Energy-Complexity Relations by Structural Complexity Methods

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Abstract. In this paper we shall review some of the most recent developments and results on work on energy-complexity relations and, if time will allow it, we shall provide an analytical proof of eq. (3) below, a fundamental relation between energy and complexity established by numerical experiments.

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INTRODUCTION

At all scales, from the mass distribution in the Universe to the intricate neural networks in our brain, structures are present, interact and evolve. From visual inspection of complex phenomena to modern visiometrics, the quest for relating aspects of structural and morphological complexity to hidden physical and biological laws has accompanied progress in science ever since. Cooperative behaviour, self-organization and functional purpose of such complex systems are often hidden by dynamics and energy localization that govern specific evolution. By using concepts and methods borrowed from differential and integral geometry, geometric and algebraic topology, and information from dynamical system analysis, there is now an unprecedented chance to make fundamental progress (Abraham & Shaw, 1992; Weickert & Hagen, 2006; Ricca, 2009b) and to develop new powerful diagnostic tools to detect and analyze complexity from both observational and computational data, by relating this complexity to fundamental properties of the system. Indeed, one of the primary tasks of structural complexity analysis (Ricca, 2005) is to develop tools and techniques to uncover connections between morphological organization of structures and functional properties. In this context, work on energy-complexity relations is of fundamental importance. In recent years there has been considerable progress in this direction and in this talk we shall review some of the most recent developments and results in the field and, if time will allow it, we shall provide an analytical proof of eq. (3) below, a fundamental relation between energy and complexity established by numerical experiments.

In this paper we shall consider examples of filamentary structures taken from studies on vortex systems in both classical and quantum fluids, and magnetic fields in astrophysical flows and solar corona. By using methods based on tangle analysis (see Figure 1) we can indeed extract geometric, topological and algebraic information to describe and classify complex morphologies, to study possible relationships between complexity and physical properties, and to understand and predict energy localization and transfer (Ricca, 2009a). These methods are of general validity and they do not depend on the particular physical context.

ENERGY-COMPLEXITY RELATION FOR VORTEX TANGLES

Numerical experiments based on the generation and growth of vortex tangles (see Barenghi *et al.*, 2001, and subsequent works) show that a power-law correspondence between complexity, measured by average crossing number $\overline{C}$, and kinetic energy E of the system holds true independently from the generating mechanism and regime. Vorticity $\omega = \nabla \times \mathbf{u}$ (where $\mathbf{u}$ is fluid velocity in $\mathbb{R}^3$, with $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{u} = 0$ at infinity) is distributed over thin filaments either generated by a background flow or freely decaying in a homogeneous turbulent state. For simplicity vorticity may be taken as a δ-function distribution on space curves. For such tangle $\mathscr{T} = \cup_i \chi_i$ of vortex lines χ_i ($i = 1, 2, \ldots, n$), average crossing number $\overline{C}$ is obtained by computing the sum of all apparent crossings at sites ε_r, where pairs of vortex

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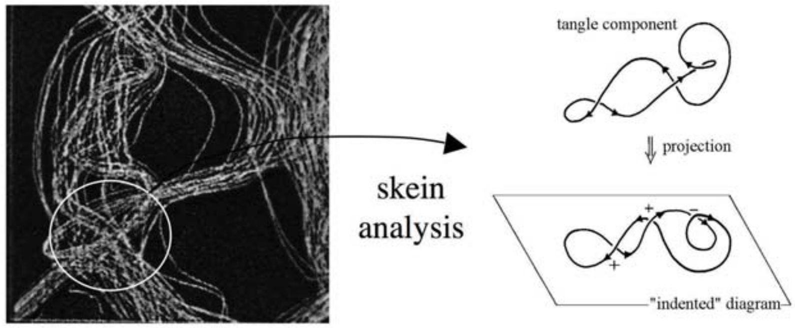

FIGURE 1. Direct numerical simulations provide data-sets domains on which we can perform tangle analysis by structural complexity methods.

lines apparently meet, averaged over all directions of sight, and by extending this counting to the whole tangle; $\overline{C}$ is defined by

$$\overline{C} = \sum_{\{\chi_i,\chi_j\}\in\mathscr{T}} \langle \sum_{r\in\chi_i\#\chi_j} \varepsilon_r \rangle \; , \tag{1}$$

where # denotes disjoint union of all apparent intersections of curve strands, including self-crossings. Kinetic energy, on the other hand, is given by

$$E = \frac{1}{2}\int_{V(\mathscr{T})} \|\mathbf{u}\|^2 \, \mathrm{d}^3\mathbf{x} \; , \tag{2}$$

where $V(\mathscr{T})$ is the total vorticity volume. During evolution these two quantities change in time t according to the following relation

$$\overline{C}(t) \propto [\bar{E}(t)]^2 \; , \tag{3}$$

where $\bar{E}(t)$ is kinetic energy normalized with respect to some reference value (for example, the energy at initial conditions). This result has been confirmed by several tests under different initial conditions and for different evolutions. This remarkable relationship can indeed be derived from analytical grounds and proof of this will be given in the talk.

TOPOLOGICAL COMPLEXITY

Foundational aspects in topological field theory call also for more work on topological complexity. Indeed in ideal conditions (i.e. ideal magnetohydrodynamics) the magnetic energy M of (zero-framed) knotted flux tubes $\mathscr{K}$ of equal and constant magnetic flux Φ and total volume $V = V(\mathscr{K})$ is bounded from below by knot complexity. In particular, if magnetic energy is given by

$$M = \int_{V(\mathscr{K})} \|\mathbf{B}\|^2 \, \mathrm{d}^3\mathbf{x} \; , \tag{4}$$

where $\mathbf{B}$ is the magnetic field confined to the tube, then, by using previous results by Arnold (1974), Moffatt (1990) and Freedman & He (1991), we can prove (Ricca, 2008) that

$$M_{\min} = \left(\frac{16}{\pi}\right)^{1/3} \frac{\Phi^2}{V^{1/3}} c_{\min} \; , \tag{5}$$

that is

$$M_{\min} \propto c_{\min} \; , \tag{6}$$

where $c_{\min}$ is the minimum number of crossings of knot type $\mathscr{K}$, a topological invariant of $\mathscr{K}$. Another important quantity, that is related to the linking number, is the magnetic helicity H, given by

$$H = \int_{V(\mathscr{K})} \mathbf{A}\cdot\mathbf{B} \, \mathrm{d}^3\mathbf{x} \; , \tag{7}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ (with $\nabla \cdot \mathbf{A} = 0$). Then, we have

$$M \geq \left(\frac{16}{\pi V}\right)^{1/3} |H| \ , \qquad M_{\min} \geq \left(\frac{16}{\pi V}\right)^{1/3} \Phi^2 \, c_{\min} \ . \tag{8}$$

Moreover, in presence of dissipation, magnetic fields reconnect and the change in topology is reflected in a change of topological complexity according to

$$H(t) \leq 2\Phi^2 \overline{C}(t) \ , \tag{9}$$

where the inequality provides an upper bound on the magnetic helicity and therefore on the total linking of the magnetic system.

CONCLUSIONS

These results shed new light also on applications. With an ever increasing computational power the implementation of these methods in effective diagnostic tools to provide real-time analysis of structural complexity properties to interpret physical and biological properties of evolutionary processes will prove useful (Ricca, 2011). Possible applications however will not be limited to the development of new diagnostic and visiometric tools, but they will prove useful to study critical properties of disordered media (phase-space transitions), neural systems (growth and organization), and in the risk assessment analysis of complex networks in the information technology, in the financial market, in ecological and social systems and in the world-wide-web social networks (Song *et al.*, 2005).

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