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Solutions to nonlinear Schrödinger equations with singular electromagnetic potential and critical exponent

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To Richard S. Palais on the occasion of his 80th birthday

Abstract. We investigate existence and qualitative behavior of solutions to nonlinear Schrödinger equations with critical exponent and singular electromagnetic potentials. We are concerned with magnetic vector potentials which are homogeneous of degree -1, including the Aharonov–Bohm class. In particular, by variational arguments we prove a result of multiplicity of solutions distinguished by symmetry properties.

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1. Introduction

In nonrelativistic quantum mechanics, the Hamiltonian associated with a charged particle in an electromagnetic field is given by $(i\nabla - A)^2 + V$, where $A : \mathbb{R}^N \to \mathbb{R}^N$ is the magnetic potential and $V : \mathbb{R}^N \to \mathbb{R}$ is the electric one. The vector field $B = \operatorname{curl} A$ has to be intended as the differential 2-form $B = da$, with a the 1-form canonically associated with the vector field A. Only in three dimensions, by duality, B is represented by another vector field.

In this paper, we are concerned with differential operators of the form

$$\left(i\,\nabla - \frac{A(\theta)}{|x|}\right)^2 - \frac{a(\theta)}{|x|^2},$$

where $A(\theta) \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R}^N)$ and $a(\theta) \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$. Notice the presence of homogeneous (Fuchsian) singularities at the origin. In some situations the potentials may also have singularities on the sphere.

This kind of magnetic potentials appear as limits of thin solenoids, when the circulation remains constant as the sequence of solenoids' radii tends to zero. The limiting vector field is then a singular measure supported in a lower-dimensional set. Though the resulting magnetic field vanishes almost everywhere, its presence still affects the spectrum of the operator, giving rise to the so-called *Aharonov–Bohm effect*.

Also from the mathematical point of view this class of operators is worth being investigated, mainly because of their critical behavior. Indeed, they share with the Laplacian the same degree of homogeneity and invariance under the Kelvin transform. Therefore, they cannot be regarded as lower-order perturbations of the Laplace operator (they do not belong to the Kato's class, see for instance [16, 17] and the references therein).

Here we shall always assume $N \geq 3$, unless specified otherwise. A quadratic form is associated with the differential operator; that is,

$$\int_{\mathbb{R}^N} \left| \left(i\,\nabla - \frac{A(\theta)}{|x|} \right) u \right|^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2}\, u^2. \tag{1}$$

Let its natural domain be the closure of compactly supported functions $C^\infty_C(\mathbb{R}^N \setminus \{0\})$ with respect to the quadratic form itself. Thanks to Hardy-type inequalities, when $N \geq 3$, this space turns out to be the same as $D^{1,2}(\mathbb{R}^N)$, provided a is suitably bounded (see [16]), while, when $N = 2$, this is a smaller space of functions vanishing at the pole of the magnetic potential. Throughout the paper, we always assume positivity of (1).

We are interested in solutions to the critical semilinear differential equations

$$\left(i\,\nabla - \frac{A(\theta)}{|x|} \right)^2 u - \frac{a(\theta)}{|x|^2} u = |u|^{2^*-2}\, u \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{2}$$

and in particular in their symmetry properties. The critical exponent appears as the natural one whenever seeking finite energy solutions: indeed, Pohozaev-type identities prevent the existence of entire solutions for power nonlinearities of different degrees.

The first existence results for equations of type (2) are given in [15] for subcritical nonlinearities. In addition, existence and multiplicity of solutions are investigated, for instance, in [8, 12, 19, 25] mainly via variational methods and concentration-compactness arguments. Some results involving critical nonlinearities are present in [2, 7]. Concerning results on semiclassical solutions we quote [10, 11]. As far as we know, not many papers are concerned with singular electromagnetic potentials, except [18], where anyway several integrability hypotheses are assumed on them, and, much more related with ours, the paper [13] that we discuss later on.

We are interested in the existence of solutions to equation (2) distinct by symmetry properties, as it happens in [27] for Schrödinger operators when magnetic vector potential is not present. To investigate these questions, we aim to extend some of the results in [27] when a singular electromagnetic potential is present.

To do this, we consider solutions which minimize the Rayleigh quotient

$$\frac{\int_{\mathbb{R}^N} \left|\left(i\nabla - \frac{A(\theta)}{|x|}\right)u\right|^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2}\, u^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}}.$$

We emphasize that, although, in general, ground states in $D^{1,2}(\mathbb{R}^N)$ to equation (2) do not exist (see Section 3), the existence of minimizers can be granted in suitable subspaces of symmetric functions.

We are concerned with Aharonov–Bohm-type potentials too. In $\mathbb{R}^2$ a vector potential associated with the Aharonov–Bohm magnetic field has the form

$$\mathcal{A}(x_1, x_2) = \alpha \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right),$$

where $\alpha \in \mathbb{R}$ stands for the circulation of $\mathcal{A}$ around the thin solenoid. Here we consider the analogous of these potentials in $\mathbb{R}^N$ for $N \geq 4$; that is,

$$\mathcal{A}(x_1, x_2, x_3) = \left(\frac{-\alpha x_2}{x_1^2 + x_2^2}, \frac{\alpha x_1}{x_1^2 + x_2^2}, 0\right), \quad (x_1, x_2) \in \mathbb{R}^2, x_3 \in \mathbb{R}^{N-2}.$$

Our main result can be stated as follows.

Theorem 1.1. *Assume $N \geq 4$, $a(\theta) \equiv a \in \mathbb{R}^-$ and $A(\theta)$ is equivariant under the group $SO(2)\times SO(N-2)$, that is, $A(g\theta) = gA(\theta)$ for all $SO(2)\times SO(N-2)$ and for all $\theta \in \mathbb{S}^N$. There exist $a^* < 0$ such that, when $a < a^*$, equation* (2) *admits at least two distinct solutions in $D^{1,2}(\mathbb{R}^N)$: one is biradially symmetric while the second one is equivariant under the action of a discrete group of rotations $\mathbb{Z}_k \times SO(N-2)$ on the first two variables.*

A similar result holds for Aharonov–Bohm-type potentials.

As we describe below, we also allow actions inducing a nontrivial angular momentum. We point out that the hypothesis on the dimension here is purely technical. By the way, in dimension $N = 3$ and in case of Aharonov–Bohm potentials, Clapp and Szulkin [13] proved the existence of at least a solution which enjoys the so-called *biradial* symmetry. However, their argument may be adapted even in further dimensions, provided a cylindrical symmetry is required for functions with respect to the second set of variables in $\mathbb{R}^{N-2}$.

The proof of our main result is based on a comparison between the different levels of the Rayleigh quotient's infima taken over different spaces of functions which enjoy certain symmetry properties. In particular, we will focus our attention on three different kinds of symmetries:

1. functions which are equivariant under the $\mathbb{Z}_k \times SO(N-2)$ action for $k \in \mathbb{N}$ and $m \in \mathbb{Z}$ defined as

$$u(z, y) \mapsto \mathrm{e}^{-i\frac{2\pi}{k}m} u\left(\mathrm{e}^{i\frac{2\pi}{k}} z, Ry\right) \qquad \text{for } z \in \mathbb{R}^2 \text{ and } y \in \mathbb{R}^{N-2},\ R \in SO(N-2),$$

$D^{1,2}_{k,m}(\mathbb{R}^N)$ will denote their vector space;

2. functions which we will call "biradial," i.e.,

$$D^{1,2}_{r_1,r_2}(\mathbb{R}^N) := \Big\{u \in D^{1,2}(\mathbb{R}^N) : u(Sz,Ty) = S^m u(z,y) \quad \forall\,(S,T) \in SO(2)\times SO(N-2)\Big\},$$

so that they have the form

$$u(z,y) = \rho(|z|\,,|y|)\,\mathrm{e}^{im\theta(z)},$$

where $\theta(z) = \arg(z)$;
3. functions which are radial, $D^{1,2}_{\mathrm{rad}}$ will be their vector space.

We will fix the notation we use throughout the paper.

Definition 1.2.

- $S^{\mathrm{birad},m}_{A,a}$ is the minimum of the Rayleigh quotient related to the magnetic Laplacian over all the biradial functions in $D^{1,2}_{r_1,r_2}(\mathbb{R}^N)$;
- $S^{\mathrm{birad},m}_{0,a}$ is the minimum of the Rayleigh quotient related to the usual Laplacian over all the biradial functions in $D^{1,2}(\mathbb{R}^N)$;
- $S^{\mathrm{rad}}_{0,a}$ is the minimum of the Rayleigh quotient related to the usual Laplacian over all the radial functions in $D^{1,2}(\mathbb{R}^N)$;
- $S^{k,m}_{0,a}$ is the minimum of the Rayleigh quotient related to the usual Laplacian over all the functions in $D^{1,2}_k(\mathbb{R}^N)$;
- $S^{k,m}_{A,a}$ is the minimum of the Rayleigh quotient related to the magnetic Laplacian over all the functions in $D^{1,2}_k(\mathbb{R}^N)$;
- S is the usual Sobolev constant for the immersion $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

In order to prove that these quantities are achieved, we use a special form of the concentration-compactness arguments due to Solimini [26]. Unfortunately, we are not able to compute the precise values of the above-mentioned infima, but we can only provide estimates in terms of the Sobolev constant S; nevertheless this is enough to our aims. By the way, it is worth noticing that in [27] a characterization is given for the radial case $S^{\mathrm{rad}}_{0,a}$: it is proved that $S^{\mathrm{rad}}_{0,a}$ is achieved and the author was able to compute its precise value. This will turn out basic when we compare it with the other infimum values in order to deduce some results about symmetry properties.

Both in case of $\frac{A(\theta)}{|x|}$-type potentials and Aharonov–Bohm-type potentials, we follow the same outline. We organize the paper as follows. First, in Section 2 we state the variational framework for our problem. Secondly, in Section 3 we provide some sufficient conditions to have the infimum of the Rayleigh quotients achieved, beginning from some simple particular cases. In Section 4 we investigate the potential symmetry of solutions. Finally, in Section 6 we deduce our main result. On the other hand, Section 5 is devoted to the study of Aharonov–Bohm-type potentials.

2. Variational setting

We let the space of compactly supported functions in $\mathbb{R}^N \setminus \{0\}$ (denoted by $C_C^\infty(\mathbb{R}^N \setminus \{0\})$) be the initial domain for the quadratic form (1). Actually, as a consequence of the following lemmas, one can consider the space $D^{1,2}(\mathbb{R}^N)$ as the maximal domain for the quadratic form (1). We recall that by definition

$$D^{1,2}(\mathbb{R}^N) = \overline{C_C^\infty(\mathbb{R}^N)}^{(\int_{\mathbb{R}^N} |\nabla u|^2)^{1/2}},$$

i.e., the completion of the compact supported functions on $\mathbb{R}^N$ under the so-called Dirichlet norm.

To prove that $D^{1,2}(\mathbb{R}^N)$ may be considered as the maximal domain for the quadratic form (1), the main tools are the following basic inequalities:

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\, dx \leq \frac{4}{(N-2)^2} \int_{\mathbb{R}^N} |\nabla u|^2\, dx \quad \text{(Hardy inequality)},$$

$$\int_{\mathbb{R}^N} |\nabla |u||^2\, dx \leq \int_{\mathbb{R}^N} \left| \left(i\,\nabla - \frac{A}{|x|} \right) u \right|^2 dx \quad \text{(diamagnetic inequality)}$$

together with the following lemmas.

Lemma 2.1. *The completion of $C_C^\infty(\mathbb{R}^N \setminus \{0\})$ under the Dirichlet norm coincides with the space $D^{1,2}(\mathbb{R}^N)$.*

Lemma 2.2. *If $A \in L^\infty(\mathbb{S}^{N-1})$, then the norm*

$$\left(\int_{\mathbb{R}^N} \left| \left(i\,\nabla - \frac{A(\theta)}{|x|} \right) u \right|^2 \right)^{1/2}$$

is equivalent to the Dirichlet norm on $C_C^\infty(\mathbb{R}^N \setminus \{0\})$.

Lemma 2.3. *The quadratic form* (1) *is equivalent to*

$$Q_A(u) = \int_{\mathbb{R}^N} \left| \left(i\,\nabla - \frac{A(\theta)}{|x|} \right) u \right|^2$$

on its maximal domain $D^{1,2}(\mathbb{R}^N)$ provided $\|a\|_\infty < (N-2)^2/4$. Moreover, it is positive definite.

We refer the reader to [16] for a deeper analysis on such relations between the quadratic forms.

We set the following variational problem:

$$S_{A,a} := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left| \left(i\,\nabla - \frac{A(\theta)}{|x|} \right) u \right|^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2} u^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}}. \tag{3}$$

Of course, $S_{A,a}$ is strictly positive since the quadratic form (1) is positive definite.

We are now proposing a lemma which will be useful later.

Lemma 2.4. *Let $\{x_n\}$ be a sequence of points such that $|x_n| \to \infty$ as $n \to \infty$. Then for any $u \in D^{1,2}(\mathbb{R}^N)$, as $n \to \infty$, we have*

$$\frac{\int_{\mathbb{R}^N} \left| \left(i\,\nabla - \frac{A(\theta)}{|x|} \right) u(\cdot + x_n) \right|^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2} \, |u(\cdot + x_n)|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}} \longrightarrow \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}}.$$

Proof. It is sufficient to prove that for all $\varepsilon > 0$ there exists an $\overline{n}$ such that $\int_{\mathbb{R}^N} \frac{|u(x+x_n)|^2}{|x|^2} dx < 2\varepsilon$ for $n \geq \overline{n}$. Let us consider $R > 0$ big enough to have

$$\int_{\mathbb{R}^N \setminus B_R(x_n)} \frac{|u(x+x_n)|^2}{|x|^2} dx < \varepsilon$$

for every $n \in \mathbb{N}$. On the other hand, when $x \in B_R(x_n)$ we have $|x| \geq |x_n| - |x - x_n| \geq |x_n| - R$ which is a positive quantity for n big enough. In this way,

$$\int_{B_R(x_n)} \frac{|u(x+x_n)|^2}{|x|^2} \, dx \leq \frac{1}{(|x_n| - R)^2} \int_{B_R(x_n)} |u(x+x_n)|^2 \, dx < \varepsilon$$

for n big enough. □

Exploiting this lemma, we can state the following proposition that holds for $S_{A,a}$.

Proposition 2.5. *If S denotes the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^N)$ in $L^{2^*}(\mathbb{R}^N)$, i.e.,*

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}}, \tag{4}$$

then $S_{A,a} \leq S$.

Proof. Lemma 2.4 shows immediately that for all $u \in D^{1,2}(\mathbb{R}^N)$,

$$S_{A,a} \leq \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \right)^{2/2^*}} + o(1).$$

If we choose a minimizing sequence for (4) in the above inequality, we see immediately that $S_{A,a} \leq S$. □

3. Attaining the infimum

Given the results due to Brezis and Nirenberg [6], one could expect that if $S_{A,a}$ is strictly less than S, then it is attained. Here we pursue this idea with concentration-compactness arguments, in the special version due to Solimini [26]. Before proceeding, it is useful to recall some definitions about the so-called *Lorentz spaces*.

Definition 3.1 (see [26]). A *Lorentz space* $L^{p,q}(\mathbb{R}^N)$ is a space of measurable functions affected by two indexes p and q which are two positive real numbers, $1 \leq p, q \leq +\infty$, like the indexes which determine the usual L^p spaces. The index p is called *principal index* and the index q is called *secondary index*. A monotonicity property holds with respect to the secondary index: if $q_1 < q_2$, then $L^{p,q_1} \subset L^{p,q_2}$. So the strongest case of a Lorentz space with principal index p is $L^{p,1}$; while the weakest case is $L^{p,\infty}$, which is equivalent to the so-called *weak* L^p *space*, or Marcinkiewicz space. Anyway, the most familiar case of Lorentz space is the intermediate case given by $q = p$, since the space $L^{p,p}$ is equivalent to the classical L^p space.

Properties 3.2 (see [26]). A basic property about the Lorentz spaces is an appropriate case of the Hölder inequality, which states that the duality product of two functions is bounded by a constant times the product of the norms of the two functions in two respective conjugate Lorentz spaces L^{p_1,q_1} and L^{p_2,q_2}, where the two pairs of indexes satisfy the relations $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1$.

Moreover, if we consider the Sobolev space $H^{1,p}(\mathbb{R}^N)$, it is well known that it is embedded in the Lebesgue space $L^{p^*}(\mathbb{R}^N)$. But this embedding is not optimal: it holds that the space $H^{1,p}(\mathbb{R}^N)$ is embedded in the Lorentz space $L^{p^*,p}$, which is strictly stronger than $L^{p^*} = L^{p^*,p^*}$.

Theorem 3.3 (Solimini [26]). *Let $(u_n)_{n\in\mathbb{N}}$ be a given bounded sequence of functions in $H^{1,p}(\mathbb{R}^N)$, with the index p satisfying $1 < p < N$. Then, replacing $(u_n)_{n\in\mathbb{N}}$ with a suitable subsequence, we can find a sequence of functions $(\phi_i)_{i\in\mathbb{N}}$ belonging to $H^{1,p}(\mathbb{R}^N)$ and, for any index n, we can find a sequence of rescalings $(\rho_n^i)_{i\in\mathbb{N}}$ in such a way that the sequence $(\rho_n^i(\phi_i))_{i\in\mathbb{N}}$ is summable in $H^{1,p}(\mathbb{R}^N)$, uniformly with respect to n, and that the sequence $(u_n - \sum_{i\in\mathbb{N}} \rho_n^i(\phi_i))_{n\in\mathbb{N}}$ converges to zero in $L(p^*, q)$ for every index $q > p$.*

Moreover, we have that, for any pair of indexes i and j, the two corresponding sequences of rescalings $(\rho_n^i)_{n\in\mathbb{N}}$ and $(\rho_n^j)_{n\in\mathbb{N}}$ are mutually diverging, that is,

$$\sum_{i=1}^{+\infty} \|\phi_i\|_{1,p}^p \leq M, \tag{5}$$

where M is the limit of $(\|u_n\|_{1,p}^p)_{n\in\mathbb{N}}$, and the sequence $(u_n - \sum_{i\in\mathbb{N}} \rho_n^i(\phi_i))_{n\in\mathbb{N}}$ converges to zero in $H^{1,p}(\mathbb{R}^N)$ if and only if (5) is an equality.

Now we can state the following result.

Theorem 3.4. *If $S_{A,a} < S$, then $S_{A,a}$ is attained.*

Proof. Let us consider a minimizing sequence $u_n \in D^{1,2}(\mathbb{R}^N)$ to $S_{A,a}$. In particular, it is bounded in $D^{1,2}(\mathbb{R}^N)$. By Theorem 3.3, up to subsequences, there exist a sequence $\phi_i \in D^{1,2}(\mathbb{R}^N)$ and a sequence of mutually divergent rescalings ρ_n^i defined as

$$\rho_n^i(u) = \left(\lambda_n^i\right)^{\frac{N-2}{2}} u\left(\lambda_n^i x + y_n^i\right)$$

such that $\sum_i \rho_n^i \phi_i \in D^{1,2}(\mathbb{R}^N)$ and $u_n - \sum_i \rho_n^i \phi_i \to 0$ in L^{2^*}. In general, the rescalings may be mutually divergent by dilation (concentration or vanishing) or by translation. We divide the proof in two different cases.

(1) Suppose there exists at least an index $\bar{j}$ such that the sequence of the corresponding translation remains bounded: $\left|y_n^{\bar{j}}\right| \le$ const for all n. Then we consider $\widetilde{u}_n := (\rho_n^{\bar{j}})^{-1}(u_n)$, which is again a minimizing sequence. The following convergence can be stated:

$$\widetilde{u}_n - \phi_{\bar{j}} + \sum_{j \neq \bar{j}} \left(\rho_n^{\bar{j}}\right)^{-1} \rho_n^j \phi_j \longrightarrow 0 \quad \text{in } L^{2^*}.$$

If we call for a moment $v_n = \sum_{j \neq \bar{j}} (\rho_n^{\bar{j}})^{-1} \rho_n^j \phi_j$, we have that $v_n \to 0$ a.e. in $\mathbb{R}^N$ because of the mutual rescalings' divergence. Then $\widetilde{u}_n \to \phi_{\bar{j}}$ a.e. in $\mathbb{R}^N$. If we assume the sequence $\widetilde{u}_n$ is normalized in L^{2^*}, the famous Brezis–Lieb lemma applies and we immediately obtain the relation

$$\|\widetilde{u}_n\|_{2^*} = \left\|\phi_{\bar{j}}\right\|_{2^*} + \|v_n\|_{2^*} + o(1) \quad \text{as } n \to \infty.$$

At the same time even

$$\|\widetilde{u}_n\|_{D^{1,2}(\mathbb{R}^N)} = \left\|\phi_{\bar{j}}\right\|_{D^{1,2}(\mathbb{R}^N)} + \|v_n\|_{D^{1,2}(\mathbb{R}^N)} + o(1) \quad \text{as } n \to \infty.$$

So that

$$S_{A,a} \leftarrow \frac{\int_{\mathbb{R}^N} \left|\nabla_A \phi_{\bar{j}}\right|^2 + \int_{\mathbb{R}^N} |\nabla_A v_n|^2 + o(1)}{\left(\int_{\mathbb{R}^N} \left|\phi_{\bar{j}}\right|^{2^*} + \int_{\mathbb{R}^N} |v_n|^{2^*} + o(1)\right)^{2/2^*}}$$

$$\ge S_{A,a} \frac{\left(\int_{\mathbb{R}^N} \left|\phi_{\bar{j}}\right|^{2^*}\right)^{2/2^*} + \left(\int_{\mathbb{R}^N} |v_n|^{2^*}\right)^{2/2^*} + o(1)}{\left(\int_{\mathbb{R}^N} \left|\phi_{\bar{j}}\right|^{2^*} + \int_{\mathbb{R}^N} |v_n|^{2^*} + o(1)\right)^{2/2^*}},$$

and in order not to fall in contradiction, the previous coefficient must tend to zero, and then $\int_{\mathbb{R}^N} |v_n|^{2^*} \to 0$, and also $\|v_n\|_{D^{1,2}(\mathbb{R}^N)} \to 0$, which in particular implies that $\phi_{\bar{j}}$ is a nontrivial function. In conclusion, we have the strong $D^{1,2}(\mathbb{R}^N)$ convergence $u_n(\cdot) - \phi_{\bar{j}} \to 0$, since we have an equality in (5) in Theorem 3.3.

(2) On the other hand, if $|y_n^j| \to \infty$ for all j as $n \to \infty$, then we argue in the following way: let us fix $m \in \mathbb{N}$ and evaluate the quadratic form over the difference $u_n - \sum_{j=1}^m \rho_n^j \phi_j$. Since it is equivalent to the $D^{1,2}(\mathbb{R}^N)$-norm,

it will be greater than or equal to zero. So that

$$\begin{aligned}
0 &\le Q_{A,a}\left(u_n - \sum_{j=1}^{m} \rho_n^j \phi_j\right) \\
&= Q_{A,a}(u_n) + Q_{A,a}\left(\sum_{j=1}^{m} \rho_n^j \phi_j\right) - 2\sum_{j=1}^{m} \int_{\mathbb{R}^N} \nabla_A u_n \cdot \nabla_A \rho_n^j \phi_j \\
&\quad - 2\sum_{j=1}^{m} \int_{\mathbb{R}^N} \frac{a}{|x|^2} u_n \, \rho_n^j \phi_j \\
&= Q_{A,a}(u_n) + Q_{A,0}\left(\sum_{j=1}^{m} \rho_n^j \phi_j\right) - 2\sum_{j=1}^{m} \int_{\mathbb{R}^N} \nabla_A u_n \cdot \nabla_A \rho_n^j \phi_j + o(1) \\
&= Q_{A,a}(u_n) + Q_{A,0}\left(\sum_{j=1}^{m} \rho_n^j \phi_j\right) - 2Q_{A,0}\left(\sum_{j=1}^{m} \rho_n^j \phi_j\right) + o(1) \\
&= Q_{A,a}(u_n) - Q_{A,0}\left(\sum_{j=1}^{m} \rho_n^j \phi_j\right) + o(1)
\end{aligned}$$

for any m, thanks to the mutual divergence of the rescalings (see also [28]). Then we have

$$\frac{Q_{A,a}(u_n)}{\|u_n\|_{2^*}^{2/2^*}} \ge \frac{Q_{A,0}\left(\sum_{j=1}^{\infty} \rho_n^j \phi_j\right)}{\left\|\sum_{j=1}^{\infty} \rho_n^j \phi_j\right\|_{2^*}^{2/2^*}} = o(1) \ge S + o(1),$$

a contradiction. □

3.1. The case $a \le 0$

In order to investigate when the infimum is attained depending on the magnetic vector potential A and the electric potential a, we start from the simplest cases. The first of them is the case $a \le 0$.

Proposition 3.5. *If $a \le 0$ but not identically zero, then $S_{A,a}$ is not achieved.*

Proof. First of all, in this case we have $S_{A,a} = S$. Indeed, by diamagnetic inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \left|\left(i\nabla - \frac{A(\theta)}{|x|}\right) u\right|^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2} u^2 &\ge \int_{\mathbb{R}^N} |\nabla |u||^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2} u^2 \\
&\ge \int_{\mathbb{R}^N} |\nabla |u||^2
\end{aligned}$$

from which we have $S_{A,a} \ge S$.

Suppose by contradiction that S_A is achieved on a function ϕ. Following the previous argument by Solimini's theorem, according to the negativity of the electric potential, we get $S_{A,a} \ge S + c$, where c is a positive constant due to the convergence of the term

$$\frac{\int_{\mathbb{R}^N} \frac{a(\theta)}{|x|^2} \phi(\cdot + x_n)^2}{\left(\int_{\mathbb{R}^N} |\phi(\cdot + x_n)|^{2^*}\right)^{2/2^*}}.$$

So we get $S_{A,a} > S$, a contradiction.

Note that here we used the considerable fact that

$$\inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla |u||^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} = S.$$

Its proof is based on the idea that S is achieved over a radial function. □

3.2. The case $a = 0$

In this case we expect in general the infimum is not achieved. Indeed, first of all we have $S_{A,a} = S$, because we have already seen in general that $S_{A,a} \leq S$, and in this case the diamagnetic inequality gives the reverse inequality. There is a simple case in which we can immediately deduce a result.

Remark 3.6. If the vector potential $A/|x|$ is a gradient of a function $\Theta \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $\nabla\Theta \in L^{N,\infty}(\mathbb{R}^N)$, then $S_{A,a}$ is achieved.

Indeed, suppose $A/|x| = \nabla\Theta$ for a function $\Theta \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that its gradient has the regularity mentioned above. The change of gauge $u \mapsto \mathrm{e}^{+i\Theta} u$ makes problem (3) equivalent to (4), so that the infimum is necessarily achieved.

Just a few words about the regularity required for $\nabla\Theta$. In order to have the minimum problem well-posed, it would be sufficient that $\nabla\Theta \in L^2$. But if we require that the function $\mathrm{e}^{-i\Theta} u \in D^{1,2}(\mathbb{R}^N)$ for any $u \in D^{1,2}(\mathbb{R}^N)$, this regularity is not sufficient any more. Rather, everything works if $\nabla\Theta \in L^{N,\infty}(\mathbb{R}^N)$.

Now, suppose the infimum $S_{A,a} = S$ is achieved on a function $u \in D^{1,2}(\mathbb{R}^N)$. Then we have

$$S = \frac{\int_{\mathbb{R}^N} \left|\left(i\nabla - \frac{A(\theta)}{|x|}\right) u\right|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} \geq \frac{\int_{\mathbb{R}^N} |\nabla |u||^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} \geq S.$$

So it is clear that the equality must hold in the diamagnetic inequality in order not to fall into a contradiction. We have the following chain of relations:

$$\begin{aligned} |\nabla |u|| &= \left|\mathrm{Re}\left(\frac{\overline{u}}{|u|}\nabla u\right)\right| = \left|\mathrm{Im}\left(i\frac{\overline{u}}{|u|}\nabla u\right)\right| \\ &= \left|\mathrm{Im}\left(i\nabla u - \frac{A}{|x|}u\right)\frac{\overline{u}}{|u|}\right| \\ &\leq \left|\left(i\nabla u - \frac{A}{|x|}u\right)\frac{\overline{u}}{|u|}\right|. \end{aligned}$$

In order that the equality holds in the last line, $\mathrm{Re}\left\{\left(i\,\nabla u - \frac{A}{|x|}u\right)\overline{u}\right\}$ must vanish. Expanding the expression, one finds that the equivalent condition is $\frac{A}{|x|} = \mathrm{Re}\left(i\,\frac{\nabla u}{u}\right)$. We can rewrite $i\frac{\nabla u}{u} = i\frac{\nabla u}{|u|^2}\overline{u}$ and get

$$\mathrm{Re}\left(i\frac{\nabla u}{u}\right) = \frac{-\mathrm{Re}(u)\nabla\left(\mathrm{Im}(u)\right) + \mathrm{Im}(u)\nabla\left(\mathrm{Re}(u)\right)}{|u|^2}$$
$$= -\nabla\left(\arctan\frac{\mathrm{Im}(u)}{\mathrm{Re}(u)}\right)$$

which is equivalent to $-\frac{A}{|x|} = \nabla\Theta$, where Θ is the phase of u.

In conclusion, we can resume our first remark together with this argument to state the following result.

Proposition 3.7. *If the electric potential $a = 0$, the infimum $S_{A,a}$ is achieved if and only if $\frac{A}{|x|} = \nabla\Theta$. In this case, Θ is the phase of the minimizing function.*

3.3. The general case: Sufficient conditions

In Theorem 3.4 we proved that a sufficient condition for the achieved infimum is $S_{A,a} < S$. In this section we look for the hypotheses on A or a which guarantee this condition.

Proposition 3.8. *Suppose there exists a small ball $B_\delta(x_0)$ centered in $x_0 \in \mathbb{S}^{N-1}$ in which*

$$a(x) - |A(x)|^2 \geq \lambda > 0 \quad a.e.\ x \in B_\delta(x_0).$$

Then $S_{A,a} < S$ and so $S_{A,a}$ is achieved.

Proof. We define the closure of compact supported functions with respect to the norm associated with the quadratic form as

$$\mathcal{H}_A(\Omega) = \overline{C_C^\infty(\Omega)}^{\left(\int_\Omega |\nabla_A u|^2\right)^{1/2}}.$$

We have the following chain of relations:

$$S_{A,a} \leq \inf_{u\in\mathcal{H}_A(B_\delta(x_0))\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla_A u|^2 - \int_{\mathbb{R}^N} \frac{a}{|x|^2}|u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}}$$
$$\leq \inf_{u\in\mathcal{H}_A(B_\delta(x_0),\mathbb{R})\setminus\{0\}} \frac{\int_{B_\delta(x_0)} |\nabla_A u|^2 - \int_{B_\delta(x_0)} \frac{a}{|x|^2}u^2}{\left(\int_{B_\delta(x_0)} |u|^{2^*}\right)^{2/2^*}}$$

since the quotient is invariant under Solimini's rescalings and we are restricted to a proper subset of functions. When we check the quotient over a real function, it reduces to

$$\frac{\int_{B_\delta(x_0)} |\nabla u|^2 - \int_{B_\delta(x_0)} \frac{|A|^2 - a}{|x|^2}u^2}{\left(\int_{B_\delta(x_0)} |u|^{2^*}\right)^{2/2^*}},$$

so the thesis follows from [6, Lemma 1.1]. □

Remark 3.9. We can resume the results reached until now: in case the magnetic vector potential $\frac{A}{|x|}$ is a gradient, the infimum $S_{A,a}$ is achieved if $a \equiv 0$ or if its essential infimum is positive and sufficiently small in a neighborhood far from the origin (we mean $\|a\|_\infty \le (N-2)^2/4$ in order to keep the quadratic form positive definite); while it is never achieved if $a \le 0$, neither in case the magnetic potential is a gradient, nor in case it is not. On the other hand, in order to achieve $S_{A,a}$, if the magnetic vector potential is not a gradient, we need to assume that it has a suitably low essential supremum somewhere in a ball far from the origin in relation to the electric potential a (see Proposition 3.8).

Anyway, what is important here is not the essential supremum of $\frac{A}{|x|}$ (or A, since we play far from the origin), but "the distance" between the magnetic vector potential and the set of gradients. Pursuing this idea, it seems possible to interpret a suitable (to be specified) norm of curl $\frac{A}{|x|}$ as a measure of this distance. In order to specify these ideas we refer the reader to [21, 5]. We recall the following definition.

Definition 3.10 (see [21]**).** Let Ω be an open set of $\mathbb{R}^N$ and let $\vec{a}, \vec{b} \in L^1_{\rm loc}(\Omega)$. We say $\vec{a}$ and $\vec{b}$ are related by a gauge transformation, $\vec{a} \sim_\Omega \vec{b}$, if there is a distribution $\lambda \in D'(\Omega)$ satisfying $\vec{b} = \vec{a} + \nabla\lambda$.

By curl $\vec{a}$ we denote the skew-symmetric, matrix-valued distribution having $\partial_i \vec{a}_j - \partial_j \vec{a}_i \in D'(\Omega)$ as matrix elements.

Lemma 3.11 (see [21]**).** *Let Ω be any open subset of $\mathbb{R}^N$, $1 \le p < +\infty$ and $\vec{a}, \vec{b} \in L^p_{\rm loc}(\Omega)$. Then every λ satisfying $\vec{b} = \vec{a} + \nabla\lambda$ belongs to $W^{1,p}(\Omega)$. If Ω is simply connected, then*

$$\vec{a} \sim_\Omega \vec{b} \iff \operatorname{curl} \vec{a} = \operatorname{curl} \vec{b}.$$

Theorem 3.12 (see [5]**).** *Let $M = (0,1)^N$ be the N-dimensional cube of $\mathbb{R}^N$ with $N \ge 2$ and $1 \le l \le N-1$. Given any X that is an l-form with coefficients in $W^{1,N}(M)$, there exists some Y that is an l-form with coefficients in $W^{1,N} \cap L^\infty(M)$ such that*

$$dY = dX$$

and

$$\|\nabla Y\|_N + \|Y\|_\infty \le C\,\|dX\|_N\,.$$

Theorem 3.12 will be very useful in our case $l = 1$, so that the external derivative is the curl of the vector field which represents the given 1-form.

Suppose $\frac{A}{|x|} \in W^{1,N}(B_\delta(x_0))$ in a ball far from the origin. Then by Theorem 3.12 there exists a vector field $Y \in L^\infty \cap W^{1,N}(B_\delta(x_0))$ such that

$$\operatorname{curl}\frac{A}{|x|} = \operatorname{curl} Y \qquad \text{and} \qquad \|Y\|_\infty \le C\left\|\operatorname{curl}\frac{A}{|x|}\right\|_N.$$

By Lemma 3.11, Y is related to $A/|x|$ by a gauge transformation, so, in the spirit of Theorem 3.8, $\| \operatorname{curl} \frac{A}{|x|} \|_N$ being not too large is sufficient to achieve $S_{A,a} < S$ and hence $S_{A,a}$.

4. Symmetry of solutions

We recall once again in general that $S_{A,a} \leq S$. When $S_{A,a} = S$ and $Q_{A,a}(u) > Q(u)$ for any $u \in D^{1,2}(\mathbb{R}^N)$, e.g., when $a \leq 0$ but not identically zero, we lose compactness since clearly $S_{A,a}$ cannot be attained. In this section, we follow the idea that introducing symmetry properties to the quadratic form can help in growing the upper bound for $S_{A,a}$, in order to increase the probability for it to be achieved.

We basically follow the ideas in [27], assuming the dimension $N \geq 4$.

Let us write $\mathbb{R}^N = \mathbb{R}^2 \times \mathbb{R}^{N-2}$ and denote $x = (z, y)$. Let us fix $k \in \mathbb{N}$, and suppose that there is a $\mathbb{Z}_k \times SO(N-2)$ group action on $D^{1,2}(\mathbb{R}^N)$, denoting the fixed point space as

$$D_k^{1,2}(\mathbb{R}^N) = \Big\{ u(z,y) \in D^{1,2}(\mathbb{R}^N) \text{ s.t. } u\big(\mathrm{e}^{i\frac{2\pi}{k}} z, Ry\big) = \mathrm{e}^{i\frac{2\pi}{k}m} u(z,y) \\ \text{for any } R \in SO(N-2) \Big\}.$$

In order to have the quadratic form invariant under this action, let us suppose that $a(\theta) \equiv a \in \mathbb{R}^-$ and

$$A\left(\frac{\mathrm{e}^{i\frac{2\pi}{k}} z, Ry}{|(z,y)|} \right) = \left(\mathrm{e}^{i\frac{2\pi}{k}} (A_1, A_2), RA_3 \right) \left(\frac{z,y}{|(z,y)|} \right) \quad \text{for any } R \in SO(N-2). \tag{6}$$

These two conditions allow us to apply the *symmetric criticality principle*, so that the minima of the problem

$$S_{A,a}^{k,m} := \inf_{u \in D_k^{1,2}(\mathbb{R}^N)} \frac{Q_{A,a}(u)}{\|u\|_{2^*}^2} \tag{7}$$

are solutions to (2).

Theorem 4.1. *If $S_{A,a}^{k,m} < k^{2/N} S$, then $S_{A,a}^{k,m}$ is achieved.*

Proof. Let us consider a minimizing sequence $\{u_n\}$. The space $D_k^{1,2}(\mathbb{R}^N)$ is a closed subspace in $D^{1,2}(\mathbb{R}^N)$, so Theorem 3.3 holds in $D_k^{1,2}(\mathbb{R}^N)$. Up to subsequences, we can find a sequence $\Phi_i \in D_k^{1,2}(\mathbb{R}^N)$ and a sequence of mutually diverging rescalings ρ_n^i such that $u_n - \sum_i \rho_n^i(\Phi_i) \to 0$ in L^{2^*}.

We can basically follow the proof of Theorem 3.4.

We just stress that the possible function $\phi_{\bar{j}}$ will be in fact of the form $\sum_{l=1}^k \phi(\cdot + x^l)$, meaning that it will enjoy the same $\mathbb{Z}_k \times SO(N-2)$ group symmetry. ◻

Remark 4.2. The above result is actually a symmetry-breaking result for the equation associated with these minimum problems. Indeed, let us consider the equation

$$-\Delta_A u = \frac{a}{|x|^2} + |u|^{2^*-2} u \quad \text{in } \mathbb{R}^N, \tag{8}$$

where $-\Delta_A$ denotes the differential operator which we have called *magnetic Laplacian*. Then the minima of (3) are the solutions to (8) and so are those of (7), thanks to the *symmetric criticality principle* (the quotient is invariant under the $\mathbb{Z}_k \times SO(N-2)$ group action). Thus, when the electric potential is constant and negative, we find a multiplicity of solutions to (8) depending on k: each of them is invariant under rotations of angle $2\pi/k$, so that two solutions u_{k_1} and u_{k_2} are distinct as soon as k_1 and k_2 are not multiples to each other.

Now we want to check when the condition $S^{k,m}_{A,a} < k^{2/N} S$ is fulfilled. Let us pick k points in $\mathbb{R}^N \setminus \{0\}$ of the form $x_j = (R\,\mathrm{e}^{\frac{2\pi i}{k} j}\xi_0, 0)$ where $|\xi_0| = 1$, and denote

$$w_j = \mathrm{e}^{\frac{2\pi i}{k} jm} \frac{\big(N(N-2)\big)^{\frac{N-2}{4}}}{\big(1 + |x - x_j|^2\big)^{\frac{N-2}{2}}}. \tag{9}$$

In this way, the sum $\sum_{j=1}^k w_j$ is an element of $D^{1,2}_k(\mathbb{R}^N)$. Additionally, we notice that w_j are minimizers of the usual Sobolev quotient, and they satisfy

$$-\Delta w_j = |w_j|^{2^*-2} w_j \quad \text{in } \mathbb{R}^N. \tag{10}$$

It is worth noticing that both

$$\frac{\int_{\mathbb{R}^N} |\nabla w_j|^2}{\big(\int_{\mathbb{R}^N} |w_j|^{2^*}\big)^{2/2^*}} = S$$

and (10) imply

$$\int_{\mathbb{R}^N} |\nabla w_j|^2 = \int_{\mathbb{R}^N} |w_j|^{2^*} = S^{N/2}. \tag{11}$$

We state the following proposition.

Proposition 4.3. *Choosing R and k large enough, the quotient evaluated over $\sum_{j=1}^k w_j$ is strictly less than $k^{2/N} S$, and so is the infimum $S^{k,m}_{A,a}$.*

In order to prove this proposition, we need some technical results whose proofs are postponed to the next subsection. We basically follow the ideas in [27].

For the sake of simplicity, we introduce the following notation:

$$\alpha = \int_{\mathbb{R}^N} \mathrm{Re}\left\{ \sum_{i\neq j} |w_i|^{2^*-2} w_i \overline{w_j} \right\},$$

$$\beta = \int_{\mathbb{R}^N} \frac{|A|^2 - a}{|x|^2} \left| \sum_{j=1}^{k} w_j \right|^2,$$

$$\gamma = \mathrm{Re}\left\{ i \int_{\mathbb{R}^N} \frac{A}{|x|} \cdot \sum_{i,j} \nabla w_i \overline{w_j} \right\}.$$

Lemma 4.4. *It holds that* $\alpha \geq 0$.

Lemma 4.5. *For every positive* δ *there exists a positive constant* K_δ *(independent of* k*) such that if*

$$\frac{|x_i - x_j|^2}{\log |x_i - x_j|} \geq K_\delta \, (k-1)^{2/(N-2)} \quad \forall i \neq j,$$

then

$$\int_{\mathbb{R}^N} \left| \sum_{j=1}^{k} w_j \right|^{2^*} \geq k\, S^{N/2} + 2^*(1-\delta) \int_{\mathbb{R}^N} \mathrm{Re}\left\{ \sum_{i\neq j} |w_i|^{2^*-2} w_i \overline{w_j} \right\}. \tag{12}$$

Lemma 4.6. *Given Lemma* 4.5*, it is possible to choose* R *and* k *in such a way that the quantity*

$$1 + \frac{1}{kS^{N/2}} \left\{ \beta - 2\gamma + \alpha\left(-1 + \delta + \frac{2-\delta}{kS^{N/2}}(2\gamma - \beta) \right) \right\}$$

is positive and strictly less than 1.

Proof of Proposition 4.3. Let us evaluate the quotient over $\sum_{j=1}^{k} w_j$:

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left| \nabla_A \left(\sum_{j=1}^{k} w_j \right) \right|^2 - \int_{\mathbb{R}^N} \frac{a}{|x|^2} \left| \sum_{j=1}^{k} w_j \right|^2 \\
&= \int_{\mathbb{R}^N} \left\{ \sum_{j=1}^{k} |\nabla w_j|^2 + \mathrm{Re}\left\{ \sum_{i\neq j} \nabla w_i \cdot \nabla \overline{w_j} \right\} + \frac{|A|^2 - a}{|x|^2} \left| \sum_{j=1}^{k} w_j \right|^2 \right. \\
&\quad \left. - 2\,\mathrm{Re}\left\{ i \frac{A}{|x|} \cdot \sum_{i,j} \nabla w_i \overline{w_j} \right\} \right\} \qquad (13) \\
&= kS^{N/2} + \int_{\mathbb{R}^N} \left\{ \mathrm{Re}\left\{ \sum_{i\neq j} |w_i|^{2^*-2} w_i \overline{w_j} \right\} + \frac{|A|^2 - a}{|x|^2} \left| \sum_{j=1}^{k} w_j \right|^2 \right. \\
&\quad \left. - 2\,\mathrm{Re}\left\{ i \frac{A}{|x|} \cdot \sum_{i,j} \nabla w_i \overline{w_j} \right\} \right\},
\end{aligned}$$

where in the last equality we have used (11) and (10). Now we use Lemma 4.5 which states the lower bound (12) for the denominator of our quotient. Thus using (13) and (12), the quotient is

$$\begin{aligned}
&\frac{Q_{A,a}\big(\sum_{j=1}^k w_j\big)}{\Big\|\big(\sum_{j=1}^k w_j\big)\Big\|_{2^*}^2} \\
&\leq \left(kS^{N/2} + \int_{\mathbb{R}^N} \left\{ \operatorname{Re}\left\{ \sum_{i\neq j} |w_i|^{2^*-2} w_i\, \overline{w_j} \right\} + \frac{|A|^2 - a}{|x|^2} \left| \sum_{j=1}^k w_j \right|^2 \right.\right. \\
&\qquad\qquad \left.\left. -2\operatorname{Re}\left\{ i\frac{A}{|x|} \cdot \sum_{i,j} \nabla w_i \overline{w_j} \right\}\right\}\right) \\
&\quad \cdot \left(kS^{N/2} + 2^*\left(1 - \frac{\delta}{2}\right) \int_{\mathbb{R}^N} \operatorname{Re}\left\{ \sum_{i\neq j} |w_i|^{2^*-2} w_i\, \overline{w_j} \right\}\right)^{-2/2^*} \\
&= k^{2/N} S\left(1 + \frac{1}{kS^{N/2}}(\alpha + \beta - 2\gamma)\right)\left(1 - \frac{2(1-\delta/2)}{kS^{N/2}}\alpha\right) + o(1),
\end{aligned}$$

where in the last line we have expanded the denominator in Taylor's series since the argument is very close to zero if R is large. Up to infinitesimal terms of higher order, the coefficient of $k^{2/N}S$ is

$$1 + \frac{1}{kS^{N/2}}\beta - \frac{2}{kS^{N/2}}\gamma + \frac{1}{kS^{N/2}}\alpha\left(-1 + \delta + \frac{2-\delta}{kS^{N/2}}(2\gamma - \beta)\right).$$

Now we invoke Lemma 4.6 to conclude the proof. □

4.1. Proofs of technical lemmas

In order to prove Lemmas 4.4, 4.5 and 4.6, we need supplementary results mainly about asymptotics of the quantities involved.

Lemma 4.7. *We have, as* $|x_i - x_j| \to +\infty$ *and* $|x_i| \to +\infty$,

$$\int_{\mathbb{R}^N} |w_i|^{2^*-2} w_i\, \overline{w_j} = O\left(\frac{1}{|x_i - x_j|^{N-2}}\right), \tag{14}$$

$$\int_{\mathbb{R}^N} |w_i\, \overline{w_j}|^{2^*/2} = O\left(\frac{\log|x_i - x_j|}{|x_i - x_j|^N}\right), \tag{15}$$

$$\int_{\mathbb{R}^N} \frac{|w_j|^2}{|x|^2} = \begin{cases} O\left(\dfrac{\log R}{R^2}\right) & \text{if } N = 4, \\ O\left(\dfrac{1}{R^2}\right) & \text{if } N \geq 5, \end{cases} \tag{16}$$

$$\int_{\mathbb{R}^N} \frac{1}{|x|} \cdot |\nabla w_j|\, |w_i| = O\left(\frac{1}{R\, |x_i - x_j|^{N-3}}\right). \tag{17}$$

Proof. For what concerns (14), (15) and (16) we refer the reader to [27].

About (17) we have

$$\begin{aligned}\int_{B_{R/2}(0)} \frac{1}{|x|}\,|\nabla w_j|\;|w_i| &= O\left(\frac{1}{R^{N-1}}\right)O\left(\frac{1}{R^{N-2}}\right)O\left(R^{N-1}\right)\\ &= O\left(\frac{1}{R^{N-2}}\right)\end{aligned}$$

since in $B_{R/2}(0)$ we have $|x-x_i|\geq R-|x|\geq R/2$ and the same holds for $|x-x_j|$;

$$\begin{aligned}\int_{B_{|x_i-x_j|/4}(x_i)} \frac{1}{|x|}\,|\nabla w_j|\;|w_i| &= O\left(\frac{1}{R}\right)O\left(\frac{1}{|x_i-x_j|^{N-1}}\right)\int_{B_{|x_i-x_j|/4}(x_i)}|w_i|\\ &= O\left(\frac{1}{R\,|x_i-x_j|^{N-3}}\right)\end{aligned}$$

since in $B_{|x_i-x_j|/4}(x_i)$ we have $|x|\geq|x_i|-|x-x_i|\geq R/2$ and $|x-x_j|\geq |x_i-x_j|-|x-x_i|\geq\frac{3}{4}\,|x_i-x_j|$;

$$\begin{aligned}&\int_{B_{|x_i-x_j|/4}(x_j)} \frac{1}{|x|}\,|\nabla w_j|\;|w_i|\\ &\qquad= O\left(\frac{1}{R}\right)O\left(\frac{1}{|x_i-x_j|^{N-2}}\right)\int_{B_{|x_i-x_j|/4}(x_j)}|\nabla w_j|\\ &\qquad= O\left(\frac{1}{R\,|x_i-x_j|^{N-3}}\right)\end{aligned}$$

since in $B_{|x_i-x_j|/4}(x_j)$ we have $|x|\geq|x_j|-|x-x_j|\geq R/2$ and $|x-x_i|\geq |x_i-x_j|-|x-x_j|\geq\frac{3}{4}\,|x_i-x_j|$; while in $\mathbb{R}^N\setminus(B_{R/2}(0)\cup B_{|x_i-x_j|/4}(x_i)\cup B_{|x_i-x_j|/4}(x_j))$ we have $|x|\geq R/2$, and via Hölder inequality,

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus(B_{R/2}(0)\cup B_{|x_i-x_j|/4}(x_i)\cup B_{|x_i-x_j|/4}(x_j))}\frac{1}{|x|}\,|\nabla w_j|\;|w_i|\\ &\qquad=\begin{cases}O\Big(\dfrac{1}{R\,|x_i-x_j|^2}\log|x_i-x_j|\Big) & \text{if } N=4,\\ O\Big(\dfrac{1}{R\,|x_i-x_j|^{2N-6}}\Big) & \text{if } N\geq 5.\end{cases}\end{aligned}$$

□

Remark 4.8. The above asymptotics in Lemma 4.7 are in terms of k and R as we note

$$\begin{aligned}|x_i-x_j|^2 &= R^2\sin^2\frac{2\pi}{k}(i-j)+R^2\Big(1-\cos\frac{2\pi}{k}(i-j)\Big)^2\\ &\sim\begin{cases}\dfrac{R^2}{k^2}+\dfrac{R^2}{k^4}=O\Big(\dfrac{R^2}{k^2}\Big) & \text{if } |i-j|\ll k,\\ R^2 & \text{otherwise.}\end{cases}\end{aligned}$$

According to the previous asymptotic, we note that we have the worst estimates in Lemma 4.7 for $|i-j| \ll k$, that is for the centers x_i, x_j quite near to each other.

Lemma 4.9. *The following asymptotic behavior holds for $k \to +\infty$ and $R \to +\infty$:*

$$\left| \int_{\mathbb{R}^N} \mathrm{Re}\left\{ i\,\frac{A}{|x|} \cdot \sum_{l,j} \nabla w_l\, \overline{w_j} \right\} \right| \le \begin{cases} O\left(\dfrac{k^2}{R^2}\right) & \text{if } N = 4, \\ O\left(\dfrac{k^2 \log k}{R^3}\right) & \text{if } N = 5, \\ O\left(\dfrac{k^{N-3}}{R^{N-2}}\right) & \text{if } N \ge 6. \end{cases}$$

Proof. First of all, we note that if $l = j$, the quantity in the statement is zero. Next,

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \mathrm{Re}\left\{ i\,\frac{A}{|x|} \cdot \sum_{l,j} \nabla w_l\, \overline{w_j} \right\} \right| \\ &\quad = \left| \sum_{l \ne j} \sin \frac{2\pi}{k} m(l-j) \int_{\mathbb{R}^N} \frac{A}{|x|} \cdot \nabla\, |w_l|\, |w_j| \right| \\ &\quad \le \frac{C}{R^{N-2}} \sum_{l \ne j} \frac{\left|\sin \frac{2\pi}{k} m(l-j)\right|}{\left(1 - \cos \frac{2\pi}{k}(l-j)\right)^{\frac{N-3}{2}}} \\ &\quad = \frac{C\,k}{R^{N-2}} \sum_{l=1}^{k-1} \frac{\left|\sin \frac{2\pi}{k} ml\right|}{\left(1 - \cos \frac{2\pi}{k} l\right)^{\frac{N-3}{2}}} \\ &\quad \le \frac{C\,m\,k}{R^{N-2}} \sum_{l=1}^{k-1} \frac{l/k}{(l/k)^{N-3}} \\ &\quad = \frac{C\,m\,k^{N-3}}{R^{N-2}} \begin{cases} k & \text{if } N = 4, \\ \log k & \text{if } N = 5, \\ O(1) & \text{if } N \ge 6. \end{cases} \end{aligned}$$

□

We recall the following result proved in [27].

Lemma 4.10. *Let $s_1, \dots, s_k \ge 0$. For every positive δ there exists a positive constant K_δ (independent of k) such that if*

$$\frac{|x_i - x_j|^2}{\log |x_i - x_j|} \ge K_\delta\, (k-1)^{2/(N-2)} \quad \forall i \ne j,$$

then

$$\int_{\mathbb{R}^N} \left(\sum_{i=1}^k s_i \right)^{2^*} \ge k\, S^{N/2} + 2^* \left(1 - \frac{\delta}{2} \right) \int_{\mathbb{R}^N} \sum_{i \ne j} s_i^{2^*-1} s_j. \tag{18}$$

Proof of Lemma 4.4. We split the sum in the definition of α in two contributions: indexes for which $\cos\frac{2\pi}{k}(j-l)\geq 0$ (we call them *j,l pos*), and indexes for which $\cos\frac{2\pi}{k}(j-l)\leq 0$ (we call them *j,l neg*). In the first case, we have $|x_j-x_l|\sim\frac{R}{k}$, whereas in the second case we have $|x_j-x_l|\sim R$. Then

$$\begin{aligned}\int_{\mathbb{R}^N}\mathrm{Re}\left\{\sum_{l,j\ pos}|w_j|^{2^*-2}w_j\,\overline{w_l}\right\}&\geq\int_{\mathbb{R}^N}\mathrm{Re}\left\{\sum_j|w_j|^{2^*-2}w_j\,\overline{w_{j+1}}\right\}\\&=k\int_{\mathbb{R}^N}\mathrm{Re}\left\{|w_2|^{2^*-2}w_2\,\overline{w_1}\right\}\\&=O\left(\frac{k^{N-1}}{R^{N-2}}\right).\end{aligned}\tag{19}$$

On the other hand,

$$\begin{aligned}&\int_{\mathbb{R}^N}\mathrm{Re}\left\{\sum_{l,j\ neg}|w_j|^{2^*-2}w_j\overline{w_l}\right\}\\&\quad\leq k^2\int_{\mathbb{R}^N}\mathrm{Re}\left\{|w_l|^{2^*-2}w_l\,\overline{w_1}\right\}\\&\quad=O\left(\frac{k^2}{R^{N-2}}\right);\end{aligned}$$

so that for k large enough we have the thesis. □

Proof of Lemma 4.5. By convexity of the function $(\cdot)^{2^*/2}$ we have

$$\begin{aligned}\left|\sum_{j=1}^k w_j\right|^{2^*}&=\left(\left|\sum_{j=1}^k w_j\right|^2\right)^{2^*/2}=\left(\sum_{i,j=1}^k\mathrm{Re}\{w_i\,\overline{w_j}\}\right)^{2^*/2}\\&=\left(\sum_{i,j}|w_i|\,|w_j|-\sum_{i,j}|w_i|\,|w_j|\left(1-\cos\left(\frac{2\pi}{k}m(i-j)\right)\right)\right)^{2^*/2}\\&\geq\left(\sum_{i,j}|w_i|\,|w_j|\right)^{2^*/2}-\frac{2^*}{2}\left(\sum_{i,j}|w_i|\,|w_j|\right)^{2^*/2-1}\\&\quad\cdot\sum_{i,j}|w_i|\,|w_j|\left(1-\cos\left(\frac{2\pi}{k}m(i-j)\right)\right).\end{aligned}\tag{20}$$

For the first term,

$$\left(\sum_{i,j}|w_i|\,|w_j|\right)^{2^*/2}=\left(\sum_j^k|w_j|\right)^{2^*},$$

we can apply directly inequality (18) in order to have

$$\int_{\mathbb{R}^N}\left(\sum_{j=1}^k|w_j|\right)^{2^*}\geq k\,S^{N/2}+2^*\left(1-\frac{\delta}{2}\right)\int_{\mathbb{R}^N}\sum_{i\neq j}|w_i|^{2^*-1}|w_j|.\tag{21}$$

We want to stress that

$$\int_{\mathbb{R}^N} \sum_{i\neq j} |w_i|^{2^*-1} |w_j| \geq \int_{\mathbb{R}^N} \sum_{j=1}^{k} |w_j|^{2^*-1} |w_{j+1}| = k \int_{\mathbb{R}^N} |w_1|^{2^*-1} |w_2|$$

(see also [27, equation (6.22)]), so that

$$\int_{\mathbb{R}^N} \sum_{i\neq j} |w_i|^{2^*-1} |w_j| \geq O\left(\frac{k^{N-1}}{R^{N-2}}\right). \tag{22}$$

Now we focus our attention on the integral of the last addendum in (20): via Hölder inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\sum_{i,j} |w_i|\,|w_j|\right)^{2^*/2-1} \sum_{i,j} |w_i|\,|w_j| \left(1-\cos\left(\frac{2\pi}{k} m(i-j)\right)\right) \\
&\quad \leq \left(\int_{\mathbb{R}^N} \left(\sum_{i,j} |w_i|\,|w_j|\right)^{2^*/2}\right)^{\frac{2^*-2}{2^*}} \\
&\qquad \cdot \left(\int_{\mathbb{R}^N} \left(\sum_{i,j} |w_i|\,|w_j| \left(1-\cos\left(\frac{2\pi}{k} m(i-j)\right)\right)\right)^{2^*/2}\right)^{2/2^*}
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} \left(\sum_{i,j} |w_i|\,|w_j|\right)^{2^*/2}\right)^{\frac{2^*-2}{2^*}} \\
&\qquad = \left(\int_{\mathbb{R}^N} \left(\sum_{j} |w_j|\right)^{2^*}\right)^{\frac{2^*-2}{2^*}} \sim (k\,S^{N/2})^{\frac{2^*-2}{2^*}},
\end{aligned}$$

thanks to inequality (18) and Lemma 4.7. On the other hand,

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} \left(\sum_{i,j} |w_i|\,|w_j| \left(1-\cos\left(\frac{2\pi}{k} m(i-j)\right)\right)\right)^{2^*/2}\right)^{2/2^*} \\
&\quad \leq \sum_{i,j} \left(\int_{\mathbb{R}^N} \left(|w_i|\,|w_j| \left(1-\cos\left(\frac{2\pi}{k} m(i-j)\right)\right)\right)^{2^*/2}\right)^{2/2^*} \\
&\quad = \sum_{i,j} \left(1-\cos\left(\frac{2\pi}{k} m(i-j)\right)\right) \frac{(\log|x_i-x_j|)^{\frac{N-2}{N}}}{|x_i-x_j|^{N-2}}
\end{aligned}$$

according to (15). Now, since $|x_i - x_j| \sim R\big(1 - \cos\big(\frac{2\pi}{k}(i-j)\big)\big)^{1/2}$, the sum

$$
\begin{aligned}
&\sum_{i,j}\Big(1-\cos\Big(\frac{2\pi}{k}m(i-j)\Big)\Big)\frac{\Big(\log\Big(R\big(1-\cos\big(\frac{2\pi}{k}(i-j)\big)\big)^{1/2}\Big)\Big)^{\frac{N-2}{N}}}{R^{N-2}\Big(1-\cos\big(\frac{2\pi}{k}(i-j)\big)\Big)^{\frac{N-2}{2}}}\\
&\quad\le C(m)\,k\sum_j\frac{\Big(\log\Big(R\big(1-\cos\big(\frac{2\pi}{k}j\big)\big)^{1/2}\Big)\Big)^{\frac{N-2}{N}}}{R^{N-2}\Big(1-\cos\big(\frac{2\pi}{k}j\big)\Big)^{N/2-2}}\\
&\quad\sim 2\,C(m)\,k^2\int_0^{1/2}\frac{\Big(\log\Big(R\big(1-\cos(2\pi x)\big)^{1/2}\Big)\Big)^{\frac{N-2}{N}}}{R^{N-2}\big(1-\cos(2\pi x)\big)^{N/2-2}}\,dx\\
&\quad\le\frac{C(m)\,k^2}{R^{N-2}}\begin{cases}O(\log R) & \text{if } N=4,\\ O(\log R\log k) & \text{if } N=5,\\ O\big((\log R\log k)^{\frac{N-2}{N}}k^{N-5}\big) & \text{if } N\ge 6\end{cases}
\end{aligned}
$$

so that the last addendum in (20) is

$$
\begin{aligned}
(20)&\le C(m)\,k^{2/N}\frac{k^2}{R^{N-2}}\begin{cases}O(\log R) & \text{if } N=4,\\ O(\log R\log k) & \text{if } N=5,\\ O\big((\log R\log k)^{\frac{N-2}{N}}k^{N-5}\big) & \text{if } N\ge 6\end{cases}\\
&=\begin{cases}O\Big(\dfrac{k^{5/2}\log R}{R^2}\Big) & \text{if } N=4,\\ O\Big(\dfrac{k^{12/5}\log R\log k}{R^3}\Big) & \text{if } N=5,\\ O\Big(\dfrac{k^{N-3+2/N}(\log R\log k)^{\frac{N-2}{N}}}{R^{N-2}}\Big) & \text{if } N\ge 6,\end{cases}
\end{aligned}
\tag{23}
$$

which can be made $o\big(\frac{k^{N-1}}{R^{N-2}}\big)$ in every dimension for a suitable choice of the parameters R and k (e.g., $k \sim R^\alpha$ with $0 < \alpha < 1$ since according to the hypothesis of the lemma itself we need $k = o(R)$).

Provided the ratio R/k is big enough, from (21), (22) and (23), we get

$$
\int_{\mathbb{R}^N}\Big|\sum_{j=1}^k w_j\Big|^{2^*} \ge k\,S^{N/2} + 2^*(1-\delta)\int_{\mathbb{R}^N}\sum_{i\ne j}|w_i|^{2^*-1}|w_j|,
$$

which in particular implies the thesis. □

Proof of Lemma 4.6. In order to have this quantity (positive) and less than 1, it is sufficient to have

(a) α/k, γ/k and β/k small,
(b) α arbitrarily greater than β,
(c) α arbitrarily greater than γ.

According to Lemmas 4.7 and 4.9 and Remark 4.8, we know that

$$\beta = \begin{cases} O\left(k^2 \dfrac{\log R}{R^2}\right) & \text{if } N = 4, \\ O\left(\dfrac{k^2}{R^2}\right) & \text{if } N \geq 5; \end{cases}$$

$$\gamma = \begin{cases} O\left(\dfrac{k^2}{R^2}\right) & \text{if } N = 4, \\ O\left(\dfrac{k^2}{R^3} \log k\right) & \text{if } N = 5, \\ O\left(\dfrac{k^{N-3}}{R^{N-2}}\right) & \text{if } N \geq 6, \end{cases}$$

$$\alpha = O\left(k^2 \frac{k^{N-2}}{R^{N-2}}\right).$$

Let us fix the condition

$$k^{(N-1)/(N-2)} = o(R) \tag{24}$$

in order to have α/k small. Consequently, we immediately find the request (a) fulfilled. Moreover, we note that this does not contradict either the hypothesis of Lemma 4.5 (rather, that is a consequence), or the conditions on equation (23).

For what concerns requests (b) and (c), we recall that equation (19) states the lower bound $\alpha \gg k^{N-1}/R^{N-2}$.

Thus, we find request (c) satisfied as soon as $k \to \infty$.

About request (b), everything works without any additional hypothesis in dimension 4. In dimension $N \geq 5$, we need $R = o(k^{(N-3)/(N-4)})$: we emphasize that this does not contradict (24), thanks to the order $\frac{N-1}{N-2} < \frac{N-3}{N-4}$. □

As a natural question, letting $k \to \infty$, we wonder if there exists any biradial solution: we mean a function belonging to the space

$$D^{1,2}_{r_1,r_2}(\mathbb{R}^N) = \Big\{u \in D^{1,2}(\mathbb{R}^N) \text{ s.t. } u(R(x_1,x_2), Sx_3) = R^m u((x_1,x_2),x_3) \\ \forall R \in SO(2)\,, \forall S \in SO(N-2)\Big\}.$$

As already pointed out, even in this case we need the magnetic potential A to be equivariant with respect to the action of the group $SO(2) \times SO(N-2)$, that is,

$$A\left(\frac{Rz, Sy}{|(z,y)|}\right) = (R(A_1, A_2), SA_3)\left(\frac{z,y}{|(z,y)|}\right)$$
$$\text{for any } (R,S) \in SO(2) \times SO(N-2),$$

in order that the *symmetric criticality principle* applies. In order to investigate the possible existence of a biradial solution, we set the problem

$$S_{A,a}^{\text{birad},m} = \inf_{u \in D^{1,2}_{r_1,r_2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \left|\left(i\nabla - \frac{A}{|x|^2}\right)u\right|^2 - \int_{\mathbb{R}^N} \frac{a}{|x|^2}|u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}},$$

and we are able to prove the following proposition.

Proposition 4.11. *There exists a biradial solution.*

Proof. As we usually do, we consider a minimizing sequence u_n to $S_{A,a}^{\text{birad},m}$ and Theorem 3.3 in $D^{1,2}_{r_1,r_2}(\mathbb{R}^N)$, since this is a closed subspace of $D^{1,2}(\mathbb{R}^N)$. As usual, we redirect ourselves to $u_n - \Phi(\cdot + x_n) \to 0$ in $D^{1,2}_{r_1,r_2}(\mathbb{R}^N)$ with $\Phi \neq 0$ and suppose by contradiction that $|x_n| \to +\infty$.

To preserve the symmetry, in Solimini's decomposition we will find all the functions obtained by Φ with a rotation of a $2\pi/k$ angle, for $k \in \mathbb{Z}$ fixed. Thus, we can write $u_n - \sum_{i=1}^k \Phi(\cdot + x_n^i) \to 0$ in $D^{1,2}_{r_1,r_2}(\mathbb{R}^N)$. Now, following the same calculations in Theorem 4.1, we obtain

$$S_{A,a}^{\text{birad},m} \geq S_{A,a}^{k,m} \geq k^{2/N} S$$

that leads to $S_{A,a}^{\text{birad},m} = +\infty$ choosing k arbitrary large: a contradiction. □

5. Aharonov–Bohm-type potentials

In dimension 2, an Aharonov–Bohm magnetic field is a δ-type magnetic field. A vector potential associated with the Aharonov–Bohm magnetic field in $\mathbb{R}^2$ has the form

$$\mathcal{A}(x_1, x_2) = \left(\frac{-\alpha x_2}{|x|^2}, \frac{\alpha x_1}{|x|^2}\right), \quad (x_1, x_2) \in \mathbb{R}^2,$$

where α is the field flux through the origin. In this contest we want to take account of Aharonov–Bohm-type potentials in $\mathbb{R}^N$, for $N \geq 4$:

$$\mathcal{A}(x_1, x_2, x_3) = \left(\frac{-\alpha x_2}{x_1^2 + x_2^2}, \frac{\alpha x_1}{x_1^2 + x_2^2}, 0\right), \quad (x_1, x_2) \in \mathbb{R}^2,\ x_3 \in \mathbb{R}^{N-2},$$

paying special attention now that the singular set is a whole subspace of $\mathbb{R}^N$ with codimension 2.

5.1. Hardy-type inequality

In order to study minimum problems and therefore the quadratic form associated with this kind of potentials, we need a Hardy-type inequality. We know by [20] that a certain Hardy-type inequality holds for Aharonov–Bohm vector potentials in $\mathbb{R}^2$; that is,

$$\int_{\mathbb{R}^2} \frac{|\varphi|^2}{|x|^2} \leq C \int_{\mathbb{R}^2} |(i\nabla - \mathcal{A})\varphi|^2 \quad \forall \varphi \in C_C^\infty(\mathbb{R}^2 \setminus \{0\}),$$

where the best constant C is

$$H = \Big(\min_{k\in\mathbb{Z}} |k - \Phi_{\mathcal{A}}| \Big)^2. \tag{25}$$

Here $\Phi_{\mathcal{A}}$ denotes the field flux around the origin:

$$\Phi_{\mathcal{A}} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}(\cos t, \sin t) \cdot (-\sin t, \cos t)\, dt.$$

One can generalize this result and gain a similar inequality to the Aharonov–Bohm potentials in $\mathbb{R}^N$, simply separating the integrals: for all $\varphi \in C_C^\infty(\mathbb{R}^N \setminus \{x_1 = x_2 = 0\})$ one has

$$\int_{\mathbb{R}^N} \frac{|\varphi|^2}{x_1^2 + x_2^2} = \int_{\mathbb{R}^{N-2}} \int_{\mathbb{R}^2} \frac{|\varphi|^2}{x_1^2 + x_2^2}\, dx_1\, dx_2\, dx_3 \leq H \int_{\mathbb{R}^N} |(i\nabla - \mathcal{A})\varphi|^2, \tag{26}$$

where H is defined in (25). Now a natural question arises: is H the best constant for inequality (26)? In other words, is H the infimum of the Rayleigh quotient?

Proposition 5.1. *The best constant for inequality* (26) *is exactly* (25).

Proof. We consider the approximating sequence u_n to (25) in $\mathbb{R}^2$. We can choose this sequence bounded in $L^2(\mathbb{R}^2)$-norm, thanks to the homogeneity of the quotient under dilation.

We claim that there exists a sequence of real-valued functions $(\eta_n)_n \subset C_C^\infty(\mathbb{R}^{N-2})$ such that $\int_{\mathbb{R}^{N-2}} |\nabla \eta_n|^2 \to 0$ and $\int_{\mathbb{R}^{N-2}} \eta_n^2 \to +\infty$ as $n \to +\infty$. We can namely consider a real radial function such that $\eta_n \equiv 1$ in $B_R(0)$ and $\eta_n \equiv 0$ in $\mathbb{R}^{N-2} \setminus B_{R+n^\alpha}(0)$, with $|\nabla \eta_n| \sim 1/n^\alpha$, for a suitable $\alpha > 0$ (e.g., $\alpha > (N-2)/2$).

Now we consider the sequence $v_n(x_1, x_2, x_3) = u_n(x_1, x_2)\eta_n(x_3)$, where x_3 as usual denotes the whole set of variables in $\mathbb{R}^{N-2}$, and we test the quotient over this sequence:

$$\frac{\int_{\mathbb{R}^N} |(i\nabla - \mathcal{A})v_n|}{\int_{\mathbb{R}^N} \frac{|v_n|^2}{x_1^2+x_2^2}} = \frac{\int_{\mathbb{R}^N} |\nabla v_n|^2 - 2\,\mathrm{Re} \int_{\mathbb{R}^N} \mathcal{A} v_n \cdot \nabla \overline{v}_n + \int_{\mathbb{R}^N} |\mathcal{A}|^2 |v_n|^2}{\int_{\mathbb{R}^{N-2}} |\eta_n|^2 \int_{\mathbb{R}^2} \frac{|u_n|^2}{x_1^2+x_2^2}},$$

where the numerator is equal to

$$\int_{\mathbb{R}^N} \eta_n^2 |\nabla u_n|^2 + \int_{\mathbb{R}^N} |\mathcal{A}|^2 |\eta_n|^2 |u_n|^2 - 2\,\mathrm{Re}\int_{\mathbb{R}^N} \eta_n^2 u_n \mathcal{A}\cdot\nabla\overline{u}_n$$
$$+ 2\,\mathrm{Re}\int_{\mathbb{R}^N} u_n\eta_n\nabla\eta_n\cdot\nabla\,\overline{u}_n + \int_{\mathbb{R}^N} u_n^2 |\nabla\eta_n|^2 - 2\,\mathrm{Re}\int_{\mathbb{R}^N} |u_n|^2 \eta_n\mathcal{A}\cdot\nabla\eta_n. \tag{27}$$

About the second line of (27) in the numerator, via Hölder inequality we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} u_n\eta_n\nabla\eta_n\cdot\nabla\,\overline{u}_n\right| \\
&\quad\le \left(\int_{\mathbb{R}^N} |u_n|^2 |\nabla\eta_n|^2\right)^{1/2}\left(\int_{\mathbb{R}^N} \eta_n^2 |\nabla u_n|^2\right)^{1/2} \\
&\quad= \left(\int_{\mathbb{R}^{N-2}} |\nabla\eta_n|^2\right)^{1/2}\left(\int_{\mathbb{R}^2} |u_n|^2\right)^{1/2}\left(\int_{\mathbb{R}^{N-2}} \eta_n^2\right)^{1/2}\left(\int_{\mathbb{R}^2} |\nabla u_n|^2\right)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \mathcal{A}u_n\eta_n\cdot\overline{u}_n\nabla\eta_n\right| \\
&\quad\le \left(\int_{\mathbb{R}^N} |\nabla\eta_n|^2 |u_n|^2\right)^{1/2}\left(\int_{\mathbb{R}^N} |\mathcal{A}|^2 |\eta_n|^2 |u_n|^2\right)^{1/2} \\
&\quad= \left(\int_{\mathbb{R}^{N-2}} |\nabla\eta_n|^2 \int_{\mathbb{R}^2} |u_n|^2\right)^{1/2}\left(\int_{\mathbb{R}^{N-2}} |\eta_n|^2 \int_{\mathbb{R}^2} |\mathcal{A}|^2 |u_n|^2\right)^{1/2}.
\end{aligned}$$

Therefore, the Rayleigh quotient is reduced to

$$\begin{aligned}
&\frac{\int_{\mathbb{R}^2} |\nabla u_n|^2 + \int_{\mathbb{R}^2} |\mathcal{A}|^2 |u_n|^2 - 2\,\mathrm{Re}\int_{\mathbb{R}^2} \mathcal{A}u_n\cdot\nabla\overline{u}_n}{\int_{\mathbb{R}^2} \frac{|u_n|^2}{x_1^2+x_2^2}} \\
&\quad+ \frac{2\,\mathrm{Re}\int_{\mathbb{R}^N} u_n\eta_n\nabla\eta_n\cdot\nabla\overline{u}_n}{\int_{\mathbb{R}^{N-2}} |\eta_n|^2 \int_{\mathbb{R}^2} \frac{|u_n|^2}{x_1^2+x_2^2}} + \frac{\int_{\mathbb{R}^2} |u_n|^2 \int_{\mathbb{R}^{N-2}} |\nabla\eta_n|^2}{\int_{\mathbb{R}^{N-2}} |\eta_n|^2 \int_{\mathbb{R}^2} \frac{|u_n|^2}{x_1^2+x_2^2}} \\
&\quad- \frac{2\,\mathrm{Re}\int_{\mathbb{R}^N} |u_n|^2 \eta_n\mathcal{A}\cdot\nabla\eta_n}{\int_{\mathbb{R}^{N-2}} |\eta_n|^2 \int_{\mathbb{R}^2} \frac{|u_n|^2}{x_1^2+x_2^2}} \\
&\quad= H + o(1),
\end{aligned}$$

thanks to the properties of the sequence η_n. □

5.2. Variational setting

We have seen before that the quadratic form associated with $\frac{A}{|x|}$-type potentials is equivalent to the Dirichlet form. On the contrary, we will see in case of Aharonov–Bohm potentials that it is stronger than the Dirichlet form, and consequently the function space is a proper subset of $D^{1,2}(\mathbb{R}^N)$.

Indeed, for any $\varphi \in C_C^\infty(\mathbb{R}^N \setminus \{x_1 = x_2 = 0\})$, we have the simple inequivalence

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla\varphi|^2 &= \int_{\mathbb{R}^N} \left|(i\nabla - \mathcal{A} + \mathcal{A})\varphi\right|^2 \\ &\le C\left(\int_{\mathbb{R}^N} \left|(i\nabla - \mathcal{A})\varphi\right|^2 + \int_{\mathbb{R}^N} |\mathcal{A}|^2 |\varphi|^2\right) \\ &\le C\int_{\mathbb{R}^N} \left|(i\nabla - \mathcal{A})\varphi\right|^2,\end{aligned}$$

thanks to Hardy-type inequality proved above.

It is immediately seen by this remark that

$$\mathcal{H}_{\mathcal{A}} \mathrel{\hat{=}} \overline{C_C^\infty(\mathbb{R}^N \setminus \{x_1 = x_2 = 0\})}^{\int_{\mathbb{R}^N}|(i\nabla-\mathcal{A})\varphi|^2} \subseteq D^{1,2}(\mathbb{R}^N).$$

To prove the strict inclusion, it is sufficient to show a function lying in $D^{1,2}(\mathbb{R}^N)$ but not in $\mathcal{H}_{\mathcal{A}}$. One can choose, for example, $\varphi(x_1, x_2, x_3) = p(x_1, x_2, x_3)\,|x|^{(-N+1)/2}$, where p is a cutoff function which is identically 0 in $B_\varepsilon(0)$ and identically 1 in $\mathbb{R}^N \setminus B_{2\varepsilon}(0)$: we have $|\nabla\varphi|^2 \sim |x|^{-N-1}$ which is integrable in $\mathbb{R}^N \setminus B_\varepsilon(0)$, whereas $\frac{\varphi}{x_1^2+x_2^2}$ is not, since φ is far from 0 near the singular set.

Remark 5.2. Of course $\mathcal{H}_{\mathcal{A}}$ is a closed subspace of $D^{1,2}(\mathbb{R}^N)$. This is a straightforward consequence of the density of $C_C^\infty(\mathbb{R}^N \setminus \{x_1 = x_2 = 0\})$ in $\mathcal{H}_A$ and the relation between the two quadratic forms. Then Theorem 3.3 holds also in this space.

Following what we did in the previous case, we state the following lemma.

Lemma 5.3. *Let x_n be a sequence of points such that $\left|({x_n}^1, {x_n}^2)\right| \to \infty$ as $n \to \infty$. Then for any $u \in \mathcal{H}_{\mathcal{A}}$, as $n \to \infty$, we have*

$$\frac{\int_{\mathbb{R}^N} \left|\,(i\nabla - \mathcal{A})\,u(\cdot + x_n)\right|^2 - \int_{\mathbb{R}^N} \frac{a(\theta)}{x_1^2+x_2^2}\,|u(\cdot + x_n)|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} \longrightarrow \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}}.$$

Proof. We can follow the proof of Lemma 2.4 noting here that the singularity involves only the first two variables. □

Therefore, we immediately have the following property for $S_{\mathcal{A},a}$.

Proposition 5.4. *If the electric potential a is invariant under translations in $\mathbb{R}^{N-2}$ (as the magnetic vector potential actually is), then the related minimum problem leads to*

$$S_{\mathcal{A},a} = \inf_{u \in \mathcal{H}_A\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |(i\nabla - \mathcal{A})u|^2 - \int_{\mathbb{R}^N} \frac{a}{x_1^2+x_2^2}\,|u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} \le S.$$

Proof. We follow the proof of Proposition 2.5 taking into account Lemma 5.3. □

5.3. Achieving the Sobolev constant

As in the case of $A(x/|x|)/|x|$-type potentials, we state the following proposition.

Proposition 5.5. *If $S_{\mathcal{A},a} < S$, then $S_{\mathcal{A},a}$ is achieved.*

Proof. The proof is essentially the same as in Theorem 3.4. □

5.4. Symmetry of solutions

We introduce the space

$$\mathcal{H}_{\mathcal{A}}^{k,m} = \left\{ u(z,y) \in \mathcal{H}_{\mathcal{A}} \text{ s.t. } u\left(e^{i\frac{2\pi}{k}} z, y\right) = e^{i\frac{2\pi}{k}m} u(z,|y|) \right\},$$

which is a closed subspace of $\mathcal{H}_{\mathcal{A}}$, so Theorem 3.3 holds in it.

We should suppose that the magnetic potential $\mathcal{A}$ is equivariant under the $\mathbb{Z}_k \times SO(N-2)$ group action on $\mathcal{H}_{\mathcal{A}}$, as in (6). But in this case, the magnetic vector potential enjoys this symmetry, thanks to its special form. On the other hand, we choose the electric potential a as a negative constant.

Following the same proof as in the previous case, we can state the following proposition.

Proposition 5.6. *If $S_{\mathcal{A},a}^{k,m} < k^{2/N} S$, then $S_{\mathcal{A},a}^{k,m}$ is achieved.*

Now we look for sufficient conditions to have $S_{\mathcal{A},a}^{k,m} < k^{2/N} S$.

The idea is again to check the quotient over a suitable sequence of test functions. We choose as well $\sum_{i=1}^k w_j$, where w_j are defined in (9) and the lines above it. Of course, we need to multiply them by a cutoff function

$$\varphi(x_1,x_2,x_3) = \varphi(x_1,x_2) = \varphi\left(\sqrt{x_1^2+x_2^2}\right) = \varphi(\rho)$$

in order to obtain the necessary integrability near the singular set.

Lemma 5.7. *Choosing R large enough in* (9), *the quotient evaluated over $\varphi \sum_{i=1}^k w_j$ is strictly less than $k^{2/N} S$, and so is the infimum $S_{\mathcal{A},a}^{k,m}$.*

Proof. Let us check the quotient over $\varphi \sum_{j=1}^k w_j$. We study

$$\int_{\mathbb{R}^N} \left| \nabla \left(\varphi \sum_{j=1}^k w_j \right) \right|^2 + \frac{\alpha^2 - a}{x_1^2 + x_2^2} \varphi^2 \left| \sum_{j=1}^k w_j \right|^2 - 2\,\mathrm{Re} \left\{ i\, \nabla \left(\varphi \sum_{j=1}^k w_j \right) \cdot \mathcal{A} \varphi \sum_{j=1}^k \overline{w}_j \right\}$$

term by term. First of all,

$$\left| \nabla \left(\varphi \sum_{j=1}^k w_j \right) \right|^2 = |\nabla \varphi|^2 \left| \sum_{j=1}^k w_j \right|^2 + \varphi^2 \left| \nabla \left(\sum_{j=1}^k w_j \right) \right|^2 + 2\,\mathrm{Re} \left\{ \varphi \nabla \varphi \cdot \sum_{j,l} \nabla w_j \overline{w}_l \right\}$$

and

$$\int_{\mathbb{R}^N} \varphi^2 \left| \nabla \left(\sum_{j=1}^k w_j \right) \right|^2 = \int_{\mathbb{R}^N} \left| \nabla \left(\sum_{j=1}^k w_j \right) \right|^2 - \int_{\mathbb{R}^N} (1-\varphi^2) \left| \nabla \left(\sum_{j=1}^k w_j \right) \right|^2$$
$$= k\, S^{N/2} + \int_{\mathbb{R}^N} \mathrm{Re} \left\{ \sum_{j \neq l} |w_j|^{2^*-2}\, w_j \overline{w}_l \right\}$$
$$- \int_{\mathbb{R}^N} (1-\varphi^2) \left| \nabla \left(\sum_{j=1}^k w_j \right) \right|^2 .$$

Secondly,

$$\nabla \left(\varphi \sum_{j=1}^k w_j \right) \cdot \mathcal{A} \varphi \sum_{j=1}^k \overline{w}_j = \varphi \nabla \varphi \cdot \mathcal{A} \left| \sum_{j=1}^k w_j \right|^2 + \varphi^2 \sum_{j,l} \nabla w_j \cdot \mathcal{A}\, \overline{w}_l .$$

So, the quadratic form is the following:

$$kS^{N/2} + \int_{\mathbb{R}^N} \mathrm{Re} \left\{ \sum_{j \neq l} |w_j|^{2^*-2}\, w_j \overline{w}_l \right\} - \int_{\mathbb{R}^N} (1-\varphi^2) \left| \nabla \left(\sum_{j=1}^k w_j \right) \right|^2$$
$$+ \int_{\mathbb{R}^N} |\nabla \varphi|^2 \left| \sum_{j=1}^k w_j \right|^2 + 2 \int_{\mathbb{R}^N} \mathrm{Re} \left\{ \varphi \nabla \varphi \cdot \sum_{j,l} \nabla w_j\, \overline{w}_l \right\}$$
$$+ \int_{\mathbb{R}^N} \frac{\alpha^2 - a}{x_1^2 + x_2^2} \varphi^2 \left| \sum_{i=1}^k w_j \right|^2 - 2 \int_{\mathbb{R}^N} \mathrm{Re} \left\{ i\, \varphi \nabla \varphi \cdot \mathcal{A} \left| \sum_{j=1}^k w_j \right|^2 \right\}$$
$$- 2 \int_{\mathbb{R}^N} \mathrm{Re} \left\{ i\, \varphi^2 \sum_{j,l} \nabla w_j \cdot \mathcal{A}\, \overline{w}_l \right\}$$
$$\leq kS^{N/2} + \int_{\mathbb{R}^N} \mathrm{Re} \left\{ \sum_{j \neq l} |w_j|^{2^*-2}\, w_j \overline{w}_l \right\} + \int_{\mathbb{R}^N} |\nabla \varphi|^2 \left| \sum_{j=1}^k w_j \right|^2$$
$$+ 2 \left(\int_{\mathbb{R}^N} \varphi^2 \left| \sum_{j=1}^k \nabla w_j \right|^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} |\nabla \varphi|^2 \left| \sum_{j=1}^k w_j \right|^2 \right)^{1/2}$$
$$+ \int_{\mathbb{R}^N} \frac{\alpha^2 - a}{x_1^2 + x_2^2} \varphi^2 \left| \sum_{j=1}^k w_j \right|^2$$
$$+ 2 \left(\int_{\mathbb{R}^N} |\nabla \varphi|^2 \left| \sum_{j=1}^k w_j \right|^2 \right)^{1/2} \left(\int_{\mathbb{R}^N} \varphi^2\, |\mathcal{A}|^2 \left| \sum_{j=1}^k w_j \right|^2 \right)^{1/2}$$
$$- 2 \int_{\mathbb{R}^N} \mathrm{Re} \left\{ i\, \varphi^2 \sum_{j,l} \nabla w_j \cdot \mathcal{A}\, \overline{w}_l \right\},$$

whereas the denominator of the Rayleigh quotient is

$$\begin{aligned}\int_{\mathbb{R}^N}\varphi^{2^*}\left|\sum_{j=1}^k w_j\right|^{2^*} &= \int_{\mathbb{R}^N}\left|\sum_{j=1}^k w_j\right|^{2^*} - \int_{\mathbb{R}^N}(1-\varphi^{2^*})\left|\sum_{j=1}^k w_j\right|^{2^*}\\ &\geq kS^{N/2} + 2^*\left(1-\frac{\delta}{2}\right)\int_{\mathbb{R}^N}\mathrm{Re}\sum_{j\neq l}|w_j|^{2^*-2}w_j\,\overline{w}_l \qquad (28)\\ &\quad - \int_{\mathbb{R}^N}(1-\varphi^{2^*})\left|\sum_{j=1}^k w_j\right|^{2^*}.\end{aligned}$$

To simplify the notation, we set $R=\sqrt{({x_j}^1)^2+({x_j}^2)^2}$ and we have

$$\alpha = \int_{\mathbb{R}^N}\mathrm{Re}\left\{\sum_{j\neq l}|w_j|^{2^*-2}w_j\,\overline{w}_l\right\} \leq O\Big(\frac{k^N}{R^{N-2}}\Big) \gg \frac{k^{N-1}}{R^{N-2}}, \qquad (29)$$

$$\beta = \int_{\mathbb{R}^N}|\nabla\varphi|^2\left|\sum_{j=1}^k w_j\right|^2 \leq O\Big(\frac{k^2}{R^{2N-4}}\Big),$$

$$\gamma = 2\left(\int_{\mathbb{R}^N}\varphi^2\left|\nabla\left(\sum_{j=1}^k w_j\right)\right|^2\int_{\mathbb{R}^N}|\nabla\varphi|^2\left|\sum_{j=1}^k w_j\right|^2\right)^{1/2} \leq O\Big(\frac{k^{3/2}}{R^{N-2}}\Big),$$

$$\eta = \int_{\mathbb{R}^N}\frac{\alpha^2-a}{x_1^2+x_2^2}\varphi^2\left|\sum_{j=1}^k w_j\right|^2 \leq \begin{cases} O\Big(\frac{k^2}{R^2}\log R\Big) & \text{if } N=4,\\ O\Big(\frac{k^2}{R^2}\Big) & \text{if } N\geq 5,\end{cases}$$

$$\begin{aligned}\xi &= 2\left(\int_{\mathbb{R}^N}|\nabla\varphi|^2\left|\sum_{j=1}^k w_j\right|^2\right)^{1/2}\left(\int_{\mathbb{R}^N}\varphi^2\,|\mathcal{A}|^2\left|\sum_{j=1}^k w_j\right|^2\right)^{1/2}\\ &\leq \begin{cases} O\Big(\frac{k^2}{R^3}\log^{1/2}R\Big) & \text{if } N=4,\\ O\Big(\frac{k^2}{R^{N-1}}\Big) & \text{if } N\geq 5,\end{cases} \qquad (30)\end{aligned}$$

$$\psi = \int_{\mathbb{R}^N}(1-\varphi^{2^*})\left|\sum_{j=1}^k w_j\right|^{2^*} \leq O\Big(\frac{k^2}{R^{2N}}\Big),$$

while for the remaining term we have

$$\left|\int_{\mathbb{R}^N}\mathrm{Re}\left\{i\,\varphi^2\sum_{j,l}\nabla w_j\cdot\mathcal{A}\overline{w}_l\right\}\right| \leq O\left(\frac{k^{N-3}}{R^{N-2}}\right) \qquad (31)$$

since Lemma 4.9 fits also in this case with the suitable modifications. In (29) the symbol $\gg$ means that α has order strictly greater than k^{N-1}/R^{N-2}.

We note that all these quantities α, β, γ, η, ξ can be chosen small simply by taking the quotient k^{N-1}/R^{N-2} small (namely, $k^{N-1}/R^{N-2} = \varepsilon$), as we can deduce from (29)–(31).

Moreover, we see $\psi = o(\alpha)$, so that we can improve estimate (28) and state

$$\int_{\mathbb{R}^N} \varphi^{2^*} \left| \sum_{j=1}^{k} w_j \right|^{2^*} \geq kS^{N/2} + 2^* \left(1 - \frac{\delta}{2}\right) \int_{\mathbb{R}^N} \mathrm{Re} \sum_{j \neq l} |w_j|^{2^*-2} w_j \, \overline{w}_l$$

for a different δ from above.

With the simplified notation, the quotient takes the form

$$\frac{kS^{N/2} + \alpha + \beta + \gamma + \eta + \xi}{\left(kS^{N/2} + 2^*(1 - \delta/2)\alpha\right)^{2/2^*}} = k^{2/N} S \, \frac{1 + \frac{1}{kS^{N/2}}(\alpha + \beta + \gamma + \eta + \xi)}{\left(1 + \frac{2^*(1-\delta/2)}{kS^{N/2}}\alpha\right)^{2/2^*}}.$$

Expanding the quotient in first order power series, we find that it is asymptotic to

$$\begin{aligned} & k^{2/N} S\Big(1 + \frac{1}{kS^{N/2}}(\alpha + \beta + \gamma + \eta + \xi)\Big)\Big(1 - \frac{2(1 - \delta/2)}{kS^{N/2}}\alpha\Big) \\ & \sim k^{2/N} S \Big\{ 1 + \frac{1}{kS^{N/2}}(\beta + \gamma + \eta + \xi) \\ & \qquad + \frac{1}{kS^{N/2}}\alpha\Big(- 1 + \delta + \frac{1}{kS^{N/2}}(\beta + \gamma + \eta + \xi)\Big)\Big\}. \end{aligned}$$

Now, in order to have the coefficient of $k^{2/N} S$ strictly less than 1, it is sufficient that β, γ, η, ξ be $o(k^{N-1}/R^{N-2})$. Taking into account (29)–(31), we see that it is sufficient to choose k as in the previous case of $\frac{A}{|x|}$-type potentials. □

As we made in the previous section, we wonder if there exists any biradial solution, that is, a function belonging to the space

$$\mathcal{H}_{\mathcal{A}}^{\mathrm{birad},m} = \Big\{ u \in \mathcal{H}_{\mathcal{A}} \text{ s.t. } u(R(x_1, x_2), Sx_3) = R^m u((x_1, x_2), x_3) \\ \forall R \in SO(2)\,, \forall S \in SO(N-2) \Big\}.$$

We note that here the suitable equivariant condition for the magnetic field is already fulfilled, thanks to the special form of Aharonov–Bohm potentials, as well as it occurs for the $\mathbb{Z}_k \times SO(N-2)$ action. In order to investigate this question, we set the problem

$$S_{\mathcal{A},a}^{\mathrm{birad},m} = \inf_{u \in \mathcal{H}_{\mathcal{A}}^{\mathrm{birad},m}} \frac{\int_{\mathbb{R}^N} |(i\nabla - A)u|^2 - \int_{\mathbb{R}^N} \frac{a}{x_1^2 + x_2^2} |u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}},$$

and we state the following proposition.

Proposition 5.8. *There exists a biradial solution.*

Proof. We follow the proof of Proposition 4.11 that fits also in this case with the suitable modifications. □

6. Symmetry breaking

In order to proceed in our analysis, we need to recall a result proved in [1].

Theorem 6.1 (see [1]). *Suppose $u = u(r_1, r_2)$ (where $r_1 = \sqrt{x_1^2 + x_2^2}$ and $r_2 = \sqrt{x_3^2 + \cdots + x_N^2}$) is a $D^{1,2}(\mathbb{R}^N)$ solution to*

$$-\Delta u - \frac{a}{|x|^2} u = f(x, u)$$

with $a \in \mathbb{R}^-$. Suppose $f : \mathbb{R}^N \times \mathbb{C} \to \mathbb{C}$ is a Carathéodory function, C^1 with respect to z, satisfying the growth restriction

$$|f'_z(x, z)| \leq C\left(1 + |z|^{2^*-2}\right)$$

for a.e. $x \in \mathbb{R}^N$ and for all $z \in \mathbb{C}$.

If the solution u has biradial Morse index $m(u) \leq 1$, then u is a radial solution, that is, $u = u(r)$ where $r = \sqrt{x_1^2 + \cdots + x_N^2}$.

We split the argument according to the value of the parameter m.

For $m = 0$, the minimizers for $S_{A,a}^{\mathrm{birad},0}$ can be chosen real valued and have in fact biradial Morse index exactly 1. Further, if the magnetic potential is not present, we are precisely under the hypothesis of the previous theorem, then the minimizers are in fact completely radial and the two levels of the quotient coincide:

$$S_{0,a}^{\mathrm{birad},0} = S_{0,a}^{\mathrm{rad}} = S\left(1 - a\frac{4}{(N-2)^2}\right),$$

where the precise value of $S_{0,a}^{\mathrm{rad}}$ has been stated, for instance, in [27]. So we can write the following chain of relations:

$$S_{A,a}^{\mathrm{birad},0} \geq S_{0,a}^{\mathrm{birad},0} = S_{0,a}^{\mathrm{rad}} = S\left(1 - a\frac{4}{(N-2)^2}\right) \geq k^{2/N} S > S_{A,a}^{k},$$

where the first inequality holds, thanks to diamagnetic inequality; the fact that $S_{0,a}^{r_1,r_2} = S_{0,a}^{\mathrm{rad}}$ is a straightforward consequence of the last theorem; the second inequality is proved in [27], Section 6 for sufficiently large values of $|a|$; and the last one is proved in Lemma 4.3.

For $m \neq 0$, the previous argument is sufficient to prove the symmetry breaking as well. Indeed, the functions in $D_{r_1,r_2}^{1,2}(\mathbb{R}^N)$ take the special form $u(z, y) = \rho(|z|, |y|)\mathrm{e}^{\mathrm{i}m\arg(z)}$, so that

$$|\nabla u|^2 = |\nabla \rho|^2 + m^2 \frac{\rho^2}{|z|^2}.$$

Then we can write the following chain of relations:

$$\begin{aligned} S_{A,a}^{\mathrm{birad},m} &\geq S_{0,a}^{\mathrm{birad},m} > S_{0,m^2-a}^{\mathrm{birad},0} \\ &= S_{0,m^2-a}^{\mathrm{rad}} = S\left(1+\frac{4(m^2-a)}{(N-2)^2}\right) \geq k^{2/N} S > S_{A,a}^{k}, \end{aligned}$$

where the first inequality holds again, thanks to the diamagnetic inequality; the second one is due to the special form of functions in $D^{1,2}_{r_1,r_2}(\mathbb{R}^N)$ (see also [1]), and the third inequality is a straightforward consequence of its analogue in the case $m = 0$.

Remark 6.2 (Symmetry breaking for Aharonov–Bohm electromagnetic potentials). We note that the same facts hold also for Aharonov–Bohm electromagnetic fields. Indeed, the diamagnetic inequivalence holds also for them with the *same* best constant, because the Hardy constant is the same (see Section 4.1); moreover, $\frac{a}{x_1^2+x_2^2} \geq \frac{a}{|x|^2}$ for $a > 0$. So we can write

$$S_{\mathcal{A},a}^{\mathrm{birad},0} \geq S_{0,a}^{\mathrm{birad},0} = S_{0,a}^{rad} \geq k^{2/N} S > S_{\mathcal{A},a}^{k},$$

where the last inequality has been proved in Lemma 5.7.

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