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Multivariate Rodriguez Copula with Applications

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MULTIVARIATE RODRIGUEZ COPULA WITH APPLICATIONS

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Abstract

In the appendix, a generalization of Polisicchio distribution (GP) allows to build bivariate distribution by mixing two independent GP's with a bivariate beta weight distribution. Unfortunately, treating this bivariate distribution is very expensive from a computational point of view. So, bivariate Burr III Rodriguez copula is generalized to multivariate case, studied and applied to some datasets. This copula seems to be very general, analytically manageable and exhibits a parsimonious number of parameters, in comparison to the copulas in literature. Moreover, its restriction to second order interactions seems to be a competitor of the multivariate normal copula. On the other hand, an application shows that second order interaction may be not sufficient to reproduce real data situations.

The Bivariate Burr III Rodriguez distribution and its copula

Let us introduce the following important distribution, proposed by Rodriguez and deeply studied by Domma (2010), to which paper the reader is sent for the main properties:

$$F_{X,Y}(x,y)=\begin{cases}\dfrac{1}{\left(1+\alpha\lambda\gamma\, x^{-\theta}y^{-\delta}+\lambda\, x^{-\theta}+\gamma\, y^{-\delta}\right)^{\varepsilon}}, & x>0,\ y>0;\quad \lambda,\gamma,\theta,\delta,\varepsilon>0;\ 0\le\alpha\le\varepsilon+1\\ 0 & otherwise\end{cases} \quad .(1)$$

For our purposes it is interesting to deduce the copula:

$$C_{XY}(u,v)=\left\{1+\alpha\left(u^{-\frac{1}{\varepsilon}}-1\right)\left(v^{-\frac{1}{\varepsilon}}-1\right)+\left(u^{-\frac{1}{\varepsilon}}-1\right)+\left(v^{-\frac{1}{\varepsilon}}-1\right)\right\}^{-\varepsilon}, \qquad (2)$$

$$0\le\alpha\le(\varepsilon+1),$$

If $\alpha=0$, we have the copula B4 in Joe p. 141, due to Kimeldorf and Simpson (1975).

The two parameters copula BB1 in Joe p. 150, formula (5.5), has a form similar to (2):

$$C(u,v)=\left\{1+\left[\left(u^{-\frac{1}{\varepsilon}}-1\right)^{\kappa}+\left(v^{-\frac{1}{\varepsilon}}-1\right)^{\kappa}\right]^{1/\kappa}\right\}^{-\varepsilon},$$

which exhibits B4 if $\kappa=1$, but (2) shows an independence situation for $\alpha=1$, which BB1 does not.

The generalized Rodriguez multivariate distribution and its copula

In order to generalize (2), building the multivariate Rodriguez distribution is necessary.

Starting from the situation of independence:

$$F_{X_i}(x_i) = \prod_{i=1}^{d} \frac{1}{\left(1+\lambda_i\ x_i^{-\vartheta_i}\right)^{\varepsilon}} = \left[\prod_{i=1}^{d} \frac{1}{\left(1+\lambda_i\ x_i^{-\vartheta_i}\right)}\right]^{\varepsilon},$$

where $\lambda_i\ x_i^{-\theta_i} = A_i$,
in the denominator there are 2^d terms:

$$\left[1+\sum_{i=1}^{d} A_i + \sum_{i>j} A_i \cdot A_j + \cdots + \prod_{i=1}^{d} A_i\right]^{-\varepsilon}.$$

It's sufficient multiply by the following interaction parameters (which are $2^d - d - 1$):

$\binom{d}{2}$ parameters α_{ij}, , $\binom{d}{3}$ parameters α_{ijk} , … , yielding:

$$\left[1+\sum_{i=1}^{d} A_i + \sum_{i>j} \alpha_{ij} \cdot A_i \cdot A_j + \sum_{i>j>k} \alpha_{ijk} \cdot A_i \cdot A_j \cdot A_k + \cdots + \alpha \cdot \prod_{i=1}^{d} A_i\right]^{-\varepsilon}, \qquad (3)$$

with restrictions on the interaction parameters $\alpha_{(.)}$.

In this way, (3) exhibits univariate Dagum margins, bivariate Rodriguez margins, etc…

Distribution (3), taking into account of epsilon and of the marginal parameters lambda and theta likewise, possesses $2^d + d$ parameters, which is exponential in *d*.

By substituting in (3) the inverse functions of the margins, the following copula is obtained:

$$\left\{1+\sum_{i}\left(u_i^{-\frac{1}{\varepsilon}}-1\right)+\sum_{i>j}\alpha_{ij}\cdot\left(u_i^{-\frac{1}{\varepsilon}}-1\right)\cdot\left(u_j^{-\frac{1}{\varepsilon}}-1\right)+\sum_{i>j>k}\alpha_{ijk}\cdot\left(u_i^{-\frac{1}{\varepsilon}}-1\right)\cdot\left(u_j^{-\frac{1}{\varepsilon}}-1\right)\cdot\left(u_k^{-\frac{1}{\varepsilon}}-1\right)+\cdots+\alpha\cdot\prod_{i=1}^{d}\left(u_i^{-\frac{1}{\varepsilon}}-1\right)\right\}^{-\varepsilon} \qquad (4)$$

Due to the exponential number of parameters $2^d - d$, reduction with respect to the (saturated) copula (4) is needed, which is possible by considering interaction levels less than $d-1$.

Definition (Joe, p.155) *A multivariate parametric family of copulas is an extension of a bivariate family if: i) all bivariate marginal copulas of the multivariate coplula are in the given family; and ii) all multivariate marginal copulas of order 3 to m-1 have the same multivariate form.*

Considering interactions of level 2 and following the definition, we obtain the following:

$$\left\{1+\sum_i\left(u_i^{-\frac{1}{\varepsilon}}-1\right)+\sum_{i>j}\alpha_{ij}\cdot\left(u_i^{-\frac{1}{\varepsilon}}-1\right)\cdot\left(u_j^{-\frac{1}{\varepsilon}}-1\right)\right\}^{-\varepsilon}, \qquad \alpha_{ij}\le\varepsilon+1 \qquad (5)$$

which has $1+d(d-1)/2$ parameters, likewise the generalized normal copula, and useful when d is large.

Considering interactions of level 3, we obtain the following:

$$\left\{1+\sum_i\left(u_i^{-\frac{1}{\varepsilon}}-1\right)+\sum_{i>j}\alpha_{ij}\cdot\left(u_i^{-\frac{1}{\varepsilon}}-1\right)\cdot\left(u_j^{-\frac{1}{\varepsilon}}-1\right)+\sum_{i>j>k}\alpha_{ijk}\cdot\left(u_i^{-\frac{1}{\varepsilon}}-1\right)\cdot\left(u_j^{-\frac{1}{\varepsilon}}-1\right)\cdot\left(u_k^{-\frac{1}{\varepsilon}}-1\right)\right\}^{-\varepsilon},$$

$$\alpha_{ijk}\le(\varepsilon+1)\cdot\min(\alpha_{ij}\cdot\alpha_{ik},\alpha_{ij}\cdot\alpha_{jk},\alpha_{ik}\cdot\alpha_{jk}) \qquad (6)$$

which has $1+d(d-1)/2+d(d-1)(d-2)/6=1+d(d-1)(d+1)/6$ parameters, and so on.

The proof of the restrictions on the parameters of formula (6), will be given in another paper.

Numerical applications suggest that the condition may be generalized, for example, when considering fourth-order interactions:

$$\alpha_{ijkl}\le(\varepsilon+1)\cdot\min(\alpha_{ijk}\cdot\alpha_{ijl}\cdot\alpha_{ikl},\alpha_{ijk}\cdot\alpha_{ijl}\cdot\alpha_{jkl},\alpha_{ijl}\cdot\alpha_{ikl}\cdot\alpha_{jikl},\alpha_{jkl}\cdot\alpha_{ijl}\cdot\alpha_{ikl})$$

Applications

In this application we considered the sample survey on household income and wealth referring to 2012, supplied by the Central Bank of Italy (Bank of Italy, 2014). The survey covers 8151 households and 20022 individuals. For each subject we consider three income sources:

- X: Payroll income;
- Y: Property income;
- Z: Pensions and Net-Transfers;

We modelled the bivariate distribution (X,Y) of individual income and household income with the bivariate copula using two types of marginal distributions described below.
First we use the Zenga three-parameters distribution (Zenga, 2010)to model the marginal distributions. The following formula describe the denisity function:

$$f(x;\alpha,\theta,\mu)=\begin{cases}\dfrac{1}{2\mu B(\alpha,\theta)}\left(\dfrac{\mu}{x}\right)^{\frac{3}{2}}\displaystyle\sum_{i=0}^{\infty}B\left(\frac{x}{\mu};\alpha+\frac{1}{2}+i,\theta\right) & 0<x<\mu\\ \dfrac{1}{2\mu B(\alpha,\theta)}\left(\dfrac{x}{\mu}\right)^{\frac{3}{2}}\displaystyle\sum_{i=0}^{\infty}B\left(\frac{\mu}{x};\alpha+\frac{1}{2}+i,\theta\right) & \mu<x\end{cases}$$

Furthermore, we model the marginal distribution with the Zenga four-parameters law (De Capitani and Zini, 2012) which is characterized by the formula:

$$f(x;\alpha,\theta,\gamma,\mu) = \begin{cases} \frac{C\sqrt{\mu}}{2} x^{-\frac{3}{2}} \int_0^{\frac{x}{\mu}} k^{\alpha-0,5}(1-k)^{\theta-2} e^{-\gamma k} dk & 0 < x < \mu \\ \frac{C\sqrt{\mu}}{2} x^{-\frac{3}{2}} \int_0^{\frac{\mu}{x}} k^{\alpha-0,5}(1-k)^{\theta-2} e^{-\gamma k} dk & \mu < x \end{cases}$$

Where

$$C = \frac{1}{B(\alpha,\theta)_1F_1(\alpha;\theta+\alpha;-\gamma)}$$

And $_1F_1$ is the Confluent Hypergeometric series.

Since it can occur some negative values in the distributions, especially in the variable Z, a new location parameters *h* was introduced to take into account the negative value.

Household income

By looking the data, we found that in the 44,74% of cases X and Y are jointly different from zero, in the 48,23% of cases X is zero and Y is clearly different from zero and in the 15% of households Y takes values between -100,00 and 100,00 euro (which correspond to interests or coupon of small investments). To take into account these situations we use a mixture of the previous models as described in the formula:

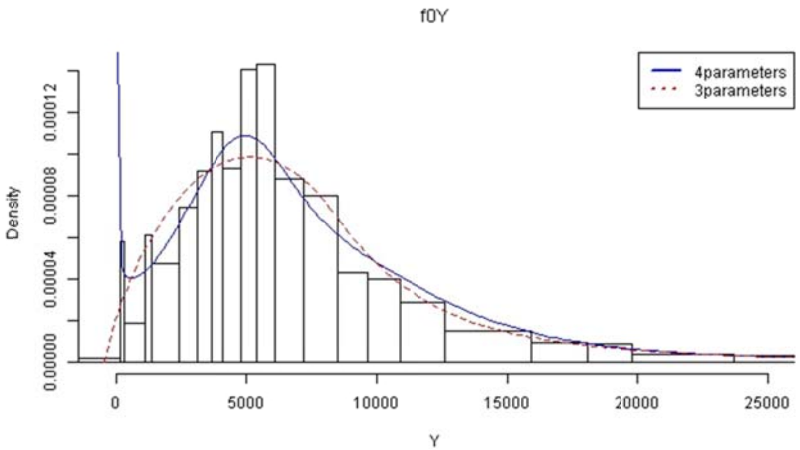
$$F_{X,Y}(x,y) = F_{X,Y}(x \neq 0, y \notin B)p(x \neq 0, y \notin B) + F_{X,Y}(x = 0, y \notin B)p(x = 0, y \notin B) + \\ +F_{X,Y}(x \neq 0, y \in B)p(x \neq 0, y \in B) + F_{X,Y}(x = 0, y \in B)p(x = 0, y \in B)$$

Where $B = \{-100; 100\}$.

The following table reports the minimum modified Chi-square estimates for household income:

Distribuzione	Tipo Marginali	μ_X	α_X	θ_X	γ_X	h_X	chi_X	μ_Y	α_Y	θ_Y	γ_Y	h_Y	chi_Y	ϵ	a	$Chi\ copula$
$F_{X,Y}(x \neq 0, y \notin B)$	3 par	24831,06	2,6218	3,6919	-	+1,5666	0,3106	7775,86	1,0716	1,7838	-	0,904	0,3522	0,3618	0,4950	0,1072
	4 par	23113,14	0,3936	15,4370	-27,1693	2275,032	0,2279	30570,47	0,3624	10,7289	-19,2779	110,6951	0,2102	0,3618	0,4950	0,1072
$F_{X,Y}(x = 0, y \notin B)$	3 par	-	-	-	-	-	-	7707,713	1,3113	2,2724	-	1,3156	0,3543	-	-	-
	4 par	-	-	-	-	-	-	8204,208	0,7117	9,9486	-15,7070	8,5521	0,3004	-	-	-
$F_{X,Y}(x \neq 0, y \in B)$	3 par	16970,09	1,5387	1,8229	-	3,3140	0,2764	-	-	-	-	-	-	-	-	-
	4 par	16844,71	0,7970	5,0690	-9,4506	19,1929	0,2227	-	-	-	-	-	-	-	-	-
$F_{X,Y}(x = 0, y \in B)$																

Figure 1: $f_{X,Y}(x = 0, y \notin B)$



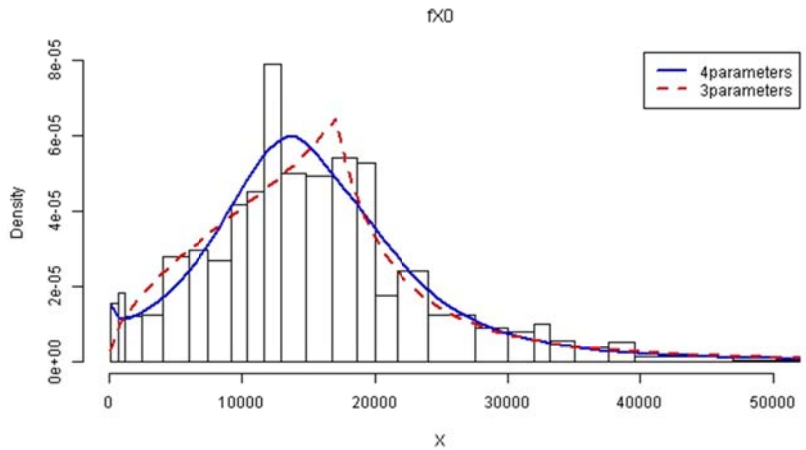


Figure 2: $f_{X,Y}(x \neq 0, y \in B)$

Individual income distribution

The same considerations hold for the individual income distribution.

Distribuzione	Tipo Marginali	μ_X	α_X	θ_X	γ_X	h_X	chi_X	μ_Y	α_Y	θ_Y	γ_Y	h_Y	chi_Y	ϵ	a	$Chi\ copula$
$F_{X,Y}(x \neq 0, y \notin B)$	Zenga	21179,03	2,2923	1,77914	-	- 2142,762	0,2476	7782,275	1,1195	1,8813	-	-119,3527	0,3456	0,3167	0,4065	0,1027
	Lucio	20957,88	2,1778	1,81743	- 0,2965	-1908,524	0,2476	7954,485	0,3439	10,5439	-19,0059	101,2418	0,1736	0,3167	0,4065	0,1027
$F_{X,Y}(x = 0, y \notin B)$	Zenga	-	-	-	-	-	-	8331,812	1,8247	2,8610	-	-479,6062	0,3054	-	-	-
	Lucio	-	-	-	-	-	-	8288,025	0,55962	11,5617	-18,5617	50,5399	0,2687	-	-	-
$F_{X,Y}(x \neq 0, y \in B)$	Zenga	16309,96	1,5899	1,6052	-	1,1907	0,3418	-	-	-	-	-	-	-	-	-
	Lucio	15885,40	0,9320	2,4452	3,7701	658,1903	0,3312	-	-	-	-	-	-	-	-	-
$F_{X,Y}(x = 0, y \in B)$																

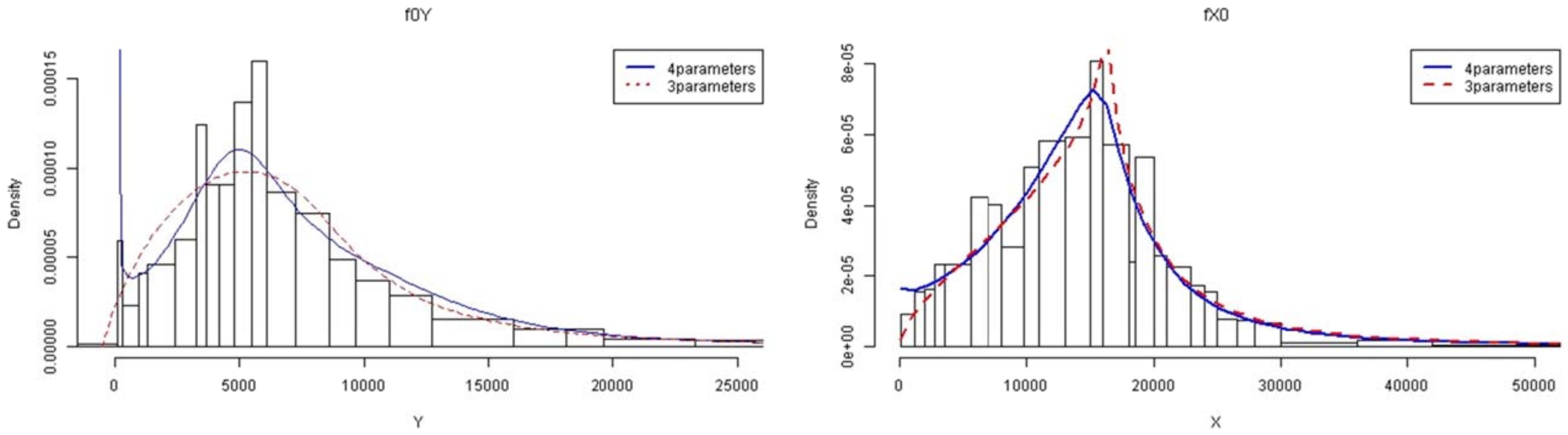
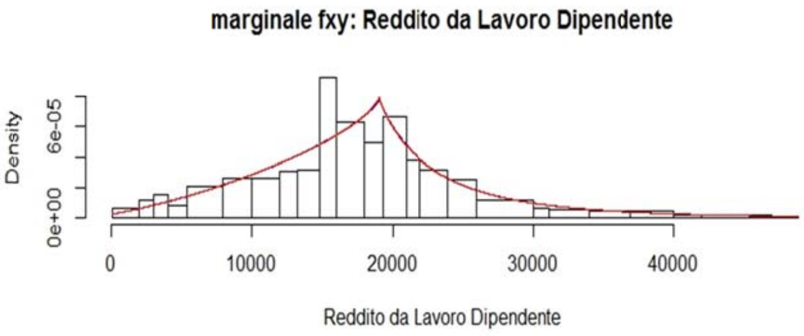


Figure 3: $f_{X,Y}(x \neq 0, y \in B)$


marginale fxy: Reddito da Lavoro Dipendente
Density
0e+00
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10000
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30000
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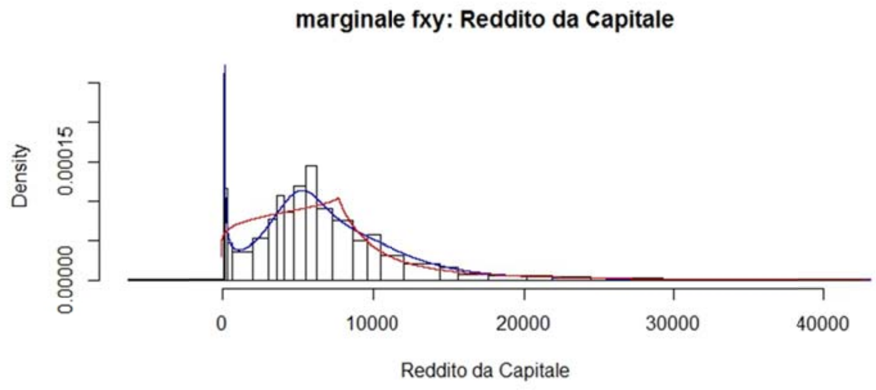
Reddito da Lavoro Dipendente


marginale fxy: Reddito da Capitale
Density
0.00000
0.00015
0
10000
20000
30000
40000
Reddito da Capitale

Trivariate copula (individual income)

In order to capture the particular aspects of this trivariate distribution we split the joint distribution to cover all cases as follows:

$$F_{XYZ}(x,y,z) = F_{XYZ}(x \neq 0, y \notin B, z \notin B)p(x \neq 0, y \notin B, z \notin B) + F_{XYZ}(x = 0, y \notin B, z \notin B)p(x = 0, y \notin B, y \notin B) + F_{XYZ}(x \neq 0, y \in B, z \notin B)p(x \neq 0, y \in B, y \notin B) + F_{XYZ}(x \neq 0, y \notin B, z \in B)p(x \neq 0, y \notin B, y \in B) + F_{XYZ}(x = 0, y \in B, z \notin B)p(x = 0, y \in B, z \notin B) + F_{XYZ}(x = 0, y \notin B, z \in B)p(x = 0, y \notin B, y \in B) + F_{XYZ}(x \neq 0, y \in B, z \in B)p(x \neq 0, y \in B, z \in B) + F_{XYZ}(x = 0, y \in B, z \in B)p(x = 0, y \in B, z \in B).$$

Where $B = \{-100; 100\}$.

We disregarded the pieces where the corresponded probability $p(x,y,z)$ assumed values lower than 0,01 because the few number of observations do not allow the estimates of three and four parameters reectively.
The table with minimum modified Chi-square estimates follows:

Distribuzione	Marginals	$F_{XYZ}(x \neq 0, y \notin B, z \notin B)$	$F_{XYZ}(x = 0, y \notin B, z \notin B)$	$F_{XYZ}(x \neq 0, y \notin B, z \in B)$	$F_{XYZ}(x = 0, y \notin B, z \in B)$	$F_{XYZ}(x \neq 0, y \in B, z \in B)$
μ_X	3 par	20292,1	-	32562,62	-	17094,22
	4 par	22193,12	-	23493,56	-	24000,2
α_X	3 par	1,8205	-	5,8836	-	1,5694
	4 par	3,4696	-	0,3689	-	6,0554
θ_X	3 par	2,2550	-	5,6389	-	1,8657
	4 par	1,1476	-	16,4129	-	2,5657
γ_X	3 par	-	-	-	-	-
	4 par	28,6040	-	-28,6762	-	4,3270
h_X	3 par	1,3486	-	-7623,658	-	3,6531
	4 par	-2053,397	-	2625,379	-	-7817,566
Chi_X	3 par	0,2808	-	0,2273	-	0,2807
	4 par	0,0196	-	0,1810	-	0,2138
μ_Y	3 par	10410,8	12183,59	7588,474	7707,003	-
	4 par	11632,28	10928,79	7720,417	7451,697	-
α_Y	3 par	1,3042	2,3140	1,0491	2,0655	-
	4 par	0,2887	0,1618	0,3221	0,6500	-
θ_Y	3 par	2,6483	6,3623	1,7314	3,1355	-
	4 par	31,4725	75,9837	10,8127	11,9648	-
γ_Y	3 par	-	-	-	-	-
	4 par	-48,5423	-99,9689	-19,9285	-19,4851	-
h_Y	3 par	1,2687	0,7767	1,5492	-497,5487	-
	4 par	130,1422	2232,484	113,4031	38,6062	-
Chi_Y	3 par	0,4041	0,3266	0,3673	0,3949	-
	4 par	0,2808	0,2323	0,1814	0,3766	-
μ_Z	3 par	19716,39	30914,51	-	-	-
	4 par	196151,85	31295,77	-	-	-
α_Z	3 par	1,1894	1,0478	-	-	-
	4 par	1,6476	0,8921	-	-	-
θ_Z	3 par	2,29645	2,1552	-	-	-
	4 par	1	9,4404	-	-	-
γ_Z	3 par		-	-	-	-
	4 par	4,1358	-12,7758	-	-	-
h_Z	3 par	1,4934	1,2036	-	-	-

	4 par	-468,95	3376,98	-	-	-
Chi_Z	3 par	0,3211	0,4718	-	-	-
	4 par	0,3057	0,4357	-	-	-
ϵ		0,6442	0,1033	0,3198	-	-
$\alpha 12$		9,5181	0,4694	0,4133	-	-
$\alpha 13$		0,9892		-	-	-
$\alpha 23$		0,5964		-	-	-
$\alpha 123$		0,2695		-	-	-
Chi		0,4236	0,2014	0,0874	-	-

Geneics Data

Another application was carry on genetic data. More precisely we have the expression of three genes ACTINA, GAPDH and H3F3A. The gene expression is a score measuring the information of the gene. We use the trivariate copula to model the dependence of the three selected genes. The marginal were modeled with both Zenga three-parameters and Zenga four-parameters. The results were listed in the table below.

Gene	Marginals	μ_X	α_X	θ_X	γ_X	chi_X
ACTINA	3 parameters	1.1890	4.3819	4.9341	-	0.1747
	4 parameters	1.1984	7.3618	1.3682	14.4973	0.1730
GAPDH	3 parameters	1.0845	3.3770	2.9870	-	0.1405
	4 parameters	1.1017	0.3953	6.4375	-15.1373	0.1156
H3F3A	3 parameters	1.0749	5.8626	5.6090	-	0.2896
	4 parameters	1.0763	1.4556	9.4828	-18.0895	0.2861

While the trivariate copula has characterized from the parameters $\alpha_{12} = 2{,}5007$, $\alpha_{13} = 2{,}6197$, $\alpha_{23} = 2{,}0959\ e^{-37}$, $\alpha_{123} = 2{,}6197$ and $\epsilon = 1{,}6196$, with a chi-square of 0,6738. By setting to zero the parameter α_{123} and let free the other parameters we obtained the following parameters $\alpha_{12} = 2{,}5001$, $\alpha_{13} = 2{,}6189$, $\alpha_{23} = 2{,}6678\ e^{-33}$ and $\epsilon = 1{,}6189$ with a chi-square of 0,6738, near the same of the unconstrained model.

Conclusions

The proposed copula seems to fit very well in many contexts of research, in particular when the marginal distributions are heavy-tailed. In future papers, we will apply it to earthquakes catalogs data.

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Appendix

A NEW BIVARIATE DISTRIBUTION AS MIXTURE OF A DOUBLE PARETO TRUNCATED DISTRIBUTION WITH BIVARIATE GENERALISED BETA WEIGHTS: A COMPARISON TO THE RODRIGUEZ DISTRIBUTION

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1. A generalisation of Polisicchio’s truncated Pareto distribution

Let us consider the following cumulative probability function, which generalizes Polisicchio’s one, by introducing the parameter *g*:

$$H(y;\mu,k) = \left(1-k^g\right)^{-1}\left[1-\left(\frac{k\mu}{y}\right)^{g/2}\right] = \sum_{j=0}^{\infty} k^{gj} - \left(\frac{\mu}{y}\right)^{g/2}\sum_{j=0}^{\infty} k^{g\left(j+\frac{1}{2}\right)},$$

$$\mu k \le y \le \frac{\mu}{k} \qquad 0<k<1,\ g>0. \tag{1}$$

Here its density function:

$$h(y;\mu,k) = \begin{cases} \frac{g}{2}\mu^{g/2}k^{g/2}\left(1-k^g\right)^{-1} y^{-1-g/2}, & \mu k \le y \le \frac{\mu}{k} \quad 0<k<1,\ g>0 \\ 0 & otherwise \end{cases}. \tag{2}$$

The moments are given by:

$$E(Y^r) = \begin{cases} \frac{g}{2r-g}\frac{\mu^r}{1-k^g}\left(k^{g-r}-k^r\right) & r<1+g/2 \\ -\frac{2r\,\mu^r\,k^r}{1-k^{2r}}\log k & g=2r \end{cases} =$$

$$= \begin{cases} \frac{g}{2r-g}\frac{\mu^r}{1-k^g}k^{g-r}\left(1-k^{2r-g}\right) & r<1+g/2 \\ -\frac{2r\,\mu^r\,k^r}{1-k^{2r}}\log k & g=2r \end{cases} =$$

$$= \begin{cases} \frac{g\,\mu^r}{2r-g}\left[\sum_{j=0}^{\infty}k^{g(j+1)-r} - \sum_{j=0}^{\infty}k^{gj+r}\right] & r<1+g/2 \\ -\frac{2r\,\mu^r\,k^r}{1-k^{2r}}\log k & g=2r \end{cases}. \tag{3}$$

Solo se il rapporto r/g è razionale ($g \neq 2r$), è ammissibile una rappresentazione come somma finita di termini.

Putting $g = 1$ and $r = 1$, (3) gives the value μ, the case of the Polisicchio's density, which does not depend on *k*. So, generalization (1) allows the mean to depend on *k* and this case allows to build a new class of bivariate densities with dependent and correlated margins.

La curva di Lorenz ha questa forma:

$$L(p) = \begin{cases} \dfrac{1}{k^{g-2}-1}\left\{\left[1-p\left(1-k^{g}\right)\right]^{1-\frac{2}{g}}-1\right\} & g \neq 2 \\ \dfrac{\log\left[1-\left(1-k^{2}\right)p\right]}{2\log k} & g = 2 \end{cases}.$$

2. A new class of univariate random variables

Integration of (1) with respect to a beta density, after reversion of integration order, yields the following density probability function:

$$g(y; g, \alpha, \vartheta, \mu) = \begin{cases} \dfrac{g\,\mu^{g/2}}{2B(\alpha,\vartheta)} x^{-1-g/2} \displaystyle\int_0^{x/\mu} \dfrac{k^{\alpha+g/2-1}(1-k)^{\vartheta-1}}{1-k^{g}}\,dk & 0 \le x \le \mu \\ \dfrac{g\,\mu^{g/2}}{2B(\alpha,\vartheta)} x^{-1-g/2} \displaystyle\int_0^{\mu/x} \dfrac{k^{\alpha+g/2-1}(1-k)^{\vartheta-1}}{1-k^{g}}\,dk & \mu < x \end{cases} =$$

$$= \begin{cases} \dfrac{g\,\mu^{g/2}}{2B(\alpha,\vartheta)} x^{-1-g/2} \displaystyle\sum_{j=0}^{\infty}\int_0^{x/\mu} k^{\alpha+g(j+\frac{1}{2})-1}(1-k)^{\vartheta-1}dk & 0 \le x \le \mu \\ \dfrac{g\,\mu^{g/2}}{2B(\alpha,\vartheta)} x^{-1-g/2} \displaystyle\sum_{j=0}^{\infty}\int_0^{\mu/x} k^{\alpha+g(j+\frac{1}{2})-1}(1-k)^{\vartheta-1}dk & \mu < x \end{cases} =$$

$$= \begin{cases} \dfrac{g\,\mu^{g/2}}{2B(\alpha,\vartheta)} x^{-1-g/2} \displaystyle\sum_{j=0}^{\infty} B\left[\dfrac{x}{\mu}; \alpha + g\left(j+\dfrac{1}{2}\right), \vartheta\right] & 0 \le x \le \mu \\ \dfrac{g\,\mu^{g/2}}{2B(\alpha,\vartheta)} x^{-1-g/2} \displaystyle\sum_{j=0}^{\infty} B\left[\dfrac{\mu}{x}; \alpha + g\left(j+\dfrac{1}{2}\right), \vartheta\right] & \mu < x \end{cases}.$$

3. A new class of bivariate random variables, with positive quadrant dependence

It's now time to construct a general class of bivariate distributions, starting from the bivariate density of two independent components of type (1), and integrating it with respect to the generalised Gauss hypergeometric beta distribution proposed by Nadarajah (2007):

$$l\left(k_1, k_2; a, b, c, d\right) = \frac{C k_1^{\,a-1}\left(1-k_1\right)^{c-a-1} k_2^{\,b-1}\left(1-k_2\right)^{c-b-1}}{(1-k_1 k_2 d)^{c}} I_{[0,1)}(k_1)\, I_{[0,1)}(k_2). \qquad (6)$$

We obtain the following cumulative probability function:

$$C(y,t)=\int_0^1\int_0^1\int_{\mu_x k_1}^{y}\int_{\mu_t k_2}^{t}\frac{Ck_1^{\,a-1}\left(1-k_1\right)^{c-a-1}k_2^{\,b-1}\left(1-k_2\right)^{c-b-1}}{(1-k_1k_2d)^c}\frac{\mu_x^{g/2}}{2}ghk_1^{\,g/2}\left(1-k_1^g\right)^{-1}x^{-1-g/2}\,\frac{\mu_t^{h/2}}{2}k_2^{\,h/2}\left(1-k_2^h\right)^{-1}s^{-1-h/2}dxdsdk_1dk_2\;=$$

$$C(y,t)=\int_0^1\int_0^1\int_{\mu_x k_1}^{y}\int_{\mu_t k_2}^{t}\frac{Ck_1^{\,a-1+g/2}\left(1-k_1\right)^{c-a-1}k_2^{\,b-1+h/2}\left(1-k_2\right)^{c-b-1}}{(1-k_1k_2d)^c\left(1-k_1^g\right)\left(1-k_2^h\right)}\frac{\mu_x^{g/2}}{2}g\,h\,x^{-1-g/2}\,\frac{\mu_t^{h/2}}{2}s^{-1-h/2}dxdsdk_1dk_2\;=$$

$$=\int_0^1\int_0^1\int_{\mu_x k_1}^{y}\int_{\mu_t k_2}^{t}\frac{Ck_1^{\,a-1+g/2}\left(1-k_1\right)^{c-a-1}k_2^{\,b-1+h/2}\left(1-k_2\right)^{c-b-1}}{(1-k_1k_2d)^c\left(1-k_1^g\right)\left(1-k_2^h\right)}\frac{\mu_x^{g/2}}{2}g\,h\,x^{-1-g/2}\,\frac{\mu_t^{h/2}}{2}s^{-1-h/2}dxdsdk_1dk_2\,.$$

Reversing the order of integration, we represent the bivariate probability density function $\frac{\partial^2}{\partial y\,\partial t}C(y,t)$ as follows:

$$\frac{\partial^2}{\partial y\,\partial t}C(y,t)=\begin{cases}A_{11}(y,t) & y\le\mu_x,\,t\le\mu_t\\ A_{12}(y,t) & y>\mu_x,\,t\le\mu_t\\ A_{21}(y,t) & y\le\mu_x,\,t>\mu_t\\ A_{22}(y,t) & y>\mu_x,\,t>\mu_t\\ 0 & y\le 0\,\cup\,t\le 0\end{cases}\,, \qquad (7)$$

with

$$A_{11}(y,t)=g\,h\frac{\mu_x^{g/2}}{2}x^{-1-g/2}\,\frac{\mu_t^{h/2}}{2}s^{-1-h/2}\int_{0_1}^{y/\mu_x}\int_0^{t/\mu_{xt}}\frac{Ck_1^{\,a-1+g/2}\left(1-k_1\right)^{c-a-1}k_2^{\,b-1+h/2}\left(1-k_2\right)^{c-b-1}}{(1-k_1k_2d)^c\left(1-k_1^g\right)\left(1-k_2^h\right)}dk_2dk_1\,.$$

$$A_{12}(y,t)=g\,h\frac{\mu_x^{g/2}}{2}x^{-1-g/2}\,\frac{\mu_t^{h/2}}{2}s^{-1-h/2}\int_{0}^{\mu_x/y}\int_0^{t/\mu_{xt}}\frac{Ck_1^{\,a-1+g/2}\left(1-k_1\right)^{c-a-1}k_2^{\,b-1+h/2}\left(1-k_2\right)^{c-b-1}}{(1-k_1k_2d)^c\left(1-k_1^g\right)\left(1-k_2^h\right)}dk_2dk_1\,.$$

$$A_{21}(y,t)=g\,h\frac{\mu_x^{g/2}}{2}x^{-1-g/2}\,\frac{\mu_t^{h/2}}{2}s^{-1-h/2}\int_{0}^{y/\mu_x}\int_0^{\mu_{xt}/t}\frac{Ck_1^{\,a-1+g/2}\left(1-k_1\right)^{c-a-1}k_2^{\,b-1+h/2}\left(1-k_2\right)^{c-b-1}}{(1-k_1k_2d)^c\left(1-k_1^g\right)\left(1-k_2^h\right)}dk_2dk_1\,.$$

$$A_{22}(y,t)=g\,h\frac{\mu_x^{g/2}}{2}x^{-1-g/2}\,\frac{\mu_t^{h/2}}{2}s^{-1-h/2}\int_{0}^{\mu_x/y}\int_0^{\mu_{xt}/t}\frac{Ck_1^{\,a-1+g/2}\left(1-k_1\right)^{c-a-1}k_2^{\,b-1+h/2}\left(1-k_2\right)^{c-b-1}}{(1-k_1k_2d)^c\left(1-k_1^g\right)\left(1-k_2^h\right)}dk_2dk_1\,.$$