SHARP ESSENTIAL SELF-ADJOINTNESS OF RELATIVISTIC SCHRÖDINGER OPERATORS WITH A SINGULAR POTENTIAL

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Abstract. This paper is devoted to the study of essential self-adjointness of a relativistic Schrödinger operator with a singular homogeneous potential. From an explicit condition on the coefficient of the singular term, we provide a sufficient and necessary condition for essential self-adjointness.

1. Introduction

The purpose of the present paper is to provide sharp essential self-adjointness of the Hamiltonian

$$H(p,x) := (p^2+m^2)^s - \frac{a(\frac{x}{|x|})}{|x|^{2s}}, \quad x \in \mathbb{R}^N, \tag{1.1}$$

with $a : \mathbb{S}^{N-1} \to \mathbb{R}$, $s \in (0,1)$, $m \geq 0$, $N > 2s$.

A symmetric densely-defined operator in a Hilbert space is said to be *essentially self-adjoint* if it has a unique self-adjoint extension. We recall that if a symmetric operator $A : D(A) \to E$, with $D(A)$ dense in the Hilbert space E, is strictly positive, i.e. if $(Au,u)_E \geq c(u,u)_E$ for all $u \in D(A)$ and some $c > 0$, then A is essentially self-adjoint if and only if its range is dense in E, see e.g. [17, Theorem X.26].

In 3-space dimension, the quantum mechanics of a spin zero relativistic particle of charge e and mass m in the Coulomb field of an infinitely heavy nucleus of charge Z is described by the Hamiltonian $H(p,x) = (p^2+m^2)^{1/2} - Ze^2|x|^{-1}$, see e.g. [9, 15]. From [9, 13] it is known that $(p^2+m^2)^{1/2} - Ze^2|x|^{-1}$ is semi-positive definite if $Ze^2 \leq 2/\pi$ and, moreover, it is essentially-self adjoint if $Ze^2 \leq 1/2$. As a particular case of the main result of the present paper, we will see that if $Ze^2 > 1/2$ then $(p^2+m^2)^{1/2} - Ze^2|x|^{-1}$ is not essentially-self adjoint.

The essential self-adjointness of the operator $A = H(i\nabla, x) = (-\Delta+m^2)^s - a(x/|x|)|x|^{-2s}$ implies uniqueness of the quantum dynamics defined by A. Next, once we know that an operator is not essentially self-adjoint, the choice of its extension to generate the quantum dynamics is dictated by the physics problem, see [17] for more explanations. Another application of essential self-adjointness is in probability. Indeed, in general, A could have several self-adjoint extensions A', yielding Markov processes with transition semigroups $p_t = e^{-tA'}$. The essential

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self-adjointness of A implies that there is only one self-adjoint extension A_F: the *Friedrichs extension.* Hence, in case of essential self-adjointness, we have a unique such semigroup and thus a unique Markov process with generator A_F.

Let $\mathbb{S}^N$ be the unit N-dimensional sphere and

$$\mathbb{S}^N_+ = \{(\theta_1, \theta_2, \dots, \theta_N) \in \mathbb{S}^N : \theta_1 > 0\}.$$

We will denote by dS (respectively dS') the volume element on N-dimensional (respectively $(N-1)$-dimensional) spheres and define $H^1(\mathbb{S}^N_+; \theta_1^{1-2s})$ as the completion of $C^\infty(\overline{\mathbb{S}^N_+})$ with respect to the norm

$$\|\psi\|_{H^1(\mathbb{S}^N_+;\theta_1^{1-2s})} = \left(\int_{\mathbb{S}^N_+} \theta_1^{1-2s} \big(|\nabla_{\mathbb{S}^N} \psi(\theta)|^2 + \psi^2(\theta) \big) dS \right)^{1/2}. \tag{1.2}$$

For every $a \in L^\infty(\mathbb{S}^{N-1})$, let

$$\mu_1(a) := \min_{\psi \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s}) \setminus \{0\}} \frac{\int_{\mathbb{S}^N_+} \theta_1^{1-2s} |\nabla \psi|^2 \, dS - \kappa_s \int_{\mathbb{S}^{N-1}} a \psi^2 \, dS'}{\int_{\mathbb{S}^N_+} \theta_1^{1-2s} \psi^2 \, dS}, \tag{1.3}$$

where

$$\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}.$$

The quantity $\mu_1(a)$ is an eigenvalue appearing from a change of polar coordinates in some Dirichlet energy defined on the half-space $\mathbb{R}^{N+1}_+$, see [4]. In [4] it is also observed that the operator $(-\Delta + m^2)^s - a(x/|x|)|x|^{-2s}$ is positive definite provided

$$\mu_1(a) + \frac{(N-2s)^2}{4} > 0, \tag{1.4}$$

see also Lemma 2.1 below. Throughout this paper, we will always assume (1.4).

The following theorem gives conditions on the coefficient a for essential self-adjointness of the operator $(-\Delta + m^2)^s - a(x/|x|)|x|^{-2s}$.

Theorem 1.1. *Assume that $s \in (0,1)$, $m \geq 0$, $N > 2s$, and $a \in L^\infty(\mathbb{S}^{N-1})$ with*

$$\mu_1(a) + \frac{(N-2s)^2}{4} > 0.$$

Then the operator

$$(-\Delta + m^2)^s - a(x/|x|)|x|^{-2s} \quad \text{with domain } C^\infty_c(\mathbb{R}^N \setminus \{0\})$$

is essentially self-adjoint in $L^2(\mathbb{R}^N)$ if and only if

$$-\mu_1(a) \leq \frac{(N-2s)^2}{4} - s^2. \tag{1.5}$$

If a is constant then $\mu_1(a)$ can be obtained implicitly from the usual Gamma function. Indeed, pick $\alpha \in \left(0, \frac{N-2s}{2}\right)$ and let

$$\lambda(\alpha) = 2^{2s} \frac{\Gamma\left(\frac{N+2s+2\alpha}{4}\right)}{\Gamma\left(\frac{N-2s-2\alpha}{4}\right)} \frac{\Gamma\left(\frac{N+2s-2\alpha}{4}\right)}{\Gamma\left(\frac{N-2s+2\alpha}{4}\right)}. \tag{1.6}$$

From [5, Proposition 2.3], we have

$$\mu_1(\lambda(\alpha)) = \alpha^2 - \left(\frac{N-2s}{2}\right)^2 \quad \text{for all } \alpha \in \left(0, \frac{N-2s}{2}\right).$$

Therefore, by combining Theorem 1.1 and [5, Proposition 2.3], we obtain the following corollary.

Corollary 1.2. *Let $\alpha \in \big(0, \frac{N-2s}{2}\big]$. Then $(-\Delta+m^2)^s - \lambda(\alpha)|x|^{-2s}$ with domain $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ is essentially self-adjoint in $L^2(\mathbb{R}^N)$ if and only if $\alpha \geq s$.*

Since the map $\alpha \mapsto \lambda(\alpha)$ is decreasing, we also obtain the following corollary.

Corollary 1.3. *Let $\beta \in \mathbb{R}$ and λ be given by (1.6). Then the operator $(-\Delta+m^2)^s - \beta|x|^{-2s}$ with domain $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ is essentially self-adjoint in $L^2(\mathbb{R}^N)$ if and only if $\beta \leq \lambda(s)$.*

Let us note that

$$\lambda(s) = 2^{2s}\frac{\Gamma(\frac{N+4s}{4})}{\Gamma(\frac{N-4s}{4})}.$$

If $s = 1/2$ and $N = 3$ then $\lambda(1/2) = 1/2$. In this case, the essential self-adjointness below the threshold $\lambda(1/2) = 1/2$ was known, see Kato [13] and Herbst [9]; moreover, the sharpness of the threshold $1/2$ in 3 dimensions was obtained in [14, Corollary 1]. In higher dimensions $N \geq 3$ and for $s = 1/2$, Ichinose in [10] proved essential self-adjointness of $(-\Delta+m^2)^s - \beta|x|^{-2s}$ provided $\beta < \frac{N-2}{2}$ using the Kato-Rellich perturbation result and the Hardy inequality. Our result in Corollary 1.3 improves the results in [10] because, for $s = 1/2$, we have $\lambda(1/2) = \frac{N-2}{2}$. In addition, we also obtain the sharpness of the threshold $\lambda(1/2) = \frac{N-2}{2}$, thus extending [14, Corollary 1] to higher dimensions.

We remark that the precise threshold for non essential self-adjointness in the local case $s = 1$ is $\lambda(1) = \frac{(N-2)^2}{4} - 1$; we refer to [7, 12, 19] for such local case. We also mention that the case of a not constant was treated in the non-relativistic case in [8].

We observe that as a direct consequence of Theorem 1.1 and Kato-Rellich Perturbation Theorem the operator $(-\Delta+m^2)^s - m^{2s} - a(x/|x|)|x|^{-2s}$ with domain $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ is essentially self-adjoint in $L^2(\mathbb{R}^N)$ if and only if $-\mu_1(a) \leq \frac{(N-2s)^2}{4} - s^2$. We refer to [2] for the study of asymptotics of the eigenstates of relativistic operators of type $(-\Delta+m^2)^s - m^{2s}$.

Our argument for proving essential self-adjointness is quite direct and it is inspired by [19]. The proof is based on a contradiction argument as follows: if $A = (-\Delta+m^2)^s - \frac{a(x/|x|)}{|x|^{2s}}$, with domain $C_c^\infty(\mathbb{R}^N \setminus \{0\})$, is not essentially self adjoint in $L^2(\mathbb{R}^N)$ then $(-\Delta)^s - \frac{a(x/|x|)}{|x|^{2s}}$, with domain $C_c^\infty(\mathbb{R}^N \setminus \{0\})$, is not essentially self adjoint in $L^2(\mathbb{R}^N)$ as well (by Kato-Rellich), see Lemma 3.1.

Therefore there exists a function $w \in L^2(\mathbb{R}^N)$, $w \neq 0$, such that $(-\Delta)^s w - a(x/|x|)|x|^{-2s}w + w = 0$ in the sense of distributions, namely

$$\int_{\mathbb{R}^N} [(-\Delta)^s\varphi - a(x/|x|)|x|^{-2s}\varphi + \varphi]w = 0, \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}). \tag{1.7}$$

The idea is now to construct appropriate test functions in (1.7) to get $w \equiv 0$ leading to a contradiction. In order to do this, we will approximate a by smooth functions $a_n \in C^\infty(\mathbb{S}^{N-1})$ such that $\mu_1(a_n) \to \mu_1(a)$.

The test functions we will consider are then solutions to the partial differential equations

$$(-\Delta)^s v_n + v_n - a_n(x/|x|)|x|^{-2s}v_n = f,$$

for arbitrary $f \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, $f \geq 0$, $f \neq 0$. Such functions v_n enjoy the following estimates at the origin and at infinity:

$$v_n \leq C|x|^{\gamma_n} \text{ in } B_{r_0}, \qquad v_n \leq C|x|^{\alpha_n} \text{ in } \mathbb{R}^N \setminus B_{r_0},$$

where $B_{r_0} = \{x \in \mathbb{R}^N : |x| < r_0\}$, $r_0 > 0$, and

$$\gamma_n = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_1(a_n)}, \quad \alpha_n = -\frac{N-2s}{2} - \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_1(a_n)}.$$

Regularity theory implies that $v_n \in C^\infty(\mathbb{R}^N \setminus \{0\})$.

We then cut off the v_n's at the origin and at infinity, in order to use them as test functions in (1.7). Then, thanks to (1.5) and some tricky integration by parts in the nonlocal framework, we end up with

$$\int_{\mathbb{R}^N} wf\,dx = \int_{\mathbb{R}^N} (a(x/|x|) - a_n(x/|x|))|x|^{-2s} v_n w\,dx.$$

Finally, from the above estimates of v_n it follows that $\int_{\mathbb{R}^N} wf\,dx = 0$, thus contradicting that $w \neq 0$. This program is elaborated in details in Section 3.

We observe that the above described arguments can be adapted to treat operators of the type $(-\Delta + m^2)^s - a(x/|x|)|x|^{-2s} + h(x)$ where $h \in L^\infty_{loc}(\mathbb{R}^N) \cap L^p(B_r)$, for some $p > N/(2s)$, $r > 0$, and h is bounded in a neighborhood of ∞, see Remark 3.5.

To prove non essential self-adjointness of a densely defined operator, it is generally inevitable to solve some partial differential equations (mostly, boundary value eigenvalue problems) for which the solutions are known explicitly or at least have some qualitative properties that can be handled. In our situation, we would like to prove that $(-\Delta + m^2)^s - a(x/|x|)|x|^{-2s}$ is not essentially self-adjoint when $-\mu_1(a) > (N-2s)^2/4 - s^2$.

To show this we argue by contradiction and assume that $(-\Delta+m^2)^s - a(x/|x|)|x|^{-2s}$ is essentially self-adjoint, which is equivalent to the density of the range of $(-\Delta+m^2)^s - a(x/|x|)|x|^{-2s} + d$ in $L^2(\mathbb{R}^N)$ for all $d > 0$.

We show in Lemma 3.1 that this is equivalent to $(-\Delta + m^2 + b)^s - a(x/|x|)|x|^{-2s}$ having dense range in $L^2(\mathbb{R}^N)$ for all $b > 0$; this circumstance is ruled out by constructing a function $f \neq 0$ solving the equation $(-\Delta + m^2 + b)^s f - a(x/|x|)|x|^{-2s} f = 0$ in $\mathbb{R}^N$ with $f \in L^2(\mathbb{R}^N)$ provided $-\mu_1(a) > \frac{(N-2s)^2}{4} - s^2$. The advantage of considering $(-\Delta + m^2 + b)^s$ instead of $(-\Delta + m^2)^s + d$ is the exponential decay at infinity of the fundamental solution of the former operator, which is crucial in our analysis and which fails for the latter operator for $m = 0$. This argument will be developed in details in Section 4.

2. Some preliminaries and Notations

We start by recalling the integral representation of $(-\Delta + m^2)^s$: for every $u \in C^2_c(\mathbb{R}^N)$

$$(-\Delta + m^2)^s u(x) = c_{N,s} m^{\frac{N+2s}{2}} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\,dy + m^{2s} u(x),$$

where $m \geq 0$ and

$$c_{N,s} = 2^{-(N+2s)/2+1} \pi^{-\frac{N}{2}} 2^{2s} \frac{s(1-s)}{\Gamma(2-s)},$$

see [4]. The kernel K_ν denotes the modified Bessel function of the second kind with order ν. We recall that, for $\nu > 0$,

$$K_\nu(r) \sim \frac{\Gamma(\nu)}{2} \left(\frac{r}{2}\right)^{-\nu} \tag{2.1}$$

as $r \to 0$ and $K_{-\nu} = K_\nu$ for $\nu < 0$, while

$$K_\nu(r) \sim \frac{\sqrt{\pi}}{\sqrt{2}} r^{-1/2} e^{-r} \tag{2.2}$$

as $r \to +\infty$, see [3]. Furthermore there holds

$$K'_\nu(r) = -\frac{\nu}{r} K_\nu(r) - K_{\nu-1}(r).$$

The Dirichlet form associated to $(-\Delta + m^2)^s$ on $C_c^\infty(\mathbb{R}^N)$ is given by

$$\begin{aligned}(u,v)_{H^s_m(\mathbb{R}^N)} :&= \int_{\mathbb{R}^N} (|\xi|^2 + m^2)^s \widehat{u}(\xi)\overline{\widehat{v}(\xi)} d\xi \\ &= \frac{c_{N,s}}{2} m^{\frac{N+2s}{2}} \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\, dx\, dy \\ &\quad + m^{2s} \int_{\mathbb{R}^N} u(x)v(x) dx,\end{aligned} \tag{2.3}$$

where $\widehat{u}$ denotes the unitary Fourier transform of u. We define $H^s_m(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm induced by the scalar product (2.3). If $m > 0$, $H^s_m(\mathbb{R}^N)$ is nothing but the standard $H^s(\mathbb{R}^N)$; then, we will write $H^s(\mathbb{R}^N)$ without the subscript "m".

The operator $(-\Delta + m^2)^s$ enjoys an extension property reminiscent of the Caffarelli-Silvestre extension [1], see [4]. Let us recall it via the Bessel Kernel which is given by

$$P_m(z) = C'_{N,s} t^{2s} m^{\frac{N+2s}{2}} |z|^{-\frac{N+2s}{2}} K_{\frac{N+2s}{2}}(m|z|), \tag{2.4}$$

with $z = (t,x) \in \mathbb{R} \times \mathbb{R}^N$ and some normalization constant $C'_{N,s}$. Pick $u \in H^s_m(\mathbb{R}^N)$ and set

$$w(t,x) = (P_m(t,\cdot) * u)(x).$$

Then, see [4], we have that $w \in H^1_m(\mathbb{R}^{N+1}_+; t^{1-2s})$ and moreover

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla w)(t,x) + m^2 t^{1-2s} w(t,x) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim_{t\to 0} t^{1-2s} \frac{\partial w}{\partial t}(t,x) = \kappa_s (-\Delta + m^2)^s u(x), & \text{on } \mathbb{R}^N, \end{cases} \tag{2.5}$$

in a weak sense, where $\mathbb{R}^{N+1}_+ = \{z = (t,x) : t \in (0,+\infty),\ x \in \mathbb{R}^N\}$. Here $H^1_m(\mathbb{R}^{N+1}_+; t^{1-2s})$ is the completion of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm $\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2\, dtdx + m^2 \int_{\mathbb{R}^N} t^{1-2s} w^2\, dtdx$. In fact, performing Fourier transform in the above equations, we can see that the Bessel Kernel $P_m(t,x)$ is the Fourier transform of $\xi \mapsto \vartheta(\sqrt{|\xi|^2 + m^2} t)$, where $\vartheta(r) = \frac{2}{\Gamma(s)} \left(\frac{r}{2}\right)^s K_s(r)$ solves

$$\begin{cases} \vartheta'' + \frac{(1-2s)}{t}\vartheta' - \vartheta = 0, \\ \vartheta(0) = 1. \end{cases}$$

This then implies that

$$\int_{\mathbb{R}^N} P_m(t,x) dx = \vartheta(mt). \tag{2.6}$$

Due to homogeneity properties of problem (2.5), we are naturally led to consider an angular eigenvalue problem. Let $H^1(\mathbb{S}^N_+; \theta_1^{1-2s})$ be defined as in (1.2). Since the weight θ_1^{1-2s} belongs to the second Muckenhoupt class, the embedding

$$H^1(\mathbb{S}^N_+; \theta_1^{1-2s}) \hookrightarrow\hookrightarrow L^2(\mathbb{S}^N_+; \theta_1^{1-2s})$$

is compact, where

$$L^2(\mathbb{S}^N_+;\theta_1^{1-2s}) := \Big\{\psi : \mathbb{S}^N_+ \to \mathbb{R} \text{ measurable such that } \textstyle\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\psi^2(\theta)\,dS < +\infty\Big\}.$$

Letting $a \in L^q(\mathbb{S}^{N-1})$, for some $q > N/(2s)$, the first eigenvalue of the angular component of the extended operator

$$\mu_1(a) = \min_{\psi\in H^1(\mathbb{S}^N_+;\theta_1^{1-2s})\setminus\{0\}} \frac{\int_{\mathbb{S}^N_+} \theta_1^{1-2s}|\nabla\psi|^2\,dS - \kappa_s\int_{\mathbb{S}^{N-1}} a\psi^2\,dS'}{\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\psi^2\,dS}$$

is attained by an eigenfunctions ψ which does not change sign and satisfies

$$\begin{cases} -\operatorname{div}_{\mathbb{S}^N}(\theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi) = \mu_1(a)\,\theta_1^{1-2s}\psi, & \text{in } \mathbb{S}^N_+,\\ -\lim_{\theta_1\to 0^+}\theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi\cdot\mathbf{e}_1 = \kappa_s a(\theta')\psi, & \text{on } \partial\mathbb{S}^N_+ = \mathbb{S}^{N-1}. \end{cases} \tag{2.7}$$

The following result is essentially contained in [4].

Lemma 2.1. *Let $q > N/(2s)$ and $a \in L^q(\mathbb{S}^{N-1})$ such that*

$$\mu_1(a) + \left(\frac{N-2s}{2}\right)^2 > 0.$$

Then there exists a constant $C_{a,N,s} > 0$ such that, for all $w \in H^1_0(\mathbb{R}^{N+1}_+;t^{1-2s})$,

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2\,dt\,dx - \kappa_s\int_{\mathbb{R}^N}\frac{a(x/|x|)}{|x|^{2s}}w^2\,dx \geq C_{a,N,s}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2\,dt\,dx.$$

Equivalently, we have that

$$\int_{\mathbb{R}^N}|\xi|^{2s}\widehat{\varphi}^2\,d\xi - \int_{\mathbb{R}^N}\frac{a(x/|x|)}{|x|^{2s}}\varphi^2\,dx \geq C_{a,N,s}\int_{\mathbb{R}^N}|\xi|^{2s}\widehat{\varphi}^2\,d\xi$$

for all $\varphi \in H^s_0(\mathbb{R}^N)$.

Remark 2.2. *It is useful to remark that the best constant $C_{a,N,s}$ in Lemma 2.1 depends continuously on a as a mapping in $L^q(\mathbb{S}^{N-1})$, see Remark 2.5 in [4].*

We will also need the following result from [6].

Lemma 2.3 ([6], Lemma 2.1)**.** *Let Ω be a bounded open set. Then there exists a positive constant $C = C(N,s,\Omega) > 0$ such that for all $\varphi \in C^2_c(\Omega)$ and for all $x \in \mathbb{R}^N$*

$$|(-\Delta)^s\varphi(x)| \leq \frac{C\|\varphi\|_{C^2(\mathbb{R}^N)}}{1+|x|^{N+2s}}.$$

3. Essentially self-adjointness

In this section we shall prove that the operator $A' = (-\Delta+m^2)^s - a(x/|x|)|x|^{-2s}$ with domain $C^\infty_c(\mathbb{R}^N\setminus\{0\})$ is essentially self-adjoint in $L^2(\mathbb{R}^N)$ provided

$$-\mu_1(a) \leq \frac{(N-2s)^2}{4} - s^2.$$

This stands to be a generalization of the case $s = 1$ by Kalf, Schmincke, Walter, Wüst [12], see also Simon [19], and of the case in which $s = 1/2$ and a is constant which was treated by Kato [13], see also Herbst [9].

For the proof, we will need some technical lemmata. Let us first observe that it is not restrictive to take $m = 0$.

Lemma 3.1. *For $s \in (0,1)$, $V \in L^2_{loc}(\mathbb{R}^N \setminus \{0\})$ and $b > 0$, let us consider $A = (-\Delta)^s - V$ and $B = (-\Delta + b)^s - V$ with domain $C_c^\infty(\mathbb{R}^N \setminus \{0\})$.*

(i) A is essentially self-adjoint on $L^2(\mathbb{R}^N)$ if and only if B is essentially self-adjoint on $L^2(\mathbb{R}^N)$.

(ii) If there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} (|\xi|^2 + b)^s \widehat{\varphi}^2(\xi)\, d\xi - \int_{\mathbb{R}^N} V(x)\varphi^2(x)\, dx \geq C\|\varphi\|^2_{H^s(\mathbb{R}^N)} \quad \text{for every } \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}), \tag{3.1}$$

then B is essentially self-adjoint on $L^2(\mathbb{R}^N)$ if and only if B has dense range in $L^2(\mathbb{R}^N)$.

Proof. To prove (i), we observe that, by Fourier transform and Parseval identity,

$$\|(B-A)u\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \left[(|\xi|^2 + b)^s - |\xi|^{2s}\right]^2 |\widehat{u}(\xi)|^2 d\xi,$$

for all $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. Using the elementary inequality $0 \leq (a+b)^s - a^s \leq b^s$, which holds for every $a, b \in [0, +\infty)$ and $s \in (0,1)$, it follows that

$$\|(B-A)u\|^2_{L^2(\mathbb{R}^N)} \leq b^{2s} \int_{\mathbb{R}^N} |\widehat{u}(\xi)|^2 d\xi = b^{2s}\|u\|^2_{L^2(\mathbb{R}^N)}.$$

Therefore, for $q \in (0,1)$ we get

$$\begin{aligned} \|(B-A)u\|_{L^2(\mathbb{R}^N)} &\leq q\|Au\|_{L^2(\mathbb{R}^N)} + b^s\|u\|_{L^2(\mathbb{R}^N)}, \\ \|(B-A)u\|_{L^2(\mathbb{R}^N)} &\leq q\|Bu\|_{L^2(\mathbb{R}^N)} + b^s\|u\|_{L^2(\mathbb{R}^N)}, \end{aligned}$$

for all $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, i.e. $B - A$ is both A-bounded and B-bounded with relative bound $q < 1$. Then by the Kato-Rellich Theorem (see e.g. [17, Theorem X.12]) it follows that if A is essentially self-adjoint then $B = A + (B - A)$ is essentially self-adjoint; in the same way, if B is essentially self-adjoint then $A = B + (A - B)$ is essentially self-adjoint, thus proving (i).

We recall (see e.g. [17, Theorem X.26]) that if a symmetric operator is strictly positive, then it is essentially self-adjoint if and only if its range is dense. Since by assumption (3.1) B is a strictly positive symmetric operator, we deduce statement (ii). □

Remark 3.2. *Let us observe that our potential $V(x) = \frac{a(x/|x|)}{|x|^{2s}}$ satisfies (3.1) for every $b > 0$ provided condition (1.4) is satisfied. Indeed, for $\varepsilon > 0$ and $u \in H^s(\mathbb{R}^N)$, we have*

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\xi|^2 + b)^s |\widehat{u}|^2(\xi) d\xi - \int_{\mathbb{R}^N} \frac{a(x/|x|)}{|x|^{2s}} u^2 dx \\ &\quad = (1-\varepsilon)\int_{\mathbb{R}^N} (|\xi|^2 + b)^s |\widehat{u}|^2(\xi) d\xi + \varepsilon \int_{\mathbb{R}^N} (|\xi|^2 + b)^s |\widehat{u}|^2(\xi) d\xi - \int_{\mathbb{R}^N} \frac{a(x/|x|)}{|x|^{2s}} u^2 dx \\ &\quad = (1-\varepsilon)\left[\int_{\mathbb{R}^N} (|\xi|^2 + b)^s |\widehat{u}|^2 d\xi - \int_{\mathbb{R}^N} \frac{a_\varepsilon(x/|x|)}{|x|^{2s}} u^2 dx\right] + \varepsilon \int_{\mathbb{R}^N} (|\xi|^2 + b)^s |\widehat{u}|^2 d\xi, \end{aligned}$$

where $a_\varepsilon = \frac{a}{1-\varepsilon}$. By continuous dependence of μ_1 on a and (1.4), there exists $\varepsilon_0 = \varepsilon_0(a, N, s) > 0$ such that

$$\mu_1(a_\varepsilon) + \frac{(N-2s)^2}{4} > 0$$

for all $\varepsilon \in (0, \varepsilon_0)$. *By Lemma 2.1, we have*

$$\int_{\mathbb{R}^N} (|\xi|^2 + b)^s \widehat{u}^2 \, d\xi - \int_{\mathbb{R}^N} \frac{a_\varepsilon(x/|x|)}{|x|^{2s}} u^2 \, dx \geq C_{a_\varepsilon, N, s} \int_{\mathbb{R}^N} |\xi|^{2s} \widehat{u}^2 \, d\xi.$$

Therefore

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\xi|^2 + b)^s |\widehat{u}|^2 d\xi - \int_{\mathbb{R}^N} \frac{a(x/|x|)}{|x|^{2s}} u^2 dx \\ &\qquad \geq (1-\varepsilon) C_{a_\varepsilon, N, s} \int_{\mathbb{R}^N} |\xi|^{2s} \widehat{u}^2 \, d\xi + \varepsilon \int_{\mathbb{R}^N} (|\xi|^2 + b)^s |\widehat{u}|^2 d\xi. \end{aligned}$$

Hence, by Parseval identity, for all $\varepsilon \in (0, \varepsilon_0)$ *we have that*

$$\int_{\mathbb{R}^N} (|\xi|^2 + b)^s |\widehat{u}|^2 d\xi - \int_{\mathbb{R}^N} \frac{a(x/|x|)}{|x|^{2s}} u^2 dx \geq (1-\varepsilon) C_{a_\varepsilon, N, s} \int_{\mathbb{R}^N} |\xi|^{2s} \widehat{u}^2 \, d\xi + \varepsilon b^s \int_{\mathbb{R}^N} u^2 dx.$$

The following uniforms decay estimates will be useful in the sequel.

Lemma 3.3. *Let* $a_n \in C^\infty(\mathbb{S}^{N-1})$ *be such that* $a_n \to a$ *in* $L^q(\mathbb{S}^{N-1})$, *for some* $q > N/(2s)$, *and* $\mu_1(a_n) \to \mu_1(a)$ *as* $n \to \infty$. *Assume that*

$$\mu_1(a) + \left(\frac{N-2s}{2}\right)^2 > 0.$$

Let $v_n \in H_0^s(\mathbb{R}^N)$ *be a sequence of functions such that* $v_n > 0$ *a.e. in* $\mathbb{R}^N$ *and* $\{v_n\}_n$ *is bounded in* $H_0^s(\mathbb{R}^N)$.

(i) If

$$(-\Delta)^s v_n - a_n(x/|x|)|x|^{-2s} v_n \leq 0, \quad \text{in } B_R,$$

for some $R > 0$, *then there exist* $C > 0$ *and* $r_0 \in (0, R)$ *(independent of* n*) such that*

$$v_n(x) \leq C|x|^{\gamma_n} \text{ for a.e. } x \in B_{r_0} \tag{3.2}$$

where

$$\gamma_n := -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_1(a_n)}.$$

(ii) If

$$(-\Delta)^s v_n - a_n(x/|x|)|x|^{-2s} v_n \leq 0, \quad \text{in } \mathbb{R}^N \setminus B_R,$$

for some $R > 0$, *then there exist* $C > 0$ *and* $r_0 > R$ *(independent of* n*) such that*

$$v_n(x) \leq C|x|^{\alpha_n} \text{ for a.e. } x \in \mathbb{R}^N \setminus B_{r_0} \tag{3.3}$$

for n *sufficiently large, where*

$$\alpha_n = -\frac{N-2s}{2} - \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_1(a_n)}.$$

Proof. To prove (i), let $w_n \in H_0^1(\mathbb{R}_+^{N+1}; t^{1-2s})$ be the Caffarelli-Silvestre extension of v_n, so that w_n solves

$$\begin{cases} \mathrm{div}(t^{1-2s} \nabla w_n) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \frac{\partial w_n}{\partial t} \leq \kappa_s a_n(x/|x|)|x|^{-2s} v_n, & \text{on } B_R. \end{cases} \tag{3.4}$$

If $r_0 \in (0, R)$, from the regularity estimates in [11] (see also [4, Proposition 3.3]), we deduce that $w_n|_{S^+_{r_0}}$ is uniformly bounded, where $S^+_{r_0} = \{z \in \mathbb{R}^{N+1}_+ : |z| = r_0\}$.

Hence there exists $C > 0$ independent of n such that $0 \le w_n \le C\tilde{w}_n$ on $S^+_{r_0}$ for n sufficiently large, where $\tilde{w}_n(z) := |z|^{\gamma_n}\psi_n(z/|z|)$ with ψ_n being the positive $L^2(\mathbb{S}^N_+; \theta_1^{1-2s})$-normalized eigenfunction corresponding to $\mu_1(a_n)$. Since $\tilde{w}_n$ solves

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla \tilde{w}_n) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\frac{\partial \tilde{w}_n}{\partial t} = \kappa_s a_n(x/|x|)|x|^{-2s}\tilde{w}_n, & \text{on } \mathbb{R}^N, \end{cases} \tag{3.5}$$

testing the difference between (3.4) and (3.5) multiplied by C with $(w_n - C\tilde{w}_n)^+$, integrating by parts, using that $v_n > 0$, and invoking Lemma 2.1, we obtain that $w_n \le C\tilde{w}_n$ a.e. in $B^+_{r_0} = \{z \in \mathbb{R}^{N+1}_+ : |z| < r_0\}$. Hence $v_n(x) \le C|x|^{\gamma_n}\psi_n(x/|x|)$ for a.e. $x \in B_{r_0}$ and the conclusion follows from an uniform upper bound of ψ_n (which follows e.g. from [4, Proposition 3.3]).

To prove (ii), we consider the Kelvin transform of v_n given by $\widetilde{v}_n = |x|^{2s-N}v_n(x/|x|^2)$. We have that $\widetilde{v}_n \in H^s_0(\mathbb{R}^N)$ with $\|\widetilde{v}_n\|_{H^s_0(\mathbb{R}^N)} = \|v_n\|_{H^s_0(\mathbb{R}^N)}$ (see [6, Lemma 2.2]) and

$$(-\Delta)^s\widetilde{v}_n - a_n(x/|x|)|x|^{-2s}\widetilde{v}_n \le 0 \quad \text{in } B_{1/R}.$$

From (i), for some $C_1 > 0$ and $r_0 > R$ (independent on n), we have

$$0 \le \widetilde{v}_n(x) \le C_1|x|^{\gamma_n}, \quad \text{for all } x \in B_{r_0} \setminus \{0\}$$

which yields

$$v_n(x) \le C|x|^{\alpha_n}, \quad \text{for all } x \in \mathbb{R}^N \setminus B_{r_0},$$

where $\alpha_n = -\frac{N-2s}{2} - \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_1(a_n)}$. □

Theorem 3.4. *Assume that $s \in (0,1)$, $N > 2s$, and $a \in L^q(\mathbb{S}^{N-1})$ for some $q > \max\left(\frac{N}{2s}, 2\right)$. Then the operator $A = (-\Delta)^s - a(x/|x|)|x|^{-2s}$ with domain $C^\infty_c(\mathbb{R}^N \setminus \{0\})$ is essentially self-adjoint in $L^2(\mathbb{R}^N)$ provided $-\mu_1(a) \le \frac{(N-2s)^2}{4} - s^2$.*

Proof. The proof of the theorem will be separated into two cases.
Case 1:

$$-\mu_1(a) < \frac{(N-2s)^2}{4} - s^2. \tag{3.6}$$

By Lemma 2.1 we have that

$$(A\varphi, \varphi)_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N}(A\varphi)(x)\varphi(x)\,dx \ge 0, \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^N \setminus \{0\}),$$

so that A is nonnegative definite in $C^\infty_c(\mathbb{R}^N \setminus \{0\})$.

From the Kato-Rellich Theorem and well-known self-adjointness criteria for positive operators (see [17, Theorem X.26]), A with domain $C^\infty_c(\mathbb{R}^N \setminus \{0\})$ is essentially self-adjoint in $L^2(\mathbb{R}^N)$ if and only if $\mathrm{Range}(A+1) = (A+1)(C^\infty_c(\mathbb{R}^N \setminus \{0\})) \subset L^2(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$.

We argue by contradiction and assume that A is not essentially self-adjoint so that

$$(A+1)(C^\infty_c(\mathbb{R}^N \setminus \{0\})) \subset L^2(\mathbb{R}^N)$$

is not dense in $L^2(\mathbb{R}^N)$. Then there exists $w \in L^2(\mathbb{R}^N)$, $w \neq 0$ such that $(w,u)_{L^2(\mathbb{R}^N)} = 0$ for all $u \in (A+1)(C_c^\infty(\mathbb{R}^N \setminus \{0\}))$. In particular

$$\int_{\mathbb{R}^N} w(x)\Big[(-\Delta)^s\varphi(x) - a\big(x/|x|\big)|x|^{-2s}\varphi(x) + \varphi(x)\Big]\,dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}), \tag{3.7}$$

i.e.

$$(-\Delta)^s w - a(x/|x|)|x|^{-2s} w + w = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^N \setminus \{0\}).$$

We will reach a contradiction by showing that $w \equiv 0$; to prove that $w \equiv 0$, we will prove that $\int_{\mathbb{R}^N} fw dx = 0$ for every $f \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, $f \geq 0$, $f \neq 0$. To this aim, let us fix

$$f \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \quad \text{such that} \quad f \geq 0, \quad f \not\equiv 0.$$

Step 1. By density, there exists a sequence $a_n \in C^\infty(\mathbb{S}^{N-1})$ such that

$$a_n \to a \quad \text{in } L^q(\mathbb{S}^{N-1}) \quad \text{as } n \to \infty. \tag{3.8}$$

Then by [4, Lemma 2.1], we have that $\mu_1(a_n) \to \mu_1(a)$. By (3.6), we have that, for some $\varepsilon_0 > 0$,

$$-s + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_1(a_n)} > \varepsilon_0 > 0 \tag{3.9}$$

for every large n.

By the Lax-Milgram theorem, for every $n \in \mathbb{N}$, there exists $v_n \in H^s(\mathbb{R}^N)$ such that

$$(-\Delta)^s v_n - a_n(x/|x|)|x|^{-2s} v_n + v_n = f. \tag{3.10}$$

Multiplying (3.10) by the negative part of v_n and using Lemma 2.1, we can see that $v_n \geq 0$ since $f \geq 0$. By the Harnack inequality $v_n > 0$ in $\mathbb{R}^N \setminus \{0\}$ (see [11]). By (3.9) and Remark 2.2, the sequence $(v_n)_n$ is bounded in $H^s(\mathbb{R}^N)$. By regularity theory $v_n \in C^\infty(\mathbb{R}^N \setminus \{0\})$ (see also [11]). Moreover Lemma 3.3, and (3.9) imply that $(-\Delta)^s v_n \in L^2(\mathbb{R}^N)$ so that $v_n \in H^{2s}(\mathbb{R}^N)$.

Step 2. Let $\eta \in C_c^\infty(\mathbb{R})$ be such that $0 \leq \eta \leq 1$, $\eta(t) = 1$ for $|t| \leq 1$ and $\eta(t) = 0$ for $|t| \geq 2$. We put $\eta_\delta(t) = \eta(\frac{t}{\delta})$ and $\eta_R(t) = \eta(\frac{t}{R})$ so that $(1-\eta_\delta)\eta_R v_n \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. We put $v_{n,\delta} = (1-\eta_\delta)v_n$. Then from (3.7) we have that

$$\int_{\mathbb{R}^N} w\Big((-\Delta)^s(\eta_R v_{n,\delta}) - a(x/|x|)|x|^{-2s}\eta_R v_{n,\delta} + \eta_R v_{n,\delta}\Big)\,dx = 0. \tag{3.11}$$

We claim that, for n and δ fixed,

$$(-\Delta)^s(\eta_R v_{n,\delta}) \to (-\Delta)^s v_{n,\delta} \quad \text{in } L^2(\mathbb{R}^N) \text{ as } R \to +\infty. \tag{3.12}$$

By direct computations, we have

$$\begin{aligned}(-\Delta)^s(\eta_R v_{n,\delta})(x) - (-\Delta)^s v_{n,\delta}(x) &= v_{n,\delta}(x)(-\Delta)^s\eta_R(x) + (\eta_R(x)-1)(-\Delta)^s v_{n,\delta}(x)\\ &\quad - c_{N,s} PV\int_{\mathbb{R}^N} \frac{(v_{n,\delta}(x)-v_{n,\delta}(y))(\eta_R(x)-\eta_R(y))}{|x-y|^{N+2s}}dy.\end{aligned}$$

Therefore, by Hölder's inequality, we get

$$\begin{aligned}&\|(-\Delta)^s(\eta_R v_{n,\delta}) - (-\Delta)^s v_{n,\delta}\|_{L^2(\mathbb{R}^N)}\\ &\qquad \leq \|v_{n,\delta}(-\Delta)^s\eta_R\|_{L^2(\mathbb{R}^N)} + \|(\eta_R-1)(-\Delta)^s v_{n,\delta}\|_{L^2(\mathbb{R}^N)}\\ &\qquad + c_{N,s}\left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{(v_{n,\delta}(x)-v_{n,\delta}(y))^2}{|x-y|^{N+2s}}dy\right)\left(\int_{\mathbb{R}^N}\frac{(\eta_R(x)-\eta_R(y))^2}{|x-y|^{N+2s}}dy\right)dx\right)^{1/2}.\end{aligned}$$

By scaling, we have, for some positive $C > 0$ independent of R,

$$|(-\Delta)^s \eta_R(x)| \le CR^{-2s} \quad \text{for all } x \in \mathbb{R}^N. \tag{3.13}$$

Next, we note that

$$c_{N,s} \int_{\mathbb{R}^N} \frac{(\eta_R(x) - \eta_R(y))^2}{|x-y|^{N+2s}} dy = -(-\Delta)^s(\eta_R^2)(x) + 2\eta_R(x)(-\Delta)^s \eta_R(x). \tag{3.14}$$

This implies, as above, that

$$\int_{\mathbb{R}^N} \frac{(\eta_R(x) - \eta_R(y))^2}{|x-y|^{N+2s}} dy \le CR^{-2s},$$

for some $C > 0$. Therefore, using the above estimate and (3.13), we get

$$\begin{aligned} &\|(-\Delta)^s(\eta_R v_{n,\delta}) - (-\Delta)^s v_{n,\delta}\|_{L^2(\mathbb{R}^N)} \\ &\qquad \le C(R^{-2s} + R^{-s})\|v_{n,\delta}\|_{H^s(\mathbb{R}^N)} + \|(\eta_R - 1)(-\Delta)^s v_{n,\delta}\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Hence (3.12) is proved. It follows that we can take the limit as $R \to \infty$ in (3.11) and use the dominated convergence theorem to obtain

$$\int_{\mathbb{R}^N} w\Big((-\Delta)^s((1-\eta_\delta)v_n) - a(x/|x|)|x|^{-2s}(1-\eta_\delta)v_n + (1-\eta_\delta)v_n\Big)\, dx = 0. \tag{3.15}$$

Step 3. We claim that

$$\{(-\Delta)^s((1-\eta_\delta)v_n)\}_{\delta\in(0,1)} \text{ is bounded in } L^2(\mathbb{R}^N). \tag{3.16}$$

As above, we have

$$\begin{aligned} \|(-\Delta)^s(1-\eta_\delta)v_n)\|_{L^2(\mathbb{R}^N)} &\le \|v_n(-\Delta)^s\eta_\delta\|_{L^2(\mathbb{R}^N)} + \|(-\Delta)^s v_n\|_{L^2(\mathbb{R}^N)} \\ &\quad + c_{N,s}\left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{(v_n(x)-v_n(y))^2}{|x-y|^{N+2s}}dy\right)\left(\int_{\mathbb{R}^N}\frac{(\eta_\delta(x)-\eta_\delta(y))^2}{|x-y|^{N+2s}}dy\right)dx\right)^{1/2}. \end{aligned}$$

Let us estimate the first term in the right hand side of the above inequality. To estimate it uniformly in δ, we use Lemma 2.3 to get

$$|(-\Delta)^s\eta_\delta| \le C_{\eta,N,s}\frac{\delta^N}{\delta^{N+2s} + |x|^{N+2s}}.$$

Then we have

$$\begin{aligned} \|v_n(-\Delta)^s\eta_\delta\|^2_{L^2(\mathbb{R}^N)} &\le C\int_{B_1} v_n^2(x)\frac{\delta^{2N}}{(\delta^{N+2s}+|x|^{N+2s})^2}dx + C\|v_n\|^2_{L^2(\mathbb{R}^N\setminus B_1)} \\ &\le C\int_{B_\delta} v_n^2\frac{\delta^{2N}}{(\delta^{N+2s}+|x|^{N+2s})^2}dx + C\int_{B_1\setminus B_\delta} v_n^2\frac{\delta^{2N}}{(\delta^{N+2s}+|x|^{N+2s})^2}dx \\ &\quad + C\|v_n\|^2_{L^2(\mathbb{R}^N)} \\ &\le C\int_{B_\delta} v_n^2\delta^{-4s}dx + C\int_{B_1\setminus B_\delta} v_n^2|x|^{2N}|x|^{-2N-4s}dx + C\|v_n\|^2_{L^2(\mathbb{R}^N)} \\ &\le C\int_{B_\delta}|x|^{-4s}v_n^2dx + C\int_{B_1\setminus B_\delta}|x|^{-4s}v_n^2dx + C\|v_n\|^2_{L^2(\mathbb{R}^N)} \\ &\le C\int_{B_1}|x|^{-4s}v_n^2dx + C\|v_n\|^2_{L^2(\mathbb{R}^N)}, \end{aligned}$$

where C is a positive constant independent on δ (varying from line to line). Hence by (3.2) and (3.9), we obtain

$$\|v_n(-\Delta)^s\eta_\delta\|^2_{L^2(\mathbb{R}^N)} \le C + C\|v_n\|^2_{L^2(\mathbb{R}^N)}. \tag{3.17}$$

In addition, by integration by parts, we have[1]

$$\begin{aligned}
&c_{N,s}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{(v_n(x)-v_n(y))^2}{|x-y|^{N+2s}}dy\right)\left(\int_{\mathbb{R}^N}\frac{(\eta_\delta(x)-\eta_\delta(y))^2}{|x-y|^{N+2s}}dy\right)dx \qquad (3.19)\\
&= \int_{\mathbb{R}^N}[-(-\Delta)^s(v_n^2)(x)+2v_n(x)(-\Delta)^s v_n(x)][-(-\Delta)^s(\eta_\delta^2)(x)+2\eta_\delta(x)(-\Delta)^s\eta_\delta(x)]dx\\
&= \int_{\mathbb{R}^N}v_n^2(-\Delta)^{2s}\eta_\delta^2 - 2\int_{\mathbb{R}^N}v_n^2(-\Delta)^s(\eta_\delta(-\Delta)^s\eta_\delta) \qquad (3.20)\\
&\quad + \int_{\mathbb{R}^N}2v_n(x)(-\Delta)^s v_n(x)[-(-\Delta)^s(\eta_\delta^2)(x)+2\eta_\delta(x)(-\Delta)^s\eta_\delta(x)]dx\\
&\le I_1+I_2+I_3,
\end{aligned}$$

where

$$I_1 = \int_{\mathbb{R}^N}v_n^2|(-\Delta)^{2s}\eta_\delta^2|dx,$$

$$I_2 = -2\int_{\mathbb{R}^N}v_n^2(-\Delta)^s(\eta_\delta(-\Delta)^s\eta_\delta)dx$$

and

$$I_3 = \left|\int_{\mathbb{R}^N}2v_n(x)(-\Delta)^s v_n(x)[-(-\Delta)^s(\eta_\delta^2)(x)+2\eta_\delta(x)(-\Delta)^s\eta_\delta(x)]dx\right|.$$

Now, for the last integral, we can use similar techniques as above to get

$$\begin{aligned}
I_3 &\le C\int_{\mathbb{R}^N}(|x|^{-2s}v_n+v_n+f)v_n\frac{\delta^N}{\delta^{N+2s}+|x|^{N+2s}}dx\\
&\le C\int_{B_1}|x|^{-4s}v_n^2dx+\int_{\mathbb{R}^N\setminus B_1}v_n^2dx + C\int_{\mathbb{R}^N}fv_ndx,
\end{aligned}$$

[1] It is worth justifying the passage from (3.19) to (3.20). If $u\in H^{2s}(\mathbb{R}^N)$ and $g\in\mathcal{S}(\mathbb{R}^N)$, the space of Schwarz functions, then

$$c_{N,s}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}dy\right)g(x)dx = -\int_{\mathbb{R}^N}u^2(x)(-\Delta)^s g(x)\,dx + 2\int_{\mathbb{R}^N}u(x)g(x)((-\Delta)^s u)(x)\,dx. \tag{3.18}$$

Indeed, we can approximate u by smooth functions $u_n=\rho_n * u$ by convolution with the standard mollifiers and consider $u_{n,R}=u_n\eta_R$ so that $u_{n,R}\to u$ in $H^{2s}(\mathbb{R}^N)$ as $n,R\to+\infty$, see (3.16). Since $(-\Delta)^s(u^2_{n,R})\in L^2(\mathbb{R}^N)$ by Lemma 2.3 we can pass to the limit in

$$\begin{aligned}
&c_{N,s}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{(u_{n,R}(x)-u_{n,R}(y))^2}{|x-y|^{N+2s}}dy\right)g(x)dx\\
&= \int_{\mathbb{R}^N}\big[-(-\Delta)^s(u^2_{n,R})(x)+2u_{n,R}(x)((-\Delta)^s u_{n,R})(x)\big]g(x)\,dx\\
&= -\int_{\mathbb{R}^N}u^2_{n,R}(x)(-\Delta)^s g(x)\,dx + 2\int_{\mathbb{R}^N}u_{n,R}(x)g(x)((-\Delta)^s u_{n,R})(x)\,dx
\end{aligned}$$

to obtain (3.18).

where $C > 0$ (varying from line to line) is independent of δ (but could depend on n and $\operatorname{Supp} f$). Hence by (3.2) and (3.9), we deduce that

$$I_3 \leq C + \int_{\mathbb{R}^N \setminus B_1} v_n^2 dx + C \int_{\mathbb{R}^N} f v_n dx. \tag{3.21}$$

We observe that $(-\Delta)^s \eta_\delta = \delta^{-2s}(-\Delta)^s \eta(\cdot/\delta) \in C^\infty(\mathbb{R}^N)$ so that $\eta_\delta(-\Delta)^s \eta_\delta \in C_c^\infty(\mathbb{R}^N)$ and $(-\Delta)^s(\eta_\delta(-\Delta)^s\eta_\delta) = \delta^{-4s}(-\Delta)^s(\eta(-\Delta)^s\eta)(\cdot/\delta)$. This implies that

$$\begin{aligned} I_2 = -2\int_{\mathbb{R}^N} v_n^2 (-\Delta)^s(\eta_\delta(-\Delta)^s\eta_\delta) dx &\leq C \int_{\mathbb{R}^N} v_n^2 \frac{\delta^{N-2s}}{\delta^{N+2s} + |x|^{N+2s}} dx \\ &\leq C \int_{B_1} |x|^{-4s} v_n^2 dx + \int_{\mathbb{R}^N \setminus B_1} v_n^2 dx \end{aligned}$$

and thus by (3.2) and (3.9), we get

$$I_2 \leq C + C\int_{\mathbb{R}^N \setminus B_1} v_n^2 dx. \tag{3.22}$$

Next, we estimate I_1. If $2s = 1$ then

$$I_1 = \int_{\mathbb{R}^N} v_n^2 |(-\Delta)^{2s}\eta_\delta^2| dx \leq C\delta^{-2} \int_{\delta \leq |x| \leq 2\delta} v_n^2 dx \leq C \int_{\mathbb{R}^N} |x|^{-2} v_n^2 dx.$$

If $2s < 1$ then (using again Lemma 2.3)

$$I_1 \leq C \int_{\mathbb{R}^N} v_n^2 \frac{\delta^N}{\delta^{N+4s} + |x|^{N+4s}} dx \leq C \int_{B_1} |x|^{-4s} v_n^2 dx + C\int_{\mathbb{R}^N \setminus B_1} v_n^2 dx.$$

Hence by (3.2) and (3.9)

$$I_1 \leq C + C\int_{\mathbb{R}^N \setminus B_1} v_n^2 dx. \tag{3.23}$$

If $2s > 1$ then $0 < 2s - 1 < 1$ so $(-\Delta)^{2s}\eta_\delta^2 = -(-\Delta)^{2s-1}(\Delta\eta_\delta^2)$ which implies (see Lemma 2.3) that

$$|(-\Delta)^{2s-1}(-\Delta\eta_\delta^2)| \leq C\delta^{-2} \frac{\delta^N}{\delta^{N+2(2s-1)} + |x|^{N+2(2s-1)}}.$$

We then have, by similar estimates as a above,

$$I_1 \leq C\int_{\mathbb{R}^N} v_n^2 \frac{\delta^{N-2}}{\delta^{N+4s-2} + |x|^{N+4s-2}} dx \leq C\int_{B_1} |x|^{-4s} v_n^2 dx + C\int_{\mathbb{R}^N \setminus B_1} v_n^2 dx$$

so that, by (3.2) and (3.9), we have

$$I_1 \leq C + C\int_{\mathbb{R}^N \setminus B_1} v_n^2 dx \tag{3.24}$$

for $2s > 1$. We thus conclude that for all $s \in (0,1)$

$$I_1 \leq C + C\int_{\mathbb{R}^N \setminus B_1} v_n^2 dx. \tag{3.25}$$

Using the estimates (3.25), (3.22) and (3.21) in (3.19), together with (3.17), we get (3.16) as claimed.

Step 4. From (3.16) it follows that $(-\Delta)^s((1-\eta_\delta)v_n) \rightharpoonup (-\Delta)^s v_n$ weakly in $L^2(\mathbb{R}^N)$ as $\delta \to 0^+$ (for any n fixed). Passing to the limit as $\delta \to 0^+$ in (3.15), we then obtain, from the Dominated Convergence Theorem, (3.2) and (3.9), that

$$\int_{\mathbb{R}^N} w\Big((-\Delta)^s v_n - a(x/|x|)|x|^{-2s}v_n + v_n\Big)\,dx = 0.$$

Therefore, recalling (3.10),

$$\int_{\mathbb{R}^N} wf\,dx = \int_{\mathbb{R}^N} (a(x/|x|) - a_n(x/|x|))|x|^{-2s}v_n w\,dx. \tag{3.26}$$

By Hölder's inequality, Fubini's theorem, and estimates (3.2) and (3.3)

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \frac{a(x/|x|) - a_n(x/|x|)}{|x|^{2s}} v_n w\right|^2 \le \|w\|^2_{L^2(\mathbb{R}^N)} \left\|\frac{a(x/|x|) - a_n(x/|x|)}{|x|^{2s}} v_n\right\|^2_{L^2(\mathbb{R}^N)}\\
&\le \|w\|^2_{L^2(\mathbb{R}^N)} \int_0^\infty r^{-4s+N-1}\left(\int_{\mathbb{S}^{N-1}} |v_n(r\theta')|^{\frac{2q}{q-2}}dS'\right)^{\frac{q-2}{q}}\left(\int_{\mathbb{S}^{N-1}} |a_n - a|^q dS'\right)^{\frac{2}{q}} dr\\
&\le \|w\|^2_{L^2(\mathbb{R}^N)} C_0^2 |\mathbb{S}^{N-1}|^{(q-2)/q}\|a_n - a\|^2_{L^q(\mathbb{S}^{N-1})}\int_0^{r_0} r^{-4s+N-1+2\gamma_n}dr\\
&+ \|w\|^2_{L^2(\mathbb{R}^N)} C_1^2 |\mathbb{S}^{N-1}|^{(q-2)/q}\|a_n - a\|^2_{L^q(\mathbb{S}^{N-1})}\int_{r_0}^{\infty} r^{-4s+N-1+2\alpha_n}dr\\
&\le C\|a_n - a\|^2_{L^q(\mathbb{S}^{N-1})}.
\end{aligned}$$

From this, (3.26) and (3.8), we deduce that

$$\int_{\mathbb{R}^N} fw\,dx = 0.$$

We have then proved that $\int_{\mathbb{R}^N} fw\,dx = 0$ for every $f \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, $f \ge 0$, $f \ne 0$. This implies that $w \equiv 0$ which leads to a contradiction.

Case 2:

$$-\mu_1(a) = \frac{(N-2s)^2}{4} - s^2. \tag{3.27}$$

As in Case 1, we argue by contradiction and assume that A is not essentially self-adjoint; as observed above, this implies that there exists $w \in L^2(\mathbb{R}^N) \setminus \{0\}$ such that

$$(-\Delta)^s w - a(x/|x|)|x|^{-2s}w + w = 0$$

in the sense of distributions in $\mathbb{R}^N \setminus \{0\}$.

Let $a_n \in C^\infty(\mathbb{S}^{N-1})$ be as in (3.8). Let $\sigma \in (0,1)$ and notice that $\mu_1(a-\sigma) > \mu_1(a)$ so that

$$-\mu_1(a-\sigma) < \frac{(N-2s)^2}{4} - s^2.$$

Given $f \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ such that $f \ge 0$, $f \not\equiv 0$, by the Lax-Milgram theorem, for every $n \in \mathbb{N}$ and $\sigma \in (0,1)$, there exists $v_n^\sigma \in H^s(\mathbb{R}^N)$ ($v_n^\sigma > 0$ in $\mathbb{R}^N \setminus \{0\}$) solution to

$$(-\Delta)^s v_n^\sigma - |x|^{-2s}(a_n(x/|x|) - \sigma)v_n^\sigma + v_n^\sigma = f.$$

It is then not difficult to check that $(v_n^\sigma)_n$ is bounded in $H^s(\mathbb{R}^N)$ and converges (weakly in $H^s(\mathbb{R}^N)$) to some v^σ weakly solving

$$(-\Delta)^s v^\sigma - |x|^{-2s}(a(x/|x|) - \sigma)v^\sigma + v^\sigma = f.$$

Arguing as in Lemma 3.3, we have that there exist $r_1, C_2 > 0$ independent on σ and n such that

$$v_n^\sigma(x)| \le C_2|x|^{\gamma_n(\sigma)}, \quad v^\sigma(x) \le C_2|x|^{\gamma(\sigma)} \quad \text{for all } x \in B_{r_1} \setminus \{0\}, \tag{3.28}$$

where

$$\gamma_n(\sigma) = -\tfrac{N-2s}{2} + \sqrt{\left(\tfrac{N-2s}{2}\right)^2 + \mu_1(a_n - \sigma)}, \quad \gamma(\sigma) = -\tfrac{N-2s}{2} + \sqrt{\left(\tfrac{N-2s}{2}\right)^2 + \mu_1(a - \sigma)}.$$

Notice that, provided n is large,

$$-s + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_1(a_n - \sigma)} > \varepsilon_0 > 0,$$

for some $\varepsilon_0 > 0$ depending on σ (but independent of n). Therefore, using similar arguments as in Case 1, we get

$$\left|\int_{\mathbb{R}^N} fwdx\right| \le \sigma \int_{\mathbb{R}^N} |x|^{-2s}|w|v^\sigma dx.$$

Hence, by Hölder's inequality

$$\left|\int_{\mathbb{R}^N} fw\,dx\right| \le \|w\|_{L^2(\mathbb{R}^N)}\sigma\||x|^{-2s}v^\sigma\|_{L^2(\mathbb{R}^N)}.$$

Since v^σ is bounded in $H^s(\mathbb{R}^N)$, so $\||x|^{-2s}v^\sigma\|_{L^2(\mathbb{R}^N\setminus B_1)}$ can be uniformly bounded in σ, we infer that

$$\left|\int_{\mathbb{R}^N} fw\,dx\right| \le \|w\|_{L^2(\mathbb{R}^N)}\sigma C_{a,N,s,f} + \|w\|_{L^2(\mathbb{R}^N)}\sigma\||x|^{-2s}v^\sigma\|_{L^2(B_1)}$$

for all $\sigma \in (0,1)$. Using (3.28) we deduce that, for all $\sigma \in (0,1)$ and for some $C > 0$ independent of σ,

$$\left|\int_{\mathbb{R}^N} fw\,dx\right| \le C\frac{\sigma}{\left(-s + \sqrt{\frac{(N-2s)^2}{4} + \mu_1(a-\sigma)}\right)^{1/2}} + C\sigma. \tag{3.29}$$

Using the variational characterization of $\mu_1(a-\sigma)$ and $\mu_1(a)$ (see also [4, Proof of Lemma 2.1] for convergences of related eigenfunctions), we deduce that

$$c_2\sigma + \mu_1(a) \le \mu_1(a-\sigma) \le \mu_1(a) + c_1\sigma,$$

where c_2, c_1 are positive constants independent on σ. Passing to the limit in (3.29) as $\sigma \to 0^+$ yields that $\int_{\mathbb{R}^N} fwdx = 0$. This then implies that $w = 0$ thus giving rise to a contradiction. □

Proof of Theorem 1.1: sufficiency of condition (1.5)**.** In view of Theorem 3.4, condition (1.5) implies that $A = (-\Delta)^s - a(x/|x|)|x|^{-2s}$ with domain $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ is essentially self-adjoint in $L^2(\mathbb{R}^N)$. Then for every $m \ge 0$ also the operator $(-\Delta + m^2)^s - a(x/|x|)|x|^{-2s}$ is essentially self-adjoint in $L^2(\mathbb{R}^N)$ by Lemma 3.1, part (i). □

Remark 3.5. *If $h \in L^\infty_{loc}(\mathbb{R}^N) \cap L^p(B_r)$, for some $p > N/(2s)$, $r > 0$ and h is bounded in a neighborhood of ∞, then the arguments proving Theorem 3.4 above can be adapted to prove that if $-\mu_1(a) \le \frac{(N-2s)^2}{4} - s^2$ and $N > 2s$ then the operator $(-\Delta + m^2)^s - a(x/|x|)|x|^{-2s} + h(x)$ with domain $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ is essentially self-adjoint in $L^2(\mathbb{R}^N)$. Indeed the estimate of Lemma 3.3 still holds for the operator $(-\Delta)^s - a(x/|x|)|x|^{-2s} + h$, see [4, Lemma 5.11].*

4. Non-essential self-adjointness

The following lemma will be crucial in our proof.

Lemma 4.1. *Let $a \in L^\infty(\mathbb{S}^{N-1})$ and $b > 0$. Let ψ_1 be an eigenfunction of problem (2.7) corresponding to first eigenvalue $\mu_1(a)$ in (1.3). Let $\nu_1 = \sqrt{(N-2s)^2/4 + \mu_1(a)}$ and assume that*

$$-\mu_1(a) > \frac{(N-2s)^2}{4} - s^2, \quad i.e.\ s > \nu_1.$$

For $z = (t,x) \in \mathbb{R}^{N+1}_+$ define $f(z) = \psi_1(z/|z|)|z|^{\frac{2s-N}{2}} K_{\nu_1}(\sqrt{b}|z|)$. Then

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla f) + t^{1-2s} bf = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\partial_t f = \kappa_s a(x/|x|)|x|^{-2s} f, & \text{on } \mathbb{R}^N \setminus \{0\}. \end{cases} \tag{4.1}$$

In addition

$$(-\Delta + b)^s f - a(x/|x|)|x|^{-2s} f = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N \setminus \{0\}), \tag{4.2}$$

i.e.

$$\int_{\mathbb{R}^N} f(0,x)\left((-\Delta+b)^s\varphi(x) - \frac{a(x/|x|)}{|x|^{2s}}\varphi(x)\right)dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}).$$

Remark 4.2. *We notice that the conclusion of the above lemma might not be true if $b = 0$. Here, we have the property that K_{ν_1} decays exponentially at infinity which plays a central role in the proof.*

Proof. Direct computations using polar coordinates, see [5], prove the first assertion.

Let $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. We now consider the extension $\Phi(t,x) = (P_{\sqrt{b}}(t,\cdot) * \varphi)(x)$, where $P_{\sqrt{b}}$ is the Bessel Kernel, see Section 2. We have that $\Phi \in H^1(\mathbb{R}^{N+1}_+; t^{1-2s})$ and

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla\Phi) + t^{1-2s} b\Phi = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ \Phi = \varphi, & \text{on } \mathbb{R}^N, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\partial_t\Phi = \kappa_s(-\Delta+b)^s\varphi, & \text{on } \mathbb{R}^N. \end{cases} \tag{4.3}$$

We multiply the first equation of (4.3) by f and integrate by parts over $\mathbb{R}^N \times (\rho, \infty)$ for $\rho > 0$ to get

$$2\int_{\mathbb{R}^N} \rho^{1-2s}\partial_t\Phi(\rho,x) f(\rho,x)dx - 2\int_{\mathbb{R}^N} \rho^{1-2s}\partial_t f(\rho,x)\Phi(\rho,x)dx = 0. \tag{4.4}$$

By (2.1) and (2.2), $|f(\rho,x)| \le C|x|^{\frac{2s-N}{2}-\nu_1}$ for all $x \in B_1$ and $|f(\rho,x)| \le Ce^{-\frac{\sqrt{b}}{2}|x|}$ for every $x \in \mathbb{R}^N \setminus B_1$ and $\rho \in (0,1)$. Since $\frac{N+2s}{2} - \nu_1 > 0$, $f(0,\cdot) \in L^1(B_1)$. Since $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have that $(-\Delta+b)^s\varphi \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. By using [4],

$$|\rho^{1-2s}\partial_t\Phi(\rho,x)| \le C, \quad \text{for all } \rho \in (0,1),\ x \in \mathbb{R}^N.$$

Therefore we can apply the dominated convergence theorem and use (4.3) to get

$$\lim_{\rho\to 0}\int_{\mathbb{R}^N} \rho^{1-2s}\partial_t\Phi(\rho,x) f(\rho,x)dx = -\kappa_s\int_{\mathbb{R}^N}(-\Delta+b)^s\varphi(x) f(0,x)dx. \tag{4.5}$$

It now remains to prove that

$$\lim_{\rho\to 0}\int_{\mathbb{R}^N} \rho^{1-2s}\partial_t f(\rho,x)\Phi(\rho,x)\,dx = \kappa_s\int_{\mathbb{R}^N} a(x/|x|)|x|^{-2s}\varphi(x) f(0,x)\,dx$$

which completes the proof. This will be done in the sequel.

By direct computations we have

$$\begin{aligned} t^{1-2s}\partial_t f(z) &= |z|^{\frac{2s-N}{2}-2s} K_{\nu_1}(\sqrt{b}|z|)\theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi_1(z/|z|)\cdot \mathbf{e}_1 \\ &\quad + [\tfrac{2s-N}{2} t^{2-2s}|z|^{\frac{2s-N}{2}-2} K_{\nu_1}(\sqrt{b}|z|) + \sqrt{b}t^{2-2s}|z|^{\frac{2s-N}{2}-1} K'_{\nu_1}(\sqrt{b}|z|)]\psi_1(z/|z|) \\ &:= H(t,x) + J(t,x), \end{aligned}$$

where, for $z=(t,x)$, we define

$$H(t,x) := |z|^{\frac{2s-N}{2}-2s} K_{\nu_1}(\sqrt{b}|z|)\theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi_1(z/|z|)\cdot \mathbf{e}_1$$

and

$$J(t,x) := \Big[\tfrac{2s-N}{2} t^{2-2s}|z|^{\frac{2s-N}{2}-2} K_{\nu_1}(\sqrt{b}|z|) + \sqrt{b}t^{2-2s}|z|^{\frac{2s-N}{2}-1} K'_{\nu_1}(\sqrt{b}|z|)\Big]\psi_1(z/|z|).$$

First we recall that $\psi_1 \in C^{0,\alpha}(\overline{\mathbb{S}^N_+})$, see [4]. Once again by (2.1) and (2.2) together with the fact that $K'_{\nu_1} = -\frac{\nu_1}{r}K_{\nu_1} - K_{\nu_1-1}$, it is plain that for $|x|\le 1$

$$|J(\rho,x)| \le c\rho^{2-2s}|(\rho,x)|^{\frac{2s-N}{2}-2-\nu_1} \tag{4.6}$$

while for $|x|\ge 1$

$$|J(\rho,x)| \le c\rho^{2-2s}e^{-\sqrt{b}/2|x|}.$$

It is then clear that

$$\lim_{\rho\to 0}\int_{\mathbb{R}^N\setminus B_1} J(\rho,x)\Phi(\rho,x)dx = 0.$$

Recalling the notations in Section 2, for $\rho\le 1$, we claim that

$$\Phi(\rho,x) = \varphi(x)\vartheta(\sqrt{b}\rho) + O(\rho^{2s}), \quad \text{for all } x\in\mathbb{R}^N. \tag{4.7}$$

To see this, we use change of variables and (2.6), to get, up some normalization constant,

$$\begin{aligned} \Phi(\rho,x) &= (P_{\sqrt{b}}(\rho,\cdot)*\varphi)(x) = \int_{\mathbb{R}^N} P_{\sqrt{b}}(\rho,y)\varphi(x+y)dy = \varphi(x)\int_{\mathbb{R}^N} P_{\sqrt{b}}(\rho,y)dy \\ &+ \int_{\mathbb{R}^N} P_{\sqrt{b}}(\rho,y)\nabla\varphi(x)\cdot y dy + \int_{\mathbb{R}^N}\int_0^1\int_0^1 \tau P_{\sqrt{b}}(\rho,y)D^2\varphi(x+\tau r y)[y,y]d\tau dr dy \\ &= \varphi(x)\vartheta(\sqrt{b}\rho) + 0 + \int_{\mathbb{R}^N}\int_0^1\int_0^1 \tau P_{\sqrt{b}}(\rho,y)D^2\varphi(x+\tau r y)[y,y]d\tau dr dy. \end{aligned}$$

Therefore

$$|\Phi(\rho,x) - \varphi(x)\vartheta(\sqrt{b}\rho)| \le C\rho^{2s}\int_{|y|\le 1}|y|^{-N-2s+2}dy + C\rho^{2s}\int_{|y|\ge 1} e^{-\sqrt{b}/2|y|}dy,$$

thus (4.7) is proved. From (4.7) together with (4.6), we deduce that

$$|J(\rho,x)\Phi(\rho,x)| \le C|\varphi(x)||x|^{\frac{2s-N}{2}-2-\nu_1} + C|(\rho,x)|^{\frac{2s-N}{2}-\nu_1} \quad \text{for all } x\in B_1.$$

Therefore

$$|J(\rho,x)\Phi(\rho,x)| \le C + C|x|^{\frac{2s-N}{2}-\nu_1} \quad \text{for all } x\in B_1.$$

The dominated convergence theorem then implies that

$$\lim_{\rho\to 0}\int_{B_1} J(\rho,x)\Phi(\rho,x)dx = 0.$$

Hence

$$\lim_{\rho\to 0}\int_{\mathbb{R}^N} J(\rho,x)\Phi(\rho,x)dx = 0.$$

It remains now to pass the limit as $\rho \to 0$ in the integral $\int_{\mathbb{R}^N} H(\rho,x)\Phi(\rho,x)dx$. To this end, we first claim that

$$\theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi_1 \cdot \mathbf{e}_1 \in L^\infty(\overline{\mathbb{S}^N_+}). \tag{4.8}$$

To prove this claim, we consider

$$g(z) = \psi_1(z/|z|)|z|^{\frac{2s-N}{2}} I_{\sqrt{(N-2s)^2/4+\mu_1(a)}}(\sqrt{b}|z|)$$

which satisfies

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla g) + t^{1-2s} b\, g = 0, & \text{in } \mathbb{R}^{N+1}_+,\\ -\lim\limits_{t\to 0^+} t^{1-2s}\partial_t g = \kappa_s a(x/|x|)|x|^{-2s} g, & \text{on } \mathbb{R}^N\setminus\{0\}, \end{cases} \tag{4.9}$$

where I is the modified Bessel function of first kind. Using its decay property near the origin (see [3]), we see that

$$g(z) \le C|z|^{\frac{2s-N}{2}+\nu_1}, \quad \text{for all } z \in \overline{B_2^+},$$

implying $|z|^{-1}g \in L^2(B_2^+;t^{1-2s})$ and $|x|^{-s}g(0,\cdot) \in L^2(B_2)$. Using standard integration by parts, we can deduce that $g \in H^1(B_2^+;t^{1-2s})$. Then by [4] it follows that $t^{1-2s}\partial_t g \in L^\infty(\overline{B^+_{3/2}\setminus B^+_{1/2}})$. As above, by direct computations,

$$\begin{aligned} t^{1-2s}\partial_t g(z) &= |z|^{\frac{2s-N}{2}-2s} I_{\nu_1}(\sqrt{b}|z|)\theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi_1(z/|z|)\cdot \mathbf{e}_1\\ &\quad + \big[\tfrac{2s-N}{2} t^{2-2s}|z|^{\frac{2s-N}{2}-2} I_{\nu_1}(\sqrt{b}|z|) + \sqrt{b}\, t^{2-2s}|z|^{\frac{2s-N}{2}-1} I'_{\nu_1}(\sqrt{b}|z|)\big]\psi_1(z/|z|). \end{aligned}$$

Evaluating at $|z| = 1$ and using the fact that $I_{\nu_1}(\sqrt{b}) \neq 0$, we see that

$$|\theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi_1(z)\cdot \mathbf{e}_1| \le C(\|\psi_1\|_{L^\infty(\mathbb{S}^N_+)} + 1), \quad \text{for all } z \in \overline{\mathbb{S}^N_+},$$

and claim (4.8) is proved.

Using the Taylor expansion (4.7), and similar arguments as above, we obtain

$$\begin{aligned} \lim_{\rho\to 0}\int_{\mathbb{R}^N} H(\rho,x)\Phi(\rho,x)dx &= -\kappa_s \int_{\mathbb{R}^N} |x|^{\frac{2s-N}{2}-2s} K_{\nu_1}(\sqrt{b}|x|) a(x/|x|)\psi_1(x/|x|)\varphi(x)dx\\ &= -\kappa_s \int_{\mathbb{R}^N} a(x/|x|)|x|^{-2s} f(0,x)\varphi(x)dx. \end{aligned}$$

Using this, (4.5) and (4.4), we get

$$\int_{\mathbb{R}^N} a(x/|x|)|x|^{-2s} f(0,x)\varphi(x)dx = \int_{\mathbb{R}^N} (-\Delta + b)^s\varphi(x) f(0,x)dx$$

which is (4.2). □

Theorem 4.3 (**Necessity of condition** (1.5) **of Theorem 1.1).** *Let $N > 2s$, $m \geq 0$, and $a \in L^\infty(\mathbb{S}^{N-1})$. Then the operator $A' = (-\Delta+m^2)^s - a(x/|x|)|x|^{-2s}$ with domain $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ is not essentially self-adjoint in $L^2(\mathbb{R}^N)$ if*

$$-\mu_1(a) > \frac{(N-2s)^2}{4} - s^2. \tag{4.10}$$

Proof. Assume by contradiction that (4.10) holds and A' is essentially self-adjoint. By part (i) of Lemma 3.1 also $(-\Delta+b)^s - a(x/|x|)|x|^{-2s}$ is essentially self-adjoint in $L^2(\mathbb{R}^N)$ for every $b > 0$; then by assumption (3.1), Remark 3.2, and part (ii) of Lemma 3.1, we have that the operator $(-\Delta+b)^s - a(x/|x|)|x|^{-2s}$ has dense range in $L^2(\mathbb{R}^N)$.

Let ψ_1 be a positive eigenfunction of problem (2.7) associated to the first eigenvalue $\mu_1(a)$ defined in (1.3). For $z = (t, x)$ let

$$f(z) = \psi_1(z/|z|)|z|^{\frac{2s-N}{2}} K_{\sqrt{(N-2s)^2/4+\mu_1(a)}}(\sqrt{b}\,|z|).$$

We observe that by (4.10), (2.1) and (2.2)

$$f(0,\cdot) \in L^2(\mathbb{R}^N). \tag{4.11}$$

Since $(-\Delta+b)^s - a(x/|x|)|x|^{-2s}$ has dense range in $L^2(\mathbb{R}^N)$ (as assumed above for the contradiction), there exists $\varphi_n \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ such that $(-\Delta+b)^s\varphi_n - a(x/|x|)|x|^{-2s}\varphi_n \to f(0,\cdot)$ in $L^2(\mathbb{R}^N)$, so that for every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that

$$\|(-\Delta+b)^s\varphi_n - a(x/|x|)|x|^{-2s}\varphi_n - f(0,\cdot)\|_{L^2(\mathbb{R}^N)} < \varepsilon \quad \text{for every } n \geq n(\varepsilon).$$

This implies that

$$-2\int_{\mathbb{R}^N} \Big((-\Delta+b)^s\varphi_n(x) - a(x/|x|)|x|^{-2s}\varphi_n(x)\Big) f(0,x)\,dx + \|f\|^2_{L^2(\mathbb{R}^N)} < \varepsilon^2$$

for every $n \geq n(\varepsilon)$. By (4.2) we obtain that for every $\varepsilon > 0$

$$\|f\|^2_{L^2(\mathbb{R}^N)} < \varepsilon^2$$

which is impossible. □

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