# Fixed point indices of central configurations

D.L. Ferrario

December 19, 2014

**Abstract**

Central configurations of $n$ point particles in $E \cong \mathbb{R}^d$ with respect to a potential function $U$ are shown to be the same as the fixed points of the normalized gradient map $F = -\nabla_M U / \|\nabla_M U\|_M$, which is an $SO(d)$-equivariant self-map defined on the intertia ellipsoid. We show that the $SO(d)$-orbits of fixed points of $F$ are all fixed points of the map induced on the quotient by $SO(d)$, and give a formula relating their indices (as fixed points) with their Morse indices (as critical points). At the end, we give an example of a non-planar relative equilibrium which is not a rotating central configuration.
*MSC Subject Class*: 55M20, 37C25, 70F10.

*Keywords*: Central configurations, relative equilibria, $n$-body problem.

## 1 Central configurations

Let $E = \mathbb{R}^d$ be the $d$-dimensional Euclidean space, and $X = E^n \smallsetminus \Delta$ the *configuration space* of $E$, defined as the space of all $n$-tuples of distinct points $\boldsymbol{q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \in E^n$ such that $\boldsymbol{q} \notin \Delta$, where $\Delta$ is the collision set

$$\Delta = \bigcup_{i<j} \{\boldsymbol{q} \in E^n : \boldsymbol{q}_i = \boldsymbol{q}_j\} \ .$$

Let $U \colon X \to \mathbb{R}$ be a regular potential function. For example, the gravitational potential

$$U(\boldsymbol{q}) = \sum_{i<j} \frac{m_i m_j}{\|\boldsymbol{q}_i - \boldsymbol{q}_j\|^\alpha}, \tag{1.1}$$

or more general potentials (charged particles, ...).

Given $n$ real non-zero numbers $m_1, \ldots, m_n$ (representing the *masses* of the $n$ interacting point particles), let $\langle -, - \rangle_M$ denote the *mass-metric* on $E^n$, defined as $\langle \boldsymbol{v}, \boldsymbol{w} \rangle_M = \sum_{j=1}^n m_j \boldsymbol{v}_j \cdot \boldsymbol{w}_j$, where $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^{dn}$ are two vectors tangent to $E^n$, and $\cdot$ denotes the standard scalar product in $E$. If the masses are positive, then the mass-metric is a non-degenerate scalar product on $E^n$, which yields both the kinetic quadratic form (on the tangent bundle) $2K = \|\dot{\boldsymbol{q}}\|_M^2 = \sum_i m_i \|\dot{\boldsymbol{q}}_i\|^2$ and the inertia form $I = \|\boldsymbol{q}\|_M^2 = \sum_i m_i \|\boldsymbol{q}_i\|^2$.

Given such $m_i$, let $\nabla_M$ denote the gradient of $U$ with respect to the bilinear product $\langle -, - \rangle_M$. Let us recall that if $dU(\boldsymbol{q})$ denotes the differential of $U$ evaluated in $\boldsymbol{q}$, then for each tangent vector $\boldsymbol{v} \in T_{\boldsymbol{q}} E^n$, $dU(\boldsymbol{q})[\boldsymbol{v}] = \langle \nabla_M U(\boldsymbol{q}), \boldsymbol{v} \rangle_M$, and therefore

$$dU(\boldsymbol{q})[\boldsymbol{v}] = \sum_j \frac{\partial U}{\partial \boldsymbol{q}_j} \cdot \boldsymbol{v}_j = \sum_j m_j \left( m_j^{-1} \frac{\partial U}{\partial \boldsymbol{q}_j} \right) \cdot \boldsymbol{v}_j,$$

from which it follows that in standard coordinates $(\nabla_M U)_j = m_j^{-1} \dfrac{\partial U}{\partial \boldsymbol{q}_j}$.

Given the mass-metric gradient $\nabla_M U$, the corresponding *Newton equations* are

$$\frac{d^2 \boldsymbol{q}}{dt^2} = \nabla_M U(\boldsymbol{q}). \tag{1.2}$$

**(1.3) Definition.** A configuration $\boldsymbol{q} \in X$ is a *central configuration* iff there exists $\lambda \in \mathbb{R}$ such that $\nabla_M U(\boldsymbol{q}) = \lambda \boldsymbol{q}$.

*(1.4) Remark.* If $U$ is homogeneous, this is equivalent to: $\boldsymbol{q}$ is a central configuration iff there is a real-valued function $\phi(t)$ such that $\varphi(t)\boldsymbol{q}$ is a solution of (1.2)[1]. If $U$ is homogeneous, the set of central configurations is a cone in $X$.

*(1.5) Remark.* Furthermore, if $U$ is invariant with respect to the group of all translations of $E$, then central configurations belong to the subspace

$$Y = \{\boldsymbol{q} \in X : \sum_j m_j \boldsymbol{q}_j = 0\} \subset X \subset E^n,$$

and $\forall \boldsymbol{q} \in X \implies \nabla_M U(\boldsymbol{q}) \in \overline{Y} \subset E^n$, where $\overline{Y}$ is the closure of $Y$ in $E^n$. Therefore, if $U$ is translation-invariant, the set of central configurations is a subset of $Y$. Sometimes central configurations are defined with the equation $\nabla_M U(\boldsymbol{q}) = \lambda(\boldsymbol{q} - \boldsymbol{c})$, where $\boldsymbol{c}$ is the center of mass $\boldsymbol{c} = (\sum_j m_j)^{-1} \sum_j m_j \boldsymbol{q}_j$ of the configuration $\boldsymbol{q}$. This equation is invariant with respect to translations (if $U$ is so).

**(1.6) Definition.** A configuration $\boldsymbol{q} \in X$ is a *relative equilibrium* iff there is a one-parameter group of rotations $\varphi^t \colon E \to E$ (around the origin, without loss of generality) such that

$$\varphi^t(\boldsymbol{q}) = (\varphi^t(\boldsymbol{q}_1), \varphi^t(\boldsymbol{q}_2), \dots, \varphi^t(\boldsymbol{q}_n))$$

satisfies the equations of motion (1.2).[2]

[1] Cf. [8], p. 61.
[2] Cf. [17], p. 47.

One-parameter subgrups of $SO(N)$ are of the form $\varphi^t(\boldsymbol{q}_1) = e^{t\Omega}\boldsymbol{q}_1$, with $\Omega$ skew-symmetric $N \times N$ (non-zero) matrix.[3] If $U$ is invariant with respect to the above-mentioned one-parameter group of rotations $\varphi^t = e^{t\Omega}$, a relative equilibrium satisfies the equation

$$\Omega^2 \boldsymbol{q} = \nabla_M U(\boldsymbol{q}) \ .$$

If follows that if $\dim(E) = 2$, then $\Omega = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$ with $\omega \in \mathbb{R}$, $\omega \neq 0$, and therefore such a relative equilibrium configuration is a central configuration, $-\omega^2 \boldsymbol{q} = \nabla_M U(\boldsymbol{q})$. Conversely, planar central configurations with $U(\boldsymbol{q}) > 0$ (if $U$ is homogeneous of negative degree) yield relative equilibria, with a suitable (angular speed) $\omega$.

If $\dim(E) = 3$, then since the non-zero $3 \times 3$ skew-symmetric matrix $\Omega$ has rank 2, $E$ can be written as $\ker \Omega \oplus E'$, where $\ker \Omega$ is the fixed direction of the rotations $\varphi^t = e^{t\Omega}$, and $E'$ is the orthogonal complement of $\ker \Omega$. In a suitable reference, $\Omega = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. For further reference, let

$$P \colon E \to E' \tag{1.7}$$

denote the orthogonal projection.

If $U$ is the homogeneous Newtonian potential of (1.1) with $\alpha > 0$ and $m_j > 0$, then it is easy to see that relative equilibrium configurations must belong to the plane $E'$. This is not true in general: it is possible to find examples of relative equilibria which are not planar – see (4.1) (and hence they are not central configurations). For more on equilibrium (and homographic) solutions: [19] (§369–§382bis at pp. 284–306), [4], [2].

Recent and non recent relevant literature on central configurations: [14], [7], [16, 17], [10, 11, 12], [8], [20], [1], [6], [3].

From now on, unless otherwise stated, assume that $U$ is invariant with respect to all isometries in $E$, all masses $m_j > 0$ are positive, and $U$ is homogeneous of negative degree $-\alpha$.

The potential $U$ is invariant with respect to a suitable subgroup of $\Sigma_n \times O(E)$, where $\Sigma_n$ is the symmetric group on $n$ elements and $O(E)$ denotes the orthogonal group on the euclidean space $E$. For example, if all masses are equal and $U$ is defined as in (1.1), then $G = \Sigma_n \times O(E)$; if all masses are distinct, then $G = \{1\} \times O(E)$.

The following proposition is a well-known characterization of the set of CC, which we will generalize to relative equilibria in (1.9).

**(1.8)** *Let $S \subset Y$ denote the inertia ellipsoid, defined as $S = \{\boldsymbol{q} \in Y : \|\boldsymbol{q}\|_M^2 = 1\}$. A point $\boldsymbol{q} \in S$ is a central configuration if and only if it is a critical point of the restriction of $U$ to $S$.*

[3] Cf. [18], p. 401.

*Proof.* Critical points of $U|_S$ are points $\boldsymbol{q} \in Y$ such that $\ker dU \supset T_{\boldsymbol{q}}S$. With respect to the (non-degenerate) bilinear form $\langle -, - \rangle_M$, this can be written as $\nabla_M U(\boldsymbol{q}) = \lambda \nabla_M (\|\boldsymbol{q}\|_M^2) = 2\lambda \boldsymbol{q}$. *q.e.d.*

**(1.9)** *Assume* $\dim(E) = 3$. *Let* $C$ *be the* vertical cylinder *defined as*

$$C = \{\boldsymbol{q} \in Y : \langle P\boldsymbol{q}, \boldsymbol{q} \rangle_M = c \ ,\}$$

*where* $P$ *is the projection of* $E$ *to* $E'$ *as in (1.7) and* $c = \langle P\overline{\boldsymbol{q}}, \overline{\boldsymbol{q}} \rangle_M$. *A configuration* $\overline{\boldsymbol{q}} \in Y$ *is a relative equilibrium configuration rotating by* $e^{t\Omega}$ *if and only if it is a critical point of* $U$ *restricted to* $C \subset X$ *and* $U(\overline{\boldsymbol{q}}) > 0$.

*Proof.* The configuration $\boldsymbol{q}$ is a relative equilibrium configuration if and only if $\Omega^2 \boldsymbol{q} = \nabla_M U(\boldsymbol{q})$; since $\Omega^2 = -\omega^2 P$ for an $\omega \neq 0$, this is equivalent to

$$\nabla_M U(\boldsymbol{q}) = -\omega^2 P\boldsymbol{q}.$$

On the other hand, $\nabla_M (\langle P\boldsymbol{q}, \boldsymbol{q} \rangle_M) = 2P\boldsymbol{q}$, hence $\boldsymbol{q} \in C$ is a critical point of the restriction $U|_C$ iff $\nabla_M U(\boldsymbol{q}) = \lambda 2 P\boldsymbol{q}$. The proof follows since, by homogeneity,

$$\langle \nabla_M U(\boldsymbol{q}), \boldsymbol{q} \rangle_M = -\alpha U(\boldsymbol{q})$$

and $\langle P\boldsymbol{q}, \boldsymbol{q} \rangle_M = \|P\boldsymbol{q}\|_M^2$. *q.e.d.*

**(1.10)** *Let* $K$ *be a subgroup of the symmetry group* $G$ *of* $U$ *on* $Y$. *Then the inertia ellipsoid* $S$ *is* $K$*-invariant, and critical points of the restriction of* $U$ *to* $S^K = \{\boldsymbol{q} \in S : K\boldsymbol{q} = \boldsymbol{q}\}$ *are precisely the critical points of* $U|_S$ *belonging to* $S^K$. *If the vertical cylinder* $C$ *is* $K$*-invariant, then critical points of the restriction of* $U$ *to* $C^K = \{\boldsymbol{q} \in C : K\boldsymbol{q} = \boldsymbol{q}\}$ *are precisely the critical points of* $U|_C$ *belonging to* $C^K$.

*Proof.* It is a consequence of Palais principle of Symmetric Criticality [9]. *q.e.d.*

# 2 Central configurations as (equivariant) fixed points.

In [5] a way to relate planar central configurations to projective classes of fixed points was introduced. We now generalize the results therein to arbitrary dimensions.

Consider a homogeneous potential $U$, as above with the further assumption that $\forall \boldsymbol{q}, U(\boldsymbol{q}) > 0$. From this it follows that $\nabla_M U(\boldsymbol{q}) \neq \boldsymbol{0}$ because $\langle \nabla_M U(\boldsymbol{q}), \boldsymbol{q} \rangle_M = -\alpha U(\boldsymbol{q})$.

**(2.1)** *The map* $F \colon S \to S$ *defined as*

$$F(\boldsymbol{q}) = -\frac{\nabla_M U(\boldsymbol{q})}{\|\nabla_M U(\boldsymbol{q})\|_M}$$

*is well-defined, and a configuration* $\bar{\boldsymbol{q}} \in S$ *is a central configuration if and only if it is a fixed point of* $F$.

*Proof.* It follows from the assumption that $\forall \boldsymbol{q}, \nabla_M U(\boldsymbol{q}) \neq \mathbf{0}$, and therefore $F$ is well-defined. A configuration $\boldsymbol{q}$ is fixed by $F$ if and only if there exists $\lambda = \|\nabla_M U(\boldsymbol{q})\|_M > 0$ such that $\nabla_M U(\boldsymbol{q}) = -\lambda \boldsymbol{q}$. Hence, if $F(\boldsymbol{q}) = \boldsymbol{q}$ then $\boldsymbol{q}$ is central. Conversely, by homogeneity of $U$, $\langle \nabla_M U(\boldsymbol{q}), \boldsymbol{q}\rangle_M = -\alpha U(\boldsymbol{q})$, and hence if $\boldsymbol{q} \in S$ is a central configuration then $\nabla_M U(\boldsymbol{q}) = \lambda \boldsymbol{q} \implies \lambda = -\alpha U(\boldsymbol{q}) < 0$ and therefore $F(\boldsymbol{q}) = \boldsymbol{q}$. *q.e.d.*

Now, in general (and without the positivity assumption) the map $F$ needs not being compactly fixed (see for example Robert's continuum [13] of central configurations with four unit masses and a fifth negative $-1/4$ mass in the origin[4]). In the graviational case (positive masses and Newtonian mutual attraction), the map $F$ turns out to be compactly fixed [5] (see also Shub's estimates [15]).

**(2.2)** *Let $G$ be the symmetry group of $U$ on $Y$, as above. Then $F$ is $G$-equivariant. For each subgroup $K \subset G$, $F$ induces a self-map $\bar{F} \colon S/K \to S/K$ on the quotient space $S/K$.*

*Proof.* For $g \in G$, $gS = S$, and for each $g$ such that $U(g\boldsymbol{q}) = U(\boldsymbol{q})$ the equality $\nabla_M U(g\boldsymbol{q}) = g\nabla_M U(\boldsymbol{q})$ holds. In fact, since $U \circ g = U$, $dU \circ g = dU$, and therefore for each vector $\boldsymbol{v}$ one has $\langle \nabla_M U(\boldsymbol{q}), \boldsymbol{v}\rangle_M = dU(\boldsymbol{q})[\boldsymbol{v}] = dU(g\boldsymbol{q})[g\boldsymbol{v}] = \langle \nabla_M U(g\boldsymbol{q}), g\boldsymbol{v}\rangle_M = \langle g^{-1}\nabla_M U(g\boldsymbol{q}), \boldsymbol{v}\rangle_M$. Thus $F$ is $G$-equivariant, and hence $K$-equivariant for each subgroup $K \subset G$. *q.e.d.*

Let $U$ be a homogeneous potential with the following property: for each orthogonal projection $p \colon E \to P$ on a plane $P$, for each $\boldsymbol{q} \in S$ there exists $j \in \{1, \dots, n\}$ such that

$$p\left(\frac{\partial U}{\partial \boldsymbol{q}_j}(\boldsymbol{q})\right) \cdot p(\boldsymbol{q}_j) \leq 0 \ . \tag{2.3}$$

Moreover, if there exists $i \in \{1, \dots, n\}$ such that $p(\boldsymbol{q}_i) \neq 0$, then the $j$ of (2.3) is such that $p(\boldsymbol{q}_j) \neq 0$.

It is easy to see that the Newtonian potential (1.1) (with positive masses and homogeneity $-\alpha$) satisfies Property (2.3): let $j$ be the index maximizing

[4] In the AMS review of [13], D. Saari states that a similar effect occurs for positive masses, but with two or more homogeneous potentials. More precisely, should the potential be a sum of homogeneous potentials, then there always is a continuum of different relative equilibria configurations.

$\|p(\boldsymbol{q}_k)\|^2$ for $k = 1, \dots, n$; then

$$p\left(\alpha \sum_{k\neq j} \frac{m_j m_k (\boldsymbol{q}_k - \boldsymbol{q}_j)}{\|\boldsymbol{q}_k - \boldsymbol{q}_j\|^{\alpha+2}}\right) \cdot p(\boldsymbol{q}_j) = \alpha \left(\sum_{k\neq j} \frac{m_j m_k (p(\boldsymbol{q}_k) - p(\boldsymbol{q}_j))}{\|\boldsymbol{q}_k - \boldsymbol{q}_j\|^{\alpha+2}}\right) \cdot p(\boldsymbol{q}_j)$$
$$= \alpha \sum_{k\neq j} \frac{m_j m_k (p(\boldsymbol{q}_k) \cdot p(\boldsymbol{q}_j) - \|p(\boldsymbol{q}_j)^2)\|}{\|\boldsymbol{q}_k - \boldsymbol{q}_j\|^{\alpha+2}}$$
$$\leq \sum_{k\neq j} \frac{m_j m_k (\|p(\boldsymbol{q}_k)\| \|p(\boldsymbol{q}_j)\| - \|p(\boldsymbol{q}_j)^2)\|}{\|\boldsymbol{q}_k - \boldsymbol{q}_j\|^{\alpha+2}}$$
$$\leq 0$$

It is trivial to see that if $U$ satisties (2.3), then the map $F\colon S \to S$ defined in (2.1) satisfies the following property: for each orthogonal projection $p\colon E \to P$ on a plane $P$, for each $\boldsymbol{q} \in S$, there exists $j \in \{1, \dots, n\}$ such that

$$p(F_j(\boldsymbol{q})) \cdot p(\boldsymbol{q}_j) \geq 0 \ . \tag{2.4}$$

**(2.5) Theorem.** *Suppose that $G \supset SO(E)$, where as above $G$ is the symmetry group of $U$ on $Y$, and $SO(E) = SO(d)$ (the group of rotations in $E$) acting diagonally on $Y$. Let $\overline{F}\colon S/SO(d) \to S/SO(d)$ be the map induced on the quotient space by (2.2). Assume that all masses are positive, $U > 0$ and (2.4) holds. Then, if $\pi\colon S \to S/SO(d)$ denotes the projection on the quotient,*

$$\pi(\mathrm{Fix}(F)) = \mathrm{Fix}(\overline{F}) \ .$$

*Proof.* If $\boldsymbol{q} \in \mathrm{Fix}(F) \subset S$ is fixed by $F$, then $\overline{F}(\pi(\boldsymbol{q})) = \pi F(\boldsymbol{q}) = \pi(\boldsymbol{q}) \implies \pi(\boldsymbol{q}) \in \mathrm{Fix}(\overline{F})$, hence $\pi(\mathrm{Fix}(F)) \subset \mathrm{Fix}(\overline{F})$. On the other hand, let $\pi(\boldsymbol{q}) \in \mathrm{Fix}(\overline{F})$. Then there exists a rotation $g \in SO(E)$ such that $F(\boldsymbol{q}) = g\boldsymbol{q}$. Without loss of generality, we can assume that $g = e^{\Omega}$, where $\Omega$ is an antisymmetric $d \times d$ matrix, with $k$ $(2\times 2)$-blocks on the diagonal $\begin{bmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{bmatrix}$, with $\theta_i \in [-\pi, \pi]$ for each $i = 1, \dots, k$, and, if $d$ is odd, a one-dimensional diagonal zero entry ($d = 2k$ or $d = 2k + 1$). We can also assume that only the first (say, $l \leq k$) blocks have $\theta_i \neq 0$, hence $\Omega$ has $l$ non-zero $(2 \times 2)$ diagonal blocks and is zero outside. Note that for each $\boldsymbol{x} \in E$, the quadratic form $(e^{\Omega}\boldsymbol{x}) \cdot (\Omega\boldsymbol{x})$ on $E$ can be written with $l$ non-singular positive-defined blocks

$$\theta_i \begin{bmatrix} \sin\theta_i & -\cos\theta_i \\ \cos\theta_i & \sin\theta_i \end{bmatrix} \sim \theta_i \begin{bmatrix} \sin\theta_i & 0 \\ 0 & \sin\theta_i \end{bmatrix}$$

on the diagonal, and hence it is non-negative. Moreover, if one writes $\boldsymbol{x} \in E \cong \mathbb{R}^d$ as $(\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_l, x_{2l+1}, \dots, x_d)$, with $\boldsymbol{z}_i \in \mathbb{R}^2$ for $i = 1, \dots, l$ and $x_i \in \mathbb{R}$, then

$$(e^{\Omega}\boldsymbol{x}) \cdot (\Omega\boldsymbol{x}) = \sum_{i=1}^{l} \theta_i \sin\theta_i \|\boldsymbol{z}_i\|^2 \ . \tag{2.6}$$

Let $p_i \colon E \to \mathbb{R}^2$ denote the projection $\boldsymbol{x} \mapsto \boldsymbol{z}_i$, for $i = 1, \dots, l$.

Since $F(e^{t\Omega}\boldsymbol{q})$ does not depend on $t \in \mathbb{R}$,

$$\begin{aligned} 0 &= \frac{d}{dt} \| F(e^{t\Omega}\boldsymbol{q}) \|^2_M |_{t=0} \\ &= 2\langle F(\boldsymbol{q}), \Omega\boldsymbol{q} \rangle_M \\ &= 2\langle e^{\Omega}\boldsymbol{q}, \Omega\boldsymbol{q} \rangle_M = \\ &= 2\sum_{j=1}^{n} m_j (e^{\Omega}\boldsymbol{q}_j) \cdot (\Omega\boldsymbol{q}_j) \end{aligned}$$

For each $j = 1, \dots, n$ the inequality $m_j > 0$ holds, and for each $\boldsymbol{x} \in E$ the inequality $(e^{\Omega}\boldsymbol{x}) \cdot (\Omega\boldsymbol{x}) \geq 0$ holds: it follows that for each $j$, $(e^{\Omega}\boldsymbol{q}_j) \cdot (\Omega\boldsymbol{q}_j) = 0$. By (2.6), this implies that, given $j$, for each $i = 1, \dots, l$ either $p_i(\boldsymbol{q}_j) = 0$ or $\theta_i \in \{\pi, -\pi\}$ (since $\theta_i \neq 0$ for $i = 1, \dots, l$). If $p_i(\boldsymbol{q}_j) = 0$ for each $j$, then actually $g\boldsymbol{q} = \boldsymbol{q}$, and therefore $\pi(\boldsymbol{q}) \in \pi(\mathrm{Fix}(F))$. So, without loss of generality one can assume that for each $i = 1, \dots, l$ there exists $j$ such that $p_i(\boldsymbol{q}_j) \neq 0$. Suppose that $l \geq 1$, and therefore $\theta_1 = \pm\pi$. By property (2.4) there exists $\bar{j}$ such that $p_1(F_{\bar{j}}(\boldsymbol{q})) \cdot p_1(\boldsymbol{q}_{\bar{j}}) \geq 0$ and $p_1(\boldsymbol{q}_{\bar{j}}) \neq 0$. But this would imply that

$$F(\boldsymbol{q}) = g\boldsymbol{q} = e^{\Omega}\boldsymbol{q} \implies -p_1(\boldsymbol{q}_{\bar{j}}) \cdot p_1(\boldsymbol{q}_{\bar{j}}) \geq 0 \implies p_1(\boldsymbol{q}_{\bar{j}}) = 0 \ ,$$

which is not possible. Therefore, if condition (2.4) holds, $l = 0$, and $g\boldsymbol{q} = \boldsymbol{q}$. The conclusion follows. *q.e.d.*

## 3 Projective fixed points and Morse indices

In this section, we finally prove the equation relating fixed point and Morse indices of central configurations.

**(3.1)** *If $U$ is homogeneous of degree $-\alpha$, then for each central configuration $\boldsymbol{q}$, up to a linear change of coordinates*

$$-\alpha U(\boldsymbol{q})(I - F'(\boldsymbol{q})) = D^2\tilde{U}(\boldsymbol{q}),$$

*where $F \colon S \to S$ is the function of (2.1), defined as $F(\boldsymbol{q}) = -\dfrac{\nabla_M U(\boldsymbol{q})}{\|\nabla_M U(\boldsymbol{q})\|_M}$, and $D^2\tilde{U}(\boldsymbol{q})$ is the Hessian of the restriction $\tilde{U}$ of $U$ to $S$, evalued at $\boldsymbol{q}$.*

*Proof.* After a linear change of coordinates in $X$, we can assume $m_i = 1$ for each $i$ and $\boldsymbol{q} = (1, 0, \dots, 0) = (1, \boldsymbol{0})$ (rescale $\boldsymbol{q}$ by a diagonal matrix with suitable $m_j$ on its diagonal, and apply a rotation - this leaves $U$ homogeneous of the same degree). Given suitable linear coordinates $\boldsymbol{x} = (x_0, x_1, \dots, x_k)$ in $Y \cong \mathbb{R}^{l+1}$, the ellipsoid $S$ has equation $\|\boldsymbol{x}\|^2 = 1$, and $F(\boldsymbol{x}) = -\dfrac{\frac{dU}{d\boldsymbol{x}}}{\|\frac{dU}{d\boldsymbol{x}}\|}$, where $\frac{dU}{d\boldsymbol{x}} = \nabla U$ in the $\boldsymbol{x}$-coordinates. Therefore, if $\boldsymbol{u} = (u_1, \ldots, u_l) \mapsto (\sqrt{1 - \|\boldsymbol{u}\|}, u_1, \ldots, u_l) \in S$ is a local chart around the central configuration $\boldsymbol{q} \sim (1, \boldsymbol{0})$,

$$\begin{aligned} \frac{\partial F_\alpha}{\partial u_\beta}(\boldsymbol{0}) &= \frac{\partial^2 U}{\partial x_\alpha \partial x_\beta}(\boldsymbol{q}) \|\nabla U(\boldsymbol{q})\|^{-1} \\ D^2_{\alpha\beta}\tilde{U}(\boldsymbol{q}) &= \frac{\partial^2 U}{\partial x_\alpha \partial x_\beta}(\boldsymbol{q}) - \delta_{\alpha\beta}\frac{\partial U}{\partial x_0} \ . \end{aligned} \tag{3.2}$$

Now, $\dfrac{\partial U}{\partial x_0} = \langle \nabla U, \boldsymbol{q} \rangle = -\alpha U(\boldsymbol{q})$, and since $\boldsymbol{q}$ is a central configuration, it is a fixed point of $F$ and therefore

$$F(\boldsymbol{q}) = \boldsymbol{q} \implies \frac{\partial U}{\partial x_\alpha} = 0, \qquad \text{for } \alpha = 1, \ldots, l$$

and

$$\frac{\partial U}{\partial x_0}(\boldsymbol{q}) = -\|\nabla U(\boldsymbol{q})\| = -\alpha U(\boldsymbol{q}) \ . \tag{3.3}$$

From (3.3) and (3.2) it follows that for $\alpha, \beta = 1, \ldots, l$,

$$\begin{aligned} D^2_{\alpha\beta}\tilde{U}(\boldsymbol{q}) + \delta_{\alpha\beta}\frac{\partial U}{\partial x_0} &= \frac{\partial^2 U}{\partial x_\alpha \partial x_\beta}(\boldsymbol{q}) = \frac{\partial F_\alpha}{\partial u_\beta}(\boldsymbol{0})\|\nabla U(\boldsymbol{q})\| \\ \implies D^2\tilde{U}(\boldsymbol{q}) &= -\alpha U(\boldsymbol{q}) I + \alpha U(\boldsymbol{q}) F'(\boldsymbol{q}) \end{aligned} \tag{3.4}$$

*q.e.d.*

**(3.5) Corollary.** *Assume the hypotheses of Theorem (2.5) hold. Then for each non-degenerate projective class of central configurations* $\boldsymbol{q} \in \mathrm{Fix}(\bar{F})$ *in the maximal isotropy stratum of* $\bar{S} = S/SO(d)$*, the fixed point index* $\mathrm{ind}(\boldsymbol{q}, \bar{F})$ *and the Morse index* $\mu_{\tilde{U}}(\boldsymbol{q})$ *of* $\tilde{U}$ *at* $\boldsymbol{q}$ *are related by the equality*

$$\mathrm{ind}(\boldsymbol{q}, \bar{F}) = (-1)^{\mu_{\tilde{U}}(\boldsymbol{q}) + \epsilon},$$

$$\text{where } \epsilon = \dim S - d(d-1)/2 + \dim(S) = d(n-1) - 1 - d(d-1)/2.$$

*Proof.* Since $\tilde{U}$ is $SO(d)$-invariant (with diagonal action), and the $SO(d)$-orbits in $S$ with maximal isotropy type have dimension $d(d-1)/2$, the Hessian $D^2\tilde{U}$ has $d(d-1)/2$-dimensional kernel, if $\boldsymbol{q}$ is non-degenerate. By proposition (3.1), if follows that $F'$ has a $d(d-1)/2$-multiple eigenvalue 1, and $\mathrm{ind}(\boldsymbol{q}, \bar{F}) = (-1)^c$, where $c$ is the number of negative eigenvalues of $-D^2\tilde{U}$, i.e. the number of positive eigenvalues of $D^2\tilde{U}$, which is equal to $\mu_{-\tilde{U}}(\boldsymbol{q}) = \dim(S - d(d-1)/2) - \mu_{\tilde{U}}(\boldsymbol{q})$ . *q.e.d.*

**(3.6) Corollary.** *Let $U$ be the Newton potential in* (1.1)*, with positive masses, $\alpha > 0$, and* $\dim(E) = 2$*. Then $X = \mathbb{C}^n \smallsetminus \Delta$, and $S/SO(2) \approx \mathbb{P}^{n-2}(\mathbb{C})_0 \subset \mathbb{P}^{n-2}(\mathbb{C})$, where $\mathbb{P}^{n-2}(\mathbb{C})_0$ is the subset of $\mathbb{P}^{n-1}(\mathbb{C})$ defined in projective coordinates as $\mathbb{P}^{n-2}(\mathbb{C})_0 \cong \{[z_1 : \dots, z_n] \in \mathbb{P}^n : \sum_j m_j z_j = 0, \forall i, j, z_i \neq z_j\}$. Then for each non-degenerate projective class of central configurations $q \in \operatorname{Fix}(\overline{F})$, with $\overline{F} \colon \mathbb{P}_0^{n-2}(\mathbb{C}) \to \mathbb{P}^{n-2}(\mathbb{C})$,*

$$\operatorname{ind}(\boldsymbol{q}, \overline{F}) = (-1)^{\mu_{\overline{U}}(\boldsymbol{q})} .$$

*Proof.* If $d = 2$, then $d(n-1) - 1 - d(d-1)/2 = 2(n-2)$. *q.e.d.*

# 4 An example of non-planar relative equilibrium

**(4.1) Example** (Non-central and non-planar equilibrium solution)**.** In $E \cong \mathbb{R}^3$, let $R_x$, $R_y$ and $R_z$ denote rotations of angle $\pi$ around the three coordinate axes. Fix three non-zero constants $c_1, c_2, c_3$. Consider the problem with 6 bodies in $E$, symmetric with respect to the group $K$ with non-trivial elements of $\Sigma_6 \times SO(3)$

$$((34)(56), R_x), ((12)(56), R_y), ((12)(34), R_z) .$$

Assume $m_j = 1$, for $j = 1, \dots, 6$, and let $U$ be the potential defined on $E^6$ by

$$U(\boldsymbol{q}) = \sum_{i<j} \frac{1 - \gamma_i \gamma_j}{\|\boldsymbol{q}_i - \boldsymbol{q}_j\|}$$

where

$$\gamma_1 = \gamma_2 = c_1, \quad \gamma_3 = \gamma_4 = c_2, \quad \gamma_5 = \gamma_6 = c_3 .$$

Now, $U$ is invariant with respect to $K$, and the vertical cylinder is $K$-invariant: it follows from (1.9) that critical points of the restriction of $U$ to $C^K$ are equilibrium configurations.

In other words, three pairs of bodies of unit masses, each pair charged with charge $c_j$, are constrained each pair to belong to one of the coordinate axes and to be symmetric with respect to the origin.

The space $X^K = Y^K$ has dimension 3, and can be parametrized by $(x, y, z)$, where $x$, $y$ and $z$ are (respectively) the coordinates along the corresponding axis of particles 1, 3 and 5. The generic configuration $\boldsymbol{q} \in X^K$ can be written as

$$\begin{aligned} \boldsymbol{q}_1 &= (x, 0, 0) & \boldsymbol{q}_2 &= (-x, 0, 0) \\ \boldsymbol{q}_3 &= (0, y, 0) & \boldsymbol{q}_4 &= (0, -y, 0) \\ \boldsymbol{q}_5 &= (0, 0, z) & \boldsymbol{q}_6 &= (0, 0, -z), \end{aligned}$$

and the potential $U$ restricted to $X^K$ in such coordinates is

$$U(x, y, z) = \frac{1 - c_1^2}{2|x|} + \frac{1 - c_2^2}{2|y|} + \frac{1 - c_3^2}{2|z|} + 4\frac{1 - c_1 c_2}{\sqrt{x^2 + y^2}} + 4\frac{1 - c_1 c_3}{\sqrt{x^2 + z^2}} + 4\frac{1 - c_2 c_3}{\sqrt{y^2 + z^2}} .$$

The vertical cylinder $C^K \subset X^K$ is

$$\begin{aligned} C^K &= \{\boldsymbol{q} \in X^K : \langle P\boldsymbol{q}, \boldsymbol{q}\rangle_M = 2\} \\ &= \{(x,y,z) \in X^K : 2x^2 + 2y^2 = 2\} \ . \end{aligned}$$

Hence an equilibrium solution is a critical point of $U(\cos t, \sin t, z)$ with positive value $U > 0$. Now, assume

$$c_1 > 1, \quad c_2 < -1, \quad c_3 < -1 \ . \tag{4.2}$$

Then $1-c_1^2 < 0$, $1-c_2^2 < 0$, $1-c_3^2 < 0$, $1-c_2c_3 < 0$; moreover, $1-c_1c_2 > 0$ and $1-c_1c_3 > 0$. The restricted potential $U$ is defined as

$$U(t,z) = \frac{1-c_1^2}{2|\cos t|} + \frac{1-c_2^2}{2|\sin t|} + \frac{1-c_3^2}{2z} + 4(1-c_1c_2) + 4\frac{1-c_1c_3}{\sqrt{\cos^2 t + z^2}} + 4\frac{1-c_2c_3}{\sqrt{\sin^2 t + z^2}},$$

and is defined on the strip $(t,z) \in T = (0, \pi/2) \times (0, +\infty)$.

If

$$1 - c_1^2 + 8(1-c_1c_3) < 0, \tag{4.3}$$

then for each $(t,z)$

$$\begin{aligned} \frac{1-c_1^2}{2|\cos t|} + 4\frac{1-c_1c_3}{\sqrt{\cos^2 t + z^2}} &< \frac{1-c_1^2}{2|\cos t|} + 4\frac{1-c_1c_3}{|\cos t|} \\ &= \frac{1}{2|\cos t|}\left(1 - c_1^2 + 8(1-c_1c_3)\right) < 0 \end{aligned}$$

and hence $U(t,z) < 4(1-c_1c_2)$ for each $(t,z) \in T$ and $U \to -\infty$ on the boundary of $T$.

Furthermore, for each $t \in (0, \pi/2)$,

$$\begin{aligned} \frac{\partial U}{\partial z} &= \frac{c_3^2-1}{2z^2} + 4z\frac{c_2c_3-1}{(z^2+\sin^2 t)^{3/2}} - 4z\frac{1-c_1c_3}{(z^2+\cos^2 t)^{3/2}} \\ &< \frac{c_3^2-1}{2z^2} + 4\frac{c_2c_3-1}{z^2} - 4z\frac{1-c_1c_3}{(z^2+1)^{3/2}} \ . \end{aligned}$$

It follows that if

$$c_3^2 - 1 + 8(c_2c_3 - 1) < 8(1-c_1c_3), \tag{4.4}$$

then for every $t \in (0, \pi/2)$ there exists $z_0$ such that $z > z_0 \implies \frac{\partial U}{\partial z}(t,z) < 0$.

Thus, whenever both (4.3) and (4.4) hold, the restriction $U(t,z)$ attains its maximum in the interior of the strip $T$. Such a maximum $(t_m, z_m)$ corresponds to an equilibrium configuration if and only if $U(t_m, z_m) > 0$, from (1.9). Note that if $t = \pi/4$ and $z = 1/2$ then

$$U(\pi/4, z) = \frac{\sqrt{2}}{2}(2 - c_1^2 - c_2^2) + 4(1-c_1c_2) + 1 - c_3^2 + \frac{4(2 - c_3(c_1+c_2))}{\sqrt{3/4}},$$

which is positive, for example, if $c1_{=}20$ and $c_2 = c_3 = -2$. Such coefficients satisfy (4.2), (4.3) and (4.4), and therefore for such a choice of $c_i$ there exist non-planar relative equilibrium configurations.

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