SINGULARITY OF EIGENFUNCTIONS AT THE JUNCTION OF SHRINKING TUBES, PART II

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Abstract. In continuation with [17], we investigate the asymptotic behavior of weighted eigenfunctions in two half-spaces connected by a thin tube. We provide several improvements about some convergences stated in [17]; most of all, we provide the exact asymptotic behavior of the implicit normalization for solutions given in [17] and thus describe the $(N-1)$-order singularity developed at a junction of the tube (where N is the space dimension).

1. Introduction and statement of the main result

The interest in the spectral analysis of thin branching domains arising in the theory of quantum graphs modeling waves in thin graph-like structures (narrow waveguides, quantum wires, photonic crystals, blood vessels, lungs), see e.g. [11, 20], motivates a large literature dealing with elliptic eigenvalue problems in varying domains; we mention among others [3, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 19, 21, 22].

In [17], the asymptotic behavior of eigenfunctions at the junction of shrinking tubes has been investigated. In a dumbbell domain which is going to disconnect, it can be shown that, generically, the mass of a given eigenfunction of the Dirichlet Laplacian concentrates in only one component of the limiting domain, while the restriction to the other domain, when suitably normalized, develops a singularity at the junction of the tube, as the channel section tends to zero. The main result of [17] states that, under a proper nondegeneracy condition, the normalized limiting profile has a singularity of order $N-1$, where N is the space dimension. The strategy developed in [17] to evaluate the rate to the singularity at the junction is based upon a sharp control of the transversal frequencies along the connecting tube, inspired by the monotonicity method introduced by Almgren [2] and then extended by Garofalo and Lin [18] to elliptic operators with variable coefficients in order to prove unique continuation properties.

In continuation with [17], we investigate the asymptotic behavior of solutions to weighted eigenvalue problems in a dumbbell domain $\Omega^\varepsilon \subset \mathbb{R}^N$, $N \geqslant 3$, formed by two half-spaces connected by a tube with length 1 and cross-section depending on ε:

$$\Omega^\varepsilon = D^- \cup \mathcal{C}_\varepsilon \cup D^+,$$

where $\varepsilon \in (0,1)$ and

$$\begin{aligned} D^- &= \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 < 0\}, \\ \mathcal{C}_\varepsilon &= \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : 0 \leqslant x_1 \leqslant 1, \ \tfrac{x'}{\varepsilon} \in \Sigma\}, \\ D^+ &= \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 > 1\}, \end{aligned}$$

and $\Sigma \subset \mathbb{R}^{N-1}$ is an open bounded set with $C^{2,\alpha}$-boundary containing 0. We refer to [17] so that assume

$$\{x' \in \mathbb{R}^{N-1} : |x'| \leqslant \tfrac{1}{\sqrt{2}}\} \subset \Sigma \subset \{x' \in \mathbb{R}^{N-1} : |x'| < 1\}. \tag{1}$$

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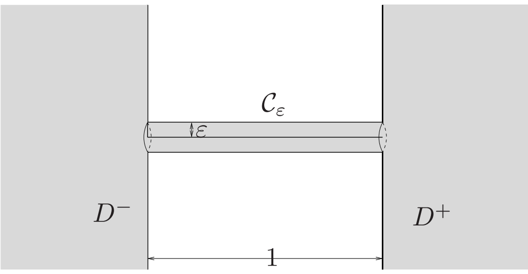

FIGURE 1. The domain Ω^ε.

We also denote, for all $t > 0$,

$$B_t^+ := D^+ \cap B(\mathbf{e}_1, t), \quad B_t^- := D^- \cap B(\mathbf{0}, t),$$

where $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^N$, $\mathbf{0} = (0, 0, \dots, 0)$, and $B(P, t) := \{x \in \mathbb{R}^N : |x - P| < t\}$ denotes the ball of radius t centered at P. Let $p \in C^1(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ be a weight satisfying

$$p \geqslant 0 \text{ a.e. in } \mathbb{R}^N, \ p \in L^{N/2}(\mathbb{R}^N), \ \nabla p(x) \cdot x \in L^{N/2}(\mathbb{R}^N), \ \frac{\partial p}{\partial x_1} \in L^{N/2}(\mathbb{R}^N), \tag{2}$$

$$p \not\equiv 0 \text{ in } D^-, \quad p \not\equiv 0 \text{ in } D^+, \quad p(x) = 0 \text{ for all } x \in B_3^- \cup \mathcal{C}_1 \cup B_3^+. \tag{3}$$

Assumption (3) is stronger than in [17]. We are confident that the present arguments apply even under the weaker assumption of [17], up to several modifications mainly concerning calculus. For reader's convenience we consider worthwhile presenting the argument in this simpler case.

For every open set $\Omega \subset \mathbb{R}^N$, we denote as $\sigma_p(\Omega)$ the set of the diverging eigenvalues $\lambda_1(\Omega) \leqslant \lambda_2(\Omega) \leqslant \cdots \leqslant \lambda_k(\Omega) \leqslant \cdots$ (where each $\lambda_k(\Omega)$ is repeated as many times as its multiplicity) of the weighted eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda p \varphi, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega. \end{cases}$$

It is easy to verify that $\sigma_p(D^- \cup D^+) = \sigma_p(D^-) \cup \sigma_p(D^+)$.

Let us assume that there exists $k_0 \geqslant 1$ such that

$$\lambda_{k_0}(D^+) \text{ is simple and the corresponding eigenfunctions} \tag{4}$$
$$\text{have in } \mathbf{e}_1 \text{ a zero of order } 1,$$

$$\lambda_{k_0}(D^+) \notin \sigma_p(D^-). \tag{5}$$

We can then fix an eigenfunction $u_0 \in \mathcal{D}^{1,2}(D^+) \setminus \{0\}$ associated to $\lambda_{k_0}(D^+)$, i.e. solving

$$\begin{cases} -\Delta u_0 = \lambda_{k_0}(D^+) p u_0, & \text{in } D^+, \\ u_0 = 0, & \text{on } \partial D^+, \end{cases} \tag{6}$$

such that

$$\frac{\partial u_0}{\partial x_1}(\mathbf{e}_1) > 0. \tag{7}$$

Here and in the sequel, for every open set $\Omega \subseteq \mathbb{R}^N$, $\mathcal{D}^{1,2}(\Omega)$ denotes the functional space obtained as completion of $C_c^\infty(\Omega)$ with respect to the Dirichlet norm $\big(\int_\Omega |\nabla u|^2 dx\big)^{1/2}$.

From [16, Example 8.2, Corollary 4.7, Remark 4.3] (see also [17, Lemma 1.1]), it follows that, letting

$$\lambda_\varepsilon = \lambda_{\bar{k}}(\Omega^\varepsilon)$$

where $\bar{k} = k_0 + \operatorname{card}\big\{j \in \mathbb{N} \setminus \{0\} : \lambda_j(D^-) \leqslant \lambda_{k_0}(D^+)\big\}$, so that $\lambda_{k_0}(D^+) = \lambda_{\bar{k}}(D^- \cup D^+)$, there holds

$$\lambda_\varepsilon \to \lambda_{k_0}(D^+) \quad \text{as } \varepsilon \to 0^+. \tag{8}$$

Furthermore, for every ε sufficiently small, λ_ε is simple and there exists an eigenfunction u_ε associated to λ_ε, i.e. satisfying

$$\begin{cases} -\Delta u_\varepsilon = \lambda_\varepsilon p u_\varepsilon, & \text{in } \Omega^\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{9}$$

such that

$$u_\varepsilon \to u_0 \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N) \quad \text{as } \varepsilon \to 0^+, \tag{10}$$

where in the above formula we mean the functions u_ε, u_0 to be trivially extended to the whole $\mathbb{R}^N$. We refer to [9, §5.2] for uniform convergence of eigenfunctions.

For all $t > 0$, let us denote as $\mathcal{H}_t^-$ the completion of $C_c^\infty(D^- \setminus B_t^-)$ with respect to the norm $\big(\int_{D^-\setminus B_t^-} |\nabla v|^2 dx\big)^{1/2}$, i.e. $\mathcal{H}_t^-$ is the space of functions with finite energy in $D^- \setminus \overline{B_t^-}$ vanishing on ∂D^-. We recall that functions in $\mathcal{H}_t^-$ satisfy the following Sobolev type inequality

$$C_S \bigg(\int_{D^-\setminus B_t^-} |v(x)|^{2^*} dx \bigg)^{2/2^*} \leqslant \int_{D^-\setminus B_t^-} |\nabla v(x)|^2 dx, \text{ for all } t > 0 \text{ and } v \in \mathcal{H}_t^-, \tag{11}$$

for some $C_S = C_S(N) > 0$ depending only on the dimension N (and independent on t), see [17, Lemma 3.2].

We also define, for all $t > 0$,

$$\Gamma_t^- = D^- \cap \partial B_t^-. \tag{12}$$

Let

$$\Psi : \mathbb{S}^{N-1} \to \mathbb{R}, \quad \Psi(\theta_1, \theta_2, \dots, \theta_N) = \frac{\theta_1}{\Upsilon_N},$$

being $\mathbb{S}^{N-1} = \{(\theta_1, \theta_2, \dots, \theta_N) \in \mathbb{R}^N : \sum_{i=1}^N \theta_i^2 = 1\}$ the unit $(N-1)$-dimensional sphere and

$$\Upsilon_N = \sqrt{\tfrac{1}{2}\textstyle\int_{\mathbb{S}^{N-1}} \theta_1^2 d\sigma(\theta)}. \tag{13}$$

Here and in the sequel, the notation $d\sigma$ is used to denote the volume element on $(N-1)$-dimensional surfaces. We notice that, letting

$$\mathbb{S}_-^{N-1} := \{\theta = (\theta_1, \theta_2, \dots, \theta_N) \in \mathbb{S}^{N-1} : \theta_1 < 0\},$$
$$\mathbb{S}_+^{N-1} := \{\theta = (\theta_1, \theta_2, \dots, \theta_N) \in \mathbb{S}^{N-1} : \theta_1 > 0\},$$

$\Psi^- = -\frac{\theta_1}{\Upsilon_N}$ is the first positive $L^2(\mathbb{S}_-^{N-1})$-normalized eigenfunction of $-\Delta_{\mathbb{S}^{N-1}}$ on $\mathbb{S}_-^{N-1}$ under null Dirichlet boundary conditions satisfying

$$-\Delta_{\mathbb{S}^{N-1}} \Psi^- = (N-1)\Psi^- \quad \text{on } \mathbb{S}_-^{N-1}, \tag{14}$$

and $\Psi^+ = \frac{\theta_1}{\Upsilon_N}$ is the first positive $L^2(\mathbb{S}_+^{N-1})$-normalized eigenfunction of $-\Delta_{\mathbb{S}^{N-1}}$ on $\mathbb{S}_+^{N-1}$ under null Dirichlet boundary conditions satisfying

$$-\Delta_{\mathbb{S}^{N-1}} \Psi^+ = (N-1)\Psi^+ \quad \text{on } \mathbb{S}_+^{N-1}. \tag{15}$$

The main results of [17] are summarized in the following theorem.

Theorem 1.1. ([17]) *Let us assume* (2)–(7) *hold and let u_ε as in* (9)*. Then there exists $\tilde{k} \in (0,1)$ such that, for every sequence $\varepsilon_n \to 0^+$, there exist a subsequence $\{\varepsilon_{n_j}\}_j$, $U \in C^2(D^-) \cup \big(\bigcup_{t>0} \mathcal{H}_t^-\big)$, $U \not\equiv 0$, and $\beta < 0$ such that*

(i) $$\frac{u_{\varepsilon_{n_j}}}{\sqrt{\int_{\Gamma_{\tilde{k}}^-} u_{\varepsilon_{n_j}}^2 d\sigma}} \to U \quad \text{as } j \to +\infty$$ *strongly in $\mathcal{H}_t^-$ for every $t > 0$ and in $C^2(\overline{B_{t_2}^- \setminus B_{t_1}^-})$ for all $0 < t_1 < t_2$;*

(ii) $$\lambda^{N-1} U(\lambda x) \to \beta \frac{x_1}{|x|^N} \quad \text{as } \lambda \to 0^+$$ *strongly in $\mathcal{H}_t^-$ for every $t > 0$ and in $C^2(\overline{B_{t_2}^- \setminus B_{t_1}^-})$ for all $0 < t_1 < t_2$.*

The aim of the present paper is twofold. On one hand, we will remove the dependence on the subsequence in the previous statement. On the other hand, the aforementioned theorem provides an implicit normalization (i.e. $\int_{\Gamma^-_{\tilde k}} u^2_{\varepsilon_{n_j}}\,d\sigma$) for the sequence of solutions to detect the limit profile; we will determine the exact behavior of this normalization, thus providing an asymptotics of eigenfunctions, which will turn out to be independent of $\widetilde{k} \in (0,1)$. To this aim, we proceed step by step, analyzing the asymptotics at succeeding points, starting at the right junction where an initial normalization is given by (10) and (7). In view of [1, Section 4], the final behavior of $\int_{\Gamma^-_{\tilde k}} u^2_{\varepsilon_{n_j}}\,d\sigma$ will depend on the particular domain's shape, which will be recognizable by some coefficients appearing in the leading term of the asymptotic expansion. More precisely, information about the geometry may be discerned in the dependence of the coefficients on the limit profiles produced by a blow-up at those points where a drastic change of geometry occurs.

We believe that from the asymptotics of eigenfunctions proved in the present paper an exact estimation of the rate of convergence of eigenvalues on the perturbed domain to eigenvalues on the limit domain could follow; this is the object of a current investigation.

Before stating our main result, let us introduce the functions describing the domain's geometry after blowing-up at each junction. Let us denote

$$\widetilde{D} = D^+ \cup T_1^-, \quad T_1^- = \{(x_1, x') : x' \in \Sigma,\ x_1 \leqslant 1\}.$$

In [17, Lemma 2.4], it is proved that there exists a unique function Φ satisfying

$$\begin{cases} \int_{T_1^- \cup B^+_{R-1}} \left(|\nabla \Phi(x)|^2 + |\Phi(x)|^{2^*}\right) dx < +\infty \text{ for all } R > 2, \\ -\Delta \Phi = 0 \text{ in a distributional sense in } \widetilde{D}, \quad \Phi = 0 \text{ on } \partial \widetilde{D}, \\ \int_{D^+} |\nabla(\Phi - (x_1 - 1))(x)|^2\,dx < +\infty. \end{cases} \tag{16}$$

Furthermore $\Phi > 0$ in $\widetilde{D}$ and, by [17, Lemma 2.9], there holds

$$\Phi(x) = (x_1 - 1)^+ + O(|x - \mathbf{e}_1|^{1-N}) \quad \text{in } D^+ \text{ as } |x - \mathbf{e}_1| \to +\infty. \tag{17}$$

Let us define

$$\widehat{D} = D^- \cup T_1^+, \quad T_1^+ = \{(x_1, x') : x' \in \Sigma,\ x_1 \geqslant 0\}, \tag{18}$$

$$T_1 = \{(x_1, x') : x_1 \in \mathbb{R},\ x' \in \Sigma\}. \tag{19}$$

We denote as $\lambda_1(\Sigma)$ the first eigenvalue of the Laplace operator on Σ under null Dirichlet boundary conditions and as $\psi_1^\Sigma(x')$ the corresponding positive $L^2(\Sigma)$-normalized eigenfunction, so that

$$\begin{cases} -\Delta_{x'} \psi_1^\Sigma(x') = \lambda_1(\Sigma) \psi_1^\Sigma(x'), & \text{in } \Sigma, \\ \psi_1^\Sigma = 0, & \text{on } \partial\Sigma, \end{cases} \tag{20}$$

being $\Delta_{x'} = \sum_{j=2}^N \frac{\partial^2}{\partial x_j^2}$, $x' = (x_2, \dots, x_N)$. We define

$$h : T_1 \to \mathbb{R}, \quad h(x_1, x') = e^{\sqrt{\lambda_1(\Sigma)}x_1} \psi_1^\Sigma(x'),$$

and observe that $h \in C^2(T_1) \cap C^0(\overline{T_1})$ satisfies

$$\begin{cases} -\Delta h = 0, & \text{in } T_1, \\ h = 0, & \text{on } \partial T_1. \end{cases}$$

In [17, Lemma 2.7] it is proved that there exists a unique function $\widehat{\Phi} : \widehat{D} \to \mathbb{R}$ such that

$$\begin{cases} \int_{D^-} \left(|\nabla \widehat{\Phi}(x)|^2 + |\widehat{\Phi}(x)|^{2^*}\right) dx < +\infty, \\ -\Delta \widehat{\Phi} = 0 \text{ in a distributional sense in } \widehat{D}, \quad \widehat{\Phi} = 0 \text{ on } \partial \widehat{D}, \\ \int_{T_1} |\nabla(\widehat{\Phi} - h)(x)|^2\,dx < +\infty. \end{cases} \tag{21}$$

Furthermore

$$\widehat{\Phi} > 0 \text{ in } \widehat{D}, \quad \widehat{\Phi} \geqslant h \text{ in } T_1, \quad \widehat{\Phi} - h \in \mathcal{D}^{1,2}(\widehat{D}), \tag{22}$$

and, by [17, Lemma 2.9],

$$\widehat{\Phi}(x) = O(|x|^{1-N}) \quad \text{as } |x| \to +\infty,\, x \in D^-. \tag{23}$$

A further limiting profile which plays a role in the asymptotic behavior of eigenfunctions u_ε at the singular junction is provided in the following lemma.

Lemma 1.2. *If* (2)–(7) *hold, there exists a unique function* $\overline{U} : D^- \to \mathbb{R}$ *such that*

$$\begin{cases} \overline{U} \in \bigcup_{R>0} C^2(\overline{D^- \setminus B_R^-}), & \overline{U} \in \bigcup_{R>0} \mathcal{H}_R^-, \\ -\Delta \overline{U} = \lambda_{k_0}(D^+) p\, \overline{U}, & \text{in } D^-, \\ \overline{U} = 0, & \text{on } \partial D^- \setminus \{\mathbf{0}\}, \\ \lambda^{N-1}\overline{U}(\lambda\theta) \underset{\lambda\to 0^+}{\longrightarrow} \Psi^-(\theta), & \text{in } C^0(\mathbb{S}^{N-1}_-). \end{cases} \tag{24}$$

Our main result is the following theorem describing the behavior as $\varepsilon \to 0^+$ of u_ε at the junction $\mathbf{0} = (0, \dots, 0)$.

Theorem 1.3. *Let us assume* (2)–(7) *hold and let* u_ε *as in* (9). *Then*

$$\frac{e^{\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}} u_\varepsilon}{\varepsilon^N} \to \left(\int_{\mathbb{S}^{N-1}_-} \widehat{\Phi}(\theta)\Psi^-(\theta)\, d\sigma \right) \left(\int_\Sigma \Phi(1,x')\psi_1^\Sigma(x')\,dx' \right) \left(\frac{\partial u_0}{\partial x}(\mathbf{e}_1) \right) \overline{U} \tag{25}$$

as $\varepsilon \to 0^+$ *strongly in* $\mathcal{H}_t^-$ *for every* $t > 0$ *and in* $C^2(\overline{B_{t_2}^- \setminus B_{t_1}^-})$ *for all* $0 < t_1 < t_2$, *where* Φ *and* $\widehat{\Phi}$ *are defined in* (16) *and* (21) *respectively, and* $\overline{U}$ *is as in Lemma 1.2.*

The paper is organized as follows. In section 2 we improve Theorem 1.1 ruling out dependance on subsequences and prove Lemma 1.2 completely classifying the limit profile at the left junction. In section 3 we describe the asymptotic behavior of the normalization of Theorem 1.1 and prove Theorem 1.3; to this aim we first evaluate the asymptotic behavior of the denominator of the Almgren quotient at a fixed point in the corridor, then at ε-distance from the left junction in the corridor, and finally at a fixed distance from the left junction in D^-.

2. Independence of the subsequence

A deep insight into [17] highlights how the dependence on the subsequences in Theorem 1.1 is *a priori* given by two different facts: on one hand, the convergence to the limit profile in the blow-up analysis at the right junction up to subsequences and, on the other hand, the possible multiplicity of the limit profiles at the left junction (named U throughout [17]). In this section we rule out both such occurrences.

2.1. **Independence in the blow-up limit on the right.** The first improvement concerns Lemma 4.1 in [17]. We recall some notation for the sake of clarity.
Let us define

$$\widetilde{u}_\varepsilon : \widetilde{\Omega}^\varepsilon \to \mathbb{R}, \quad \widetilde{u}_\varepsilon(x) = \frac{1}{\varepsilon} u_\varepsilon\big(\mathbf{e}_1 + \varepsilon(x - \mathbf{e}_1)\big), \tag{26}$$

where

$$\widetilde{\Omega}^\varepsilon := \mathbf{e}_1 + \frac{\Omega^\varepsilon - \mathbf{e}_1}{\varepsilon} = \{x \in \mathbb{R}^N : \mathbf{e}_1 + \varepsilon(x - \mathbf{e}_1) \in \Omega^\varepsilon\}. \tag{27}$$

For all $R > 1$, let $\mathcal{H}_R^+$ be the completion of $C_c^\infty\big(((-\infty,1) \times \mathbb{R}^{N-1}) \cup \overline{B_R^+}\big)$ with respect to the norm $\big(\int_{((-\infty,1]\times\mathbb{R}^{N-1})\cup \overline{B_R^+}} |\nabla v|^2 dx\big)^{1/2}$, i.e. $\mathcal{H}_R^+$ is the space of functions with finite energy in $((-\infty,1] \times \mathbb{R}^{N-1}) \cup \overline{B_R^+}$ vanishing on $\{(1,x') \in \mathbb{R} \times \mathbb{R}^{N-1} : |x'| \geqslant R\}$.

Lemma 2.1. ([17, Lemma 4.1 and Corollary 4.4]) *For every sequence* $\varepsilon_n \to 0^+$ *there exist a subsequence* $\{\varepsilon_{n_k}\}_k$ *and a constant* $\widetilde{C} > 0$ *such that* $\widetilde{u}_{\varepsilon_{n_k}} \to \widetilde{C}\Phi$ *strongly in* $\mathcal{H}_R^+$ *for every* $R > 2$ *and in* $C^2(\overline{B_{r_2}^+ \setminus B_{r_1}^+})$ *for all* $1 < r_1 < r_2$, *where* Φ *is the unique solution to problem* (16).

We are now able to prove that the limit $\widetilde{C}\Phi$ does not depend on the subsequence.

Lemma 2.2. *Let Φ, $\Psi^+ = \frac{\theta_1}{\Upsilon_N}$, and Υ_N be as in* (16)*,* (15)*, and* (13) *respectively. Then, for every $r > 1$, the following identity holds true*

$$\frac{r^N}{r^N - 1}\left(\frac{1}{r}\int_{\mathbb{S}^{N-1}_+} \Phi(\mathbf{e}_1 + r\theta)\Psi^+(\theta)\, d\sigma(\theta) - \int_{\mathbb{S}^{N-1}_+} \Phi(\mathbf{e}_1 + \theta)\Psi^+(\theta)\, d\sigma(\theta)\right)$$
$$= \Upsilon_N - \int_{\mathbb{S}^{N-1}_+} \Phi(\mathbf{e}_1 + \theta)\Psi^+(\theta)\, d\sigma(\theta).$$

Proof. For all $r > 1$, let us define

$$v(r) = \int_{\mathbb{S}^{N-1}_+} \Phi(\mathbf{e}_1 + r\theta)\Psi^+(\theta)\, d\sigma(\theta).$$

From (16) and Lemma A.1, v satisfies

$$\left(r^{N+1}\left(\frac{v}{r}\right)'\right)' = 0, \quad \text{in } (1, +\infty), \tag{28}$$

hence, by integration, there exists $C \in \mathbb{R}$ such that

$$\frac{v(r)}{r} = v(1) + \frac{C}{N}(1 - r^{-N}), \quad \text{for all } r \in (1, +\infty). \tag{29}$$

From (17) we deduce that $\frac{v(r)}{r} \to \int_{\mathbb{S}^{N-1}_+} \theta_1 \Psi^+(\theta)\, d\sigma(\theta) = \frac{1}{\Upsilon_N}\int_{\mathbb{S}^{N-1}_+} \theta_1^2\, d\sigma(\theta) = \Upsilon_N$ as $r \to +\infty$. Hence, passing to the limit as $r \to +\infty$ in (29), we obtain that $\Upsilon_N = v(1) + \frac{C}{N}$, i.e. $\frac{C}{N} = \Upsilon_N - v(1)$. Then (29) becomes

$$\frac{v(r)}{r} = v(1)r^{-N} + \Upsilon_N(1 - r^{-N}), \quad \text{for all } r \in (1, +\infty),$$

which directly gives the conclusion. □

Proposition 2.3. *Let $\{\varepsilon_n\}_n$, $\{\varepsilon_{n_k}\}_k$, and $\widetilde{C}$ be as in Lemma 2.1. Then*

$$\widetilde{C} = \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1).$$

Proof. For all $r \in (\varepsilon, 3)$ let us define

$$\varphi_\varepsilon(r) = \int_{\mathbb{S}^{N-1}_+} u_\varepsilon(\mathbf{e}_1 + r\theta)\Psi^+(\theta)\, d\sigma(\theta).$$

From (9), (3), and Lemma A.1 it follows that φ_ε satisfies

$$\left(r^{N+1}\left(\frac{\varphi_\varepsilon(r)}{r}\right)'\right)' = 0, \quad \text{in } (\varepsilon, 3),$$

hence there exists a constant c_ε (depending on ε but independent of r) such that

$$\left(\frac{\varphi_\varepsilon(r)}{r}\right)' = \frac{c_\varepsilon}{r^{N+1}}, \quad \text{in } (\varepsilon, 3),$$

Integration of the previous equation in $(R\varepsilon, 1)$ for a fixed $R \in (1, 1/\varepsilon)$ yields

$$\frac{\varphi_\varepsilon(R\varepsilon)}{R\varepsilon} = \varphi_\varepsilon(1) + \frac{c_\varepsilon}{N}\left(1 - (R\varepsilon)^{-N}\right). \tag{30}$$

On the other hand, integration over $(\varepsilon, \xi\varepsilon)$ provides

$$\frac{\varphi_\varepsilon(\xi\varepsilon)}{\xi\varepsilon} - \frac{\varphi_\varepsilon(\varepsilon)}{\varepsilon} = \frac{c_\varepsilon}{N\varepsilon^N}\left(1 - \xi^{-N}\right), \quad \text{for all } \xi \in \left(1, \frac{3}{\varepsilon}\right). \tag{31}$$

From (26) it follows that

$$\frac{\varphi_\varepsilon(\xi\varepsilon)}{\xi\varepsilon} = \frac{1}{\xi}\int_{\mathbb{S}^{N-1}_+} \widetilde{u}_\varepsilon(\mathbf{e}_1 + \xi\theta)\Psi^+(\theta)\, d\sigma(\theta)$$

and (31) becomes

$$\frac{1}{\xi}\int_{\mathbb{S}^{N-1}_+} \widetilde{u}_\varepsilon(\mathbf{e}_1+\xi\theta)\Psi^+(\theta)\,d\sigma(\theta) - \int_{\mathbb{S}^{N-1}_+} \widetilde{u}_\varepsilon(\mathbf{e}_1+\theta)\Psi^+(\theta)\,d\sigma(\theta) = \frac{c_\varepsilon}{N\varepsilon^N}\left(1-\xi^{-N}\right).$$

Then, from Lemma 2.1,

$$\begin{aligned}\frac{c_{\varepsilon_{n_k}}}{N\varepsilon_{n_k}^N} \underset{k\to+\infty}{\longrightarrow} \frac{\widetilde{C}\xi^N}{\xi^N-1}\left(\frac{1}{\xi}\int_{\mathbb{S}^{N-1}_+}\Phi(\mathbf{e}_1+\xi\theta)\Psi^+(\theta)\,d\sigma - \int_{\mathbb{S}^{N-1}_+}\Phi(\mathbf{e}_1+\theta)\Psi^+(\theta)\,d\sigma\right)\\ = \widetilde{C}\left(\Upsilon_N - \int_{\mathbb{S}^{N-1}_+}\Phi(\mathbf{e}_1+\theta)\Psi^+(\theta)\,d\sigma\right),\end{aligned}$$

where the last identity is a consequence of Lemma 2.2. Therefore, passing to the limit along the subsequence ε_{n_k} in (30) and exploiting Lemma 2.1 and (10), we obtain that

$$\begin{aligned}\frac{\widetilde{C}}{R}\int_{\mathbb{S}^{N-1}_+}\Phi(\mathbf{e}_1+R\theta)\Psi^+(\theta)\,d\sigma = \int_{\mathbb{S}^{N-1}_+} u_0(\mathbf{e}_1+\theta)\Psi^+(\theta)\,d\sigma\\ - \frac{\widetilde{C}}{R^N}\left(\Upsilon_N - \int_{\mathbb{S}^{N-1}_+}\Phi(\mathbf{e}_1+\theta)\Psi^+(\theta)\,d\sigma\right)\end{aligned}$$

for every $R > 1$. In view of Lemma 2.2, the previous identity becomes

$$\widetilde{C}\Upsilon_N = \int_{\mathbb{S}^{N-1}_+} u_0(\mathbf{e}_1+\theta)\Psi^+(\theta)\,d\sigma. \tag{32}$$

For all $r \in (0,3)$, let us define

$$w(r) = \int_{\mathbb{S}^{N-1}_+} u_0(\mathbf{e}_1+r\theta)\Psi^+(\theta)\,d\sigma(\theta).$$

From (6) and Lemma A.1, w satisfies

$$\left(r^{N+1}\left(\frac{w}{r}\right)'\right)' = 0, \quad \text{in } (0,3),$$

hence, by integration, there exist $c, d \in \mathbb{R}$ such that

$$w(r) = c\,r + d\,r^{1-N}, \quad \text{for all } r \in (0,3).$$

The fact that $u_0 \in \mathcal{D}^{1,2}(D^+)$ implies that $d = 0$. Hence

$$c = \frac{w(r)}{r} = \frac{1}{r}\int_{\mathbb{S}^{N-1}_+} u_0(\mathbf{e}_1+r\theta)\Psi^+(\theta)\,d\sigma(\theta), \quad \text{for all } r \in (0,3).$$

Moreover

$$\lim_{r\to0^+}\frac{u_0(\mathbf{e}_1+r\theta)}{r} = \nabla u_0(\mathbf{e}_1)\cdot\theta = \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)\theta_1 = \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)\Upsilon_N\Psi^+(\theta),$$

thus implying that $c = \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)\Upsilon_N$ and hence

$$\frac{1}{r}\int_{\mathbb{S}^{N-1}_+} u_0(\mathbf{e}_1+r\theta)\Psi^+(\theta)\,d\sigma(\theta) = \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)\Upsilon_N, \quad \text{for all } r \in (0,3).$$

Replacing this last relation into (32) we conclude that $\widetilde{C} = \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)$. □

Combining Lemma 2.1 and Proposition 2.3, we obtain the convergence of $\widetilde{u}_\varepsilon$ to its limit profile as $\varepsilon \to 0^+$.

Lemma 2.4. *Let $\widetilde{u}_\varepsilon$ be defined in* (26)*. Then*

$$\widetilde{u}_\varepsilon \to \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)\Phi, \quad \text{as } \varepsilon \to 0^+,$$

strongly in $\mathcal{H}^+_R$ for every $R > 2$ and in $C^2(\overline{B^+_{r_2}\setminus B^+_{r_1}})$ for all $1 < r_1 < r_2$, where Φ is the unique solution to problem (16).

2.2. **Independence in the limit profile at the left.** The second improvement about independence on subsequences concerns Proposition 6.1 in [17] and the convergence of the normalized eigenfunctions

$$U_\varepsilon(x) = \frac{u_\varepsilon(x)}{\sqrt{\int_{\Gamma^-_{\tilde k}} u_\varepsilon^2 d\sigma}} \tag{33}$$

to a universal profile (not depending on subsequences), with $\tilde k$ as in Theorem 1.1 and $\Gamma^-_{\tilde k}$ as in (12). We notice that, for ε small, U_ε solves

$$\begin{cases} -\Delta U_\varepsilon = \lambda_\varepsilon p\, U_\varepsilon, & \text{in } \Omega^\varepsilon, \\ U_\varepsilon = 0, & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{34}$$

and

$$\int_{\Gamma^-_{\tilde k}} U_\varepsilon^2 d\sigma = 1. \tag{35}$$

The following proposition summarizes the results of [17, Propositions 6.1 and 6.5].

Proposition 2.5. ([17, Propositions 6.1 and 6.5]) *For every sequence $\varepsilon_n \to 0^+$ there exist a subsequence $\{\varepsilon_{n_k}\}_k$, a function $U \in C^2(D^-) \cup \big(\bigcup_{t>0} \mathcal{H}^-_t\big)$, and $\beta < 0$ such that*

(i) *$U_{\varepsilon_{n_k}} \to U$ strongly in $\mathcal{H}^-_t$ for all $t > 0$ and in $C^2(\overline{B^-_{t_2} \setminus B^-_{t_1}})$ for all $0 < t_1 < t_2$;*
(ii) *$\int_{\Gamma^-_{\tilde k}} U^2 d\sigma = 1$;*
(iii) *U solves*

$$\begin{cases} -\Delta U(x) = \lambda_{k_0}(D^+) p(x) U(x), & \text{in } D^-, \\ U = 0, & \text{on } \partial D^- \setminus \{\mathbf{0}\}; \end{cases} \tag{36}$$

(iv) *$\lambda^{N-1} U(\lambda x) \to \beta \frac{x_1}{|x|^N}$ as $\lambda \to 0^+$ strongly in $\mathcal{H}^-_t$ for every $t > 0$ and in $C^2(\overline{B^-_{t_2} \setminus B^-_{t_1}})$ for all $0 < t_1 < t_2$.*

To prove that the limit profile U in Proposition 2.5 does not depend on the subsequence, we are going to show that it is necessarily a multiple of the universal profile $\overline{U}$ provided by Lemma 1.2; normalization (35) will univocally determine the multiplicative constant.

A key tool in the proof of Lemma 1.2 is the following uniform coercivity type estimate for the quadratic form associated to equation (36), whose validity is strongly related to the nondegeneracy condition (5). We denote

$$\Omega_r := D^- \setminus \overline{B^-_{-r}} \quad \text{for all } r < 0. \tag{37}$$

Lemma 2.6. *Let $u \in C^2(D^-) \cup \big(\bigcup_{t>0} \mathcal{H}^-_t\big)$ be a solution to the problem*

$$\begin{cases} -\Delta u(x) = \lambda_{k_0}(D^+) p(x) u(x), & \text{in } D^-, \\ u = 0, & \text{on } \partial D^- \setminus \{\mathbf{0}\}, \end{cases} \tag{38}$$

where $p \in L^{N/2}(D^-) \setminus \{0\}$ and $\lambda_{k_0}(D^+) \notin \sigma_p(D^-)$. For any $f \in L^{N/2}(D^-)$ and $M > 0$ there exists $R_{M,f} > 0$ such that, for every $r \in (0, R_{M,f})$,

$$\int_{\Omega_{-r}} |\nabla u(x)|^2 \, dx \geqslant M \int_{\Omega_{-r}} |f(x)| \, u^2(x) \, dx. \tag{39}$$

Proof. The proof is similar to the proof of Lemma 3.6 in [17] and hence is omitted. □

We are now in position to prove Lemma 1.2.
Proof of Lemma 1.2. The existence of a solution to (24) follows from Proposition 2.5. To prove uniqueness, we argue by contradiction and assume that there exist U_1, U_1 solutions to (24)

such that $U_1 \neq U_2$. The difference $V = U_1 - U_2$ satisfies

$$\begin{cases} V \in \bigcup_{R>0} C^2(\overline{D^- \setminus B_R^-}), & V \in \bigcup_{R>0} \mathcal{H}_R^-, \\ -\Delta V = \lambda_{k_0}(D^+)pV, & \text{in } D^-, \\ V = 0, & \text{on } \partial D^- \setminus \{\mathbf{0}\}, \\ \lambda^{N-1} V(\lambda\theta) \to 0, & \text{in } C^0(\mathbb{S}_-^{N-1}). \end{cases} \tag{40}$$

Let us fix $\delta > 0$. From Lemma 2.6, there exists $R_\delta > 0$ such that

$$\|2p + x \cdot \nabla p\|_{L^{3N}\left(B^-_{R_\delta}\right)} \leqslant \left(\frac{2N}{\omega_{N-1}}\right)^{\frac{5}{3N}} \frac{C_S \delta}{8\lambda_{k_0}(D^+)}, \tag{41}$$

$$\int_{\Omega_r} \Big(|\nabla V|^2 - \lambda_{k_0}(D^+)pV^2\Big) dx \geqslant \frac{1}{2}\int_{\Omega_r} |\nabla V|^2 dx, \tag{42}$$

$$\int_{\Omega_r} \Big(|\nabla V|^2 - \lambda_{k_0}(D^+)pV^2\Big) dx \geqslant \tfrac{4\lambda_{k_0}(D^+)}{\delta}\int_{\Omega_r} |2p + x\cdot\nabla p| V^2 dx, \tag{43}$$

for all $r \in (-R_\delta, 0)$, with C_S as in (11). More precisely, (42) is obtained first choosing $M = 2$ and $f = \lambda_{k_0}(D^+)p$ in Lemma 2.6; then by taking $M = \frac{8\lambda_{k_0}(D^+)}{\delta}$ and $f = |2p + x \cdot \nabla p|$ in Lemma 2.6 and using (42), we obtain

$$\frac{8\lambda_{k_0}(D^+)}{\delta}\int_{\Omega_r} |2p + x\cdot\nabla p| V^2 dx \leqslant \int_{\Omega_r} |\nabla V|^2 dx \leqslant 2\int_{\Omega_r} \Big(|\nabla V|^2 - \lambda_{k_0}(D^+)pV^2\Big) dx$$

thus proving (43).

For all $t > 0$, let us define

$$D_V(t) = \frac{1}{t^{N-2}}\int_{\Omega_{-t}} \Big(|\nabla V(x)|^2 - \lambda_{k_0}(D^+)p(x)V^2(x)\Big) dx, \tag{44}$$

$$H_V(t) = \frac{1}{t^{N-1}}\int_{\Gamma_t^-} V^2(x)\, d\sigma = \int_{\mathbb{S}_-^{N-1}} V^2(t\theta)\, d\sigma(\theta). \tag{45}$$

Direct calculations (see [17, Lemma 3.15] for details) yield

$$D_V'(t) = -\frac{2}{t^{N-2}}\int_{\Gamma_t^-} \left|\frac{\partial V}{\partial \nu}\right|^2 d\sigma - \frac{\lambda_{k_0}(D^+)}{t^{N-1}}\int_{\Omega_{-t}} (2p(x) + x\cdot\nabla p(x))V^2(x) dx.$$

From (40), we have that

$$\int_{\Omega_{-t}} \Big(|\nabla V(x)|^2 - \lambda_{k_0}(D^+)p(x)V^2(x)\Big) dx = -\int_{\Gamma_t^-} V\,\frac{\partial V}{\partial \nu}\, d\sigma$$

$\nu = \nu(x) = \frac{x}{|x|}$ which, by Schwarz's inequality, Lemmas 2.6, and the Poincaré type inequality

$$\frac{1}{t^{N-2}}\int_{\Omega_{-t}} |\nabla v(x)|^2 dx \geqslant \frac{N-1}{t^{N-1}}\int_{\Gamma_t^-} v^2 d\sigma, \quad \text{for all } t > 0 \text{ and } v \in \mathcal{H}_t^- \tag{46}$$

proved in [17, Lemma 3.4], for every $t \in (0, R_\delta)$, up to shrinking $R_\delta > 0$, yields

$$\begin{aligned} \int_{\Gamma_t^-} \left|\frac{\partial V}{\partial \nu}\right|^2 d\sigma &\geqslant \frac{\int_{\Omega_{-t}}\big(|\nabla V|^2 - \lambda_{k_0}(D^+)pV^2\big)dx}{\int_{\Gamma_t^-} V^2\, d\sigma}\int_{\Omega_{-t}} \Big(|\nabla V|^2 - \lambda_{k_0}(D^+)pV^2\Big) dx \\ &\geqslant \frac{(1-\frac{\delta_0}{2})(N-1)}{t}\int_{\Omega_{-t}} \Big(|\nabla V|^2 - \lambda_{k_0}(D^+)pV^2\Big) dx, \end{aligned} \tag{47}$$

with $\delta_0 = \frac{2N-5}{4(N-1)} \in (0,1)$. From Lemma 2.6 and (47), up to shrinking R_δ, there holds

$$\begin{aligned} -\frac{d}{dt} D_V(t) &\geqslant \frac{2(1-\delta_0)(N-1)}{t^{N-1}}\int_{\Omega_{-t}} \Big(|\nabla V(x)|^2 - \lambda_{k_0}(D^+)p(x)V^2(x)\Big) dx \\ &= \frac{2(1-\delta_0)(N-1)}{t} D_V(t) \end{aligned}$$

for all $t \in (0, R_\delta)$. Integrating the above inequality, we obtain that

$$D_V(t_1) \geqslant \left(\frac{t_2}{t_1}\right)^{N+\frac{1}{2}} D_V(t_2) \quad \text{for every } t_1, t_2 \in (0, R_{\delta_0}) \text{ such that } t_1 < t_2. \tag{48}$$

Let us define $\mathcal{N}_V : (-\infty, 0) \to \mathbb{R}$ as

$$\mathcal{N}_V(r) := \frac{(-r)\int_{\Omega_r} \Big(|\nabla V(x)|^2 - \lambda_{k_0}(D^+)p(x)V^2(x)\Big)dx}{\int_{\Gamma^-_{-r}} V^2(x)\,d\sigma}.$$

Direct calculations (see [17, Lemmas 3.15 and 6.2] for details in a similar case) yield

$$\frac{d}{dr}\mathcal{N}_V(r) = \nu_1(r) + \nu_2(r), \quad r \in (-\infty, 0),$$

where

$$\nu_1(r) = -2r \frac{\Big(\int_{\Gamma^-_{-r}} \big|\frac{\partial V}{\partial \nu}\big|^2 d\sigma\Big)\Big(\int_{\Gamma^-_{-r}} V^2(x)\,d\sigma\Big) - \Big(\int_{\Gamma^-_{-r}} V \frac{\partial V}{\partial \nu} d\sigma\Big)^2}{\Big(\int_{\Gamma^-_{-r}} V^2(x)\,d\sigma\Big)^2} \geqslant 0$$

by Schwarz's inequality and

$$\nu_2(r) = \lambda_{k_0}(D^+)\frac{\int_{\Omega_r}(2p(x) + x\cdot\nabla p(x))V^2(x)dx}{\int_{\Gamma^-_{-r}} V^2(x)d\sigma}.$$

Hence, for all $r \in (-R_\delta, 0)$,

$$\frac{\mathcal{N}_V'(r)}{\mathcal{N}_V(r)} \geqslant -\mathcal{I}(r)$$

where $\mathcal{I}(r) = \frac{\lambda_{k_0}(D^+)}{-r}\big(I(-r) + II(-r)\big)$ with

$$I(t) = \frac{\int_{\Omega_{-t}\setminus\Omega_{-t^{3/5}}} |2p + x\cdot\nabla p|V^2 dx}{\int_{\Omega_{-t}}\Big(|\nabla V|^2 - \lambda_{k_0}(D^+)pV^2\Big)dx}, \quad II(t) = \frac{\int_{\Omega_{-t^{3/5}}} |2p + x\cdot\nabla p|V^2 dx}{\int_{\Omega_{-t}}\Big(|\nabla V|^2 - \lambda_{k_0}(D^+)pV^2\Big)dx}.$$

By Hölder inequality, (42), (11), and (41), $I(t)$ can be estimated as

$$\begin{aligned} I(t) &\leqslant \|2p + x\cdot\nabla p\|_{L^{3N}\left(B^-_{t^{3/5}}\right)} \Big|\Omega_{-t}\setminus\Omega_{-t^{3/5}}\Big|^{\frac{5}{3N}} \frac{\Big(\int_{\Omega_{-t}} V^{2^*}dx\Big)^{2/2^*}}{\int_{\Omega_{-t}}\Big(|\nabla V|^2 - \lambda_{k_0}(D^+)pV^2\Big)dx} \\ &\leqslant \frac{2}{C_S}\left(\frac{\omega_{N-1}}{2N}\right)^{\frac{5}{3N}} \|2p + x\cdot\nabla p\|_{L^{3N}\left(B^-_{\tilde{R}_\delta}\right)} t \leqslant \frac{\delta}{4\lambda_{k_0}(D^+)}\,t \end{aligned}$$

for all $t \in (0, R_\delta^{5/3})$. On the other hand, from (43) and (48)

$$\begin{aligned} II(t) &= \frac{\int_{\Omega_{-t^{3/5}}} |2p + x\cdot\nabla p|V^2 dx}{\int_{\Omega_{-t^{3/5}}}\big(|\nabla V|^2 - \lambda_{k_0}(D^+)pV^2\big)dx}\,\frac{\int_{\Omega_{-t^{3/5}}}\big(|\nabla V|^2 - \lambda_{k_0}(D^+)pV^2\big)dx}{\int_{\Omega_{-t}}\big(|\nabla V|^2 - \lambda_{k_0}(D^+)pV^2(x)\big)dx} \\ &\leqslant \frac{\delta}{4\lambda_{k_0}(D^+)}\,t^{-\frac{2}{5}(N-2)}\frac{D_V(t^{3/5})}{D_V(t)} \leqslant \frac{\delta}{4\lambda_{k_0}(D^+)}\,t \end{aligned}$$

for all $t \in (0, R_\delta^{5/3})$. Combining the previous estimates we obtain that $\mathcal{I}(r) \leqslant \delta$ and hence

$$\frac{\mathcal{N}_V'(r)}{\mathcal{N}_V(r)} \geqslant -\delta$$

for all $r \in (-R_\delta^{5/3}, 0)$.

Since $\mathcal{N}_V(r) \geqslant 0$ for r close enough to 0 by (42), we obtain $\mathcal{N}_V'(r) + \delta\mathcal{N}_V(r) \geqslant 0$, therefore the function $e^{\delta r}\mathcal{N}_V(r)$ is monotone nondecreasing in a left neighborhood of the origin, so that in particular $\mathcal{N}_V(r)$ admits a limit as $r \to 0$. We observe that Lemma 2.6 and (46) ensure that $\lim_{r\to 0^-} \mathcal{N}_V(r) \geqslant N-1$. We claim that

$$\lim_{r\to 0^-} \mathcal{N}_V(r) = N-1. \tag{49}$$

To prove claim (49), we assume by contradiction that there exist $\delta > 0$ and $\bar{r} < 0$ such that $\mathcal{N}_V(r) \geqslant N-1+\delta$ for all $r \in (\bar{r}, 0)$. If we integrate the inequality

$$\frac{H_V'(t)}{H_V(t)} = -\frac{2}{t}\mathcal{N}_V(-t) \leqslant -\frac{2}{t}(N-1+\delta)$$

over $(t, -\bar{r})$, we obtain

$$t^{2(N-1+\delta)} H(t) \geqslant const > 0. \tag{50}$$

On the other hand, from (45) and (40) it follows that $H_V(t) = o(t^{2(1-N)})$ as $t \to 0^+$, thus contradicting (50) and proving claim (49).

From (49), arguing as in [17, Lemma 6.2, Lemma 6.4, Proposition 6.5] one can prove that, if $V \not\equiv 0$, then V would satisfy

$$\lambda^{N-1} V(\lambda\theta) \to c\,\Psi^-(\theta) \quad \text{as } \lambda \to 0^+ \text{ in } C^0(\mathbb{S}_-^{N-1})$$

for some $c \neq 0$, thus contradicting (40). Then $V \equiv 0$. □

Remark 2.7. The previous proof does not require assumption (3). More generally, the same argument applies replacing p with any $L^{N/2}(D^-)$-function and $\lambda_{k_0}(D^+)$ with any $\lambda_0 \notin \sigma_p(D^-)$.

Combining Proposition 2.5 with Lemma 1.2, we can prove that, due to the universality of the limit profile, the convergence of U_ε does not depend on subsequences.

Proposition 2.8. *Let U_ε be defined in* (33) *and $\overline{U}$ as in Lemma 1.2. Then*

$$U_\varepsilon \to \frac{\overline{U}}{\sqrt{\int_{\Gamma_{\tilde{k}}^-} \overline{U}^2 d\sigma}}, \quad \text{as } \varepsilon \to 0^+,$$

strongly in $\mathcal{H}_t^-$ for every $t > 0$ and in $C^2(\overline{B_{t_2}^- \setminus B_{t_1}^-})$ for all $0 < t_1 < t_2$.

Proof. Let $\varepsilon_n \to 0^+$. From Proposition 2.5, there exist a subsequence $\{\varepsilon_{n_k}\}_k$, $\beta < 0$, and a function $U \in C^2(D^-) \cup \big(\bigcup_{t>0} \mathcal{H}_t^-\big)$ such that $U_{\varepsilon_{n_k}} \to U$ strongly in $\mathcal{H}_t^-$ for all $t > 0$ and in $C^2(\overline{B_{t_2}^- \setminus B_{t_1}^-})$ for all $0 < t_1 < t_2$, U solves (36), and $\lambda^{N-1}U(\lambda\theta) \to -\beta\,\Psi^-$ in $C^0(\mathbb{S}_-^{N-1})$. From Lemma 1.2 it follows that $\frac{U}{-\beta} = \overline{U}$, whereas part (ii) of Proposition 2.5 implies that $\beta = -\big(\int_{\Gamma_{\tilde{k}}^-} \overline{U}^2 d\sigma\big)^{-1/2}$. Hence the limit U depends neither on the sequence $\{\varepsilon_n\}_n$ nor on the subsequence $\{\varepsilon_{n_k}\}_k$, thus concluding the proof. □

3. Asymptotic behavior of the normalization

As already mentioned, in this section the technique is proceeding by steps. Starting from the right, where we can exploit the strong convergence (10), we first evaluate the asymptotic behavior of the denominator of the Almgren quotient at a fixed point in the corridor, then at ε-distance from the left junction in the corridor, and finally at a fixed distance from the left junction in D^-.

Following [17], for every $r \in (0,1)$ and $t > \varepsilon$ we define

$$\widetilde{H}_\varepsilon(r) := \int_\Sigma u_\varepsilon^2(r, \varepsilon x')dx' = \varepsilon^{1-N}\int_{\Sigma_\varepsilon} u_\varepsilon^2(r,x')dx' = \varepsilon^{1-N} H_\varepsilon^c(r), \tag{51}$$

$$H_\varepsilon^-(t) := \frac{1}{t^{N-1}}\int_{\Gamma_t^-} u_\varepsilon^2 d\sigma, \tag{52}$$

where Γ_t^- is defined in (12) and $\Sigma_\varepsilon := \{x' \in \mathbb{R}^{N-1} : \frac{x'}{\varepsilon} \in \Sigma\}$. We also define, for every $r > 0$,

$$\widehat{\Omega}_r = D^- \cup \{(x_1, x') \in T_1^+ : x_1 < r\}$$

and $\mathcal{H}_r$ as the completion of

$$\mathcal{D}_r := \big\{ v \in C^\infty(\overline{\widehat{\Omega}_r}) : \operatorname{supp} v \Subset \widehat{D} \big\}$$

with respect to the norm $\big(\int_{\widehat{\Omega}_r} |\nabla v|^2 dx\big)^{1/2}$, i.e. $\mathcal{H}_r$ is the space of functions with finite energy in $\widehat{\Omega}_r$ vanishing on $\{(x_1, x') \in \partial\widehat{\Omega}_r : x_1 < r\}$.

Lemma 3.1. *Let us fix $x_0 \in (0,1)$ and define*

$$w_\varepsilon : \widetilde{\Omega}_{x_0,\varepsilon} \to \mathbb{R}, \quad w_\varepsilon(x_1, x') = \frac{u_\varepsilon\big(\varepsilon(x_1 - 1) + x_0, \varepsilon x'\big)}{\big(\widetilde{H}_\varepsilon(x_0)\big)^{1/2}} \tag{53}$$

where

$$\begin{aligned} \widetilde{\Omega}_{x_0,\varepsilon} := &\left\{ (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : \ x_1 < 1 - \frac{x_0}{\varepsilon} \right\} \\ &\cup \left\{ (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : \ 1 - \frac{x_0}{\varepsilon} \leqslant x_1 \leqslant 1 + \frac{1 - x_0}{\varepsilon}, \ x' \in \Sigma \right\} \\ &\cup \left\{ (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : \ x_1 > 1 + \frac{1 - x_0}{\varepsilon} \right\}. \end{aligned}$$

Then

$$w_\varepsilon(x_1, x') \to e^{\sqrt{\lambda_1(\Sigma)}(x_1 - 1)} \psi_1^\Sigma(x') \quad \text{in } C^2_{\rm loc}(\overline{T_1}) \text{ and in } \mathcal{H}_r \text{ for every } r \in \mathbb{R},$$

as $\varepsilon \to 0^+$, where $\lambda_1(\Sigma)$ the first eigenvalue of $-\Delta_{x'}$ on Σ under null Dirichlet boundary conditions, ψ_1^Σ is the corresponding positive $L^2(\Sigma)$-normalized eigenfunction (see (20)*), and T_1 is defined in* (19).

Proof. For every $r \in \mathbb{R}$, we define

$$\widetilde{\Omega}_r^{x_0,\varepsilon} := \{(x_1, x') \in \widetilde{\Omega}_{x_0,\varepsilon} : \ x_1 < r\}.$$

Let us fix $r > 1$. Then, for ε sufficiently small, $\varepsilon(r-1) + x_0 \in (0,1)$. By direct computations we have that

$$\frac{\int_{\widetilde{\Omega}_r^{x_0,\varepsilon}} \big(|\nabla w_\varepsilon(x)|^2 - \varepsilon^2 \lambda_\varepsilon p(\varepsilon(x - \mathbf{e}_1) + x_0 \mathbf{e}_1) w_\varepsilon^2(x)\big)\, dx}{\int_\Sigma w_\varepsilon^2(r, x')\, dx'} = \mathcal{N}_\varepsilon(\varepsilon(r-1) + x_0) \tag{54}$$

where, for all $t \in (0,1)$,

$$\mathcal{N}_\varepsilon(t) = \frac{\varepsilon \int_{\{(x_1,x') \in \Omega^\varepsilon : x_1 < t\}} \Big(|\nabla u_\varepsilon(x)|^2 - \lambda_\varepsilon p(x) u_\varepsilon^2(x)\Big) dx}{H_\varepsilon^c(t)}.$$

From [17, Lemma 3.21, Lemma 4.5, and Corollary 2.6] it follows that for every $\delta > 0$ there exists $\varepsilon_{\delta,r,x_0} > 0$ (depending on δ, r, and x_0) such that

$$\mathcal{N}_\varepsilon(\varepsilon(r-1) + x_0) \leqslant (1 + \delta)\sqrt{\lambda_1(\Sigma)} \quad \text{for all } \varepsilon \in (0, \varepsilon_{\delta,r,x_0}). \tag{55}$$

Furthermore, [17, Lemma 3.6] implies that, up to shrinking $\varepsilon_{\delta,r,x_0} > 0$, for all $\varepsilon \in (0, \varepsilon_{\delta,r,x_0})$,

$$\begin{aligned} \lambda_\varepsilon \varepsilon^2 \int_{\widetilde{\Omega}_r^{x_0,\varepsilon}} p(\varepsilon(x - \mathbf{e}_1) + x_0 \mathbf{e}_1) |w_\varepsilon(x)|^2\, dx &= \frac{\varepsilon^{2-N} \lambda_\varepsilon}{\widetilde{H}_\varepsilon(x_0)} \int_{\Omega^\varepsilon_{\varepsilon(r-1)+x_0}} p(x) u_\varepsilon^2(x) dx \\ &\leqslant \delta \frac{\varepsilon^{2-N}}{\widetilde{H}_\varepsilon(x_0)} \int_{\Omega^\varepsilon_{\varepsilon(r-1)+x_0}} |\nabla u_\varepsilon(x)|^2 dx = \delta \int_{\widetilde{\Omega}_r^{x_0,\varepsilon}} |\nabla w_\varepsilon(x)|^2\, dx, \end{aligned} \tag{56}$$

where $\Omega^\varepsilon_{\varepsilon(r-1)+x_0} = \{(x_1, x') \in \Omega^\varepsilon : x_1 < \varepsilon(r-1) + x_0\}$. Collecting (54), (55), and (56), and recalling (51), we obtain that

$$\int_{\widetilde{\Omega}_r^{x_0,\varepsilon}} |\nabla w_\varepsilon(x)|^2\, dx \leqslant \frac{(1+\delta)\sqrt{\lambda_1(\Sigma)}}{1 - \delta} \frac{\widetilde{H}_\varepsilon(\varepsilon(r-1) + x_0)}{\widetilde{H}_\varepsilon(x_0)} \tag{57}$$

for all $\varepsilon \in (0, \varepsilon_{\delta,r,x_0})$. From [17, Lemma 3.20] it follows that

$$\frac{\widetilde{H}'_\varepsilon(t)}{\widetilde{H}_\varepsilon(t)} = \frac{2}{\varepsilon}\mathcal{N}_\varepsilon(t), \quad \text{for all } t \in (0,1).$$

Integrating the above identity and using again Lemma 3.21 of [17] and (55), it follows that, up to shrinking $\varepsilon_{\delta,r,x_0} > 0$, for all $\varepsilon \in (0, \varepsilon_{\delta,r,x_0})$,

$$\frac{\widetilde{H}_\varepsilon(\varepsilon(r-1)+x_0)}{\widetilde{H}_\varepsilon(x_0)} \leqslant e^{2(r-1)e^\delta(1+\delta)\sqrt{\lambda_1(\Sigma)}}. \tag{58}$$

In view of (57) and (58), we have proved that for every $r > 1$ there exists $\varepsilon_{r,x_0} > 0$ such that

$$\{w_\varepsilon\}_{\varepsilon \in (0,\varepsilon_{r,x_0})} \text{ is bounded in } \mathcal{H}_r. \tag{59}$$

Let $\varepsilon_n \to 0^+$. From (59) and a diagonal process, we deduce that there exist a subsequence $\varepsilon_{n_k} \to 0^+$ and some $w \in \bigcup_{r>1} \mathcal{H}_r$ such that $w_{\varepsilon_{n_k}} \rightharpoonup w$ weakly in $\mathcal{H}_r$ for every $r > 1$. In particular $w_{\varepsilon_{n_k}} \to w$ a.e., so that $w \equiv 0$ in $\mathbb{R}^N \setminus T_1$. Passing to the weak limit in

$$\begin{cases} -\Delta w_\varepsilon = \varepsilon^2 \lambda_\varepsilon p(\varepsilon(x-\mathbf{e}_1)+x_0\mathbf{e}_1)w_\varepsilon, & \text{in } \widetilde{\Omega}_{x_0,\varepsilon}, \\ w_\varepsilon = 0, & \text{on } \partial\widetilde{\Omega}_{x_0,\varepsilon}, \end{cases} \tag{60}$$

along the subsequence ε_{n_k} we obtain that w satisfies

$$\begin{cases} -\Delta w = 0, & \text{in } T_1, \\ w = 0, & \text{on } \partial T_1. \end{cases} \tag{61}$$

By classical elliptic estimates, we also have that $w_{\varepsilon_{n_k}} \to w$ in $C^2_{\rm loc}(\overline{T_1})$. Therefore, multiplying (61) by w and integrating in $T_{1,r}$ where $T_{1,r} := \{(x_1,x') \in T_1 : x_1 < r\}$, we obtain

$$\begin{aligned} \int_\Sigma \frac{\partial w_{\varepsilon_{n_k}}}{\partial x_1}(r,x')w_{\varepsilon_{n_k}}(r,x')\,dx' &\to \int_\Sigma \frac{\partial w}{\partial x_1}(r,x')w(r,x')\,dx' \\ &= \int_{T_{1,r}} |\nabla w(x)|^2 dx \quad \text{as } k \to +\infty. \end{aligned} \tag{62}$$

On the other hand, multiplication of (60) by $w_{\varepsilon_{n_k}}$ and integration by parts over $\widetilde{\Omega}_r^{x_0,\varepsilon_{n_k}}$ yield

$$\begin{aligned} \int_{\widetilde{\Omega}_r^{x_0,\varepsilon_{n_k}}} |\nabla w_{\varepsilon_{n_k}}(x)|^2 dx &= \int_\Sigma \frac{\partial w_{\varepsilon_{n_k}}}{\partial x_1}(r,x')w_{\varepsilon_{n_k}}(r,x')\,dx' \\ &\quad + \lambda_{\varepsilon_{n_k}}\varepsilon_{n_k}^2 \int_{\widetilde{\Omega}_r^{x_0,\varepsilon_{n_k}}} p(\varepsilon(x-\mathbf{e}_1)+x_0\mathbf{e}_1)|w_{\varepsilon_{n_k}}(x)|^2\,dx. \end{aligned} \tag{63}$$

From (56) it follows that

$$\varepsilon_{n_k}^2 \int_{\widetilde{\Omega}_r^{x_0,\varepsilon_{n_k}}} p(\varepsilon(x-\mathbf{e}_1)+x_0\mathbf{e}_1)|w_{\varepsilon_{n_k}}(x)|^2\,dx \to 0 \quad \text{as } k \to +\infty, \tag{64}$$

which, in view of (62) and (63), implies that $\|w_{\varepsilon_{n_k}}\|_{\mathcal{H}_r} \to \|w\|_{\mathcal{H}_r}$ and then $w_{\varepsilon_{n_k}}$ converges to w strongly in $\mathcal{H}_r$ for every $r > 1$. Then, from (54), (64), and (55) we deduce that, for all $r > 1$ and $\delta > 0$,

$$\frac{\int_{T_{1,r}} |\nabla w(x)|^2\,dx}{\int_\Sigma w^2(r,x')\,dx'} \leqslant (1+\delta)\sqrt{\lambda_1(\Sigma)},$$

which implies that, for all $r > 1$,

$$\frac{\int_{T_{1,r}} |\nabla w(x)|^2\,dx}{\int_\Sigma w^2(r,x')\,dx'} \leqslant \sqrt{\lambda_1(\Sigma)}.$$

Hence, from [17, Lemma 2.5] it follows that $w(x_1,x') = C\,e^{\sqrt{\lambda_1(\Sigma)}(x_1-1)}\psi_1^\Sigma(x')$ for some constant $C \neq 0$. Thanks to the definition of w_ε, we have that

$$\int_\Sigma {w_\varepsilon}^2(1,x')dx' = 1,$$

and then $C^2 = 1$.

It remains to prove that $C > 0$ so that $C = 1$. We assume by contradiction that $C < 0$. By the convergence $w_{\varepsilon_{n_k}}(1, x') \to w(1, x')$ in $C^2(\Sigma)$, it follows that, if k is sufficiently large, then $u_{\varepsilon_{n_k}}(x_0, x') < 0$ for every $x' \in \mathbb{R}^{N-1}$ such that $\frac{x'}{\varepsilon} \in \Sigma$.

From [17, Corollary 1.3], for every r small, there exists $\varepsilon_r > 0$ such that $u_\varepsilon > 0$ on Γ_r^- for all $\varepsilon \in (0, \varepsilon_r)$. Hence there exists a sub-subsequence $\{\varepsilon_{n_{k_j}}\}_j$ such that $u_{\varepsilon_{n_{k_j}}} > 0$ on $\Gamma^-_{1/j}$. Therefore, for j large, the functions

$$v_j = \begin{cases} u_{\varepsilon_{n_{k_j}}}, & \text{in } D^- \setminus B^-_{1/j},\\ u^+_{\varepsilon_{n_{k_j}}}, & \text{in } B^-_{1/j} \cup \{(x_1, x') \in \mathcal{C}_\varepsilon : x_1 \leqslant x_0\},\\ 0, & \text{in } \{(x_1, x') \in \Omega^{\varepsilon_{n_{k_j}}} : x_1 > x_0\}, \end{cases} \quad \widetilde{v}_j := \frac{v_j}{\big(\int_{\Omega^{\varepsilon_{n_{k_j}}}} p\, v_j^2 dx\big)^{\frac12}}$$

are well-defined and belong to $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (if trivially extended to the whole $\mathbb{R}^N$).

Let $A_j = D^- \setminus B^-_{1/j}$. Then $\widetilde{v}_j$ satisfies

$$\begin{cases} -\Delta \widetilde{v}_j = \lambda_{\varepsilon_{n_{k_j}}} p \widetilde{v}_j, & \text{in } A_j,\\ \widetilde{v}_j = 0, & \text{on } \partial A_j \cap \partial D^-,\\ \int_{\mathbb{R}^N} p\, \widetilde{v}_j^2\, dx = 1. \end{cases}$$

Testing equation (9) for $\varepsilon = \varepsilon_{n_{k_j}}$ with v_j we obtain

$$\int_{\{(x_1,x')\in\Omega^{\varepsilon_{n_{k_j}}} : x_1 \leqslant x_0\}} |\nabla v_j|^2 dx = \lambda_{\varepsilon_{n_{k_j}}} \int_{\{(x_1,x')\in\Omega^{\varepsilon_{n_{k_j}}} : x_1 \leqslant x_0\}} p v_j^2 dx,$$

hence

$$\int_{\mathbb{R}^N} p \widetilde{v}_j^2 dx = \int_{\{(x_1,x')\in\Omega^{\varepsilon_{n_{k_j}}} : x_1 \leqslant x_0\}} p \widetilde{v}_j^2 dx = 1,$$
$$\int_{\mathbb{R}^N} |\nabla \widetilde{v}_j|^2 dx = \int_{\{(x_1,x')\in\Omega^{\varepsilon_{n_{k_j}}} : x_1 \leqslant x_0\}} |\nabla \widetilde{v}_j|^2 dx = \lambda_{\varepsilon_{n_{k_j}}}.$$

Hence $\{\widetilde{v}_j\}_j$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and, along a subsequence, $\mathcal{D}^{1,2}(\mathbb{R}^N)$-weakly converges to some $\widetilde{v} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^n} p \widetilde{v}^2\, dx = 1$, $\operatorname{supp} \widetilde{v} \subset D^-$, and

$$\begin{cases} -\Delta \widetilde{v} = \lambda_{k_0}(D^+) p \widetilde{v}, & \text{in } D^-,\\ \widetilde{v} = 0, & \text{on } \partial D^-, \end{cases}$$

thus implying that $\lambda_{k_0}(D^+) \in \sigma_p(D^-)$, in contradiction with assumption (5).

We have then proved that the limit w depends neither on the sequence $\{\varepsilon_n\}_n$ nor on the subsequence $\{\varepsilon_{n_k}\}_k$, thus concluding the proof. □

Let us define

$$\phi_\varepsilon : (0, 1) \to \mathbb{R}, \quad \phi_\varepsilon(t) = \int_\Sigma u_\varepsilon(t, \varepsilon x') \psi_1^\Sigma(x') dx'. \tag{65}$$

As a consequence of Lemma 3.1, the following result holds.

Corollary 3.2. *For every* $x_0 \in (0, 1)$

$$\lim_{\varepsilon\to 0^+} \frac{\phi_\varepsilon(x_0)}{\sqrt{\widetilde{H}_\varepsilon(x_0)}} = 1,$$

where ϕ_ε *is defined in* (65) *and* $\widetilde{H}_\varepsilon$ *in* (51).

Proof. From Lemma 3.1 it follows that $w_\varepsilon(1, x') \to \psi_1^\Sigma(x')$ in $C^2(\Sigma)$ as $\varepsilon \to 0^+$, i.e.

$$\frac{u_\varepsilon(x_0, \varepsilon x')}{\sqrt{\widetilde{H}_\varepsilon(x_0)}} = w_\varepsilon(1, x') \to \psi_1^\Sigma(x'), \text{ in } C^2(\Sigma),$$

which easily implies the conclusion. □

Proposition 3.3. *For every $x_0 \in (0,1)$*

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-1} e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(x_0-1)} \sqrt{\widetilde{H}_\varepsilon(x_0)} = \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1) \int_\Sigma \Phi(1,x')\psi_1^\Sigma(x')dx',$$

being u_0 as in (6) *and Φ the unique solution to problem* (16).

Proof. By virtue of Corollary 3.2, it is sufficient to prove that

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-1} e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(x_0-1)} \phi_\varepsilon(x_0) = \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1) \int_\Sigma \Phi(1,x')\psi_1^\Sigma(x')dx'. \tag{66}$$

Recalling that, from (9) and (3), u_ε is harmonic in the corridor, ϕ_ε satisfies

$$(\phi_\varepsilon)''(t) = \frac{\lambda_1(\Sigma)}{\varepsilon^2}\phi_\varepsilon(t), \quad \text{for all } t \in (0,1),$$

which can be rewritten as

$$\left(e^{\frac{2}{\varepsilon}\sqrt{\lambda_1(\Sigma)}(t-1)}\left(e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(t-1)}\phi_\varepsilon(t)\right)'\right)' = 0 \quad \text{in } (0,1).$$

Hence

$$\left(e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(t-1)}\phi_\varepsilon(t)\right)' = C_\varepsilon e^{-\frac{2}{\varepsilon}\sqrt{\lambda_1(\Sigma)}(t-1)}, \quad \text{for all } t \in (0,1) \tag{67}$$

$$\phi_\varepsilon(t) = A_\varepsilon e^{\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(t-1)} + B_\varepsilon e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(t-1)}, \quad \text{for all } t \in (0,1), \tag{68}$$

being C_ε, A_ε and B_ε some real constants depending on ε (and independent of t). The proof of the proposition is divided in several steps.

Step 1: we claim that

$$C_\varepsilon = -2\sqrt{\lambda_1(\Sigma)}\,\frac{B_\varepsilon}{\varepsilon}. \tag{69}$$

Indeed, fixing $h > 1$ and integrating (67) over $(1-h\varepsilon, 1-\varepsilon)$, we obtain that

$$\begin{aligned}
\frac{C_\varepsilon \varepsilon}{2\sqrt{\lambda_1(\Sigma)}}\left(e^{2h\sqrt{\lambda_1(\Sigma)}} - e^{2\sqrt{\lambda_1(\Sigma)}}\right) &= \int_{1-h\varepsilon}^{1-\varepsilon}\left(e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(t-1)}\phi_\varepsilon(t)\right)' dt \\
&= \int_{1-h\varepsilon}^{1-\varepsilon}\left(A_\varepsilon + B_\varepsilon e^{-2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(t-1)}\right)' dt \\
&= B_\varepsilon\left(e^{2\sqrt{\lambda_1(\Sigma)}} - e^{2h\sqrt{\lambda_1(\Sigma)}}\right).
\end{aligned}$$

Step 2: we claim that, for every $R, h > 0$,

$$e^{\sqrt{\lambda_1(\Sigma)}R}\int_\Sigma \Phi(1-R,x')\psi_1^\Sigma(x')\,dx' - e^{\sqrt{\lambda_1(\Sigma)}h}\int_\Sigma \Phi(1-h,x')\psi_1^\Sigma(x')\,dx' = 0.$$

Let us define

$$\phi : (-\infty, 1] \to \mathbb{R}, \quad \phi(t) = \int_\Sigma \Phi(t,x')\psi_1^\Sigma(x')dx'.$$

Since Φ is harmonic on its domain, ϕ solves

$$\left(e^{-\sqrt{\lambda_1(\Sigma)}(t-1)}\phi(t)\right)' = Ce^{-2\sqrt{\lambda_1(\Sigma)}(t-1)}, \quad \text{for all } t \in (-\infty, 1],$$

being C a real constant (independent of t). Integrating the above equation over $(1-\rho, 1)$, we obtain

$$\phi(1) - e^{\rho\sqrt{\lambda_1(\Sigma)}}\phi(1-\rho) = \frac{C}{2\sqrt{\lambda_1(\Sigma)}}\left(e^{2\rho\sqrt{\lambda_1(\Sigma)}} - 1\right) \text{ for all } \rho > 0$$

and then

$$C = \frac{2\sqrt{\lambda_1(\Sigma)}}{e^{2\rho\sqrt{\lambda_1(\Sigma)}} - 1}\left(\phi(1) - e^{\rho\sqrt{\lambda_1(\Sigma)}}\phi(1-\rho)\right) \quad \text{for all } \rho > 0. \tag{70}$$

From [17, Lemma 2.9 (ii)], $\Phi(x_1,x') = O(e^{\sqrt{\lambda_1(\Sigma)}\frac{x_1-1}{2}})$ as $x_1 \to -\infty$. Hence

$$\phi(1-\rho) = \int_\Sigma \Phi(1-\rho,x')\psi_1^\Sigma(x')\,dx' = O(e^{-\frac{\rho}{2}\sqrt{\lambda_1(\Sigma)}})$$

as $\rho \to +\infty$. Therefore, letting $\rho \to +\infty$ in (70) implies that $C = 0$. This yields that

$$e^{\rho\sqrt{\lambda_1(\Sigma)}}\phi(1-\rho) = \phi(1) \tag{71}$$

for any $\rho > 0$, thus proving the claim.

Step 3: we claim that $C_\varepsilon \to 0$ as $\varepsilon \to 0$. Indeed, fixing $h > R > 0$ and integrating (67) over $(1-h\varepsilon, 1-R\varepsilon)$, we obtain that

$$e^{R\sqrt{\lambda_1(\Sigma)}}\frac{\phi_\varepsilon(1-R\varepsilon)}{\varepsilon} - e^{h\sqrt{\lambda_1(\Sigma)}}\frac{\phi_\varepsilon(1-h\varepsilon)}{\varepsilon} = \frac{C_\varepsilon}{2\sqrt{\lambda_1(\Sigma)}}\left(e^{2h\sqrt{\lambda_1(\Sigma)}} - e^{2R\sqrt{\lambda_1(\Sigma)}}\right),$$

from which, thank to Lemma 2.4 and Step 2, it follows that

$$\begin{aligned} C_\varepsilon &= \frac{2\sqrt{\lambda_1(\Sigma)}}{e^{2h\sqrt{\lambda_1(\Sigma)}} - e^{2R\sqrt{\lambda_1(\Sigma)}}}\left(e^{R\sqrt{\lambda_1(\Sigma)}}\frac{\phi_\varepsilon(1-R\varepsilon)}{\varepsilon} - e^{h\sqrt{\lambda_1(\Sigma)}}\frac{\phi_\varepsilon(1-h\varepsilon)}{\varepsilon}\right) \\ &\xrightarrow{\varepsilon\to0^+} \frac{2\sqrt{\lambda_1(\Sigma)}\frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)}{e^{2h\sqrt{\lambda_1(\Sigma)}} - e^{2R\sqrt{\lambda_1(\Sigma)}}}\Bigg(e^{R\sqrt{\lambda_1(\Sigma)}}\int_\Sigma \Phi(1-R,x')\psi_1^\Sigma(x')\,dx' \\ &\qquad\qquad - e^{h\sqrt{\lambda_1(\Sigma)}}\int_\Sigma \Phi(1-h,x')\psi_1^\Sigma(x')\,dx'\Bigg) = 0 \end{aligned}$$

thanks to the previous step.

Step 4: we claim that

$$\frac{A_\varepsilon}{\varepsilon} \to \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)\int_\Sigma \Phi(1,x')\psi_1^\Sigma(x')\,dx' \quad \text{as } \varepsilon \to 0^+. \tag{72}$$

For all $R > 0$, the convergence

$$\frac{\phi_\varepsilon(1-R\varepsilon)}{\varepsilon} \to \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)\int_\Sigma \Phi(1-R,x')\psi_1^\Sigma(x')\,dx', \quad \text{as } \varepsilon \to 0^+,$$

which is a consequence of Lemma 2.4, implies, in view of (68), that

$$\frac{A_\varepsilon}{\varepsilon}e^{-R\sqrt{\lambda_1(\Sigma)}} + \frac{B_\varepsilon}{\varepsilon}e^{R\sqrt{\lambda_1(\Sigma)}} \to \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)\int_\Sigma \Phi(1-R,x')\psi_1^\Sigma(x')\,dx' \text{ as } \varepsilon \to 0^+.$$

Claim (72) follows from the above convergence, Steps 1 and 3, which imply that

$$\frac{B_\varepsilon}{\varepsilon} = -\frac{C_\varepsilon}{2\sqrt{\lambda_1(\Sigma)}} = o(1)$$

as $\varepsilon \to 0^+$, and (71).

Step 5: we claim that, for every $x_0 \in (0,1)$,

$$B_\varepsilon\varepsilon^{-1}e^{-2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(x_0-1)} \to 0 \quad \text{as } \varepsilon \to 0^+. \tag{73}$$

Let us fix $\overline{x} \neq 1$. Then, taking into account (68), Lemma 3.1 and Corollary 3.2 imply that

$$\begin{aligned} e^{\sqrt{\lambda_1(\Sigma)}(\overline{x}-1)} &= \lim_{\varepsilon\to0^+}\frac{\phi_\varepsilon(\varepsilon(\overline{x}-1)+x_0)}{\phi_\varepsilon(x_0)} \\ &= \lim_{\varepsilon\to0^+}\frac{\frac{A_\varepsilon}{B_\varepsilon}e^{2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(x_0-1)}e^{\sqrt{\lambda_1(\Sigma)}(\overline{x}-1)} + e^{-\sqrt{\lambda_1(\Sigma)}(\overline{x}-1)}}{\frac{A_\varepsilon}{B_\varepsilon}e^{2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(x_0-1)} + 1}. \end{aligned} \tag{74}$$

By contradiction, let us assume that (73) is not true and hence that there exist $\alpha > 0$ and a sequence $\{\varepsilon_n\}_n \to 0^+$ such that

$$\left|B_{\varepsilon_n}\varepsilon_n^{-1}e^{-2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon_n}(x_0-1)}\right| \geqslant \alpha > 0, \quad \text{for all } n,$$

which, in view of (72), implies that

$$\frac{A_{\varepsilon_n}}{B_{\varepsilon_n}} e^{2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon_n}(x_0-1)} = O(1) \quad \text{as } n \to +\infty.$$

Then there exist $\ell \in \mathbb{R}$ and a subsequence $\{\varepsilon_{n_k}\}_k$ such that

$$\frac{A_{\varepsilon_{n_k}}}{B_{\varepsilon_{n_k}}} e^{2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon_{n_k}}(x_0-1)} \to \ell.$$

If $\ell \neq -1$, then from (74) it follows that

$$e^{\sqrt{\lambda_1(\Sigma)}(\overline{x}-1)} = \frac{\ell\, e^{\sqrt{\lambda_1(\Sigma)}(\overline{x}-1)} + e^{-\sqrt{\lambda_1(\Sigma)}(\overline{x}-1)}}{\ell+1},$$

thus contradicting the fact that $\bar{x} \neq 1$. On the other hand, if $\ell = -1$ the limit at the second line of (74) is $\pm\infty$, giving again rise to a contradiction.

We are now in position to conclude the proof. From (68), (72), and (73) it follows that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \varepsilon^{-1} e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(x_0-1)} \phi_\varepsilon(x_0) \\
&\quad = \lim_{\varepsilon\to 0^+} \varepsilon^{-1} e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(x_0-1)} \left(A_\varepsilon e^{\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(x_0-1)} + B_\varepsilon e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(x_0-1)} \right) \\
&\quad = \lim_{\varepsilon\to 0^+} \left\{ \frac{A_\varepsilon}{\varepsilon} + \frac{B_\varepsilon}{\varepsilon} e^{-2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(x_0-1)} \right\} = \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1) \int_\Sigma \Phi(1,x')\psi_1^\Sigma(x')\,dx',
\end{aligned}$$

thus completing the proof. □

In order to come to a further step in our analysis, we find useful to recall some basic facts in [17] concerning the blow-up limit at the left junction. We define

$$\widehat{u}_\varepsilon : \widehat{\Omega}^\varepsilon \to \mathbb{R}, \quad \widehat{u}_\varepsilon(x) = \frac{u_\varepsilon(\varepsilon x)}{\sqrt{\int_\Sigma u_\varepsilon^2(\varepsilon, \varepsilon x')\,dx'}}, \tag{75}$$

where

$$\widehat{\Omega}^\varepsilon := D^- \cup \{(x_1,x') \in T_1 : 0 \leqslant x_1 \leqslant 1/\varepsilon\} \cup \{(x_1,x') : x_1 > 1/\varepsilon\}.$$

We observe that $\widehat{u}_\varepsilon$ solves

$$\begin{cases} -\Delta \widehat{u}_\varepsilon(x) = \varepsilon^2 \lambda_\varepsilon p(\varepsilon x) \widehat{u}_\varepsilon(x), & \text{in } \widehat{\Omega}^\varepsilon, \\ \widehat{u}_\varepsilon = 0, & \text{on } \partial\widehat{\Omega}^\varepsilon. \end{cases}$$

We let $\widehat{D}$ as in (18), and consider, for all $r > 0$, $\mathcal{H}_r$ as defined at page 12. The change of variable $y' = \varepsilon x'$ yields

$$\int_\Sigma \widehat{u}_\varepsilon^2(1,x')\,dx' = 1. \tag{76}$$

In [17, Lemma 5.2 and Corollary 5.5], the following result is proved.

Proposition 3.4. *([17]) For every sequence $\varepsilon_n \to 0^+$, there exist a subsequence $\{\varepsilon_{n_k}\}_k$ and $\widehat{C} \in \mathbb{R} \setminus \{0\}$ such that $\widehat{u}_{\varepsilon_{n_k}} \to \widehat{C}\widehat{\Phi}$ strongly in $\mathcal{H}_r$ for every $r > 1$, in $C^2(\overline{B_{r_2}^- \setminus B_{r_1}^-})$ for all $1 < r_1 < r_2$, and in $C^2(\{(x_1,x') : t_1 \leqslant x_1 \leqslant t_2,\ x' \in \overline{\Sigma}\})$ for all $0 < t_1 < t_2$, where $\widehat{\Phi}$ is the unique solution to* (21)*.*

The following lemma ensures that the constant $\widehat{C}$ in Proposition 3.4 is positive.

Lemma 3.5. *Let $\{\varepsilon_n\}_n$, $\{\varepsilon_{n_k}\}_k$ and $\widehat{C}$ as in Proposition 3.4. Then $\widehat{C} > 0$.*

Proof. Let us assume by contradiction that $\widehat{C} < 0$. Since $\widehat{u}_{\varepsilon_{n_k}} \to \widehat{C}\widehat{\Phi}$ strongly in $C^2(\Gamma_2^-)$, we have that, if k is sufficiently large, then $u_{\varepsilon_{n_k}} < 0$ on $\Gamma^-_{2\varepsilon_{n_k}}$.

From [17, Corollary 1.3], there exists a sub-subsequence $\{\varepsilon_{n_{k_j}}\}_j$ such that

$$2\varepsilon_{n_{k_j}} < \frac{1}{j}, \quad u_{\varepsilon_{n_{k_j}}} > 0 \text{ on } \Gamma^-_{1/j}, \quad \text{and} \quad u_{\varepsilon_{n_{k_j}}} < 0 \text{ on } \Gamma^-_{2\varepsilon_{n_{k_j}}}.$$

Therefore, for j large, the functions

$$v_j = \begin{cases} u_{\varepsilon_{n_{k_j}}}, & \text{in } D^- \setminus B^-_{1/j}, \\ u^+_{\varepsilon_{n_{k_j}}}, & \text{in } B^-_{1/j} \setminus B^-_{2\varepsilon_{n_{k_j}}}, \\ 0, & \text{in } B^-_{2\varepsilon_{n_{k_j}}} \cup \{(x_1,x') \in \Omega^{\varepsilon_{n_{k_j}}} : x_1 \geqslant 0\}, \end{cases} \qquad \widetilde{v}_j := \frac{v_j}{\left(\int_{\mathbb{R}^N} p\, v_j^2 dx\right)^{\frac{1}{2}}}$$

are well-defined and belong to $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (if trivially extended to the whole $\mathbb{R}^N$).

Let $A_j = D^- \setminus B^-_{1/j}$. Then $\widetilde{v}_j$ satisfies

$$\begin{cases} -\Delta \widetilde{v}_j = \lambda_{\varepsilon_{n_{k_j}}} p \widetilde{v}_j, & \text{in } A_j, \\ \widetilde{v}_j = 0, & \text{on } \partial A_j \cap \partial D^-, \\ \int_{\mathbb{R}^N} p\, \widetilde{v}_j^2\, dx = 1. \end{cases}$$

Testing equation (9) for $\varepsilon = \varepsilon_{n_{k_j}}$ with v_j we obtain

$$\int_{D^- \setminus B^-_{2\varepsilon_{n_{k_j}}}} |\nabla v_j|^2 dx = \lambda_{\varepsilon_{n_{k_j}}} \int_{D^- \setminus B^-_{2\varepsilon_{n_{k_j}}}} p v_j^2 dx,$$

hence

$$\int_{\mathbb{R}^N} p\widetilde{v}_j^2 dx = \int_{D^- \setminus B^-_{2\varepsilon_{n_{k_j}}}} p\widetilde{v}_j^2 dx = 1, \quad \int_{\mathbb{R}^N} |\nabla \widetilde{v}_j|^2 dx = \int_{D^- \setminus B^-_{2\varepsilon_{n_{k_j}}}} |\nabla \widetilde{v}_j|^2 dx = \lambda_{\varepsilon_{n_{k_j}}}.$$

Hence $\{\widetilde{v}_j\}_j$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and, along a subsequence, $\mathcal{D}^{1,2}(\mathbb{R}^N)$-weakly converges to some $\widetilde{v} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^n} p\widetilde{v}^2\, dx = 1$, $\operatorname{supp} \widetilde{v} \subset D^-$, and

$$\begin{cases} -\Delta \widetilde{v} = \lambda_{k_0}(D^+) p \widetilde{v}, & \text{in } D^-, \\ \widetilde{v} = 0, & \text{on } \partial D^-, \end{cases}$$

thus implying that $\lambda_{k_0}(D^+) \in \sigma_p(D^-)$ and contradicting assumption (5). □

We can conclude that the convergence in Proposition 3.4 is not up to subsequences, since the limit can be univocally characterized by virtue of (76) and Lemma 3.5. Indeed, passing to the limit in (76), we obtain that

$$\widehat{C}^2 = \left(\int_\Sigma \widehat{\Phi}^2(1,x') dx' \right)^{-1},$$

and hence by Lemma 3.5 we conclude that

$$\widehat{C} = \frac{1}{\sqrt{\int_\Sigma \widehat{\Phi}^2(1,x')dx'}}. \tag{77}$$

Therefore we can improve Proposition 3.4 as follows.

Proposition 3.6. *As* $\varepsilon \to 0^+$,

$$\widehat{u}_\varepsilon \to \frac{\widehat{\Phi}}{\sqrt{\int_\Sigma \widehat{\Phi}^2(1,x')dx'}}$$

strongly in $\mathcal{H}_r$ *for every* $r > 1$, *in* $C^2(\{(x_1,x') : t_1 \leqslant x_1 \leqslant t_2,\ x' \in \overline{\Sigma}\})$ *for all* $0 < t_1 < t_2$, *and in* $C^2(\overline{B^-_{r_2} \setminus B^-_{r_1}})$ *for all* $1 < r_1 < r_2$, *where* $\widehat{\Phi}$ *is the unique solution to* (21).

As a further step in our analysis, we evaluate the asymptotic behavior as $\varepsilon \to 0^+$ of the function $\widetilde{H}_\varepsilon$ defined in (51) at ε-distance from the left junction in the corridor. To this aim, the following lemma is required.

Lemma 3.7. *Let $\widehat{\Phi}$ be the unique solution to* (21)*. Then*

$$\widehat{\Phi}(x_1, x') = e^{\sqrt{\lambda_1(\Sigma)}x_1}\psi_1^\Sigma(x') + O(e^{-\sqrt{\lambda_1(\Sigma)}\frac{x_1}{2}})$$

as $x_1 \to +\infty$ uniformly with respect to $x' \in \Sigma$.

Proof. Let $g : T_1^+ \to \mathbb{R}$, $g(x_1, x') = \widehat{\Phi}(x_1, x') - e^{\sqrt{\lambda_1(\Sigma)}x_1}\psi_1^\Sigma(x')$. From (21) and (22) it follows that $\int_{T_1^+}(|\nabla g|^2 + |g|^{2^*}) < +\infty$, $g \geqslant 0$ in T_1^+, $g = 0$ on $\{(x_1, x') : x_1 > 0, x' \in \partial\Sigma\}$, and g weakly solves $-\Delta g = 0$ in T_1^+. Let $f(x_1, x') = e^{-\sqrt{\lambda_1(\Sigma)}\frac{x_1}{2}}\psi_1^\Sigma(x'/2)$; we notice that f is harmonic and strictly positive in T_1^+, bounded from below away from 0 on $\{(x_1, x') : x_1 = 0, x' \in \overline{\Sigma}\}$, and $\int_{T_1^+}(|\nabla f|^2 + |f|^{2^*}) < +\infty$. Hence, from the Maximum Principle we deduce that $g(x) \leqslant \mathrm{const}\, f(x)$ in T_1^+, thus implying the conclusion. □

We are now in position to provide the asymptotics of $\widetilde{H}_\varepsilon$ at ε-distance from the left junction in the corridor.

Proposition 3.8. *Let $\widetilde{H}_\varepsilon$ be as in* (51)*. Then*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}e^{\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}}\sqrt{\widetilde{H}_\varepsilon(\varepsilon)} = \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)\bigg(\int_\Sigma \Phi(1, x')\psi_1^\Sigma(x')dx'\bigg)\sqrt{\int_\Sigma \widehat{\Phi}^2(1, x')dx'}, \tag{78}$$

being u_0 as in (6)*, Φ the unique solution to* (16)*, and $\widehat{\Phi}$ the unique solution to* (21)*.*

Proof. Let ϕ_ε as in (65), C_ε as in (67), and $A_\varepsilon, B_\varepsilon$ as in (68). We proceed by steps.

Step 1: we claim that

$$\lim_{\varepsilon\to 0^+}\varepsilon^{-1}e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(\varepsilon-1)}\phi_\varepsilon(\varepsilon) - \frac{B_\varepsilon}{\varepsilon}e^{-2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(\varepsilon-1)} = \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)\int_\Sigma \Phi(1, x')\psi_1^\Sigma(x')dx'. \tag{79}$$

To prove (79), let us fix $x_0 \in (0, 1)$ and integrate (67) over (ε, x_0). For ε sufficiently small, $x_0 > \varepsilon$. We obtain that

$$\varepsilon^{-1}e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(x_0-1)}\phi_\varepsilon(x_0) - \varepsilon^{-1}e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(\varepsilon-1)}\phi_\varepsilon(\varepsilon) = \frac{C_\varepsilon}{2\sqrt{\lambda_1(\Sigma)}}\left(e^{-2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(\varepsilon-1)} - e^{-2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(x_0-1)}\right).$$

Hence (79) follows from (69), (73), and (66).

Step 2: we claim that, for every $h > 0$,

$$e^{-h\sqrt{\lambda_1(\Sigma)}}\hat{\phi}(h) = \widehat{C} - \widehat{C}e^{-2h\sqrt{\lambda_1(\Sigma)}} + \hat{\phi}(0)e^{-2h\sqrt{\lambda_1(\Sigma)}} \tag{80}$$

where $\widehat{C}$ is as in (77) and

$$\hat{\phi} : [0, +\infty) \to \mathbb{R}, \quad \hat{\phi}(t) = \widehat{C}\int_\Sigma \widehat{\Phi}(t, x')\psi_1^\Sigma(x')dx'. \tag{81}$$

Since $\widehat{\Phi}$ is harmonic on its domain, $\hat{\phi}$ solves

$$\left(e^{-\sqrt{\lambda_1(\Sigma)}t}\hat{\phi}(t)\right)' = Ce^{-2\sqrt{\lambda_1(\Sigma)}t}, \quad \text{for all } t \in [0, +\infty),$$

being C a real constant (independent of t). Integrating the above equation over $(0, h)$, we obtain

$$e^{-h\sqrt{\lambda_1(\Sigma)}}\hat{\phi}(h) - \hat{\phi}(0) = \frac{C}{2\sqrt{\lambda_1(\Sigma)}}\left(1 - e^{-2h\sqrt{\lambda_1(\Sigma)}}\right) \text{ for all } h > 0$$

and then

$$C = \frac{2\sqrt{\lambda_1(\Sigma)}}{1 - e^{-2h\sqrt{\lambda_1(\Sigma)}}} \left(e^{-h\sqrt{\lambda_1(\Sigma)}} \hat{\phi}(h) - \hat{\phi}(0) \right) \quad \text{for all } h > 0. \tag{82}$$

From Lemma 3.7 it follows that $e^{-h\sqrt{\lambda_1(\Sigma)}} \hat{\phi}(h) = \widehat{C} + o(1)$ as $h \to +\infty$, then letting $h \to +\infty$ in (82) we obtain that $C = 2\sqrt{\lambda_1(\Sigma)} \left(\widehat{C} - \hat{\phi}(0) \right)$ and then

$$e^{-h\sqrt{\lambda_1(\Sigma)}} \hat{\phi}(h) = \big(\widehat{C} - \hat{\phi}(0) \big) \Big(1 - e^{-2h\sqrt{\lambda_1(\Sigma)}} \Big) + \hat{\phi}(0) \quad \text{for all } h > 0$$

which yields claim (80).

Step 3: we claim that

$$\lim_{\varepsilon \to 0^+} \frac{B_\varepsilon e^{\frac{\sqrt{\lambda_1((\Sigma)}}{\varepsilon}}}{\big(\widetilde{H}_\varepsilon(\varepsilon) \big)^{1/2}} = \hat{\phi}(0) - \widehat{C}. \tag{83}$$

To prove (83) we follow the scheme of Proposition 3.3. We fix $k > 1$ and integrate (67) over $(\varepsilon, k\varepsilon)$, thus obtaining

$$\begin{aligned} &e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(k\varepsilon - 1)} \phi_\varepsilon(k\varepsilon) - e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(\varepsilon - 1)} \phi_\varepsilon(\varepsilon) \\ &\qquad = \frac{C_\varepsilon \varepsilon}{2\sqrt{\lambda_1(\Sigma)}} \left(e^{-2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(\varepsilon - 1)} - e^{-2\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}(k\varepsilon - 1)} \right) \end{aligned}$$

i.e., in view of (69),

$$e^{-\sqrt{\lambda_1(\Sigma)} k} \frac{\phi_\varepsilon(k\varepsilon)}{\sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} - e^{-\sqrt{\lambda_1(\Sigma)}} \frac{\phi_\varepsilon(\varepsilon)}{\sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} = \frac{B_\varepsilon e^{\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}}}{\sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} \left(e^{-2k\sqrt{\lambda_1(\Sigma)}} - e^{-2\sqrt{\lambda_1(\Sigma))}} \right),$$

which, in view of (65), Proposition 3.6, (81), and (80), implies that

$$\begin{aligned} \frac{B_\varepsilon e^{\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}}}{\sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} \left(e^{-2k\sqrt{\lambda_1(\Sigma)}} - e^{-2\sqrt{\lambda_1(\Sigma))}} \right) &\to e^{-\sqrt{\lambda_1(\Sigma)} k} \hat{\phi}(k) - e^{-\sqrt{\lambda_1(\Sigma)}} \hat{\phi}(1) \\ &= \left(e^{-2k\sqrt{\lambda_1(\Sigma)}} - e^{-2\sqrt{\lambda_1(\Sigma))}} \right) \big(\hat{\phi}(0) - \widehat{C} \big) \end{aligned}$$

as $\varepsilon \to 0^+$, thus proving claim (83).

In order to conclude the proof, we observe that from (79) it follows

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \sqrt{\widetilde{H}_\varepsilon(\varepsilon)} \varepsilon^{-1} e^{\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}} \left(e^{-\sqrt{\lambda_1(\Sigma)}} \frac{\phi_\varepsilon(\varepsilon)}{\sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} - e^{-2\sqrt{\lambda_1(\Sigma)}} \frac{B_\varepsilon e^{\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}}}{\sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} \right) \\ &\qquad = \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1) \int_\Sigma \Phi(1, x') \psi_1^\Sigma(x') dx' \end{aligned}$$

which is sufficient to conclude in view of (65), (75), Proposition 3.6, (81), (83) and (80). □

We are now in position to derive an asymptotics for the normalization $(\int_{\Gamma_{\tilde{k}}^-} u_{\varepsilon_{n_j}}^2 d\sigma)^{1/2}$.

Proposition 3.9. *Let $\tilde{k}$ as in Theorem 1.1. Then*

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} e^{\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}} \varepsilon^{-N} \sqrt{\int_{\Gamma_{\tilde{k}}^-} u_\varepsilon^2 \, d\sigma} \\ &\qquad = \sqrt{\int_{\Gamma_{\tilde{k}}^-} \overline{U}^2 d\sigma} \left(\int_{\mathbb{S}_-^{N-1}} \widehat{\Phi}(\theta) \Psi^-(\theta) d\sigma(\theta) \right) \left(\int_\Sigma \Phi(1, x') \psi_1^\Sigma(x') dx' \right) \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1). \end{aligned} \tag{84}$$

Proof. We divide the proof in several steps.
Step 1: we claim that, for every $h > k > 1$,

$$\frac{\hat{v}(h)}{h^{1-N}} = \frac{\hat{v}(k)}{k^{1-N}} = \hat{v}(1) \tag{85}$$

where

$$\hat{v} : [1, +\infty) \to \mathbb{R}, \quad \hat{v}(r) := \widehat{C} \int_{\mathbb{S}^{N-1}_-} \widehat{\Phi}(r\theta) \Psi^-(\theta) d\sigma(\theta),$$

with $\widehat{\Phi}$ being the unique solution to (21) and $\widehat{C}$ as in (77). Since $\widehat{\Phi}$ is harmonic on its domain, by Lemma A.1 $\hat{v}$ solves

$$\left(r^{N+1} \left(\frac{\hat{v}}{r} \right)' \right)' = 0, \quad \text{in } (1, +\infty),$$

hence, by integration, there exists $C \in \mathbb{R}$ (independent of h, k) such that

$$\frac{\hat{v}(h)}{h} = \frac{\hat{v}(k)}{k} + \frac{C}{N}(k^{-N} - h^{-N}), \quad \text{for all } 1 < k < h.$$

Hence

$$C = \frac{N}{k^{-N} - h^{-N}} \left(\frac{\hat{v}(h)}{h} - \frac{\hat{v}(k)}{k} \right), \quad \text{for all } 1 < k < h.$$

From (23), it follows that $\frac{\hat{v}(h)}{h} \to 0$ as $h \to +\infty$. Hence $C = -N \frac{\hat{v}(k)}{k^{1-N}}$ and claim (85) is proved.
Step 2: for all $r \in (\varepsilon, 3)$ let us define

$$\varphi_\varepsilon^-(r) = \int_{\mathbb{S}^{N-1}_-} u_\varepsilon(r\theta) \Psi^-(\theta) \, d\sigma(\theta).$$

From (9), (3), and Lemma A.1 it follows that φ_ε^- satisfies

$$\left(r^{N+1} \left(\frac{\varphi_\varepsilon^-(r)}{r} \right)' \right)' = 0, \quad \text{in } (\varepsilon, 3), \tag{86}$$

and hence there exists a constant d_ε (depending on ε but independent of r) such that

$$\left(\frac{\varphi_\varepsilon^-(r)}{r} \right)' = \frac{d_\varepsilon}{r^{N+1}}, \quad \text{in } (\varepsilon, 3). \tag{87}$$

We claim that, for every $k > 1$,

$$\lim_{\varepsilon \to 0^+} \frac{d_\varepsilon}{N \varepsilon^{N-1} \sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} = -\frac{\hat{v}(k)}{k^{1-N}}. \tag{88}$$

Integration of (87) in $(k\varepsilon, h\varepsilon)$ for $h > k > 1$ yields

$$\frac{\varphi_\varepsilon^-(h\varepsilon)}{h} - \frac{\varphi_\varepsilon^-(k\varepsilon)}{k} = \frac{d_\varepsilon}{N \varepsilon^{N-1}} \left(k^{-N} - h^{-N} \right), \quad \text{for all } 1 < k < h < \frac{3}{\varepsilon}, \tag{89}$$

and then

$$\frac{d_\varepsilon}{N \varepsilon^{N-1} \sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} = \frac{1}{k^{-N} - h^{-N}} \left(\frac{1}{h} \frac{\varphi_\varepsilon^-(h\varepsilon)}{\sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} - \frac{1}{k} \frac{\varphi_\varepsilon^-(k\varepsilon)}{\sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} \right). \tag{90}$$

Since $\frac{\varphi_\varepsilon^-(r\varepsilon)}{\sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} = \int_{\mathbb{S}^{N-1}_-} \widehat{u}_\varepsilon(r\theta) \Psi^-(\theta) \, d\sigma(\theta)$ for all $r > 1$, from Proposition 3.6 it follows that

$$\lim_{\varepsilon \to 0^+} \frac{\varphi_\varepsilon^-(r\varepsilon)}{\sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} = \hat{v}(r), \quad \text{for all } r > 1,$$

hence passing to the limit as $\varepsilon \to 0^+$ in (90) we obtain

$$\lim_{\varepsilon \to 0^+} \frac{d_\varepsilon}{N \varepsilon^{N-1} \sqrt{\widetilde{H}_\varepsilon(\varepsilon)}} = \frac{1}{k^{-N} - h^{-N}} \left(\frac{\hat{v}(h)}{h} - \frac{\hat{v}(k)}{k} \right).$$

which yields claim (88) in view of (85).

Step 3: we claim that

$$\lim_{\varepsilon\to 0^+} \frac{d_\varepsilon}{\sqrt{\int_{\Gamma^-_{\tilde k}} u_\varepsilon^2\, d\sigma}} = -\frac{N}{\sqrt{\int_{\Gamma^-_{\tilde k}} \overline{U}^2 d\sigma}}. \tag{91}$$

From (86) it follows that there exist $\alpha_\varepsilon, \beta_\varepsilon \in \mathbb{R}$ (depending on ε but independent of r) such that

$$\varphi^-_\varepsilon(r) = \alpha_\varepsilon r + \beta_\varepsilon r^{1-N}, \quad \text{for all } r \in (\varepsilon, 3).$$

From (87) it follows that

$$\beta_\varepsilon = -\frac{d_\varepsilon}{N}. \tag{92}$$

From Proposition 2.8 we have that

$$\frac{u_\varepsilon(x)}{\sqrt{\int_{\Gamma^-_{\tilde k}} u_\varepsilon^2\, d\sigma}} \to \frac{\overline{U}}{\sqrt{\int_{\Gamma^-_{\tilde k}} \overline{U}^2 d\sigma}}$$

as $\varepsilon \to 0^+$, strongly in $\mathcal{H}^-_t$ for every $t > 0$ and in $C^2(\overline{B^-_{t_2} \setminus B^-_{t_1}})$ for all $0 < t_1 < t_2$. Hence, for all $r \in (0, 3)$,

$$\lim_{\varepsilon\to 0^+} \frac{\varphi^-_\varepsilon(r)}{\sqrt{\int_{\Gamma^-_{\tilde k}} u_\varepsilon^2\, d\sigma}} = \frac{\overline{\varphi}(r)}{\sqrt{\int_{\Gamma^-_{\tilde k}} \overline{U}^2 d\sigma}} \tag{93}$$

where

$$\overline{\varphi}(r) := \int_{\mathbb{S}^{N-1}_-} \overline{U}(r\theta)\Psi^-(\theta)\, d\sigma(\theta).$$

Since $\overline{U}$ is harmonic in B^-_3, it is easy to prove that there exist $a, b \in \mathbb{R}$ such that

$$\overline{\varphi}(r) = a\, r + b\, r^{1-N}, \quad \text{for all } r \in (0, 3).$$

From (24) it follows that $b = 1$. Hence (93) can be rewritten as

$$\lim_{\varepsilon\to 0^+} \frac{\alpha_\varepsilon r + \beta_\varepsilon r^{1-N}}{\sqrt{\int_{\Gamma^-_{\tilde k}} u_\varepsilon^2\, d\sigma}} = \frac{a\, r + r^{1-N}}{\sqrt{\int_{\Gamma^-_{\tilde k}} \overline{U}^2 d\sigma}}, \quad \text{for all } r \in (0, 3). \tag{94}$$

We claim that $\frac{\alpha_\varepsilon}{\beta_\varepsilon} = O(1)$ as $\varepsilon \to 0^+$; to prove this, we assume by contradiction that along a sequence $\varepsilon_n \to 0^+$ there holds $\lim_{n\to+\infty} \frac{\beta_{\varepsilon_n}}{\alpha_{\varepsilon_n}} = 0$. Then from (94) there would follow, for all $r \in (0, 3)$,

$$\lim_{n\to+\infty} \frac{\alpha_{\varepsilon_n}}{\sqrt{\int_{\Gamma^-_{\tilde k}} u_{\varepsilon_n}^2\, d\sigma}} = \lim_{n\to+\infty} \frac{\alpha_{\varepsilon_n} r + \beta_{\varepsilon_n} r^{1-N}}{\sqrt{\int_{\Gamma^-_{\tilde k}} u_{\varepsilon_n}^2\, d\sigma}} \frac{1}{\left(r + \frac{\beta_{\varepsilon_n}}{\alpha_{\varepsilon_n}} r^{1-N}\right)} = \frac{a + r^{-N}}{\sqrt{\int_{\Gamma^-_{\tilde k}} \overline{U}^2 d\sigma}},$$

thus giving rise to a contradiction since different values of r yield different limits for the same sequence.

From the fact that $\{\frac{\alpha_\varepsilon}{\beta_\varepsilon}\}_\varepsilon$ is bounded, it follows that there exist a sequence $\varepsilon_n \to 0^+$ and some $\ell \in \mathbb{R}$ such that $\lim_{n\to+\infty} \frac{\alpha_{\varepsilon_n}}{\beta_{\varepsilon_n}} = \ell$. Hence (94) implies that

$$\lim_{n\to+\infty} \frac{\beta_{\varepsilon_n}}{\sqrt{\int_{\Gamma^-_{\tilde k}} u_{\varepsilon_n}^2\, d\sigma}} = \lim_{n\to+\infty} \frac{\alpha_{\varepsilon_n} r + \beta_{\varepsilon_n} r^{1-N}}{\sqrt{\int_{\Gamma^-_{\tilde k}} u_{\varepsilon_n}^2\, d\sigma}} \frac{1}{\left(\frac{\alpha_{\varepsilon_n}}{\beta_{\varepsilon_n}} r + r^{1-N}\right)} = \frac{1}{\sqrt{\int_{\Gamma^-_{\tilde k}} \overline{U}^2 d\sigma}} \frac{a\, r + r^{1-N}}{\left(\ell r + r^{1-N}\right)}$$

for all $r \in (0, 3)$, hence necessarily $\ell = a$ (otherwise different values of r would yield different limits for the same sequence). In particular the limit $\lim_{n\to+\infty} \frac{\alpha_{\varepsilon_n}}{\beta_{\varepsilon_n}}$ does not depend on the sequence $\{\varepsilon_n\}_n$, thus implying that

$$\lim_{\varepsilon\to 0^+} \frac{\alpha_\varepsilon}{\beta_\varepsilon} = a.$$

Hence (94) implies that, for all $r \in (0,3)$,

$$\lim_{\varepsilon\to0^+} \frac{\beta_\varepsilon}{\sqrt{\int_{\Gamma^-_{\tilde k}} u_\varepsilon^2\,d\sigma}} = \lim_{\varepsilon\to0^+} \frac{\alpha_\varepsilon r + \beta_\varepsilon r^{1-N}}{\sqrt{\int_{\Gamma^-_{\tilde k}} u_\varepsilon^2\,d\sigma}} \frac{1}{\big(\frac{\alpha_\varepsilon}{\beta_\varepsilon} r + r^{1-N}\big)}$$
$$= \frac{1}{\sqrt{\int_{\Gamma^-_{\tilde k}} \overline{U}^2 d\sigma}} \frac{a\,r + r^{1-N}}{\big(ar + r^{1-N}\big)} = \frac{1}{\sqrt{\int_{\Gamma^-_{\tilde k}} \overline{U}^2 d\sigma}}$$

which yields claim (91) in view of (92).

Combining (91), (88), and Proposition 3.8, we finally obtain

$$\lim_{\varepsilon\to0^+} e^{\frac{\sqrt{\lambda_1(\Sigma)}}{\varepsilon}} \varepsilon^{-N} \sqrt{\int_{\Gamma^-_{\tilde k}} u_\varepsilon^2\,d\sigma}$$
$$= \sqrt{\int_{\Gamma^-_{\tilde k}} \overline{U}^2 d\sigma} \bigg(\int_{\mathbb{S}^{N-1}_-} \widehat{\Phi}(\theta)\Psi^-(\theta) d\sigma(\theta)\bigg)\bigg(\int_\Sigma \Phi(1,x')\psi_1^\Sigma(x')dx'\bigg) \frac{\partial u_0}{\partial x_1}(\mathbf{e}_1)$$

thus completing the proof. □

Remark 3.10. We would like to stress that, in the flavor of [1], the asymptotic behavior of solutions is affected by the domain's geometry: the constants at the left hand side of (84) depend on the solutions of the relative blow-up limits, in addition to the initial normalization u_0. It is interesting to notice that the geometry of the left-hand side already appears in the asymptotics of Proposition 3.8, even if the solution has not crossed the left junction yet.

Proof of Teorema 1.3. It follows combining Propositions 2.8 and 3.9. □

Appendix A. Appendix

We state in this appendix a simple technical lemma which contains a trick used many times throughout the paper.

Lemma A.1. *Let u be a harmonic function in $B_R^+ \setminus B_r^+$ for some $0 < r < R$ such that $u = 0$ on $\partial\big(B_R^+ \setminus B_r^+\big) \cap \partial D^+$. Let $\varphi_u : (r,R) \to \mathbb{R}$, $\varphi_u(r) := \int_{\mathbb{S}^{N-1}_+} u(\mathbf{e}_1 + r\theta)\Psi^+(\theta)\,d\sigma(\theta)$, where Ψ^+ is the first positive $L^2(\mathbb{S}^{N-1}_+)$-normalized Dirichlet eigenfunction on the hemisphere $\mathbb{S}^{N-1}_+$ (i.e. Ψ^+ satisfies (15)). Then φ_u satisfies*

$$\bigg(r^{N+1}\Big(\frac{\varphi_u}{r}\Big)'\bigg)' = 0, \quad \text{in } (r,R).$$

Proof. The proof follows by direct calculations. □

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