R} &= at^{-1}\eta_R^2\mathcal{G} - 4a\eta_R^2\sum_{i,j=1}^d Q_{ij}\langle (D_jQ^{1/2})\nabla_x(D_iu_R), Q^{1/2}\nabla_x u_R\rangle\\
&- 4a\eta_R^2\sum_{i,j=1}^d Q_{ij}\langle (D_jQ^{1/2})\nabla_x u_R, Q^{1/2}\nabla_x(D_iu_R)\rangle\\
&+ 4a\eta_R^2\langle Q^{1/2}\mathrm{Tr}((\nabla_x Q^{1/2})Q^{1/2}(D^2u_R)), Q^{1/2}\nabla_x u_R\rangle,
\end{aligned}
$$

We are going to prove that there exists a positive constant c, independent of R, such that $g_R(t,x) \le cv_R(t,x)$, for any $(t,x) \in [0,T]\times\overline{B(R)}$.

The terms $g_{1,R}$ and $g_{2,R}$ are crucial in the estimate of g_R, since they allow us to control all the other terms $g_{i,R}$, $i = 3,4,5$.

We get

$$
\begin{aligned}
g_{1,R}(t,x) &\le -2t^{-1}\mathcal{G}(t,x) - 2a\eta_R^2\nu(x)\mathcal{H}(t,x) - 2a\eta_R\langle B,\nabla_x\eta_R\rangle\mathcal{G}(t,x)\\
&\le -2t^{-1}\mathcal{G}(t,x) - 2a\eta_R^2\nu(x)\mathcal{H}(t,x) + 2aR\eta_R^2\mathcal{K}_R B_0(x),\\
g_{2,R} &= 2a\eta_R^2\langle \mathcal{M}Q^{1/2}\nabla_x u_R, Q^{1/2}\nabla_x u_R\rangle \le 2a\eta_R^2\omega\mathcal{G},\\
g_{3,R} &\le 0.
\end{aligned}
$$

$g_{4,R}$ is the awkward term. We have to pay particular attention to the way we estimate its addends which we want to compare with $g_{1,R}$ and $g_{2,R}$.

As far as the first addend is concerned, taking advantage of (4.15)(ii) and of the well known Young's inequality $ab \le (\varepsilon/2)a^2 + (2\varepsilon)^{-1}b^2$, which holds true for any $a,b,\varepsilon > 0$, we get

$$
\begin{aligned}
\left|2a\eta_R\sum_{i,j=1}^d Q_{ij}(D_{ij}\eta_R)\mathcal{G}\right| &\le \frac{a}{\varepsilon}\mathcal{G} + a\varepsilon\eta_R^2|\mathrm{Tr}(QD^2\eta_R)|^2\mathcal{G}\\
&\le \frac{a}{\varepsilon}\mathcal{G} + ca\varepsilon\eta_R^2\nu^2\mathcal{G}.
\end{aligned}
$$

As far as the second term in the definition of $g_{4,R}$ is concerned, we have

$$\begin{aligned}
&\left|8a\eta_R \sum_{i,j=1}^{d} Q_{ij}(D_i\eta_R)\langle \left(D_j Q^{1/2}\right) \nabla_x u_R, Q^{1/2}\nabla_x u_R\rangle\right| \\
&= \left|8a\eta_R \sum_{i,j=1}^{d} Q_{ij}(D_i\eta_R)\langle \left(D_j Q^{1/2}\right) Q^{-1/2}Q^{1/2}\nabla_x u_R, Q^{1/2}\nabla_x u_R\rangle\right| \\
&\leq 8a\eta_R|Q\nabla\eta_R||(\nabla_x Q^{1/2})Q^{-1/2}|\mathcal{G} \\
&\leq acR^2\eta_R^2\mathcal{K}_R\nu\gamma_1\mathcal{G}.
\end{aligned}$$

The last term in the definition of $g_{4,R}$ is the worst one because we need to estimate the growth of both $\mathcal{G}$ and $\mathcal{H}_i$, with $i = 1,\dots,d$. We split it using the following inequality, which follows applying twice the Young's inequality, and holds for any $b, c, h, \varepsilon > 0$:

$$bch \leq \frac{1}{4}\left(\frac{1}{\varepsilon}b^4 + \frac{1}{\varepsilon}c^4 + 2\varepsilon h^2\right).$$

We set

$$\begin{aligned}
b_k &:= a^{3/8}\eta_R^{1/6}|(Q\nabla\eta_R)_k|^{1/2}\mathcal{G}^{1/4}, \\
c &:= a^{1/8}\mathcal{G}^{1/4}, \\
h_{j,k} &= a^{1/2}\eta_R^{5/6}|(Q\nabla\eta_R)_k|^{1/2}\mathcal{H}_j^{1/2},
\end{aligned}$$

for any $j, k = 1,\dots,d$, and we observe that, since $\mathcal{K}_R\eta_R \leq cR^{-2}\eta_R^{1/3}$, from (4.15)($i$) we get

$$|(Q\nabla\eta_R)_k|^2 \leq cR^4\eta_R^2\mathcal{K}_R^2\nu^2 \leq cR^2\eta_R^{4/3}\mathcal{K}_R\nu^2.$$

The particular split into b, c and e arises from the necessity of having suitable coefficients of $\mathcal{H}_j$, $j = 1,\dots,d$, and $\mathcal{G}$, which we can estimate with $g_{1,R}$ and $g_{2,R}$. Straightforward computations yield to

$$\begin{aligned}
&\left|8a\eta_R \sum_{i,j=1}^{d} Q_{ij}(D_i\eta_R)\langle Q^{1/2}\nabla_x(D_j u_R), Q^{1/2}\nabla_x u_R\rangle\right| \leq 8\sum_{j,k=1}^{d} b_k c h_{j,k} \\
&\leq 8\sum_{j,k=1}^{d}\left(\frac{1}{4\varepsilon}b_k^4 + \frac{1}{4\varepsilon}c^4 + \frac{1}{2}\varepsilon h_{j,k}^2\right) \\
&\leq \frac{2a^{1/2}d}{\varepsilon}\mathcal{G} + \frac{a^{3/2}c}{\varepsilon}R^2\eta_R^2\mathcal{K}_R\nu^2\mathcal{G} + ac\varepsilon\eta_R^2\nu\mathcal{H}.
\end{aligned}$$

The last term that we need to estimate is $g_{5,R}$. Applying the Young's inequality we get

$$\left|4a\eta_R^2 \sum_{i,j=1}^{d} Q_{ij}\langle \left(D_j Q^{1/2}\right) Q^{-1/2}Q^{1/2}\nabla_x(D_i u_R), Q^{1/2}\nabla_x u_R\rangle\right|$$

$$\leq 4a\eta_R^2\gamma_1\gamma_2\sum_{i=1}^d |Q^{1/2}\nabla_x(D_iu_R)||Q^{1/2}\nabla_x u_R|$$
$$\leq \frac{2a}{\varepsilon}\eta_R^2\nu^{-1}\gamma_1^2\gamma_2^2\mathcal{G}+2a\varepsilon\eta_R^2\nu\mathcal{H},$$

$$|4a\eta_R^2\langle Q^{1/2}\mathrm{Tr}(\left(\nabla_x Q^{1/2}\right)Q^{1/2}(D^2u_R)),Q^{1/2}\nabla_x u_R\rangle|$$
$$=4a\eta_R^2\left|\sum_{i,j,l,m=1}^d Q_{ij}^{1/2}D_jQ_{lm}^{1/2}(Q^{1/2}\nabla_x D_ju_R)_m(Q^{1/2}\nabla u_R)_i\right|$$
$$\leq 4aR\eta_R^2\gamma_1\gamma_2\sum_{i=1}^d|Q^{1/2}\nabla_x D_iu_R||Q^{1/2}\nabla_x u_R|$$
$$\leq \frac{2a}{\varepsilon}\eta_R^2\nu^{-1}\gamma_1^2\gamma_2^2\mathcal{G}+2a\varepsilon\eta_R^2\nu\mathcal{H},$$

and so

$$|g_{5,R}|\leq \frac{4a}{\varepsilon}\nu^{-1}\eta_R^2\gamma_1^2\gamma_2^2\mathcal{G}+4a\varepsilon\eta_R^2\nu\mathcal{H}.$$

Hence, collecting the similar terms, we deduce that

$$g_R\leq I_1\mathcal{G}+I_2\mathcal{H},$$

where, for any $x\in\overline{B(R)}$ and $t\in[0,T]$, we have

$$I_1(t,x)=-1+\frac{a}{\varepsilon}+\frac{2a^{1/2}td}{\varepsilon}+at\eta_R^2(x)(2\omega(x)+ca\varepsilon\nu^2(x)+4\frac{1}{\varepsilon}\nu^{-1}\gamma_1^2(x)\gamma_2^2(x))$$
$$+atR\eta_R^2(x)\mathcal{K}_R(2B_0(x)+cR\nu(x)\gamma_1(x)+\frac{a^{1/2}c}{\varepsilon}R\nu^2(x))$$

and

$$I_2(t,x)=at\eta_R^2(x)\nu(x)(-2+\varepsilon(c+4)).$$

Choosing $\varepsilon<2/(c+4)$ we obtain that I_2 is negative in $[0,T]\times\overline{B(R)}$.

Moreover, Hypotheses 4.2$(ii)-(iv)$ and a suitable choice of ε and a imply that I_1 is bounded from above. Hence, there exists a positive constant c, indipendent of R, such that $|g_R|\leq cv_R$. From the classical maximum principle we deduce that $v_R\leq c\|\varphi\|_\infty$. Taking the limit as $R\to\infty$, estimate (4.16) follows. □

Remark 4.6. *By the semigroup property, it easily follows that, for any $\omega>0$, there exists $C=C(\omega)>0$ such that*

$$\|Q^{1/2}\nabla_x S(t)\varphi\|_\infty\leq\frac{Ce^{\omega t}}{t^{1/2}}\|\varphi\|_\infty, \tag{4.19}$$

for any $t>0$ and any $\varphi\in C_b(\mathbb{R}^d)$.

Indeed, for any $\omega > 0$, we can choose $\sigma = \sigma(\omega)$ such that $e^{\omega t} t^{-1/2} > 1$, for any $t > \sigma$. If $t > \sigma$ we can estimate (using (4.16) *and recalling that $\{S(t)\}_{t \geq 0}$ is a contraction semigroup)*

$$\begin{aligned}\|Q^{1/2} \nabla_x S(t)\varphi\|_\infty = \|Q^{1/2} \nabla_x S(\sigma) S(t-\sigma)\varphi\|_\infty &\leq \frac{C_\sigma}{\sigma^{1/2}} \|S(t-\sigma)\varphi\|_\infty \\ &\leq \frac{C_\sigma}{\sigma^{1/2}} \|\varphi\|_\infty \leq \frac{C_\sigma e^{\omega t}}{\sigma^{1/2} t^{1/2}} \|\varphi\|_\infty,\end{aligned}$$

and therefore (4.19) *holds with $C = \max\{C_\sigma, \sigma^{-1/2} C_\sigma\}$.*

Proposition 4.7. *Under the same assumptions of Theorem* 4.5*, if $\varphi \in C_b^1(\mathbb{R}^d)$, then the function*

$$(t, x) \mapsto Q^{1/2}(x) \nabla_x S(t)\varphi(x)$$

is bounded in $[0, T] \times \mathbb{R}^d$.

Proof. The proof is quite similar to the one of Theorem 4.5, hence we just sketch it. We fix $R \geq 1$, and denote by u_R the solution to the Dirichlet Cauchy problem (4.17). Further we set

$$v_R(t, x) = u_R(t, x)^2 + a\eta_R^2 |Q^{1/2}(x) \nabla_x u_R(t, x)|^2, \quad t \in (0, T], \quad x \in \overline{B(R)}.$$

Function v_R is continuous in its domain and it solves the Cauchy problem

$$\begin{cases} D_t v_R(t, x) - A v_R(t, x) = \tilde{g}_R(t, x), & t \in [0, T], \quad x \in B(R), \\ v_R(t, x) = 0, & t \in [0, T], \quad x \in \partial B(R), \\ v_R(0, x) = (\eta_R \varphi)^2(x), & \qquad\qquad x \in \overline{B(R)}, \end{cases}$$

where $\tilde{g}_R(t, x) = \tilde{g}_{1,R}(t, x) + \sum_{i=2}^{5} g_{i,R}(t, x)$,

$$\tilde{g}_{1,R} = -2\mathcal{G} - 2a\eta_R^2 \sum_{i,j=1}^{d} Q_{ij} \langle Q^{1/2} \nabla_x (D_i u_R), Q^{1/2} \nabla_x (D_j u_R) \rangle - 2a\eta_R \langle B, \nabla \eta_R \rangle \mathcal{G},$$

and $g_{i,R}$, $i = 2, 3, 4, 5$, have been defined in Theorem 4.5. Repeating the computations of Theorem 4.5, we see that

$$\tilde{g}_R \leq I_1 \mathcal{G} + I_2 \mathcal{H},$$

where, for any $x \in \overline{B(R)}$ and $t \in [0, T]$, we have

$$\begin{aligned} I_1(t, x) = -2 + \frac{a}{\varepsilon} + \frac{2a^{1/2} d}{\varepsilon} + a\eta_R^2(x)(2\omega(x) + ca\varepsilon\nu^2(x) + 4\frac{1}{\varepsilon}\nu^{-1}\gamma_1^2(x)\gamma_5(x)) \\ + aR\eta_R^2(x)\mathcal{K}_R(2B_0(x) + cR\nu(x)\gamma(x) + \frac{a^{1/2} c}{\varepsilon} R\nu^2(x)) \end{aligned}$$

and

$$I_2(t,x) = a\eta_R^2(x)\nu(x)(-2+\varepsilon(c+4)).$$

With a suitable choice of the parameters a and ε, there exists a positive constant c, independent of R, such that $\tilde{g}_R \le cv_R$ (see the proof of Theorem 4.5). An application of the classical maximum principle implies that $|v_R| \le c\|\varphi\|_\infty$, and taking the limit as $R \to \infty$ we obtain the desired estimate. □

4.2.2 Existence and uniqueness of a mild solution to the semilinear Cauchy Problem

In this subsection we will prove that the operator Γ, defined for any $u \in \mathcal{K}_T$ by

$$(\Gamma u)(t,x) := S(T-t)\varphi(x) - \int_t^T (S(r-t)F(r,u))(x)dr,$$

for any $t \in [0,T]$ and $x \in \mathbb{R}^d$, admits a unique fixed point. We call a *mild solution* of problem (4.3) any fixed point $v \in \mathcal{K}_T$ of the operator Γ.

Remark 4.8. *If ψ satisfies Hypothesis* 4.1*, then (see* (4.13)*)*

$$\begin{aligned} &(i)\quad \|F(s,u)-F(s,v)\|_\infty \le L_\psi (T-s)^{-1/2}[u-v]_{\mathcal{K}_T}, \quad s\in[0,T), x\in\mathbb{R}^d,\\ &(ii)\quad \|F(s,u)\|_\infty \le L_\psi\left(1+(T-s)^{-1/2}[u]_{\mathcal{K}_T}\right), \end{aligned} \tag{4.20}$$

for any $u,v \in \mathcal{K}_T$. Moreover, if $u \in \mathcal{K}_T$, $F(\cdot,u)(\cdot) : [0,T)\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ belongs to $C([0,T)\times\mathbb{R}^d)$.

The next lemma is a general result of misurability for the supremum norm of continuous functions. We will repeatedly use it in the follow.

Lemma 4.9. *Let $f \in C([0,T]\times\mathbb{R}^d)$ such that, for any $t\in[0,T]$, $f(t,\cdot) \in C_b(\mathbb{R}^d)$. Then the function $t \mapsto \|f(t,\cdot)\|_\infty$ is a measurable function.*

Proof. Let $n \in \mathbb{N}$. For any $t,s \in [0,T]$, we have

$$\|f(s,\cdot)\|_{L^\infty(B(n))} \le \|f(s,\cdot)-f(t,\cdot)\|_{L^\infty(B(n))} + \|f(t,\cdot)\|_{L^\infty(B(n))}$$

and

$$\|f(t,\cdot)\|_{L^\infty(B(n))} \le \|f(t,\cdot)-f(s,\cdot)\|_{L^\infty(B(n))} + \|f(s,\cdot)\|_{L^\infty(B(n))}.$$

Obviously, f is uniformly continuous in $[0,T]\times\overline{B(n)}$; hence, letting $s \to t$, from the uniform continuity of f we get

$$\lim_{s\to t}\|f(s,\cdot)\|_{L^\infty(B(n))} = \|f(t,\cdot)\|_{L^\infty(B(n))},$$

i.e., the map $t \mapsto \|f(s,\cdot)\|_{L^\infty(B(n))}$ is continuous in $[0,T]$. To conclude, it is enough to observe that, for any $t \in [0,T]$,

$$\|f(t,\cdot)\|_\infty = \lim_{n\to\infty} \|f(t,\cdot)\|_{L^\infty(B(n))},$$

which means that $t \mapsto \|f(t,\cdot)\|_\infty$ is measurable, being pointwise limit of measurable functions. □

Hereafter, we will apply Lemma 4.9 without mentioning it.

The following proposition shows some continuity and boundedness properties of the functions which belong to $\mathcal{K}_\delta$, for some $\delta > 0$.

Proposition 4.10. *If $u \in \mathcal{K}_\delta$, for some $\delta > 0$, F satisfies* (4.20) *and*

$$\sup_{t\in(T-\delta,T)} (T-t)^{1/2}\|Q^{1/2}\nabla_x u(t,\cdot)\|_\infty < \infty,$$

then the functions

$$(t,x) \mapsto \tilde{F}(t,x) := \int_t^T (S(r-t)F(r,u))(x)dr$$

and

$$(t,x) \mapsto Q^{1/2}(x)\nabla_x \tilde{F}(t,x)$$

are continuous and bounded in $[T-\delta,T]\times\mathbb{R}^d$.

Proof. For any fixed $t \in [T-\delta,T]$, the functions

$$x \mapsto \tilde{F}(t,x) := \int_t^T (S(r-t)F(r,u))(x)dr, \quad x \mapsto Q^{1/2}(x)\nabla_x \tilde{F}(t,x)$$

are continuous in $\mathbb{R}^d$. Hence it is enough to show that these functions are continuous with respect to t, locally uniformly with respect to x.

Let $(t_0,x_0) \in (T-\delta,T)\times\mathbb{R}^d$, $B = B(x_0,1) \in \mathbb{R}^d$, and fix $t \in (t_0-\delta, t_0+\delta)$, where $0 < \delta < \min\{T-t_0, \delta+t_0-T\}$. We will only prove the continuity from the right with respect to time, uniformly with respect to x, since the continuity from the left can be proved arguing in the same way. Hence we consider $t \in (t_0, t_0+\delta)$. We have

$$\begin{aligned}
|\tilde{F}(t_0,x) - \tilde{F}(t,x)| &\le \int_t^T |(S(r-t_0)F(r,u))(x) - (S(r-t)F(r,u))(x)|dr \\
&\quad + \int_{t_0}^t |(S(r-t_0)F(r,u))(x)|dr \\
&= \int_{t_0}^T |(S(r-t_0)F(r,u))(x) - (S(r-t)F(r,u))(x)|\chi_{(t,T)}(r)dr \\
&\quad + \int_{t_0}^t |(S(r-t_0)F(r,u))(x)|dr \\
&=: I_1(t,x) + I_2(t,x).
\end{aligned}$$

Since $\|S(r-t_0)F(r,u)\|_\infty \le C$, for any $r \in (t_0, t_0+\delta)$, I_2 tends to 0, as t tends to t_0, uniformly with respect to $x \in B$.

Now we consider I_1. Since $u \in \mathcal{K}_\delta$, we can estimate the function under the integral sign as follows:

$$\begin{aligned}\|S(r-t_0)F(r,u) - S(r-t)F(r,u)\|_\infty \chi_{(t,T)}(r) &\le 2M_0\|F(r,u)\|_\infty \\ &\le 2M_0 L_\psi \left(1 + (T-r)^{-1/2}[u]_{\mathcal{K}_\delta}\right),\end{aligned}$$

for any $r \in (t_0, T)$, and the last function is integrable in (t_0, T).

Finally, for any $r \in (0,T)$, $F(r,u) \in C_b(\mathbb{R}^d)$ by (4.20). Hence $(S(\cdot)F(r,u))(\cdot)$ belongs to $C([0,\infty)\times\mathbb{R}^d)$, and

$$\lim_{t\downarrow t_0} |(S(r-t_0)F(r,u))(x) - (S(r-t)F(r,u))(x)| = 0,$$

uniformly with respect to $x \in B$, for any $r \in (0,T)$.

By dominated convergence we can conclude that I_1 tends to 0 as t approaches t_0, uniformly with respect to $x \in B$.

Proving the continuity of the gradient is a bit more complicated. Let t_0, x_0, t, B, δ be as above; we have

$$\begin{aligned}&|Q^{1/2}(x)\nabla_x \tilde{F}(t_0,x) - Q^{1/2}(x)\nabla_x \tilde{F}(t,x)| \\ &\quad\le \int_t^T |Q^{1/2}(x)\nabla_x(S(r-t_0)F(r,u))(x) - Q^{1/2}(x)\nabla_x(S(r-t)F(r,u))(x)|dr \\ &\quad\quad+ \int_{t_0}^t |Q^{1/2}(x)\nabla_x(S(r-t_0)F(r,u))(x)|dr \\ &\quad\quad=: \tilde{I}_1(t,x) + \tilde{I}_2(t,x).\end{aligned}$$

By Theorem 4.5, there exists a positive constant C such that

$$\|Q^{1/2}\nabla_x S(r-t_0)F(r,u)\|_\infty \le (r-t_0)^{-1/2}C,$$

for any $r \in (t_0, t_0+\delta)$. Hence, $\tilde{I}_2$ tends to zero as t tends to t_0, uniformly with respect to $x \in B$.

The term $\tilde{I}_1$ should be analyzed differently. Fix $\varepsilon > 0$ and $t \in (t_0, t_0+\delta)$ such that $t - t_0 < \varepsilon^2$. Now we split the integral:

$$\begin{aligned}&\tilde{I}_1(t,x) \\ &\quad= \int_{t_0+\varepsilon^2}^T |Q^{1/2}(x)\nabla_x(S(r-t_0)F(r,u))(x) - Q^{1/2}(x)\nabla_x(S(r-t)F(r,u))(x)|dr \\ &\quad+ \int_{t_0}^{t_0+\varepsilon^2} |Q^{1/2}(x)\nabla_x(S(r-t_0)F(r,u))(x) - Q^{1/2}(x)\nabla_x(S(r-t)F(r,u))(x)|\chi_{(t,T)}(r)dr \\ &\quad=: J_1(t,x) + J_2(t,x).\end{aligned}$$

Easy computations show that there exists a positive constant $C > 0$, independent of t, x, such that

$$J_2(t,x) \le C\varepsilon, \quad t \in (t_0, t_0+\varepsilon^2), \quad x \in B.$$

For J_1, it is enough to observe that the function under the integral sign converges to 0 pointwise with respect to t, locally uniformly with respect to x, and that the function h, defined by

$$h(r) = C_T L_\psi \left(1 + (T-r)^{-1/2}[u]_{\mathcal{K}_\delta}\right)\left((r-t_0)^{-1/2} + (r-t_0-\varepsilon^2)^{-1/2}\right)$$

is independent on t and x and bounds J_1 from above. Dominated convergence allows us to conclude that $J_1(t,x)$ vanishes to 0 as t tends to t_0, locally uniformly with respect to x. Hence, there exists $c_\varepsilon \le \varepsilon^2$ such that, if $t_0 - t < c_\varepsilon$ and $x \in B$, then $J_1(t,x) \le \varepsilon$. It means that there exists a suitable $C > 0$ such that $\tilde{I}_1(t,x) \le C\varepsilon$ for any $t > t_0 - c_\varepsilon$ and $x \in B$. □

We now look for a solution to problem (4.3) in $\mathcal{K}_T$. At first, we show that, if u is a mild solution of (4.3) in $\mathcal{K}_\delta$, for some $\delta \in (0,T)$, then it is the unique mild solution in such a space.

Proposition 4.11 (Uniqueness)**.** *If problem* (4.3) *admits a mild solution in* $\mathcal{K}_\delta$*, then it is unique.*

Proof. Let $u, v \in \mathcal{K}_\delta$ be two mild solutions of (4.3). Then, taking (4.5) and (4.16) into account, for any $t \in [T-\delta, T]$ we get

$$\begin{aligned}
\|Q^{1/2}\nabla_x(u-v)(t,\cdot)\|_\infty &\le \|\int_t^T Q^{1/2}\nabla_x S(r-t)(F(r,u)-F(r,v))dr\|_\infty \\
&\le C_T L_\psi \int_t^T (r-t)^{-1/2}\|Q^{1/2}\nabla_x(u-v)(r,\cdot)\|_\infty dr \\
&\le C_T^2 L_\psi^2 \int_t^T (r-t)^{-1/2}dr\Big(\int_r^T (s-r)^{-1/2} \\
&\quad \times \|Q^{1/2}\nabla_x(u-v)(s,\cdot)\|_\infty ds\Big) \\
&= C_T^2 L_\psi^2 \int_t^T \|Q^{1/2}\nabla_x(u-v)(s,\cdot)\|_\infty ds \\
&\quad \times \left(\int_t^s (r-t)^{-1/2}(s-r)^{-1/2}dr\right) \\
&= C_T^2 L_\psi^2 \pi \int_t^T \|Q^{1/2}\nabla_x(u-v)(s,\cdot)\|_\infty ds.
\end{aligned}$$

Hence, by the Gronwall Lemma we deduce that $\|Q^{1/2}\nabla_x(u-v)(t,\cdot)\|_\infty = 0$, for any

$t \in [T-\delta, T)$. To conclude, it is enough to observe that

$$\begin{aligned}\|u-v\|_\infty &\le \|\int_t^T S(r-t)(F(r,u)-F(r,v))dr\|_\infty \\ &\le L_\psi \int_t^T \|Q^{1/2}\nabla_x(u(r,\cdot)-v(r,\cdot))\|_\infty dr \\ &= 0.\end{aligned}$$

□

Now, we prove the existence of a mild solution of problem (4.3).

Theorem 4.12. *There exists $\delta < T$ such that the operator Γ, defined by*

$$(\Gamma v)(t,x) = S(T-t)\varphi(x) - \int_t^T (S(r-t)F(r,v))(x)dr, \tag{4.21}$$

$(t,x) \in (T-\delta, T] \times \mathbb{R}^d$, for any $v \in \mathcal{K}_\delta$, admits a unique fixed point.

Proof. Set

$$\mathcal{K}_{\delta,R} = \left\{ \begin{array}{l} h \in C_b\left([T-\delta,T]\times\mathbb{R}^d\right) \cap C^{0,1}\left([T-\delta,T)\times\mathbb{R}^d\right) : \\ \\ \|h\|_{\mathcal{K}_\delta} \le R \end{array} \right\},$$

endowed with the norm $\|\cdot\|_{\mathcal{K}_\delta}$ (see (4.12)). Since $\mathcal{K}_{\delta,R} \subset \mathcal{K}_\delta$, Proposition 4.11 shows that if Γ is a contraction in $\mathcal{K}_{\delta,R}$ then its unique fixed point is the unique mild solution to problem (4.3) which belongs to $\mathcal{K}_\delta$.

Hence, we prove that $\Gamma(v) \in \mathcal{K}_{\delta,R}$ for any $v \in \mathcal{K}_{\delta,R}$, and there exists $c<1$ such that

$$\|(\Gamma u)-(\Gamma v)\|_{\mathcal{K}_{\delta,R}} \le c\|u-v\|_{\mathcal{K}_{\delta,R}}, \quad u,v \in \mathcal{K}_{\delta,R}.$$

For this purpose, we set

$$C_T := \sup_{t\in(0,T]} t^{1/2}\|Q^{1/2}\nabla_x S(t)\|$$

and recall that $\sup_{t\in[0,T]}\|S(t)\| \le 1$ since $\{S(t)\}_{t\ge0}$ is a contraction semigroup.

Then by the second inequality in (4.20) we have

$$\begin{aligned}\|(\Gamma v)(t,\cdot)\|_\infty &\le \|\varphi\|_\infty + \left\|\int_t^T S(r-t)F(r,v)dr\right\|_\infty \\ &\quad + \left\|\int_t^T S(r-t)F(r,0)dr\right\|_\infty \\ &\le \|\varphi\|_\infty + 2L_\psi(T-t)^{1/2}\|v\|_{\mathcal{K}_{\delta,R}} + (T-t)L_\psi \\ &\le \|\varphi\|_\infty + 2L_\psi\delta^{1/2}\|v\|_{\mathcal{K}_{\delta,R}} + \delta L_\psi.\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}(T-t)^{1/2}&\|Q^{1/2}\nabla_x(\Gamma v)(t,\cdot)\|_\infty\\ &\le C_T\|\varphi\|_\infty+(T-t)^{1/2}C_TL_\psi\int_t^T(r-t)^{-1/2}\\ &\quad\times\left(\|Q^{1/2}\nabla_x v(r,\cdot)\|_\infty+1\right)dr\\ &\le C_T\|\varphi\|_\infty+C_TL_\psi(T-t)^{1/2}\|v\|_{\mathcal{K}_{\delta,R}}\int_t^T(r-t)^{-1/2}(T-r)^{-1/2}dr\\ &\quad+2(T-t)C_TL_\psi\\ &\le C_T\|\varphi\|_\infty+\pi C_TL_\psi(T-t)^{1/2}\|v\|_{\mathcal{K}_{\delta,R}}+2(T-t)C_TL_\psi\\ &\le C_T\|\varphi\|_\infty+\pi\delta^{1/2}C_TL_\psi\|v\|_{\mathcal{K}_{\delta,R}}+2\delta C_TL_\psi.\end{aligned}\tag{4.23}$$

Moreover,

$$\begin{aligned}\|(\Gamma u)(t,\cdot)-(\Gamma v)(t,\cdot)\|_\infty&\le\int_t^T\|S(r-t)\left(F(r,u)-F(r,v)\right)\|_\infty dr\\ &\le 2L_\psi\delta^{1/2}\|u-v\|_{\mathcal{K}_{\delta,R}}\end{aligned}\tag{4.24}$$

and

$$\begin{aligned}(T-t)^{1/2}&\|Q^{1/2}\nabla_x(\Gamma u)(t,\cdot)-Q^{1/2}\nabla_x(\Gamma v)(t,\cdot)\|_\infty\\ &\le(T-t)^{1/2}C_TL_\psi\int_t^T(r-t)^{-1/2}\|Q^{1/2}\nabla_x u(r,\cdot)-Q^{1/2}\nabla_x v(r,\cdot)\|_\infty dr\\ &\le(T-t)^{1/2}C_TL_\psi\|u-v\|_{\mathcal{K}_{\delta,R}}\int_t^T(r-t)^{-1/2}(T-r)^{-1/2}dr\\ &\le\pi(T-t)^{1/2}C_TL_\psi\|u-v\|_{\mathcal{K}_{\delta,R}}\\ &\le\pi\delta^{1/2}C_TL_\psi\|u-v\|_{\mathcal{K}_{\delta,R}}.\end{aligned}\tag{4.25}$$

Now we have to choose δ and R. Set

$$\delta=(4L_\psi+2\pi C_TL_\psi)^{-2}\wedge T$$

in (4.24) and (4.25); it immediately follows that

$$\begin{aligned}\|(\Gamma u)-(\Gamma v)\|_{\mathcal{K}_{\delta,R}}&\le 2L_\psi\delta^{1/2}\|u-v\|_{\mathcal{K}_{\delta,R}}+\delta^{1/2}\pi C_TL_\psi\|u-v\|_{\mathcal{K}_{\delta,R}}\\ &=\delta^{1/2}\left(2L_\psi+\pi C_TL_\psi\right)\|u-v\|_{\mathcal{K}_{\delta,R}}\\ &\le\frac{1}{2}\|u-v\|_{\mathcal{K}_{\delta,R}},\end{aligned}$$

and so Γ is a 1/2-contraction. To show that Γ maps $X_{\delta,R}$ into itself, it is sufficient to take

$$R=2\left(1+2C_T\right)\left(\|\varphi\|_\infty+\delta L_\psi\right).$$

Indeed, substituting in (4.22) and (4.23), we get

$$\begin{aligned}\|(\Gamma v)\|_{\mathcal{K}_{\delta,R}} &\leq \|\varphi\|_\infty + 2L_\psi \delta^{1/2}\|u\|_{\mathcal{K}_{\delta,R}} + \delta L_\psi \\ &\quad + C_T\|\varphi\|_\infty + \delta^{1/2}\pi C_T L_\psi \|v\|_{\mathcal{K}_{\delta,R}} + 2\delta C_T L_\psi \\ &\leq (1+2C_T)\left(\|\varphi\|_\infty + \delta L_\psi\right) + \delta^{1/2}\left(2L_\psi + \pi C_T L_\psi\right)\|v\|_{\mathcal{K}_{\delta,R}} \\ &\leq \frac{R}{2} + \frac{R}{2} \leq R.\end{aligned}$$

□

Remark 4.13. *If $\varphi \in C_b^1(\mathbb{R}^d)$ the same arguments as in the proof of Proposition* 4.11 *and Theorem* 4.12 *show that the operator Γ in* (4.21) *admits a unique fixed point in the space $\mathcal{K}_\delta$ defined by*

$$\mathcal{K}_\delta = \left\{ \begin{array}{l} h \in C_b\left([T-\delta, T] \times \mathbb{R}^d\right) \cap C^{0,1}\left([T-\delta,T]\times\mathbb{R}^d\right): \\ \\ \displaystyle\sup_{(t,x)\in(T-\delta,T)\times\mathbb{R}^d} |Q^{1/2}(x)\nabla h(t,x)| < \infty. \end{array} \right\}$$

for some $\delta > 0$.

Now, we can construct the maximally defined solution of (4.3). Set

$$\begin{cases} \tau(\varphi) = \inf\{0 < a < T: \text{ problem (4.3) has a mild solution } v_a \text{ in } \mathcal{K}_a\}, \\ \\ v(t,x) = v_a(t,x), \quad \text{if } t \geq T-a. \end{cases}$$

The function v is well defined, thanks to Theorem 4.12, in the interval

$$I(\varphi) = \cup\{[T-a, T]: \text{ problem (4.3) has a mild solution } v_a \text{ in } \mathcal{K}_a\},$$

and we have $\tau(\varphi) = \inf I(\varphi)$.

Proposition 4.14. *If $\varphi \in C_b(\mathbb{R}^d)$ is such that $I(\varphi) \neq [0,T]$, and F satisfies* (4.20)*, then the function*

$$t \mapsto (T-t)^{1/2}\|Q^{1/2}\nabla_x v(t,\cdot)\|_\infty$$

is unbounded in $I(\varphi)$.

Proof. Even if the proof is rather classical, for the reader's convenience we provide the details. Let us suppose that the function

$$t \mapsto (T-t)^{1/2}\|Q^{1/2}\nabla_x v(t,\cdot)\|_\infty$$

is bounded in $I(\varphi)$, and let v be the maximally defined solution to (4.3). Moreover, we set $\tau(\varphi) = \tau$. $S(\cdot)\varphi$ is continuous in $(0,\infty)\times\mathbb{R}^d$, and by Proposition 4.10 the function

$$(t,x) \mapsto \int_t^T (S(r-t)F(r,v))(x)dr$$

is continuous and bounded in $[\tau, T] \times \mathbb{R}^d$. Hence, we can extend v up to $t = \tau$, defining

$$v(\tau, x) := S(\tau)\varphi(x) - \int_\tau^T (S(r-\tau)F(r,v))(x)dr.$$

Since $v(\tau, \cdot) \in C_b(\mathbb{R}^d)$, by Theorem 4.12 the Cauchy problem

$$\begin{cases} w(t,x) + Aw(t,x) = \psi(x, Q^{1/2}\nabla_x w(t,x)), & t < \tau, \quad x \in \mathbb{R}^d, \\ \\ w(\tau, x) = v(\tau, x), & \qquad\quad\;\; x \in \mathbb{R}^d, \end{cases}$$

admits a unique mild solution in $[\tau - \delta, \tau]$, for some $\delta > 0$. If we define

$$z(t,x) = \begin{cases} w(t,x), & \tau - \delta \le t \le \tau, \quad x \in \mathbb{R}^d, \\ \\ v(t,x), & \tau \le t \le T, \quad x \in \mathbb{R}^d, \end{cases}$$

then z is a mild solution of (4.3) in $[\tau - \delta, T] \times \mathbb{R}^d$ which extends v, and it contradicts the maximality of v. □

Proposition 4.15. *If F satisfies* (4.1)*, then the mild solution v of problem* (4.3) *exists in* $[0, T] \times \mathbb{R}^d$.

Proof. By Proposition 4.14, it is enough to show that the function

$$(t, x) \mapsto (T-t)^{1/2} Q^{1/2}(x) \nabla_x v(t,x)$$

is bounded in $I(\varphi) \times \mathbb{R}^d$.

For the sake of simplicity, we set

$$l(t) := \|Q^{1/2}\nabla_x v(t, \cdot)\|_\infty,$$

where v is the maximally defined solution of problem (4.3). Then, for any $t \in I(\varphi)$ and

$x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&(T-t)^{1/2} l(t) \\
&\quad \leq C_T \|\varphi\|_\infty + L_\psi \int_t^T (T-t)^{1/2} (r-t)^{-1/2} \left(1 + l(r)\right) dr \\
&\quad \leq C_T \|\varphi\|_\infty + 2T L_\psi \\
&\qquad + L_\psi (T-t)^{1/2} \int_t^T (r-t)^{-1/2} (T-r)^{-1/2} (T-r)^{1/2} l(r) dr \\
&\leq C_T \|\varphi\|_\infty + 2T L_\psi \\
&\qquad + L_\psi (T-t)^{1/2} \int_t^T (r-t)^{-1/2} (T-r)^{-1/2} \left(C_T \|\varphi\|_\infty + 2T L_\psi\right) dr \\
&\qquad + L_\psi^2 (T-t)^{1/2} \int_t^T (r-t)^{-1/2} \left(\int_r^T (s-r)^{-1/2} (T-s)^{-1/2} (T-s)^{1/2} l(s) ds \right) dr \\
&\quad \leq (C_T \|\varphi\|_\infty + 2T L_\psi)(1 + T^{1/2} \pi L_\psi) \\
&\qquad + \pi L_\psi^2 (T-t)^{1/2} \int_t^T (T-s)^{-1/2} (T-s)^{1/2} l(s) ds.
\end{aligned}$$

The generalized Gronwall Lemma guarantees that the function

$$(t,x) \mapsto (T-t)^{1/2} Q^{1/2}(x) \nabla_x v(t,x)$$

is bounded in $I(\varphi) \times \mathbb{R}^d$, and the thesis follows. □

Remark 4.16. *Since the problem* (4.3) *is autonomous, in Propositions* 4.14 *and* 4.15 *we can replace* $[0,T]$ *with* $(-\infty, T]$.

Remark 4.17. *Under the Hypotheses of Proposition* 4.15*, if* $\varphi \in C_b^1(\mathbb{R}^d)$ *then the mild solution* v *of problem* (4.3) *exists in* $(-\infty, T] \times \mathbb{R}^d$*, it belongs to* $C^{0,1}((-\infty, T] \times \mathbb{R}^d)$ *and it is bounded in* $(a, T] \times \mathbb{R}^d$*, for any* $a < T$.

4.3 The Forward Backward Stochastic Differential Equation Associated to the Semi-Linear PDE

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $(W_t)_{t \geq 0}$ a $\mathbb{R}^d-$valued standard Brownian motion and $\mathcal{N}$ be the family of elements of $\mathcal{F}$ of probability 0. We define as $\mathcal{F}_t^W$ the natural filtration with respect to W_t, completed by the $\mathbb{P}-$null set of $\mathcal{F}$, i.e.

$$\mathcal{F}_t^W := \sigma\{W_s : 0 \leq s \leq t,\ \mathcal{N}\}.$$

In this setting we study the Forward Backward Stochastic Differential Equation

$$\begin{cases} dY_\tau = \psi(X_\tau, Z_\tau)d\tau + Z_\tau dW_\tau, & \tau \in [t,T], \\ dX_\tau = B(X_\tau)d\tau + G(X_\tau)dW_\tau, & \tau \in [t,T], \\ Y_T = \varphi(X_T), & \\ X_t = x, & x \in \mathbb{R}^d, \end{cases} \tag{FBSDE}$$

where

$$\psi : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}, \quad \varphi : \mathbb{R}^d \longrightarrow \mathbb{R},$$

are given Borel functions, and

$$B : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad G : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d},$$

are Borel measurable. We will assume further hypotheses on these functions.

Hypotheses 4.18. *There exist $C > 0$ such that, for any $x, x' \in \mathbb{R}^d$, we have*

$$|B(x) - B(x')| + |G(x) - G(x')| \le C|x - x'|.$$

For any $p \in [1, \infty)$, let $\mathbb{H}^p$ be the space of progressively measurable with respect to $\mathcal{F}_t^W$ random processes X_t such that

$$\|X\|_{\mathbb{H}^p} := \mathbb{E} \sup_{t\in[0,T]} |X_t|^p < \infty,$$

and let $\mathbb{K}$ be the space of $(\mathcal{F}_t^W)-$progressively measurable processes (Y, Z) such that

$$\|(Y,Z)\|^2_{cont} := \mathbb{E} \sup_{t\in[0,T]} |Y_t|^2 + \mathbb{E}\int_0^T |Z_\sigma|^2 d\sigma < \infty.$$

Moreover, we denote by $Y(s,t,x)$ and $Z(s,t,x)$ the solution to (FBSDE).

If Hypothesis 4.18 are satisfied, then system (FBSDE) admits a unique solution (X, Y, Z), where $X \in \mathbb{H}^p$, for any $p \in [1, \infty)$, and $(Y, Z) \in \mathbb{K}$ (see [89]). Henceforth, X denotes the solution to the forward equation in (FBSDE).

Example 4.19. *If we consider the operator A with coefficients*

$$Q(x) = \begin{pmatrix} 1 & 0 \\ 0 & (1+|x|^2)^m \end{pmatrix}, \quad B(x) = -x,$$

for any $x \in \mathbb{R}^2$, then Hypotheses 4.2 and 4.18 are satisfied if $m < 1/2$.

We will use the result of previous section to show that the solution (X, Y, Z) of (FBSDE) can be written in terms of the mild solution v of (4.3). This result is well known if B, G, ψ, φ satisfy the following conditions (see Chapter 3, Section 3.3).

Hypotheses 4.20. **(i)** *B and G are of class C^1 and their derivatives of order 1 are bounded;*

(ii) *φ is of class C^1 and it has polynomial growth together with its derivatives of order 1;*

(iii) *$\psi(t,\cdot,\cdot,\cdot)$ is of class C^1, for all $t \in [0,T]$;*

(iv) *$|\nabla_x \psi(t,x,y,z)| \leq K(1+|z|)(1+|x|+|y|)^\mu$ for suitable constants $K, \mu \geq 0$;*

(v) *$\nabla_y \psi$ and $\nabla_z \psi$ are bounded with respect to y and z.*

We want to relax the regularity conditions on ψ and φ, and the growth conditions on B and G, and show that the identification formulae

$$Y(s,t,x) = v(s, X(s,t,x)), \quad Z(s,t,x) = G(X(s,t,x))\nabla_x v(s, X(s,t,x)), \tag{4.26}$$

which hold under Hypotheses 4.20, are still true.

Let us assume that G, B, ψ satisfy Hypotheses 4.1 and 4.2. Moreover, we suppose that $\varphi \in BUC(\mathbb{R}^d)$. Hence, by Theorem 4.12 and Proposition 4.15, there exists a unique solution v to (4.3) which belongs to $\mathcal{K}_T$ (see Definition 4.4).

We approximate the functions φ, ψ by convolution: let $(\rho_n^d)_{n\in\mathbb{N}}$ and $(\rho_n^{d,d})_{n\in\mathbb{N}}$ be a standard sequence of mollifiers in $\mathbb{R}^d$ and in $\mathbb{R}^{d\times d}$, respectively, and set

$$\varphi_n = \varphi \star \rho_n^d, \quad \psi_n(x,z) = \theta_n(z)(\psi \star \rho_n^{d,d})(x,z),$$

where $\chi_{B(n)} \leq \theta_n \leq \chi_{B(n+1)}$. ψ_n and φ_n, are smooth functions and φ_n are bounded.

Lemma 4.21. *For any $n \in \mathbb{N}$ we have that $\|\varphi_n\|_\infty \leq \|\varphi\|_\infty$ and for any $n, m \in \mathbb{N}$, $n \leq m$, and $x, z_1, z_2 \in \mathbb{R}^d$, it holds that*

$$\begin{aligned}
&|\psi_n(x,z_1) - \psi(x,z_2)| \\
&\quad \leq \chi_{B(n)}(|z_1|)L_\psi\left(|z_1 - z_2| + \frac{2 + \frac{1}{n} + |z_1| + |z_2|}{n}\right) \\
&\quad + \chi_{B(n)^c}(|z_1|)L_\psi(2 + |z_1| + |z_2|), \qquad\qquad (4.27)\\
&|\psi_n(x,z_1) - \psi_m(x,z_2)| \\
&\quad \leq \chi_{B(n)}(|z_1|)L_\psi\left(|z_1 - z_2| + (\frac{1}{n} + \frac{1}{m})(3 + |z_1| + |z_2|)\right) \\
&\quad \chi_{B(n)^c}(|z_1|)\chi_{B(m)}(|z_2|)L_\psi\left(|z_1 - z_2| + \frac{3 + |z_1| + |z_2|}{m}\right. \\
&\quad \left. + 2 + |z_1| + |z_2|\right) + 2\chi_{B(m)^c}(|z_2|)L_\psi(2 + |z_1| + |z_2|). \qquad\qquad (4.28)
\end{aligned}$$

Proof. The above inequalities follow from the definition of ψ_n and the properties of $\theta_n, \rho_n, \varphi$ and ψ. □

For any $n \in \mathbb{N}$, let us consider the approximated Cauchy problem

$$\begin{cases} D_t v_n(t,x) + A v_n(t,x) = \psi_n(x, Q^{1/2}(x)\nabla_x v_n(t,x)), & t \in [0,T), \quad x \in \mathbb{R}^d, \\ v_n(T,x) = \varphi_n(x), & x \in \mathbb{R}^d, \end{cases} \tag{4.29}$$

whose mild solution is given by (see Theorem 4.12)

$$\begin{aligned} v_n(t,x) &= S(T-t)\varphi_n(x) - \int_t^T S(r-t)\psi_n(x, Q^{1/2}(x)\nabla_x v_n(r,x))dr \\ &= S(T-t)\varphi_n(x) - \int_t^T (S(r-t)F_n(r,v_n))(x)dr, \end{aligned} \tag{4.30}$$

where

$$F_n : (0,T) \times \mathcal{K}_T \longrightarrow C_b(\mathbb{R}^d), \quad F_n(t,u)(x) := \psi_n(x, Q^{1/2}(x)\nabla_x u(t,x)).$$

Remarks 4.13 and 4.17 guarantee that $v_n \in C_b([0,T] \times \mathbb{R}^d) \cap C^{0,1}([0,T] \times \mathbb{R}^d)$ and $\|Q^{1/2}\nabla_x v_n(t,\cdot)\|_\infty \le C_n$, for any $t \in (0,T)$ and any $n \in \mathbb{N}$.

Moreover, from Hypotheses 4.1, 4.2(i) and 4.18, it easily follows that $B, G = Q^{1/2}, \psi_n$ and φ_n satisfy Hypotheses 4.20. It means that (4.26) hold, provided that we replace Y, Z, v by Y_n, Z_n, v_n, and (X, Y_n, Z_n) is solution to

$$\begin{cases} dY^n_\tau = \psi_n(X_\tau, Z^n_\tau)d\tau + Z^n_\tau dW_\tau, & \tau \in [t,T], \\ dX_\tau = B(X_\tau)d\tau + G(X_\tau)dW_\tau, & \tau \in [t,T], \\ Y^n_T = \varphi_n(X_T), & \\ X_t = x, & x \in \mathbb{R}^d. \end{cases} \tag{4.31}$$

Now we need to study how v_n and $G\nabla_x v_n$ converge to v and $G\nabla_x v$, respectively. We claim that, for any fixed $t \in [0,T)$, $v_n(t,\cdot)$ and $G\nabla_x v_n(t,\cdot)$ converge uniformly. Then, we can define

$$Y(s,t,x) := v(s, X(s,t,x)), \quad Z(s,t,x) := G(X(s,t,x))\nabla_x v(s, X(s,t,x)), \tag{4.32}$$

for any $t \in [0,T]$, $t \le s < T$, and $x \in \mathbb{R}^d$. We will show that (X,Y,Z) is a solution to (FBSDE).

To prove the above claim, we need an intermediate result, contained in the following lemma.

Lemma 4.22. $[v_n]_{\mathcal{K}_T}$ *is uniformly bounded, and there exists a positive constant* K *such that* $\|v_n\|_{\mathcal{K}_T} \le K$*, for any* $n \in \mathbb{N}$.

Proof. Let $t \in [0,T)$. Since $|\psi_n(x,0)| \le L_\psi$, the same computations of Proposition 4.11 yield to the thesis. ☐

Theorem 4.23. *Suppose that Hypotheses* 4.1, 4.2 *and* 4.18 *hold. Moreover, let* $\varphi \in BUC(\mathbb{R}^d)$. *Then, for any* $t \in [0,T)$, $v_n(t,\cdot)$ *and* $G\nabla_x v_n(t,\cdot)$ *converge uniformly to* $v(t,\cdot)$ *and* $G(\cdot)\nabla_x v(t,\cdot)$ *respectively. Moreover,* $(X,Y,Z) \in \mathbb{H}^p \times \mathbb{K}$ *and it is a solution to* (FBSDE), *where* Y *and* Z *are defined by* (4.32).

Proof. As usual, at first we prove the convergence of $G\nabla_x v_n$, since it is involved in the definition of v_n. To simplify the notations, we set

$$h_n(t) := \|G\nabla_x v_n(t,\cdot) - G\nabla_x v(t,\cdot)\|_\infty,$$

from which we deduce

$$\begin{aligned} h_n(t) &= \|G\nabla_x (v_n - v)(t,\cdot)\| \\ &\le \|G\nabla_x S(T-t)(\varphi_n - \varphi)\| + \left\| G\nabla_x \int_t^T S(T-t)\left(F_n(r,v_n) - F(r,v)\right) dr \right\|. \end{aligned}$$

We have

$$\begin{aligned} &(T-t)^{1/2} h_n(t) \\ &\le C_T\|\varphi_n - \varphi\|_\infty + C_T(T-t)^{1/2}\int_t^T (r-t)^{-1/2}\|F_n(r,v_n) - F(r,v)\|_\infty dr \\ &\le C_T\|\varphi_n - \varphi\|_\infty + C_T(T-t)^{1/2}\int_t^T (r-t)^{-1/2}\|F_n(r,v_n) - F(r,v_n)\|_\infty dr \\ &\quad + C_T(T-t)^{1/2}\int_t^T (r-t)^{-1/2}\|F(r,v_n) - F(r,v)\|_\infty ds \\ &=: I_1^n + I_2^n(t) \\ &\quad + C_T L_\psi (T-t)^{1/2}\int_t^T (r-t)^{-1/2}(T-r)^{-1/2}(T-r)^{1/2} h_n(t) dr. \end{aligned}$$

Now we use the estimate

$$I_2^n(t) \le C_T \frac{L_\psi}{n}\int_t^T (r-t)^{-1/2} dr = 2C_T\frac{L_\psi}{n}T^{1/2},$$

which follows from (4.27) with $z_1 = z_2$ and holds for any $t \in [0,T)$. Hence

$$\begin{aligned} &\le I_1^n + 2C_T\frac{L_\psi}{n}T^{1/2} \\ &+ C_T L_\psi (T-t)^{1/2}\int_t^T (r-t)^{-1/2}(T-r)^{-1/2}\left(I_1^n + 2C_T\frac{L_\psi}{n}T^{1/2}\right) dr \\ &+ C_T^2 L_\psi^2 (T-t)^{1/2}\int_t^T (r-t)^{-1/2}\Big(\int_r^T (r-s)^{-1/2}(T-s)^{-1/2} \\ &\times (T-s)^{1/2} h_n(r) ds\Big) dr \\ &\le \left(I_1^n + 2C_T\frac{L_\psi}{n}T^{1/2}\right)(1 + \pi C_T L_\psi T^{1/2}) \\ &+ \pi C_T^2 L_\psi^2 (T-t)^{1/2}\int_t^T (T-s)^{-1/2}(T-s)^{1/2} h_n(r) ds. \end{aligned}$$

Since $\varphi \in BUC(\mathbb{R}^d)$, I_1^n tends to zero, as $n \to +\infty$. Clearly, also

$$2C_T \frac{L_\psi}{n} T^{1/2}$$

vanishes as $n \to \infty$.

Now we apply the generalized Gronwall Lemma to the function

$$(T-t)^{1/2} h_n = (T-t)^{1/2} \| G\nabla v_n(t,\cdot) - G\nabla v(t,\cdot)\|_\infty.$$

We obtain

$$(T-t)^{1/2} h_n \leq \left(I_1^n + 2C_T \frac{L_\psi}{n} T^{1/2} \right) (1 + \pi C_T L_\psi T^{1/2}) \exp\left(\pi C_T^2 L_\psi^2 T\right),$$

and the right-hand side tends to zero, as $n \to +\infty$, which means that

$$G\nabla_x v_n(t,\cdot) \to G\nabla_x v(t,\cdot)$$

as $n \to \infty$, uniformly with respect to x.

Using the fact that $[v_n - v]_{\mathcal{K}_T}$ tends to zero, similar computations yield the uniformly convergence of $v_n(t,\cdot)$ to $v(t,\cdot)$, for any $t \in [0,T]$.

Finally, we prove that the processes Y, Z defined in (4.32) are solutions to (FBSDE). Since Y_n, Z_n are solutions of (4.31), and the equalities hold $\mathbb{P}-$a.s., there exists a family $\{\Omega_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{F}$, such that each of them has zero measure. Moreover, if we set $\tilde{\Omega} = \cup_n \Omega_n$, then $\mathbb{P}(\tilde{\Omega}) = 0$, and in $\tilde{\Omega}^c$ (4.31) pointwise holds, for any $n \in \mathbb{N}$.

Now we fix $x \in \mathbb{R}^d$, $t \in [0,T]$, set $X_\tau := X(\tau,t,x)$, and define

$$\begin{aligned} Y_\tau &= v(\tau, X_\tau), \quad & Y_\tau^n &= v_n(\tau, X_\tau), \\ Z_\tau &= G(X_\tau)\nabla_x v(\tau, X_\tau), \quad & Z_\tau^n &= G(X_\tau)\nabla_x v_n(\tau, X_\tau), \end{aligned}$$

for any $\tau \in [t,T]$. The previous estimates guarantee that

$$Y_\tau^n \longrightarrow Y_\tau, \quad \varphi_n(X_T) \longrightarrow \varphi(X_T),$$

uniformly in Ω, and

$$\int_\tau^T \psi_n(X_\sigma, Z_\sigma^n) d\sigma \longrightarrow \int_\tau^T \psi(X_\sigma, Z_\sigma) d\sigma.$$

Indeed, by (4.27) we deduce that

$$\begin{aligned} |\psi_n(X_\sigma, Z_\sigma^n) - \psi(X_\sigma, Z_\sigma)| &\leq \chi_{B(n)}(|Z_\sigma^n|) L_\psi \left(|Z_\sigma^n - Z_\sigma| + \frac{3 + |Z_\sigma^n| + |Z_\sigma|}{n} \right) \\ &= L_\psi \left(|Z_\sigma^n - Z_\sigma| + \frac{3 + 2K(T-\sigma)^{-1/2}}{n} \right), \\ |\psi(X_\sigma, Z_\sigma)|, \ |\psi_n(X_\sigma, Z_\sigma^n)| &\leq L_\psi C(1 + K(T-\sigma)^{-1/2}), \end{aligned}$$

for any $x \in \mathbb{R}^d$, any $\sigma \in [\tau, T)$ and $n > K(T-\sigma)^{-1/2}$, where K has been introduced in Lemma 4.22. $|Z^n_\sigma - Z_\sigma|$ tends to zero uniformly in Ω, as $n \to +\infty$ since, for any $\omega \in \Omega$,

$$\begin{aligned}|Z^n_\sigma(\omega) - Z_\sigma(\omega)| &\le |G(X_\sigma(\omega))\nabla_x(v_n - v)(\sigma, X_\sigma(\omega))|\\ &\le \|G\nabla_x(v_n - v)(\sigma, \cdot)\|\\ &\to 0, \quad n \to \infty.\end{aligned}$$

Moreover, since $|\psi(X_\sigma, Z_\sigma)|$, $|\psi_n(X_\sigma, Z^n_\sigma)|$ can be estimated by an integrable function, we can apply dominated convergence to the integral term.

It remains to prove the convergence of $\int_\tau^T Z^n_\sigma dW_\sigma$ to $\int_\tau^T Z_\sigma dW_\sigma$. At first, we prove that $\int_\tau^T Z_\sigma dW_\sigma$ makes sense, since this is not guaranteed by previous estimates, which show only that the growth Z_σ can be estimated by $(T-\sigma)^{-1/2}$, which is not square integrable in T.

We are going to show that $\{Z^n_\tau\}$ is a Cauchy sequence in $L^2(\Omega \times (0,T))$, the space of the square integrable processes V, endowed with the norm $\mathbb{E}\int_0^T |V_\sigma|^2 d\sigma$. Since this is a Hilbert space, $\{Z^n_\tau\}$ converges to a process $\tilde{Z}_\tau$ which is square integrable, and so, up to a subsequence, $\{Z^n_\tau\}$ converges to $\tilde{Z}_\tau$ $[0,T] \otimes \mathbb{P}-$a.s. But $\{Z^n_\tau\}$ converges to Z_τ uniformly, hence pointwise, for any $\tau \in [0,T]$. Therefore, $\tilde{Z}_\tau = Z_\tau$ $\mathbb{P}-$a.s., for almost every $\tau \in [0,T]$. This means that Z_σ is a square integrable process.

For the reader's convenience, we introduce some new notations:

$$\begin{aligned}\overline{Y}^{n,m}_\sigma &:= Y^n_\sigma - Y^m_\sigma,\\ \overline{Z}^{n,m}_\sigma &:= Z^n_\sigma - Z^m_\sigma,\\ \overline{\varphi}^{n,m}_\sigma &:= \varphi_n(X_\sigma) - \varphi_m(X_\sigma),\\ \overline{\psi}^{n,m}_\sigma &:= \psi_n(X_\sigma, Z^n_\sigma) - \psi_m(X_\sigma, Z^m_\sigma),\end{aligned}$$

for any $n, m \in \mathbb{N}$, $\sigma \in [0,T]$. By the Itô formula, we get

$$d|\overline{Y}^{n,m}_\tau|^2 = -2\overline{Y}^{n,m}_\tau \overline{\psi}^{n,m}_\tau d\tau - 2\overline{Y}^{n,m}_\tau \overline{Z}^{n,m}_\tau dW_\tau + |\overline{Z}^{n,m}_\tau|^2 d\tau,$$

and, recalling that $\overline{Y}^{n,m}_T = \overline{\varphi}^{n,m}_T$, we obtain

$$|\overline{Y}^{n,m}_\tau|^2 + \int_\tau^T |\overline{Z}^{n,m}_\sigma|^2 d\sigma = |\overline{\varphi}^{n,m}_T|^2 - 2\int_\tau^T \overline{Y}^{n,m}_\sigma \overline{\psi}^{n,m}_\sigma d\sigma - 2\int_\tau^T \overline{Y}^{n,m}_\sigma \overline{Z}^{n,m}_\sigma dW_\sigma.$$

Let us estimate the terms in the right-hand side. Note that $(Y^n, Z^n), (Y^m, Z^m) \in \mathbb{K}$, since they are solutions of a backward stochastic differential equation. Hence, the process $I_\tau = \int_0^\tau \overline{Y}^{n,m}_\sigma \overline{Z}^{n,m}_\sigma dW_\sigma$ is a martingale and, in particular, $\mathbb{E}I_\tau = 0$, for any τ. Computing the expectation, we get

$$\mathbb{E}|\overline{Y}^{n,m}_\tau|^2 + \mathbb{E}\int_\tau^T |\overline{Z}^{n,m}_\sigma|^2 d\sigma = \mathbb{E}|\overline{\varphi}^{n,m}_T|^2 - 2\mathbb{E}\int_\tau^T \overline{Y}^{n,m}_\sigma \overline{\psi}^{n,m}_\sigma d\sigma. \tag{4.33}$$

Moreover, by (4.28), the last term in the right-hand side of (4.33) can be estimated as follows:

$$
\begin{aligned}
\mathbb{E}\int_\tau^T |\overline{Y}_\sigma^{n,m}\overline{\psi}_\sigma^{n,m}|d\sigma &\leq \mathbb{E}\left(\sup_{\tau\in[0,T]}|\overline{Y}_\tau^{n,m}|\int_\tau^T|\overline{\psi}_\sigma^{n,m}|d\sigma\right)\\
&\leq L_\psi \sup_{n\in\mathbb{N}}\|v_n\|_\infty\int_\tau^T|\overline{\psi}_\sigma^{n,m}|d\sigma\\
&\leq L_\psi K\int_\tau^T|\overline{\psi}_\sigma^{n,m}|d\sigma.
\end{aligned}
$$

From Lemmas 4.21 and 4.22 we have

$$
|\overline{\psi}_\sigma^{n,m}| \leq |\overline{Z}_\sigma^{n,m}| + \left(\frac{1}{m}+\frac{1}{n}\right)(3+|Z_\sigma^n|+|Z_\sigma^m|),
$$

for any fixed $\sigma\in(\tau,T)$ and $n,m\geq K(T-\sigma)^{-1/2}$, and

$$
|Z_\sigma^n|,|Z_\sigma^m|\leq K(T-\sigma)^{-1/2}.
$$

Moreover,

$$
|\overline{Z}_\sigma^{n,m}|\leq |Z_\sigma^n - Z_\sigma| + |Z_\sigma^m - Z_\sigma|,
$$

and the addends in the right-hand side vanish as n,m go to $+\infty$, for any fixed $\sigma\in(\tau,T)$. Hence, by dominated convergence, there exists $\bar{n}\in\mathbb{N}$ such that $\mathbb{E}\int_\tau^T|\overline{\psi}_\sigma^{n,m}|d\sigma\leq\varepsilon$, for any $n,m\geq\bar{n}$.

The same arguments can be applied to $\overline{\varphi}_T^{n,m}$. Indeed, recalling that φ is uniformly continuous, for any $\varepsilon>0$ there exists $\bar{n}\in\mathbb{N}$ such that $\mathbb{E}|\overline{\varphi}_T^{n,m}|^2\leq\varepsilon$, for any $n,m\geq\bar{n}$.

Hence $\{Z_\tau^n\}$ is a Cauchy sequence in $L^2(\Omega\times(0,T))$, and this implies that $\int_\tau^T Z_\sigma dW_\sigma$ makes sense. Moreover, since Z^n converges to Z in $L^2(\Omega\times(0,T))$, we see that

$$
\mathbb{E}\Big|\int_\tau^T(Z_\sigma^n - Z_\sigma)dW_\sigma\Big|^2 \longrightarrow 0,\quad n\to\infty.
$$

We can conclude that $\int_\tau^T Z_\sigma^n dW_\sigma$ tends to $\int_\tau^T Z_\sigma dW_\sigma$ $\mathbb{P}-$a.s., and passing to the limit (4.31), we obtain that the processes (X,Y,Z) are a solution to (FBSDE) $\mathbb{P}-$a.s. □

4.4 An application to the Stochastic Optimal Control in Weak Formulation

In this section we consider the controlled equation

$$
\begin{cases}
d_\tau X_\tau = B(X_\tau)d\tau + G(X_\tau)r(X_\tau,u_\tau)d\tau + G(X_\tau)dW_\tau, & \tau\in[t,T],\\
X_t = x\in\mathbb{R}^d,
\end{cases}
\tag{4.34}
$$

and the cost functional

$$\mathbb{E}\int_0^T l(X_t,u_t)dt + \mathbb{E}\varphi(X_T), \tag{4.35}$$

where u is a progressive measurable stochastic process with values in some specified set $\mathcal{U}\subset\mathbb{R}^m$, $r:\mathbb{R}^d\times\mathcal{U}\longrightarrow\mathbb{R}^d$, W is a $\mathbb{R}^d$−valued cylindrical Wiener process, and $l:\mathbb{R}^d\times\mathcal{U}\longrightarrow\mathbb{R}$. Our purpose is to minimize over all admissible controls the cost functional.

We assume the following hypotheses on l and r:

Hypotheses 4.24. *There exists $C>0$ such that for all $x,x'\in\mathbb{R}^d, t\in[0,T], u,u'\in\mathcal{U}$, we have*

$$\begin{aligned}|l(x,u)-l(x',u')|+|r(x,u)-r(x',u')| &\le C\left(|x-x'|+|u-u'|\right),\\ |l(x,u)|+|r(x,u)| &\le C.\end{aligned} \tag{4.36}$$

Definition 4.25. *An admissible control system (acs) $\mathbb{U}$ is the set*

$$\mathbb{U}=(\widehat{\Omega},\widehat{\mathcal{F}},(\widehat{\mathcal{F}}_t)_{t\ge0},\widehat{\mathbb{P}},\widehat{u},\widehat{W},\widehat{X}),$$

where $(\widehat{\Omega},\widehat{\mathcal{F}},\mathbb{P})$ is a probability space, the filtration $(\widehat{\mathcal{F}}_t)_{t\ge0}$ verifies the usual conditions, the process $\widehat{W}:[0,T]\times\widehat{\Omega}\longrightarrow\mathbb{R}^d$ is a Wiener process with respect to $(\widehat{\mathcal{F}}_t)_{t\ge0}$, $\widehat{u}$ is progressive measurable with respect to the filtration $(\widehat{\mathcal{F}}_t)_{t\ge0}$, and $\widehat{X}_\tau$ is a solution to

$$\widehat{X}_\tau = x+\int_t^\tau B(\widehat{X}_\sigma)d\sigma+\int_t^\tau G(\widehat{X}_\sigma)r(\widehat{X}_\sigma,\widehat{u}_\sigma)d\sigma+\int_t^\tau G(\widehat{X}_\sigma)d\widehat{W}_\sigma, \quad \tau\in[t,T].$$

In this setting, the cost functional has the form

$$J(t,x,\mathbb{U})=\widehat{\mathbb{E}}\int_t^T l(\widehat{X}_\sigma,\widehat{u}_\sigma)d\sigma+\widehat{\mathbb{E}}\varphi(\widehat{X}_T). \tag{4.37}$$

An acs is called *optimal* for the control problem starting from x at the time t, if it minimizes $J(t,x,\cdot)$, and the minimum value of the cost is called the *optimal cost.* Finally, we introduce the value function $V:[0,T]\times\mathbb{R}^d\to\mathbb{R}$, defined by

$$V(t,x):=\inf_{u\in\mathbb{U}} J(t,x,u), \tag{4.38}$$

for any $t\in[0,T]$ and $x\in\mathbb{R}^d$.

The Hamiltonian function of the problem, defined below, is crucial in the analysis of the stochastic control problem.

Definition 4.26. *The function $\psi:\mathbb{R}^d\times\mathbb{R}^d\longrightarrow\mathbb{R}$, defined by*

$$\psi(x,z)=\inf_{u\in\mathcal{U}}\{l(x,u)+zr(x,u)\}, \tag{4.39}$$

is called Hamiltonian function.

Lemma 4.27. *There exists a positive constant c such that*

$$\begin{aligned}|\psi(x,0)| &\leq c,\\ |\psi(x,z)-\psi(x',z')| &\leq c|z-z'| + c|x-x'|\left(1+|z|+|z'|\right),\end{aligned}$$

for any $x, x', z, z' \in \mathbb{R}^d$.

Proof. The result is well known, we report the proof for the reader's convenience. We prove only the second inequality, since the first one is a trivial consequence of Hypothesis (4.1). For all $u \in \mathcal{U}$ we have

$$\begin{aligned} l(x,u) + zr(x,u) &\leq l(x',u) + z'r(x',u) + |l(x,u) - l(x',u)| \\ &\quad + |zr(x,u) - z'r(x',u)| \\ &\leq l(x',u) + z'r(x',u) + |l(x,u) - l(x',u)| \\ &\quad + |zr(x,u) - z'r(x,u)| + |z'r(x,u) - z'r(x',u)| \\ &\leq l(x',u) + z'r(x',u) + c|x-x| + c|z-z'| + c|x-x'||z'|. \end{aligned}$$

Taking the infimum over u and exchanging x, z with x', z' we get the conclusion. □

We introduce the possibly empty set

$$\Gamma(x,z) := \{u \in \mathcal{U} : \varphi(x,z) = l(x,u) + zr(x,u)\}, \tag{4.40}$$

for any $x, z \in \mathbb{R}^d$.

To prove the main theorem of this section, we need the following hypothesis:

Hypothesis 4.28. *The set Γ is non-empty.*

Remark 4.29. *Hypothesis 4.28 is satisfied if $\mathcal{U}$ is a compact set.*

Remark 4.30. *From Hypothesis 4.28 it follows that (see [6, Thm 8.2.10]) there exists a measurable map $\gamma : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow U$ such that*

$$\psi(x,z) = l(x,\gamma(x,z)) + zr(x,\gamma(x,z)), \tag{4.41}$$

for any $(x,z) \in \mathbb{R}^d \times \mathbb{R}^d$.

Section 4.2 assures that the Hamilton Jacobi Bellman equation, associated to the problem (4.34) and (4.35), admits a unique solution v in the space $\mathcal{K}_T$. We stress that this solution has a good regularity, but not the optimal one; hence, we can not use the Itô formula. However, the BSDE's techniques enable us to prove that v is indeed the value function of the problem, and has enough regularity to identify the optimal feedback law.

Theorem 4.31. *Let Hypotheses* 4.1, 4.2, 4.18, 4.28 *and* 4.36 *hold. Moreover, let $\varphi \in BUC(\mathbb{R}^d)$. Then the following properties are satisfied:*

(i) *there exists a unique solution v of HJB such that $v \in \mathcal{K}_T$. Hence, $G(x)\nabla_x v(t,x)$ is defined for any $t \in [0,T), x \in \mathbb{R}^d$;*

(ii) $v(t,x) \le V(t,x)$, *for any* $t \in [0,T], x \in \mathbb{R}^d$;

(iii) $v(t,x) = V(t,x)$ *if and only if there exists an acs* $\mathbb{U}^*$ *such that*

$$\psi(X_t^{\mathbb{U}^*}, Z_t) = l(X_t^{\mathbb{U}^*}, u_t^*) + Z_t r(X_t^{\mathbb{U}^*}, u_t^*), \tag{4.42}$$

where $X_t^{\mathbb{U}^*}$ *is the solution to* (4.34), *with* $u = u^*$;

(iv) *there exists an acs* $U^{\#}$ *such that* (4.42) *is satisfied.*

Proof. For the reader's convenience we report the proof, which is close to the one in [41].

(i): since the HJB equation associated to (4.34) and (4.35) is (4.3), the existence and uniqueness of the mild solution follow from Section 4.2.

(ii): we fix an acs $\mathbb{U}$, $t \in [0,T]$, $x \in \mathbb{R}^d$, and consider the equation

$$X_\tau^{\mathbb{U}} = x + \int_t^\tau B(X_\sigma^{\mathbb{U}})d\sigma + \int_t^\tau G(X_\sigma^{\mathbb{U}})r(X_\sigma^{\mathbb{U}}, u_\sigma)d\sigma + \int_t^\tau G(X_\sigma^{\mathbb{U}})dW_\sigma, \quad \tau \in [t,T].$$

Since r is bounded, by Girsanov theorem there exists a probability measure $\widetilde{\mathbb{P}}$ such that

$$\widetilde{W}_\tau = W_\tau + \int_t^{t\wedge\tau} r(X_\sigma^{\mathbb{U}}, u_\sigma)d\sigma$$

is a Wiener process with respect to $\widetilde{\mathbb{P}}$, and $X^{\mathbb{U}}$ is a solution to

$$X_\tau^{\mathbb{U}} = x + \int_t^\tau B(X_\sigma^{\mathbb{U}})d\sigma + \int_t^\tau G(X_\sigma^{\mathbb{U}})d\widetilde{W}_\sigma, \quad \tau \in [t,T].$$

Notice that $X^{\mathbb{U}}$ is measurable with respect to the $\sigma-$field generated by $\widetilde{W}$. Now we introduce the backward equation

$$\widetilde{Y}_\tau + \int_t^\tau \widetilde{Z}_\sigma d\widetilde{W}_\sigma = \varphi(X_T^{\mathbb{U}}) + \int_t^\tau \psi(X_\sigma^{\mathbb{U}}, \widetilde{Z}_\sigma)d\sigma.$$

By the Theorem 4.23 there exists a unique solution $(\widetilde{Y}, \widetilde{Z})$ of this equation. Writing the backward equation with respect to W, we get

$$\widetilde{Y}_\tau + \int_\tau^T \widetilde{Z}_\sigma dW_\sigma + \int_\tau^T \widetilde{Z}_\sigma r(X_\sigma^{\mathbb{U}}, u_\sigma)d\sigma = \varphi(X_T^{\mathbb{U}}) + \int_\tau^T \psi(X_\sigma^{\mathbb{U}}, \widetilde{Z}_\sigma)d\sigma. \tag{4.43}$$

By easy computations, we have that $\mathbb{E}\left(\int_0^T |\widetilde{Z}_t|^2 dt\right)^{1/2} < \infty$. Hence, taking the expectation in (4.43) with respect to $\mathbb{P}$ and $\tau = t$, we obtain

$$\widetilde{Y}_t = \mathbb{E}\varphi(X_T^{\mathbb{U}}) + \mathbb{E}\int_t^T \left[\psi(X_\sigma^{\mathbb{U}}, \widetilde{Z}_\sigma) - \widetilde{Z} r(X_\sigma^{\mathbb{U}}, u_\sigma)\right] d\sigma.$$

Adding and subtracting $\mathbb{E}\int_t^T l(X_\sigma^{\mathbb{U}}, u_\sigma)d\sigma$, and recalling that $v(t,x) = \widetilde{Y}(t,t,x)$, we get

$$v(t,x) = J(t,x,\mathbb{U}) + \mathbb{E}\int_t^T \left[\psi(X_\sigma^{\mathbb{U}}, \widetilde{Z}_\sigma) - \widetilde{Z}_\sigma r(X_\sigma^{\mathbb{U}}, u_\sigma) - l(X_\sigma^{\mathbb{U}}, u_\sigma)\right] d\sigma. \tag{4.44}$$

From the definition of ψ, the term in square brackets is non positive. Hence $v(t,x) \leq J(y,x,\mathbb{U})$ for any acs $\mathbb{U}$, and taking the minimum we deduce that

$$v(t,x) \leq V(t,x), \quad t \in [0,T], \quad x \in \mathbb{R}^d.$$

(iii): from (4.44), it is clear that $v(t,x) = J(t,x,\mathbb{U}^*)$ if and only if the acs $\mathbb{U}^*$ satisfies (4.42). In this case, the integral term in (4.44) is zero; hence

$$v(t,x) \leq V(t,x) \leq J(t,x,\mathbb{U}^*) = v(t,x).$$

(iv): from Remark 4.30 and (4.32), it is natural to define

$$\widetilde{\gamma}(x) = \gamma(x, G(x)\nabla_x v(t,x)),$$

for any $t \in [0,T)$ and $x \in \mathbb{R}^d$.

Notice that $\widetilde{\gamma}$ is, a priori, not regular. Let W be a $d-$dimensional Brownian Motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$, and $X^\#$ be the solution to

$$\begin{cases} dX_\tau^\# = B(X_\tau^\#)d\tau + G(X_\tau^\#)dW_\tau, \quad \tau \in [t,T], \\ \\ X(t) = x \in \mathbb{R}^d. \end{cases}$$

For any $\tau \in [t,T]$, we set

$$W_\tau^\# = W_\tau - \int_t^{t\wedge\tau} r(X_\sigma^\#, \widetilde{\gamma}(X_\sigma^\#))d\sigma;$$

then $X^\#$ satisfies the close-loop equation

$$X_\tau^\# = x + \int_t^\tau B(X_\sigma^\#)d\sigma + \int_t^\tau G(X_\sigma^\#)r(X_\sigma^\#, \widetilde{\gamma}(X_\sigma^\#))d\sigma + \int_t^\tau G(X_\sigma^\#)dW_\sigma^\#,$$

for any $\tau \in [t,T]$. Clearly, $\mathbb{U}^\# = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P}, \widetilde{\gamma}(X^\#), X^\#, w^\#)$ is an acs with $u^\# = \widetilde{\gamma}(X^\#)$. Moreover, $u^\#$ satisfies (4.42): indeed

$$\begin{aligned} \psi(X_\tau^\#, Z_\tau^\#) &= l(X_\tau^\#, \gamma(X_\tau^\#, Z_\tau^\#)) + Z_\tau^\# r(X_\tau^\#, \gamma(X_\tau^\#, Z_\tau^\#)) \\ &= l(X_\tau^\#, \widetilde{\gamma}(X_\tau^\#)) + Z_\tau^\# r(X_\tau^\#, \widetilde{\gamma}(X_\tau^\#)) \\ &= l(X_\tau^\#, u_\tau^\#) + Z_\tau^\# r(X_\tau^\#, u_\tau^\#), \end{aligned}$$

where $Z_\tau^\# = G(X_\tau^\#)\nabla_x v(\tau, X_\tau^\#)$. Hence $\mathbb{U}^\#$ is an optimal control system for the problem. □

Chapter 5

Systems of Parabolic Equations

5.1 Introduction

In this chapter we deal with systems of parabolic differential equations

$$D_t\mathbf{u}(t,x) = (\mathbf{A}(t)\mathbf{u})(t,x),\ \ t > s,\ \ x \in \mathbb{R}^d, \tag{5.1}$$

where $\mathbf{A}(t)$ is the elliptic operator defined on smooth vector-valued functions $\mathbf{v}$

$$(\mathbf{A}(t)\mathbf{v})(x) = \sum_{i,j=1}^{d} q_{ij}(t,x)D^2_{ij}\mathbf{v}(x) + \sum_{j=1}^{d} B_j(t,x)D_j\mathbf{v}(x) + C(t,x)\mathbf{v}(x), \tag{5.2}$$

for any $(t,x) \in I \times \mathbb{R}^d$, with possible unbounded coefficients. Here, I is a right halfline, possibly $I = \mathbb{R}$. Note that the equations are coupled both at zero and first order.

In particular, we study the Cauchy problem

$$\begin{cases} D_t\mathbf{u}(t,x) = (\mathbf{A}(t)\mathbf{u})(t,x), & t > s, \quad x \in \mathbb{R}^d, \\ \mathbf{u}(s,x) = \mathbf{f}(x), & \qquad\qquad x \in \mathbb{R}^d, \end{cases} \tag{5.3}$$

where $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and provide sufficient conditions in order to get a unique classical solution $\mathbf{u}$ to (5.3). This is the starting point of our investigation; indeed, throughout this classical solution we introduce the evolution operator $\{\mathbf{G}(t,s)\}_{t\geq s\in I}$ on $C_b(\mathbb{R}^d;\mathbb{R}^m)$, and the rest of the chapter is devoted to the study of continuity and compactness properties of $\{\mathbf{G}(t,s)\}_{t\geq s\in I}$.

The chapter is organized as follows. In Section 5.2 we prove the existence and uniqueness of the classical solution to (5.3). In Subsection 5.2.1 we show some a priori local estimates of solution of the systems (5.1).

Subsection 5.2.2 is devoted to the uniqueness of a classical solution. We combine the techniques in [55, Chp. 8], where a maximum modulus principle for systems in bounded domains and smooth coefficients has proved, and a Lyapunov method for a suitable scalar differential operator. In particular, we require that there exists a positive constant $\varepsilon > 0$ and a function $\kappa : I \times \mathbb{R}^d \to \mathbb{R}$, bounded from above by a constant κ_0 such that

$$\begin{aligned}\mathcal{K}(t,x,\eta) := \sum_{i,j=1}^{d} & a_{ij}(t,x)[\langle B_i(t,x)\eta,\eta\rangle\langle B_j(t,x)\eta,\eta\rangle - \langle B_i(t,x)^*\eta, B_j(t,x)^*\eta\rangle] \\ & - 4\langle C(t,x)\eta,\eta\rangle + 4\varepsilon\kappa(t,x) \geq 0,\end{aligned}$$

where $Q(t,x)^{-1} = [a_{ij}(t,x)]$, and for any bounded interval $J \subset I$ there exist a constant λ_J and a positive function $\varphi_J \in C^2(\mathbb{R}^d)$, blowing up as $|x| \to +\infty$, such that

$$\sup_{\eta\in\partial B(1)} \sup_{(t,x)\in J\times\mathbb{R}^d} (\tilde{A}_\eta(t)\varphi_J)(x) - \lambda\varphi_J(x)) < +\infty, \tag{5.4}$$

where $\tilde{A}_\eta = \mathrm{Tr}(QD^2) + \sum_{j=1}^d b_{\eta,j}D_j + 2\varepsilon\kappa$ and $b_{\eta,j} = \langle B_j\eta,\eta\rangle$.

In such a this way we prove that, if a classical solution to (5.3) exists, it satisfies

$$\|\mathbf{u}(t,\cdot)\|_\infty \leq e^{\varepsilon\kappa_0(t-s)}\|\mathbf{f}\|_\infty, \qquad t > s,$$

and uniqueness immediately follows.

The existence of a classical solution is the content of Subsection 5.2.3. Here, we construct a function $\mathbf{u}$ as limit in $C^{1,2}(K;\mathbb{R}^m)$ of the solutions $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ of the Cauchy problems

$$\begin{cases} D_t\mathbf{u}_n(t,x) = (\mathbf{A}\mathbf{u}_n)(t,x), & t\in(s,+\infty), \quad x\in B(n), \\ \mathbf{u}_n(t,x) = 0 & t\in(s,+\infty), \quad x\in\partial B(n), \\ \mathbf{u}(s,x) = \mathbf{f}(x), & \qquad\qquad\qquad\quad x\in B(n), \end{cases}$$

for any compact set $K \subset (s,+\infty)\times\mathbb{R}^d$. To conclude, we show that $\mathbf{u}$ is continuous up to s and $\mathbf{u}(s,\cdot) \equiv \mathbf{f}$. This result has been obtained using a localization method: we fix $M \in \mathbb{N}$, consider a function ϑ such that $\chi_{B(M-1)} \leq \vartheta \leq \chi_{B(M)}$ and $\mathbf{v}_k := \vartheta\mathbf{u}_k$, and study the Cauchy problem

$$\begin{cases} D_t\mathbf{v}_k(t,x) = (\mathbf{A}\mathbf{v}_k)(t,x) + \mathbf{g}_k(t,x), & t\in(s,T], \quad x\in B(M), \\ \mathbf{v}_k(t,x) = 0 & t\in(s,T], \quad x\in\partial B(M), \\ \mathbf{v}_k(s,x) = (\vartheta\mathbf{f})(x), & \qquad\qquad x\in\overline{B(M)}, \end{cases}$$

which $\mathbf{v}_k$ solves, where $\mathbf{g}_k = -\mathrm{Tr}(QD^2\vartheta)\mathbf{u}_{n_k} - 2(J_x\mathbf{u}_{n_k})Q\nabla\vartheta - \sum_{j=1}^d(B_j\mathbf{u}_{n_k})D_j\vartheta$.

Throughout this solution we can define a family $\{\mathbf{G}(t,s)\}_{t>s\in I}$ as follows: for any $f\in C_b(\mathbb{R}^d;\mathbb{R}^m)$ we consider the classical solution $\mathbf{u}$ of (5.3) and, for any $t > s \in I$ and $x\in\mathbb{R}^d$, we set $\mathbf{G}(t,s)\mathbf{g}(x) := \mathbf{u}(t,x)$. Moreover, we set $\mathbf{G}(t,t)\mathbf{g} = \mathbf{g}$. $\mathbf{G}(t,s)$ turns out to be an evolution operator of bounded linear operators on $C_b(\mathbb{R}^d;\mathbb{R}^m)$

In Section 5.3 we deal with the continuity properties of $\{\mathbf{G}(t,s)\}_{t<s\in I}$. In particular, we prove that, if $\{\mathbf{f}_n\}_{n\in\mathbb{N}} \subset C_b(\mathbb{R}^d;\mathbb{R}^m)$ is uniformly bounded and pointwise converges to $f\in C_b(\mathbb{R}^d)$, then $\{\mathbf{G}(\cdot,s)\mathbf{f}_n\}_{n\in\mathbb{N}}$ converges to $\mathbf{G}(\cdot,s)\mathbf{f}$ locally uniformly in $(s,+\infty)\times\mathbb{R}^d$.

Moreover, representation Riesz Theorem implies that there exists a family of finite Borel signed measures $\{p_{ij}(t,s,x,dy) : t > s \in I,\ x \in \mathbb{R}^d\ \ i,j = 1,\ldots,m\}$ such that

$$((\mathbf{G}(t,s)\mathbf{f})(x))_i = \sum_{j=1}^{m} \int_{\mathbb{R}^d} p_{ij}(t,s,x,dy) f_j(y), \tag{5.5}$$

where f_j denotes the $j-$th component of $\mathbf{f}$. Moreover, the signed functions $p_{ij}(t,s,x,dy)$ are absolutely continuous with respect the Lebesgue measure. As a byproduct, we deduce that $\{G(t,s)\}_{t>s\in I}$ is Strong Feller.

Finally, in Section 5.4 we assume further hypotheses in order to link the compactness of $\{\mathbf{G}(t,s)\}_{t>s\in J}$ and $\{G(t,s)\}_{t>s\in J}$, where $j \Subset I$ and $\{G(t,s)\}_{t>s\in J}$ in the scalar evolution operator generated by a suitable linear operator $\mathcal{A}$.

First of all, we obtain pointwise estimates which relate the vector-valued evolution operator to the scalar one. Easily follows that the compactness of $\{G(t,s)\}_{t>s\in J}$ implies the compactness of $\{\mathbf{G}(t,s)\}_{t>s\in J}$.

Proving that the compactness of the vector-valued evolution operator implies the compactness of the vector-valued one is much more complicated, and we obtain it under some additional conditions related to the growth of the coefficients of operator $\mathbf{A}$. The critical point consists in proving that

$$\begin{aligned}(\mathbf{G}(t,s)\mathbf{f})_j(x) = (G(t,s)f_j)(x) + \int_s^t \left(G(t,r) \sum_i \langle (\tilde{B}_i)_{j\cdot}, D_i(\mathbf{G}(r,s)\mathbf{f})\rangle \right)(x)dr \\ + \int_s^t (G(t,r)\langle C_{j\cdot}, \mathbf{G}(r,s)\mathbf{f}\rangle)(x)dr,\end{aligned} \tag{5.6}$$

for some $j \in \{1,\ldots,m\}$. Then, we conclude adapting the procedure in [31, Thm. 3.6] to our situation.

We observe that the last integral makes sense if the vector $C_{j\cdot} = (C_{j1},\ldots,C_{jm})$ is bounded, for some j. The first one is more difficult to treat, since, in general, the function under the integral sign is not bounded.

To overcome this problem we prove the following weighted gradient estimates

$$(t-s)\sum_{j=1}^{m} \|Q^{1/2}(t,\cdot)\nabla_x(\mathbf{G}(t,s)\mathbf{f})_j\|_\infty^2 \le C\|\mathbf{f}\|_\infty^2,$$

for $\mathbf{G}(t,s)\mathbf{f}$, which are obtained with techniques similar to those used to prove (4.16). Hence, from an approximation argument we get (5.6).

5.2 The evolution operator $\mathbf{G}(t,s)$

Let I be an open right-halfline (possibly $I = \mathbb{R}$). In this section we prove that we can associate an evolution operator in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ with the system of elliptic operators $\mathbf{A}(t)$

($t \in I$), defined on smooth functions $\mathbf{v}$ by

$$(\mathbf{A}(t)\mathbf{v})(x) = \sum_{i,j=1}^{d} q_{ij}(t,x)D^2_{ij}\mathbf{v}(x) + \sum_{j=1}^{d} B_j(t,x)D_j\mathbf{v}(x) + C(t,x)\mathbf{v}(x), \tag{5.7}$$

for any $(t,x) \in I \times \mathbb{R}^d$. For any $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$, $\mathbf{G}(t,s)\mathbf{f}$ will be defined as the value at t of the unique classical solution to the Cauchy problem

$$\begin{cases} D_t\mathbf{u}(t,x) = (\mathbf{A}\mathbf{u})(t,x), & t > s, \quad x \in \mathbb{R}^d, \\ \mathbf{u}(s,x) = \mathbf{f}(x), & \qquad\quad\;\; x \in \mathbb{R}^d \end{cases} \tag{5.8}$$

which is bounded in all the strips $[s,T] \times \mathbb{R}^d$. Throughout this chapter we assume the following standing assumptions.

Hypotheses 5.1. (i) *For any $i,j = 1,\dots,d$, the coefficients q_{ij} and the entries of the matrices B_j and C belong to $C^{\alpha/2,\alpha}_{\rm loc}(I \times \mathbb{R}^d)$.*

(ii) *The matrix $Q = [q_{ij}]$ is uniformly elliptic, i.e., there exist a function ν with positive infimum ν_0 such that*

$$\langle Q(t,x)\xi,\xi\rangle \ge \nu(t,x)|\xi|^2, \qquad t \in I,\ \ x,\xi \in \mathbb{R}^d; \tag{5.9}$$

(iii) *there exist $\varepsilon > 0$ and a function $\kappa : I \times \mathbb{R}^d \to \mathbb{R}$, bounded from above by a constant κ_0, such that $\mathcal{K}(t,x,\eta) \ge 0$ for any $(t,x) \in I \times \mathbb{R}^d$ and any $\eta \in \partial B(1)$, where*

$$\begin{aligned}\mathcal{K}(t,x,\eta) = &\sum_{i,j=1}^{d} a_{ij}(t,x)[\langle B_i(t,x)\eta,\eta\rangle\langle B_j(t,x)\eta,\eta\rangle - \langle B_i(t,x)^*\eta, B_j(t,x)^*\eta\rangle] \\ &- 4\langle C(t,x)\eta,\eta\rangle + 4\varepsilon\kappa(t,x),\end{aligned} \tag{5.10}$$

and $Q(t,x)^{-1} = [a_{ij}(t,x)]$.

(iv) *for any bounded interval $J \subset I$ there exist a constant λ_J and a positive function $\varphi_J \in C^2(\mathbb{R}^d)$ blowing up as $|x| \to +\infty$ such that*

$$\sup_{\eta\in\partial B(1)} \sup_{(t,x)\in J\times\mathbb{R}^d} (\tilde{A}_\eta(t)\varphi_J)(x) - \lambda\varphi_J(x)) < +\infty, \tag{5.11}$$

where $\tilde{A}_\eta = \mathrm{Tr}(QD^2) + \sum_{j=1}^d b_{\eta,j}D_j + 2\varepsilon\kappa$ and $b_{\eta,j} = \langle B_j\eta,\eta\rangle$.

In Subsection 5.2.2 we will prove that problem (5.8) admits at most a unique classical solution, then in Subsection 5.2.3 we will show that problem (5.8) is solvable with a classical solution which is bounded in each strip $[s,T] \times \mathbb{R}^d$.

At first, we present a priori estimates which we will widely use in the continuation.

5.2.1 A priori estimates for solutions to parabolic systems

Here, we prove some a priori estimates for classical solutions to the Cauchy problem (5.8). To enlighten the notation, throughout this subsection we set $\|\cdot\|_{h,x_0,R} = \|\cdot\|_{C_b^h(B(R);\mathbb{R}^k)}$ for any $h \in \mathbb{N}$ and $k \in \mathbb{N}\cup\{0\}$, $R > 0$. We simply write $\|\cdot\|_{k,R}$ when $x_0 = 0$. Moreover, for any $\alpha, \beta \geq 0$, we denote by $\|\cdot\|_{\alpha,\beta}$ the norm of the space $C^{\alpha,\beta}([s,T]\times\mathbb{R}^d)$.

We recall that, for any $0 \leq \alpha < \theta$ and any bounded domain Ω of class C^θ, there exists a positive constant c such that

$$\|\mathbf{f}\|_{C^\alpha(\overline{\Omega};\mathbb{R}^k)} \leq c\|\mathbf{f}\|_\infty^{1-\frac{\alpha}{\theta}}\|\mathbf{f}\|^{\frac{\alpha}{\theta}}, \tag{5.12}$$

for any $\mathbf{f} \in C^\theta(\overline{\Omega};\mathbb{R}^k)$ $(k \geq 1)$. Moreover, if $T \in L(C(\overline{\Omega};\mathbb{R}^k); C^\beta(\overline{\Omega};\mathbb{R}^k)) \cap L(C_0^\theta(\overline{\Omega};\mathbb{R}^k); C^\beta(\overline{\Omega};\mathbb{R}^k))$ for some $\beta, \theta > 0$, then T is bounded from $C_0^\alpha(\overline{\Omega};\mathbb{R}^k)$ to $C^\beta(\overline{\Omega};\mathbb{R}^k)$, for any $\alpha \in (0,\theta)\setminus\mathbb{N}$, and

$$\|T\|_{L(C_0^\theta(\overline{\Omega};\mathbb{R}^k);C^\beta(\overline{\Omega};\mathbb{R}^k))} \leq \|T\|^{1-\frac{\alpha}{\theta}}_{L(C(\overline{\Omega};\mathbb{R}^k);C^\beta(\overline{\Omega};\mathbb{R}^k))}\|T\|_{L(C_0^\theta(\overline{\Omega};\mathbb{R}^k);C^\beta(\overline{\Omega};\mathbb{R}^k))}. \tag{5.13}$$

Estimate (5.12) holds true also when Ω is replaced by $\mathbb{R}^d$.

Proposition 5.2. *Let Ω be an open set and $\mathbf{u} \in C_b([s,s+1]\times\overline{\Omega}) \cap C^{1,2}((s,s+1)\times\Omega)$ satisfy $D_t\mathbf{u} - \mathbf{A}(\cdot)\mathbf{u} = \mathbf{h}$ in $(s,s+1)\times\Omega$, where $h \in C^{\alpha/2,\alpha}([s,s+1]\times\bar{\Omega};\mathbb{R}^m)$. Further, assume that the function $t \mapsto (t-s)\|\mathbf{u}(t,\cdot)\|_{C_b^2(\Omega)}$ is bounded in $(s,s+1)$. Then, for any $R_1 > 0$ and any $x_0 \in \Omega$, such that $B(x_0,R_1) \Subset \Omega$, there exists a positive constant $c = c(R_1)$ such that*

$$\begin{aligned}(t-s)\|D_x^2\mathbf{u}(t,\cdot)\|_{L^\infty(B(x_0,R_1))} &+ \sqrt{t-s}\|J_x\mathbf{u}(t,\cdot)\|_{L^\infty(B(x_0,R_1))} \\ &\leq c(\|\mathbf{u}\|_{C_b([s,s+1]\times\overline{\Omega})} + \|\mathbf{h}\|_{C^{\alpha/2,\alpha}([s,s+1]\times\overline{\Omega});\mathbb{R}^m}),\end{aligned} \tag{5.14}$$

for any $t \in (s,s+1)$.

Proof. Throughout the proof, we denote by $\|\cdot\|_\infty$ the sup-norm over $(s,s+1)\times\Omega$, by $\|\mathbf{h}\|_{\alpha/2,\alpha}$ the norm $\|\mathbf{h}\|_{C^{\alpha/2,\alpha}([s,s+1]\times\overline{\Omega});\mathbb{R}^m}$, and by c a positive constant, which can vary from line to line and it is independent of n. We fix x_0 and R_1 as in the statement, and R_2 such that $B(x_0,R_2) \Subset \Omega$. Then, we define $r_n := 2R_1 - R_2 + (R_2 - R_1)\sum_{k=0}^n 2^{-k}$ for any $n \in \mathbb{N}\cup\{0\}$. Further, we consider a sequence $(\vartheta_n) \subset C_c^\infty(\mathbb{R}^d)$ of functions such that $0 \leq \vartheta_n \leq 1$ in $\mathbb{R}^d$, $\vartheta_n \equiv 1$ on $B(r_n)$ and $\vartheta_n \equiv 0$ on $\mathbb{R}^d\setminus B(r_{n+1})$, for any $n \in \mathbb{N}$. As it is easily seen, $\|\vartheta_n\|_{C_b^k(\mathbb{R}^d)} \leq 2^{kn}c$ for any $k = 0,1,2,3$. Let us set $\mathbf{u}_n := \vartheta_n\mathbf{u}$ and observe that, for any $n \in \mathbb{N}$, the function $\mathbf{u}_n$ solves the Dirichlet-Cauchy problem

$$\begin{cases} D_t\mathbf{u}_n(t,x) = (\tilde{\mathbf{A}}(t)\mathbf{u}_n)(t,x) + \mathbf{g}_n(t,x), & t \in (s,s+1), \quad x \in B(x_0,r_{n+1}), \\ \mathbf{u}_n(t,x) = 0, & t \in [s,s+1], \quad x \in \partial B(x_0,r_{n+1}), \\ \mathbf{u}_n(s,x) = (\vartheta_n\mathbf{f})(x), & \qquad\qquad\quad\;\; x \in \overline{B(x_0,r_{n+1})}, \end{cases}$$

where

$$\begin{aligned}\mathbf{g}_n = &- \mathrm{Tr}(QD^2\vartheta_n)\mathbf{u} - \sum_{j=1}^{d} D_j\vartheta_n B_j\mathbf{u} - 2\sum_{i,j=1}^{d} q_{ij}D_i\vartheta_n D_j\mathbf{u}\\ &+ \vartheta_n \sum_{i=1}^{d}\sum_{j=1}^{m}(B_i)_j D_i u_j + \vartheta_n\sum_{j=1}^{m} C_j u_j,\end{aligned}$$

and $\tilde{\mathbf{A}} = \sum_{i,j=1}^{d} q_{ij}D_{ij}^2 I_m$. Since the coefficients of the operator $\tilde{\mathbf{A}}(\cdot)$ are smooth and bounded in $[s,s+1]\times\overline{B(x_0,R_{n+1})}$, we can associate an evolution operator $\mathbf{G}_{n+1}(t,s)$ to its realization in $C(\overline{B(R_{n+1})})$ with homogeneous Dirichlet boundary conditions. In view of the variation-of-constants-formula it thus follows that

$$\mathbf{u}(t,x) = (\mathbf{G}_{n+1}(t,s)\vartheta_n\mathbf{u}(s,\cdot))(x) + \int_s^t (\mathbf{G}_{n+1}(t,r)\mathbf{g}_n(r,\cdot))(x)dr,$$

for any $t\in[s,s+1]$, $x\in B(R_n)$. By classical results (see e.g. [70, Chp. 7])

$$(t-s)\|\mathbf{G}_{n+1}(t,s)\mathbf{k}\|_{2,x_0,r_{n+1}} \le c\|\mathbf{k}\|_{0,r_{n+1}}, \qquad \mathbf{k}\in C(\overline{B(x_0,r_{n+1})};\mathbb{R}^m)$$

and

$$(t-s)^{\frac{1}{2}}\|\mathbf{G}_{n+1}(t,s)\tilde{\mathbf{k}}\|_{2,r_{n+1}} \le c\|\tilde{\mathbf{k}}\|_{1,r_{n+1}}, \qquad \tilde{\mathbf{k}}\in C_0^1(\overline{B(x_0,r_{n+1})};\mathbb{R}^m),$$

for any $t\in(s,s+1)$ and any $n\in\mathbb{N}$. Note that the constant c in the previous two estimates is independent of n since it depends on the ellipticity constant of the operator $\tilde{\mathbf{A}}(\cdot)$ and the norms of its coefficients in $(s,s+1)\times B(x_0,r_{n+1})$, which can be estimated in terms of the same norms taken in $(s,s+1)\times B(x_0,R_2)$.

From estimate (5.13), with $\theta=1$ and $\beta=2$, we deduce that

$$(t-s)^{1-\frac{\alpha}{2}}\|\mathbf{G}_{n+1}(t,s)\mathbf{g}\|_{2,x_0,r_{n+1}} \le c\|\mathbf{g}\|_{\alpha,x_0,r_{n+1}},$$

for any $\mathbf{g}\in C_0^\alpha(B(r_{n+1});\mathbb{R}^m)$. Since, for any $\sigma\in(s,s+1)$ each function $\mathbf{g}_n(\sigma,\cdot)$ satisfies these properties, we can thus estimate

$$(t-s)\|\mathbf{u}_n(t,\cdot)\|_{2,x_0,r_n} \le c\|\mathbf{u}\|_\infty + c\int_s^t (t-\sigma)^{-1+\alpha/2}\|\mathbf{g}(\sigma,\cdot)\|_{\alpha,x_0,r_{n+1}}dr \tag{5.15}$$

for any $t\in(s,s+1)$. Note that

$$\|\mathbf{g}_n(\sigma,\cdot)\|_{\alpha,x_0,r_{n+1}} \le c\|\vartheta_n\|_{2+\alpha,x_0,r_{n+1}}(\|\mathbf{u}(\sigma,\cdot)\|_{1+\alpha,x_0,r_{n+1}} + \|\mathbf{h}\|_{\alpha/2,\alpha}), \tag{5.16}$$

for any $\sigma\in(s,s+1)$. Using (5.12), we can estimate, for any $\sigma\in(s,s+1)$ and $n\in\mathbb{N}$

$$\|\mathbf{u}(\sigma,\cdot)\|_{1,x_0,r_{n+1}} \le c\|\mathbf{u}(\sigma,\cdot)\|_\infty^{\frac{1}{2}}\|\mathbf{u}(\sigma,\cdot)\|_{2,x_0,r_{n+1}}^{\frac{1}{2}}$$

$$
\begin{aligned}
&\leq c(\sigma-s)^{-\frac{1}{2}}\|\mathbf{u}\|_\infty^{\frac{1}{2}}\left(\sup_{\sigma\in(s,s+1)}(\sigma-s)\|\mathbf{u}(\sigma,\cdot)\|_{2,x_0,r_{n+1}}\right)^{\frac{1}{2}}\\
&=: c(\sigma-s)^{-\frac{1}{2}}\|\mathbf{u}\|_\infty^{\frac{1}{2}}\zeta_{n+1}^{\frac{\alpha}{2}} \qquad (5.17)
\end{aligned}
$$

and

$$
\|J_x\mathbf{u}(\sigma,\cdot)\|_{\alpha,x_0,r_{n+1}} \leq c(\sigma-s)^{-\frac{\alpha+1}{2}}\|\mathbf{u}\|_\infty^{\frac{1-\alpha}{2}}\zeta_{n+1}^{\frac{1+\alpha}{2}}. \qquad (5.18)
$$

Using Young inequality $a^\theta b^{1-\theta}\leq \varepsilon a + c_\theta\varepsilon^{-\theta/(1-\theta)}$, which holds for any $a,b,\varepsilon>0$, any $\theta\in(0,1)$ and some positive constant c_θ, from (5.17) and (5.18) we deduce that

$$
\|\mathbf{u}(\sigma,\cdot)\|_{1,x_0,r_{n+1}} \leq (\sigma-s)^{-\frac{1}{2}}\left(c\varepsilon^{-1}\|\mathbf{u}\|_\infty+\varepsilon\zeta_{n+1}\right), \qquad (5.19)
$$

$$
\|J_x\mathbf{u}(\sigma,\cdot)\|_{\alpha,x_0,r_{n+1}} \leq (\sigma-s)^{-\frac{\alpha+1}{2}}\left(c\varepsilon^{-\frac{1+\alpha}{1-\alpha}}\|\mathbf{u}\|_\infty+\varepsilon\zeta_{n+1}\right), \qquad (5.20)
$$

for any $\sigma\in(s,s+1)$ and $\varepsilon>0$. From (5.16), (5.19), (5.20) and observing that $\|\vartheta_n\|_{C_b^{2+\alpha}(\mathbb{R}^d)}\leq c8^n$, for any $n\in\mathbb{N}$, we can estimate

$$
\|\mathbf{g}_n(\sigma,\cdot)\|_{\alpha,x_0,r_{n+1}} \leq 8^n(\sigma-s)^{-\frac{\alpha+1}{2}}\left(c\varepsilon^{-\frac{1+\alpha}{1-\alpha}}+\varepsilon\zeta_{n+1}\right)+8^nc\|\mathbf{h}\|_{\alpha/2,\alpha},
$$

for any $\sigma\in(s,s+1)$, any $\varepsilon>0$. Replacing this estimate into (5.15) yields

$$
\begin{aligned}
(t-s)\|\mathbf{u}_n(t,\cdot)\|_{2,x_0,r_n} &\leq c\|\mathbf{u}\|_\infty + 8^nc\left(\varepsilon^{-\frac{1+\alpha}{1-\alpha}}\|\mathbf{u}\|_\infty+\varepsilon\zeta_{n+1}+\|\mathbf{h}\|_{\alpha/2,\alpha}\right)\\
&\qquad\times(t-s)\int_s^t(t-\sigma)^{-1+\alpha/2}(\sigma-s)^{-(\alpha+1)/2}dr\\
&\leq c\|\mathbf{u}\|_\infty + 8^nc\left(\varepsilon^{-\frac{1+\alpha}{1-\alpha}}\|\mathbf{u}\|_\infty+\varepsilon\zeta_{n+1}+\|\mathbf{h}\|_{\alpha/2,\alpha}\right)\\
&\qquad\times(t-s)^{1/2}\int_0^1(1-\tau)^{-1+\alpha/2}\tau^{-(\alpha+1)/2}da
\end{aligned}
$$

Hence,

$$
\zeta_n \leq c\|\mathbf{u}\|_\infty + 8^nc\left(\varepsilon^{-\frac{1+\alpha}{1-\alpha}}\|u\|_\infty+\varepsilon\zeta_{n+1}+\|\mathbf{h}\|_{\alpha/2,\alpha}\right), \qquad n\in\mathbb{N},\ \varepsilon>0, \qquad (5.21)
$$

Let us fix $\eta\in(0,64^{-1/(1-\alpha)})$ and choose $\varepsilon=\varepsilon_n>0$ such that $8^nc\varepsilon=\eta$. With this choice of ε, from (5.21) we deduce that

$$
\zeta_n \leq c64^{\frac{n}{1-\alpha}}\|\mathbf{u}\|_\infty + c8^n\|\mathbf{h}\|_{\alpha/2,\alpha}+\eta\zeta_{n+1}, \qquad n\in\mathbb{N}. \qquad (5.22)
$$

Multiplying both sides of (5.22) by η^n we get

$$
\eta^n\zeta_n-\eta^{n+1}\zeta_{n+1} \leq c64^{\frac{n}{1-\alpha}}\eta^n\|\mathbf{u}\|_\infty + c8^n\eta^n\|\mathbf{h}\|_{\alpha/2,\alpha}, \qquad n\in\mathbb{N},
$$

which implies that

$$\zeta_0 - \eta^{n+1}\zeta_{n+1} = \sum_{k=0}^{n}(\eta^k\zeta_k - \eta^{k+1}\zeta_{k+1}) \le c\|\mathbf{u}\|_\infty \sum_{k=0}^{n} 64^{\frac{n}{1-\alpha}}\eta^n + c\|\mathbf{h}\|_{\alpha/2,\alpha}\sum_{k=0}^{n}(8\eta)^n$$
$$\le (c\|\mathbf{u}\|_\infty + \|\mathbf{h}\|_{\alpha/2,\alpha}),$$

for any $n \in \mathbb{N}$, since $\sum_{k=0}^{+\infty} 64^{\frac{n}{1-\alpha}}\eta^n, \sum_{k=0}^{\infty}(8\eta)^n < +\infty$ due to our choice of η. To conclude, we observe that $\eta^{n+1}\zeta_{n+1}$ tends to 0 as $n \to +\infty$. Indeed, by assumptions, ζ_{n+1} is bounded, uniformly with respect to n. It thus follows that $\eta^{n+1}\zeta_{n+1}$ vanishes as $n \to +\infty$. We have so proved that

$$(t-s)\|\mathbf{u}(t,\cdot)\|_{2,x_0,R_1} \le c(\|\mathbf{u}\|_\infty + \|\mathbf{h}\|_{\alpha/2,\alpha}), \qquad t \in (s, s+1). \tag{5.23}$$

Again, estimate (5.12) implies that

$$\|J_x\mathbf{u}(t,\cdot)\|_{0,x_0,R_1} \le c\|\mathbf{u}(t,\cdot)\|_{0,x_0,R_1}^{\frac12}\|\mathbf{u}(t,\cdot)\|_{2,x_0,R_1}^{\frac12}, \qquad t \in (s,s+1),$$

which, combined with (5.23), allows us to complete the proof of (5.14). □

We now prove some interior Schauder estimates for classical solutions to problem (5.8).

Theorem 5.3. *Fix $T > s \in I$ and let $\mathbf{u} \in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,T]\times\mathbb{R}^d;\mathbb{R}^m)$ satisfy the differential equation $D_t\mathbf{u} = \mathbf{A}(\cdot)\mathbf{u} + \mathbf{h}$ in $(s,T]\times\mathbb{R}^d$, where $h \in C^{\alpha/2,\alpha}((s,T]\times\mathbb{R}^d;\mathbb{R}^m)$. Then, for any $\tau \in (0, T-s)$ and any pair of bounded open sets Ω_1 and Ω_2 such that $\Omega_1 \Subset \Omega_2$, there exists a positive constant c, depending on $\Omega_1, \Omega_2, \tau, s, T$, but being independent of $\mathbf{u}$, such that*

$$\|\mathbf{u}\|_{C^{1+\alpha/2,2+\alpha}((s+\tau,T)\times\Omega_1;\mathbb{R}^m)} \le c(\|\mathbf{u}\|_{C_b((s+\tau/2,T)\times\Omega_2;\mathbb{R}^m)}$$
$$+ \|\mathbf{h}\|_{C^{\alpha/2,\alpha}((s+\tau/2,T)\times\Omega_2;\mathbb{R}^m)}). \tag{5.24}$$

Proof. The main step of the proof consists in showing that, for any $x_0 \in \overline{\Omega_1}$ and any $r > 0$, such that $B(x_0, 2r) \Subset \Omega_2$, there exists a positive constant $c > 0$, independent of $\mathbf{u}$, such that

$$\|\mathbf{u}\|_{C^{1+\alpha/2,2+\alpha}((s+\tau,T)\times B(x_0,r);\mathbb{R}^m)} \le C(\|\mathbf{u}\|_{C((s+\tau/2,T)\times\overline{B(x_0,2r)};\mathbb{R}^m)}$$
$$+ \|\mathbf{h}\|_{C^{\alpha/2,\alpha}((s+\tau/2,T)\times\overline{B(x_0,2r)};\mathbb{R}^m)}). \tag{5.25}$$

Indeed, once (5.25) is proved, a covering argument allow us to obtain easily estimate (5.24). So, let us prove (5.25). In the sequel, we denote by c a positive constant, independent of $\mathbf{u}$ and n, which may vary from line to line. Fix $x_0 \in \overline{\Omega_1}$, $\tau \in (0, T-s)$, $r > 0$ such that $B(x_0, 2r) \Subset \Omega_2$, and, for any $n \in \mathbb{N}$, any $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, set

$$\varphi_n(t) = \varphi\left(1 + \frac{t-s-t_{n+1}}{t_n - t_{n+1}}\right), \qquad \vartheta_n(x) = \vartheta\left(1 + \frac{|x-x_0| - r_n}{r_{n+1} - r_n}\right),$$

where $\varphi,\vartheta\in C^\infty(\mathbb{R})$ satisfy $\chi_{[2,\infty)}\le\varphi\le\chi_{[1,\infty)}$ and $\chi_{(-\infty,1]}\le\vartheta\le\chi_{(-\infty,2]}$, $r_n=(2-2^{-n})r$ and $t_n=(2^{-1}+2^{-n-1})\tau$, for any $n\in\mathbb{N}$. The function $\mathbf{v}_n$, defined by $\mathbf{v}_n(t,x)=\mathbf{u}(t,x)\varphi_n(t)\vartheta_n(x)$ for any $(t,x)\in[s,T]\times\mathbb{R}^d$ and any $n\in\mathbb{N}$, vanishes at $t=s$ and satisfies $D_t\mathbf{v}_n=\hat{\mathbf{A}}(\cdot)\mathbf{v}_n+\mathbf{g}_n$, where

$$\mathbf{g}_n=-\varphi_n\mathbf{u}\mathrm{Tr}(QD^2\vartheta_n)-\vartheta_n\sum_{j=1}^d D_j\vartheta_nB_j\mathbf{u}-2\varphi_n\sum_{i,j=1}^d q_{ij}D_i\vartheta_nD_j\mathbf{u}$$
$$+\varphi_n'\vartheta_n\mathbf{u}+\varphi_n\vartheta_n\mathbf{h},$$

and $\hat{\mathbf{A}}$ is a nonautonomous elliptic operator with bounded and smooth coefficients, which coincides with $\mathbf{A}$ in $[s,T]\times B(x_0,2r)$ (recall that $\mathbf{v}_n$ is compactly supported in $[s,T]\times B(x_0,2r)$).

By well known results and a straightforward computation we get

$$\begin{aligned}\|\mathbf{v}_n\|_{\alpha/2,\alpha}\le& c\|\mathbf{g}_n\|_{1+\alpha/2,2+\alpha(\alpha/2,\alpha)}\\
\le& c\|\vartheta_n\|_3\|\varphi\|_2\Big(\|\mathbf{u}\|_{C^{\alpha/2,\alpha}((s+t_{n+1},T)\times B(x_0,r_{n+1});\mathbb{R}^m)}\\
&+\|\nabla\mathbf{u}\|_{C^{\alpha/2,\alpha}((s+t_{n+1},T)\times B(x_0,r_{n+1});\mathbb{R}^m)}\\
&+\|\mathbf{h}\|_{C^{\alpha/2,\alpha}((s+t_{n+1},T)\times B(x_0,r_{n+1});\mathbb{R}^m)}\Big)\\
\le& 2^{5n}c\big(\|\mathbf{v}_{n+1}\|_{\alpha/2,\alpha}+\|J_x\mathbf{v}_{n+1}\|_{\alpha/2,\alpha}\\
&+\|\mathbf{h}\|_{C^{\alpha/2,\alpha}((s+t_{n+1},T)\times B(x_0,r_{n+1});\mathbb{R}^m)}\big).\end{aligned}\tag{5.26}$$

Now, using (5.12) we can estimate

$$\begin{aligned}\|\boldsymbol{\zeta}\|_\alpha&\le C\left(\varepsilon^{-\alpha/2}\|\boldsymbol{\zeta}\|_\infty+\varepsilon\|\boldsymbol{\zeta}\|_{2+\alpha}\right),\quad \boldsymbol{\zeta}\in C^{2+\alpha}_b(\mathbb{R}^d;\mathbb{R}^m),\\
\|\boldsymbol{\zeta}\|_{1+\alpha}&\le C\left(\varepsilon^{-(1+\alpha)}\|\boldsymbol{\zeta}\|_\infty+\varepsilon\|\boldsymbol{\zeta}\|_{2+\alpha}\right),\quad \boldsymbol{\zeta}\in C^{2+\alpha}_b(\mathbb{R}^d;\mathbb{R}^m),\\
\|\boldsymbol{\psi}\|_{\alpha/2}&\le C\left(\varepsilon^{-\alpha/2}\|\boldsymbol{\psi}\|_\infty+\varepsilon\|\boldsymbol{\psi}\|_{1+\alpha/2}\right),\quad \boldsymbol{\psi}\in C^{1+\alpha/2}_b([s,T];\mathbb{R}^m),\end{aligned}$$

for any $\varepsilon(0,1)$. Applying these estimates to $\mathbf{v}_{n+1}$, we deduce that

$$\|\mathbf{v}_{n+1}\|_{\alpha/2,\alpha}+\|J_x\mathbf{v}_{n+1}\|_{0,\alpha}\le c\left(\varepsilon\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha}+\varepsilon^{-(1+\alpha)}\|\mathbf{v}_{n+1}\|_\infty\right),\tag{5.27}$$

for any $\varepsilon>0$ and any $n\in\mathbb{N}$.

To estimate the $\alpha/2$-Hölder norm of the function $\mathbf{v}_{n+1}(\cdot,x)$ for any $x\in\mathbb{R}^d$, we observe that $\mathbf{v}_{n+1}\in\mathrm{Lip}([s,T],C^\alpha_b(\mathbb{R}^n;\mathbb{R}^m))$ and $\|\mathbf{v}_{n+1}\|_{\mathrm{Lip}}\le c\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha}$. Indeed,

$$\mathbf{v}_{n+1}(t,x)-\mathbf{v}_{n+1}(r,x)=\int_r^t D_t\mathbf{v}_{n+1}(\sigma,x)d\sigma,\qquad r,t\in[s,T],\ \ x\in\mathbb{R}^d,$$

which implies that

$$\mathbf{v}_{n+1}(t,x)-\mathbf{v}_{n+1}(r,x)-\mathbf{v}_{n+1}(t,y)+\mathbf{v}_{n+1}(r,y)$$

$$= \int_r^t (D_t\mathbf{v}_{n+1}(\sigma,x) - D_t\mathbf{v}_{n+1}(\sigma,y))d\sigma,$$

for any r,t as above and $x,y\in\mathbb{R}^d$. From these formulas, we immediately deduce that $\|\mathbf{v}_{n+1}(t\cdot)-\mathbf{v}_{n+1}(r,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}\le c\|D_t\mathbf{v}_{n+1}\|_{0,\alpha}$.

Since the function $\mathbf{v}_{n+1}$ is bounded in $[s,T]$ with values in $C_b^{2+\alpha}(\mathbb{R}^d)$, by interpolation we can estimate

$$\begin{aligned}
&\|\mathbf{v}_{n+1}(t_1,\cdot)-\mathbf{v}_{n+1}(t_2,\cdot)\|_1\\
\le& c\|\mathbf{v}_{n+1}(t_1,\cdot)-\mathbf{v}_{n+1}(t_2,\cdot)\|_\alpha^{\frac{1+\alpha}{2}}\|\mathbf{v}_{n+1}(t_1,\cdot)-\mathbf{v}_{n+1}(t_2,\cdot)\|_{2+\alpha}^{\frac{1-\alpha}{2}}\\
\le& c\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha}|t_2-t_1|^{\frac{1+\alpha}{2}}.
\end{aligned}$$

This shows that $J_x\mathbf{v}_{n+1}\in C^{(1+\alpha)/2,0}((s,T)\times\mathbb{R}^d;\mathbb{R}^{md})$ and

$$\|J_x\mathbf{v}_{n+1}\|_{(1+\alpha)/2,0}\le c\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha}. \tag{5.28}$$

Using the interpolative estimate

$$\|\boldsymbol{\psi}\|_{\alpha/2}\le c\left(\|\boldsymbol{\psi}\|_{\varepsilon(1+\alpha)/2}+\varepsilon^{-\alpha}\|\boldsymbol{\psi}\|_\infty\right),\qquad \boldsymbol{\psi}\in C_b^{(1+\alpha)/2}([s,T];\mathbb{R}^m),$$

which holds for any $\varepsilon\in(0,1)$, and (5.27) and (5.28) we obtain that

$$\|J_x\mathbf{v}_{n+1}\|_{\alpha/2,\alpha}\le c\left(\varepsilon\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha}+\varepsilon^{-(1+\alpha)}\|\mathbf{v}_{n+1}\|_\infty+\varepsilon^{-\alpha}\|J_x\mathbf{v}_{n+1}\|_\infty\right). \tag{5.29}$$

Now,

$$\|J_x\mathbf{v}_{n+1}\|_\infty\le\delta\|\mathbf{v}_{n+1}\|_{0,2+\alpha}+\delta^{-\frac{1}{1+\alpha}}\|\mathbf{v}_{n+1}\|_\infty,\qquad \delta\in(0,1).$$

Choosing $\delta=\varepsilon^{1+\alpha}$ and replacing this estimate in (5.29), we get

$$\|J_x\mathbf{v}_{n+1}\|_{\alpha/2,\alpha}\le c\left(\varepsilon\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha}+\varepsilon^{-(1+\alpha)}\|\mathbf{v}_{n+1}\|_\infty\right).$$

From (5.26), (5.27) and (5.29) it follows that

$$\begin{aligned}
\|\mathbf{v}_n\|_{1+\alpha/2,2+\alpha}\le& 2^{5n}c\big(\varepsilon\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha}+\varepsilon^{-(1+\alpha)}\|\mathbf{v}_{n+1}\|_\infty\\
&+\|\mathbf{h}\|_{C^{\alpha/2,\alpha}((s+t_{n+1},T)\times B(x_0,r_{n+1});\mathbb{R}^m)}\big),
\end{aligned} \tag{5.30}$$

for any $\varepsilon\in(0,1)$. Now, we are almost. Indeed, if we fix $\eta\in(0,2^{-5(2+\alpha)})$ and choose $\varepsilon=\varepsilon_n=2^{-5n}c^{-1}\eta$, from (5.30) we obtain

$$\begin{aligned}
\|\mathbf{v}_n\|_{1+\alpha/2,2+\alpha}\le&\big(\eta\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha}+2^{5n(2+\alpha)}c\|\mathbf{v}_{n+1}\|_\infty\\
&\|\mathbf{h}\|_{C^{\alpha/2,\alpha}((s+t_{n+1},T)\times B(x_0,r_{n+1});\mathbb{R}^m)}\big)\\
\le&\big(\eta\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha}+2^{5n(2+\alpha)}c\|\mathbf{u}\|_{C_b((s+\tau/2]\times B(x_0,2r))}\\
&+\|\mathbf{h}\|_{C^{\alpha/2,\alpha}((s+\tau/2,T)\times B2r,x_0);\mathbb{R}^m)}\big),
\end{aligned}$$

and can proceed as in the proof of Proposition 5.2 with $\zeta_n=\|\mathbf{v}_n\|_{1+\alpha/2,2+\alpha}$. $\square$

5.2.2 Uniqueness of the classical solution to problem (5.8)

The uniqueness of the classical solution to problem (5.8) which is bounded in any strip $[s,T]\times\mathbb{R}^d$, $s<T$, is a straightforward consequence of the following result, whose proof is an adaption to our situation of the methods in [55, Thm. 8.6] which deals with the case of smooth coefficients in bounded domains.

Proposition 5.4. *Let $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and let $\mathbf{u}$ be a classical solution to problem* (5.8) *which is bounded in the strip $[s,T]\times\mathbb{R}^d$ for any $s,T\in I$, $s<T$. Moreover, suppose that Hypothesis 5.1 holds. Then,*

$$\|\mathbf{u}(t,\cdot)\|_\infty\le e^{\varepsilon\kappa_0(t-s)}\|\mathbf{f}\|_\infty,\qquad t>s,\tag{5.31}$$

where κ_0 is defined in Hypotheses 5.1(*iii*).

Proof. Fix $T>s$ and set $J=[s,T]$. Up to replacing λ_J with a larger constant if needed, we can assume that

$$\sup_{\eta\in\partial B(1)}\sup_{(t,x)\in J\times\mathbb{R}^d}((\mathcal{A}_\eta(t)\varphi_J)(x)-\lambda_J(x))<0.$$

and $\lambda_J>2\varepsilon\kappa_0$. To enlighten the notation, from now on we simply write λ and φ.

For any $t\in[s,T]$ any $x\in\mathbb{R}^d$ and $n\in\mathbb{N}$, we set

$$v_n(t,x):=e^{-\lambda(t-s)}|\mathbf{u}(t,x)|^2-e^{-(\lambda-2\varepsilon\kappa_0)(t-s)}\|\mathbf{f}\|_\infty^2-\frac{\varphi(x)}{n}.$$

As it is immediately seen,

$$\begin{aligned}D_tv_n(t,\cdot)=&-\lambda e^{-\lambda(t-s)}|\mathbf{u}(t,\cdot)|^2+(\lambda-2\varepsilon\kappa_0)e^{-(\lambda-2\varepsilon\kappa_0)(t-s)}\|\mathbf{f}\|_\infty^2\\&+e^{-\lambda(t-s)}\bigg(\mathcal{A}_0(t)|\mathbf{u}(t,\cdot)|^2-2\sum_{i,j=1}^d q_{ij}(t,\cdot)\langle D_i\mathbf{u}(t,\cdot),D_j\mathbf{u}(t,\cdot)\rangle\\&\qquad+2\sum_{j=1}^d\langle B_j(t,\cdot)D_j\mathbf{u}(t,\cdot),\mathbf{u}(t,\cdot)\rangle+2\langle C(t,\cdot)\mathbf{u}(t,\cdot),\mathbf{u}(t,\cdot)\rangle\bigg)\\=&-\lambda e^{-\lambda(t-s)}|\mathbf{u}(t,\cdot)|^2+(\lambda-2\varepsilon\kappa_0)e^{-(\lambda-2\varepsilon\kappa_0)(t-s)}\|\mathbf{f}\|_\infty^2\\&+(\mathcal{A}_0(t)+2\varepsilon\kappa(t,\cdot))(e^{-\lambda(t-s)}|\mathbf{u}(t,\cdot))|^2\\&-2e^{-\lambda(t-s)}V(t,\cdot,D_1\mathbf{u}(t,\cdot),\dots,D_d\mathbf{u}(t,\cdot),\mathbf{u}(t,\cdot)),\end{aligned}$$

for any $t\in(s,T]$ and $x\in\mathbb{R}^d$, where $\mathcal{A}_0$ is the principal part of the operator $\mathbf{A}$ and

$$V(\cdot,\cdot,\xi^1,\dots,\xi^d,\zeta):=\sum_{i,j=1}^d q_{ij}\langle\xi^i,\xi^j\rangle-\sum_{j=1}^d\langle B_j\xi^j,\zeta\rangle-\langle(C-\varepsilon\kappa)\zeta,\zeta\rangle,$$

for any $\xi^1,\dots,\xi^d,\zeta\in\mathbb{R}^m$. Equivalently, we can write

$$D_tv_n(t,\cdot)-(\mathcal{A}_0(t)+2\varepsilon\kappa(t,\cdot)-\lambda)v_n(t,\cdot)-2\varepsilon(\kappa(t,\cdot)-\kappa_0)e^{-(\lambda-2\varepsilon\kappa_0)(t-s)}\|\mathbf{f}\|_\infty$$

$$=\frac{1}{n}(\mathcal{A}_0(t)+2\varepsilon\kappa(t,\cdot)-\lambda)\varphi-2e^{-\lambda(t-s)}V(t,\cdot,D_1\mathbf{u}(t,\cdot),\ldots,D_d\mathbf{u}(t,\cdot),\mathbf{u}(t,\cdot)), \tag{5.32}$$

for any $t\in(s,T]$ and $x\in\mathbb{R}^d$.

Our aim consists in proving that $v_n\le 0$ in $[s,T]\times\mathbb{R}^d$ for any $n\in\mathbb{N}$. Indeed in this case letting $n\to\infty$ and recalling that T has been arbitrarily fixed, we will obtain $e^{-2\varepsilon\kappa_0(t-s)}|u(t,x)|^2-\|\mathbf{f}\|_\infty^2\le 0$ for any $t\in[s,T]$ and $x\in\mathbb{R}^d$ i.e, (5.31) will follow from the arbitrariness of $T>s$.

Since v_n tends to $-\infty$ as $|x|\to+\infty$, it has a maximum attained at some point $(t_0,x_0)\in[s,T]\times\mathbb{R}^d$. If $t_0=s$, then we are done since $v_n(s,\cdot)<0$. Suppose that $t_0>s$ and assume, by contradiction, that $v_n(t_0,x_0)>0$. Then, $D_tv_n(t_0,x_0)\ge 0$. Moreover, since $2\varepsilon\kappa(t_0,x_0)-\lambda\le 2\varepsilon\kappa_0-\lambda<0$, it follows that $(\mathcal{A}_0(t_0)+2\varepsilon\kappa(t_0,x_0)-\lambda)v_n(t_0,x_0)<0$. Hence, the left-hand side of (5.32) is strictly positive at (t_0,x_0).

Let us prove that the right-hand side of (5.32) is nonpositive at (t_0,x_0). This will lead us to a contradiction and we will conclude that $v_n\le 0$ in $[s,T]\times\mathbb{R}^d$.

Since $\nabla v_n(t_0,x_0)=0$, we deduce that

$$\langle D_j\mathbf{u}(t_0,x_0),\mathbf{u}(t_0,x_0)\rangle=\frac{e^{\lambda(t_0-s)}}{2n}D_j\varphi(x_0),\qquad j=1,\ldots,d.$$

Therefore, to prove that the right-hand side of (5.32) is nonpositive it suffices to show that the maximum of the function

$$F_{n,\zeta}(\xi^1,\ldots,\xi^d):=\frac{1}{n}(\mathcal{A}_0(t_0)+2\varepsilon\kappa(t,\cdot)-\lambda)\tilde\varphi(x_0)-2V(t_0,x_0,\xi^1,\ldots,\xi^d,\zeta),$$

in the set $\Sigma=\left\{(\xi^1,\ldots,\xi^d)\in\mathbb{R}^{md}:\langle\xi^j,\zeta\rangle=\frac{1}{2n}D_j\tilde\varphi(x_0),\ j=1,\ldots,d\right\}$ is nonpositive, where $\tilde\varphi=e^{\lambda(t_0-s)}\varphi$. Note that, for any fixed $\zeta\in\mathbb{R}^m$, the function $(\xi^1,\ldots\xi^d)\mapsto V(t_0,x_0,\xi^1,\ldots,\xi^d,\zeta)$ tends to $+\infty$ as $\|(\xi^1,\ldots,\xi^d)\|\to+\infty$. Hence, $F_{n,\zeta}$ has a maximum in Σ attained at some point $(\xi_0^1,\ldots,\xi_0^d)$. Applying the Lagrange multipliers theorem, we easily see that $(\xi_0^1,\ldots,\xi_0^d)$ satisfies

$$2\sum_{k=1}^d q_{jk}(t_0,x_0)\xi_{0,i}^k-\sum_{k=1}^m(B_j(t_0,x_0))_{ki}\zeta_k-\mu_j\zeta_i=0,\qquad i=1,\ldots,d,\ j=1,\ldots,m, \tag{5.33}$$

for some real numbers $\mu_1,\ldots,\mu_d$, where $\xi_{0,i}^k$ and ζ_i $(i=1,\ldots,m)$ denote, respectively, the components of the vectors ξ_0^k and ζ. Multiplying both sides of (5.33) by ζ_i and summing over i, we get

$$\begin{aligned}0=&2\sum_{k=1}^d q_{jk}(t_0,x_0)\langle\xi_0^k,\zeta\rangle-\langle B_j(t_0,x_0)\zeta,\zeta\rangle-\mu_j|\zeta|^2\\=&\frac{1}{n}(Q(t_0,x_0)\nabla\tilde\varphi(x_0))_j-\langle B_j(t_0,x_0)\zeta,\zeta\rangle-\mu_j|\zeta|^2,\end{aligned}$$

for any $j=1,\ldots,m$. Hence,

$$\mu_j=|\zeta|^{-2}\left[\frac{1}{n}(Q(t_0,x_0)\nabla\tilde\varphi(x_0))_j-\langle B_j(t_0,x_0)\zeta,\zeta\rangle\right],\qquad j=1,\ldots,m.$$

Replacing the expression of μ_j in (5.33) we deduce that

$$\xi_0^j = \frac{1}{2n}|\zeta|^{-2}\zeta D_j\tilde{\varphi}(x_0) + \frac{1}{2}\sum_{k=1}^d a_{jk}(t,x)\left[B_k(t_0,x_0)^*\zeta - |\zeta|^{-2}\langle B_k(t_0,x_0)\zeta,\zeta\rangle\zeta\right],$$

for $j = 1,\dots,d$, where we recall that $(Q^{-1}(t_0,x_0))_{jk} = [a_{jk}(t_0,x_0)]$. We can now compute the value of $V(t_0,x_0,\xi_0^1,\dots,\xi_0^d,\zeta)$. For this purpose, we observe that

$$\begin{aligned}\langle \xi_0^i,\xi_0^j\rangle =& \frac{1}{4n^2|\zeta|^2}D_i\tilde{\varphi}(x_0)D_j\tilde{\varphi}(x_0) + \frac{1}{4}\sum_{h,k=1}^d a_{ih}a_{jk}\langle B_h(t_0,x_0)^*\zeta, B_k(t_0,x_0)^*\zeta\rangle \\ &- \frac{1}{4|\zeta|^2}\sum_{h,k=1}^d a_{ih}a_{jk}\langle B_h(t_0,x_0)\zeta,\zeta\rangle\langle B_k(t_0,x_0)\zeta,\zeta\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{i,j=1}^d q_{ij}\langle \xi_0^i,\xi_0^j\rangle =& \frac{1}{4n^2|\zeta|^2}\langle Q(t_0,x_0)\nabla\tilde{\varphi}(x_0),\nabla\tilde{\varphi}(x_0)\rangle \\ &+ \frac{1}{4}\sum_{i,h=1}^d a_{ih}\langle B_i^*(t_0,x_0)\zeta, B_h^*(t_0,x_0)\zeta\rangle \\ &- \frac{1}{4|\zeta|^2}\sum_{i,h=1}^d a_{ih}\langle B_i(t_0,x_0)\zeta,\zeta\rangle\langle B_h(t_0,x_0)\zeta,\zeta\rangle,\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^d\langle B_j(t_0,x_0)\xi_0^j,\zeta\rangle =& \frac{1}{2n|\zeta|^2}\sum_{j=1}^d D_j\tilde{\varphi}(x_0)\langle B_j(t_0,x_0)\zeta,\zeta\rangle \\ &+ \frac{1}{2}\sum_{j,k=1}^d a_{jk}\langle B_j^*(t_0,x_0)\zeta, B_k^*(t_0,x_0)\zeta\rangle \\ &- \frac{1}{2|\zeta|^2}\sum_{j,k=1}^d a_{jk}\langle B_j(t_0,x_0)\zeta,\zeta\rangle\langle B_k(t_0,x_0)\zeta,\zeta\rangle\end{aligned}$$

The previous formulas show that

$$\begin{aligned}V(t_0,x_0,\xi_0^1,\dots,\xi_0^d) =& \frac{1}{4n^2|\zeta|^2}\langle Q(t_0,x_0)\nabla\tilde{\varphi}(x_0),\nabla\tilde{\varphi}(x_0)\rangle \\ &- \frac{1}{4}\sum_{i,h=1}^d a_{ih}\langle B_i^*(t_0,x_0)\zeta, B_h^*(t_0,x_0)\zeta\rangle \\ &+ \frac{1}{4|\zeta|^2}\sum_{i,h=1}^d a_{ih}\langle B_i(t_0,x_0)\zeta,\zeta\rangle\langle B_h(t_0,x_0)\zeta,\zeta\rangle\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2n|\zeta|^2}\sum_{j=1}^d D_j\tilde\varphi(x_0)\langle B_j(t_0,x_0)\zeta,\zeta\rangle \\
&- \langle (C(t_0,x_0)-\varepsilon\kappa(t_0,x_0))\zeta,\zeta\rangle .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\max_{\Sigma} F_{n,\zeta} =& F_{n,\zeta}(\xi_0^1,\dots,\xi_0^d)\\
=& \frac{1}{n}(\tilde{\mathcal{A}}_{\zeta/|\zeta|}(t_0)\tilde\varphi(x_0)-\lambda\tilde\varphi(x_0))\\
&- \frac{1}{2n^2|\zeta|^2}\langle Q(t_0,x_0)\nabla\tilde\varphi(x_0),\nabla\tilde\varphi(x_0)\rangle - \frac{1}{2}|\zeta|^2\mathcal{K}(t_0,x_0,|\zeta|^{-1}\zeta). \qquad (5.34)
\end{aligned}
$$

By Hypothesis 5.1(iv) and the choice of λ, the last side of (5.34) is nonnegative. The proof is now complete. $\square$

Remark 5.5. Hypothesis 5.1(iii) can be replaced with the weaker condition $\mathcal{K} \geq -c_J$ in $J\times\mathbb{R}^d\times\partial B(1)$, for any bounded interval $J\subset I$ and for a suitable positive constant c_J. Indeed, if $\mathbf{u}$ is a classical solution to (5.8), then we define $\mathbf{v}(t,x) := e^{-c_J(t-s)/4}\mathbf{u}(t,x)$. $\mathbf{v}$ is a regular function which satisfies the Cauchy problem

$$
\begin{cases}
D_t\mathbf{v}(t,x) = \left(\mathbf{A}-\dfrac{c_J}{4}\right)\mathbf{v}(t,x), & t\in[s,T], \quad x\in\mathbb{R}^d,\\
\mathbf{v}(s,x)=\mathbf{f}(x), & x\in\mathbb{R}^d.
\end{cases}
$$

Hence, condition (5.10) for $\mathbf{v}$ is satisfied if $\mathcal{K}\geq -c_J$, and the uniqueness of $\mathbf{v}$ is equivalent to the uniqueness of $\mathbf{u}$.

Remark 5.6. (i) Weakly coupled systems of elliptic operators have been considered in [31] in the autonomous case. In this case, the equation are coupled only in the zero-order terms. More precisely, $\mathbf{A}$ is given by (5.7) with $B_j(x) = b_j(x)\mathrm{Id}$ for any $x\in\mathbb{R}^d$, any $j=1,\dots,d$ and some functions $b_j: I\times\mathbb{R}^d\to\mathbb{R}$. We claim that the conditions in [31] coincide with Hypotheses 5.1 in this situation. The regularity conditions in [31] are the same that we are assuming here. Moreover, Hypothesis 5.1(iii) reduces to the condition $\langle C(x)\eta,\eta\rangle\leq\varepsilon\kappa(x)$ for any $x\in\mathbb{R}^d$, which is the same condition assumed in [31]. Similarly, since $\tilde{\mathcal{A}}_\eta = \mathrm{Tr}(QD^2)+\sum_{j=1}^d b_jD_j+2\varepsilon\kappa$, Hypothesis 5.1(iv) coincides with Hypothesis 2.1(iv) in [31].

(ii) In the particular case when $m=1$ i.e., in the scalar case when the elliptic operator in (5.7) is $\mathcal{A}=\mathrm{Tr}(QD^2)+\langle b,\nabla_x\rangle + c$, if we take $\varepsilon=1$ and $\kappa=c$ then Hypothesis 5.1(iii) is immediately satisfied. Moreover, in the scalar case, to guarantee the uniqueness of a classical solution to problem (5.8) (which is bounded in every strip $[s,T]\times\mathbb{R}^d$), typically one assumes that c is bounded from above and that there exist $\lambda\in\mathbb{R}$ and a function $\varphi\in C^2(\mathbb{R}^d)$, blowing up as $|x|\to+\infty$, such that $\mathcal{A}\varphi-\lambda\varphi\leq 0$. When this is the case, with the previous choices of ε and κ, Hypothesis 5.1(iv) is clearly satisfied by the same function φ.

We now provide some examples of operators which satisfy Hypothesis 5.1.

Example 5.7. Let $\mathbf{A}(t)$ be as in (5.7), with

$$q_{ij}(t,x)=\delta_{ij},\quad B_j(t,x)=-x_j(1+|x|^2)^l g(t)\hat{B}_j,$$
$$C(t,x)=-|x|^2(1+|x|^2)^p h(t)\hat{C},$$

for any $(t,x)\in I\times\mathbb{R}^d$, $i,j=1,\dots,d$ where $\hat{B}_j$ $(j=1,\dots,d)$ and $\hat{C}$ are constant and positive definite matrices, $g,h\in C^{\alpha/2}_{\rm loc}(I)$ have positive infimum, and $p>2l\geq 0$. We observe that

$$\begin{aligned}\mathcal{K}'(t,x,\eta):=&(1+|x|^2)^{2l}(g(t))^2\sum_{i=1}^d x_i^2\left(\langle\hat{B}_i\eta,\eta\rangle^2-|\hat{B}_i^*\eta|^2\right)\\&+4|x|^2(1+|x|^2)^p h(t)\langle\hat{C}\eta,\eta\rangle\\ \geq&(1+|x|^2)^{2l}(g(t))^2\sum_{i=1}^d x_1^2\inf_{\eta\in\partial B(1)}\left(\langle\hat{B}_i\eta,\eta\rangle^2-|\hat{B}_i^*\eta|^2\right)\\&+4|x|^2(1+|x|^2)^p h_0c_0,\end{aligned}$$

for any $t\in I$, $x\in\mathbb{R}^d$, $\eta\in\partial B(1)$, where h_0 denotes the positive infimum of the function h and c_0 is the minimum eigenvalue of the matrix $\hat{C}$. Since $p>2l$, the function $\mathcal{K}'(t,\cdot,\eta)$ tends to $+\infty$ as $|x|\to+\infty$, uniformly with respect to $t\in I$ and $\eta\in\partial B(1)$. Therefore, we can find a positive constant κ such that $\mathcal{K}(t,x,\eta)=\mathcal{K}'(t,x,\eta)+4\kappa\geq 0$ in $I\times\mathbb{R}^d$ and, Hypothesis 5.1(iii) is satisfied with $\varepsilon=1$. On the other hand, the function φ, defined by $\varphi(x)=1+|x|^2$, for any $x\in\mathbb{R}^d$, satisfies Hypothesis 5.1(iv), with $\varepsilon=1$, for any $\lambda>0$.

Example 5.8. Let $\mathbf{A}(t)$ be as in (5.7), with

$$\begin{aligned}&q_{ij}(t,x)=g(t)(1+|x|^2)^k\mathrm{Id},\\&B_j(t,x)=-h(t)x_j(1+|x|^2)^r\mathrm{Id}+\ell(t)(1+|x|^2)^p\hat{B}_j,\qquad j=1,\dots,d,\\&\langle C(t,x)\xi,\xi\rangle\leq K_1|\xi|^2,\quad \xi\in\mathbb{R}^m,\end{aligned}$$

for any $(t,x)\in I\times\mathbb{R}^d$, where $g,h\in C^{\alpha/2}_{\rm loc}(I)$ have a positive infimum, $\ell\in C^{\alpha/2}_{\rm loc}(I)$, $\hat{B}_j$ $(j=1,\dots,d)$ are constant $(m\times m)$-matrices, the entries of the matrix valued function C belong to $C^{\alpha/2,\alpha}_{\rm loc}(I\times\mathbb{R}^d)$ and K_1 is a positive constant. Finally $k,p,r\in[0,+\infty)$ satisfy $2p\leq k<r+1$.

As in the previous example, to check Hypothesis 5.1(iii), we just need to show that the function $\mathcal{K}'$, given by the first three terms in the right-hand side of (5.10) is bounded from below in $I\times\mathbb{R}^d$. It turns out that

$$\mathcal{K}'(t,x,\eta)=\frac{(\ell(t))^2}{g(t)}(1+|x|^2)^{-k+2p}\sum_{j=1}^d\left(\langle\hat{B}_j\eta,\eta\rangle^2-|\hat{B}_j^*\eta|^2\right)-4\langle C(t,x)\eta,\eta\rangle$$

$$\geq \frac{(\ell(t))^2}{g(t)}(1+|x|^2)^{-k+2p}\sum_{j=1}^d \inf_{\eta\in\partial B(1)}\left(\langle \hat{B}_j\eta,\eta\rangle^2-|\hat{B}_j^*\eta|^2\right)-4K_1,$$

for any $(t,x)\in I\times\mathbb{R}^d$ and $\eta\in\partial B(1)$. Since $k\geq 2p$ and g has a positive infimum, the last side of the previous formula is bounded in $I\times\mathbb{R}^d$. Then we can find a constant κ such that the function $\mathcal{K}=\mathcal{K}'+4\kappa$ satisfies Hypothesis 5.1(iii), with $\varepsilon=1$.

Moreover, the function φ, defined by $\varphi(x)=1+|x|^2$ for any $x\in\mathbb{R}^d$, satisfies

$$(\tilde{\mathcal{A}}(t)\varphi)(x)-\lambda\varphi(x)=2dg(t)(1+|x|^2)^k-2h(t)|x|^2(1+|x|^2)^r + 2\ell(t)(1+|x|^2)^p\sum_{j=1}^d x_j\langle\hat{B}_j\eta,\eta\rangle-\lambda(1+|x|^2),$$

for any $\lambda>0$, $(t,x)\in I\times\mathbb{R}^d$ and $\eta\in\partial B(1)$. Our assumptions on k,p,r and on the functions g,h,l reveal that the leading term in the last side of the previous formula is the second one. Therefore, for any $\lambda>0$ and any bounded interval $J\subset\mathbb{R}$, the function $\tilde{\mathcal{A}}_\eta(t)\varphi-\lambda\varphi$ tends to $-\infty$ as $|x|\to+\infty$, uniformly with respect to $t\in J$ and $\eta\in\partial B(1)$. This immediately implies that Hypothesis 5.1(iv) is satisfied for any $\lambda>0$.

5.2.3 Existence of a solution to problem (5.8)

Here, we prove the existence of a classical solution $\mathbf{u}$ to problem (5.8) which belongs to $C_b([s,T]\times\mathbb{R}^d;\mathbb{R}^m)$, for any $s,T\in I$, $s<T$.

Theorem 5.9. *For any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and any $s\in I$, the Cauchy problem* (5.8) *admits a* (*unique*) *classical solution $\mathbf{u}$ which belongs to $C([s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)\cap C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)$. Moreover,*

$$\|\mathbf{u}(t,\cdot)\|_\infty\leq e^{\varepsilon\kappa_0(t-s)}\|\mathbf{f}\|_\infty,\qquad t>s, \tag{5.35}$$

where ε and κ_0 are defined in Hypotheses 5.1(*iii*).

Proof. Fix $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and let $\mathbf{u}_n$ be the unique classical solution to the Cauchy-Dirichlet problem

$$\begin{cases} D_t\mathbf{u}_n(t,x)=(\mathbf{A}\mathbf{u}_n)(t,x), & t\in(s,+\infty), & x\in B(n),\\ \mathbf{u}_n(t,x)=0 & t\in(s,+\infty), & x\in\partial B(n),\\ \mathbf{u}(s,x)=\mathbf{f}(x), & & x\in B(n),\end{cases} \tag{5.36}$$

(see [63, Thm. IV.5.5]). By classical solution we mean a function which belongs to $C^{1,2}((s,+\infty)\times B(n))$ which is continuous in $([s,+\infty)\times\overline{B(n)})\setminus(\{s\}\times\partial B(n))$.

Let us prove that the sequence $(\mathbf{u}_n)$ converges to a solution to problem (5.8) which satisfies the properties in the statement. The same arguments as in the proof of Proposition 5.4 show that

$$\|\mathbf{u}_n(t,\cdot)\|_\infty\leq e^{\varepsilon\kappa_0(t-s)}\|\mathbf{f}\|_\infty,\qquad t>s. \tag{5.37}$$

Hence, the interior Schauder estimates in Theorem 5.3 guarantee that, for any compact set $K \subset (s,+\infty)\times\mathbb{R}^d$ and large n, the sequence $\|\mathbf{u}_n\|_{C^{1+\alpha/2,2+\alpha}(K;\mathbb{R}^m)}$ is bounded by a constant independent of n. The Ascoli-Arzelà Theorem, a diagonal argument and the arbitrariness of K show that there exists a subsequence $(\mathbf{u}_{n_k})$ which converges to a function $\mathbf{u}\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)$ in $C^{1,2}(K;\mathbb{R}^m)$ for any K as above. Clearly, $\mathbf{u}$ satisfies the differential equation in (5.8) as well as the estimate (5.35), as it is easily seen letting $n\to+\infty$ in (5.37); we just need to show that $\mathbf{u}$ is continuous in $t=s$ and it therein equals the function $\mathbf{f}$. As a byproduct, we will deduce that the whole sequence $(\mathbf{u}_n)$ converges in $C^{1,2}(K;\mathbb{R}^m)$, for any compact set K as above, since any subsequence of $(\mathbf{u}_n)$ has a subsequence which converges in $C^{1,2}(K;\mathbb{R}^m)$.

Fix $M\in\mathbb{N}$ and let ϑ be any smooth function such that $\chi_{B(M-1)}\le\vartheta\le\chi_{B(M)}$. For any $n_k>M$ the function $\mathbf{v}_k:=\vartheta\mathbf{u}_{n_k}$ belongs to $C([s,T]\times\overline{B(M)};\mathbb{R}^m)\cap C^{1,2}((s,T]\times B(M);\mathbb{R}^m)$ and solves the Dirichlet-Cauchy problem

$$\begin{cases} D_t\mathbf{v}_k(t,x)=(\mathbf{A}\mathbf{v}_k)(t,x)+\mathbf{g}_k(t,x), & t\in(s,T],\quad x\in B(M),\\ \mathbf{v}_k(t,x)=0 & t\in(s,T],\quad x\in\partial B(M),\\ \mathbf{v}_k(s,x)=(\vartheta\mathbf{f})(x), & \qquad\qquad\; x\in\overline{B(M)},\end{cases}$$

where $\mathbf{g}_k=-{\rm Tr}(QD^2\vartheta)\mathbf{u}_{n_k}-2(J_x\mathbf{u}_{n_k})Q\nabla\vartheta-\sum_{j=1}^d(B_j\mathbf{u}_{n_k})D_j\vartheta$, for any $n_k>M$. Note that

$$|\mathbf{g}_k(t,x)|\le K_M\left(e^{\varepsilon\kappa_0(t-s)}\|\mathbf{f}\|_{L^\infty(B(M))}+\|J_x\mathbf{u}_{n_k}(t,\cdot)\|_{L^\infty(B(M))}\right),$$

for any $t\in(s,T]$, $x\in B(M)$ and some positive constant K_M independent of k, where we have used (5.37).

Since the function $t\mapsto(t-s)\|\mathbf{u}_{n_k}(t,\cdot)\|_{C^2_b(B(M))}$ is bounded in $(s,s+1)$ (by a constant depending on k) we can apply Proposition 5.2 and, taking (5.37) into account, we obtain that $\|J_x\mathbf{u}_{n_k}\|_{L^\infty(B(M))}\le c(t-s)^{-1/2}\|\mathbf{f}\|_\infty$, for any $t\in(s,s+1)$ and some constant $c>0$, independent of k. Therefore we can estimate $|\mathbf{g}_k(t,x)|\le K'_M(1+(t-s)^{-1/2})\|\mathbf{f}\|_\infty$, for any $(t,x)\in(s,s+1)\times B(M)$ and any $n_k>M$ where K'_M is a positive constant independent of k. We can thus represent $\mathbf{v}_k$ by means of the variation-of-constants formula

$$\mathbf{v}_k(t,x)=(\mathbf{G}_M(s,t)(\vartheta\mathbf{f})(x)+\int_s^t(\mathbf{G}_M(t,r)\mathbf{g}_k(r,\cdot))(x)dr,$$

where $\mathbf{G}_M(s,t)$ is the evolution family associated to the realization of $\mathbf{A}(\cdot)$ in $C_b(\overline{B(M)});\mathbb{R}^m)$ with homogeneous Dirichlet boundary conditions. Since $\mathbf{v}_k\equiv\mathbf{u}_{n_k}$ in $B(M-1)$, it follows that

$$|\mathbf{u}_{n_k}(t,x)-\mathbf{f}(x)|\le|\mathbf{G}_M(t,s)(\vartheta\mathbf{f})(x)-\mathbf{f}(x)|+K''_M\|\mathbf{f}\|_\infty\int_s^t(1+(r-s)^{-1/2})dr,$$

for any $t\in(s,s+1)$, $x\in B(M-1)$ and some positive constant K''_M independent of k. Letting k tend to $+\infty$ we get

$$|\mathbf{u}(t,x)-\mathbf{f}(x)|\le\|\mathbf{G}_M(s,t)(\vartheta\mathbf{f})-\vartheta\mathbf{f}\|_{L^\infty(B(M-1))}+K'_M\|\mathbf{f}\|_\infty\int_s^t(1+(r-s)^{-1/2})dr,$$

which shows that $\mathbf{u}$ is continuous at $t=s$ for any $x\in B(M-1)$, Since $M\in\mathbb{N}$ is arbitrary, we conclude that $\mathbf{u}\in C([s,T]\times\mathbb{R}^d;\mathbb{R}^m)$ and $\mathbf{u}(s,\cdot)=\mathbf{f}$. □

5.3 The evolution operator $\{\mathbf{G}(t,s)\}_{t\geq s}$

For any $t>s\in I$ and any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, let $\mathbf{G}(t,s)\mathbf{f}$ denote the value at t of the unique classical solution to the Cauchy problem (5.8). The uniqueness of the solution to such a problem shows that the family $\{\mathbf{G}(t,s)\}_{t\geq s\in I}$ is an evolution operator in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. Moreover, estimate (5.31) shows that

$$\|\mathbf{G}(t,s)\mathbf{f}\|_\infty\leq e^{\varepsilon\kappa_0(t-s)}\|\mathbf{f}\|_\infty,\qquad \mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m),\ \ t>s\in I. \tag{5.38}$$

We are interested in studying some properties of this evolution operator which, from now on, will be denoted simply by $\mathbf{G}(t,s)$.

Proposition 5.10. *Let $(\mathbf{f}_n)$ be a bounded sequence of functions in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. Then, the following properties are satisfied:*

(i) *if $\mathbf{f}_n$ converges pointwise to $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, then $\mathbf{G}(\cdot,s)\mathbf{f}_n$ converges to $\mathbf{G}(\cdot,s)\mathbf{f}$ in $C^{1,2}(D)$ for any compact set $D\subset(s,+\infty)\times\mathbb{R}^d$;*

(ii) *if $\mathbf{f}_n$ converges to $\mathbf{f}$ locally uniformly in $\mathbb{R}^d$, then $\mathbf{G}(\cdot,s)\mathbf{f}_n$ converges to $\mathbf{G}(\cdot,s)\mathbf{f}$ locally uniformly in $[s,+\infty)\times\mathbb{R}^d$.*

Proof. (i). From (5.35) and the interior Schauder estimates in Theorem 5.3 we deduce that, for any compact set $D\subset(s,+\infty)\times\mathbb{R}^d$, it holds that

$$\sup_{n\in\mathbb{N}}\|\mathbf{G}(\cdot,s)\mathbf{f}_n\|_{C^{1+\alpha/2,2+\alpha}(D)}<+\infty.$$

Therefore, using the same arguments as in the proof of Theorem 5.9, we can prove that there exists a function $\mathbf{v}\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$ and a subsequence $G_{n_k}(\cdot,s)\mathbf{f}_n$ which converges to $\mathbf{v}$ in $C^{1,2}(D)$ as $k\to+\infty$, for any D as above. Clearly, $D_t\mathbf{v}-\mathbf{A}(\cdot)\mathbf{v}=0$ in $(s,+\infty)\times\mathbb{R}^d$.

To complete the proof, we need to show that $\mathbf{v}$ can be extended by continuity on $\{s\}\times\mathbb{R}^d$ and $\mathbf{v}(s,\cdot)=\mathbf{f}$. Indeed, once this property is proved, we can conclude that $\mathbf{v}$ is a classical solution to problem (5.8), which is bounded in each strip $[s,T]\times\mathbb{R}^d$. Hence, by Proposition 5.4, we conclude that $\mathbf{v}\equiv\mathbf{G}(\cdot,s)\mathbf{f}$. Since this argument can be applied to any subsequence of $(\mathbf{G}(\cdot,s)\mathbf{f}_n)$ which converges in $C^{1,2}((s,+\infty)\times\mathbb{R}^d)$, and the limit is $\mathbf{G}(\cdot,s)\mathbf{f}$, we conclude that the whole sequence $(\mathbf{G}(\cdot,s)\mathbf{f}_n)$ converges to $\mathbf{G}(\cdot,s)\mathbf{f}$ locally uniformly in $(s,+\infty)\times\mathbb{R}^d$.

To prove that $\mathbf{v}$ can be extended by continuity at $t=s$, we fix $m,M\in\mathbb{N}$, with $m>M$. From the proof of Theorem 5.9, with its notation, and recalling that $\sup_{n\in\mathbb{N}}\|\mathbf{f}_n\|_\infty<+\infty$, we deduce that

$$|(\mathbf{G}_m(t,s)\mathbf{f}_n)(x)-\mathbf{f}_n(x)|\leq|\mathbf{G}_M(t,s)(\vartheta\mathbf{f}_n)(x)-\vartheta(x)\mathbf{f}_n(x)|+c_M\sqrt{t-s},$$

for any $(t,x)\in(s,s+1)\times B(M-1)$, and some positive constant c_M independent of m, We can let $m\to+\infty$ and conclude that

$$|(\mathbf{G}(t,s)\mathbf{f}_n)(x)-\mathbf{f}_n(x)|\leq|\mathbf{G}_M(t,s)(\vartheta\mathbf{f}_n)(x)-\vartheta(x)\mathbf{f}_n(x)|+c_M\sqrt{t-s}, \tag{5.39}$$

for any $(t,x)\in(s,s+1)\times B(M-1)$. Next step consists in letting $n\to+\infty$. Clearly, the left-hand side of (5.39) converges to $|\mathbf{v}(t,x)-\mathbf{f}(x)|$ for any $(t,x)\in(s,+\infty)\times\mathbb{R}^d$. As far as the right-hand side is concerned, we observe that Riesz's representation theorem (see [2, Rem. 1.57]) shows that there exists a family $\{p_{ij}^M(t,s,x,dy): t>s,\ x\in B(M),\ i,j=1,\ldots m\}$ of Borel finite measures such that

$$(\mathbf{G}_M(t,s)\mathbf{g})_i(x)=\sum_{j=1}^m\int_{\mathbb{R}^d}g_j(y)p_{ij}^M(t,s,x,dy),\qquad \mathbf{g}\in C_0(B(M);\mathbb{R}^m),$$

for any $t>s$, $x\in\mathbb{R}^d$, $i=1,\ldots,m$. Since each function $\vartheta\mathbf{f}_n$ is compactly supported in $B(M)$, from the previous representation formula it follows that $\mathbf{G}_M(\cdot,s)(\vartheta\mathbf{f}_n)$ converges to $\mathbf{G}_M(\cdot,s)(\vartheta\mathbf{f})$ pointwise in $[s,+\infty)\times\mathbb{R}^d$, as $n\to+\infty$. Hence, we can let $n\to+\infty$ in (5.39) and obtain

$$|\mathbf{v}(t,x)-\mathbf{f}(x)|\leq|\mathbf{G}_M(t,s)(\vartheta\mathbf{f})(x)-\vartheta(x)\mathbf{f}(x)|+c_M\sqrt{t-s},$$

for any $(t,x)\in(s,s+1)\times B(M-1)$. Now, we are done. Indeed, the function $\mathbf{G}_M(\cdot,s)(\vartheta\mathbf{f})$ is continuous in $[s,+\infty)\times B(M)$. Hence, letting $t\to s^+$ we conclude that $\mathbf{v}$ can be extended by continuity to $\{s\}\times B(M-1)$. The arbitrariness of M shows that $\mathbf{v}$ is continuous on $\{s\}\times\mathbb{R}^d$ and $\mathbf{v}(s,\cdot)=\mathbf{f}$.

(ii). Fix $T>s\in I$. In view of property (i), we just need to prove that, for any compact set $K\subset\mathbb{R}^d$ and $\varepsilon>0$, there exists $\delta>0$ such that

$$\limsup_{n\to+\infty}\|\mathbf{G}(\cdot,s)\mathbf{f}_n-\mathbf{G}(\cdot,s)\mathbf{f}\|_{C([s,s+\delta]\times K;\mathbb{R}^m)}\leq\varepsilon. \tag{5.40}$$

Indeed, we can estimate

$$\begin{aligned}\|\mathbf{G}(\cdot,s)\mathbf{f}_n-\mathbf{f}_n\|_{C([s,T]\times K;\mathbb{R}^m)}\leq&\|\mathbf{G}(\cdot,s)\mathbf{f}_n-\mathbf{f}_n\|_{C([s,s+\delta]\times K;\mathbb{R}^m)}\\&+\|\mathbf{G}(\cdot,s)\mathbf{f}_n-\mathbf{f}_n\|_{C([s+\delta,T]\times K;\mathbb{R}^m)}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\limsup_{n\to+\infty}\|\mathbf{G}(\cdot,s)\mathbf{f}_n-\mathbf{f}_n\|_{C([s,T]\times K;\mathbb{R}^m)}\\\leq&\limsup_{n\to+\infty}\|\mathbf{G}(\cdot,s)\mathbf{f}_n-\mathbf{f}_n\|_{C([s,s+\delta]\times K;\mathbb{R}^m)}+\lim_{n\to+\infty}\|\mathbf{G}(\cdot,s)\mathbf{f}_n-\mathbf{f}_n\|_{C([s+\delta,T]\times K;\mathbb{R}^m)}\\\leq&2\varepsilon,\end{aligned}$$

due to the property (i) and (5.40). The arbitrariness of $\varepsilon>0$ shows that $\mathbf{G}(\cdot,s)\mathbf{f}_n$ converges to $\mathbf{G}(\cdot,s)\mathbf{f}$, uniformly in $[s,T]\times K$, and we are done.

To prove (5.40), we fix $M \in \mathbb{N}$, such that $K \subset B(M-1)$, and we observe that estimate (5.39) shows that

$$\|\mathbf{G}(\cdot,s)\mathbf{f}_n - \mathbf{f}_n\|_{C([s,s+\delta]\times\overline{B(M-1)})} \leq \|\mathbf{G}_M(\cdot,s)(\vartheta\mathbf{f}_n) - \vartheta\mathbf{f}_n\|_{C([s,s+\delta]\times\overline{B(M-1)})} + c_M\sqrt{\delta}. \tag{5.41}$$

Since $\vartheta\mathbf{f}_n$ converges to $\vartheta\mathbf{f}$, uniformly in $B(M)$, from estimate (5.37) we deduce that $\mathbf{G}_M(\cdot,s)(\vartheta\mathbf{f}_n)$ converges to $\mathbf{G}_M(\cdot,s)(\vartheta\mathbf{f})$ uniformly in $[s,s+1]\times\overline{B(M-1)}$. Letting $n\to+\infty$ in (5.41) it follows that

$$\limsup_{n\to+\infty}\|\mathbf{G}(\cdot,s)\mathbf{f}_n - \mathbf{f}_n\|_{C([s,s+\delta]\times\overline{B(M-1)};\mathbb{R}^m)} \leq \|\mathbf{G}_M(\cdot,s)(\vartheta\mathbf{f}) - \vartheta\mathbf{f}\|_{C([s,s+\delta]\times\overline{B(M-1)};\mathbb{R}^m)} + c_M\sqrt{\delta}.$$

We are almost done. Indeed, we can estimate

$$\begin{aligned}&\|\mathbf{G}(\cdot,s)\mathbf{f}_n - \mathbf{G}(\cdot,s)\mathbf{f}\|_{C([s,s+\delta]\times\overline{B(M-1)};\mathbb{R}^m)}\\ \leq&\|\mathbf{G}(\cdot,s)\mathbf{f}_n - \mathbf{f}_n\|_{C([s,s+\delta]\times\overline{B(M-1)};\mathbb{R}^m)}\\ &\|\mathbf{f}_n - \mathbf{f}\|_{C(\overline{B(M-1)})} + \|\mathbf{G}(\cdot,s)\mathbf{f} - \mathbf{f}\|_{C([s,s+\delta]\times\overline{B(M-1)};\mathbb{R}^m)}.\end{aligned}$$

Letting $n\to+\infty$ and recalling that $\mathbf{f}_n$ tends to $\mathbf{f}$, locally uniformly in $\mathbb{R}^d$, we conclude that

$$\begin{aligned}&\limsup_{n\to+\infty}\|\mathbf{G}(\cdot,s)\mathbf{f}_n - \mathbf{G}(\cdot,s)\mathbf{f}\|_{C([s,s+\delta]\times\overline{B(M-1)};\mathbb{R}^m)}\\ \leq&\|\mathbf{G}_M(\cdot,s)(\vartheta\mathbf{f}) - \vartheta\mathbf{f}\|_{C([s,s+\delta]\times\overline{B(M-1)};\mathbb{R}^m)} + c_M\sqrt{\delta}\\ &+\|\mathbf{G}(\cdot,s)\mathbf{f} - \mathbf{f}\|_{C([s,s+\delta]\times\overline{B(M-1)};\mathbb{R}^m)}.\end{aligned}$$

Since the functions $\mathbf{G}_M(\cdot,s)(\vartheta\mathbf{f})$ and $\mathbf{G}(\cdot,s)\mathbf{f}$ are continuous in $[s,s+1]\times\overline{B(M-1)}$, from the previous estimate, it follows immediately that, for any $\varepsilon>0$, we can find $\delta>0$ such that (5.40) holds true. □

Theorem 5.11. *There exists a family $\{p_{ij}(t,s,x,dy) : t>s\in I,\ x\in\mathbb{R}^d,\ i,j=1,\dots m\}$ of finite Borel measures, which are absolutely continuous with respect to the Lebesgue measure, such that*

$$(\mathbf{G}(t,s)\mathbf{f})_i(x) = \sum_{j=1}^m\int_{\mathbb{R}^d} f_j(y)p_{ij}(t,s,x,dy), \qquad \mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m), \tag{5.42}$$

for any $t>s$, $x\in\mathbb{R}^d$, $i=1,\dots,m$. Moreover, through formula (5.42), the evolution operator $\mathbf{G}(t,s)$ extends to $B_b(\mathbb{R}^d;\mathbb{R}^m)$ with a strong Feller evolution operator. Actually, $\mathbf{G}(\cdot,s)\mathbf{f}\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$ for any $\mathbf{f}\in B_b(\mathbb{R}^d;\mathbb{R}^m)$ and $s\in I$.

Proof. Throughout the proof, s is arbitrarily fixed in I. Since, for any $(t,x)\in(s,+\infty)\times\mathbb{R}^d$, the map $\mathbf{f}\mapsto(\mathbf{G}(t,s)\mathbf{f})(x)$ is bounded from $C_0(\mathbb{R}^d;\mathbb{R}^m)$ into $\mathbb{R}$, from the Riesz's Representation Theorem (see e.g., [2, Rem. 1.57]) it follows that there exists a family $\{p_{ij}(t,s,x,dy): t>s\in I,\ x\in\mathbb{R}^d,\ i,j=1,\dots m\}$ of finite Borel measures such that (5.42) is satisfied by any $\mathbf{f}\in C_0(\mathbb{R}^d;\mathbb{R}^m)$. To extend the previous formula to any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, we consider a bounded sequence $(\mathbf{f}_n)\subset C_0(\mathbb{R}^d;\mathbb{R}^m)$ converging to $\mathbf{f}$, locally uniformly in $\mathbb{R}^d$. Writing (5.42), with $\mathbf{f}$ being replaced by $\mathbf{f}_n$, and using Proposition 5.10(ii) and the dominated convergence theorem, applied to the positive and negative parts of the measures $p_{ij}(t,s,x,dy)$, we conclude that (5.42) is satisfied also by $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$.

It is clear that formula (5.42) allows us to extend the evolution operator $\mathbf{G}(t,s)$ to $B_b(\mathbb{R}^d;\mathbb{R}^m)$.

Let us now prove that each measure $p_{ij}(t,s,x,dy)$ is absolutely continuous with respect to the Lebesgue measure. Equivalently, we can limit ourselves to proving that, for any $(t,x)\in(s,+\infty)\times\mathbb{R}^d$ and any $i,j=1,\dots,m$, the positive and negative parts of $p_{ij}(t,s,x,dy)$ are absolutely continuous with respect to the Lebesgue measure. For this purpose, we recall that, the Hahn decomposition theorem (see e.g., [96, Thm. 6.14]) shows that, for any $(t,x)\in(s,+\infty)\times\mathbb{R}^d$, there exist two Borel sets $P=P(t,s,x)$ and $N=N(t,s,x)$ such that the maps $p^+_{ij}(t,s,x,dy)$ and $p^-_{ij}(t,s,x,dy)$, defined, respectively, by $p^+_{ij}(t,s,x,A)=p_{ij}(t,s,x,A\cap P)$ and $p^-_{ij}(t,s,x,A)=-p_{ij}(t,s,x,A\cap N)$ for any Borel set $A\subset\mathbb{R}^d$, are positive measures and $p_{ij}(t,s,x,dy)=p^+_{ij}(t,s,x,dy)-p^-_{ij}(t,s,x,dy)$.

Being rather long, we split the proof into several steps.

Step 1. In view of formula (5.42) we can extend $\mathbf{G}(t,s)$ to $B_b(\mathbb{R}^d;\mathbb{R}^m)$. We claim that, for any $\mathbf{f}\in B_b(\mathbb{R}^d)$ and any $j=1,\dots,m$, $\mathbf{G}(\cdot,s)(f\mathbf{e}_j)\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$, $D_t\mathbf{G}(\cdot,s)(f\mathbf{e}_j)-\mathbf{A}\mathbf{G}(\cdot,s)(f\mathbf{e}_j)=0$ in $(s,+\infty)\times\mathbb{R}^d$ and $\|\mathbf{G}(t,s)(f\mathbf{e}_j)\|_\infty\leq e^{\varepsilon\kappa_0(t-s)}\|f\|_\infty$ for any $t>s$. Since $\mathbf{G}(\cdot,s)\mathbf{f}=\sum_{j=1}^m\mathbf{G}(\cdot,s)(f_j\mathbf{e}_j)$ for any $\mathbf{f}\in B_b(\mathbb{R}^d,\mathbb{R}^m)$, from the claim it follows immediately that the function $\mathbf{G}(\cdot,s)\mathbf{f}$ has the claimed regularity properties in the statement of the proposition and $\|\mathbf{G}(t,s)\mathbf{f}\|_\infty\leq\sqrt{d}e^{\varepsilon\kappa_0(t-s)}\|\mathbf{f}\|_\infty$ for any $t>s$. (This estimate will be improved in Corollary 5.12, removing the constant $\sqrt{d}$.).

To prove the claim, we begin by recalling that the space $B(\mathbb{R}^d)$ of all the real valued Borel functions coincides with the set $B^{\omega_1}(\mathbb{R}^d)=\bigcup_{\eta<\omega_1}B^\eta(\mathbb{R}^d)$, where, throughout this step, we denote by η the ordinal numbers and ω_1 is the first nonnumerable ordinal number. The sets $B^\eta(\mathbb{R}^d)$ are defined as follows: $B^0(\mathbb{R}^d)=C(\mathbb{R}^d)$ and, if $\eta>0$, the definition of $B^\eta(\mathbb{R}^d)$ depends on the fact that $\eta+1$ is a successor ordinal or not. In the first case, $B^\eta(\mathbb{R}^d)$ is the set of the pointwise limits, everywhere in $\mathbb{R}^d$, of sequences of functions in $B^{\tilde\eta}(\mathbb{R}^d)$, where $\tilde\eta+1=\eta$; in the second one, $B^\eta(\mathbb{R}^d)=\bigcup_{\eta_0<\eta}B^{\eta_0}(\mathbb{R}^d)$. Hence, any Borel function belongs to $B^\eta(\mathbb{R}^d)$ for some ordinal less than ω_1. We refer the reader to [61, Chpt. 30] [84, Introduction] and [99] for further details.

We fix $j\in\{1,\dots,m\}$ and, for any ordinal $\eta<\omega_1$, we set $\mathcal{P}_j(\eta)=\{f\in B^\eta_b(\mathbb{R}^d):\ \mathbf{G}(\cdot,s)(f\mathbf{e}_j)\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d),\ D_t\mathbf{G}(\cdot,s)(f\mathbf{e}_j)-\mathbf{A}\mathbf{G}(\cdot,s)(f\mathbf{e}_j)=0,\ \|\mathbf{G}(\cdot,s)(f\mathbf{e}_j)\|_\infty\leq e^{\varepsilon\kappa_0(t-s)}\|f\|_\infty\}$, where, as usually, the subscript "b" means that we are considering bounded functions.

We use the transfinite induction to prove that $\mathcal{P}_j(\eta)=B_b^\eta(\mathbb{R}^d)$ for any ordinal less than ω_1. In view of Theorem 5.9, $\mathcal{P}_j(0)=B_b^0(\mathbb{R}^d)=C_b(\mathbb{R}^d)$. Fix now an ordinal η and suppose that $\mathcal{P}_j(\beta)=B_b^\eta(\mathbb{R}^d)$ for any ordinal $\beta<\eta$. We first assume that $\eta+1$ is a successor ordinal. In such a case, f is the pointwise limit, everywhere in $\mathbb{R}^d$, of a sequence $(f_n)\in B_b^\eta(\mathbb{R}^d)$. Since, by assumptions, f is bounded, up to replacing f_n by $f_n\wedge\|f\|_\infty$, we can assume that $\|f_n\|_\infty\leq\|f\|_\infty$ for any $n\in\mathbb{N}$. Note that $f_n\wedge\|f\|_\infty\in B_b^\alpha(\mathbb{R}^d)$ for any $n\in\mathbb{N}$, as it can be easily checked. Since the function $\mathbf{G}(\cdot,s)(f_n\mathbf{e}_j)$ belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$ for any $n\in\mathbb{N}$, using the interior Schauder estimates in Theorem 5.3, as in the proof of Theorem 5.9, we can prove that, up to a subsequence, $\mathbf{G}(t,s)(f_n\mathbf{e}_j)$ converges in $C^{1,2}(K)$, for any compact set $K\subset(s,+\infty)\times\mathbb{R}^d$, to a function $\mathbf{v}\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$. Moreover, since $\|\mathbf{G}(t,s)(f_n\mathbf{e}_j)\|_\infty\leq e^{\varepsilon\kappa_0(t-s)}\|f_n\|_\infty\leq e^{\varepsilon\kappa_0(t-s)}\|f\|_\infty$, for any $t>s$ and any $n\in\mathbb{N}$, it holds that $\|\mathbf{v}(t,\cdot)\|_\infty\leq e^{\varepsilon\kappa_0(t-s)}\|f\|_\infty$ for any $t>s$, so that $\mathbf{v}\in C_b([s,T]\times\mathbb{R}^d;\mathbb{R}^m)$ for any $T>s$. The representation formula (5.42) reveals that $\mathbf{v}=\mathbf{G}(\cdot,s)(f\mathbf{e}_j)$. Moreover, since $D_t\mathbf{G}(\cdot,s)(f_n\mathbf{e}_j)-\mathbf{A}\mathbf{G}(\cdot,s)(f_n\mathbf{e}_j)=0$ in $(s,+\infty)\times\mathbb{R}^d$, letting $n\to+\infty$, we immediately deduce that $D_t\mathbf{G}(\cdot,s)(f\mathbf{e}_j)-\mathbf{A}\mathbf{G}(\cdot,s)(f\mathbf{e}_j)=0$ in $(s,+\infty)\times\mathbb{R}^d$. Hence, $f\in B_b^{\eta+1}(\mathbb{R}^d)$. Suppose now that $\eta+1$ is a limit ordinal. Then, $f\in B_b^{\eta+1}(\mathbb{R}^d)$ means that $f\in B_b^\beta(\mathbb{R}^d)$ for some ordinal β less than η. Since $\mathcal{P}(\beta)=B_b^\beta(\mathbb{R}^d)$, then $\mathbf{G}(\cdot,s)(f\mathbf{e}_j)\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$, $D_t\mathbf{G}(\cdot,s)(f\mathbf{e}_j)-\mathbf{A}\mathbf{G}(\cdot,s)(f\mathbf{e}_j)$ and $\|\mathbf{G}(\cdot,s)(f\mathbf{e}_j)\|\leq e^{\varepsilon\kappa_0(t-s)}\|f\|_\infty$. Therefore, $f\in\mathcal{P}_j(\eta)$, and we are done also in this case. The claim is thus proved.

Step 2. Here, we prove that, for any $M>0$, there exists a positive constant c, depending on M but being independent of t and $\mathbf{f}$, such that

$$\|\mathbf{G}_M(t,s)(f\mathbf{e}_j)\|_{C_b(B(M);\mathbb{R}^m)}\leq\|\tilde{G}_M(t,s)f\|_{C_b(B(M))}+c\sqrt{t-s}\,\|f\|_\infty,\tag{5.43}$$

for any $t\in(s,s+1)$, $f\in C^{2+\alpha}_c(B(M))$ and $j\in\{1,\ldots,m\}$. Here, $\mathbf{G}_M(t,s)$ and $\tilde{G}_M(t,s)$ denote, respectively, the evolution operator associated with the realization in $C_b(B(M);\mathbb{R}^m)$ of the operator $\mathbf{A}$ in (5.7) and the elliptic operator $\tilde{\mathcal{A}}={\rm Tr}(QD^2_x)$. Both these operators are endowed with homogeneous Dirichlet boundary conditions.

Throughout this and the next step, we denote by c a positive constant, which is independent of $t\in(s,s+1)$, $x\in B(M)$ and may vary from line to line. By [38, Chp. 3, Sec. 7, Thm. 16], $\tilde{G}_M(t,s)$ can be extendend to any function $\mathbf{f}\in B_b(B(M))$. Moreover, $\|\tilde{G}_M(t,s)f\|_{C_b(B(M))}\leq c(t-s)^{-1/2}\|f\|_\infty$ for any $f\in C_b(B(M))$ and any $t\in(s,s+1)$ (see [70]).

Fix $f\in C^{2+\alpha}_c(B(M))$ and $j\in\{1,\ldots,m\}$. Since the function $\mathbf{u}=\mathbf{G}_M(\cdot,s)(f\mathbf{e}_j)$ solves the Cauchy problem

$$\begin{cases} D_t\mathbf{u}(t,x)=(\tilde{\mathbf{A}}\mathbf{u})(t,x)+\mathbf{g}(t,x), & t\in(s,s+1),\quad x\in B(M),\\ \mathbf{u}(t,x)=0, & t\in(s,s+1),\quad x\in B(M),\\ \mathbf{u}(s,x)=f(x)\mathbf{e}_j, & \qquad\qquad\qquad\quad x\in B(M),\end{cases}$$

where $\tilde{\mathbf{A}}$ is the diagonal operator all whose components coincide with the operator $\mathcal{A}$

and $\mathbf{g}=-\sum_{i=1}^{d}B_jD_j\mathbf{u}-C\mathbf{u}$, the variation-of-constants-formula shows that

$$\mathbf{u}(t,x)=(\tilde{\mathbf{G}}_M(t,s)(f\mathbf{e}_j)(x)+\int_s^t(\tilde{\mathbf{G}}_M(r,s)\mathbf{g}(r,\cdot))(x)dr,\qquad (t,x)\in(s,+\infty)\times B(M), \tag{5.44}$$

where $(\tilde{\mathbf{G}}_M(t,s)\mathbf{g})_k=\tilde{G}_M(t,s)g_k$ for any $k=1,\dots,m$. Therefore,

$$\nabla_x\mathbf{u}(t,x)=(\nabla_x\tilde{\mathbf{G}}_M(t,s)(f\mathbf{e}_j))(x)+\int_s^t(\nabla_x\tilde{\mathbf{G}}_M(r,s)\mathbf{g}(r,\cdot))(x)dr, \tag{5.45}$$

for any $(t,x)\in(s,+\infty)\times B(M)$. Taking the norms of both the sides of (5.45) we can estimate

$$\begin{aligned}&\|\nabla_x\mathbf{u}(t,\cdot)\|_{C_b(B(M);\mathbb{R}^m)}\\ \leq&\frac{c}{\sqrt{t-s}}\|f\|_\infty+c\int_s^t\frac{1}{\sqrt{r-s}}\left(\|\nabla_x\mathbf{u}(r,\cdot)\|_{C_b(B(M);\mathbb{R}^m)}+\|\mathbf{u}(r,\cdot)\|_{C_b(B(M);\mathbb{R}^m)}\right)dr\\ \leq&\frac{c}{\sqrt{t-s}}\|f\|_\infty+c\int_s^t\frac{1}{\sqrt{r-s}}\|\nabla_x\mathbf{u}(r,\cdot)\|_{C_b(B(M);\mathbb{R}^m)}dr,\end{aligned}$$

for any $t\in(s,s+1)$. To get the previous estimate we took advantage of the fact that $\|\mathbf{u}(t,\cdot)\|_\infty\leq c\|f\|_\infty$ for any $t\in(s,s+1)$. The generalized Gronwall lemma (see [46]) shows that

$$\|\nabla_x\mathbf{u}(t,\cdot)\|_{C_b(B(M);\mathbb{R}^m)}\leq\frac{c}{\sqrt{t-s}}\|f\|_\infty,\qquad t\in(s,s+1). \tag{5.46}$$

In view of (5.46) we can estimate $\|\mathbf{g}(t,\cdot)\|_{C_b(B(M);\mathbb{R}^m)}\leq c(t-s)^{-1/2}\|f\|_\infty$ for any $t\in(s,s+1)$. Taking the norms in both the sides of (5.44) we get

$$\begin{aligned}\|\mathbf{G}_M(t,s)(f\mathbf{e}_j)\|_{C_b(B(M);\mathbb{R}^m)}\leq&\|\tilde{G}_M(t,s)f\|_{C_b(B(M))}+c\|f\|_\infty\int_s^t(r-s)^{-\frac{1}{2}}dr\\ =&\|\tilde{G}_M(t,s)f\|_{C_b(B(M))}+c\sqrt{t-s}\|f\|_\infty,\end{aligned}$$

for any $t\in(s,s+1)$. Estimate (5.43) follows.

Step 3. Here, we prove that, for any Borel set $\mathcal{O}\subset\mathbb{R}^d$ with zero Lebesgue measure, any $M>0$ any $j\in\{1,\dots,m\}$ and any $t\in(s,s+1)$, it holds that

$$|(\mathbf{G}(t,s)\chi_{\mathcal{O}}\mathbf{e}_j)(x)|\leq c\sqrt{t-s},\qquad t\in(s,s+1),\ \ x\in B(M/2). \tag{5.47}$$

To prove this inequality, it suffices to prove that

$$\|\mathbf{G}_M(t,s)(f\mathbf{e}_j)\|_{C_b(B(M);\mathbb{R}^m)}\leq\|\tilde{G}_M(t,s)f\|_{C_b(B(M))}+c\sqrt{t-s}\,\|f\|_\infty, \tag{5.48}$$

for any $t\in(s,s+1)$ any $f\in B_b(\mathbb{R}^d)$ and $j\in\{1,\dots,m\}$. Indeed, it is well known that, if $\mathcal{O}$ has null Lebesgue measure, then $\tilde{G}_M(t,s)\chi_{\mathcal{O}}=0$ (see [38, Chp. 3, Sec. 7, Thm. 16]).

Estimate (5.48) can be proved by transfinite induction, arguing as in Step 1. For this purpose, let us prove that, if $f \in B_b(\mathbb{R}^d)$ is the pointwise limit everywhere in $\mathbb{R}^d$ of a sequence $(f_n) \subset B_b(\mathbb{R}^d)$ of functions, which satisfy (5.48) and $\|f_n\|_\infty \leq \|f\|_\infty$ for any $n \in \mathbb{N}$, then f satisfies (5.48) as well. This property will imply in particular that all the functions in $C_b(\mathbb{R}^d)$ satisfy (5.48) and gives the argument to make the transfinite induction work.

Fix $M > 0$, f and (f_k) as above and a function $\vartheta \in C_c^\infty(\mathbb{R}^d)$ such that $\chi_{B(M/4)} \leq \vartheta \leq \chi_{B(M/2)}$. By the proof of Theorem 5.9, we know that $\mathbf{G}(\cdot,s)(f_k\mathbf{e}_j)$ is the local uniform limit in $(s,+\infty)\times\mathbb{R}^d$ of the unique classical solution $\mathbf{u}_k^n$ to the Cauchy problem (5.36), with $\mathbf{f}$ being replaced by $\vartheta f_k \mathbf{e}_j$, As it is easily seen, the function $\mathbf{v}_k^n := \vartheta\mathbf{u}_k^n$ solves the Cauchy problem

$$\begin{cases} D_t\mathbf{v}_k^n(t,x) = (\mathbf{A}\mathbf{v}_k^n)(t,x) + \mathbf{g}_k^n(t,x), & t>s, \quad x \in B(M),\\ \mathbf{v}(t,x) = 0, & t>s, \quad x\in\partial B(M),\\ \mathbf{v}_k^n(s,x) = \vartheta(x) f_k(x)\mathbf{e}_j, & \qquad\quad x \in B(M),\end{cases}$$

where

$$\mathbf{g}_k^n = -\mathrm{Tr}(QD^2\vartheta)\mathbf{u}_k^n - 2(J_x\mathbf{u}_k^n)Q\nabla\vartheta - \sum_{j=1}^d (B_j\mathbf{u}_k^n)D_j\vartheta.$$

The variation-of-constants formula yields that

$$\mathbf{v}_k^n(t,x) = (\mathbf{G}_M(t,s)(\vartheta f\mathbf{e}_j))(x) + \int_s^t (\mathbf{G}_M(t,r)\mathbf{g}_k^n(r,\cdot))(x)dr,$$

for any $s<t$, and $x \in B(M)$. From estimate (5.14) and recalling that $\|f_k\|_\infty \leq \|f\|_\infty$ for any $k\in\mathbb{N}$, it follows that there exist two positive constants c_1 and c_M, independent of k and n, such that

$$\|\mathbf{G}_M(t,r)\mathbf{g}_k^n(r,\cdot)\|_{L^\infty(B(M))} \leq c_1\|\mathbf{g}_k^n(r,\cdot)\|_{L^\infty(B(M/2))} \leq c_M\|f\|_\infty(1+(t-r)^{-\frac{1}{2}}),$$

for any $t\in(r,s+1)$. Hence, taking (5.43) into account, we can estimate

$$|\mathbf{v}_k^n(t,x)| \leq |(\tilde{\mathbf{G}}_M(t,s)(\vartheta f_k\mathbf{e}_j))(x)| + c_M\|f\|_\infty\sqrt{t-s}, \qquad t\in(s,s+1),\ x\in B(M).$$

Letting $n\to+\infty$ we get

$$|(\mathbf{G}(t,s)(f_k\mathbf{e}_j))(x)| \leq |\tilde{\mathbf{G}}_M(t,s)(\vartheta f_k\mathbf{e}_j)(x)| + c_M\|f\|_\infty\sqrt{t-s}, \tag{5.49}$$

for any $t\in(s,s+1)$ and $x\in B(M/2)$. Clearly, ϑf_k converges to ϑf pointwise in $\mathbb{R}^d$ as $k\to+\infty$. The same arguments as in the proof of (ii) reveal that $\tilde{\mathbf{G}}_M(\cdot,s)(\vartheta f_k\mathbf{e}_j)$ converges to $\tilde{\mathbf{G}}_M(\cdot,s)(\vartheta f\mathbf{e}_j)$ pointwise in $(s,s+1)\times B(M)$. Hence, letting $k\to+\infty$ in (5.49) we get (5.48).

Step 4. We fix $i,j\in\{1,\ldots,m\}$, $t_0>s$, $x_0\in\mathbb{R}^d$ and prove that the measures $p_{ij}^+(t_0,s,x_0,dy)$ and $p_{ij}^-(t_0,s,x_0,dy)$ are absolutely continuous with respect to the Lebesgue measure. We begin by considering the measure $p^+_{ij}(t_0,s,x_0,dy)$ and we fix a bounded Borel set A with null Lebesgue measure. Clearly $A\cap P$ has null Lebesgue measure. Therefore, from estimate (5.47) it follows that

$$|(\mathbf{G}(t,s)(\chi_{A\cap P}\mathbf{e}_j))(x)|\leq c\sqrt{t-s},\qquad t\in(s,s+1),\ \ x\in B(M/2).$$

It thus follows that $\mathbf{G}(t,s)(\chi_{A\cap P}\mathbf{e}_j)$ tends to 0 uniformly in $B(M/2)$ as $t\to s^+$. By the arbitrariness of M, we deduce that $\mathbf{G}(t,s)(\chi_{A\cap P}\mathbf{e}_j)$ tends to 0, locally uniformly in $\mathbb{R}^d$, as $t\to s^+$. From Step 1, the function $\mathbf{v}$, defined by $\mathbf{v}(s,\cdot)=0$ and $\mathbf{v}(t,\cdot)=\mathbf{G}(t,s)(\chi_{A\cap P}\mathbf{e}_j)$, if $t>s$, belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)\cap C_b([s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)$ and is a classical solution to the Cauchy problem

$$\begin{cases} D_t\mathbf{u}(t,x)=(\mathbf{A}\mathbf{u})(t,x), & t>s,\quad x\in\mathbb{R}^d,\\ \mathbf{u}(s,x)=0, & \qquad\quad\ x\in\mathbb{R}^d.\end{cases}$$

By Proposition 5.4, it follows that $\mathbf{v}\equiv 0$. Thus, we conclude that $\mathbf{G}(\cdot,s)(\chi_{A\cap P}\mathbf{e}_j)\equiv 0$ in $(s,\infty)\times\mathbb{R}^d$. In particular, $(\mathbf{G}(\cdot,s)(\chi_{A\cap P}\mathbf{e}_j))(x_0)=0$ which implies that

$$\begin{aligned}0=&(\mathbf{G}(t_0,s)(\chi_{A\cap P}\mathbf{e}_j))_i(x_0)=\int_{\mathbb{R}^d}\chi_{A\cap P}p_{ij}(t_0,s,x_0,dy)\\ =&p_{ij}(t_0,s,x_0,A\cap P)=p^+_{ij}(t_0,s,x_0,A)\end{aligned}$$

and we are done.

In the same way, one can show that $p^-_{ij}(t,s,x,dy)=0$ is absolutely continuous with respect to the Lebesgue measure. The proof is complete. □

Corollary 5.12. *The following properties are satisfied.*

(i) *Estimate* (5.38) *is satisfied by any* $\mathbf{f}\in B_b(\mathbb{R}^d;\mathbb{R}^m)$.

(ii) *Proposition* 5.10(i) *holds true for any bounded sequence* $(\mathbf{f}_n)$ *of Borel functions which converge pointwise* (*almost everywhere in* $\mathbb{R}^d$) *to a Borel measurable function* $\mathbf{f}$.

Proof. (i) Since any function $\mathbf{f}\in B_b(\mathbb{R}^d)$ is the almost everywhere pointwise limit in $\mathbb{R}^d$ of a bounded sequence $(\mathbf{f}_n)\subset C_b(\mathbb{R}^d)$, and the measures $p_{ij}(t,s,x,dy)$ are absolutely continuous with respect to the Lebesgue measure, for any $t>s\in I$ and any $x\in\mathbb{R}^d$, by formula (5.42) and the dominated convergence theorem, we deduce that $\mathbf{G}(t,s)\mathbf{f}_n$ converges to $\mathbf{G}(t,s)\mathbf{f}$ pointwise everywhere in $\mathbb{R}^d$, as $n\to+\infty$. Clearly, without loss of generality we can assume that $\|f_n\|_\infty\leq\|\mathbf{f}\|_\infty$ for any $n\in\mathbb{N}$. From (5.38) it follows that $|\mathbf{G}(t,s)\mathbf{f}_n)(x)|\leq e^{\varepsilon\kappa_0(t-s)}\|\mathbf{f}_n\|_\infty\leq e^{\varepsilon\kappa_0(t-s)}\|\mathbf{f}_n\|_\infty$ for any $t>s\in I$ and any $x\in\mathbb{R}^d$. Letting $n\to+\infty$, we conclude the proof of (i).

(ii) Fix $s\in I$ and $(\mathbf{f}_n),(\mathbf{f})$ as in the statement. For any $\varepsilon>0$ the functions $\mathbf{G}(s+\varepsilon,s)\mathbf{f}_n$ and $\mathbf{G}(s+\varepsilon,s)\mathbf{f}$ are bounded and continuous in $\mathbb{R}^d$, thanks to property (ii). Moreover, the representation formula (5.42) shows that $\mathbf{G}(s+\varepsilon,s)\mathbf{f}_n$ converges pointwise in $\mathbb{R}^d$ to $\mathbf{G}(s+\varepsilon,s)\mathbf{f}$ as $n\to+\infty$. The evolution law which allows us us to split $\mathbf{G}(t,s)\mathbf{f}_n=\mathbf{G}(t,s+\varepsilon)\mathbf{G}(s+\varepsilon)\mathbf{f}_n$, for any $n\in\mathbb{N}$ and Proposition 5.10(i) allow us to conclude the proof. □

5.4 Compactness

In this section we provide sufficient conditions ensuring that the evolution operator $\mathbf{G}(t,s)$ is compact in $\mathcal{L}(C_b(\mathbb{R}^d;\mathbb{R}^m))$. Besides Hypotheses 5.1(i), (ii) we assume the following additional conditions on the coefficients of the operator $\mathbf{A}$.

Hypotheses 5.13. (i) *For any $i=1,\dots,d$, there exist $b_i \in C^{\alpha/2,\alpha}_{\rm loc}(I\times\mathbb{R}^d)$ and $\tilde{B}_i \in C^{\alpha/2,\alpha}_{\rm loc}(I\times\mathbb{R}^d;\mathbb{R}^{m^2})$ such that $B_i(t,x) := b_i(t,x)Id_m + \tilde{B}_i(t,x)$ for any $i=1,\dots,d$ and any $(t,x)\in I\times\mathbb{R}^d$. Further, for any bounded interval $J\subset I$ there exists a positive constant Ξ_J such that $|(\tilde{B}_i)_{jk}| \le \Xi_J\sqrt{\nu}$ in $J\times\mathbb{R}^d$, for any $j,k=1,\dots,m$, $i=1,\dots,d$ where $(\tilde{B}_i)_{jk}$ denotes the jk-th element of the matrix $\tilde{B}_i$;*

(ii) *for any bounded interval $J\subset I$, there exists a constant $c_J\in\mathbb{R}$ such that $\langle C(t,x)\eta,\eta\rangle \le c_J$, for any $(t,x)\in J\times\mathbb{R}^d$ and any $\eta\in\partial B_1$;*

(iii) *for any bounded interval $J\subset I$ there exist a constant λ_J and a positive function $\varphi_J\in C^2(\mathbb{R}^d)$ blowing up as $|x|\to+\infty$ such that*

$$\sup_{(t,x)\in J\times\mathbb{R}^d}(\mathcal{A}(t)\varphi_J)(x) - \lambda_J\varphi_J(x)) < +\infty,$$

where $\mathcal{A} = \mathrm{Tr}(QD^2_x) + \langle b,\nabla_x\rangle$ and $b=(b_1,\dots,B_m)$.

Since we no longer assume Hypotheses 5.1(iii), (iv), we can not apply Proposition 5.4 to guarantee the uniqueness of the solution to the Cauchy problem (5.8). The role of the following proposition twofold. First, it replaces Proposition 5.4, and, combined with Theorem 5.9, it shows that the Cauchy problem (5.8) admits a unique solution $\mathbf{u}\in C([s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)\cap C^{1+\alpha/2,2+\alpha}((s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)$, which is bounded in each strip $[s,T]\times\mathbb{R}^d$. This allows us to define the evolution operator $\mathbf{G}(t,s)$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$, as we did in Section 5.3, and all the results therein proved still hold true, since they are mainly based on Schauder estimates and on the arguments in the proof of Theorem 5.9. Secondly, Proposition 5.14 shows that, for any $\mathbf{f}\in B_b(\mathbb{R}^d;\mathbb{R}^m)$, the function $|\mathbf{G}(\cdot,s)\mathbf{f}|^2$ can be estimated pointwise in terms of $G(t,s)|\mathbf{f}|^2$, where $G(t,s)$ denotes the evolution operator in $C_b(\mathbb{R}^d)$ associated with the operator $\mathcal{A}$, whose existence has been proved in [60]. This is the first step to provide sufficient conditions for the compactness of the evolution operator $\mathbf{G}(t,s)$.

Throughout this section, for any interval $J\subset I$ we set $\Lambda_J = \{(t,s)\in J\times J : t>s\}$.

Proposition 5.14. *Suppose that Hypotheses 5.1(i), 5.1(ii) and 5.13 hold true. Then, for any $s\in I$ and any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, the Cauchy problem (5.8) admits a unique solution $\mathbf{u}\in C([s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)\cap C^{1+\alpha/2,2+\alpha}((s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)$, which is bounded in each strip $[s,T]\times\mathbb{R}^d$. Setting $\mathbf{G}(\cdot,s)\mathbf{f} := \mathbf{u}$, we define an evolution operator in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ which extends to $B_b(\mathbb{R}^d;\mathbb{R}^m)$. Moreover, for any $T>s\in I$ there exists a positive constant $c=c(s,T)$ such that*

$$|(\mathbf{G}(t,s)\mathbf{f})(x)|^2 \le c(G(t,s)|\mathbf{f}|^2)(x), \tag{5.50}$$

for any $(t,x)\in[s,T]\times\mathbb{R}^d$ and any bounded Borel measurable function $\mathbf{f}$.

Proof. The core of the proof consists in showing that any solution $\mathbf{u}$ to problem (5.8) corresponding to $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and enjoying the regularity properties in the statement of the proposition, satisfies the estimate

$$|\mathbf{u}(t,x)|^2 \le c(G(t,s)|\mathbf{f}|^2)(x), \qquad (t,x) \in (s,T) \times \mathbb{R}^d, \tag{5.51}$$

for any $T > s$ and some positive constant $c = c(s,T)$. Clearly, this estimate gives the uniqueness of the solution to problem (5.8). Moreover, the arguments here below can also be applied to prove that the solution $\mathbf{u}_n$ to the Cauchy problem (5.36) satisfies (5.51), with $\mathbb{R}^d$ being replaced by $B(n)$. This estimate replaces (5.37) and allows us to repeat verbatim the proof of Theorem 5.9. We can thus define the evolution operator $\mathbf{G}(t,s)$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ and, then extend it to $B_b(\mathbb{R}^d;\mathbb{R}^m)$. To extend (5.50) to functions in $B_b(\mathbb{R}^d;\mathbb{R}^m)$, we approximate any function $\mathbf{f} \in B_b(\mathbb{R}^d;\mathbb{R}^m)$ by a sequence $(\mathbf{f}_n) \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ converging to $\mathbf{f}$ almost everywhere in $\mathbb{R}^d$, as $n \to +\infty$. Writing (5.50) with $\mathbf{f}$ being replaced by $\mathbf{f}_n$ and using the result in Proposition 5.10(i) and the dominated convergence theorem to let $n \to +\infty$, we get (5.50) also for bounded and Borel measurable functions.

So, let us assume that $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$. For any $T > s \in I$, we consider the interval $J = [s,T]$ and we set $K = c_J + m^2\Xi_J^2 d$, where the constants c_J and Ξ_J are defined in Hypotheses 5.13. Moreover, we consider the function $v : [s,T] \times \mathbb{R}^d \to \mathbb{R}$ defined by $v(t,x) := e^{-2K(t-s)}|\mathbf{u}(t,x)|^2 - w(t,x)$ for any $(t,x) \in J \times \mathbb{R}^d$, where $w = G(\cdot,s)|\mathbf{f}|^2$. The function v belongs to $C_b([s,T] \times \mathbb{R}^d) \cap C^{1,2}((s,T) \times \mathbb{R}^d)$ and, taking Hypothesis 5.13(i) into account, a straightforward computation yields

$$\begin{aligned} D_t v(t,x) = (\mathcal{A}v)(t,x) + 2e^{-2K(t-s)}\bigg(&\sum_{i=1}^d \langle \tilde{B}_i(t,x) D_i\mathbf{u}(t,x), \mathbf{u}(t,x)\rangle \\ &- \mathrm{Tr}(J_x\mathbf{u}(t,x)Q(t,x)(J_x\mathbf{u}(t,x))^T) \\ &+ \langle C(t,x)\mathbf{u}(t,x), \mathbf{u}(t,x)\rangle - K|\mathbf{u}(t,x)|^2\bigg), \end{aligned}$$

for any $(t,x) \in (s,T] \times \mathbb{R}^d$. From estimate (5.9), it follows that $-\mathrm{Tr}(J_x\mathbf{u}Q(J_x\mathbf{u})^T) \le -\nu|J_x\mathbf{u}|^2$ in $J \times \mathbb{R}^d$.

Young, Cauchy-Schwarz inequalities and Hypothesis 5.13(i) imply that

$$\begin{aligned} 2\sum_{i=1}^d \langle \tilde{B}_i D_i\mathbf{u}, \mathbf{u}\rangle \le& 2\sum_{i=1}^d \|\tilde{B}_i\||D_i\mathbf{u}||\mathbf{u}| \le 2m\Xi_J\sqrt{\nu}|\mathbf{u}|\sum_{i=1}^d |D_i\mathbf{u}| \\ \le& \varepsilon m^2\Xi_J^2 d\nu|J_x\mathbf{u}|^2 + \frac{1}{\varepsilon}|\mathbf{u}|^2, \end{aligned}$$

Now, choosing $\varepsilon = (m^2\Xi_J^2 d)^{-1}$ and taking Hypothesis 5.13(ii) and our choice of K into account, we get

$$D_t v(t,x) - (\mathcal{A}v)(t,x) \le 2e^{-2K(t-s)}(c_J + m^2\Xi_J^2 d - K)|\mathbf{u}(t,x)|^2 = 0,$$

for any $(t,x) \in (s,T] \times \mathbb{R}^d$. Moreover, since $v(s,\cdot) = 0$ in $\mathbb{R}^d$, the maximum principle in [60, Thm. 2.1] shows that $v \le 0$ in $[s,T] \times \mathbb{R}^d$ which is the claim with $c = e^{K(T-s)}$. □

Remark 5.15. If the constants c_J and Ξ_J in Hypotheses 5.13(i) and 5.13(ii) are independent of the bounded interval $J \subset I$, then the proof of Proposition 5.14 reveals that

$$|(\mathbf{G}(t,s)\mathbf{f})(x)|^2 \le e^{(c+m^2\Xi^2 d)(t-s)}(G(t,s)|\mathbf{f}|^2)(x), \qquad t > s \in I,\ x \in \mathbb{R}^d.$$

We can now prove the following result, which allows us to provide sufficient conditions for the compactness of $\mathbf{G}(t,s)$.

Theorem 5.16. *Let $J \subset I$ be an interval. If the family $\{\mathbf{G}(t,s) : (t,s) \in \Lambda_J\}$ consists of compact operators in $C_b(\mathbb{R}^d;\mathbb{R}^m)$, then, for any $(t,s) \in \Lambda_J$ and $i,j = 1,\dots,m$ the family of measures $\{p_{ij}(t,s,x,dy) : x \in \mathbb{R}^d\}$ is tight. If, in addition, Hypotheses* 5.13 *hold true and $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$ for every $(t,s) \in \Lambda_J$, then $\mathbf{G}(t,s)$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ for every $(t,s) \in \Lambda_J$.*

Proof. Let J be as in the statement, fix an index $j_0 \in \{1,\dots,m\}$, $(t,s) \in \Lambda_J$, and set $\mathbf{f}_n := \chi_{\mathbb{R}^d\setminus B(n)}\mathbf{e_{j_0}}$ for any $n \in \mathbb{N}$. Clearly $\mathbf{f}_n$ vanishes locally uniformly in $\mathbb{R}^d$ as $n \to +\infty$ and, by formula (5.42), it is easy to deduce that $(\mathbf{G}(t,s)\mathbf{f}_n)_i$ vanishes pointwise in $\mathbb{R}^d$ as $n \to +\infty$ for any $i = 1,\dots,m$. Fix an arbitrary point $r \in (s,t)$. The previous argument can be applied to show that the sequence $(\mathbf{G}(r,s)\mathbf{f}_n)_{n\in\mathbb{N}}$ (which consists of bounded and continuous functions by Corollary 5.12(i)) converges pointwise to zero as $n \to +\infty$. Since $\mathbf{G}(t,r)$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$, there exists a sequence $(\mathbf{G}(r,s)\mathbf{f}_{n_k}) \subset (\mathbf{G}(r,s)\mathbf{f}_n)$ such that $\mathbf{G}(t,s)\mathbf{f}_{n_k} = \mathbf{G}(t,r)\mathbf{G}(r,s)\mathbf{f}_{n_k}$ vanishes uniformly in $\mathbb{R}^d$ as $k \to +\infty$. As a byproduct, we deduce that, for any $i \in \{1,\dots,m\}$, the whole sequence $(\mathbf{G}(t,s)\mathbf{f}_n)_i$ converges to 0 uniformly in $\mathbb{R}^d$ as $n \to +\infty$. By formula (5.42) it follows that $(\mathbf{G}(t,s)\mathbf{f}_n)_i = p_{ij_0}(t,s,\cdot,\mathbb{R}^d \setminus B(n))$ for any $n \in \mathbb{N}$ and $i \in \{1,\dots,m\}$. Hence, the tightness of the family $\{p_{ij}(t,s,x,dy) : x \in \mathbb{R}^d\}$ follows from the arbitrariness of i and j_0.

Now, let us assume that Hypotheses 5.13 are satisfied. We fix $(t,s) \in \Lambda_J$, $r \in (s,t)$ and, for any $n \in \mathbb{N}$, we consider the operator $\mathbf{R}_n := \mathbf{G}(t,r)(\chi_{B(n)}\mathbf{G}(r,s))$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. Since $\mathbf{G}(t,r)$ is strong Feller (see again Corollary 5.12(i)), each operator $\mathbf{R}_n$ is bounded in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. Moreover, from estimate (5.50) it follows that

$$\begin{aligned}
|(\mathbf{G}(t,s)\mathbf{f})(x) - (\mathbf{R}_n\mathbf{f})(x)|^2 =& |(\mathbf{G}(t,r)(\chi_{\mathbb{R}^d\setminus B(n)}\mathbf{G}(r,s)\mathbf{f}))(x)|^2\\
\le& c(G(t,r)(\chi_{\mathbb{R}^d\setminus B(n)}|\mathbf{G}(r,s)\mathbf{f}|^2))(x)\\
=& c\int_{\mathbb{R}^d\setminus B(n)} |(\mathbf{G}(r,s)\mathbf{f})(y)|^2 g_{t,r}(x,dy)\\
\le& ce^{2\varepsilon k_0(r-s)}\|\mathbf{f}\|_\infty^2 g_{t,r}(x,\mathbb{R}^d \setminus B(n)),
\end{aligned} \tag{5.52}$$

for any $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ where $g_{t,s}$ are the transition kernels associated to $G(t,s)$ in $C_b(\mathbb{R}^d)$. Since the compactness of $G(t,s)$ is equivalent to the tightness of the family $\{g_{t,s}(x,dy) : x \in \mathbb{R}^d\}$, (see [71, Thm 3.3]), from (5.52) we deduce that $\mathbf{R}_n$ tends to $\mathbf{G}(t,s)$ in $\mathcal{L}(C_b(\mathbb{R}^d;\mathbb{R}^m))$ as $n \to +\infty$. Therefore, we can limit ourselves to proving that each operator $\mathbf{R}_n$ is compact. To this aim, let $(\mathbf{f}_k)$ be a bounded sequence in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. From the interior Schauder estimates in Theorem 5.3 it follows that the sequence $(\mathbf{G}(r,s)\mathbf{f}_k)$ is bounded in $C^{2+\alpha}(B(n);\mathbb{R}^m)$. Hence, there exists a subsequence $(\mathbf{G}(r,s)\mathbf{f}_{k_j})$ converging uniformly in $B(n)$ to some function $\mathbf{g}$ as $j \to +\infty$. As a byproduct, $\chi_{B(n)}\mathbf{G}(r,s)\mathbf{f}_{k_j}$ converges to $\chi_{B(n)}\mathbf{g}$ uniformly in $\mathbb{R}^d$ as $j \to +\infty$. Since the estimate (5.38) holds true also for bounded Borel functions (see Theorem 5.11), we conclude that $\mathbf{R}_n\mathbf{f}_{k_j}$ converges uniformly in $\mathbb{R}^d$ to $\mathbf{G}(t,r)(\chi_{B(n)}\mathbf{g})$ as $j \to +\infty$. Hence, $\mathbf{R}_n$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. □

In view of Theorem 5.16 and [71, Thm 3.3], some sufficient conditions in order to get compactness of $\mathbf{G}(t,s)$ can be provided.

Corollary 5.17. *Suppose that there exist a C^2 function $W : \mathbb{R}^d \to \mathbb{R}$ such that $\lim_{|x|\to\infty} W(x) = +\infty$, a number $R > 0$ and a convex increasing function $g : [0,+\infty) \to \mathbb{R}$ such that $1/g$ belongs to $L^1((a,+\infty))$ for large a and $(\mathcal{A}(t)W)(x) \leq -g(W(x))$ for any $t \in I$ and any $|x| \geq R$. Then $\mathbf{G}(t,s)$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ for any $t > s \in I$.*

In Theorem 5.16, we have proved that the compactness of the scalar evolution operator $G(t,s)$ in $C_b(\mathbb{R}^d)$ implies the compactness of the evolution operator $\mathbf{G}(t,s)$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. As in [31], we are also interested in providing sufficient conditions which guarantee that the compactness of $\mathbf{G}(t,s)$ in $C_b(\mathbb{R}^d,\mathbb{R}^m)$ implies the compactness of $G(t,s)$ in $C_b(\mathbb{R}^d)$. The main step in this direction, consists in writing, by means of the variation-of-constants formula, the components of $\mathbf{G}(t,s)\mathbf{f}$ in terms of the scalar evolution operator introduced in Proposition 5.14. More precisely, under suitable assumptions, we will prove that

$$(\mathbf{G}(t,s)\mathbf{f})_{\bar{k}}(x) = (G(t,s)f_{\bar{k}})(x) + \int_s^t (G(t,r)\Phi_{\mathbf{f},\bar{k}}(r,\cdot))(x)\,dr, \tag{5.53}$$

for any $(t,s) \in \Lambda_J$, any $x \in \mathbb{R}^d$, and some $\bar{k} \in \{1,\ldots,m\}$, where $J \subset I$ is an interval and

$$\Phi_{\mathbf{f},\bar{k}} = \sum_{i=1}^d \langle row_{\bar{k}}\tilde{B}_i, D_i\mathbf{G}(\cdot,s)\mathbf{f}\rangle + \langle row_{\bar{k}}C, \mathbf{G}(\cdot,s)\mathbf{f}\rangle.$$

To give a meaning to formula (5.53) we need to guarantee that the function $\Phi_{\mathbf{f},\bar{k}}(r,\cdot)$ is bounded in $\mathbb{R}^d$ for any $r \in (s,t)$. Indeed, the results in [60] show that $G(t,s)$ is well defined in $B_b(\mathbb{R}^d)$. In the weakly coupled case considered in [31], $\tilde{B}_i \equiv 0$ for any $i = 1,\ldots,d$. Hence, the boundedness of $row_{\bar{k}}C$ was enough to guarantee the existence of the integral term in (5.53). In our situation things are much more difficult since we have to guarantee that also the function $\sum_{i=1}^d \langle row_{\bar{k}}\tilde{B}_i(r,\cdot), D_i\mathbf{G}(r,s)\mathbf{f}\rangle$ is bounded in $\mathbb{R}^d$ for any $r \in (s,t)$, in order to apply the evolution operator $G(t,r)$ to such a function. This is proved as a byproduct of some weighted gradient estimate. To prove them, besides Hypotheses 5.13 we consider the following stronger conditions on the coefficients q_{ij} and b_i.

Hypotheses 5.18. (i) *The coefficients q_{ij} belong to $C^{1,2+\alpha}_{\rm loc}(I\times\mathbb{R}^d)$ for some $\alpha \in (0,1)$ and any $i,j = 1,\ldots,d$;*

(ii) *the coefficients of the vector* b *and the entries of the matrices* $\tilde{B}_i$ $(i = 1, \dots, d)$ *and* C *belongs to* $C^{0,1+\alpha}_{\rm loc}(I \times \mathbb{R}^d)$*; further,* $\langle b(t,x), x\rangle \le b_0(t,x)|x|$ *for any* $t \in I$ *any* $x \in \mathbb{R}^d$ *and some negative function* b_0*;*

(iii) *there exist a function* $K_0 : I \to \mathbb{R}_+$ *and positive functions* $\psi_j : I \times \mathbb{R}^d \to \mathbb{R}$ $(j = 1, \dots, 6)$ *such that*

$$\langle Q(t,x), x\rangle| \le K_0(t)(1 + |x|^2)\nu(t,x), \qquad x \in \mathbb{R}^d, \tag{5.54}$$

$$|\nabla_x Q^{1/2}||Q^{-1/2}| \le \psi_1, \qquad |Q^{1/2}\nabla_x C_{ij}| \le \psi_2, \quad i,j = 1, \dots, d, \tag{5.55}$$

$$|Q^{1/2}\nabla_x(\tilde{B}_i)_{jk}Q^{-1/2}_{ih}| \le \psi_3, \qquad i,h = 1, \dots, d, \ j,k = 1, \dots, m, \tag{5.56}$$

$$\langle C\xi, \xi\rangle \le -\psi_4|\xi|^2, \qquad \xi \in \mathbb{R}^d, \tag{5.57}$$

$$|Q^{-1/2}D_t Q Q^{-1/2}| \le \psi_5, \qquad |Q| \le \psi_6, \qquad |\tilde{B}| \le \psi_7, \tag{5.58}$$

in $I \times \mathbb{R}^d$*;*

(iv) *the functions* ψ_1*,* ψ_3*,* ψ_4*,* ψ_5*,* ψ_6 *and* ψ_7 *satisfy the following conditions:*

$$\begin{aligned}&\lim_{|x|\to+\infty} \frac{\psi_5(t,x)}{\psi_4(t,x) - \omega(t,x)} = \lim_{|x|\to+\infty} \frac{\psi_3(t,x)}{\psi_4(t,x) - \omega(t,x)}\\ =&\lim_{|x|\to+\infty} \frac{(\nu(t,x))^{-1}(\psi_1(t,x))^2(\psi_6(t,x))^2}{\psi_4(t,x) - \omega(t,x)} = \lim_{|x|\to+\infty} \frac{\psi_7(t,x)\nu^{-1}}{\psi_4(t,x) - \omega(t,x)} = 0,\end{aligned} \tag{5.59}$$

uniformly with respect to t *in bounded intervals* $J \subset I$*, where the function* $\omega : I \times \mathbb{R}^d \to \mathbb{R}$ *is a function which bounds from above the quadratic form associated with the matrix*

$$Q^{1/2}(J_x b)^T Q^{-1/2} - \sum_{j=1}^d b_j(D_j Q^{1/2})Q^{-1/2} - \sum_{i,j=1}^d q_{ij}(D_{ij}Q^{1/2})Q^{-1/2}.$$

Moreover,

$$0 \le \sup_{t\in J}\limsup_{|x|\to+\infty} \frac{(\nu(t,x))^2 + (\psi_2(t,x))^2}{\psi_4(t,x) - \omega(t,x)} < +\infty, \tag{5.60}$$

$$\inf_{t\in J}\lim_{|x|\to+\infty} \frac{|x|\nu(t,x)\psi_1(t,x)}{b_0(t,x)} = 0, \tag{5.61}$$

$$\inf_{t\in J}\liminf_{|x|\to+\infty} \frac{|x|(\nu(t,x))^2}{b_0(t,x)} > -\infty, \tag{5.62}$$

for any J *as above.*

Remark 5.19. Note that, in the particular case when Q is independent of x (e.g., when $Q = \mathrm{Id}$) the matrix $\mathcal{M}$ reduces to the matrix $J_x b$. Therefore, we are assuming a bound on the growth as $|x| \to +\infty$ of the quadratic form associated with the matrix $J_x b$, i.e., we are assuming a dissipativity condition on the diagonal part of the drift of $\mathbf{A}$. In the scalar case, this is an hypothesis typically assumed in the literature to prove gradient estimates both in the autonomous and nonautonomous setting. See e.g., [4, 14, 15, 60, 66, 72].

Here, we provide an example of operator $\mathbf{A}$ whose coefficients satisfy Hypotheses 5.13 and 5.18.

Example 5.20. *We consider the operator* $\mathbf{A}$ *with coefficients*

$$\begin{aligned} q_{ij}(t,x) &= q(t)q_{ij}(1+|x|^2)^l, \quad i,j=1,\dots,m,\\ (b(t,x))_i &= -x_i b(t)(1+|x|^2)^p, \quad i=1,\dots,d,\\ (\tilde{B}_i(t,x))_{jk} &= \tilde{b}(t)(\tilde{B}_i)_{jk}(1+|x|^2)^r, \quad i=1,\dots,d,\ j,k=1,\dots,m,\\ (C(t,x))_{jk} &= -c(t)C_{jk}(1+|x|^2)^s, \quad j,k=1,\dots,m, \end{aligned}$$

where l,p,r,s are positive numbers. We assume that the following conditions hold true:

- *$q\in C^1_b(I)$ and there exists a positive constant $\bar{q}$ such that $q(t)\geq \bar{q}$, for any $t\in I$;*
- *$b,\tilde{b},c\in C_b(I)$ and there exists a positive constant D such that $b(t),c(t)\geq E$, for any $t\in I$;*
- *the matrices $Q=[q_{jk}]$ and $C=[C_{jk}]$ are positive definite, i.e., there exists a positive constant ν_0 such that $\langle Q\xi,\xi\rangle,\langle C\eta,\eta\rangle\geq \nu_0|\xi|^2$, for any $\xi\in\mathbb{R}^d$ and $\eta\in\mathbb{R}^m$.*

Under above assumptions, in Hypotheses 5.18$(ii)-(iii)$ *we can choose as b_0,K_0,ψ_i, $i=1,\dots,4$ the following functions:*

$$\begin{aligned} b_0(t,x) &:= -E|x|(1+|x|^2)^{p_2},\\ K_0(t) &:= \frac{q(t)|Q|}{\bar{q}\nu_0},\\ \psi_1(t,x) &:= l(1+|x|^2)^{-1/2},\\ \psi_2(t,x) &:= 2s\sqrt{q(t)}\|c\|_\infty \max_{j,k=1,\dots,m}|C_{jk}|(1+|x|^2)^{l/2+s-1/2},\\ \psi_3(t,x) &:= 2r\|\tilde{b}\|_\infty \max_{i=1,\dots,d,j,k=1,\dots,m}|(\tilde{B}_i)_{jk}|(1+|x|^2)^{r-1/2},\\ \psi_4(t,x) &:= \nu_0 E(1+|x|^2)^s,\\ \psi_5(t,x) &:= \frac{|q'(t)|}{\bar{q}},\\ \psi_6(t,x) &:= \|q\|_\infty|Q|(1+|x|^2)^l,\\ \psi_7(t,x) &\leq \|\tilde{b}\|_\infty \sup_{i=1,\dots,d}|\tilde{B}_i|(1+|x|^2)^r, \end{aligned}$$

for any $t\in I$ and $x\in\mathbb{R}^d$. Moreover, long but straightforward computations show that

$$\begin{aligned} \omega(x) \leq{}& -b(t)(1+|x|^2)^p - l\nu_0\bar{q}(1+|x|^2)^{l-2}(d+(2l+d-1)|x|^2)\\ &+ lb(t)\sum_{j=1}^d x_j^2(1+|x|^2)^{p-1}. \end{aligned}$$

Finally, Hypotheses 5.13(i), 5.13(iii) *and* 5.18(iv) *are satisfied if we assume that the following inequalities hold:*

$$r \le l/2, \quad l < 1, \quad p > \max\{2l, l + 2s - 1\}.$$

Now, we prove a weighted gradient estimate satisfied by $\mathbf{G}(t,s)\mathbf{f}$ which allows us to deduce that the first integral in (5.53) is well defined.

Proposition 5.21. *Assume that Hypotheses* 5.13 *and* 5.18 *are satisfied. Then, for any* $j = 1, \dots, d$, *the function* $(t,x) \mapsto Q^{1/2}(t,x)\nabla_x u_j(t,x)$ *is continuous and bounded in* $J \times \mathbb{R}^d$ *for any* $J \Subset (s,+\infty)$. *Moreover, for any* $T > s \in I$ *there exists a positive constant* $C = C(s,T)$ *such that*

$$(t-s)\|Q^{1/2}(t,\cdot)(J_x\mathbf{G}(t,s)\mathbf{f})^T\|_\infty^2 \le C\|\mathbf{f}\|_\infty^2, \tag{5.63}$$

for any $t \in (s,T]$ *and* $\mathbf{f} \in C_b(\mathbb{R}^d, \mathbb{R}^m)$.

Proof. To simplify the notation we set $\mathbf{u} := \mathbf{G}(\cdot,s)\mathbf{f}$, $\mathbf{u}_n = \mathbf{G}_n(t,s)\mathbf{f}_n$, where $\mathbf{G}_n(t,s)$ is the evolution operator associated with the realization of the operator $\mathbf{A}(t)$ in $C_b(B(n),\mathbb{R}^m)$ with homogenous Dirichlet boundary conditions Further, we denote by $u_{n,j}$ the j-th component of $\mathbf{u}_n$. Finally, we set $\mathcal{F}_n := \sum_{i=1}^d \sum_{j=1}^m |Q^{1/2}\nabla_x(D_i u_{n,j})|^2$, $\mathcal{G}_n := \sum_{j=1}^m |Q^{1/2}\nabla_x u_{j,n}|^2$ and, throughout the proof, we denote by c a positive constant, which may vary from line to line, may depend on s and T and is independent of n. Let us consider the function $v_n := |\mathbf{u}_n|^2 + a(\cdot - s)\eta_n^2\mathcal{G}_n$, where a is positive parameter to be fixed later on, $\eta_n(x) = \eta(|x|/n)$ for any $x \in \mathbb{R}^d$ and $\eta(t) = \chi_{[0,1/2]}(t) + \exp\left(-\frac{(4t-2)^3}{1-(4t-2)^3}\right)\chi_{(1/2,3/4)}(t)$ for any $t \in \mathbb{R}$. Clearly, $\eta_n \in C_c^2(\mathbb{R}^d)$ and $\chi_{B(n/2)} \le \eta_n \le \chi_{B(3n/4)}$. Easy computations show that $D_i\eta_n(x) = -x_i\eta_n(x)\mathcal{K}_n(x)$ for any $x \in \mathbb{R}^d$, where

$$\mathcal{K}_n(x) = \frac{12(4|x|/n - 2)^2}{|x|n\left(1 - (4|x|/n - 2)^3\right)^2}\chi_{[\frac{n}{2},\frac{3}{4}n)}(|x|), \qquad x \in \mathbb{R}^d.$$

Moreover, as it is easy to see computing the first and the second order derivatives of η_n, we have

$$(a)\ |Q\nabla\eta_n| \le cn^2\nu\mathcal{K}_n\eta_n, \qquad (b)\ |\mathrm{Tr}(QD^2\eta_n)| \le c\nu, \tag{5.64}$$

Long but straightforward computations show that

$$\begin{aligned} D_t v_n =& \mathcal{A}|\mathbf{u}_n|^2 + (a\eta_n^2 - 2)\mathcal{G}_n + 2\sum_{i=1}^d \langle \tilde{B}_i\mathbf{u}_n, D_i\mathbf{u}_n\rangle + 2\langle C\mathbf{u}_n, \mathbf{u}_n\rangle \\ &+ 2(\cdot - s)a\eta_n^2 \sum_{j=1}^m \langle D_tQ\nabla_x u_{n,j}, \nabla_x u_{n,j}\rangle \\ &+ 2(\cdot - s)a\eta_n^2 \sum_{i,k=1}^d D_{ik}^2 u_{n,j}\langle Q^{1/2}\nabla_x q_{ik}, Q^{1/2}\nabla_x u_{n,j}\rangle \end{aligned}$$

$$
\begin{aligned}
&+ 2(\cdot - s)a\eta_n^2 \sum_{i,k=1}^{d} q_{ik}\langle Q^{1/2}\nabla_x D^2_{ik} u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle \\
&+ 2(\cdot - s)a\eta_n^2 \sum_{i=1}^{d}\sum_{j=1}^{m} \langle Q^{1/2}(J_x b)^T \nabla_x u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle \\
&+ 2(\cdot - s)a\eta_n^2 \sum_{i=1}^{d}\sum_{j,k=1}^{m} D_i u_{n,k}\langle Q^{1/2}\nabla_x (\tilde{B}_i)_{jk}, Q^{1/2}\nabla_x u_{n,j}\rangle \\
&+ 2(\cdot - s)a\eta_n^2 \sum_{i=1}^{d}\sum_{j=1}^{m} b_i\langle Q^{1/2}\nabla_x D_i u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle \\
&+ 2(\cdot - s)a\eta_n^2 \sum_{i=1}^{d}\sum_{j,k=1}^{m} (\tilde{B}_i)_{jk}\langle Q^{1/2}\nabla_x D_i u_{n,k}, Q^{1/2}\nabla_x u_{n,j}\rangle \\
&+ 2(\cdot - s)a\eta_n^2 \sum_{j,k=1}^{m} u_{n,k}\langle Q^{1/2}\nabla_x C_{jk}, Q^{1/2}\nabla_x u_{n,j}\rangle \\
&+ 2(\cdot - s)a\eta_n^2 \sum_{j,k=1}^{m} C_{jk}\langle Q^{1/2}\nabla_x u_{n,k}, Q^{1/2}\nabla_x u_{n,j}\rangle.
\end{aligned}
$$

Then we consider the spacial derivatives; we get

$$
\begin{aligned}
\langle b, \nabla_x v_n\rangle =& \langle b, \nabla_x |\mathbf{u}_n|^2\rangle + 2(\cdot - s)a\eta_n\langle b, \nabla\eta_n\rangle \mathcal{G}_n \\
&+ 2(\cdot - s)a\eta_n^2 \sum_{i=1}^{d}\sum_{j=1}^{m} b_i\langle D_i Q^{1/2}\nabla_x u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle \\
&+ 2(\cdot - s)a\eta_n^2 \sum_{i=1}^{d}\sum_{j=1}^{m} b_i\langle Q^{1/2}\nabla_x D_i u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Tr}(QD^2 v_n) =& \mathrm{Tr}(QD^2_x|\mathbf{u}_n|^2) + 4(\cdot - s)a\eta_n^2 \sum_{i,k=1}^{d}\sum_{j=1}^{m} q_{ik}\langle D_i Q^{1/2}\nabla_x D_k u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle \\
&+ 2(\cdot - s)a\left(\langle Q\nabla\eta_n, \nabla\eta_n\rangle + \eta_n \mathrm{Tr}(QD^2\eta_n)\right)\mathcal{G}_n \\
&+ 8(\cdot - s)a\eta_n \sum_{i,k=1}^{d}\sum_{j=1}^{m} q_{ik} D_i\eta_n\langle D_k Q^{1/2}\nabla_x u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle \\
&+ 8(\cdot - s)a\eta_n \sum_{i,k=1}^{d}\sum_{j=1}^{m} q_{ik} D_i\eta_n\langle Q^{1/2}\nabla_x D_k u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle \\
&+ 4(\cdot - s)a\eta_n^2 \sum_{i,k=1}^{d}\sum_{j=1}^{m} q_{ik}\langle D_i Q^{1/2}\nabla_x u_{n,j}, Q^{1/2}\nabla_x D_k u_{n,j}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(\cdot - s)a\eta_n^2 \sum_{i,k=1}^{d}\sum_{j=1}^{m} q_{ik}\langle D^2_{ik}Q^{1/2}\nabla_x u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle\\
&+ 2(\cdot - s)a\eta_n^2 \sum_{i,k=1}^{d}\sum_{j=1}^{m} q_{ik}\langle Q^{1/2}\nabla_x D^2_{ik}u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle\\
&+ 2(\cdot - s)a\eta_n^2 \sum_{i,k=1}^{d}\sum_{j=1}^{m} q_{ik}\langle D_i Q^{1/2}\nabla_x u_{n,j}, D_k Q^{1/2}\nabla_x u_{n,j}\rangle\\
&+ 2(\cdot - s)a\eta_n^2 \sum_{i,k=1}^{d}\sum_{j=1}^{m} q_{ik}\langle Q^{1/2}\nabla_x D_i u_{n,j}, Q^{1/2}\nabla_x D_k u_{n,j}\rangle.
\end{aligned}
$$

Hence, v_n is the unique classical solution to the Cauchy problem

$$
\begin{cases}
D_t v_n(t,x) - A v_n(t,x) = g_n(t,x), & t\in(s,T], \quad x\in B(n),\\
v_n(t,x) = 0, & t\in(s,T], \quad x\in\partial B(n),\\
v_n(s,x) = |\mathbf{f}(x)|^2, & \qquad\qquad x\in\overline{B(n)},
\end{cases}
$$

where $g_n = \sum_{i=1}^{5} g_{i,n}$ with

$$
\begin{aligned}
g_{1,n} =& - 2\left[1 + (\cdot - s)a\langle Q\nabla\eta_n, \nabla\eta_n\rangle\right]\mathcal{G}_n\\
&- 2(\cdot - s)a\eta_n^2 \sum_{i,k=1}^{d}\sum_{j=1}^{m} q_{ik}\langle Q^{1/2}\nabla_x D_i u_{n,j}, Q^{1/2}\nabla_x D_k u_{n,j}\rangle,\\
g_{2,n} =& 2(\cdot - s)a\eta_n^2 \sum_{j=1}^{m}\langle \mathcal{M}Q^{1/2}\nabla_x u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle\\
&- 2(\cdot - s)a\eta_n^2 \sum_{i,k=1}^{d}\sum_{j=1}^{m} q_{ik}\langle D_i Q^{1/2}\nabla_x u_{n,j}, D_k Q^{1/2}\nabla_x u_{n,j}\rangle\\
&+ 2(\cdot - s)a\eta_n^2 \sum_{j,k=1}^{m} C_{jk}\langle Q^{1/2}\nabla_x u_{n,k}, Q^{1/2}\nabla_x u_{n,j}\rangle,\\
g_{3,n} =& - 2(\cdot - s)a\eta_n \mathrm{Tr}(QD^2\eta_n)\mathcal{G}_n - 2(\cdot - s)a\eta_n\langle b, \nabla\eta_n\rangle\mathcal{G}_n\\
&- 8(\cdot - s)a\eta_n \sum_{k=1}^{d}\sum_{j=1}^{m}(Q\nabla\eta_n)_k\langle D_k Q^{1/2}\nabla_x u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle\\
&- 8(\cdot - s)a\eta_n \sum_{k=1}^{d}\sum_{j=1}^{m}(Q\nabla\eta_n)_k\langle Q^{1/2}\nabla_x D_k u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle,
\end{aligned}
$$

$$
\begin{aligned}
g_{4,n} =&2(\cdot - s)a\eta_n^2 \sum_{i,k=1}^{d}\sum_{j=1}^{m} D_{ik}^2 u_{n,j}\langle Q^{1/2}\nabla_x q_{ik}, Q^{1/2}\nabla_x u_{n,j}\rangle\\
&- 4(\cdot - s)a\eta_n^2 \sum_{i,k=1}^{d}\sum_{j=1}^{m} q_{ik}\langle D_i Q^{1/2}\nabla_x D_k u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle\\
&- 4(\cdot - s)a\eta_n^2 \sum_{i,k=1}^{d}\sum_{j=1}^{m} q_{ik}\langle D_i Q^{1/2}\nabla_x u_{n,j}, Q^{1/2}\nabla_x D_k u_{n,j}\rangle,
\end{aligned}
$$

$$
\begin{aligned}
g_{5,n} =&2\sum_{i=1}^{d}\langle \tilde{B}_i \mathbf{u}_n, D_i \mathbf{u}_n\rangle + 2\langle C\mathbf{u}_n, \mathbf{u}_n\rangle + (\cdot - s)a\eta_n^2 \sum_{j=1}^{m}\langle D_t Q^{1/2}\nabla_x u_{n,j}, Q^{1/2}\nabla_x u_{n,j}\rangle\\
&+ a\eta_n^2 \mathcal{G}_n + 2(\cdot - s)a\eta_n^2 \sum_{i=1}^{d}\sum_{j,k=1}^{m} (\tilde{B}_i)_{jk}\langle Q^{1/2}\nabla_x D_i u_{n,k}, Q^{1/2}\nabla_x u_{n,j}\rangle\\
&+ 2(\cdot - s)a\eta_n^2 \sum_{i=1}^{d}\sum_{j,k=1}^{m} D_i u_{n,k}\langle Q^{1/2}\nabla_x (\tilde{B}_i)_{jk}, Q^{1/2}\nabla_x u_{n,j}\rangle\\
&+ 2(\cdot - s)a\eta_n^2 \sum_{j,k=1}^{m} u_{n,k}\langle Q^{1/2}\nabla_x C_{jk}, Q^{1/2}\nabla_x u_{n,j}\rangle.
\end{aligned}
$$

Let us estimate the function g_n. Recalling that, for any pair of nonnegative definite matrices M_1 and M_2, it holds that $\mathrm{Tr}(M_1M_2) \geq \lambda_{\min}(M_1)\mathrm{Tr}(M_2)$, where $\lambda_{\min}(M_1)$ is the minimum eigenvalue of M_1, we conclude that $g_{1,n} \leq -2\mathcal{G}_n - 2(\cdot - s)a\eta_n^2\nu\mathcal{F}_n$.

The assumptions on the matrices $\mathcal{M}$ and C allow us to estimate $g_{2,n} \leq 2(\cdot - s)a\eta_n^2(\omega - \psi_4)\mathcal{G}_n$.

Let us now consider the function $g_{3,n}$. Using both the estimate in (5.64) and and Young inequality we get

$$
\begin{aligned}
\left|\sum_{k=1}^{3} g_{3,k,n}\right| \leq& 2(\cdot - s)ac\eta_n\nu\mathcal{G}_n + 2(\cdot - s)a\mathcal{K}_n\eta_n^2\langle b(t,x), x\rangle\mathcal{G}_n\\
&+ 8(\cdot - s)ac\nu\eta_n^2 n^2\mathcal{K}_n \sum_{j=1}^{m}\sum_{k=1}^{d} |D_k Q^{1/2}Q^{-1/2}||Q^{1/2}\nabla_x u_j|^2\\
\leq& \frac{\cdot - s}{\varepsilon}ac\mathcal{G}_n + (\cdot - s)\varepsilon ac\eta_n^2\nu^2\mathcal{G}_n + (\cdot - s)an\mathcal{K}_n\eta_n^2(b_0 + 8cn\nu\psi_1)\mathcal{G}_n.
\end{aligned}
$$

Finally, to estimate $g_{3,4,n}$ we observe that, using (5.64)(a) and the estimate $\mathcal{K}_n\eta_n \leq cn^{-2}\eta_n^{1/3}$, we get $|(Q\nabla_x\eta_n)_k|^2 \leq cn^4\nu^2\mathcal{K}_n^2\eta_n^2 \leq cn^2\nu^2\mathcal{K}_n\eta_n^{4/3}$. Using this estimate, (5.64)(a) and the estimate

$$
\alpha\beta\gamma \leq \frac{1}{4\varepsilon}(\alpha^4 + \beta^4) + \frac{1}{2}\varepsilon\gamma^2,
$$

which holds for any $\varepsilon > 0$, with

$$\begin{aligned}\alpha_{j,k} &= a^{3/8}\eta_n^{1/6}|(Q\nabla\eta_n)_k|^{1/2}|Q^{1/2}\nabla_x u_j|^{1/2},\\ \beta_j &= a^{1/8}|Q^{1/2}\nabla_x u_j|^{1/2},\\ \gamma_{j,k} &= \sqrt{a}\eta_n^{5/6}|(Q\nabla_x\eta_n)_k|^{1/2}|Q^{1/2}\nabla_x D_k u_j|,\end{aligned}$$

to get

$$\begin{aligned}|g_{3,4,n}| \le& 8(\cdot - s)\sum_{j=1}^m\sum_{k=1}^d \alpha_{jk}\beta_j\gamma_{jk}\\ \le& 8(\cdot - s)\sum_{j=1}^m\sum_{k=1}^d\left(\frac{1}{4\varepsilon}\beta_j^4 + \frac{1}{4\varepsilon}\alpha_{jk}^4 + \frac{1}{2}\varepsilon\gamma_{jk}^2\right)\\ =& \frac{2}{\varepsilon}(\cdot - s)d\sqrt{a}\mathcal{G}_n + 4(\cdot - s)\varepsilon a\eta_n^{5/3}\sum_{j=1}^m\sum_{k=1}^d |(Q\nabla_x\eta_n)_k||Q^{1/2}\nabla_x D_k u_j|^2\\ &+ \frac{2}{\varepsilon}(\cdot - s)a^{3/2}\eta_n^{2/3}\sum_{j=1}^m\sum_{k=1}^d |(Q\nabla_x\eta_n)_k|^2|Q^{1/2}\nabla_x u_j|^2\\ \le& \frac{2}{\varepsilon}(\cdot - s)\sqrt{a}\left(d + acn^2\mathcal{K}_n\eta_n^2\nu^2\right)\mathcal{G}_n + 4(\cdot - s)\varepsilon ac\eta_n^2\nu\mathcal{F}_n.\end{aligned}$$

Similarly, we split $g_{4,n} = g_{4,1,n} + g_{4,2,n} + g_{4,3,n}$. To estimate $g_{4,1,n}$ we observe that $\sum_{i,k=1}^d \nabla_x q_{ik} D_{ik} u_j = 2\sum_{i,k=1}^d \nabla_x q_{ik}^{1/2}(Q^{1/2}\nabla_x D_i u_j)_k$. Hence,

$$\begin{aligned}&\left|\sum_{i,k=1}^d\sum_{j=1}^m D_{ik}^2 u_j\langle Q^{1/2}\nabla_x q_{ik}, Q^{1/2}\nabla_x u_j\rangle\right|\chi_{[n/2,3n/4)}\\ \le& 2\left|\sum_{i,k=1}^d\sum_{j=1}^m\langle Q^{1/2}\nabla_x Q_{ik}^{1/2}(Q^{1/2}\nabla_x D_i u_j)_k, Q^{1/2}\nabla_x u_j\rangle\right|\chi_{[n/2,3n/4)}\\ \le& c\psi_1\psi_6\sum_{i,k=1}^d\sum_{j=1}^m |(Q^{1/2}\nabla_x D_i u_j)_k||Q^{1/2}\nabla_x u_j|\\ \le& \varepsilon\nu\mathcal{F}_n + \varepsilon^{-1}c\nu^{-1}\psi_1^2\psi_6^2\mathcal{G}_n.\end{aligned}$$

To estimate the other two terms, we write $D_i Q^{1/2}\nabla_x = (D_i Q^{1/2}Q^{-1/2})Q^{1/2}\nabla_x$ and argue similarly. Collecting everything together, we get

$$|g_{4,n}| \le (\cdot - s)a\varepsilon\eta_n^2\nu\mathcal{F}_n + \frac{\cdot - s}{\varepsilon}ac\eta_n^2\nu^{-1}\psi_1^2\psi_6^2\mathcal{G}_n.$$

Finally, taking Hypothesis 5.13 into account and writing

$$\langle D_t Q\nabla_x u_j, Q^{1/2}\nabla_x u_j\rangle = \langle (Q^{-1/2}D_t Q Q^{-1/2})Q^{1/2}\nabla_x u_j, Q^{1/2}\nabla_x u_j\rangle$$

we can estimate

$$
\begin{aligned}
|g_{5,n}| \leq & c\left(1+\frac{\cdot-s}{\varepsilon}a\right)|\mathbf{u}_n|^2+2(\cdot-s)a\varepsilon\eta_n^2\nu\mathcal{F}_n+\left(\frac{1}{2}+a\right)\mathcal{G}_n \\
& +(\cdot-s)ac\eta_n^2\left(\psi_3+\varepsilon\psi_2^2+\psi_5+\frac{\psi_7\nu^{-1}}{\varepsilon}\right)\mathcal{G}_n.
\end{aligned}
$$

Then, collecting all the terms, we get that

$$
g_n \leq c\left(1+\frac{t-s}{\varepsilon}a\right)|\mathbf{u}_n|^2+2(\cdot-s)a\eta_n^2\nu(c\varepsilon-1)\mathcal{F}_n+\mathcal{I}\mathcal{G}_n,
$$

where

$$
\begin{aligned}
\mathcal{I}(t,x)= & -\frac{3}{2}+a+2\frac{t-s}{\varepsilon}d\sqrt{a}+\frac{t-s}{\varepsilon}ac \\
& +(t-s)a(\eta_n(x))^2\mathcal{I}_1(t,x)+(t-s)a(\eta_n(x))^2n\mathcal{K}_n(x)\mathcal{I}_2(t,x),
\end{aligned}
$$

for any $(t,x)\in[s,T]\times\mathbb{R}^d$ and

$$
\begin{aligned}
\mathcal{I}_1(t,x)= & 2\omega(t,x)-2\psi_4(t,x)+\frac{c}{\varepsilon}(\nu(t,x))^{-1}(\psi_1(t,x))^2(\psi_6(t,x))^2 \\
& +c\left(\varepsilon(\nu(t,x))^2+\psi_3(t,x)+\varepsilon(\psi_2(t,x))^2+\psi_5(t,x)+\frac{\psi_7(t,x)\nu^{-1}}{\varepsilon}\right), \\
\mathcal{I}_2(t,x)= & b_0(t,x)+c|x|\nu(t,x)\psi_1(t,x)+\frac{\sqrt{a}}{\varepsilon}c|x|(\nu(t,x))^2.
\end{aligned}
$$

Clearly, the coefficient in front of $\mathbf{u}_n$ is bounded in $[s,T]$ for any choice of a and ε. Moreover, the coefficient in front of $\mathcal{F}_n$ tends to $-\nu$ as $\varepsilon\to 0^+$. Therefore, there exists $\varepsilon_0>0$ such that this coefficients is negative for any $\varepsilon\in(0,\varepsilon_0)$ and any $a>0$.

Let us consider the term $\mathcal{I}$. As far as $\mathcal{I}_1$ is concerned, we get

$$
\begin{aligned}
\mathcal{I}_1(t,x)= & 2\omega(t,x)-2\psi_4(t,x)+\frac{c}{\varepsilon}(\nu(t,x))^{-1}(\psi_1(t,x))^2(\psi_6(t,x))^2+\frac{c\psi_7(t,x)\nu^{-1}}{\varepsilon} \\
& +c\left\{\varepsilon[(\nu(t,x))^2+(\psi_2(t,x))^2]+\psi_3(t,x)+\psi_5(t,x)\right\},
\end{aligned}
$$

From (5.59) and (5.60) it follows immediately that $\mathcal{I}_1$ is bounded from above in $[s,T]\times\mathbb{R}^d$ provided $\varepsilon>0$ is properly fixed. Finally, taking a small enough and using conditions (5.61) and (5.62) we deduce that $\mathcal{I}_2$ is nonpositive as well.

Summing up, we have shown that $|g_n|\leq c\left(|\mathbf{u}_n|^2+a(\cdot-s)\eta_n^2\mathcal{G}_n\right)=cv_n$ in $[s,T]\times\mathbb{R}^d$, and we can invoke the classical maximum principle to infer that $|v_n|\leq c\|\mathbf{f}\|_\infty$, i.e., $\|\mathbf{G}_n(t,s)\mathbf{f}\|_\infty+(t-s)^{1/2}\|\eta_nQ^{1/2}(J_x\mathbf{G}_n(t,s)\mathbf{f})^T\|_\infty\leq c\|\mathbf{f}\|_\infty$ for any $t\in(s,T]$. As the proof of Theorem 5.9 shows, $\mathbf{G}_n(t,s)\mathbf{f}$ converges to $\mathbf{G}(t,s)\mathbf{f}$ in $C^2(B(M))$ for any $M>0$. Therefore, letting $n\to+\infty$ in the previous estimate for $\mathbf{G}_n(t,s)\mathbf{f}$, inequality (5.63) follows at once.

□

Now, we show that, under suitable assumptions, it is possible to write a component of the vector-valued evolution operator $\mathbf{G}(t,s)$ by means of the scalar evolution operator $G(t,s)$. A sufficient condition in order to prove this fact is the following additional assumption.

Hypotheses 5.22. *There exists* $\bar{k} \in \{1,\dots,m\}$ *such that all the entries of* $row_{\bar{k}}C$ *belong to* $C_b(I \times \mathbb{R}^d;\mathbb{R}^m)$.

Proposition 5.23. *Assume that Hypotheses* 5.13*,* 5.18 *and* 5.22 *hold true. Then, for any* $(t,s) \in \Lambda_I$*,* $x \in \mathbb{R}^d$ *and any* $\mathbf{f} \in B_b(\mathbb{R}^d;\mathbb{R}^m)$*, formula* (5.53) *holds true.*

Proof. Let us fix $T > s \in I$ and $x \in \mathbb{R}^d$. We prove (5.53) for any $t \in (s,T]$. The arbitrariness of $T > s$ will allow us to complete the proof. Being rather long, we split the proof into four steps and, to simplify the notation, as usually we set $\mathbf{u} := \mathbf{G}(\cdot,s)\mathbf{f}$ and, for any $n \in \mathbb{N}$, $\mathbf{u}_n := \mathbf{G}_n(\cdot,s)\mathbf{f}$, where $\mathbf{G}_n(t,s)$ is the evolution operator in $C_b(B(n))$ associated with the operator $\mathbf{A}$ with homogeneous Dirichlet boundary conditions. Finally, $\mathcal{A} = \mathrm{Tr}(QD_x^2) + \langle b, \nabla_x\rangle$.

Step 1. Fix $\mathbf{f} \in C_c^{2+\alpha}(\mathbb{R}^d;\mathbb{R}^m)$ and let n_0 be the smallest integer such that $\mathrm{supp}\,\mathbf{f} \subset B(n_0)$. For any $n \in \mathbb{N} \cap [n_0,+\infty)$, we consider the classical solution $\mathbf{w}_n$ of the Cauchy-Dirichlet system

$$\begin{cases} D_t\mathbf{v}(t,x) = (\tilde{\mathbf{A}}\mathbf{v})(t,x) + \mathbf{\Phi}_n(t,x), & t \in (s,T), \quad x \in B(n), \\ \mathbf{v}(t,x) = 0, & t \in (s,T), \quad x \in \partial B(n), \\ \mathbf{v}(s,x) = \mathbf{f}(x), & \qquad\qquad\quad x \in B(n), \end{cases} \tag{5.65}$$

where, as in the proof of Theorem 5.11, $\tilde{\mathbf{A}}$ is the diagonal operator with all the components which coincide with the operator $\mathcal{A}$, $\Phi_{n,j} = \eta_n \sum_{i=1}^d \langle row_j\tilde{B}_i, D_i\mathbf{u}_n\rangle + \langle row_jC, \mathbf{u}_n\rangle$, for any $j = 1,\dots,m$, $(\eta_n) \subset C_c^\infty(\mathbb{R}^d)$ is the same sequence of cut-off functions considered in the proof of Proposition 5.21 and $\mathbf{u}_n$ is the classical solution to the Cauchy-Dirichlet system (5.36).

We claim that $\mathbf{u}$ is the limit of sequence $(\mathbf{w}_n)$.

Since $\mathbf{u}_n \in C^{1+\alpha/2,2+\alpha}((s,T) \times B(n);\mathbb{R}^m)$ (see [63, Thm. VII.4.1]) and Hypotheses 5.1(i) and 5.13(i) are satisfied, $\mathbf{\Phi}_n$ belongs to $C^{\alpha/2,\alpha}((s,+\infty) \times B(n);\mathbb{R}^m)$. Hence, from [63, Thm. IV.5.5] it follows that there exists a unique classical solution $\mathbf{w}_n \in C^{1+\alpha/2,2+\alpha}((s,T) \times B(n))$ to the problem (5.65). Moreover, the variation-of-constants formula yields

$$w_{n,k}(t,x) = (G_n(t,s)f_k)(x) + \int_s^t (G_n(t,r)\Phi_{n,k}(r,\cdot))(x)dr, \tag{5.66}$$

for any $t \in (s,T)$, any $x \in \overline{B(n)}$ and $k = 1,\dots,m$, where $G_n(t,s)$ denotes the evolution operator associated to $\mathcal{A}(t)$ in $C_b(B(n))$. Recalling that $\mathbf{u}_n$ is bounded in $C_{\rm loc}^{1+\alpha/2,2+\alpha}(K)$ for any compact $K \subset (s,T) \times B(n)$ (see the proof of Theorem 5.3, the function $\mathbf{\Phi}_n$ belongs to $C_{\rm loc}^{1+\alpha/2,2+\alpha}((s,T) \times B(n))$. Hence, by the classical Schauder estimates in Theorem 5.3, there exists a positive constant C independent of n such that $\|\mathbf{w}_n\|_{C^{1+\alpha/2,2+\alpha}(K;\mathbb{R}^m)} \leq C$

for any compact set $K \subset (s,T)\times\mathbb{R}^d$ and n large enough. The Ascoli-Arzelà Theorem and a diagonal argument guarantee the existence of a subsequence $(\mathbf{w}_{n_j})$ which converges in $C^{1,2}((s+1/R,T)\times B(R);\mathbb{R}^m)$, for any $R>0$, to some function $\mathbf{w}\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,T)\times\mathbb{R}^d)$, as $j\to+\infty$. Since $\mathbf{u}_n$ converges to $\mathbf{u}$ in $C^{1,2}((s,s+1/R)\times B(R);\mathbb{R}^m)$ for any $R>0$ (see the proof of Theorem 5.9), we immediately obtain that $\mathbf{w}$ satisfies the equation $D_t\mathbf{v} = \tilde{\mathbf{A}}\mathbf{v}+\boldsymbol{\Phi}$ in $(s,T)\times\mathbb{R}^d$ where $\Phi_k = \sum_{i=1}^d\langle row_k\tilde{B}_i, D_i\mathbf{u}\rangle + \langle row_k C,\mathbf{u}\rangle$ for any $k=1,\dots,m$. To claim that $\mathbf{w}=\mathbf{u}$, it suffices to show that $\mathbf{w}$ can be extended by continuity at $t=s$, where it equals $\mathbf{f}$. For this purpose, we argue as in the proof of Theorem 5.9. We fix $M\in\mathbb{N}$ and a function $\vartheta\in C^\infty_c(\mathbb{R}^d)$ such that $\chi_{B(M-1)}\le\vartheta\le\chi_{B(M)}$. For any $n_j>M$ the function $\mathbf{v}_{n_j}=\vartheta\mathbf{w}_{n_j}$ belongs to $C^{1+\alpha/2,2+\alpha}((s,T)\times B(M);\mathbb{R}^m)$ and

$$\begin{cases} D_t\mathbf{v}_{n_j}(t,x) = (\tilde{\mathbf{A}}\mathbf{v}_{n_j})(t,x)+\mathbf{g}_{n_j}(t,x), & t\in(s,T],\quad x\in B(M),\\ \mathbf{v}_{n_j}(t,x)=0 & t\in(s,T],\quad x\in\partial B(M),\\ \mathbf{v}_{n_j}(s,x)=(\vartheta\mathbf{f})(x), & \qquad\qquad x\in\overline{B(M)},\end{cases}$$

where $\mathbf{g}_{n_j} = -\mathbf{w}_{n_j}\mathcal{A}\vartheta - 2J_x\mathbf{w}_{n_j}(Q\nabla\vartheta)+\vartheta\boldsymbol{\Phi}_{n_j}$, for any $n_j>M$. Clearly, $\mathbf{g}_{n_j}\in C^{\alpha/2,\alpha}((s,T)\times B(M))$. Therefore, we can represent $\mathbf{v}_{n_j}$ by means of the variation-of-constants formula

$$\mathbf{v}_{n_j}(t,x) = \mathbf{G}_M(t,s)(\vartheta\mathbf{f})(x)+\int_s^t(\mathbf{G}_M(t,r)\mathbf{g}_{n_j}(r,\cdot))(x)dr, \tag{5.67}$$

for any $t\in(s,T)$ and $x\in\overline{B(M)}$, where $\mathbf{G}_M(t,s)$ is the evolution operator associated to the realization of $\tilde{\mathbf{A}}$ in $C_b(B(M);\mathbb{R}^m)$ with homogeneous Dirichlet boundary conditions.

To estimate the sup-norm of the function $\mathbf{g}_{n_j}(r,\cdot)$, we begin by recalling that the proof of Proposition 5.21 shows that $\eta_n^2|Q^{1/2}(t,x)(J_x\mathbf{u}_n(t,x))^T|^2\le C(t-s)^{-1}\|\mathbf{f}\|_\infty$ for any $t\in(s,T)$, any $x\in B(n)$, any $n\in\mathbb{N}$ and some positive constant C, independent of t, n and $\mathbf{f}$. Arguing as in the proof of (5.70) we deduce that

$$\eta_n^2\bigg|\sum_{i=1}^d\langle row_j\tilde{B}_i(r,\cdot), D_i\mathbf{u}_n(r,\cdot)\rangle\bigg|^2\le Cd\Xi^2_{[s,T]}(r-s)^{-1}\|\mathbf{f}\|^2_\infty,$$

for any $r\in(s,T]$ and $j=1,\dots,m$. Moreover, since the evolution operator $G_n(t,s)$ is contractive, it follows that $\|\boldsymbol{\Phi}_{n_j}(r,\cdot)\|_{C_b(B(M))}\le c_M(r-s)^{-1/2}\|\mathbf{f}\|_\infty$ and, consequently, using (5.66) we deduce that $\|\mathbf{w}_{n_j}(r,\cdot)\|_{C_b(B(M))}\le c_M(r-s)^{1/2}\|\mathbf{f}\|_\infty$ for some positive constant c_M independent of j. Differentiating formula (5.66) with respect to x and using the interior Schauder estimates in Theorem 5.3, we obtain that

$$\begin{aligned}\|J_x\mathbf{w}_{n_j}(t,\cdot)\|_{C_b(B(M))}&\le c\left((t-s)^{-1/2}\|\mathbf{f}\|_\infty+\int_s^t(t-r)^{-1/2}\|\boldsymbol{\Phi}_{n_j}(r,\cdot)\|_{C_b(B(M))}dr\right)\\ &\le c\|\mathbf{f}\|_\infty((t-s)^{-1/2}+\sqrt{\pi}),\end{aligned}$$

for any $t\in(s,T]$ and some positive constant c, independent of j. We have so proved that $|\mathbf{g}_{n_j}(t,x)|\le K_M(t-s)^{-1/2}\|\mathbf{f}\|_\infty$ for any $t\in(s,s+1)$, $x\in B(M)$ and some positive constant K_M independent of j. Arguing as above, we deduce that the integral term in (5.67) can be controlled from above by $c(t-s)^{1/2}\|\mathbf{f}\|_\infty$ for any $(t,x)\in(s,s+1)\times B(M)$ and some positive constant c, independent of t and $\mathbf{f}$.

Since $\mathbf{v}_{n_j}\equiv\mathbf{w}_{n_j}$ in $B(M-1)$, it follows that

$$|\mathbf{w}_{n_j}(t,x)-\mathbf{f}(x)|\le|(\mathbf{G}_M(t,s)(\vartheta\mathbf{f}))(x)-\mathbf{f}(x)|+K_M\sqrt{t-s}\,\|\mathbf{f}\|_\infty, \tag{5.68}$$

for any $t\in(s,s+1)$, $x\in B(M-1)$ and some positive constant K_M, independent of j. Thus, letting j tend to $+\infty$ in (5.68) we get

$$\|\mathbf{w}(t,\cdot)-\mathbf{f}\|_{C_b(B(M-1))}\le\|\mathbf{G}_M(t,s)(\vartheta\mathbf{f})-\vartheta\mathbf{f}\|_{C_b(B(M-1))}+K_M\sqrt{t-s}\,\|\mathbf{f}\|_\infty,$$

for any $t\in(s,s+1)$. This shows that $\mathbf{w}(t,\cdot)$ tends to $\mathbf{f}$ as $t\to s^+$, uniformly in $B(M-1)$ and, by the arbitrariness of $M\in\mathbb{N}$, we obtain that $\mathbf{w}$ can be the extended by continuity to $[s,T]\times\mathbb{R}^d$ by setting $\mathbf{w}(s,\cdot)=\mathbf{f}$. We have so proved that $\mathbf{u}=\mathbf{w}$.

Step 2. Here, we prove that there exists a positive constant c, independent of n, t and $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$ such that

$$\|\Phi_{n,\bar{k}}(r,\cdot)\|_{C_b(B(n))}\le c(r-s)^{-1/2}\|\mathbf{f}\|_\infty,\qquad t\in(s,T). \tag{5.69}$$

For this purpose, we observe that the proof of Proposition 5.21 shows that

$$\|\mathbf{u}_n(t,\cdot)\|_{C_b(B(n))}+\sqrt{t-s}\|\eta_nQ^{1/2}(J_x\mathbf{u}_n(t,\cdot))^T\|_{C_b(B(n))}\le c\|f\|_\infty,\qquad t\in(s,T),$$

for any $n\in\mathbb{N}$ and some positive constant c, independent of t, n and $\mathbf{f}$. It thus follows that $\|\langle row_{\bar{k}}C,\mathbf{u}\rangle\|_{C([s,T]\times\overline{B(n)})}\le c\|row_{\bar{k}}C\|_\infty\|\mathbf{f}\|_\infty$. Moreover,

$$\begin{aligned}\left|\sum_{i=1}^d\langle row_{\bar{k}}\tilde{B}_i(r,\cdot),D_i\mathbf{u}_n(r,\cdot)\rangle\right|^2&\le\sum_{i=1}^d|row_{\bar{k}}\tilde{B}_i(r,\cdot)|^2\sum_{i=1}^d|D_i\mathbf{u}_n(r,\cdot)|^2\\&\le md\Xi^2_{[s,T]}\nu|J_x\mathbf{u}_n(r,\cdot)|^2\\&\le md\Xi^2_{[s,T]}|Q^{1/2}(r,\cdot)(J_x\mathbf{u}_n(r,\cdot))^T|^2\\&\le Cmd\Xi^2_{[s,T]}(r-s)^{-1}\|\mathbf{f}\|^2_\infty,\end{aligned} \tag{5.70}$$

for any $r\in[s,T]$. Estimate (5.69) now follows at once.

Step 3. Here, we prove formula (5.53) for functions $\mathbf{f}\in C^{2+\alpha}_c(\mathbb{R}^d;\mathbb{R}^m)$. We observe that formula (5.66) yields that

$$w_{n_j,\bar{k}}(t,x)=(G_{n_j}(t,s)f_{\bar{k}})(x)+\int_s^t(G_{n_j}(t,r)\Phi_{n_j,\bar{k}}(r,\cdot))(x)dr, \tag{5.71}$$

for any $(t,x)\in(s,T)\times\overline{B(n_j)}$. To let j tend to $+\infty$ in (5.71), we begin by observing that, by [60, Thm. 2.2], $(G_{n_j}(t,s)f_{\bar{k}})(x)$ increases to $(G(t,s)f_{\bar{k}})(x)$ as $j\to\infty$ for any $x\in\mathbb{R}^d$. As far as the convolution term in (5.71) is concerned, we claim that $G_{n_j}(t,r)\Phi_{n_j,\bar{k}}$ converges pointwise in $\mathbb{R}^d$ to $G(t,r)\Phi_{\mathbf{f},\bar{k}}$ as $j\to+\infty$. Indeed,

$$|G_{n_j}(t,r)\Phi_{n_j,\bar{k}}(r,\cdot)-G(t,r)\Phi_{\mathbf{f},\bar{k}}(r,\cdot)|$$

$$\leq G_{n_j}(t,r)|\Phi_{n_j,\bar{k}}(r,\cdot)-\Phi_{\mathbf{f},\bar{k}}(r,\cdot)|+|G_{n_j}(t,r)\Phi_{\mathbf{f},\bar{k}}(r,\cdot)-G(t,r)\Phi_{\mathbf{f},\bar{k}}(r,\cdot)|$$
$$\leq G(t,r)|\Phi_{n_j,\bar{k}}(r,\cdot)-\Phi_{\mathbf{f},\bar{k}}(r,\cdot)|+|G_{n_j}(t,r)\Phi_{\mathbf{f},\bar{k}}(r,\cdot)-G(t,r)\Phi_{\mathbf{f},\bar{k}}(r,\cdot)|, \tag{5.72}$$

for all $r\in(s,t]$. Clearly, $G_{n_j}(t,r)\Phi_{\mathbf{f},\bar{k}}(r,\cdot)$ converges to $G(t,r)\Phi_{\mathbf{f},\bar{k}}(r,\cdot)$ for any $r\in(s,t]$, as $j\to+\infty$ and, since $\Phi_{n_j,\bar{k}}(r,\cdot)$ converges to $\Phi_{\mathbf{f},\bar{k}}(r,\cdot)$ locally uniformly in $\mathbb{R}^d$ for any $r\in(s,T]$, $G(t,r)|\Phi_{n_j,\bar{k}}(r,\cdot)-\Phi_{\mathbf{f},\bar{k}}(r,\cdot)|$ converges to 0 locally uniformly in $\mathbb{R}^d$ for any $r\in(s,T]$ (see [60, Prop.3.1]). Therefore, from (5.72) we deduce that $G_{n_j}(t,r)\Phi_{n_j,\bar{k}}$ converges, locally uniformly in $\mathbb{R}^d$, to $G(t,r)\Phi_{\mathbf{f},\bar{k}}$ as $j\to+\infty$. Moreover, since $G_{n_j}(t,s)$ is a contractive evolution operator for any $j\in\mathbb{N}$, from (5.69) it follows that $\|G_{n_j}(t,r)\Phi_{n_j,\bar{k}}(r,\cdot)\|\leq c(r-s)^{-1/2}\|\mathbf{f}\|_\infty$ for any $r\in(s,t]$. Thus by the dominated convergence theorem we conclude that the integral in (5.71) converges to $\int_s^t(G(t,r)\Phi_{\mathbf{f},\bar{k}}(r,\cdot))(x)dr$ as $j\to+\infty$. Thus, letting $j\to+\infty$ in (5.71) we obtain (5.53) for any $\mathbf{f}\in C_c^{2+\alpha}(\mathbb{R}^d;\mathbb{R}^d)$.

Step 4. Now, we extend (5.53) to any $\mathbf{f}\in B_b(\mathbb{R}^d;\mathbb{R}^m)$. For this purpose, let us fix such a function $\mathbf{f}$ and a sequence $(\mathbf{f}_n)\in C_c^{2+\alpha}(\mathbb{R}^d;\mathbb{R}^m)$ converging to $\mathbf{f}$ pointwise almost everywhere in $\mathbb{R}^d$, as $n\to+\infty$, and satisfying $\|\mathbf{f}_n\|_\infty\leq\|\mathbf{f}\|_\infty$ for any $n\in\mathbb{N}$. By Step 3,

$$(\mathbf{G}(t,s)\mathbf{f}_n)_{\bar{k}}(x)=(G(t,s)\mathbf{f}_n)_{\bar{k}})(x)+\int_s^t(G(t,r)\Phi_{\mathbf{f}_n,\bar{k}}(r,\cdot))(x)\,dr, \tag{5.73}$$

for any $t\in(s,T]$, $x\in\mathbb{R}^d$ and $n\in\mathbb{N}$. From Corollary 5.12(ii), $(\mathbf{G}(r,s)\mathbf{f}_n)_{\bar{k}}$ and $G(r,s)f_{n,\bar{k}}$ converge, respectively, to $(\mathbf{G}(r,s)\mathbf{f})_{\bar{k}}$ and $G(r,s)f_{\bar{k}}$ in $C^2(B(R))$, for any $R>0$ and any $r\in(s,T)$, as $n\to+\infty$. Similarly, $\Phi_{\mathbf{f}_n,\bar{k}}(r,\cdot)$ converges locally uniformly in $\mathbb{R}^d$ to $\Phi_{\mathbf{f},\bar{k}}(r,\cdot)$ as $n\to+\infty$ for any $r\in(s,T)$. To conclude the proof, let us observe that, since the constant c in (5.69) is independent of n, we can estimate

$$\|G(t,r)\Phi_{\mathbf{f}_n,\bar{k}}(r,\cdot)\|_\infty\leq c(t-r)^{-1/2}\|\mathbf{f}_n\|_\infty\leq c(t-r)^{-1/2}\|\mathbf{f}\|_\infty,\qquad r\in(s,t).$$

Now, taking into account that $G(t,s)$ is a contractive evolution operator in $C_b(\mathbb{R}^d)$, we can let $n\to+\infty$ in (5.73), and the representation formula (5.53) is proved in its full generality. □

Proposition 5.24. *Assume that Hypotheses* 5.13*,* 5.18 *and* 5.22 *hold true and that* $\mathbf{G}(t,s)$ *is compact in* $C_b(\mathbb{R}^d;\mathbb{R}^m)$ *for any* $(t,s)\in\Lambda_{[a,b]}$ *and some interval* $[a,b]\subset I$*. Then* $G(t,s)$ *is compact in* $C_b(\mathbb{R}^d)$ *for any* $(t,s)\in\Lambda_{[a,b]}$.

Proof. Let $t>s\in[a,b]$. We consider a sequence $(f_n)\subset C_b(\mathbb{R}^d)$ such that $\|f_n\|_\infty\leq M$ for any $n\in\mathbb{N}$ and we set $\mathbf{f}_n=f_n\mathbf{e}_{\bar{k}}$. Let $s_0\in(s,t]$ satisfy $s_0-s\leq(8e^{\varepsilon\kappa_0^+(t-s)}\Xi_{[a,b]}\sqrt{dmC}M)^{-2}$, where ε and κ_0 are defined in Hypotheses 5.1 and $C=C(a,b)$ is the constant appearing in (5.63).

Since $\mathbf{G}(s_0,s)$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$, there exists a sequence $(\mathbf{G}(s_0,s)\mathbf{f}_{j_n^0})\subset(\mathbf{G}(s_0,s)\mathbf{f}_n)$ converging uniformly in $\mathbb{R}^d$ as $j\to+\infty$ to some function $\mathbf{g}_{s_0}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$. Formula (5.53) yields that

$$(G(t,s)f_{j_n^0})(x)=(\mathbf{G}(t,s)\mathbf{f}_{j_n^0,\bar{k}})(x)-\int_s^t(G(t,r)\Phi_{\mathbf{f}_{j_n^0},\bar{k}}(r,\cdot))(x)\,dr,$$

for any $x \in \mathbb{R}^d$ and any $n_j \in \mathbb{N}$. Clearly, $(\mathbf{G}(t,s)\mathbf{f}_{j_n^0})_{\bar{k}}$ converges uniformly to the $\bar{k}$-th component of $G(t,s_0)\mathbf{g}_{s_0}$. As far as the second term is concerned we have

$$\begin{aligned}&\int_s^t (G(t,r)\Phi_{\mathbf{f}_{j_n^0,\bar{k}}-\mathbf{f}_{j_m^0,\bar{k}}}(r,\cdot))(x)dr\\ &=\int_s^t (G(t,r)\langle row_{\bar{k}}\tilde{B}_i(r,\cdot), D_i\mathbf{G}(r,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0})\rangle)(x)dr\\ &\quad+\int_s^t (G(t,r)\langle row_{\bar{k}}C(r,\cdot), \mathbf{G}(r,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0})\rangle)(x)dr.\end{aligned}$$

The contractiveness of the evolution operator $G(t,s)$ implies that

$$\|G(t,r)\langle row_{\bar{k}}C(r,\cdot), \mathbf{G}(r,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0})\rangle\|_\infty \le \|row_{\bar{k}}C\|_\infty\|\mathbf{G}(r,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0})\|_\infty$$

Moreover, we have

$$\begin{aligned}&\int_s^t (G(t,r)\langle row_{\bar{k}}\tilde{B}_i(r,\cdot), D_i\mathbf{G}(r,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0})\rangle)(x)\,dr\\ &=\int_s^{s_0} (G(t,r)\langle row_{\bar{k}}\tilde{B}_i(r,\cdot), D_i\mathbf{G}(r,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0})\rangle)(x)\,dr\\ &\quad+\int_{s_0}^t (G(t,r)\langle row_{\bar{k}}\tilde{B}_i(r,\cdot), D_i\mathbf{G}(r,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0})\rangle)(x)\,dr\\ &=I_{1,j}+I_{2,j}.\end{aligned}$$

Using again the contractiveness of $G(t,s)$, taking (5.63) and the choice of s_0 into account, and arguing as in estimate (5.70) with $\mathbf{u}$ being replaced respectively by $\mathbf{G}(r,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0})$ and by $\mathbf{G}(r,s_0)\mathbf{G}(s_0,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0})$, we obtain

$$I_{1,j} \le e^{\varepsilon\kappa_0^+(t-s)}\Xi_{[a,b]}\sqrt{dmC}\|\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0}\|_\infty\int_s^{s_0}(r-s)^{-1/2}dr \le \frac{1}{2}.$$

and

$$I_{2,j} \le c\|\mathbf{G}(s_0,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0})\|_\infty,$$

where $c = 2e^{\varepsilon\kappa_0^+(t-s)}\Xi_{[a,b]}\sqrt{dmC}(b-a)^{1/2}$. Summing up we deduce that

$$\|G(t,s)(f_{j_n^0}-f_{j_m^0})\|_\infty \le \frac{1}{2}+\|(\mathbf{G}(t,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0}))_{\bar{k}}\|_\infty + c\|\mathbf{G}(s_0,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0})\|_\infty. \tag{5.74}$$

By the assumptions the last two terms in the right hand side of (5.74) vanishes as $j \to +\infty$. Therefore, there exists $N_0 \in \mathbb{N}$ such that $\|G(t,s)(f_{j_n^0}-f_{j_m^0})\|_\infty \le 1$ for any $n,m \ge N_0$.

Now, we fix $s_1 \in (s,t)$ such that $s-s_1 \le (16e^{\varepsilon\kappa_0^+(t-s)}\Xi_{[a,b]}\sqrt{dmC}M)^{-2}$. Since $G(s_1,s)$ is a compact operator, there exists a subsequence $(f_{j_n^1}) \subset (f_{j_n^0})$ such that $\mathbf{G}(s_1,s)\mathbf{f}_{j_n^1}$

converges uniformly in $\mathbb{R}^d$ to some function $\mathbf{g}_{s_1}$. The same arguments as above reveal that

$$\|G(t,s)(f_{j_n^1} - f_{j_m^1})\|_\infty \leq \frac{1}{4} + \|(\mathbf{G}(t,s)(\mathbf{f}_{j_n^1} - \mathbf{f}_{j_m^1}))_{\bar{k}}\|_\infty + c\|\mathbf{G}(s_1,s)(\mathbf{f}_{j_n^1} - \mathbf{f}_{j_m^1})\|_\infty.$$

Hence, we can determine $N_1 \in \mathbb{N}$ such that

$$\|G(t,s)(f_{j_n^1} - f_{j_m^1})\|_\infty \leq \frac{1}{2}, \qquad m,n \geq N_1.$$

Iterating this argument, for any $h \in \mathbb{N}$ we can determine a subsequence $(f_{j_n^h}) \subset (f_{j_n^{h-1}})$ and an integer N_h such that

$$\|G(t,s)(f_{j_n^h} - f_{j_m^h})\|_\infty \leq 2^{-h}, \qquad m,n \geq N_h. \tag{5.75}$$

Now, we are almost done and to conclude the proof we consider the diagonal sequence (ψ_n) with $\psi_n = f_{j_n^n}$ for any $n \in \mathbb{N}$. Of course (ψ_n) is a subsequence of (f_n). We claim that $G(t,s)\psi_n$ converges uniformly in $\mathbb{R}^d$. For this purpose, we fix $\varepsilon > 0$ and $h \in \mathbb{N}$ such $2^{-h} \leq \varepsilon$. We also set $N = \max\{h, N_h\}$. With this choice of N, and recalling that $\psi_n, \psi_m \in (f_{j_p^h})$ if $n, m \geq h$, from (5.75) we deduce that

$$\|G(t,s)(\psi_n - \psi_m)\|_\infty \leq \varepsilon, \qquad m,n \geq N,$$

which, clearly, shows that $(G(t,s)\psi_n)$ is a Cauchy sequence. $\square$

Chapter 6

Semilinear System and Applications to Differential Games

6.1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $(W_t)_{t\geq 0}$ be an $\mathbb{R}^d-$valued standard Brownian motion and $\mathcal{N}$ be the family of elements of $\mathcal{F}$ of probability 0. We define as $\mathcal{F}_t^W$ the natural filtration with respect to W_t, completed by the $\mathbb{P}-$null set of $\mathcal{F}$, i.e.

$$\mathcal{F}_t^W := \sigma\{W_s : 0 \leq s \leq t,\ \mathcal{N}\}.$$

We analyze a nonzero-sum stochastic differential game (NZSDG) which is described as follows. Suppose that we have m players, which intervene on a system. For each of them, we introduce the space of admissible controls and the space of admissible strategies.

Definition 6.1. *For any player i, with $i = 1, \dots, m$, we fix $V^i \subset \mathbb{R}^m$ and define*

$$U^i := \{u : [0,T] \times \Omega \longrightarrow V^i : u \text{ is a predictable process}\}.$$

U^i is called Space of admissible controls, for any $i = 1, \dots, m$, and the set

$$U := \prod_{i=1}^{m} U^i,$$

is the Space of admissible strategies. Clearly, every player i will choose its strategy u^i which belongs to U^i.

Let $T > 0$, and $0 \leq t < T$. When m players make use of a strategy $u := (u^1, \dots, u^m)$, the dynamics of the controlled system is described by the Controlled Stochastic Differential

Equation (SDE)

$$\begin{cases} dX_\tau^{(u)} = b(X_\tau^{(u)})d\tau + G(X_\tau^{(u)})r(X_\tau^{(u)}, u_\tau) + G(X_\tau^{(u)})dW_\tau^{(u)}, & \tau \in [t, T], \\ X_t^{(u)} = x, & x \in \mathbb{R}^d, \end{cases} \tag{6.1}$$

where

$$b : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad G : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d},$$

are Borel measurable functions, $r : \mathbb{R}^d \times U \longrightarrow \mathbb{R}^d$ is a measurable and bounded function, $u \in U$ and $X^{(u)} = (X^{(u)}(\tau, t, x), \Omega, \mathcal{F}^{(u)}, \mathbb{P}^{(u)}, W^{(u)})$ is the weak solution to (6.1).

If we define

$$\tilde{W}_\tau := W_\tau^{(u)} + \int_t^{\tau\wedge t} r(X_s, u_s)ds,$$

by the Girsanov Theorem there exists a probability measure $\tilde{\mathbb{P}}$ such that $\tilde{W}_\tau$ is an $\mathbb{R}^d-$valued $\tilde{\mathbb{P}}-$Brownian motion.

Now we associate a functional cost to any player i, which will depend on the strategies of the whole players. This means that the cost for the $i - th$ player will derive not only by its choice u^i, but it will be a consequence of the choice u^j of any other player j, with $j \neq i$. In this setting, the cost functionals have the following form:

$$J^i(u) = \mathbb{E}^{(u)}\left[\int_0^T h^i(X_s, u_s)ds + g^i(X_T)\right], \quad i = 1, \dots, m, \tag{6.2}$$

where $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$ and $\mathbf{h} : \mathbb{R}^d \times U \longrightarrow \mathbb{R}^m$ and $\mathbf{g} : \mathbb{R}^d \longrightarrow \mathbb{R}^m$ are Borel measurable and bounded functions. Here, $\mathbf{h}$ is the *running cost*, while $\mathbf{g}$ is the *terminal cost*, and h^i and g^i denote the $i-$th component of $\mathbf{h}$ and $\mathbf{g}$, respectively.

Definition 6.2. *We say that $\tilde{u} = \left(\tilde{u}^1, \dots, \tilde{u}^m\right)$, $\tilde{u} \in U$, is a Nash equilibrium if, for any $i = 1, \dots, m$, any $u^i \in U^i$, we have*

$$J^i(\tilde{u}) \leq J^i\left(\tilde{u}^1, \dots, \tilde{u}^{i-1}, u^i, \tilde{u}^{i+1}, \dots, u^m\right). \tag{6.3}$$

The above definition means that if $\tilde{u}$ is a Nash equilibrium, for $i = 1, \dots, m$, the player i has no earn changing its control $\tilde{u}^i$, if the other $m - 1$ players choose the strategy $(\tilde{u}_1, \dots, \tilde{u}_{i-1}, \tilde{u}_{i+1}, \dots, \tilde{u}_m)$.

We follow the approach of [47], where the case of bounded coefficients of the controlled system has been considered, and the diffusion does not depend on the control. Here, the authors prove the existence of a solution to a Backward Stochastic Differential Equation (BSDE for short) and, consequently, the existence of a Nash equilibrium for an $N-$players NZSDG. The result of [47] has been extended in the infinite dimensional setting in [39], where the drift term of the BSDE has been considered only continuous. Moreover, in [48] authors has proved the existence of a Nash equilibrium, relaxing as much as possible the boundedness of the drift and diffusion coefficients of the controlled system.

We prove the existence of a Nash equilibrium in a more general setting by analytic methods, considering the case of unbounded coefficients for the controlled system. In particular, we link the NZSDG with the mild solution to the backward semilinear Cauchy problem

$$\begin{cases} D_t\mathbf{v}(t,x) + \mathbf{A}\mathbf{v}(t,x) = \psi(x, Q^{1/2}(x)\nabla_x \mathbf{v}(t,x)), & t \in [0,T), \quad x \in \mathbb{R}^d, \\ \mathbf{v}(T,x) = \mathbf{f}(x), & \qquad\qquad x \in \mathbb{R}^d, \end{cases} \tag{SL-CP}$$

where $\mathbf{v}$ is an $\mathbb{R}^m-$valued function, $\mathbf{A}$ is the autonomous elliptic operator defined on smooth $\mathbb{R}^m-$valued functions ϕ by

$$(\mathbf{A}\phi)_j(x) = Tr[Q(x)D^2\phi_j(x)] + \sum_{k=1}^{m} \langle (B)_{jk}(x), \nabla \phi_k \rangle, \quad j = 1, \dots, m,$$

and $\psi : \mathbb{R}^d \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}$ is a continuous function which satisfies the following conditions:

$$\begin{aligned} |\psi(x_1, z_1) - \psi(x_2, z_2)| &\le C(1 + |z_1| + |z_2|)\Big(|x_1 - x_2|^\alpha + |z_1 - z_2|^\alpha\Big), \\ |\psi(x, z)| &\le C(1 + |z|). \end{aligned}$$

This connection is possible by means of solution to the System of Forward Backward Stochastic Differential Equations

$$\begin{cases} d\mathbf{Y}_\tau = \mathbf{H}(X_\tau, \mathbf{Z}_\tau)d\tau + \mathbf{Z}_\tau dW_\tau, & \tau \in [t,T], \\ dX_\tau = b(X_\tau)d\tau + G(X_\tau)dW_\tau, & \tau \in [t,T], \\ \mathbf{Y}_T = \mathbf{g}(X_T), & \\ X_t = x, & x \in \mathbb{R}^d, \end{cases}$$

where $\mathbf{H}$ is the Hamiltonian function of the controlled system. In particular, we want to prove that both $\mathbf{Y}$ and $\mathbf{Z}$ can be expressed in terms of $\mathbf{v}$ and $Q^{1/2}\nabla_x \mathbf{v}$, respectively.

The chapter is organized as follows. In Section 6.2 we show the existence of a mild solution to the semilinear Cauchy problem (SL-CP). This result is proved in Subsections 6.2.1 and 6.2.2. In the first one, we repeat the procedure of Chapter 4 to get the existence of a mild solution $\mathbf{v}$ when ψ satisfies stronger assumptions. In Subsection 6.2.2, we approximate ψ by Lipschitz continuous functions $\{\psi_n\}_{n\in\mathbb{N}}$, and prove that the sequence of the corresponding mild solutions $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ converges to a function $\mathbf{v}$ which is a mild solution to (SL-CP), which means that $\mathbf{v}$ satisfies

$$\mathbf{v}(t,x) = \mathbf{T}(T-t)\mathbf{f}(x) - \int_t^T (\mathbf{T}(s-t)F(s,\mathbf{v}))(x)ds,$$

for any $t \in [0,T]$ and $x \in \mathbb{R}^d$, where

$$F(s,\mathbf{w})(x) := \psi(x, Q^{1/2}\nabla_x \mathbf{w}(s,x)).$$

Section 6.3 is devoted to prove of the identification formulae

$$\mathbf{Y}(s,t,x) := \mathbf{v}(s, X(s,t,x)), \quad \mathbf{Z}(s,t,x) := G(X(s,t,x))\nabla_x \mathbf{v}(s, X(s,t,x)),$$

where $\mathbf{v}$ has been introduced above and $(X, \mathbf{Y}, \mathbf{Z})$ is the predictable solution to

$$\begin{cases} d\mathbf{Y}_\tau = \mathbf{H}(X_\tau, \mathbf{Z}_\tau)d\tau + \mathbf{Z}_\tau dW_\tau, & \tau \in [t,T], \\ dX_\tau = b(X_\tau)d\tau + G(X_\tau)dW_\tau, & \tau \in [t,T], \\ \mathbf{Y}_T = \mathbf{g}(X_T), & \\ X_t = x, & x \in \mathbb{R}^d. \end{cases}$$

Finally, throughout the identification formulae, in Section 6.4 we are able to prove that a Nash equilibrium for (6.1) and (6.2) is reached, and it can be written in terms of the mild solution $\mathbf{v}$ of (SL-CP).

6.2 The Semilinear System

In this section we deal with the system of backward autonomous semilinear parabolic equations

$$\begin{cases} D_t\mathbf{u}(t,x) + \mathbf{A}\mathbf{u}(t,x) = \psi(x, Q^{1/2}(x)\nabla \mathbf{u}(t,x)), & t \in [0,T), \quad x \in \mathbb{R}^d, \\ \mathbf{u}(T,x) = \mathbf{f}(x), & x \in \mathbb{R}^d, \end{cases} \tag{6.4}$$

where $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and $\mathbf{A}$ is the vectorial operator defined on $\mathbb{R}^m$−valued smooth functions ϕ by

$$(\mathbf{A}\phi)_j(x) = Tr[Q(x)D^2\phi_j(x)] + \sum_{i=1}^{d}\sum_{k=1}^{m} \langle (B_i)_{jk}(x), \nabla \phi_k \rangle, \quad j = 1, \dots, m, \tag{6.5}$$

and $\psi : \mathbb{R}^d \times \mathbb{R}^{d\times m} \longrightarrow \mathbb{R}^m$ is a continuous function which satisfies the following assumptions.

Hypotheses 6.3. *ψ is a continuous function and*

$$\begin{aligned} |\psi(x_1, z_1) - \psi(x_2, z_2)| &\leq C(1 + |z_1| + |z_2|)\Big(|x_1 - x_2|^\alpha + |z_1 - z_2|^\alpha\Big), \\ |\psi(x, z)| &\leq C(1 + |z|), \end{aligned} \tag{6.6}$$

for any $x, x_1, x_2 \in \mathbb{R}^d$ and $z, z_1, z_2 \in \mathbb{R}^{d\times m}$, for some positive constant C and $\alpha \in (0,1)$.

Throughout this chapter, we assume the following standing assumptions (see Chapter 5), which are related to the existence and uniqueness of a classical solution to the linear Cauchy problem

$$\begin{cases} D_t\mathbf{u}(t,x) = \mathbf{A}\mathbf{u}(t,x), & t\in[0,T), \quad x\in\mathbb{R}^d, \\ \mathbf{u}(T,x) = \mathbf{f}(x), & \qquad\qquad\quad x\in\mathbb{R}^d. \end{cases} \tag{6.7}$$

Hypotheses 6.4. (i) *For any $i,j=1,\dots,d$, the coefficients q_{ij} and the entries of the matrices B_j belong to $C^\alpha_{\rm loc}(\mathbb{R}^d)$, for some $\alpha\in(0,1)$;*

(ii) *the matrix $Q=[q_{ij}]$ is uniformly elliptic, i.e., there exists a function ν with positive infimum ν_0 such that*

$$\langle Q(x)\xi,\xi\rangle \ge \nu(x)|\xi|^2, \qquad x\in\mathbb{R}^d,\ \ \xi\in\mathbb{R}^d; \tag{6.8}$$

(iii) *for any $i=1,\dots,d$, there exist $b_i\in C^\alpha_{\rm loc}(\mathbb{R}^d)$ and $\tilde B_i\in C^\alpha_{\rm loc}(\mathbb{R}^d;\mathbb{R}^{m\times m})$ such that*

$$B_i(x) := b_i(x)Id_m + \tilde B_i(x),$$

for any $i=1,\dots,d$ and any $x\in\mathbb{R}^d$. Further, there exists a positive constant C such that $|(\tilde B_i)_{jk}|^2\le C\nu$ in $\mathbb{R}^d$, for any $j,k=1,\dots,m$, $i=1,\dots,d$ where $(\tilde B_i)_{jk}$ denotes the jk-th element of the matrix $\tilde B_i$;

(iv) *there exists a constant $c\in\mathbb{R}$ such that $\langle C(x)\eta,\eta\rangle\le c$, for any $x\in\mathbb{R}^d$ and any $\eta\in\partial B_1$;*

(v) *there exist a constant λ and a positive function $\varphi\in C^2(\mathbb{R}^d)$ blowing up as $|x|\to+\infty$ such that*

$$\sup_{x\mathbb{R}^d}(\mathcal{A}(t)\varphi)(x)-\lambda\varphi(x)) < +\infty,$$

where $\mathcal{A}={\rm Tr}(QD^2_x)+\langle b,\nabla_x\rangle$ and $b=(b_1,\dots,b_m)$.

Remark 6.5. *Hypotheses 6.4 guarantee that the linear Cauchy problem* (6.7) *admits a unique classical solution* $\mathbf{u}$, *for any* $T>0$ *(compare with Hypotheses 5.13 and see Theorem 5.9 and Proposition 5.14). We denote by $\{\mathbf{T}(\cdot)\}_{t\ge0}$ the semigroup on $C_b(\mathbb{R}^d;\mathbb{R}^m)$ defined by $\mathbf{T}(t)\mathbf{f}(x):=\mathbf{u}(t,x)$, where $\mathbf{u}$ is the unique classical solution to* (6.7). *$\{\mathbf{T}(t)\}_{t\ge0}$ is a semigroup of bounded linear operators, and there exists a positive constant $K=K(T)$ such that*

$$|(\mathbf{T}(t)\mathbf{f})(x)|^2 \le e^{Kt}(S(t)|\mathbf{f}|^2)(x), \tag{6.9}$$

for any $(t,x)\in[0,T]\times\mathbb{R}^d$ and any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, where $\{S(t)\}_{t\ge0}$ is the semigroup associated to $\mathcal{A}$ in $C_b(\mathbb{R}^d)$ (see Proposition 5.14).

To go further, besides Hypotheses 6.4, we will consider the following stronger conditions on the coefficients q_{ij} and b_i.

Hypotheses 6.6. (i) *The coefficients* q_{ij} *belong to* $C^{2+\alpha}_{\rm loc}(\mathbb{R}^d)$ *for some* $\alpha \in (0,1)$ *and any* $i,j = 1,\dots,d$*;*

(ii) *the coefficients of the vector* b *and the matrices* $\tilde{B}_i$ $(i = 1,\dots,d)$ *belong to* $C^{1+\alpha}_{\rm loc}(\mathbb{R}^d)$*; further,* $\langle b(x), x\rangle \le b_0(x)|x|$ *for any* $x \in \mathbb{R}^d$ *and some negative function* b_0*;*

(iii) *there exist a positive constant* K_0 *and positive functions* $\psi_j : \mathbb{R}^d \to \mathbb{R}$ $(j = 1,2,3)$ *such that*

$$|\langle Q(x), x\rangle| \le K_0(1 + |x|^2)\nu(x), \qquad x \in \mathbb{R}^d, \tag{6.10}$$

$$|\nabla_x(Q_{ij}^{1/2})Q^{-1/2}| \le \psi_1, \qquad |Q| \le \psi_2; \tag{6.11}$$

(iv) *the functions* ψ_1*,* ψ_2*,* ψ_3 *satisfy the following conditions:*

$$\lim_{|x|\to+\infty} \frac{\psi_3(x)}{\omega(x)} = \lim_{|x|\to+\infty} \frac{(\nu(x))^{-1}(\psi_1(x))^2(\psi_2(x))^2}{\omega(x)} = 0, \tag{6.12}$$

where the function $\omega : \mathbb{R}^d \to \mathbb{R}$ *is a function which bounds from above the quadratic form associated with the matrix* $Q^{1/2}(J_x b)^T Q^{-1/2} - \sum_{j=1}^d b_j(D_j Q^{1/2})Q^{-1/2} - \sum_{i,j=1}^d q_{ij}(D_{ij}Q^{1/2})Q^{-1/2}$*. Moreover,*

$$\liminf_{|x|\to+\infty} \frac{(\nu(x))^2}{\omega(x)} > -\infty, \tag{6.13}$$

$$\lim_{|x|\to+\infty} \frac{|x|\nu(x)\psi_1(x)}{b_0(x)} = 0, \tag{6.14}$$

$$\liminf_{|x|\to+\infty} \frac{|x|(\nu(x))^2}{b_0(x)} > -\infty. \tag{6.15}$$

We recall a result of the previous chapter (see Proposition 5.21), which will be useful in order to prove the existence of a solution to (6.4).

Proposition 6.7. *Assume that Hypotheses* 6.6 *are satisfied. Then, for any* $j = 1,\dots,d$*, the function* $(t,x) \mapsto Q^{1/2}(x)\nabla_x u_j(t,x)$ *is continuous and bounded in* $J \times \mathbb{R}^d$ *for any* $J \Subset (0,+\infty)$*. Moreover, for any* $T > 0$ *there exists a positive constant* $C = C(T)$ *such that*

$$t\sum_{j=1}^m \|Q^{1/2}\nabla_x(\mathbf{T}(t)\mathbf{f})_j\|_\infty^2 \le C\|\mathbf{f}\|_\infty^2, \tag{6.16}$$

for any $t \in (0,T]$ *and* $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$*.*

The goal of this section is to provide sufficient conditions to prove the existence of a mild solution $\mathbf{v}$ of (6.4), i.e., a function $\mathbf{v} \in \mathbf{K}_T$ which satisfies

$$\mathbf{v}(t,x) = \mathbf{T}(T-t)\mathbf{f}(x) - \int_t^T (\mathbf{T}(s-t)F(s,\mathbf{v}))(x)ds, \tag{6.17}$$

for any $t \in [0,T]$ and $x \in \mathbb{R}^d$, where $F(s,\mathbf{w})(x) := \psi(s, Q^{1/2}(x)\nabla_x \mathbf{w}(x))$, for any $s \in [0,T) \times \mathbf{K}_T$, and

$$\mathbf{K}_\delta := \left\{ \begin{array}{l} \mathbf{h} \in C_b\left([T-\delta,T]\times\mathbb{R}^d;\mathbb{R}^m\right) \cap C^{0,1}\left([T-\delta,T)\times\mathbb{R}^d;\mathbb{R}^m\right) : \\ \\ \|\mathbf{h}\|_{\mathbf{K}_\delta} < \infty \end{array} \right\},$$

$$\|\mathbf{h}\|_{\mathbf{K}_\delta} := \|\mathbf{h}\|_\infty + [\mathbf{h}]_{\mathbf{K}_\delta}, \ [\mathbf{h}]_{\mathbf{K}_\delta} := \sup_{t\in[T-\delta,T)} (T-t)^{1/2} \sum_{j=1}^m \|Q^{1/2}(\cdot)\nabla_x \mathbf{h}_j(t,\cdot)\|_\infty,$$

for any $\delta > 0$.

At this stage, (6.17) is only formal, because, in general, $\{\mathbf{T}(t)\}_{t\ge0}$ can be applied only on bounded and continuous functions. Hence, we prove that it is possible to apply $\mathbf{T}(t)$ to the function $F(s,\mathbf{w})$, provided that $\mathbf{w}$ belongs to a suitable space of functions. Then we show that, if ψ is Lipschitz continuous, a classical fixed point argument shows that there exists a unique mild solution $\mathbf{v}$. Finally, if ψ satisfies Hypothesis 6.3, then we can define a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ of Lipschitz continuous functions such that the sequence of associated solutions $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ converges to a function $\mathbf{v} \in \mathbf{K}_T$ which satisfies (6.17).

6.2.1 Existence and uniqueness of a mild solution to (SL-CP) when ψ is a Lipschitz continuous function

We introduce the following space of functions: for any $\delta > 0$ and $R > 0$ we set

$$\mathbf{K}_{\delta,R} := \{\mathbf{w} \in \mathbf{K}_\delta : \|\mathbf{w}\|_{\mathbf{K}_\delta} \le R\}.$$

Moreover, throughout this subsection we assume a further assumption on ψ.

Hypothesis 6.8. *There exists a positive constant $C > 0$ such that*

$$|\psi(x,z_1) - \psi(x,z_2)| \le C|z_1 - z_2|,$$

for any $x \in \mathbb{R}^d$ and $z_1, z_2 \in \mathbb{R}^{m\times d}$.

In this subsection we will prove that the operator Γ, defined for any $\mathbf{u} \in \mathbf{K}_T$ by

$$(\Gamma\mathbf{u})(t,x) := \mathbf{T}(T-t)\mathbf{f}(x) - \int_t^T (\mathbf{T}(s-t)F(s,\mathbf{u}))(x)ds,$$

for any $t \in [0,T]$ and $x \in \mathbb{R}^d$, admits a unique fixed point in $\mathbf{K}_T$. Clearly, any fixed point of Γ in $\mathbf{K}_T$ is a mild solution to problem (SL-CP).

Remark 6.9. *If ψ satisfies Hypothesis 6.8, then*

$$\begin{aligned} &(i) \quad \|F(s,\mathbf{u}) - F(s,\mathbf{v})\|_\infty \le C(T-s)^{-1/2}[\mathbf{u}-\mathbf{v}]_{\mathbf{K}_T}, \\ &(ii) \quad \|\big(F(s,\mathbf{u})\|_\infty \le C\left(1 + (T-s)^{-1/2}[\mathbf{u}]_{\mathbf{K}_T}\right), \end{aligned} \tag{6.18}$$

for any $s \in [0,T)$, any $\mathbf{u},\mathbf{v} \in \mathbf{K}_T$. Moreover, if $\mathbf{u} \in \mathbf{K}_T$, then the function $F(\cdot,\mathbf{u})(\cdot) : [0,T)\times\mathbb{R}^d \longrightarrow \mathbb{R}^m$ belongs to $C([0,T)\times\mathbb{R}^d;\mathbb{R}^m)$.

As in Chapter 4, we begin showing that the function $t \mapsto \|\mathbf{f}(t,\cdot)\|_\infty$ is measurable, for any $\mathbf{f} \in C([0,T]\times\mathbb{R}^d;\mathbb{R}^m)$ such that $\mathbf{f}(t,\cdot)$ is bounded, for any $t \in [0,t]$.

Lemma 6.10. *Let $f \in C([0,T]\times\mathbb{R}^d;\mathbb{R}^m)$ such that, for any $t \in [0,T]$, $f(t,\cdot) \in C_b(\mathbb{R}^d;\mathbb{R}^m)$. Then the function $t \mapsto \|f(t,\cdot)\|_\infty$ is a measurable function.*

From now on, we won't refer to Lemma 6.10, but directly use it.

The following proposition shows some continuity and boundedness properties of the function F. These results and their proofs are analogous to those of Proposition 4.10.

Proposition 6.11. *If $u \in \mathbf{K}_\delta$, for some $\delta > 0$, F satisfies (6.18) and*

$$\sup_{t\in(T-\delta,T)} (T-t)^{1/2}\sum_{j=1}^m \|Q^{1/2}\nabla_x \mathbf{u}_j(t,\cdot)\|_\infty < \infty,$$

then the functions

$$(t,x) \mapsto \tilde{F}(t,x) := \int_t^T (\mathbf{T}(s-t)F(s,\mathbf{u}))(x)ds$$

and

$$(t,x) \mapsto Q^{1/2}(x)\nabla_x \tilde{F}(t,x),$$

are continuous and bounded in $[T-\delta,T]\times\mathbb{R}^d$ and $[T-\delta,T)\times\mathbb{R}^d$, respectively.

Now we prove the existence and uniqueness of a mild solution $\mathbf{v}$ to (6.4). We follow the same reasoning of Chapter 4, Subsection 4.2.2; at first we show the uniqueness of the mild solution to problem (SL-CP).

Proposition 6.12. *If problem (6.4) admits a mild solution in $\mathbf{K}_\delta$, then it is unique.*

Proof. Let $\mathbf{u}, \mathbf{v} \in \mathbf{K}_\delta$ be two mild solutions of (6.4). Then, taking (6.16) and (6.18) into account, for any $t \in [T-a,T)$ we get

$$\begin{aligned}
&\|Q^{1/2}\nabla_x(\mathbf{u}-\mathbf{v})(t,\cdot)\|_\infty \\
&\quad\le \left\| \int_t^T Q^{1/2}\nabla_x\big(\mathbf{T}(s-t)(F(s,\mathbf{u})-F(s,\mathbf{v}))(\cdot)\big)ds \right\|_\infty \\
&\quad\le C_T C\int_t^T (s-t)^{-1/2}\|Q^{1/2}\nabla_x(\mathbf{u}-\mathbf{v})(s,\cdot)\|_\infty ds \\
&\quad\le C_T^2C^2\int_t^T (s-t)^{-1/2}ds\left(\int_s^T (r-s)^{-1/2}\|Q^{1/2}\nabla_x(\mathbf{u}-\mathbf{v})(r,\cdot)\|_\infty dr\right) \\
&\quad= C_T^2C^2\int_t^T \|Q^{1/2}\nabla_x(\mathbf{u}-\mathbf{v})(r,\cdot)\|_\infty ds\left(\int_t^r (s-t)^{-1/2}(r-s)^{-1/2}ds\right) \\
&\quad= C_T^2C^2\pi\int_t^T \|Q^{1/2}\nabla_x(\mathbf{u}-\mathbf{v})(r,\cdot)\|_\infty dr.
\end{aligned}$$

Hence, by the Gronwall Lemma we deduce that $\|Q^{1/2}\nabla_x(\mathbf{u}-\mathbf{v})(t,\cdot)\|_\infty = 0$, for any $t \in [T-\delta, T)$. To conclude, it is enough to observe that

$$\begin{aligned}\|(\mathbf{u}-\mathbf{v})(t,\cdot)\|_\infty &\leq \left\| \int_t^T \big(\mathbf{T}(s-t)(F(s,\mathbf{u})-F(s,\mathbf{v}))(x)\big)ds \right\|_\infty \\ &\leq C\int_t^T \|Q^{1/2}\nabla_x(\mathbf{u}(s,\cdot)-\mathbf{v}(s,\cdot))\|_\infty ds \\ &= 0.\end{aligned}$$

□

The existence of a mild solution to (6.4) is a byproduct of the Banach fixed point theorem, applied to the space $\mathbf{K}_{\delta,R}$, with suitable δ and R.

Proposition 6.13. *There exist $\delta < T$ and $R > 0$ such that the operator Γ, defined by*

$$(\Gamma\mathbf{v})(t,x) = \mathbf{T}(T-t)\mathbf{f}(x) - \int_t^T (\mathbf{T}(s-t)F(s,\mathbf{v}))(x)ds, \tag{6.19}$$

$(t,x) \in (T-\delta, T] \times \mathbb{R}^d$, for any $\mathbf{v} \in \mathbf{K}_{\delta,R}$, admits a unique fixed point in $\mathbf{K}_{\delta,R}$.

Proof. We prove that Γ is a contraction on $\mathbf{K}_{\delta,R}$, endowed with the norm $\|\cdot\|_{\mathbf{K}_\delta}$, i.e., $\Gamma(\mathbf{K}_{\delta,R}) \subset \mathbf{K}_{\delta,R}$ and there exists a positive constant $c < 1$, such that

$$\|(\Gamma\mathbf{v}) - (\Gamma\mathbf{u})\|_{\mathbf{K}_\delta} \leq c\|\mathbf{v}-\mathbf{u}\|_{\mathbf{K}_\delta},$$

for any $\mathbf{u}, \mathbf{v} \in \mathbf{K}_{\delta,R}$.

We set $M := \sup_{t\in[0,T]} \|\mathbf{T}(t)\|_{\mathcal{L}(C_b(\mathbb{R}^d;\mathbb{R}^m))}$ and

$$C_T := \sup_{t\in[0,T)} (T-t)^{1/2} \sum_{j=1}^m \|Q^{1/2}\nabla_x \mathbf{T}(t)\|_{\mathcal{L}(C_b(\mathbb{R}^d;\mathbb{R}^m))}.$$

The following computations are similar to those in the proof of Theorem 4.12, hence we skip some details. From estimates (6.18) we deduce

$$\begin{aligned}\|\Gamma(\mathbf{v}(t,\cdot))\|_\infty &\leq M\|\mathbf{f}\|_\infty + \left\| \int_t^T (\mathbf{T}(s-t)F(s,\mathbf{v})(\cdot))ds \right\|_\infty \\ &\leq M\|\mathbf{f}\|_\infty + 2MC\delta^{1/2}\|\mathbf{v}\|_{\mathbf{K}_\delta} + \delta MC.\end{aligned} \tag{6.20}$$

and

$$\begin{aligned}&(T-t)^{1/2}\sum_{j=1}^m \|Q^{1/2}\nabla_x((\Gamma\mathbf{v})(t,\cdot))_j\|_\infty \\ &\leq C_T\|\mathbf{f}\|_\infty + (T-t)^{1/2}C_TC\int_t^T (s-t)^{-1/2}\left(\|Q^{1/2}\nabla_x\mathbf{v}(s,\cdot)\|_\infty + 1\right)ds \\ &\leq C_T\|\mathbf{f}\|_\infty + \pi\delta^{1/2}C_TC\|\mathbf{v}\|_{\mathbf{K}_\delta} + 2\delta C_TC.\end{aligned} \tag{6.21}$$

Moreover,

$$\begin{aligned}\|(\Gamma\mathbf{u})(t,\cdot)) - (\Gamma\mathbf{v})(t,\cdot))\|_\infty &\leq \int_t^T \|\big(\mathbf{T}(s-t)(F(s,\mathbf{u}) - F(s,\mathbf{v}))\big)\|_\infty ds \\ &\leq 2MC\delta^{1/2}\|\mathbf{u}-\mathbf{v}\|_{\mathbf{K}_\delta}\end{aligned} \tag{6.22}$$

and

$$\begin{aligned}&(T-t)^{1/2}\sum_{j=1}^m \|Q^{1/2}\nabla_x((\Gamma\mathbf{u})(t,\cdot))_j - Q^{1/2}\nabla_x((\Gamma\mathbf{v})(t,\cdot))_j\|_\infty \\ &\quad\leq (T-t)^{1/2}C_T C\int_t^T (s-t)^{-1/2}\|Q^{1/2}\nabla_x\mathbf{u}(s,\cdot) - Q^{1/2}\nabla_x\mathbf{v}(s,\cdot)\|_\infty ds \\ &\quad\leq \pi\delta^{1/2}C_T C\|\mathbf{u}-\mathbf{v}\|_{\mathbf{K}_\delta}.\end{aligned} \tag{6.23}$$

Thus, choosing

$$\begin{aligned}\delta &= \min\left\{\frac{1}{(4MC+2\pi C_T C)^2}, T\right\}, \\ R &= 2(M+C_T)(\|\mathbf{f}\|_\infty + 2\delta C),\end{aligned}$$

we obtain

$$\begin{aligned}\|\Gamma\mathbf{v}\|_{\mathbf{K}_\delta} &\leq R, \\ \|\Gamma\mathbf{v} - \Gamma\mathbf{u}\|_{\mathbf{K}_\delta} &\leq \frac{1}{2}\|\mathbf{v}-\mathbf{u}\|_{\mathbf{K}_\delta}.\end{aligned}$$

Therefore, Γ is a contraction on $\mathbf{K}_{\delta,R}$ and it follows that there exists a unique $\mathbf{v} \in \mathbf{K}_{\delta,R}$ such that $\Gamma(\mathbf{v}) \equiv \mathbf{v}$ in $[T-\delta, T] \times \mathbb{R}^d$. □

Propositions 6.12 and 6.13 easily imply the following theorem.

Theorem 6.14. *There exists a unique* $\mathbf{v} \in \mathbf{K}_\delta$ *which satisfies* (6.17)*.*

Proof. From Proposition 6.13 there exists a unique $\mathbf{v} \in \mathbf{K}_{\delta,R}$ which satisfies $\Gamma\mathbf{v} \equiv \mathbf{v}$. Since $\mathbf{K}_{\delta,R} \subset \mathbf{K}_\delta$, using Proposition 6.12 we conclude that $\mathbf{v}$ is the unique element of $\mathbf{K}_\delta$ such that

$$\mathbf{v}(t,x) = \mathbf{T}(T-t)\mathbf{f}(x) - \int_t^T (\mathbf{T}(s-t)F(s,\mathbf{v}))(x)ds,$$

for any $t \in [T-\delta, T]$ and $x \in \mathbb{R}^d$. □

In the last part of this subsection we show that it is possible to extend $\mathbf{v}$ to the whole interval $[0,T]$. We follow the procedure of Subsection 4.2.2, hence we just state the main result.

Proposition 6.15. *If* F *satisfies* (6.18)*, then the mild solution* $\mathbf{v}$ *of problem* (6.4) *exists in* $[0,T] \times \mathbb{R}^d$*.*

6.2.2 Convergence of Mild Solutions

In this subsection we prove that, if ψ only satisfies Hypothesis 6.3, then the Cauchy problem (6.4) admits a mild solution $\mathbf{v} \in \mathbf{K}_T$. We approximate ψ by suitable more regular functions ψ_n, and we consider the mild solution $\mathbf{v}_n$ of the approximate Cauchy problem with data ψ_n, for any $n \in \mathbb{N}$. We want to prove that, up to a subsequence, $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ converges to a function $\mathbf{v} \in \mathbf{K}_T$ and

$$\mathbf{v}(t,x) = \mathbf{T}(T-t)\mathbf{f}(x) - \int_t^T (\mathbf{T}(s-t)F(s,\mathbf{v}))(x)ds,$$

for any $(t,x) \in [0,T] \times \mathbb{R}^d$. Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a standard sequence of mollifiers in $\mathbb{R}^{m\times d}$ and, for any $n \in \mathbb{N}$, let $\theta_n \in C_c^\infty(\mathbb{R}^{m\times d})$ satisfy $\chi_{B(n)} \le \theta_n \le \chi_{B(n+1)}$. We set

$$\psi_n(x,z) := \theta_n(z)(\rho_n \star_z \psi)(x,z), \quad n \in \mathbb{N}, \tag{6.24}$$

for any $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^{m\times d}$, where $\star_z$ denotes the convolution only with respect to the variable z.

Lemma 6.16. *For any $k,n \in \mathbb{N}$, $k \le n$, any $x \in \mathbb{R}^d$ and $z_1, z_2 \in \mathbb{R}^{m\times d}$, we have*

$$(i)\quad |\psi_n(x,z) - \psi(x,z)| \le \frac{C(2+|z|)}{n^\alpha}\chi_{B(n)}(|z|) + C\left(2+2|z|+\frac{1}{n}\right)\chi_{B(n)^c}(|z|),$$

$$\begin{aligned}(ii)\quad & |\psi_n(x,z) - \psi_k(x,z)| \\ & \le \left(\frac{C(2+|z|)}{n^\alpha} + \frac{C(2+|z|)}{k^\alpha}\right)\chi_{B(k)}(|z|) \\ & + \left(\frac{C(2+|z|)}{n^\alpha} + C\left(2+2|z|+\frac{1}{k}\right)\right)\chi_{B(n)\cap B(k)^c}(|z|) \\ & + C\left(4+4|z|+\frac{1}{n}+\frac{1}{k}\right)\chi_{B(n)^c}(|z|),\end{aligned} \tag{6.25}$$

$$(iii)\quad |\psi_n(x,z)| \le C(1+|z|), \tag{6.26}$$

$$(iv)\quad |\psi_n(x,z_1) - \psi_n(x,z_2)| \le C_n|z_1 - z_2|, \tag{6.27}$$

where C has been defined in (6.6) *and C_n is a suitable positive constant which depends on n and blows up as $n \to \infty$.*

Proof. (i) follows from

$$\begin{aligned}|\psi_n(x,z) - \psi(x,z)| =& \Big|\int_{\mathbb{R}^{m\times d}} \rho_n(y)(\theta_n(z)\psi(x,z-y) - \psi(x,z))dy\Big| \\ \le & \chi_{B(n)}(z)\int_{\mathbb{R}^{m\times d}} \rho_n(y)|\psi(x,z-y) - \psi(x,z)|dy\Big| \\ & + C\chi_{B(n)^c}(z)\int_{\mathbb{R}^{m\times d}} \rho_n(y)|\theta_n(z)\psi(x,z-y) - \psi(x,z)|dy\Big| \\ \le & \chi_{B(n)}(z)\int_{\mathbb{R}^{m\times d}} \rho_n(y)(1+|z|+|z-y|)|y|ds\end{aligned}$$

$$+ C\chi_{B(n)^c}(z) \int_{\mathbb{R}^{m\times d}} \rho_n(y)(2 + |z| + |z-y|)ds. \tag{6.28}$$

(ii) is a byproduct of (6.28) and the fact that

$$|\psi_n(x,z) - \psi(x,z)| + |\psi(x,z) - \psi_k(x,z)|.$$

□

For any $n \in \mathbb{N}$, we consider the approximate problem

$$\begin{cases} D_t \mathbf{v}_n(t,x) + \mathbf{A}\mathbf{v}_n(t,x) = \psi_n(x, Q^{1/2}(x)\nabla_x \mathbf{v}_n(t,x)), & t \in [0,T), \quad x \in \mathbb{R}^d, \\ \mathbf{v}_n(T,x) = \mathbf{f}(x), & x \in \mathbb{R}^d. \end{cases} \tag{6.29}$$

From (6.27), ψ_n is Lipschitz in $\mathbb{R}^{m\times d}$. Hence, Theorem 6.14 implies that there exists a unique mild solution $\mathbf{v}_n \in \mathbf{K}_T$ to (6.29), i.e., there exists a unique function $\mathbf{v}_n$ which belongs to $\mathbf{K}_T$ and satisfies

$$\mathbf{v}_n(t,x) = \mathbf{T}(T-t)\mathbf{f}(x) - \int_t^T (\mathbf{T}(s-t)F_n(s,\mathbf{v}_n))(x)ds,$$

for any $t \in [0,T)$, any $x \in \mathbb{R}^d$, where

$$F_n(s,\mathbf{u})(x) := \psi_n(x, Q^{1/2}(x)\nabla_x \mathbf{u}(s,x)),$$

for any $\mathbf{u} \in \mathbf{K}_T$, any $s \in [0,T)$ and $x \in \mathbb{R}^d$.

Remark 6.17. *For any $n \in \mathbb{N}$, any $\mathbf{u} \in \mathbf{K}_T$ and $s \in [0,T)$, we have*

$$\|F_n(s,\mathbf{u})\|_\infty \le C(1 + \|\mathbf{u}\|_{\mathbf{K}_T}(T-s)^{-1/2}).$$

Lemma 6.18. *There exists a positive constant K such that*

$$\|\mathbf{v}_n\|_{\mathbf{K}_T} \le K, \tag{6.30}$$

for any $n \in \mathbb{N}$.

Proof. We set $\mathbf{h}_n(t) := \|Q^{1/2}\nabla_x \mathbf{v}_n(t,\cdot)\|_\infty$, for any $t \in [0,T)$. It follows that

$$\begin{aligned} (T-t)^{1/2}\mathbf{h}_n(t) &\le C_T\|\mathbf{f}\|_\infty + (T-t)^{1/2}\int_t^T C_T C(s-t)^{-1/2}(1+\mathbf{h}_n(s))ds \\ &\le C_T\|\mathbf{f}\|_\infty + (T-t)^{1/2}C_T C\int_t^T (s-t)^{-1/2}ds \\ &\quad + (T-t)^{1/2}C_T C\int_t^T (s-t)^{-1/2}\mathbf{h}_n(s)ds = (*). \end{aligned} \tag{6.31}$$

If we write $\mathbf{h}_n(s) = (T-s)^{-1/2}(T-s)^{1/2}\mathbf{h}_n(s)$, and we replace $(T-s)^{1/2}\mathbf{h}_n(s)$ with the right-hand side of (6.31), then we get

$$\begin{aligned}(*) &\leq C_T\|\mathbf{f}\|_\infty + TC_TC + T^{1/2}C_TC\int_t^T (s-t)^{-1/2}(T-s)^{-1/2}C_T\|\mathbf{f}\|_\infty ds \\ &\quad + T^{1/2}C_T^2C^2\int_t^T (s-t)^{-1/2}\left(\int_s^T (r-s)^{-1/2}(1+\mathbf{h}_n(r))dr\right)ds \\ &\leq C_T\|\mathbf{f}\|_\infty + TC_TC + T^{1/2}(C_T)^2C\pi\|\mathbf{f}\|_\infty + 2T^{3/2}(C_TC)^2 \\ &\quad + T^{1/2}(C_TC)^2\pi\int_t^T (T-r)^{-1/2}(T-r)^{1/2}\mathbf{h}_n(r)dr,\end{aligned}$$

where

$$C_T := \sup_{t\in[0,T)}(T-t)^{1/2}\sum_{j=1}^m \|Q^{1/2}\nabla_x\mathbf{T}(t)\|_{\mathcal{L}(C_b(\mathbb{R}^d;\mathbb{R}^m))}.$$

The generalized Gronwall lemma implies that

$$\begin{aligned}(T-t)^{1/2}\mathbf{h}_n(t) \leq &C_T(\|\mathbf{f}\|_\infty + TC + T^{1/2}C_TC\pi\|\mathbf{f}\|_\infty + 2T^{3/2}C_TC^2) \\ &\times\exp(2T^{3/2}(C_TC)^2\pi).\end{aligned} \tag{6.32}$$

Further, we have

$$\begin{aligned}\|\mathbf{v}_n\|_\infty &\leq M\|\mathbf{f}\|_\infty + MC\int_t^T (1+\|\mathbf{h}(s)\|_\infty)ds \\ &\leq M\|\mathbf{f}\|_\infty + MC\int_t^T (1+(T-s)^{-1/2}(T-s)^{1/2}\|\mathbf{h}(s)\|_\infty)ds \\ &\leq \tilde{C}_T,\end{aligned} \tag{6.33}$$

by virtue of (6.32).

Finally, combining (6.32) and (6.33), we obtain the thesis, with

$$\begin{aligned}K = &(C_T\|\mathbf{f}\|_\infty + T(C_T)^2C\pi\|\mathbf{f}\|_\infty + (TC_TC)^2\pi)(1+2MCT^{1/2}) \\ &\times\exp(2T^{3/2}(C_TC)^2\pi) + M\|f\|_\infty + TMC.\end{aligned}$$

□

To go further, we need an intermediate result. At first, for any $l \in \mathbb{N}$, we introduce the space of functions

$$X_l := C_b([0, T-1/\hat{l}] \times \overline{B(\hat{l})}; \mathbb{R}^m) \cap C^{0,1}([0, T-1/\hat{l}] \times B(l); \mathbb{R}^m),$$

where $\hat{l} := [1/T] + l$. Then we define

$$\Phi_k^n(\mathbf{u})(t,x) := (\mathbf{T}(T-t)\mathbf{f})(x) - \int_{t+1/\hat{n}}^T (\mathbf{T}(s-t)F_k(s,\mathbf{u}))(x)ds,$$

for any $n \in \mathbb{N}$, any $(t,x) \in [0, T-1/\tilde{n}] \times \mathbb{R}^d$, any $\mathbf{u} \in \mathbf{K}_T$ and $k, n \in \mathbb{N}$. We want to prove that Φ_k^n is compact from $\mathbf{K}_T$ in X_l.

Remark 6.19. *from Lemma 6.18 it follows that* $\|\Phi_k^n(\mathbf{v}_m)\|_{\mathbf{K}_T} \leq \tilde{K}$*, for any* $k, m, n \in \mathbb{N}$ *and some positive constant* $\tilde{K}$*.*

Proposition 6.20. Φ_k^n *is compact from* $\mathbf{K}_T$ *in* X_l*, for any* $l \geq n$*.*

Proof. We fix $k, l, n \in \mathbb{N}$, $l \geq n$. We consider a uniformly bounded subset $W \subset \mathbf{K}_T$, i.e., such that $\|\mathbf{w}\|_{\mathbf{K}_T} \leq H$ for any $\mathbf{w} \in W$ and some $H > 0$. We observe that we can limit ourselves to consider only the integral term of Φ_k^n, and we define

$$\mathbf{g}_{\mathbf{w}}(t, x) := \int_{t+1/\hat{n}}^{T} (\mathbf{T}(s-t)F_k(s, \mathbf{w}))(x)ds,$$

for any $t \in [0, T - 1/\hat{l}]$ and $x \in B(\hat{l})$. We claim that $\{\mathbf{g}_{\mathbf{w}}\}_{\mathbf{w}\in W}$ is equibounded and equicontinuous in X_l. The equiboundness is trivial, since

$$\begin{aligned}
\|\mathbf{g}_{\mathbf{w}}\|_\infty &\leq \int_t^T MC(1 + H\nu_0^{-1/2}(T-s)^{-1/2})ds \\
\|\nabla_x \mathbf{g}_{\mathbf{w}}\|_\infty &\leq \int_t^T C_T C(s-t)^{-1/2}(1 + H\nu_0^{-1/2}(T-s)^{-1/2})ds,
\end{aligned}$$

where ν_0 has been defined in Hypothesis 5.1(ii). To prove that it is equicontinuous, we recall that $\mathbf{T}(\cdot)\mathbf{f}$ is the locally uniform limit of a sequence of functions $\{\mathbf{u}_r\}_{r\in\mathbb{N}}$, which $r-$th element is solution to

$$\begin{cases}
D_t \mathbf{u}_r(t, x) = (\mathbf{A}\mathbf{u}_r)(t, x), & t \in (0, +\infty), \quad x \in B(r), \\
\mathbf{u}_r(t, x) = 0, & t \in (0, +\infty), \quad x \in \partial B(r), \\
\mathbf{u}_r(0, x) = \mathbf{f}(x), & x \in B(r),
\end{cases}$$

for any $r \in \mathbb{N}$ (see Theorem 5.9), and, from Theorem 5.3, for any compact set $K \subset (0, +\infty) \times \mathbb{R}^d$, there exists a positive constant C_K such that

$$\|\mathbf{u}_r\|_{C^{1+\alpha/2, 2+\alpha}(K;\mathbb{R}^m)} \leq C_K \|\mathbf{f}\|_\infty.$$

Hence

$$\|\mathbf{T}(\cdot)\mathbf{f}\|_{C^{1+\alpha/2, 2+\alpha}(K;\mathbb{R}^m)} \leq C_K \|\mathbf{f}\|_\infty, \tag{6.34}$$

for any $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$.

We go back to our problem. For any $t, r \in [0, T - 1/\hat{l}]$, $t > r$, and $x, y \in B(\hat{l})$, we have

$$\begin{aligned}
|\mathbf{g}_{\mathbf{w}}(t, x) - \mathbf{g}_{\mathbf{w}}(r, y)| &\leq |\mathbf{g}_{\mathbf{w}}(t, x) - \mathbf{g}_{\mathbf{w}}(r, x)| + |\mathbf{g}_{\mathbf{w}}(r, x) - \mathbf{g}_{\mathbf{w}}(r, y)| \\
&\leq \left| \int_{t+1/\hat{n}}^{T} \big((\mathbf{T}(s-t) - \mathbf{T}(s-r))F_k(s, \mathbf{w})\big)(x)ds \right| \\
&\quad + \left| \int_{r+1/\hat{n}}^{t+1/\hat{n}} (\mathbf{T}(s-r)F_k(s, \mathbf{w}))(x)ds \right| + |\mathbf{g}_{\mathbf{w}}(r, x) - \mathbf{g}_{\mathbf{w}}(r, y)| \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

The estimate of I_1 is a byproduct of above results; indeed, for any $s \in (t + 1/\hat{n}, T)$ the function under the integral sign is uniformly bounded by an integrable function, i.e.,

$$\begin{aligned}
&\big|\big((\mathbf{T}(s-t) - \mathbf{T}(s-r))F_k(s,\mathbf{w})\big)(x)\big| \\
&\qquad \le \sup_{r \in (1/\hat{n}, T),\ x \in B(\hat{l}),} |D_t(\mathbf{T}(r)F_k(s,\mathbf{w}))(x)|(t-r) \\
&\qquad \le C_{n,l}\|F_k(s,\mathbf{w})\|_\infty (t-r) \\
&\qquad \le C_{n,l}C(1 + H(T-s)^{-1/2})(t-r),
\end{aligned}$$

for any $t, r \in [0, T - 1/\hat{l}]$ and $x \in B(\hat{l})$, where $C_{n,l}$ is a positive constant which denotes the constant C_K in (6.34), related to the compact set $K = [1/\hat{n}, T] \times \overline{B(\hat{l})}$. Hence

$$I_1 \le C_{n,l}C[(T - t - 1/\hat{n}) + 2H(T - t - 1/\hat{n})^{1/2}](t-r).$$

The estimate of I_2 is trivial, since

$$|(\mathbf{T}(s-r)F_k(s,\mathbf{w}))(x)| \le MC(1 + H(T-s)^{-1/2}),$$

and so

$$\begin{aligned}
I_2 \le & MC\int_{r+1/\hat{n}}^{t+1/\hat{n}} (\sqrt{T} + 2)(1 + H(T-s)^{-1/2})ds \\
\le & MC\big(t - r + 2H(t + 1/\hat{n})^{1/2} - 2H(r + 1/\hat{n})^{1/2}\big) \\
\le & MC(T^{1/2} + 2H)(t-r)^{1/2}.
\end{aligned}$$

Finally, for I_3 we note that, for any $s \in (r + 1/\hat{n}, T)$ and $x \in B(\hat{l})$ we have

$$|\nabla_x(\mathbf{T}(s-r)F_k(s,\mathbf{w}))(x)| \le C_{n,l}\|F_k(s,\mathbf{w})\|_\infty \le C_{n,l}(1 + H(T-s)^{-1/2}).$$

Therefore

$$\begin{aligned}
&|\mathbf{g_w}(r,x) - \mathbf{g_w}(r,y)| \\
&= \Big|\int_{r+1/\hat{n}}^{T} \big((\mathbf{T}(s-r)F_k(s,\mathbf{w}))(x) - (\mathbf{T}(s-r)F_k(s,\mathbf{w}))(y)\big)ds\Big| \\
&\le |x-y|C_{n,l}\int_r^T (1 + H(T-s)^{-1/2})ds \\
&\le |x-y|C_{n,l}(2\sqrt{T} + 2H(T-r)^{1/2}).
\end{aligned}$$

We get

$$\begin{aligned}
|\mathbf{g_w}(t,x) - \mathbf{g_w}(r,y)| \le & (t-r)C_{n,l}C[T - t - 1/\hat{n} + 2H(T - t - 1/\hat{n})^{1/2}] \\
& + (t-r)^{1/2}MC(T^{1/2} + 2H) \\
& + |x-y|C_{n,l}(2\sqrt{T} + 2H(T-r)^{1/2}).
\end{aligned}$$

Next, we observe that

$$\begin{aligned}
&|\nabla_x \mathbf{g}_{\mathbf{w}}(t,x) - \nabla_x \mathbf{g}_{\mathbf{w}}(r,y)| \\
&\leq |\nabla_x \mathbf{g}_{\mathbf{w}}(t,x) - \nabla_x \mathbf{g}_{\mathbf{w}}(r,x)| + |\nabla_x \mathbf{g}_{\mathbf{w}}(r,x) - \nabla_x \mathbf{g}_{\mathbf{w}}(r,y)| \\
&\leq \left| \int_{t+1/\hat{n}}^{T} \nabla_x \big((\mathbf{T}(s-t) - \mathbf{T}(s-r)) F_k(s,\mathbf{w}) \big)(x) ds \right| \\
&+ \left| \int_{r+1/\hat{n}}^{t+1/\hat{n}} \nabla_x (\mathbf{T}(s-r) F_k(s,\mathbf{w}))(x) ds \right| \\
&+ |\nabla_x \mathbf{g}_{\mathbf{w}}(r,x) - \nabla_x \mathbf{g}_{\mathbf{w}}(r,y)| \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

By (6.34), for any compact set $K \subset (0,+\infty) \times \mathbb{R}^d$ we have

$$[\nabla_x T(\cdot)\mathbf{f}]_{C^{\alpha/2,\alpha}(K;\mathbb{R}^m)} \leq C_K \|\mathbf{f}\|_\infty. \tag{6.35}$$

Hence, arguing as in I_1, for J_1 we get

$$\nabla_x \big((\mathbf{T}(s-t) - \mathbf{T}(s-r)) F_k(s,\mathbf{w}) \big)(x) \leq C_{n,l} C (1 + H(T-s)^{1/2})(t-r)^{\alpha/2},$$

and so

$$J_1 \leq (t-r)^{\alpha/2} C_{n,l} C [T - t - 1/\hat{n} + 2H(T-t-1/\hat{n})^{1/2}].$$

As far as J_2 is concerned, we have

$$|\nabla_x (\mathbf{T}(s-r) F_k(s,\mathbf{w}))(x)| \leq \frac{C_T C}{\nu_0} \frac{(1 + H(T-s)^{-1/2})}{(s-r)^{1/2}}.$$

Therefore

$$\begin{aligned}
J_2 \leq & \frac{C_T C}{\nu_0} \int_{r+1/\hat{n}}^{t+1/\hat{n}} (s-r)^{-1/2} (1 + H(T-s)^{-1/2}) ds \\
\leq & (t-r)^{1/2} \sqrt{\hat{n}} \frac{C_T C (\sqrt{T} + 2H)}{\nu_0}.
\end{aligned}$$

By (6.35), in J_3 we have

$$\begin{aligned}
J_3 &\leq \Big| \int_{r+1/\hat{n}}^{T} \Big(\nabla_x (\mathbf{T}(s-r) F_k(s,\mathbf{w}))(x) - \nabla_x (\mathbf{T}(s-r) F_k(s,\mathbf{w}))(y) \Big) ds \Big| \\
&\leq |x-y|^\alpha C_{n,l} C \int_{r+1/\hat{n}}^{T} \left(1 + H(T-s)^{-1/2} \right) ds \\
&\leq |x-y|^\alpha C_{n,l} C \sqrt{T} \left(\sqrt{T} + \pi H \right).
\end{aligned}$$

Hence

$$|\nabla_x \mathbf{g}_{\mathbf{w}}(t,x) - \nabla_x \mathbf{g}_{\mathbf{w}}(r,y)| \leq (t-r)^{\alpha/2} C_{n,l} C [T - t - 1/\hat{n} + 2H(T-t-1/\hat{n})^{1/2}]$$

$$
\begin{aligned}
&+(t-r)^{1/2}\sqrt{\hat{n}}\frac{C_T C(\sqrt{T}+2H)}{\nu_0}\\
&+|x-y|^{\alpha}C_{n,l}C\sqrt{T}\left(\sqrt{T}+\pi H\right).
\end{aligned}
$$

Therefore $\{\mathbf{g_w}\}_{\mathbf{w}\in W}$ and $\{\nabla_x\mathbf{g_w}\}_{\mathbf{w}\in W}$ are equicontinuous in $[0,T-1/\hat{l}]\times B(\hat{l})$. By Ascoli-Arzelà Theorem it follows that Φ_k^n is compact from $\mathbf{K}_T$ in X_l, for any $k,l,n\in\mathbb{N}$, $n\geq l$. □

Proposition 6.21. *Suppose that Hypothesis* 6.3 *holds. Then, up to a subsequence,* $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ *and* $\{\nabla_x\mathbf{v}_n\}_{n\in\mathbb{N}}$ *converge locally uniformly in* $[0,T]\times\mathbb{R}^d$ *and* $[0,T)\times\mathbb{R}^d$*, respectively. If we denote by* $\mathbf{v}$ *the limit of* $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$*, then* $\mathbf{v}\in\mathbf{K}_T$ *and it is a mild solution of* (6.4).

Proof. We recall that, from Remark 6.19, we have that $\|\Phi_k^n(\mathbf{v}_m)\|_{\mathbf{K}_T}\leq\tilde{K}$, for any $k,m,n\in\mathbb{N}$. Moreover, to simplify the proof, we consider at first the case $T>1$, from which it follows that $\hat{n}=n$. The general case can be obtained with identical proof, provided that one replaces n with $\hat{n}$.

STEP 1**: convergence of** $\{\Phi_k^n(\mathbf{v}_m)\}_m$.

Since $\{\Phi_1^1(\mathbf{v}_m)\}_{m\in\mathbb{N}}$ is compact in X_1 by Lemma 6.18 and Proposition 6.20, there exists a subsequence $\{\mathbf{v}_{(1,1)_m}\}_{m\in\mathbb{N}}\subset\{\mathbf{v}_m\}_{m\in\mathbb{N}}$ such that $\{\Phi_1^1(\mathbf{v}_{(1,1)_m})\}_{m\in\mathbb{N}}$ converges in X_1. We define

$$\xi_{(1,1)}(t,x):=\lim_{m\to\infty}\Phi_1^1(\mathbf{v}_{(1,1)_m})(t,x).$$

Clearly

$$D_i\xi_{(1,1)}(t,x)=\lim_{m\to\infty}D_i\Phi_1^1(\mathbf{v}_{(1,1)_m})(t,x),$$

for any $i=1,\ldots,d$ and $(t,x)\in[0,T-1]\times B(1)$.

Now we consider the sequence $\{\Phi_1^2(\mathbf{v}_{(1,1)_m})\}_{m\in\mathbb{N}}$. Arguing as above, we deduce that there exists a subsequence $\{\mathbf{v}_{(2,1)_m}\}_{m\in\mathbb{N}}\subset\{\mathbf{v}_{(1,1)_m}\}_{m\in\mathbb{N}}$ such that $\{\Phi_1^2(\mathbf{v}_{(2,1)_m})\}_{m\in\mathbb{N}}$ converges in X_2.

Also for $\{\Phi_2^2(\mathbf{v}_{(2,1)_m})\}_{m\in\mathbb{N}}$ we can find a sequence $\{\mathbf{v}_{(2,2)_m}\}_{m\in\mathbb{N}}\subset\{\mathbf{v}_{(2,1)_m}\}_{m\in\mathbb{N}}$ such that $\{\Phi_2^2(\mathbf{v}_{(2,2)_m})\}_{m\in\mathbb{N}}$ converges in X_2. We set

$$\xi_{(2,j)}(t,x):=\lim_{m\to\infty}\Phi_j^2(\mathbf{v}_{(2,j)_m})(t,x),\quad j=1,2,$$

for any $(t,x)\in[0,T-1/2]\times B(2)$.

Iterating the above argument, we get that, for any $k,n\in\mathbb{N}$, $k\leq n$, there exists a sequence $\{\mathbf{v}_{(n,k)_m}\}_{m\in\mathbb{N}}\subset\{\mathbf{v}_{(l,j)_m}\}_{m\in\mathbb{N}}$, $l<n$ or $l=n$ and $j\leq k$, such that $\{\Phi_k^n(\mathbf{v}_{(n,k)_m})\}_{m\in\mathbb{N}}$ converges in X_n to $\xi_{n,k}$.

Let us observe that

$$\xi_{(l,j)}(t,x):=\lim_{m\to\infty}\Phi_j^l(\mathbf{v}_{(n,k)_m})(t,x),\tag{6.36}$$

$$D_i\xi_{(l,j)}(t,x):=\lim_{m\to\infty}D_i\Phi_j^l(\mathbf{v}_{(n,k)_m})(t,x),\tag{6.37}$$

for any $l < n$ or $l = n$ and $k \geq j$, for any $(t,x) \in [0, T - 1/l] \times B(l)$.

Now, for any $n \in \mathbb{N}$, we set

$$\xi_n := \xi_{(n,n)}, \quad \mathbf{w}_n := \mathbf{v}_{(n,n)_n}. \tag{6.38}$$

STEP 2: convergence of $\{\Phi_n^n(\mathbf{w}_n)\}_{n\in\mathbb{N}}$ and $\{D_i\Phi_n^n(\mathbf{w}_n)\}_{n\in\mathbb{N}}$, i=1,. . . ,d. For any $k,m,n \in \mathbb{N}$, $n \geq k$, and any $(t,x) \in [0, T - 1/k] \times B(k)$, we have

$$\begin{aligned}
&\Phi_n^n(\mathbf{w}_m)(t,x)\\
&\quad= (\mathbf{T}(T-t)\mathbf{f})(x) - \int_{t+1/n}^{T} (\mathbf{T}(s-t)F_n(s,\mathbf{w}_m))(x)ds\\
&\quad= (\mathbf{T}(T-t)\mathbf{f})(x) - \int_{t+1/k}^{T} (\mathbf{T}(s-t)F_k(s,\mathbf{w}_m))(x)ds\\
&\qquad+ \int_{t+1/k}^{T} \Big((\mathbf{T}(s-t)F_k(s,\mathbf{w}_m))(x) - (F_n(s,\mathbf{w}_m))(x) \Big) ds\\
&\qquad- \int_{t+1/n}^{t+1/k} (\mathbf{T}(s-t)F_n(s,\mathbf{w}_m))(x)ds\\
&\quad= \Phi_k^k(\mathbf{w}_m)(t,x) - \int_{t+1/n}^{t+1/k} (\mathbf{T}(s-t)F_n(s,\mathbf{w}_m))(x)ds\\
&\qquad+ \int_{t+1/k}^{T} \Big((\mathbf{T}(s-t)F_k(s,\mathbf{w}_m))(x) - (\mathbf{T}(s-t)F_n(s,\mathbf{w}_m))(x) \Big) ds.
\end{aligned} \tag{6.39}$$

We consider the sequence $\{\Phi_n^n(\mathbf{w}_n)\}_{n\in\mathbb{N}}$. Fix $l \in \mathbb{N}$, $(t,x) \in [0, T - 1/l) \times B(l)$, and $k,n,m \in \mathbb{N}$ such that $n,m \geq k > l$. Hence, for any fixed $\varepsilon > 0$, we have

$$\begin{aligned}
&= |\Phi_n^n(\mathbf{w}_n)(t,x) - \Phi_m^m(\mathbf{w}_m)(t,x)| \leq |\Phi_k^k(\mathbf{w}_n)(t,x) - \Phi_k^k(\mathbf{w}_m)(t,x)|\\
&\quad+ \left| \int_{t+1/k}^{T} \Big((\mathbf{T}(s-t)F_k(s,\mathbf{w}_n))(x) - (\mathbf{T}(s-t)F_n(s,\mathbf{w}_n))(x) \Big) ds \right|\\
&\quad+ \left| \int_{t+1/k}^{T} \Big((\mathbf{T}(s-t)F_k(s,\mathbf{w}_m))(x) - (\mathbf{T}(s-t)F_m(s,\mathbf{w}_m))(x) \Big) ds \right|\\
&\quad+ \left| \int_{t+1/n}^{t+1/k} (\mathbf{T}(s-t)F_n(s,\mathbf{w}_n))(x)ds \right| + \left| \int_{t+1/m}^{t+1/k} (\mathbf{T}(s-t)F_m(s,\mathbf{w}_m))(x)ds \right|\\
&=: A_{n,m,k}(t,x) + B_{n,k}(t,x) + B_{m,k}(t,x) + C_{n,k}(t,x) + C_{m,k}(t,x),
\end{aligned}$$

where

$$\begin{aligned}
A_{n,m,k}(t,x) &= |\Phi_k^k(\mathbf{w}_n)(t,x) - \Phi_k^k(\mathbf{w}_m)(t,x)|,\\
B_{n,k}(t,x) &= \left| \int_{t+1/k}^{T} \Big((\mathbf{T}(s-t)(F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)))(x) \Big) ds \right|,
\end{aligned} \tag{6.40}$$

$$C_{n,k}(t,x) = \left| \int_{t+1/n}^{t+1/k} (\mathbf{T}(s-t)F_n(s,\mathbf{w}_n))(x)ds \right|. \tag{6.41}$$

Since $\{\mathbf{w}_n\}_{n\geq k}, \{\mathbf{w}_m\}_{m\geq k} \subset \{\mathbf{v}_{(k,k)_j}\}_{j\in\mathbb{N}}$, from (6.36) it follows that $\{\Phi_k^k(\mathbf{w}_n)(t,x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^m$. Hence, there exists $N_k \in \mathbb{N}$ such that

$$A_{n,m,k}(t,x) \leq \frac{\varepsilon}{5}, \quad n,m \geq N_k.$$

We fix $0 < \tilde{\varepsilon} < 1/l$ such that

$$2MC(\tilde{\varepsilon} + 2K\sqrt{\tilde{\varepsilon}}) \leq \frac{\varepsilon}{10},$$

and we observe that

$$\begin{aligned} M\int_{T-\tilde{\varepsilon}}^{T} \|F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)\|_\infty ds &\leq 2M\int_{T-\tilde{\varepsilon}}^{T} C(1+K(T-s)^{-1/2})ds \\ &\leq 2MC(\tilde{\varepsilon} + 2K\sqrt{\tilde{\varepsilon}}). \end{aligned} \tag{6.42}$$

Hence,

$$\begin{aligned} B_{n,k}(t,x) &\leq M\int_t^T \|F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)\|_\infty ds \\ &\leq M\int_{T-\tilde{\varepsilon}}^T \|F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)\|_\infty ds \\ &\quad + M\int_t^{T-\tilde{\varepsilon}} \|F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)\|_\infty ds \\ &\leq \frac{\varepsilon}{10} + M\int_t^{T-\tilde{\varepsilon}} \|F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)\|_\infty ds. \end{aligned} \tag{6.43}$$

From Lemma 6.18, for any fixed $s \in [0,T)$ we have $\|Q^{1/2}\nabla\mathbf{w}_n(s,\cdot)\|_\infty \leq K(T-s)^{-1/2}$. Hence, if $k \geq K(T-s)^{-1/2}$, from Lemma 6.16 we deduce that

$$\|F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)\|_\infty \leq \frac{2C(2+K(T-s)^{-1/2})}{k^\alpha}. \tag{6.44}$$

Indeed, in the right-hand side of (6.25) it remains only the first addend, since the others vanish when $|z| \geq k$, and

$$\begin{aligned} |F_k(s,\mathbf{w}_n)(x) - F_n(s,\mathbf{w}_n)(x)| &\leq C(2+|Q^{1/2}(x)\nabla\mathbf{w}_n(s,x)|)\left(\frac{1}{n^\alpha} + \frac{1}{k^\alpha}\right) \\ &\leq \frac{2C(2+|Q^{1/2}(x)\nabla\mathbf{w}_n(s,x)|)}{k^\alpha}, \end{aligned} \tag{6.45}$$

for any $(s,x) \in [0,T) \times \mathbb{R}^d$ and $n \geq k$. From estimate (6.44) we obtain

$$B_{n,k}(t,x) \leq \frac{\varepsilon}{10} + \frac{2C(2+K/\sqrt{\tilde{\varepsilon}})}{k^\alpha}\int_t^{T-\tilde{\varepsilon}} ds$$

$$\leq \frac{\varepsilon}{10} + 2TC(2 + K/\sqrt{\tilde{\varepsilon}})\frac{1}{k^\alpha}, \tag{6.46}$$

for any $k \geq K/\sqrt{\tilde{\varepsilon}}$. This follows from the fact that $s \leq T - \tilde{\varepsilon}$ implies $(T-s)^{-1/2} \leq (\tilde{\varepsilon})^{-1/2}$. Hence, there exists $\bar{k}_1 = \bar{k}_1(\tilde{\varepsilon})$ such that

$$B_{n,k}(t,x) \leq \varepsilon/5,$$

for any $n \geq \bar{k}_1$. Obviously, we also have

$$B_{m,k}(t,x) \leq \varepsilon/5,$$

for any $m \geq \bar{k}_1$.

Finally, by (6.26) and

$$\begin{aligned}(T-s)^{-1/2} &\leq (T-t-1/k)^{-1/2} \leq (1/l - 1/k)^{-1/2} \\ &\leq (1/l - 1/(l+1))^{-1/2} \leq l+1,\end{aligned}$$

for any $s \leq t + 1/k$ and $k \geq l+1$, it follows that

$$\begin{aligned}C_{n,k}(t,x) &\leq \int_{t+1/n}^{t+1/k} MC(1 + K(T-s)^{-1/2})ds \\ &\leq MC\left(1 + K\left(\frac{1}{l} - \frac{1}{k}\right)^{-1/2}\right)\left(\frac{1}{k} - \frac{1}{n}\right) \\ &\leq \frac{MC(1 + K(l+1))}{k},\end{aligned}$$

for any $n > k \geq l+1$. Hence, there exists $\bar{k}_2 \in \mathbb{N}$ such that

$$C_{n,k}(t,x), C_{m,k}(t,x) \leq \varepsilon/5,$$

for any $n, m \leq \bar{k}_2$. Therefore, for any $n, m \geq \max\{N_k, \bar{k}_1, \bar{k}_2\}$, we have

$$|\Phi_n^n(\mathbf{w}_n)(t,x) - \Phi_m^m(\mathbf{w}_m)(t,x)| \leq \varepsilon,$$

for any $(t,x) \in [0, T - 1/(l)) \times B(l)$.

It follows that $\{\Phi_n^n(\mathbf{w}_n)(t,x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^m$. In a similar way we can prove that $\{D_i\Phi_n^n(\mathbf{w}_n)(t,x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, for all $i = 1, \ldots, d$. Indeed, we have

$$\begin{aligned}|D_i\Phi_n^n(\mathbf{w}_n)(t,x) - D_i\Phi_m^m(\mathbf{w}_m)(t,x)| \leq{}& \tilde{A}_{n,m,k}(t,x) + \tilde{B}_{n,k}(t,x) + \tilde{B}_{m,k}(t,x) \\ &+ \tilde{C}_{n,k}(t,x) + \tilde{C}_{m,k}(t,x),\end{aligned}$$

where

$$\tilde{A}_{n,m,k}(t,x) = |D_i\Phi_k^k(\mathbf{w}_n)(t,x) - D_i\Phi_k^k(\mathbf{w}_m)(t,x)|,$$

$$\tilde{B}_{n,k}(t,x) = \left| \int_{t+1/k}^{T} D_i\big(\mathbf{T}(s-t)(F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n))\big)(x)ds \right|, \tag{6.47}$$

$$\tilde{C}_{n,k}(t,x) = \left| \int_{t+1/n}^{t+1/k} D_i(\mathbf{T}(s-t)F_n(s,\mathbf{w}_n))(x)ds \right|. \tag{6.48}$$

From (6.37) we know that there exists $N_k \in \mathbb{N}$ such that, for any $n, m \geq N_k$, it holds that

$$\tilde{A}_{n,m,k}(t,x) \leq \frac{\varepsilon}{5}.$$

$\tilde{B}_{n,k}$ can be estimated as above. We recall that

$$\begin{aligned}
&\|D_i\big(\mathbf{T}(s-t)(F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n))\big)\|_\infty \\
&\quad \leq \frac{C_T\|F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)\|_\infty}{\sqrt{\nu_0}(s-t)^{1/2}},
\end{aligned}$$

and $(t,x) \in [0, T-1/l] \times B(l)$. Hence, from (6.45), fixed $0 < \tilde{\varepsilon} < 1/(l+1)$ which satisfies

$$\frac{2C_T C(l+1)}{\sqrt{\nu_0}}(\tilde{\varepsilon} + 2K\sqrt{\tilde{\varepsilon}}) \leq \frac{\varepsilon}{10},$$

we obtain

$$\begin{aligned}
\tilde{B}_{n,k}(t,x) &\leq \frac{C_T}{\sqrt{\nu_0}} \int_t^T \frac{\|F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)\|_\infty}{(s-t)^{1/2}}ds \\
&= \frac{C_T}{\sqrt{\nu_0}} \int_{T-\tilde{\varepsilon}}^T \frac{\|F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)\|_\infty}{(s-t)^{1/2}}ds \\
&\quad + \frac{C_T}{\sqrt{\nu_0}} \int_t^{T-\tilde{\varepsilon}} \frac{\|F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)\|_\infty}{(s-t)^{1/2}}ds.
\end{aligned}$$

The first addend can be estimated as follows:

$$\begin{aligned}
&\frac{C_T}{\sqrt{\nu_0}} \int_{T-\tilde{\varepsilon}}^T \frac{\|F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)\|_\infty}{(s-t)^{1/2}}ds \\
&\qquad \leq \frac{2C_T C}{\sqrt{\nu_0}}(T-\tilde{\varepsilon}-t)^{-1/2} \int_{T-\tilde{\varepsilon}}^T (1 + (T-s)^{-1/2})ds \\
&\quad \leq \frac{2C_T C(l^2+l)^{1/2}}{\sqrt{\nu_0}}(\tilde{\varepsilon} + 2K\sqrt{\tilde{\varepsilon}}) \\
&\quad \leq \frac{\varepsilon}{10}.
\end{aligned}$$

As far as the second one is concerned, we get

$$\frac{C_T}{\sqrt{\nu_0}} \int_t^{T-\tilde{\varepsilon}} \frac{\|F_k(s,\mathbf{w}_n) - F_n(s,\mathbf{w}_n)\|_\infty}{(s-t)^{1/2}}ds$$

$$\leq \frac{2C_T C(2+K/\sqrt{\tilde{\varepsilon}})}{\sqrt{\nu_0}} \frac{1}{k^\alpha} \int_t^T (s-t)^{-1/2} ds$$
$$\leq \frac{2\sqrt{T} C_T C(2+K/\sqrt{\tilde{\varepsilon}})}{\sqrt{k^\alpha \nu_0}},$$

for any $k \geq K/\sqrt{\tilde{\varepsilon}}$. Hence, there exists $\bar{k}_1 = \bar{k}_1(\tilde{\varepsilon})$ such that

$$\tilde{B}_{n,k}(t,x) \leq \varepsilon/5,$$

for any $n \geq \bar{k}_1$. Obviously, we also have

$$\tilde{B}_{m,k}(t,x) \leq \varepsilon/5,$$

for any $m \geq \bar{k}_1$.

Finally, we consider $\tilde{C}_{n,k}(t,x)$. From (6.26) it follows that

$$\tilde{C}_{n,k}(t,x) \leq \int_{t+1/n}^{t+1/k} \frac{C_T C(1+K(T-s)^{-1/2})}{\sqrt{\nu_0}(s-t)^{1/2}} ds$$
$$\leq \frac{C_T C(1+K(l+1))}{\sqrt{\nu_0}} \int_{t+1/n}^{t+1/k} (s-t)^{-1/2} ds$$
$$\leq \frac{2C_T C(1+K(l+1))}{\sqrt{\nu_0}} k^{-1/2}.$$

Hence, there exists $\bar{k}_2 \in \mathbb{N}$ such that

$$\tilde{C}_{n,k}(t,x), \tilde{C}_{m,k}(t,x) \leq \varepsilon/5,$$

for any $n, m \leq \bar{k}_2$.

We have so proved that $\{\Phi_n^n(\mathbf{w}_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X_l for any $l \in \mathbb{N}$. We set

$$\mathbf{v}(t,x) := \lim_{n\to\infty} \Phi_n^n(\mathbf{w}_n)(t,x),$$

for any $t \in [0,T)$, any $x \in \mathbb{R}^d$. Of course, $\mathbf{v} \in X_l$ for any $l \in \mathbb{N}$.

STEP 3: convergence of $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$.

Now we show that also $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ converges in X_l, for any $l \in \mathbb{N}$. Indeed, from (6.39) we have

$$\mathbf{w}_n(t,x) = \Phi_{(n,n)_n}^n(\mathbf{w}_n)(t,x) - \int_t^{t+1/n} (\mathbf{T}(s-t)F_{(n,n)_n}(s,\mathbf{w}_n))(x) ds$$
$$= \Phi_n^n(\mathbf{w}_n)(t,x) - B_{(n,n)_n,n}(t,x) - D_n(t,x),$$

where $B_{n,k}$ has been defined in (6.40), and

$$D_n(t,x) := \int_t^{t+1/n} (\mathbf{T}(s-t)F_{(n,n)_n}(s,\mathbf{w}_n))(x) ds.$$

$B_{(n,n)_n,n}(t,x)$ goes to 0, as $n \to \infty$. This easily follows from the computations in (6.42) - (6.46) about $B_{n,k}(t,x)$. Also $D_n(t,x)$ vanishes, as $n \to \infty$, because the function under the integral sign is bounded in a neighborhood of t; indeed,

$$\|\mathbf{T}(s-t)F_{(n,n)_n}(s,\mathbf{w}_n)\|_\infty \le MC(1+K(T-s)^{-1/2}).$$

Hence

$$\lim_{n\to\infty} \mathbf{w}_n(t,x) = \lim_{n\to\infty} \Phi_n^n(\mathbf{w}_n)(t,x) = \mathbf{v}(t,x),$$

for any $(t,x) \in [0,T) \times \mathbb{R}^d$.

The same reasoning holds for $\{D_i\mathbf{w}_n\}_{n\in\mathbb{N}}$, and so we have that

$$\nabla_x \mathbf{v}(t,x) = \lim_{n\to\infty} D_i\mathbf{w}_n(t,x), \quad i = 1,\dots,d,$$

for any $(t,x) \in [0,T) \times \mathbb{R}^d$, and both the convergences are locally uniformly in $[0,T) \times \mathbb{R}^d$.

STEP 4: v is a mild solution of (6.4) and $\mathbf{v} \in \mathbf{K}_T$.

For any $x, y \in \mathbb{R}^d$, for any $t, s \in [0,T)$, we have

$$\begin{aligned} &|\mathbf{v}(s,y)| + (T-t)^{1/2}|Q^{1/2}(x)\nabla_x\mathbf{v}(t,x)| \\ &= \lim_{n\to\infty}\Big(|\mathbf{w}_n(s,y)| + (T-t)^{1/2}|Q^{1/2}(x)\nabla_x\mathbf{w}_n(t,x)|\Big) \\ &\le \limsup_{n\to\infty}\Big(\|\mathbf{w}_n\|_\infty + (T-t)^{1/2}\|Q^{1/2}(\cdot)\nabla_x\mathbf{w}_n(t,\cdot)\|_\infty\Big) \\ &\le K. \end{aligned} \tag{6.49}$$

We recall that

$$\mathbf{w}_n(t,x) = (\mathbf{T}(T-t)\mathbf{f})(x) - \int_t^T (\mathbf{T}(s-t)F_{(n,n)_n}(s,\mathbf{w}_n))(x)ds,$$

that $\nabla_x\mathbf{w}_n(s,\cdot)$ converges to $\nabla_x\mathbf{v}(s,\cdot)$ locally uniformly, and $F_k(s,\mathbf{u})(x) = \psi_k(x, Q^{1/2}(x)\nabla_x\mathbf{u}(s,x))$. By the properties of $\mathbf{T}$ and ψ_k, and the hypotheses on ψ, we deduce that the function under the integral sign converges to

$$(\mathbf{T}(s-t)F(s,\mathbf{v}))(x)(s),$$

pointwise with respect to s. Moreover,

$$\left\|(\mathbf{T}(s-t)F_{(n,n)_n}(s,\mathbf{w}_n))\right\|_\infty \le MC\left(1 + \frac{K}{(T-s)^{1/2}}\right).$$

The dominated convergence theorem implies that the integral in the right-hand side converges to

$$\int_t^T (\mathbf{T}(s-t)F(s,\mathbf{v}))(x)(s)ds,$$

as $n \to \infty$. Also $\mathbf{w}_n(t,x)$ converges to $\mathbf{v}(t,x)$. Hence we can conclude that

$$\mathbf{v}(t,x) = (\mathbf{T}(T-t)\mathbf{f})(x) - \int_t^T (\mathbf{T}(s-t)F(s,\mathbf{v}))(x)ds,$$

for any $t \in [0,T)$, any $x \in \mathbb{R}^d$. Moreover, we can extend by continuity $\mathbf{v}$ in T setting $\mathbf{v}(T,\cdot) = \mathbf{f}$. Putting together with (6.49), we conclude that $\mathbf{v} \in \mathbf{K}_T$. □

6.3 The System of Forward Backward Stochastic Differential Equations

We consider the System of Forward Backward Stochastic Differential Equations (S-FBSDE)

$$\begin{cases} d\mathbf{Y}_\tau = \mathbf{H}(X_\tau, \mathbf{Z}_\tau)d\tau + \mathbf{Z}_\tau dW_\tau, & \tau \in [t,T], \\ dX_\tau = b(X_\tau)d\tau + G(X_\tau)dW_\tau, & \tau \in [t,T], \\ \mathbf{Y}_T = \mathbf{g}(X_T), & \\ X_t = x, & x \in \mathbb{R}^d, \end{cases} \tag{6.50}$$

where

$$\mathbf{H} : \mathbb{R}^d \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^m,$$

is a given Borel function, and $b, G, \mathbf{g}$ have been defined above. The processes $\mathbf{Y}$ and $\mathbf{Z}$ take values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively, and the first and the third equation are considered component by component.

For any $p \in [1, \infty)$, let $\mathbb{H}^p$ be the space of progressively measurable with respect to $\mathcal{F}_t^W$ random processes X_t such that

$$\|X\|_{\mathbb{H}^p} := \mathbb{E} \sup_{t\in[0,T]} |X_t|^p < \infty,$$

and let $\mathbb{K}$ be the space of $(\mathcal{F}_t^W)-$progressively measurable processes $\mathbf{Y}, \mathbf{Z}$ such that

$$\|(\mathbf{Y}, \mathbf{Z})\|_{cont}^2 := \mathbb{E} \sup_{t\in[0,T]} |\mathbf{Y}_t|^2 + \mathbb{E} \int_0^T |\mathbf{Z}_\sigma|^2 d\sigma < \infty.$$

Moreover, we denote by $\mathbf{Y}(\tau, t, x)$ and $\mathbf{Z}(\tau, t, x)$ the solution to (6.50).

We introduce the differential operator $\mathcal{A}$ defined on smooth functions ϕ by

$$(\mathcal{A}\phi)(x) = Tr[Q(x)D^2\phi_j(x)] + \langle b(x), \nabla_x \phi\rangle,$$

where $G = Q^{1/2}$.

We assume the following hypotheses on $\mathbf{H}$.

Hypotheses 6.22. (i) *There exist* $(\gamma_i)_{jk} \in C^\alpha_{loc}(\mathbb{R}^d)$, $i = 1, \dots, d$, $j, k = 1, \dots, m$, *and a function* $\psi \in C(\mathbb{R}^d \times \mathbb{R}^m; \mathbb{R}^m)$ *such that*

$$\begin{aligned} &\mathbf{H}_j(x,z) = \sum_{i=1}^d \sum_{k=1}^m (\gamma_i)_{jk}(x) z_k^i + \psi_j(x,z), \quad j = 1, \dots, m, \\ &|\mathbf{H}(x,z)| \le C_1(1 + |z|), \end{aligned} \tag{6.51}$$

for any $x \in \mathbb{R}^d$, $z \in \mathbb{R}^{m\times d}$, *and*

$$\left|\sum_{l=1}^{d}(\gamma_l)_{jk}(x)Q_{li}^{1/2}(x)\right|^2 \le C_2\nu(x), \quad i=1,\dots,d,\ j=1,\dots,m, \tag{6.52}$$

for some positive constants C_1, C_2*;*

(ii) ψ *satisfies*

$$|\psi(x_1,z_1)-\psi(x_2,z_2)| \le C(1+|z_1|\vee|z_2|)\left(|x_1-x_2|^\alpha+|z_1-z_2|^\alpha\right),$$

for any $x\in\mathbb{R}^d$*, any* $z_1,z_2\in\mathbb{R}^{d\times m}$*, some* $\alpha\in(0,1)$ *and some positive constant* C*.*

Remark 6.23. *The functions* $(\gamma_i)_{jk}$ *are bounded, for any* $j,k=1,\dots,m$*. Indeed we have*

$$\begin{aligned}\sum_{i=1}^{d}|(\gamma_i)_{jk}(x)| &= \sum_{i=1}^{d}\left|\sum_{h,l=1}^{d}Q_{ih}^{-1/2}(x)Q_{hl}^{1/2}(x)(\gamma_l)_{jk}(x)\right|^2\\ &= \sum_{i=1}^{d}\left|\sum_{h=1}^{d}Q_{ih}^{-1/2}(x)\sum_{l=1}^{d}\left(Q_{hl}^{1/2}(x)(\gamma_l)_{jk}(x)\right)\right|^2\\ &= \left|Q^{-1/2}(x)\left(Q^{1/2}(x)(\gamma)_{jk}(x)\right)\right|^2\\ &\le \frac{1}{\nu(x)}\left|Q^{1/2}(x)(\gamma)_{jk}(x)\right|^2\\ &\le C_2^2.\end{aligned}$$

Remark 6.24. *It easy to see that* (6.51) *and* (6.52)

$$|\psi(t,z)| \le C(1+|z|),$$

for some $C>0$.

Remark 6.25. *We split* $\mathbf{H}$ *into the sum of different terms in order to generalize the condition on its smoothness. Indeed, a coefficient of a linear term with respect to* z *in* $\mathbf{H}$ *may have the Hölder's constant which depends on* x*, but in this case it has to satisfy the growth condition* (6.52)*. Otherwise, if its Hölder's constant depends only on* z*, it has to fulfill only* (6.51)*.*

Hereafter, we assume the following additional assumptions on b and G.

Hypothesis 6.26. *There exists* $C>0$ *such that, for all* $x,x'\in\mathbb{R}^d$*, we have*

$$|b(x)-b(x')|+|G(x)-G(x')| \le C|x-x'|. \tag{6.53}$$

If Hypothesis 6.26 is satisfied and

$$|\mathbf{g}(x)| + |\mathbf{H}(x,0)| \leq C(1+|x|^k), \quad x \in \mathbb{R}^d.$$

for some $k \in \mathbb{R}^+$, then system (6.50) admits a unique solution $(X, \mathbf{Y}, \mathbf{Z})$, where $X \in \mathbb{H}^p$, for any $p \in [1, \infty)$, and $(\mathbf{Y}, \mathbf{Z}) \in \mathbb{K}$ (see [89]). Henceforth, X denotes the solution to the forward equation in (6.50).

Now we introduce the system of differential equations

$$D_t\mathbf{u}(t,x) + A\mathbf{u}(t,x) = \mathbf{H}(x, Q^{1/2}(x)\nabla_x \mathbf{u}(t,x)).$$

By (6.51) we obtain

$$D_t\mathbf{u}(t,x) + \mathbf{A}\mathbf{u}(t,x) = \psi(x, Q^{1/2}(x)\nabla_x \mathbf{u}(t,x)),$$

where

$$(\mathbf{A}\phi)_j(x) = Tr[Q(x)D^2\phi_j(x)] + \sum_{k=1}^{m} \langle (B)_{jk}(x), \nabla_x \phi_k \rangle, \quad j = 1, \dots, m,$$

$$(B_i)_{jk}(x) = -\sum_{l=1}^{d} (\gamma_l)_{jk}(x) Q_{li}^{1/2}(x) + b_i(x)\delta_{jk}.$$

Hence the growth of the drift term satisfies Hypothesis 6.4(iii), with $C = C_2$.

This means that the Cauchy problem

$$\begin{cases} D_t\mathbf{u}(t,x) + A\mathbf{u}(t,x) = \mathbf{H}(x, Q^{1/2}(x)\nabla_x \mathbf{u}(t,x)), & t \in [0,T), \quad x \in \mathbb{R}^d, \\ \mathbf{u}(T,x) = \mathbf{g}(x), & x \in \mathbb{R}^d, \end{cases} \tag{6.54}$$

and

$$\begin{cases} D_t\mathbf{u}(t,x) + \mathbf{A}\mathbf{u}(t,x) = \psi(x, Q^{1/2}(x)\nabla_x \mathbf{u}(t,x)), & t \in [0,T), \quad x \in \mathbb{R}^d, \\ \mathbf{u}(T,x) = \mathbf{g}(x), & x \in \mathbb{R}^d, \end{cases} \tag{6.55}$$

are equivalent.

(6.54) (or, equivalently, (6.55)) is strictly connected to (6.50). Indeed, if $\mathbf{u} \in C^{1,2}([0,T] \times \mathbb{R}^d; \mathbb{R}^m)$ is a classical solution to (6.55), then $\mathbf{u}(t,x) = \mathbf{Y}(t,t,x)$. Conversely, if $\mathbf{H}, \mathbf{g}, b, G$, satisfy the following hypotheses, then, setting $\mathbf{u}(t,x) = \mathbf{Y}(t,t,x)$, it turns out that $\mathbf{u} \in C^{0,1}([0,T] \times \mathbb{R}^d; \mathbb{R}^m)$ and it is a mild solution to (6.55) (see [89]).

Hypotheses 6.27. **(i)** *b and G are of class C^1 and their derivatives of order 1 are bounded;*

(ii) *$\mathbf{g}$ is of class C^1 and it has polynomial growth together with its derivatives of order 1;*

(iii) *$\mathbf{H}(t,\cdot,\cdot,\cdot)$ is of class C^1, for all $t \in [0,T]$;*

(iv) $|\nabla_x \mathbf{H}(t,x,y,z)| \leq K(1+|z|)(1+|x|+|y|)^\mu$ *for suitable constants* $K, \mu \geq 0$*;*

(v) $\nabla_y \mathbf{H}$ *and* $\nabla_z \mathbf{H}$ *are bounded with respect to* y *and* z*.*

We want to relax regularity conditions on $\mathbf{H}$ and $\mathbf{g}$, and growth conditions on b and G, and prove that $\mathbf{u}$ is still a solution to (6.55). For this purpose, we will use the results in Section 6.2.

Now we approximate both $\mathbf{g}$ and $\mathbf{H}$ by means of convolution; let $\{\rho_n^d\}_{n\in\mathbb{N}}$, $\rho_n^{m\times,d}$ be a standard sequence of mollifiers in $\mathbb{R}^d$, $\mathbb{R}^{m\times d}$, respectively. We set

$$\mathbf{g}_n := \mathbf{g} \star \rho_n^d, \quad (\mathbf{H}_n)_j(x,z) := \theta_n(z)(\mathbf{H}_j \star (\rho_n^d \rho_n^{m\times d}))(x,z), \quad n \in \mathbb{N},$$

where $\theta_n \in C_c^\infty(\mathbb{R}^{m\times d})$ and $\chi_{B(n)} \leq \theta_n \leq \chi_{B(n+1)}$. For any $n \in \mathbb{N}$ we have

$$\begin{aligned}(\mathbf{H}_n)_j(x,z) &= \theta_n(z)\sum_{i=1}^d \sum_{k=1}^m z_k^i \int_{\mathbb{R}^d} \rho_n^d(w)(\gamma_i)_{jk}(x-w)dw \\ &\quad + \theta_n(z)(\psi_j \star (\rho_n^d \rho_n^{m\times d}))(x,z) \\ &= \theta_n(z)\sum_{i=1}^d \sum_{k=1}^m ((\gamma_n)_i)_{jk}(x) z_k^i + (\tilde{\psi}_n)_j(x,z),\end{aligned}$$

where

$$\begin{aligned}\tilde{\psi}_n(x,z) &:= \int_{\mathbb{R}^d \times \mathbb{R}^{m\times d}} \rho_n^d(w)\rho_n^{m\times d}(y)\psi(x-w, z-y)dwdy, \\ ((\gamma_n)_i)_{jk}(x) &:= \int_{\mathbb{R}^d} \rho_n^d(w)(\gamma_i)_{jk}(x-w)dw.\end{aligned}$$

Lemma 6.28. $\{((\gamma_n)_i)_{jk}\}_{n\in\mathbb{N}}$ *converges to* $(\gamma_i)_{jk}$ *locally uniformly.*

Proof. The proof is trivial, since $(\gamma_i)_{jk} \in C_{\text{loc}}^\alpha(\mathbb{R}^d)$, for any $i = 1,\ldots,d$ and $j,k = 1,\ldots,m$. □

Remark 6.29. *For any* $n \in \mathbb{N}$*,* $\mathbf{g}_n$ *and* $\mathbf{H}_n$ *satisfy Hypotheses* 6.27*(ii)* − *(iv).*

The statement of the following lemma is a byproduct of straightforward computations, hence we skip the proof.

Lemma 6.30. *For any* $n, k \in \mathbb{N}$*,* $n \leq m$*, and* $x \in \mathbb{R}^d$*,* $z_1, z_2 \in \mathbb{R}^{m\times d}$*, we have*

$$\begin{aligned}|\tilde{\psi}_n(x,z_1) - \tilde{\psi}_n(x,z_2)| &\leq C_n|z_1 - z_2|, \\ |\tilde{\psi}_n(x,0)| &\leq C, \\ |\tilde{\psi}_n(x,z) - \psi(x,z)| &\leq \frac{2C(2+|z|)}{n^\alpha}\chi_{B(n)}(|z|) + 2C(2+|z|)\chi_{B(n)^c}(|z|), \\ |\tilde{\psi}_n(x,z) - \tilde{\psi}_k(x,z)| &\leq \left(\frac{2C(2+|z|)}{n^\alpha} + \frac{2C(2+|z|)}{k^\alpha}\right)\chi_{(B(k))}(|z|)\end{aligned}$$

$$
\begin{aligned}
& + \left(\frac{2C(2+|z|)}{n^\alpha} + 2C(2+|z|)\right)\chi_{(B(n)\cap B(k)^c)}(|z|) \\
& + 4C(2+|z|)\chi_{B(n)^c}(|z|), \qquad (6.56) \\
|\tilde{\psi}_n(x,z_1) - \psi(x,z_2)| \le {} & \chi_{B(n)}(|z_1|)C\Big(2+|z_1|+|z_2|\Big)\big(n^{-\alpha} + |z_1-z_2|^\alpha\big) \\
& + \chi_{B(n)^c}(|z_1|)C(2+|z_1|+|z_2|), \qquad (6.57) \\
|\tilde{\psi}_n(x,z_1) - \tilde{\psi}_k(x,z_2)| \le {} & \chi_{B(n)}(|z_1|)C\Big(2+|z_1|+|z_2|\Big)\big(n^{-\alpha} + k^{-\alpha} + 2|z_1-z_2|^\alpha\big) \\
& + \chi_{B(n)^c}(|z_1|)\chi_{B(k)}(|z_2|)C\big(k^{-\alpha} + |z_1-z_2|^\alpha + 1\big) \\
& + 2\chi_{B(k)^c}(|z_2|)C(2+|z_1|+|z_2|), \qquad (6.58)
\end{aligned}
$$

for some positive constants C and C_n.

For any $n \in \mathbb{N}$, we consider the approximate problem

$$
\begin{cases}
D_t \mathbf{v}_n(t,x) + \mathbf{A}\mathbf{v}_n(t,x) = \psi_n(x, Q^{1/2}(x)\nabla_x \mathbf{v}_n(t,x)), & t \in [0,T), \quad x \in \mathbb{R}^d, \\
\mathbf{v}_n(T,x) = \mathbf{g}_n(x), & x \in \mathbb{R}^d.
\end{cases}
\qquad (6.59)
$$

and its unique mild solution (see Subsection 6.2.1)

$$
\mathbf{v}_n(t,x) = (\mathbf{T}(T-t)\mathbf{g}_n)(x) - \int_t^T \big(\mathbf{T}(s-t)(\psi_n(\cdot, Q^{1/2}(\cdot)\nabla_x \mathbf{v}_n(s,\cdot)))\big)(x)ds,
$$

for any $t \in [0,T)$, any $x \in \mathbb{R}^d$.

Moreover, Since $\mathbf{g} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$, Proposition 5.10 implies that $\mathbf{T}(T-t)\mathbf{g}_n$ converges to $\mathbf{T}(T-t)\mathbf{g}$ locally uniformly. Hence, we can apply the same procedure of Subsection 6.2.2 in order to prove that, up to a subsequence, $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ locally uniformly converges to a function $\mathbf{v} \in \mathbf{K}_T$ and

$$
\mathbf{v}(t,x) = (\mathbf{T}(T-t)\mathbf{g})(x) - \int_t^T \big(\mathbf{T}(s-t)(\psi(\cdot, Q^{1/2}(\cdot)\nabla_x \mathbf{v}(s,\cdot)))\big)(x)ds,
$$

for any $t \in [0,T)$, any $x \in \mathbb{R}^d$.

From Remark 6.29 we have that

$$
\mathbf{v}_n(t,x) = \mathbf{Y}^n(t,t,x), \quad G(x)\nabla_x \mathbf{v}_n(t,x) = \mathbf{Z}^n(t,t,x),
$$

for any $t \in [0,T)$, $x \in \mathbb{R}^d$, where $(X, \mathbf{Y}^n, \mathbf{Z}^n) \in \mathbb{H}^p \times \mathbb{K}$ is the unique solution to

$$
\begin{cases}
d(\mathbf{Y}^n)_\tau = \mathbf{H}_n(X_\tau, \mathbf{Z}^n_\tau)d\tau + \mathbf{Z}^n_\tau dW_\tau, & \tau \in [t,T], \\
dX_\tau = b(X_\tau)d\tau + G(X_\tau)dW_\tau, & \tau \in [t,T], \\
(\mathbf{Y}^n)_T = \mathbf{g}_n(X_T), & \\
X_t = x, & x \in \mathbb{R}^d,
\end{cases}
\qquad (6.60)
$$

Hence, if for any $0 \le t \le \tau \le T$ and $x \in \mathbb{R}^d$ we define $\mathbf{Y}(\tau,t,x) := \mathbf{v}(\tau, X(\tau,t,x))$ and $\mathbf{Z}(\tau,t,x) := G(X_\tau(\tau,t,x))\mathbf{v}(\tau,X(\tau,t,x))$, then we have the following result.

Theorem 6.31. *If Hypotheses* 6.4, 6.6, 6.22 *and* 6.26 *hold, then* $(\mathbf{Y},\mathbf{Z}) \in \mathbb{K}$ *and* $(X,\mathbf{Y},\mathbf{Z})$ *is a solution to* (6.50).

Proof. We prove that $(X,\mathbf{Y},\mathbf{Z})$ is a solution to (6.50). Since $\mathbf{Y}^n, \mathbf{Z}^n$ are solutions of (6.60), and the equalities hold $\mathbb{P}-$a.s., there exists a family of elements of $\mathcal{F}$, $\{\Omega_n\}$, such that each of them has zero measure. Moreover, if we set $\tilde{\Omega} = \cup_n \Omega_n$, then $\mathbb{P}(\tilde{\Omega}) = 0$, and in $\tilde{\Omega}^c$ (6.60) holds, for any $n \in \mathbb{N}$.

Now we fix $x \in \mathbb{R}^d$, $t \in [0,T]$, set $X_\tau := X(\tau,t,x)$, and define

$$\mathbf{Y}_\tau = \mathbf{v}(\tau,X_\tau),\ \mathbf{Y}^n_\tau = \mathbf{v}_n(\tau,X_\tau),\ \mathbf{Z}_\tau = G(X_\tau)\nabla_x \mathbf{v}(\tau,X_\tau),\ \mathbf{Z}^n_\tau = G(X_\tau)\nabla_x \mathbf{v}_n(\tau,X_\tau),$$

for any $\tau \in [t,T]$. Since $\mathbf{v}_n$ and $D_i\mathbf{v}_n$, for any $i = 1,\dots,d$, converge locally uniformly, we have

$$\mathbf{Y}^n_\tau \longrightarrow \mathbf{Y}_\tau, \quad \mathbf{g}_n(X_T) \longrightarrow \mathbf{g}(X_T),$$

and

$$\int_\tau^T \mathbf{H}_n(X_\sigma,\mathbf{Z}^n_\sigma)d\sigma \longrightarrow \int_\tau^T \mathbf{H}(X_\sigma,\mathbf{Z}_\sigma)d\sigma,$$

pointwise in Ω.

Indeed, from Remark 6.23 and Lemma 6.30 we deduce that

$$\begin{aligned}
|\mathbf{H}_n(X_\sigma,\mathbf{Z}^n_\sigma) - \mathbf{H}(X_\sigma,Z_\sigma)| &= |\tilde{\psi}_n(X_\sigma,\mathbf{Z}^n_\sigma) - \psi(X_\sigma,\mathbf{Z}_\sigma)| \\
&\quad + |\theta_n(\mathbf{Z}^n)((\gamma_n))_j(X_\sigma)\mathbf{Z}^n - (\gamma)_j(X_\sigma)\mathbf{Z}| \\
&\le C\Big(2 + |\mathbf{Z}^n_\sigma| + |\mathbf{Z}_\sigma|\Big)\big(n^{-\alpha} + |\mathbf{Z}^n_\sigma - \mathbf{Z}_\sigma|^\alpha\big) \\
&\quad + |\big(((\gamma_n))_j(X_\sigma) - (\gamma)_j(X_\sigma)\big)\mathbf{Z}_\sigma| + C_2^2|\mathbf{Z}^n_\sigma - \mathbf{Z}_\sigma|, \\
|\psi(X_\sigma,\mathbf{Z}_\sigma)|,\ |\tilde{\psi}_n(X_\sigma,\mathbf{Z}^n_\sigma)| &\le C(1 + K(T-\sigma)^{-1/2}), \\
\mathbf{Z}^n_\sigma, \mathbf{Z}_\sigma &\le K(T-\sigma)^{-1/2}, \\
\|((\gamma_n)_i)_{jk}\|_\infty, \|(\gamma_i)_{jk}\|_\infty &\le C_2^2,
\end{aligned}$$

for any $x \in \mathbb{R}^d$ and $\sigma \in [\tau,T)$ and $n > K(T-s)^{-1/2}$, where K has been defined in Lemma 6.18. $|\mathbf{Z}^n_\sigma - \mathbf{Z}_\sigma|$ tends to zero uniformly in Ω, as $n \to +\infty$, since, for any $\omega \in \Omega$,

$$\begin{aligned}
|\mathbf{Z}^n_\sigma(\omega) - \mathbf{Z}_\sigma(\omega)| &\le |G(X_\sigma(\omega))\nabla(\mathbf{u}_n - \mathbf{u})(\sigma,X_\sigma(\omega))| \\
&\to 0, \quad n \to \infty.
\end{aligned}$$

Moreover, from Lemma 6.28 deduce that, for any $\sigma \in [0,T)$ and $\omega \in \Omega$,

$$|((\gamma_n))_j(X_\sigma) - (\gamma)_j(X_\sigma)| \to 0,$$

as $n \to \infty$.

Since in (**??**) $|\psi(X_\sigma, \mathbf{Z}_\sigma)|$, $|\psi_n(X_\sigma, \mathbf{Z}^n_\sigma)|$ can be estimated by an integrable function, we can apply dominated convergence to the integral term.

It remains to prove the convergence of $\int_\tau^T \mathbf{Z}^n_\sigma dW_\sigma$ to $\int_\tau^T \mathbf{Z}_\sigma dW_\sigma$. At first, we prove that $\int_\tau^T Z_\sigma dW_\sigma$ makes sense, since this is not guaranteed by previous estimates, which show only that the growth $\mathbf{Z}_\sigma$ can be estimated by $(T-\sigma)^{-1/2}$, which is not square integrable in T.

We are going to show that $\{\mathbf{Z}^n_\tau\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $L^2(\Omega \times (0,T))$, the space of the square integrable processes V, endowed with the norm $\mathbb{E}\int_0^T |V_\sigma|^2 d\sigma$. Since this is a Hilbert space, $\{\mathbf{Z}^n_\tau\}$ converges to a process $\tilde{\mathbf{Z}}_\tau$ which is square integrable, and so, up to a subsequence, $\{\mathbf{Z}^n_\tau\}$ converges to $\tilde{\mathbf{Z}}_\tau$ $[0,T] \otimes \mathbb{P}-$a.s. But $\{\mathbf{Z}^n_\tau\}$ converges to $\mathbf{Z}_\tau$ uniformly, hence pointwise, for any $\tau \in [0,T]$. Therefore, $\tilde{\mathbf{Z}}_\tau = \mathbf{Z}_\tau$ $\mathbb{P}-$a.s., for almost every $\tau \in [0,T]$. This means that $\mathbf{Z}_\sigma$ is a square integrable process.

For the reader's convenience, we introduce some new notations:

$$
\begin{aligned}
\overline{\mathbf{Y}}^{n,k}_\sigma &:= \mathbf{Y}^n_\sigma - \mathbf{Y}^k_\sigma, \\
\overline{\mathbf{Z}}^{n,k}_\sigma &:= \mathbf{Z}^n_\sigma - \mathbf{Z}^k_\sigma, \\
\overline{\mathbf{g}}^{n,k}_\sigma &:= \mathbf{g}_n(X_\sigma) - \mathbf{g}_k(X_\sigma), \\
\overline{\mathbf{H}}^{n,k}_\sigma &:= \mathbf{H}_n(X_\sigma, \mathbf{Z}^n_\sigma) - \mathbf{H}_k(X_\sigma, \mathbf{Z}^k_\sigma),
\end{aligned}
$$

for any $n,k \in \mathbb{N}$, $\sigma \in [0,T]$. By the Itô formula, we get

$$
\begin{aligned}
d|\overline{\mathbf{Y}}^{n,k}_\tau|^2 = &-2\langle \overline{\mathbf{Y}}^{n,k}_\tau, \overline{\mathbf{H}}^{n,k}_\tau \rangle d\tau - 2\langle \overline{\mathbf{Y}}^{n,k}_\tau, \overline{\mathbf{Z}}^{n,k}_\tau \rangle dW_\tau \\
&+ |\overline{\mathbf{Z}}^{n,k}_\tau|^2 d\tau,
\end{aligned}
$$

and, recalling that $\overline{\mathbf{Y}}^{n,k}_T = \overline{\mathbf{g}}^{n,k}_T$, we obtain

$$
\begin{aligned}
|\overline{\mathbf{Y}}^{n,k}_\tau|^2 + \int_\tau^T |\overline{\mathbf{Z}}^{n,k}_\sigma|^2 d\sigma = &|\overline{\mathbf{g}}^{n,k}_T|^2 - 2\int_\tau^T \langle \overline{\mathbf{Y}}^{n,k}_\sigma, \overline{\mathbf{H}}^{n,k}_\sigma \rangle d\sigma \\
&- 2\int_\tau^T \langle \overline{\mathbf{Y}}^{n,k}_\sigma, \overline{\mathbf{Z}}^{n,k}_\sigma \rangle dW_\sigma.
\end{aligned}
$$

Let us estimate the terms in the right-hand side. Note that $(\mathbf{Y}^n, \mathbf{Z}^n), (\mathbf{Y}^k, \mathbf{Z}^k) \in \mathbb{K}$, since they are solutions of a backward stochastic differential equation. Hence, the process $I_\tau = \displaystyle\int_0^\tau \langle \overline{\mathbf{Y}}^{n,k}_\sigma, \overline{\mathbf{Z}}^{n,k}_\sigma \rangle dW_\sigma$ is a martingale and, in particular, $\mathbb{E}I_\tau = 0$, for any τ. Computing the expectation, we get

$$
\mathbb{E}|\overline{\mathbf{Y}}^{n,k}_\tau|^2 + \mathbb{E}\int_\tau^T |\overline{\mathbf{Z}}^{n,k}_\sigma|^2 d\sigma = \mathbb{E}|\overline{\mathbf{g}}^{n,k}_T|^2 - 2\mathbb{E}\int_\tau^T \langle \overline{\mathbf{Y}}^{n,k}_\sigma, \overline{\mathbf{H}}^{n,k}_\sigma \rangle d\sigma. \tag{6.61}
$$

Moreover, the last term in the right-hand side of (6.61) can be estimated as follows.

$$\begin{aligned}\mathbb{E}\int_\tau^T |\overline{\mathbf{Y}}^{n,k}_\sigma \overline{\mathbf{H}}^{n,k}_\sigma| d\sigma &\le \mathbb{E}\left(\sup_{\tau\in[0,T]} |\overline{\mathbf{Y}}^{n,k}_\tau| \int_\tau^T |\overline{\mathbf{H}}^{n,k}_\sigma| d\sigma\right)\\ &\le \sup_{n\in\mathbb{N}} \|\mathbf{u}_n\|_\infty \mathbb{E}\int_\tau^T |\overline{\mathbf{H}}^{n,k}_\sigma| d\sigma,\end{aligned}$$

and

$$\begin{aligned}|\overline{\mathbf{H}}^{n,k}_\sigma| &= |\tilde{\psi}_n(X_\sigma, \mathbf{Z}^n_\sigma) - \tilde{\psi}_k(X_\sigma, \mathbf{Z}^k_\sigma)| + |\theta_n(\mathbf{Z}^n_\sigma)\gamma^j(x)\mathbf{Z}^n_\sigma - \theta_m(\mathbf{Z}^m_\sigma)\gamma^j(x)\mathbf{Z}^m_\sigma|\\ &\le \Big(2 + |\mathbf{Z}^n_\sigma| + |\mathbf{Z}^k_\sigma|\Big)\big(n^{-\alpha} + k^{-\alpha} + 2|\mathbf{Z}^n_\sigma - \mathbf{Z}^k_\sigma|^\alpha\big)\\ &\quad + |\big(((\gamma_n))_j(X_\sigma) - (\gamma_k)_j(X_\sigma)\big)\mathbf{Z}^n_\sigma| + C_2^2|\mathbf{Z}^n_\sigma - \mathbf{Z}^k_\sigma|,\end{aligned}$$

for any $n, m > K(T-s)^{-1/2}$.

By the definitions of $\mathbf{Z}^n, \overline{\mathbf{Z}}^{n,k}, ((\gamma_n)_i)_{jk}$ and the above estimates, it is easy to prove, using dominated convergence, that, for any $\varepsilon > 0$, there exists $\bar{n} \in \mathbb{N}$ such that $\mathbb{E}\int_0^T |\overline{\mathbf{Z}}^{n,k}_\sigma| d\sigma \le \varepsilon$, for any $n, k \ge \bar{n}$.

The same arguments can be applied to $\overline{\mathbf{g}}^{n,k}_T$. Indeed, recalling that $\mathbf{g}$ is uniformly continuous, for any $\varepsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ such that $\mathbb{E}|\overline{\mathbf{g}}^{n,k}_T|^2 \le \varepsilon$, for any $n, k \ge \bar{n}$.

Hence $\{\mathbf{Z}^n_\tau\}$ is a Cauchy sequence in $L^2(\Omega \times (0,T))$, and this implies that $\int_\tau^T \mathbf{Z}_\sigma dW_\sigma$ makes sense. Moreover, since $\mathbf{Z}^n$ converges to $\mathbf{Z}$ in $L^2(\Omega \times (0,T))$, we see that

$$\mathbb{E}\left|\int_\tau^T (\mathbf{Z}^n_\sigma - \mathbf{Z}_\sigma) dW_\sigma\right|^2 \longrightarrow 0, \quad n \to \infty.$$

We can conclude that $\int_\tau^T \mathbf{Z}^n_\sigma dW_\sigma$ tends to $\int_\tau^T \mathbf{Z}_\sigma dW_\sigma$ $\mathbb{P}-$a.s., and passing to the limit (6.60), we obtain that the processes $(X, \mathbf{Y}, \mathbf{Z})$ are a solution to (6.50) $\mathbb{P}-$a.s. □

Corollary 6.32. *For any $t \in [0,T]$, the law of the process $\{\mathbf{Y}_\tau\}_{\tau\in[t,T]}$, obtained as limit of the sequence $\{\mathbf{Y}^m_\tau\}_{\tau\in[t,T]}$, is uniquely determined. This means that if $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{\mathbb{P}})$ are two probability spaces, and $\{\mathbf{Y}_\tau\}_{\tau\in[t,T]}$ and $\{\tilde{\mathbf{Y}}_\tau\}_{\tau\in[t,T]}$ are the random processes of Theorem* 6.31*, in such a these spaces, then $\{\mathbf{Y}_\tau\}_{\tau\in[t,T]}$ and $\{\tilde{\mathbf{Y}}_\tau\}_{\tau\in[t,T]}$ have the same law.*

Proof. The result is a straightforward consequence of the uniqueness in law for the solutions $\{\mathbf{Y}^m\}_{m\in\mathbb{N}}$ of approximated problems (6.60), and of the $\mathbb{P}-$a.s. convergence of $\{\mathbf{Y}^m\}_{m\in\mathbb{N}}$ to $\mathbf{Y}$. □

6.4 Existence of a Nash Equilibrium for Stochastic Differential Games

We recall that, for any fixed $u \in U$, $X_t^{(u)}$ is the weak solution to

$$\begin{cases} dX_\tau^{(u)} = b(X_\tau^{(u)})d\tau + G(X_\tau^{(u)})r(X_\tau^{(u)}, u_\tau)d\tau + G(X_\tau^{(u)})dW_\tau^{(u)}, \quad \tau \in [t,T], \\ X_t = x \in \mathbb{R}^d, \end{cases} \tag{6.62}$$

and that the cost functional associated to the $i-$th player is

$$J^i(u) = \mathbb{E}^{(u)}\left[\int_0^T h^i(X_s^{(u)}, u_s)ds + g^i(X_T)\right], \quad i = 1, \dots, m. \tag{6.63}$$

We provide sufficient conditions in order to get a control $\tilde{u} \in U$ which realize a Nash equilibrium. At first, we define the Hamiltonian function associated to the (6.62) with cost functionals (6.63).

Definition 6.33. *We define the function* $\tilde{\mathbf{H}} : \mathbb{R}^d \times \mathbb{R}^{m\times d} \times U \longrightarrow \mathbb{R}^m$ *as*

$$\tilde{\mathbf{H}}^i(x,z,u) := \langle z^i, r(x,u)\rangle + h^i(x,u), \tag{6.64}$$

for any $x \in \mathbb{R}^d$, $z \in \mathbb{R}^{m\times d}$, $u \in U$.

We assume the following hypotheses on r and h^i, $i = 1, \dots, m$.

Hypotheses 6.34. *There exists* $L > 0$ *and two measurable functions* $r_1 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $r_2 : \mathbb{R}^d \times U \longrightarrow \mathbb{R}^d$, *such that* $r(x,u) = r_1(x) + r_2(x,u)$ *for any* $x \in \mathbb{R}^d$ *and* $u \in U$, *and for all* $x, x' \in \mathbb{R}^d, u, u' \in U$, *for any* $i = 1, \dots, m$, *we have*

$$\begin{gathered} |h^i(x,u) - h^i(x',u')| + |r_2(x,u) - r_2(x',u')| \le L\left(|x-x'|^\gamma + |u-u'|^\gamma\right), \\ |h^i(x,u)| + |r_1(x)| + |r_2(x,u)| \le L, \end{gathered} \tag{6.65}$$

for some $\gamma \in (0,1)$. *Moreover,* $r_1 \in C^\gamma_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$.

We require that $\tilde{\mathbf{H}}$ satisfies the following hypothesis.

Hypotheses 6.35. *We assume that* $\tilde{H}$ *satisfies the generalized minimax condition (GMC for short), i.e., for any* $i = 1, \dots, m$ *there exists a continuous function* $\tilde{u} : \mathbb{R}^d \times \mathbb{R}^{m\times d} \longrightarrow U$ *such that for any* $x \in \mathbb{R}^d$, $z \in \mathbb{R}^{m\times d}$ *and* $u^i \in U^i$, *for any* $i = 1, \dots, m$, $\mathbb{P}-$*a.s the following inequality holds:*

$$\tilde{\mathbf{H}}^i(x,z,\tilde{u}(x,z)) \le \tilde{\mathbf{H}}^i(x,z,\tilde{u}^1(x,z), \dots, \tilde{u}^{i-1}(x,z), u^i, \tilde{u}^{i+1}(x,z), \dots, \tilde{u}^m(x,z)). \tag{6.66}$$

Moreover, there exist a positive constant K *and* $\beta \in (0,1)$ *such that*

$$|\tilde{u}(x_1,z_1) - \tilde{u}(x_2,z_2)| \le K\left(|x_1-x_2|^\beta + |z_1-z_2|^\beta\right), \tag{6.67}$$

for any $x_1, x_2 \in \mathbb{R}^d$ *and any* $z_1, z_2 \in \mathbb{R}^{m\times d}$.

Definition 6.36. *If set* $\mathbf{H}(x,z) := \tilde{\mathbf{H}}^i(x,z,\tilde{u}(x,z))$ *and* $\psi(x,z) := \langle z^i, r_2(x,\tilde{u}(x,z))\rangle + h^i(x,\tilde{u}(x,z))$*, then we can write*

$$\mathbf{H}^i(x,z) = \langle z^i, r_1(x)\rangle + \psi^i(x,z),$$

for any $x \in \mathbb{R}^d$ *and* $z \in \mathbb{R}^{m\times d}$.

Lemma 6.37. *The function* ψ *satisfies the following inequalities:*

$$\begin{aligned} |\psi(x_1,z_1)-\psi(x_2,z_2)| &\le C(1+|z_1|\vee|z_2|)\left(|x_1-x_2|^\alpha+|z_1-z_2|^\alpha\right), \\ & x\in\mathbb{R}^d,\quad z_1,z_2\in\mathbb{R}^{d\times m}, \\ |\psi(x,z)| &\le C(1+|z|),\quad x\in\mathbb{R}^d,\ z\in\mathbb{R}^{d\times m}, \end{aligned} \tag{6.68}$$

for a suitable positive constant C *and* $\alpha = \beta\gamma$*, where* γ *and* β *have been defined in Hypotheses 6.34 and 6.35, respectively.*

The above Lemma shows that the function $\mathbf{H}$ satisfies Hypotheses 6.22, with $(\gamma_i)_{jk}(x) = (r_1(x))_i\delta_{jk}$, for any $i = 1,\dots,d$ and $j,k = 1,\dots,m$. Therefore the results of previous section hold, and in particular we will use the fact that the system (6.50) admits a solution $(X,\mathbf{Y},\mathbf{Z})$ to prove the existence of a Nash equilibrium.

Theorem 6.38. *There exists a Nash equilibrium for* (6.62) *and* (6.63).

Proof. We set

$$\tilde{u}_t^{-1} := (u_t^1, \tilde{u}_t^2, \dots, \tilde{u}_t^d).$$

Hence $X_\tau^{(\tilde{u}^{-1})}$ is solution

$$\begin{aligned} X_\tau^{(\tilde{u}^{-1})} &= x + \int_t^\tau b(X_\sigma^{(\tilde{u}^{-1})})d\sigma + \int_t^\tau G(X^{(\tilde{u}^{-1})})r(X_\sigma^{(\tilde{u}^{-1})},\tilde{u}_\sigma^{-1})d\sigma \\ &\quad + \int_t^\tau G(X_\sigma^{(\tilde{u}^{-1})})dW_\sigma^{(u)}, \quad \tau\in[t,T] \\ &= x + \int_t^\tau b(X_\sigma^{(\tilde{u}^{-1})})d\sigma + \int_t^\tau G(X_\sigma^{(\tilde{u}^{-1})})d\tilde{W}_\sigma, \end{aligned}$$

where

$$\tilde{W}_\tau := W_\tau^{(u^{-1})} + \int_t^{t\wedge\tau} r(X_\sigma^{(\tilde{u}^{-1})},\tilde{u}_\sigma^{-1})d\sigma,$$

and by $\tilde{\mathbb{P}}$ we denote the probability with respect to $\tilde{W}$ is a Brownian motion. Further, $X^{(\tilde{u}^{-1})}$ is measurable with respect to the $\sigma-$field generated by $\tilde{W}$.

We introduce the backward system

$$\mathbf{Y}_\tau + \int_t^\tau \mathbf{Z}_\sigma d\tilde{W}_\sigma = \mathbf{g}(X_T^{(\tilde{u}^{-1})}) + \int_t^\tau \tilde{\mathbf{H}}(X_\sigma^{(\tilde{u}^{-1})},\mathbf{Z}_\sigma,\tilde{u}_\sigma^{-1})d\sigma.$$

By Theorem 6.31 there exists a solution $(X^{(\tilde{u}^{-1})},\mathbf{Y},\mathbf{Z})$ of this system.

Writing the backward system with respect to $W^{(\tilde{u}^{-1})}$, we get

$$\begin{aligned}\mathbf{Y}_\tau + \int_\tau^T \mathbf{Z}_\sigma dW_\sigma^{(\tilde{u}^{-1})} + \int_\tau^T \mathbf{Z}_\sigma r(X_\sigma^{(\tilde{u}^{-1})}, \tilde{u}_\sigma^{-1}) d\sigma = \\ \mathbf{g}(X_T^{(\tilde{u}^{-1})}) + \int_\tau^T \mathbf{H}(X_\sigma^{(\tilde{u}^{-1})}, \mathbf{Z}_\sigma) d\sigma.\end{aligned} \tag{6.69}$$

We have that $\mathbb{E}^{(\tilde{u}^{-1})} \left(\int_0^T |\mathbf{Z}_t|^2 dt \right)^{1/2} < \infty$. Indeed, we have

$$\left| \int_\tau^T \mathbf{Z}_\sigma dW_\sigma^{(\tilde{u}^{-1})} \right| \leq 2 \sup_{\tau \in [0,T]} |\mathbf{Y}_\tau| + \int_\tau^T \big(|\mathbf{Z}_\sigma| + |\mathbf{H}(X_\sigma^{(\tilde{u}^{-1})}, \mathbf{Z}_\sigma)| \big) d\sigma.$$

Taking into account the Burkholder-Davis-Gundy inequalities, we get

$$\begin{aligned}
&\mathbb{E}^{(\tilde{u}^{-1})} \left(\int_0^T |\mathbf{Z}_t|^2 dt \right)^{1/2} \\
&\quad \leq c\mathbb{E}^{(\tilde{u}^{-1})} \sup_{\tau \in [0,T]} |\mathbf{Y}_\tau| + c\mathbb{E}^{(\tilde{u}^{-1})} \int_\tau^T \big(|\mathbf{Z}_\sigma| + |\mathbf{H}(X_\sigma^{(\tilde{u}^{-1})}, \mathbf{Z}_\sigma)| \big) d\sigma \\
&\quad \leq c\mathbb{E}^{(\tilde{u}^{-1})} \sup_{\tau \in [0,T]} |\mathbf{Y}_\tau| + c\mathbb{E}^{(\tilde{u}^{-1})} \int_\tau^T \big(|\mathbf{Z}_\sigma| + |\mathbf{H}(X_\sigma^{(\tilde{u}^{-1})}, 0)| \big) d\sigma \\
&\quad + c\mathbb{E}^{(\tilde{u}^{-1})} \int_\tau^T |\mathbf{Z}_\sigma|^{1+\alpha} d\sigma \\
&\quad \leq c(\tilde{\mathbb{E}}\rho^2)^{1/2} \Big(\tilde{\mathbb{E}} \sup_{\tau \in [0,T]} |\mathbf{Y}_\tau| + \tilde{\mathbb{E}} \int_\tau^T \big(|\mathbf{Z}_\sigma|^2 + |\mathbf{H}(X_\sigma^{(\tilde{u}^{-1})}, 0)|^2 \big) d\sigma \Big)^{1/2} \\
&\quad + c(\tilde{\mathbb{E}}\rho^{2/(1-\alpha)})^{(1-\alpha/)2} \Big(\tilde{\mathbb{E}} \int_\tau^T |\mathbf{Z}_\sigma|^2 d\sigma \Big)^{2/(1+\alpha)} \\
&\quad < \infty,
\end{aligned}$$

where $\rho = d\mathbb{P}^{(\tilde{u}^{-1})}/d\tilde{\mathbb{P}}$ is the Girsanov density, and $\tilde{\mathbb{E}}$ denotes the conditional expectation with respect to $\tilde{\mathbb{P}}$.

Hence, taking the conditional expectation in (6.69) with respect to $\mathbb{P}^{(\tilde{u}^{-1})}$ and $\tau = t$, we obtain

$$\begin{aligned}
\mathbf{Y}_t = &\mathbb{E}^{(\tilde{u}^{-1})} \left[\varphi(X_T^{(\tilde{u}^{-1})}) | \mathcal{F}_t \right] \\
&+ \mathbb{E}^{(\tilde{u}^{-1})} \left[\int_t^T \Big(\mathbf{H}(X_\sigma^{(\tilde{u}^{-1})}, \mathbf{Z}_\sigma, \tilde{u}_\sigma^{-1}) - \mathbf{Z}_\sigma r(X_\sigma^{(\tilde{u}^{-1})}, \tilde{u}_\sigma^{-1}) \Big) d\sigma | \mathcal{F}_t \right].
\end{aligned}$$

Adding and subtracting $\mathbb{E}^{(\tilde{u}^{-1})}\left[\int_t^T \mathbf{h}(X_\sigma^{(\tilde{u}^{-1})}, \tilde{u}_\sigma^{-1}) d\sigma | \mathcal{F}_t\right]$, and setting $t = 0$, we get

$$\begin{aligned}
\mathbf{Y}_0 = \mathbf{J}(\tilde{u}^{-1}) + \mathbb{E}^{(\tilde{u}^{-1})} & \left[\int_t^T \Big(\mathbf{H}(X_\sigma^{(\tilde{u}^{-1})}, \mathbf{Z}_\sigma,) - \mathbf{Z}_\sigma r(X_\sigma^{(\tilde{u}^{-1})}, \tilde{u}_\sigma^{-1}) \right. \\
& \left. -\mathbf{h}(X_\sigma^{(\tilde{u}^{-1})}, \tilde{u}_\sigma^{-1})\Big)\, d\sigma\right] \qquad (6.70) \\
= \mathbf{J}(\tilde{u}^{-1}) + \mathbb{E}^{(\tilde{u}^{-1})} & \left[\int_t^T \Big(\mathbf{H}(X_\sigma^{(\tilde{u}^{-1})}, \mathbf{Z}_\sigma) - \tilde{\mathbf{H}}(X^{(\tilde{u}^{-1})}, \mathbf{Z}_\sigma, \tilde{u}_\sigma^{-1})\Big)\, d\sigma\right],
\end{aligned}$$

where $\mathbf{J} = (J^1, \ldots, J^m)$, and considering the first component we conclude that

$$\mathbf{Y}_0^1 \leq J^1(\tilde{u}^{-1}).$$

The thesis follows because we can apply the same reasoning to any $i = 2, \ldots, m$. □

Appendix A

Basic Stochastic Calculus

For the results of this appendix, we refer to the monograph [101].

A.1 Introduction

Definition A.1. *Let I be a nonempty index set and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family $\{X(t) : t \in I\}$ of random variables from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}^m$ is called Stochastic process. For any $\omega \in \Omega$, the map $\omega \mapsto X(t, \omega)$ is called Sample path.*

Hereafter, we will consider $I = [0, T]$ or $I = [0, \infty)$. Moreover, we recall that a Polish space is a separable complete metric space.

Now, we provide some useful definitions for the continuous.

Definition A.2. *Let $X(t), \bar{X}(t)$ be two stochastic processes. We say that X and $\bar{X}$ are stochastically equivalent if*

$$X(t) = \bar{X}(t), \quad \mathbb{P} - a.s., \quad \forall t \in I. \tag{A.1}$$

In this case, one is called a modification of the other.

Definition A.3. *The process $X(t)$ is said to be stochastically continuous at $s \in I$ if for any $\epsilon > 0$ we have*

$$\lim_{t \to s} \mathbb{P}\{\omega \in \Omega : |X(t, \omega) - X(s, \omega)| > \epsilon\} = 0.$$

Moreover, $X(t)$ is said to be continuous if there exists a $\mathbb{P}-$null set $\mathcal{N}$ such that $t \mapsto X(t, \omega)$ is continuous for any $\omega \in \Omega \setminus \mathcal{N}$.

We introduce an increasing family of $\sigma-$fields $\mathcal{F}_t \subset \mathcal{F}$, for any $t \in I$, with $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ for any $t_1, t_2 \in I$, $t_1 \leq t_2$. Such a family is called *filtration.* Set $\mathcal{F}_{t+} = \cap_{s>t} \mathcal{F}_s$, we say that $\{\mathcal{F}_t\}_{t \in I}$ is *right continuous* if $\mathcal{F}_{t+} = \mathcal{F}_t$. We call $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ a *filtered probability space.*

Definition A.4. *We say that* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *satisfies the usual condition if* $(\Omega, \mathcal{F}, \mathbb{P})$ *is complete,* $\mathcal{F}_0$ *contains all the* $\mathbb{P}-$*null sets in* $\mathcal{F}$ *and* $\{\mathcal{F}_t\}_{t\in I}$ *is right continuous.*

Definition A.5. *Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$ *be a filtered probability space and* $X(t)$ *be a process taking values in a metric space* (U, d)*. The process* $X(t)$ *is said to be:*

(i) *measurable if the map* $(t, \omega) \mapsto X(t, \omega)$ *is* $(B(I) \times \mathcal{F}, B(U))-$*measurable;*

(ii) $\{\mathcal{F}_t\}_{t\in I}-$*adapted if for any* $t \in I$*, the map* $\omega \mapsto X(t, \omega)$ *is* $(\mathcal{F}_t, B(U)-$*measurable;*

(iii) $\{\mathcal{F}_t\}_{t\in I}-$*progressively measurable if for any* $t \in I$ *the map* $(s, \omega) \mapsto X(s, \omega)$ *is* $(B([0, t]) \times \mathcal{F}_t, B(U))-$*measurable.*

It is clear that if $X(t)$ is $\{\mathcal{F}_t\}_{t\in I}-$progressively measurable, then it is also $\{\mathcal{F}_t\}-$adapted. Conversely, on a filtered probability space we have the following result.

Proposition A.6. *Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P}))$ *be a filtered probability space, and let* $X(t)$ *be an* $\{\mathcal{F}_t\}_{t\in I}-$*adapted process. Then there exists an* $\{\mathcal{F}_t\}_{t\in I}-$*progressively measurable process* $\bar{X}(t)$ *which is stochastically equivalent to* X*. Moreover, if* $X(t)$ *is also right continuous, then* $X(t)$ *itself is* $\{\mathcal{F}_t\}_{t\in I}-$*progressively measurable.*

Next we set

$$\mathbb{W}^n := C([0, \infty); \mathbb{R}^n), \tag{A.2}$$

and define the following spaces:

$$\begin{aligned} \mathbb{W}^n_t &:= \{\xi(\cdot \wedge t) | \xi \in \mathbb{W}^n\}, \quad \forall t \in [0, \infty), \\ B_t(\mathbb{W}^n) &:= \sigma(B(\mathbb{W}^n_t)), \quad \forall t \in [0, \infty), \\ B_{t+}(\mathbb{W}^n) &:= \bigcap_{s>t} \sigma(B(\mathbb{W}^n_s)), \quad \forall t \in [0, \infty). \end{aligned} \tag{A.3}$$

$B_t(\mathbb{W}^n)$ and $B_{t+}(\mathbb{W}^n)$ contain $\mathbb{W}^n$, for any $t \in [0, \infty)$, and clearly both

$$\begin{aligned} &(\mathbb{W}^n, B(\mathbb{W}^n), \{B_t(\mathbb{W}^n)\}_{t\geq 0}), \\ &(\mathbb{W}^n, B(\mathbb{W}^n), \{B_{t+}(\mathbb{W}^n)\}_{t\geq 0}), \end{aligned}$$

are filtered measurable spaces.

Moreover, for any Polish space U let $A^n(U)$ be the set of all $\{B_{t+}(\mathbb{W}^n)\}_{t\geq 0}-$progressive measurable processes $\eta : [0, \infty) \times \mathbb{W}^n \longrightarrow U$. Then we have the following result.

Theorem A.7. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a completely probability space and* (U, d) *a Polish space. Let* $\xi : [0, \infty) \times \Omega \longrightarrow \mathbb{R}^n$ *be a continuous process and* $\mathcal{F}^\xi_t := \sigma\{\xi(s) | 0 \leq s \leq t\}$*. Then* $\varphi : [0, \infty) \times \Omega \longrightarrow U$ *is* $\{\mathcal{F}_t\}_{t\geq 0}-$*adapted if and only if there exists* $\eta \in A^n(U)$ *such that*

$$\eta(t, \xi(\cdot \wedge t, \omega)) = \varphi(t, \omega), \quad \mathbb{P} - a.s. - \omega \in \Omega, \ \forall t \in [0, \infty).$$

Finally, it is possible defining a metric $\hat{\rho}$ on $\mathbb{W}^n$ under which $\mathbb{W}^n$ is a Polish space. This metric is defined by

$$\hat{\rho}(w, \tilde{w}) = \sum_{j=1}^{\infty} 2^{-j} [|w - \tilde{w}|_{C([0,j];\mathbb{R}^n)} \wedge 1], \quad \forall w, \tilde{w} \in \mathbb{W}^n. \tag{A.4}$$

A.2 Stochastic Differential Equations

The goal of this section is the study of Stochastic Differential Equations (SDE's for short) which can be considered as an extension of Ordinary Differential Equations (ODE's for short). An SDE of Itô type has the form

$$\begin{cases} dX_t = b(t, X)dt + \sigma(t, X)dW_t, \quad t > 0, \\ X_0 = \xi, \end{cases} \tag{A.5}$$

where $b \in A^n(\mathbb{R}^n)$ is the drift term, $\sigma \in A^n(\mathbb{R}^{n\times m})$ is the diffusion one and ξ is a random variable. In the above equation the unknown is the process X, and it is understood in the integral sense, which means that we want to solve the integral equation

$$X(t) = x + \int_0^t b(s, X)ds + \int_0^t \sigma(s, X)dW(s). \tag{A.6}$$

At first, we deal with the measurability of the drift term.

Lemma A.8. *Let $b \in A^n(\mathbb{R}^n)$ and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space which satisfies the usual condition. If X is a continuous $\mathbb{R}^n-$valued $\{\mathcal{F}_t\}_{t\geq 0}-$adapted process, then the process $t \mapsto b(t, X(\cdot, \omega))$ is $\{\mathcal{F}_t\}_{t\geq 0}-$adapted.*

Equation (A.5) admits different notions of solutions. They depend on different roles that the underlying filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and the Brownian motion W are playing.

A.3 Strong Solutions

Definition A.9. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space which satisfies the usual condition, W an $m-$dimensional standard Brownian motion on such space, and ξ an $\mathcal{F}_0-$measurable random variable. An $\{\mathcal{F}_t\}_{t\geq 0}-$adapted process X is called strong solution of* (A.5) *if*

$$X(0) = \xi, \quad \mathbb{P} - a.s.,$$

$$\int_0^t \{|b(s, X)|^2 + |\sigma(s, X)|^2\}ds < \infty, \quad t \geq 0, \ \mathbb{P} - a.s., \tag{A.7}$$

$$X(t) = x + \int_0^t b(s, X)ds + \int_0^t \sigma(s, X)dW(s), \quad t \geq 0, \ \mathbb{P} - a.s. \tag{A.8}$$

If for any two strong solutions X and Y of (A.5) *defined on any given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ along with any $\{\mathcal{F}_t\}_{t\geq 0}-$Brownian motion W we have*

$$\mathbb{P}\{\omega | X(t, \omega)\} = Y(t, \omega), \ t \geq 0\} = 1, \tag{A.9}$$

then we say that the strong solution is unique or, similarly, that strong uniqueness holds.

Now we provide sufficient conditions in order to get existence and uniqueness of strong solutions to (A.5).

Hypotheses A.10. *Let* $b \in A^n(\mathbb{R}^n), \sigma A^n(\mathbb{R}^{n\times m})$*. Moreover, suppose that there exists* $L > 0$ *such that for any* $t \in [0,\infty)$*,* $x, y \in \mathbb{W}^n$*,*

$$\begin{cases} |b(t,x(\cdot)) - b(t,y(\cdot))| \le L|x(\cdot) - y(\cdot)|_{\mathbb{W}^n}, \\ |\sigma(t,x(\cdot)) - \sigma(t,y(\cdot))| \le L|x(\cdot) - y(\cdot)|_{\mathbb{W}^n}, \\ |b(\cdot,0)| + |\sigma(\cdot,0)| \in L^2(0,T;\mathbb{R}), \quad \forall T > 0, \end{cases} \tag{A.10}$$

We say that b *and* σ *are Lipschitz continuous in* $X(\cdot)$ *if the first two inequalities hold.*

Theorem A.11. *Assume Hypothesis A.*10 *holds. Then, for any* $\xi \in L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n), p \ge 1$*, the problem* (A.5) *admits a unique strong solution* X *such that, for any* $T > 0$*, there exists a positive constant* K_T *such that*

$$\begin{cases} \mathbb{E}\sup_{0\le s\le T}|X(s)|^p \le K_T(1+\mathbb{E}|\xi|^p), \\ \mathbb{E}|X(s) - X(t)|^p \le K_T(1+\mathbb{E}|\xi|^p)|t-s|^{p/2}, \quad \forall s,t \in [0,T]. \end{cases} \tag{A.11}$$

Moreover, if $\hat{\xi} \in L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n), p \ge 1$ *is another random variable and* $\hat{X}$ *the corresponding solution to* (A.5)*, for any* $T > 0$ *there exists a positive constant* K_T *such that*

$$\mathbb{E}\sup_{0\le s\le T}|X(s) - \hat{X}(s)|^p \le K_T\mathbb{E}|\xi - \hat{\xi}|^p. \tag{A.12}$$

In a special case of SDE's, we consider the functions $b : [0,\infty)\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\sigma : [0,\infty)\times\mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$. Then the maps $(t,\omega) \mapsto b(t,\omega(t)), \sigma(t,\omega(t))$ are progressively measurable maps from $[0,\infty)\times\mathbb{W}^n$ to $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, respectively. (A.5) becomes

$$\begin{cases} dX_t = b(t,X(t))dt + \sigma(t,X(t))dW_t, \quad t > 0, \\ X_0 = \xi, \end{cases} \tag{A.13}$$

Such an equation is said of *Markovian type.* If, in addition, b and σ do not depend on time, (A.13) is called *time homogeneous Markovian SDE.* In this case we assume the following hypotheses on the coefficients b and σ.

Hypotheses A.12. $b : [0,\infty)\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ *and* $\sigma : [0,\infty)\times\mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$ *are measurable with respect to* $t \in [0,\infty)$*. Moreover, there exists* $L > 0$ *such that for any* $t \in [0,\infty)$*,*

$$\begin{cases} |b(t,x) - b(t,y)| \le L|x-y|, & \forall x,y \in \mathbb{R}^n, \\ |\sigma(t,x) - \sigma(t,y)| \le L|x-y|, & \forall x,y \in \mathbb{R}^n, \\ |b(\cdot,0)| + |\sigma(\cdot,0)| \in L^2(0,T;\mathbb{R}), \quad \forall T > 0, \end{cases} \tag{A.14}$$

Under this conditions, we get the existence and uniqueness of strong solutions to problem (A.13).

Corollary A.13. *If Hypothesis A.12 holds, then problem (A.13) admits a unique strong solution.*

A.4 Weak Solutions

Definition A.14. *A 6−tuple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W, X)$ is called weak solution of (A.5) if*

(i) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *is a filtered probability space satisfying the usual condition;*

(ii) *W is an $m-$dimensional standard $\{\mathcal{F}_t\}_{t\geq 0}-$Brownian motion and X is an $\{\mathcal{F}_t\}_{t\geq 0}-$adapted and continuous random process;*

(iii) *$X(0)$ has the same distribution of ξ;*

(iv) (A.7) *and* (A.8) *hold.*

The main difference between strong and weak solution is that, in the last one, also the probability space is part of the solution, while in the strong formulation $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and the Brownian motion W are a priori fixed.

Definition A.15. *If for any two weak solutions $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W, X)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{X})$ of (A.5) with*

$$\mathbb{P}(X(0) \in B) = \tilde{\mathbb{P}}(\tilde{X}(0) \in B), \quad \forall B \in B(\mathbb{R}^n),$$

we have

$$\mathbb{P}(X \in A) = \tilde{\mathbb{P}}(\tilde{X} \in A), \quad \forall B \in B(\mathbb{W}^n),$$

then we say that the weak solution to (A.5) is unique in law.

Definition A.16. *If*

$$\mathbb{P}(X(t) = \tilde{X}(t), \quad 0 \leq t < \infty) = 1,$$

for any two weak solutions $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W, X)$ and $(\Omega, \mathcal{F}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \mathbb{P}, W, \tilde{X})$ of (A.5) with

$$\mathbb{P}(X(0) = \tilde{X}(0)) = 1,$$

then we say that the weak solutions have pathwise uniqueness.

The following two theorems clarify the relationships between strong and weak solutions, and weak and pathwise uniqueness.

Theorem A.17. *Let $b \in A^n(\mathbb{R}^n)$ and $\sigma \in A^n(\mathbb{R}^{n\times m})$. Then (A.5) admits a unique strong solution if and only if for any probability measure μ on $(\mathbb{R}^n, B(\mathbb{R}^n))$, (A.5) admits a unique weak solution with the initial distribution μ and pathwise uniqueness holds for (A.5).*

Theorem A.18. *Pathwise uniqueness implies weak uniqueness.*

We conclude with the general existence result of weak solutions for equations with only continuous coefficients

Theorem A.19. *Let $b \in A^n(\mathbb{R}^n)$ and $\sigma \in A^n(\mathbb{R}^{n\times m})$ be bounded and continuous. Then there exists a weak solution to* (A.5).

A.5 Other Types of SDE's

Here, we discuss of another type of equations that play a crucial rule in establishing a right formulation for optimal stochastic control problems. We consider the following SDE:

$$\begin{cases} dX_t = b(t,X,W)dt + \sigma(t,X,W)dW_t, \quad t>0, \\ X_0 = \xi, \end{cases} \tag{A.15}$$

where b and σ are defined on $[0,\infty)\times\mathbb{W}^n\mathbb{W}^m$. If we define $Y(t)\equiv W(t)$, then the above equation is equivalent to

$$\begin{cases} dX_t = b(t,X,Y)dt + \sigma(t,X,Y)dW_t, \quad t>0, \\ DY(t) = dW(t), \qquad t>0, \\ X_0 = \xi, \quad Y(0)=0. \end{cases} \tag{A.16}$$

This is a special case of (A.5). Thus, we can extend all the notions of strong and weak solutions and of strong, weak and pathwise uniqueness to (A.15). In particular, we assume that the following holds true.

Hypotheses A.20. *Let $b \in A^{n+m}(\mathbb{R}^n)$ and $\sigma \in A^{n+m}(\mathbb{R}^{n\times m})$, and suppose that there exists a positive constant L such that, for any $t>0$, any $x(\cdot),y(\cdot)\in\mathbb{W}^n$ and any $w(\cdot)\in\mathbb{W}^m$, we have*

$$\begin{cases} |b(t,x(\cdot),w(\cdot)) - b(t,x(\cdot),w(\cdot))| \le L|x(\cdot)-y(\cdot)|_{\mathbb{W}^n}, \\ |\sigma(t,x(\cdot),w(\cdot)) - \sigma(t,x(\cdot),w(\cdot))| \le L|x(\cdot)-y(\cdot)|_{\mathbb{W}^n}, \\ |b(t,x(\cdot),w(\cdot))| + |\sigma(t,x(\cdot),w(\cdot))| \le L(1+|x(\cdot)|_{\mathbb{W}^n}). \end{cases}$$

Theorem A.21. *Under Hypotheses A.20, equation* (A.15) *admits a unique strong solution, which means that both pathwise and weak uniqueness and existence hold.*

Appendix B

List of symbols

Number sets and vector spaces	
$\mathbb{N}, \mathbb{R}$,	set of natural and real numbers
I	right halfline of real numbers (possible $I = \mathbb{R}$)
$\mathbb{R}^n$	set of all real n-tuples
$a \wedge b, a \vee b$	minimum and maximum of a and b
$\lvert\alpha\rvert$	the length of the multi-index α, i.e. $\lvert\alpha\rvert = \alpha_1 + \cdots + \alpha_n$
Topological and metric space notation	
$\overline{E}$	topological closure of E
∂E	topological boundary of E
E^c	the complementary set of E in a domain Ω or in $\mathbb{R}^n$
$E \Subset F$	$\overline{E} \subset F, \overline{E}$ compact
$B(x,r)$	open ball with center x and radius r
$B(r)$	$B(0,r)$
$\mathcal{L}(X,Y)$	set of bounded and linear operators from X to Y
$\mathcal{L}(X)$	$\mathcal{L}(X,X)$
Matrix and linear algebra	
Id	the identity matrix
$\mathrm{det}B$	the determinant of the matrix B
e_i	i-th vector of the canonical basis of $\mathbb{R}^n$
$\mathrm{Tr}B$	the trace of the matrix B
$\lVert B\rVert_\infty$	the Euclidean norm of the matrix B, i.e. $(\sum_{i,j=1}^n b_{ij}^2)^{1/2}$
$\langle\cdot,\cdot\rangle$ or $x \cdot y$	the Euclidean inner product between the vectors $x, y \in \mathbb{R}^n$

Function spaces: let $f : X \to Y$

χ_E	characteristic function of the set E
u_t	partial derivative with respect to t
D_i or $\frac{\partial}{\partial x_i}$	partial derivative with respect to x_i
D^2_{ij} or $\frac{\partial^2}{\partial x_i^2}$	$D_i D_j$
Du or $\nabla_x u$	space gradient of a real-valued function u
$D^2 u$	Hessian matrix of a real-valued function u
Δu	$\mathrm{Tr}(D^2 u)$
$B(X;Y)$	space of Borel measurable functions from X into Y
$C(X;Y)$	space of continuous functions from X into Y
$B(\Omega)$	space of $\mathbb{R}-$valued Borel measurable functions
$B_b(\Omega)$	space of $\mathbb{R}-$valued bounded Borel measurable functions
$C(\Omega)$	space of $\mathbb{R}-$valued continuous functions
$C_c(\Omega)$	functions in $C(\Omega)$ with compact support in Ω
$C_0(\Omega)$	closure in the sup norm of $C_c(\Omega)$
$BUC(\Omega)$	space of the uniformly continuous and bounded functions on Ω
$C^k_b(\overline{\Omega})$	space of k-times differentiable functions with $D^m f$ for $\vert m\vert \leq k$ bounded and continuous up to the boundary
$C^\alpha(\Omega)$	space of α-Hölder continuous functions, $\alpha \in (0,1)$
$C^{k,\alpha}(\Omega)$	space of $f \in C^k(\Omega)$ with $D^m f \in C^\alpha(\Omega)$ for $\vert m\vert \leq k$ and $\alpha \in (0,1)$
$[u]_{C^\alpha(\Omega)}$	the seminorm $\sup_{x,y\in\Omega} \frac{\vert u(x)-u(y)\vert}{\vert x-y\vert^\alpha}$
$\Vert \cdot \Vert_{L^\infty(\Omega)}$	sup norm
$C(I \times X;Y)$	space of functions u from $I \times X$ into Y
$C_b(I \times X;Y)$	space of bounded functions u from $I \times X$ into Y
$[u]^K_{\alpha/2,\alpha}$	$\sup_{(t,x),(t',x),(t,x')\in K} \frac{\vert u(t',x)-u(t,x)\vert}{\vert t'-t\vert^{\alpha/2}} + \frac{\vert u(t,x')-u(t,x)\vert}{\vert x'-x\vert^\alpha}$, with $\alpha \in (0,1)$, for any $K \Subset I \times X$
$C^{\alpha/2,\alpha}_{\mathrm{loc}}(I \times X;Y)$	$u \in C(I \times X;Y)$ such that $[u]^K_{\alpha/2,\alpha} < \infty$, for any $K \Subset I \times X$
$C^{0,1}(I \times X;Y)$	$u \in C(I \times X;Y)$ such that $D_i u \in C(I \times X;Y)$
$C^{1,2}(I \times X;Y)$	$u \in C(I \times X;Y)$ such that $u_t, D_i u, D^2_{ij} \in C(I \times X;Y)$
$C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}(I \times X;Y)$	$u \in C^{1,2}(I \times X;Y)$ such that $u_t, D^2_{ij} \in C^{\alpha/2,\alpha}_{\mathrm{loc}}(I \times X;Y)$
$(L^p(\Omega), \Vert \cdot \Vert_{L^p(\Omega)})$	usual Lesbegue space

Probability	
$(\Omega, \mathcal{F}, \mathbb{P})$	Complete probability Space
$\{W(t)\}_{t\in I}$ or $\{W_t\}_{t\in I}$	Standard Brownian motion defined in $(\Omega, \mathcal{F}, \mathbb{P})$ in the interval I
$\mathcal{N}$	Family of elements of $\mathcal{F}$ with probability 0
$\{\mathcal{F}_t^W\}_{t\in I}$	Filtration generated by $\{W_s : s \in I, s \leq t\} \cup \mathcal{N}$
$\mathbb{E}$	Expectation with respect $\mathbb{P}$, i.e., for any random variable Y $\mathbb{E}[Y] := \int_\Omega Y(\omega) d\mathbb{P}(\omega)$
$f(Y_t)dW_t$	Itó integral with respect W, i.e., $\int_0^t f(Y_s)dW_s$
$\mathbb{E}[Y \vert \mathcal{G}]$	Conditional Expectation of Y with respect to $\mathcal{G}$

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