ON SEMILINEAR ELLIPTIC EQUATIONS WITH BORDERLINE HARDY POTENTIALS

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Abstract. In this paper we study the asymptotic behavior of solutions to an elliptic equation near the singularity of an inverse square potential with a coefficient related to the best constant for the Hardy inequality. Due to the presence of a borderline Hardy potential, a proper variational setting has to be introduced in order to provide a weak formulation of the equation. An Almgren-type monotonicity formula is used to determine the exact asymptotic behavior of solutions.

1. Introduction

On a domain $\Omega \subseteq \mathbb{R}^N$, $N \geqslant 3$, containing the origin, let us consider the following problem

$$-\Delta u - \left(\frac{N-2}{2}\right)^2 \frac{u}{|x|^2} = h(x)u + f(x,u), \quad \text{in } \Omega, \tag{1}$$

where h is possibly singular at the origin but negligible with respect to the Hardy potential and f is a nonlinearity subcritical with respect to the critical Sobolev exponent. Looking at equation (1), one may observe that the best constant for the classical Hardy inequality appears in front of the inverse square potential; this can be considered as a borderline situation for several points of view, from the variational setting to the existence and qualitative behavior of solutions. Recent papers were devoted to equations and differential inequalities involving elliptic operators with inverse square potentials in the borderline situation, see [5, 11, 12, 13, 20, 28, 29].

In [13] the authors study necessary conditions for the existence of nonnegative distributional solutions of the differential inequality

$$-\Delta u - \left(\frac{N-2}{2}\right)^2 \frac{u}{|x|^2} \geqslant \alpha \frac{u}{|x|^2 \log^2 |x|} \quad \text{in } \mathcal{D}'(B_R \setminus \{0\}),$$

where B_R denotes the ball of radius R centered at the origin. The logarithmic term appearing in the above inequality is related to an improved version of the Hardy inequality, see for example [1, 3, 10, 22].

In [11], the author studies existence of positive distributional solutions of the nonlinear elliptic equation

$$-\Delta u - \lambda \frac{u}{|x|^2} + b(x)h(u) = 0 \quad \text{in } \Omega \setminus \{0\},$$

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satisfying some prescribed asymptotic behaviors at the origin, where $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$ is a domain containing the origin and $\lambda \in (-\infty, (N-2)^2/4]$. These prescribed asymptotic behaviors are related to the following functions

$$\Phi_\lambda^+(x) = |x|^{-\frac{N-2}{2}-\sqrt{\frac{(N-2)^2}{4}-\lambda}}, \quad \Phi_\lambda^-(x) = |x|^{-\frac{N-2}{2}+\sqrt{\frac{(N-2)^2}{4}-\lambda}},$$
$$\Psi^+(x) = |x|^{-\frac{N-2}{2}} \log(1/|x|), \quad \Psi^-(x) = |x|^{-\frac{N-2}{2}}$$

which are solutions to

$$-\Delta u - \lambda \frac{u}{|x|^2} = 0 \qquad \text{in } \mathbb{R}^N \setminus \{0\}$$

respectively in the cases $\lambda \in (-\infty, (N-2)^2/4)$ and $\lambda = (N-2)^2/4$. Similar results were obtained in [4] for equations with elliptic operators in divergence form. The analysis performed in [11] highlights how the asymptotics at the isolated singularity of positive solutions is sensitive to the interplay of many factors, such as the space dimension, the mass of the singularity, the behavior of the nonlinearity and of its coefficient. The present paper means to provide a classification of the behavior at the singularity of all (not only positive) finite energy solutions and to relate such behavior to the limit of an Almgren type frequency function.

In [12] the author studies a singular elliptic Dirichlet problem with a power type nonlinearity and a forcing term:

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2}\, u = u^p + tf & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a domain containing the origin, $p > 1$, $t > 0$, f a smooth, bounded, nonnegative function, and $c \in (0, (N-2)^2/4]$. In [12], the author provides a classification of different kind of solutions of problem (2), both of distributional and variational type. In the present paper, we are going to introduce an analogous terminology for solutions to (1), a classification of which will be provided as a byproduct of our main result, see section 2 for details. In particular, in [12] three types of solution are defined and discussed: weak solutions, which provide a good setting for proving non-existence results, H_0^1–solutions, for which uniqueness results can be established, and strong solutions, which have the optimal regularity. The classification of solutions proposed in [12] motivates the study of the relation between the different kinds of solutions performed in the present paper: a deeper knowledge of the link between the different notions of solutions allows drawing a more complete picture on existence, non-existence, and regularity.

We also mention that, in [5] the authors study existence and nonexistence of solutions of the equation in (2) with $t = 0$.

In the spirit of [12], in the present paper we concentrate our attention on local solutions to (1) belonging to a suitable functional space related to the borderline case of the Hardy inequality. To this purpose, in section 2 we introduce the Hilbert space $\mathcal{H}(\omega)$ defined as completion of $C_c^\infty(\overline{\omega} \setminus \{0\})$ with respect to a scalar product related to the Hardy potential appearing in (1) (see (4)). Here ω represents a bounded domain with $\partial\omega \in C^1$.

The purpose of this paper is to classify the possible asymptotic behaviors of solutions to (1) near the singularity of the Hardy potential. Some results in this direction were obtained in [14, 15, 16, 17] for different kinds of problems: in [15, 17] Schrödinger equations with electromagnetic potentials, in

[16] Schrödinger equations with inverse square many-particle potentials; finally in [14] the authors study the asymptotic behavior of solutions of a singular elliptic equation near a corner of the boundary. For other results concerning elliptic equations with singular inverse square potentials see also [18, 19, 23, 24, 27].

The results of the present paper are closely related to the ones obtained in [15, 17]. If we drop the magnetic part of the electromagnetic potential, the equation studied in [15, 17] becomes

$$-\Delta u - \frac{a(x/|x|)}{|x|^2}u = h(x)u + f(x,u) \quad \text{in } \Omega, \tag{3}$$

where $a \in L^\infty(\mathbb{S}^{N-1})$. In [15, 17] the quadratic form associated to the linear operator $-\Delta - \frac{a(x/|x|)}{|x|^2}$ is assumed to satisfy a coercivity type condition. More precisely, it is required that the first eigenvalue $\mu_1(0,a)$ of the spherical operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ satisfies $\mu_1(0,a) > -\left(\frac{N-2}{2}\right)^2$.

Due to this coercivity, it was quite natural in that setting looking for H^1-solutions to (3), i.e. functions $u \in H^1(\Omega)$ satisfying (3) in a variational sense, whereas, in the borderline situation considered in the present paper, it is reasonable to replace the classical H^1 Sobolev space with the above mentioned $\mathcal{H}$ space.

In the proof of our main result (Theorem 2.1 below) we perform an Almgren-type monotonicity procedure (see [2, 21]) and provide a characterization of the leading term in the asymptotic expansion by means of a Cauchy's integral type representation formula.

As an application of the main result, we also prove an a priori estimate and a unique continuation principle for solutions to (1); see [25] for questions related to unique continuation principles for elliptic equations with singular potentials.

This paper is organized as follows. In Section 2 we introduce the assumptions of the main result and explain in details what we mean by a $\mathcal{H}$-solution of (1). In Section 3 we describe the main properties of the space $\mathcal{H}$, while in Section 4 we reformulate (1) in cylindrical variables, introducing an auxiliary equation in a cylinder of $\mathbb{R}^{N+1}$. In Section 5 we study the Almgren-type function associated to the problem, which is combined in Section 6 with a blow-up argument to characterize the leading term in the asymptotic expansion of solutions of (1) near the origin, thus proving the main theorem.

2. Assumptions and main results

We first introduce the assumptions on the potential h and the nonlinearity f. We assume that h satisfies

(H) $\quad h \in L^\infty_{\rm loc}(\Omega \setminus \{0\}), \quad |h(x)| \leqslant C_h|x|^{-2+\varepsilon} \quad \text{in } \Omega \setminus \{0\}$ for some $C_h > 0$ and $\varepsilon > 0$.

It is not restrictive to assume that $\varepsilon \in (0,2)$. Let f satisfy

(F) $$\begin{cases} f \in C^0(\Omega \times \mathbb{R}), \quad F \in C^1(\Omega \times \mathbb{R}), \quad s \mapsto f(x,s) \in C^1(\mathbb{R}) \quad \text{for a.e. } x \in \Omega, \\ |f(x,s)s| + |f'_s(x,s)s^2| + |\nabla_x F(x,s) \cdot x| \leqslant C_f(|s|^2 + |s|^p) \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}, \end{cases}$$

where $F(x,s) = \int_0^s f(x,t)\,dt$, $2 < p < 2^* = \frac{2N}{N-2}$, $C_f > 0$ is a constant independent of $x \in \Omega$ and $s \in \mathbb{R}$, and $f'_s(x,s) = \frac{\partial f}{\partial s}(x,s)$.

In order to state the main result of this paper, a suitable variational formulation for solutions of (1) has to be introduced (see also [12, 20, 29]). For any bounded domain $\omega \subset \mathbb{R}^N$ containing the origin and satisfying $\partial\omega \in C^1$, let us define $\mathcal{H}(\omega)$ as the completion of the space $C_c^\infty(\overline{\omega}\setminus\{0\})$ with respect to the scalar product

$$(4)\qquad (u,v)_{\mathcal{H}(\omega)} := \int_\omega \nabla u(x)\cdot\nabla v(x)\,dx - \left(\frac{N-2}{2}\right)^2\int_\omega \frac{u(x)v(x)}{|x|^2}\,dx + \int_\omega u(x)v(x)\,dx + \frac{N-2}{2}\int_{\partial\omega}\frac{u(x)v(x)}{|x|^2}\,(x\cdot\nu(x))\,dS,\quad u,v\in C_c^\infty(\overline{\omega}\setminus\{0\}).$$

The form in (4) is actually a scalar product on $C_c^\infty(\overline{\omega}\setminus\{0\})$ as detailed in Section 3.

For any domain $\Omega\subseteq\mathbb{R}^N$ satisfying $0\in\Omega$ (with $\partial\Omega$ not necessarily in C^1), we define the space $\mathcal{H}_{\rm loc}(\Omega)$ as the space of functions $u\in H^1_{\rm loc}(\Omega\setminus\{0\})$ such that $u_{|\omega}\in\mathcal{H}(\omega)$ for any domain $\omega\Subset\Omega$ with $\partial\omega\in C^1$.

We are ready to provide a rigorous definition for solutions to (1). Let h, f satisfy respectively (**H**) and (**F**): by a solution of (1) we mean a function $u\in\mathcal{H}_{\rm loc}(\Omega)$ such that

$$(5)\qquad \int_\Omega \nabla u(x)\cdot\nabla\varphi(x)\,dx - \left(\frac{N-2}{2}\right)^2\int_\Omega\frac{u(x)}{|x|^2}\,\varphi(x)\,dx = \int_\Omega\big(h(x)u(x)+f(x,u(x))\big)\varphi(x)\,dx$$

for any $\varphi\in C_c^\infty(\Omega\setminus\{0\})$. We observe that every term in the above identity is well-defined in view of Proposition 3.2 and Proposition 3.5.

The above notion of solution corresponds to the notion of $\mathcal{H}(\Omega)$-solution introduced in [12, Section 6], as we will prove in Proposition 3.7. In other words, if $u\in\mathcal{H}_{\rm loc}(\Omega)$ is an $\mathcal{H}$-solution of (1), then for any $\omega\Subset\Omega$ with $\partial\omega\in C^1$ we have

$$(u,v)_{\mathcal{H}(\omega)} = \int_\omega (h(x)+1)u(x)v(x)\,dx + \int_\omega f(x,u(x))v(x)\,dx \quad \text{for any } v\in\mathcal{H}_0(\omega),$$

where $\mathcal{H}_0(\omega)$ is the closure in $\mathcal{H}(\omega)$ of the space $C_c^\infty(\omega\setminus\{0\})$.

In [12], the following notion of strong solution is also discussed. By a strong solution to (1) we mean a function $u\in C^2(\Omega\setminus\{0\})$ which solves (1) in the classical sense and satisfies the following pointwise estimate: for any $R>0$ there exists a constant $C=C(N,h,f,u,\Omega,R)$ depending only on N,h,f,u,Ω,R but independent of x such that

$$|u(x)|\leqslant C|x|^{-\frac{N-2}{2}}\quad\text{for any } x\in(\Omega\cap B_R)\setminus\{0\}.$$

Before giving the statement of our main result, we recall that the eigenvalues of the Laplace Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ are given by

$$\lambda_\ell = (N-2+\ell)\ell,\quad \ell=0,1,2,\dots,$$

having the ℓ-th eigenvalue λ_ℓ multiplicity

$$(6)\qquad m_\ell = \frac{(N-3+\ell)!(N+2\ell-2)}{\ell!(N-2)!},$$

and the eigenfunctions coincide with the usual spherical harmonics. For every $\ell\geqslant 0$, let $\{Y_{\ell,m}\}_{m=1}^{m_\ell}$ be a $L^2(\mathbb{S}^{N-1})$-orthonormal basis of the eigenspace of $-\Delta_{\mathbb{S}^{N-1}}$ associated to λ_ℓ with $Y_{\ell,m}$ being spherical harmonics of degree ℓ.

Theorem 2.1. *Let $N\geqslant 3$ and assume* (**H**), (**F**). *Let $u\in\mathcal{H}_{\rm loc}(\Omega)$ be a nontrivial $\mathcal{H}$-solution to (1). Then there exist $\ell_0\in\mathbb{N}$ and $\beta_{\ell_0,1},\dots,\beta_{\ell_0,m_{\ell_0}}\in\mathbb{R}$ such that $(\beta_{\ell_0,1},\dots,\beta_{\ell_0,m_{\ell_0}})\neq(0,\dots,0)$*

and, for any $\alpha \in (0,1)$,

$$r^{\frac{N-2}{2}-\sqrt{\lambda_{\ell_0}}}\,u(r,\theta) \to \sum_{m=1}^{m_{\ell_0}} \beta_{\ell_0,m} Y_{\ell_0,m}(\theta) \quad \text{in } C^{1,\alpha}(\mathbb{S}^{N-1}) \quad \text{as } r \to 0^+$$

and

$$r^{\frac{N}{2}-\sqrt{\lambda_{\ell_0}}}\nabla u(r,\theta) \to \sum_{m=1}^{m_{\ell_0}} \beta_{\ell_0,m}\left[\left(-\tfrac{N-2}{2}+\sqrt{\lambda_{\ell_0}}\right)Y_{\ell_0,m}(\theta)\theta + \nabla_{\mathbb{S}^{N-1}}Y_{\ell_0,m}(\theta)\right]$$

in $C^{0,\alpha}(\mathbb{S}^{N-1})$ *as* $r \to 0^+$*, where* $u(r,\theta)$ *is the representation on* u *in polar coordinates* $r \in (0,+\infty)$*,* $\theta \in \mathbb{S}^{N-1}$*. Moreover, the coefficients* $\beta_{\ell_0,1},\dots,\beta_{\ell_0,m_{\ell_0}}$ *admit the following representation*

$$\beta_{\ell_0,m} = \int_{\mathbb{S}^{N-1}}\left[\frac{u(R,\theta)}{R^{\widetilde{\gamma}}} + \int_0^R \frac{h(s,\theta)u(s,\theta)+f(s\theta,u(s,\theta))}{2\widetilde{\gamma}+N-2}\left(s^{-\widetilde{\gamma}+1} - \frac{s^{\widetilde{\gamma}+N-1}}{R^{2\widetilde{\gamma}+N-2}}\right)ds\right]Y_{\ell_0,m}(\theta)\,dS(\theta)$$

for any $R>0$ *such that* $B_R := \{x \in \mathbb{R}^N : |x| < R\} \subset \Omega$*, where* $\widetilde{\gamma} := -\frac{N-2}{2} + \sqrt{\lambda_{\ell_0}}$.

As a consequence of Theorem 2.1, the following pointwise estimates hold true.

Corollary 2.2. *Let* $N \geqslant 3$ *and assume* (**H**), (**F**)*. If* $u \in \mathcal{H}_{\rm loc}(\Omega)$ *is a nontrivial* $\mathcal{H}$*-solution to (1), then there exists* $\ell_0 \in \mathbb{N}$ *such that*

$$|u(x)| = O\Big(|x|^{-\frac{N-2}{2}+\sqrt{\lambda_{\ell_0}}}\Big) \quad \text{and} \quad |\nabla u(x)| = O\Big(|x|^{-\frac{N}{2}+\sqrt{\lambda_{\ell_0}}}\Big) \quad \text{as } |x| \to 0.$$

As observed in [12], from elliptic regularity theory, it follows easily that if a $\mathcal{H}$-solution u satisfies the decay condition $|u(x)| = O\big(|x|^{-\frac{N-2}{2}}\big)$, then u is necessarily a *strong solution* in the sense of [12]. Corollary 2.2 gives a stronger information; indeed, Corollary 2.2 implies that any nontrivial $\mathcal{H}$-solution u satisfies the decay condition $|u(x)| = O\big(|x|^{-\frac{N-2}{2}}\big)$ as $|x| \to 0$. Hence, from classical elliptic regularity theory, it follows that if u is an $\mathcal{H}$-solution to (1) and h, f are smooth outside 0, then u is a *strong solution* in the sense of [12].

We also observe that Corollary 2.2 implies that, if u changes sign in a neighborhood of 0, then $u \in H^1_{\rm loc}(\Omega)$.

As another byproduct of Theorem 2.1, we also have the following version of the *Strong Unique Continuation Principle* for an elliptic equation with a singular coefficient.

Corollary 2.3. *Let* $N \geqslant 3$ *and assume* (**H**), (**F**)*. Let* $u \in \mathcal{H}_{\rm loc}(\Omega)$ *be a* $\mathcal{H}$*-solution of (1). If* $u(x) = O(|x|^k)$ *as* $|x| \to 0^+$ *for any* $k \in \mathbb{N}$*, then* $u \equiv 0$ *in* Ω.

The proof of Theorem 2.1 is based on a monotonicity method. We give below a brief description of our argument. Using the Emden-Fowler transformation

$$v(t,\theta) = e^{-\frac{N-2}{2}t}u(e^{-t}\theta), \quad (t,\theta) \in \mathcal{C} := \mathbb{R} \times \mathbb{S}^{N-1},$$

see (7), equation (1) in a ball centered at 0 can be rewritten as

$$-\Delta_{\mathcal{C}} v = e^{-2t}\widetilde{h}v + e^{-2t}\widetilde{f}(t,\theta,v), \quad \text{in } (\bar{t},+\infty) \times \mathbb{S}^{N-1} \subset \mathcal{C},$$

for some $\bar{t}$, where $\Delta_{\mathcal{C}}$ is the Laplace-Beltrami operator on the cylinder $\mathcal{C}$ and $\widetilde{h}, \widetilde{f}$ are defined in (24). The proof of our results is based on the study at ∞ of the *Almgren frequency function* associated to v, which is defined as

$$\mathcal{N}(t) = \frac{D(t)}{H(t)}$$

where

$$D(t) := \int_{(t,+\infty)\times\mathbb{S}^{N-1}} \Big(|\nabla_{\mathcal{C}} v|^2 - e^{-2s}\widetilde{h}v^2 - e^{-2s}\widetilde{f}(s,\theta,v)v\Big)ds\,dS(\theta),$$

$$H(t) := \int_{\mathbb{S}^{N-1}} v^2(t,\theta)dS(\theta).$$

The first step of our strategy consists in proving that $\lim_{t\to+\infty}\mathcal{N}(t) := \gamma$ exists and it is finite, see Lemma 5.8. Next, we perform a blow-up argument by translating at infinity v (i.e. zooming around the origin the solution u) and normalizing by $\sqrt{H}$. More precisely, we define $w_\lambda(t,\theta) := \frac{v(t+\lambda,\theta)}{\sqrt{H(\lambda)}}$ and show that w_λ converges as $\lambda \to +\infty$ (in some Hölder and Sobolev spaces) to some w solving the limiting equation $-\Delta_{\mathcal{C}} w = 0$ on the cylinder $(0,+\infty)\times\mathbb{S}^{N-1}$: to this aim, assumptions (**H**) and (**F**) requiring negligibility of h with respect to the Hardy potential and at most criticality of f with respect to the Sobolev exponent play a crucial role. We refer to Lemma 6.1 for details.

The main point is that the Almgren's frequency for w satisfies $\mathcal{N}_w(t) = \lim_{\lambda\to+\infty}\mathcal{N}(t+\lambda) = \gamma$, i.e. $\mathcal{N}_w$ is constant; hence $w(t,\cdot)$ and $\frac{\partial w}{\partial s}(t,\cdot)$ are proportional in $L^2(\mathbb{S}^{N-1})$. Therefore $w(t,\theta) = \varphi(t)\psi(\theta)$. By separating variables on the cylinder, we deduce that ψ must be an eigenfunction of the Laplace Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ associate to some eigenvalue λ_{ℓ_0}. Since w has finite energy at ∞, we deduce that $\varphi(t)$ is proportional to $e^{-\lambda_{\ell_0}t}$ and that $\gamma = \sqrt{\lambda_{\ell_0}}$. The final step relies in deriving the exact asymptotics of the normalization of the blow-up family, i.e. in proving that $\lim_{t\to+\infty} e^{2\gamma t}H(t) > 0$, see Lemma 6.3.

Notation.

- For all $r > 0$, B_r denotes the open ball $\{x \in \mathbb{R}^N : |x| < r\}$ in $\mathbb{R}^N$ with center at 0 and radius r.
- $C_c^\infty(A)$ denotes the space of $C^\infty(A)$-functions whose support is compact in A.
- For any open set $\Omega \subseteq \mathbb{R}^N$, $\mathcal{D}'(\Omega)$ denotes the space of distributions on Ω.
- dS denotes the volume element on the spheres ∂B_r, $r > 0$.
- For any $N \geqslant 1$ we put $\omega_{N-1} := \int_{\mathbb{S}^{N-1}} dS$.

3. On $\mathcal{H}$-solutions to (1)

In this section we describe the main properties of the space $\mathcal{H}$ and of $\mathcal{H}$-solutions to (1). In the sequel, ω denotes a bounded domain in $\mathbb{R}^N$ satisfying $\partial\omega \in C^1$ and $0 \in \omega$. In order to reformulate (1) in cylindrical variables, we let

$$\Phi : \mathbb{R}^N \setminus \{0\} \to \mathcal{C} := \mathbb{R}\times\mathbb{S}^{N-1} \subset \mathbb{R}^{N+1}$$

be the diffeomorphism *(Emden-Fowler transformation)* defined as

$$\Phi(x) := \left(-\log|x|, \frac{x}{|x|}\right) \quad \text{for any } x \in \mathbb{R}^N \setminus \{0\}, \tag{7}$$

see [9], and let $\mathcal{C}_\omega := \Phi(\omega \setminus \{0\}) \subseteq \mathcal{C}$. Let us introduce the linear operator

$$\begin{aligned} &T : C_c^\infty(\overline{\omega}\setminus\{0\}) \to C_c^\infty(\overline{\mathcal{C}}_\omega), \\ &Tu(t,\theta) := e^{-\frac{N-2}{2}t}u(e^{-t}\theta), \quad \text{for any } (t,\theta)\in\mathcal{C},\ u \in C_c^\infty(\overline{\omega}\setminus\{0\}). \end{aligned} \tag{8}$$

Clearly T is an isomorphism between vector spaces. Let us denote by μ the standard volume measure on the cylinder $\mathcal{C}$, by $\nabla_{\mathcal{C}}$ the gradient associated with the standard Riemannian metric of $\mathcal{C}$, and by (t,θ) the generic element of $\mathcal{C}$.

We observe that $(\cdot,\cdot)_{\mathcal{H}(\omega)}$ as defined in (4) is actually a scalar product on $C^\infty_c(\overline{\omega}\setminus\{0\})$ since the following identities hold for any $u\in C^\infty_c(\overline{\omega}\setminus\{0\})$, see [8, 9, 30]:

$$\int_\omega |\nabla u(x)|^2dx-\left(\frac{N-2}{2}\right)^2\int_\omega \frac{u^2(x)}{|x|^2}\,dx + \frac{N-2}{2}\int_{\partial\omega}\frac{u^2(x)}{|x|^2}\,(x\cdot\nu(x))\,dS=\int_{\mathcal{C}_\omega}|\nabla_{\mathcal{C}}(Tu)|^2d\mu\geqslant 0, \tag{9}$$

$$\int_\omega u^2(x)\,dx=\int_{\mathcal{C}_\omega} e^{-2t}(Tu)^2d\mu. \tag{10}$$

We also define, for any ω as above, the weighted Sobolev space $H_\mu(\mathcal{C}_\omega)$ as the completion of $C^\infty_{\rm c}(\overline{\mathcal{C}}_\omega)$ with respect to the norm

$$\|w\|_{H_\mu(\mathcal{C}_\omega)}:=\left(\int_{\mathcal{C}_\omega}|\nabla_{\mathcal{C}}w|^2d\mu+\int_{\mathcal{C}_\omega}e^{-2t}w^2d\mu\right)^{1/2}.$$

By density and continuity it is possible to extend T as a linear and continuous operator from $\mathcal{H}(\omega)$ to $H_\mu(\mathcal{C}_\omega)$. In this way $T:\mathcal{H}(\omega)\to H_\mu(\mathcal{C}_\omega)$ becomes an isometric isomorphism.

The following proposition relates $\mathcal{H}(\omega)$ with the classical Sobolev space $H^1(\omega)$.

Proposition 3.1. *Let $\omega\subset\mathbb{R}^N$ be a bounded domain satisfying $0\in\omega$ and $\partial\omega\in C^1$. Then $H^1(\omega)\subset\mathcal{H}(\omega)$ with continuous embedding.*

We omit the proof of Proposition 3.1 which can be easily obtained by classical density arguments. We observe the inclusion $H^1(\omega)\subset\mathcal{H}(\omega)$ is actually strict, since the function $|x|^{-\frac{N-2}{2}}$ belongs to $\mathcal{H}(\omega)$ but not to $H^1(\omega)$.

Proposition 3.2. *Let $\omega\subset\mathbb{R}^N$ be a bounded domain satisfying $0\in\omega$ and $\partial\omega\in C^1$. Then $\mathcal{H}(\omega)\subseteq H^1_{\rm loc}(\overline{\omega}\setminus\{0\})\cap L^2(\omega)$ where by $H^1_{\rm loc}(\overline{\omega}\setminus\{0\})$ we mean the space of functions which belong to $H^1(A)$ for any open set A satisfying $\overline{A}\subseteq\overline{\omega}\setminus\{0\}$.*

Proof. Let $u\in\mathcal{H}(\omega)$ and let $\{u_n\}\subset C^\infty_c(\overline{\omega}\setminus\{0\})$ be a sequence such that $u_n\to u$ in $\mathcal{H}(\omega)$. Then, for any n, Tu_n belongs to $H_\mu(\mathcal{C}_\omega)$. By (8), (9), (10) and direct calculations, for any open set A such that $\overline{A}\subseteq\overline{\omega}\setminus\{0\}$, we have that, denoting $V_{n,m}=T(u_n-u_m)$,

$$\begin{aligned}\|u_n-u_m\|^2_{H^1(A)}&=\int_{\mathcal{C}_A}\left[|\nabla_{\mathcal{C}}V_{n,m}|^2+\left(\tfrac{N-2}{2}\right)^2V^2_{n,m}\right]d\mu+\int_{\mathcal{C}_A}\left[\tfrac{N-2}{2}\,\partial_t(V^2_{n,m})+e^{-2t}V^2_{n,m}\right]d\mu\\ &\leqslant\int_{\mathcal{C}_A}\left[|\nabla_{\mathcal{C}}V_{n,m}|^2+\left(\tfrac{N-2}{2}\right)^2V^2_{n,m}\right]d\mu+\int_{\mathcal{C}_A}\left[\tfrac{N-2}{2}|\partial_tV_{n,m}|^2+\tfrac{N-2}{2}V^2_{n,m}+e^{-2t}V^2_{n,m}\right]d\mu\\ &\leqslant K_A\left(\int_{\mathcal{C}_A}|\nabla_{\mathcal{C}}V_{n,m}|^2d\mu+\int_{\mathcal{C}_A}e^{-2t}V^2_{n,m}d\mu\right)\\ &\leqslant K_A\left(\int_{\mathcal{C}_\omega}|\nabla_{\mathcal{C}}V_{n,m}|^2d\mu+\int_{\mathcal{C}_\omega}e^{-2t}V^2_{n,m}d\mu\right)=K_A\|u_n-u_m\|^2_{\mathcal{H}(\omega)}\end{aligned}\tag{11}$$

where $K_A := \max\left\{\frac{N}{2}, 1 + \frac{N(N-2)}{4}\sup_{(t,\theta)\in\mathcal{C}_A} e^{2t}\right\}$. Then $\{u_n\}$ is a Cauchy sequence in $H^1(A)$ for any A as above and hence $u \in \mathcal{H}(\omega)$ may be seen as a function in $H^1_{\rm loc}(\omega \setminus \{0\})$. Moreover, by (4), (9) and (10) it is clear that

$$\|u_n - u_m\|_{L^2(\omega)} \leqslant \|u_n - u_m\|_{\mathcal{H}(\omega)}$$

and hence $\{u_n\}$ is a Cauchy sequence in $L^2(\omega)$. This completes the proof of the proposition. □

In [7, Extension 4.3] the following Poincaré-Sobolev type inequality was proved.

Proposition 3.3. [7, Extension 4.3] *Let $\omega \subset \mathbb{R}^N$ be a bounded domain satisfying $0 \in \omega$ and let $1 \leqslant q < \frac{2N}{N-2}$. Then there exists a constant $C(\omega, q)$ such that*

$$\left(\int_\omega |u(x)|^q\,dx\right)^{2/q} \leqslant C(\omega,q)\left[\int_\omega |\nabla u(x)|^2 dx - \left(\frac{N-2}{2}\right)^2\int_\omega \frac{u^2(x)}{|x|^2}\,dx\right] \tag{12}$$

for any $u \in C_c^\infty(\omega \setminus \{0\})$.

Let us consider the space $\mathcal{H}_0(\omega)$ defined in Section 2 as the closure in $\mathcal{H}(\omega)$ of $C_c^\infty(\omega \setminus \{0\})$, see also [12, Section 6]. If we define the scalar product

$$(u,v)_{\mathcal{H}_0(\omega)} := \int_\omega \nabla u(x)\cdot\nabla v(x)\,dx - \left(\frac{N-2}{2}\right)^2\int_\omega \frac{u(x)v(x)}{|x|^2}\,dx, \quad \text{for any } u,v \in C_c^\infty(\omega\setminus\{0\}), \tag{13}$$

then by (12) with $q = 2$ we deduce that the norms $\|\cdot\|_{\mathcal{H}(\omega)}$ and $\|\cdot\|_{\mathcal{H}_0(\omega)}$ are equivalent on $C_c^\infty(\omega \setminus \{0\})$. Hence $\mathcal{H}_0(\omega)$ may be endowed with the equivalent scalar product obtained by density, extending the scalar product $(\cdot,\cdot)_{\mathcal{H}_0(\omega)}$ defined in (13) to the whole $\mathcal{H}_0(\omega) \times \mathcal{H}_0(\omega)$.

By Proposition 3.3 and the definition of $\mathcal{H}_0(\omega)$, the following Sobolev type embedding follows.

Proposition 3.4. *Let $\omega \subset \mathbb{R}^N$ be a bounded domain satisfying $0 \in \omega$ and let $1 \leqslant q < \frac{2N}{N-2}$. Then $\mathcal{H}_0(\omega) \subset L^q(\omega)$ with continuous embedding.*

Actually the continuous embedding $\mathcal{H}(\omega) \subset L^q(\omega)$, $1 \leqslant q < 2N/(N-2)$, also holds true as shown Proposition 3.5.

Proposition 3.5. *Let $\omega \subset \mathbb{R}^N$ be a bounded domain satisfying $0 \in \omega$ and let $1 \leqslant q < \frac{2N}{N-2}$. Then $\mathcal{H}(\omega) \subset L^q(\omega)$ with continuous embedding.*

Proof. Let $u \in \mathcal{H}(\omega)$. Then by Proposition 3.2 we deduce that $u \in L^q(\omega \setminus \overline{B}_\delta)$ for any $\delta > 0$ such that $\overline{B}_\delta \subset \omega$. Moreover, arguing as in (11), we infer that there exists a constant $C(N, q, \delta)$ depending only on N, q, δ, ω such that

$$\|u\|_{L^q(\omega\setminus B_\delta)} \leqslant C(N,q,\delta)\|u\|_{\mathcal{H}(\omega)}\,. \tag{14}$$

Let us prove that for some fixed $\delta > 0$ chosen as above $u \in L^q(B_\delta)$. To this purpose let $\eta \in C_c^\infty(\omega)$ be a radial function such that $0 \leqslant \eta \leqslant 1$ and $\eta \equiv 1$ in $\overline{B}_\delta$. Let $\{u_n\} \subset C_c^\infty(\overline{\omega} \setminus \{0\})$ be a sequence such that $u_n \to u$ in $\mathcal{H}(\omega)$. Then by (9), (10), (12) we obtain

$$\begin{aligned}(15)\quad \left(\int_\omega |(u_n-u_m)\eta|^q dx\right)^{\frac{2}{q}} &\leqslant C(\omega,q)\left[\int_\omega |\nabla((u_n-u_m)\eta)|^2 dx - \left(\tfrac{N-2}{2}\right)^2\int_\omega \frac{|(u_n-u_m)\eta|^2}{|x|^2}\,dx\right]\\ &= C(\omega,q)\int_{\mathcal{C}_\omega} |\nabla_{\mathcal{C}}(T((u_n-u_m)\eta))|^2 d\mu\\ &\leqslant 2\int_{\mathcal{C}_\omega} \eta^2(e^{-t}\theta)|\nabla_{\mathcal{C}}(T(u_n-u_m))|^2 d\mu + 2\int_{\mathcal{C}_\omega} |T(u_n-u_m)|^2|\nabla_{\mathcal{C}}(\eta(e^{-t}\theta))|^2 d\mu\\ &\leqslant 2\int_{\mathcal{C}_\omega} |\nabla_{\mathcal{C}}(T(u_n-u_m))|^2 d\mu + 2\|\nabla\eta\|^2_{L^\infty(\omega)}\int_{\mathcal{C}_\omega} e^{-2t}|T(u_n-u_m)|^2 d\mu\\ &\leqslant 2(1+\|\nabla\eta\|^2_{L^\infty(\omega)})\|u_n-u_m\|^2_{\mathcal{H}(\omega)}\,.\end{aligned}$$

This shows that $\{u_n\eta\}$ is a Cauchy sequence in $L^q(\omega)$. Since $u_n\eta \to u\eta$ pointwise then $u\eta \in L^q(\omega)$. In particular $u \in L^q(B_\delta)$. Moreover proceeding as in (15) we also have that

$$\begin{aligned}(16)\quad \|u\|_{L^q(B_\delta)} &\leqslant \|\eta u\|_{L^q(\omega)} = \lim_{n\to+\infty}\|\eta u_n\|_{L^q(\omega)}\\ &\leqslant \lim_{n\to+\infty}[2(1+\|\nabla\eta\|^2_{L^\infty(\omega)})]^{1/2}\|u_n\|_{\mathcal{H}(\omega)} = [2(1+\|\nabla\eta\|^2_{L^\infty(\omega)})]^{1/2}\|u\|_{\mathcal{H}(\omega)}\,.\end{aligned}$$

Combining (14) and (16) we conclude that $\mathcal{H}(\omega) \subset L^q(\omega)$ with continuous embedding. □

From Propositions 3.2 and 3.5 we infer that, if $u \in \mathcal{H}_{\rm loc}(\Omega)$, then $u \in H^1_{\rm loc}(\Omega\setminus\{0\}) \cap L^q_{\rm loc}(\Omega)$ for all $1 \leqslant q < 2N/(N-2)$.

Remark 3.6. From Proposition 3.2, we have that if $u \in \mathcal{H}_{\rm loc}(\Omega)$ is a solution to (1) in the sense of (5), then u is a weak H^1-solution in $\Omega\setminus\{0\}$. Hence, classical Brezis-Kato [6] estimates, bootstrap, and elliptic regularity theory, imply that $u \in H^2_{\rm loc}(\Omega\setminus\{0\}) \cap C^{1,\alpha}_{\rm loc}(\Omega\setminus\{0\})$ for any $\alpha \in (0,1)$.

From (5) we deduce the following characterizations of solutions to (1).

Proposition 3.7. *Let h satisfy* (**H**), *f satisfy* (**F**) *and $u \in \mathcal{H}_{\rm loc}(\Omega)$ be a solution to (1) in the sense of (5). Then u solves (1) in the sense of distributions in Ω, i.e.*

$$(17)\quad -\int_\Omega u(x)\Delta\varphi(x)\,dx - \left(\frac{N-2}{2}\right)^2\int_\Omega \frac{u(x)}{|x|^2}\,\varphi(x)\,dx = \int_\Omega \big(h(x)u(x)+f(x,u(x))\big)\varphi(x)\,dx$$

for any $\varphi \in C^\infty_c(\Omega)$. Moreover, for any bounded domain ω with $\partial\omega \in C^1$ and $\overline{\omega} \subset \Omega$, we have that

$$(18)\quad (u,v)_{\mathcal{H}(\omega)} = \int_\omega (h(x)+1)u(x)v(x)\,dx + \int_\omega f(x,u(x))v(x)\,dx$$

for all $v \in \mathcal{H}_0(\omega)$ and

$$(19)\quad (u,v)_{\mathcal{H}(\omega)} = \int_\omega (h+1)uv\,dx + \int_\omega f(x,u)v\,dx + \int_{\partial\omega}\frac{\partial u}{\partial\nu}v\,dS + \frac{N-2}{2}\int_{\partial\omega}\frac{uv}{|x|^2}\,(x\cdot\nu)\,dS$$

for all $v \in \mathcal{H}(\omega)$.

Proof. Let $\varphi \in C^\infty_c(\Omega)$ and let ω be an open domain with smooth boundary satisfying $\overline{\omega} \subset \Omega$ and $\operatorname{supp}\varphi \subset \omega$. Since $u \in \mathcal{H}_{\rm loc}(\Omega)$ then $u \in \mathcal{H}(\omega)$ and there exists a sequence $\{u_n\} \subset C^\infty_c(\overline{\omega}\setminus\{0\})$

such that $u_n \to u$ in $\mathcal{H}(\omega)$. Since $\varphi \in H^1_0(\omega)$, Proposition 3.1 implies that there exists a sequence $\{\varphi_m\} \subset C^\infty_c(\omega \setminus \{0\})$ such that $\varphi_m \to \varphi$ in $H^1(\omega)$ and in $\mathcal{H}(\omega)$ as $m \to +\infty$. Hence we have

$$
\begin{aligned}
(u,\varphi)_{\mathcal{H}(\omega)} &= \lim_{n\to+\infty}\left(\lim_{m\to+\infty}(u_n,\varphi_m)_{\mathcal{H}(\omega)}\right) \\
&= \lim_{n\to+\infty}\left\{\lim_{m\to+\infty}\left[\int_\omega \nabla u_n\cdot\nabla\varphi_m\,dx - \left(\frac{N-2}{2}\right)^2\int_\omega \frac{u_n\varphi_m}{|x|^2}\,dx + \int_\omega u_n\varphi_m\,dx\right]\right\} \\
&= \lim_{n\to+\infty}\left[\int_\omega \nabla u_n\cdot\nabla\varphi\,dx - \left(\frac{N-2}{2}\right)^2\int_\omega \frac{u_n\varphi}{|x|^2}\,dx + \int_\omega u_n\varphi\,dx\right] \\
&= \lim_{n\to+\infty}\left[-\int_\omega u_n\Delta\varphi\,dx - \left(\frac{N-2}{2}\right)^2\int_\omega \frac{u_n\varphi}{|x|^2}\,dx + \int_\omega u_n\varphi\,dx\right].
\end{aligned} \tag{20}
$$

By Proposition 3.5 we also have that $u_n \to u$ in $L^q(\omega)$ for any $1 \leqslant q < 2^*$. By Hölder inequality with $\frac{N}{N-2} < p < \frac{2N}{N-2}$ we have that $\frac{u\varphi}{|x|^2} \in L^1(\omega)$ and

$$
\left|\int_\omega \frac{u_n(x)\varphi(x)}{|x|^2}\,dx - \int_\omega \frac{u(x)\varphi(x)}{|x|^2}\,dx\right| \leqslant \|\varphi\|_{L^\infty(\omega)}\left(\int_\omega |u_n(x)-u(x)|^q dx\right)^{1/q}\left(\int_\omega |x|^{-\frac{2q}{q-1}}\,dx\right)^{\frac{q-1}{q}}
$$

and hence passing to the limit in (20) we obtain

$$
(u,\varphi)_{\mathcal{H}(\omega)} = -\int_\omega u(x)\Delta\varphi(x)\,dx - \left(\frac{N-2}{2}\right)^2\int_\omega \frac{u(x)\varphi(x)}{|x|^2}\,dx + \int_\omega u(x)\varphi(x)\,dx\,. \tag{21}
$$

On the other hand, by the convergence $u_n \to u$ in $H^1_{\rm loc}(\overline{\omega}\setminus\{0\})$, see Proposition 3.2, we obtain

$$
\begin{aligned}
(u,\varphi)_{\mathcal{H}(\omega)} &= \lim_{m\to+\infty}\left(\lim_{n\to+\infty}(u_n,\varphi_m)_{\mathcal{H}(\omega)}\right) \\
&= \lim_{m\to+\infty}\left\{\lim_{n\to+\infty}\left[\int_\omega \nabla u_n\cdot\nabla\varphi_m\,dx - \left(\frac{N-2}{2}\right)^2\int_\omega \frac{u_n\varphi_m}{|x|^2}\,dx + \int_\omega u_n\varphi_m\,dx\right]\right\} \\
&= \lim_{m\to+\infty}\left[\int_\omega \nabla u\cdot\nabla\varphi_m\,dx - \left(\frac{N-2}{2}\right)^2\int_\omega \frac{u\varphi_m}{|x|^2}\,dx + \int_\omega u\varphi_m\,dx\right] \\
&= \lim_{m\to+\infty}\left[\int_\omega (h+1)u\varphi_m\,dx + \int_\omega f(x,u)\varphi_m\,dx\right] = \int_\omega (h+1)u\varphi\,dx + \int_\omega f(x,u)\varphi\,dx
\end{aligned} \tag{22}
$$

where the last identity follows from assumptions (**H**) and (**F**) and the fact that $\varphi_m \to \varphi$ in $L^q(\omega)$ for any $1 \leqslant q < \frac{2N}{N-2}$. Combining (21) and (22) obtain (17).

The proof of (18) follows by the following density argument: let $\{v_m\} \subset C^\infty_c(\omega\setminus\{0\})$ such that $v_m \to v$ in $\mathcal{H}(\omega)$. Now it is enough to pass to the limit as $m \to +\infty$ in (22) with v_m in place of φ.

It remains to prove (19). By elliptic regularity estimates $u \in C^1(\overline{\omega}\setminus\{0\})$ (see Remark 3.6) and hence the normal derivative of u on $\partial\omega$ is continuous. Let $v \in \mathcal{H}(\omega)$ and let $\{v_m\} \subset C^\infty_c(\overline{\omega}\setminus\{0\})$

be such that $v_m \to v$ in $\mathcal{H}(\omega)$. Therefore we are allowed to integrate by parts to obtain

$$(u, v_m)_{\mathcal{H}(\omega)} = \int_\omega \nabla u \cdot \nabla v_m \, dx - \left(\frac{N-2}{2}\right)^2 \int_\omega \frac{u}{|x|^2} v_m \, dx + \int_\omega u v_m \, dx + \frac{N-2}{2} \int_{\partial\omega} \frac{u v_m}{|x|^2} (x \cdot \nu) \, dS$$

$$= \int_\omega -(\Delta u) v_m \, dx + \int_{\partial\omega} \frac{\partial u}{\partial \nu} v_m \, dS - \left(\frac{N-2}{2}\right)^2 \int_\omega \frac{u}{|x|^2} v_m \, dx + \int_\omega u v_m \, dx + \frac{N-2}{2} \int_{\partial\omega} \frac{u v_m}{|x|^2} (x \cdot \nu) \, dS$$

$$= \int_\omega h u v_m \, dx + \int_\omega f(x, u) v_m \, dx + \int_{\partial\omega} \frac{\partial u}{\partial \nu} v_m \, dS + \int_\omega u v_m \, dx + \frac{N-2}{2} \int_{\partial\omega} \frac{u v_m}{|x|^2} (x \cdot \nu) \, dS \, .$$

The proof of (19) follows passing to the limit as $m \to +\infty$. □

4. An equivalent problem on the cylinder $\mathcal{C}$

Reformulation of (5) in cylindrical variables yields the following characterizations of solutions to (1).

Proposition 4.1. *Let h satisfy* (**H**)*, f satisfy* (**F**)*, and $u \in \mathcal{H}_{\rm loc}(\Omega)$ be a solution to (1) in the sense of (5). If ω is a bounded domain with $\partial\omega \in C^1$ and $\overline{\omega} \subset \Omega$, then the function $v := Tu \in H_\mu(\mathcal{C}_\omega)$ is a weak solution of the equation*

$$-\Delta_{\mathcal{C}} v(t, \theta) = e^{-2t} \widetilde{h}(t, \theta) v(t, \theta) + e^{-2t} \widetilde{f}(t, \theta, v(t, \theta)), \quad \text{in } \mathcal{C}_\omega, \tag{23}$$

where $\Delta_{\mathcal{C}}$ denotes the Laplace-Beltrami operator on $\mathcal{C}$ and

$$\widetilde{h}(t, \theta) := h(e^{-t}\theta), \quad \widetilde{f}(t, \theta, s) := e^{-\frac{N-2}{2}t} f\Big(e^{-t}\theta, e^{\frac{N-2}{2}t} s\Big), \quad \text{for any } (t, \theta) \in \mathcal{C}_\omega, \tag{24}$$

in the sense that

$$\int_{\mathcal{C}_\omega} \nabla_{\mathcal{C}} v \cdot \nabla_{\mathcal{C}} w \, d\mu = \int_{\mathcal{C}_\omega} e^{-2t} \big(\widetilde{h} v + \widetilde{f}(t, \theta, v)\big) w \, d\mu, \quad \text{for every } w \in H_{\mu,0}(\mathcal{C}_\omega) := T(\mathcal{H}_0(\omega)). \tag{25}$$

Moreover

$$\int_{\mathcal{C}_\omega} \nabla_{\mathcal{C}} v \cdot \nabla_{\mathcal{C}} w \, d\mu = \int_{\partial\mathcal{C}_\omega} (\nabla_{\mathcal{C}} v \cdot \nu_{\partial\mathcal{C}_\omega}) \, w \, dS + \int_{\mathcal{C}_\omega} e^{-2t} \big(\widetilde{h} v + \widetilde{f}(t, \theta, v)\big) w \, d\mu, \tag{26}$$

for every $w \in H_\mu(\mathcal{C}_\omega)$, where $\nu_{\partial\mathcal{C}_\omega}$ denotes the exterior normal vector to $\partial\mathcal{C}_\omega$ on $\mathcal{C}$.

The following corollary is an immediate consequence of (26).

Corollary 4.2. *Let h satisfy* (**H**)*, let f satisfy* (**F**)*, and let $u \in \mathcal{H}_{\rm loc}(\Omega)$ be a solution to (1) in the sense of (5). For any $t \in \mathbb{R}$, let*

$$\mathcal{C}_t := \{(s, \theta) \in \mathcal{C} : s > t, \theta \in \mathbb{S}^{N-1}\}, \quad \Gamma_t := \{(t, \theta) \in \mathcal{C} : \theta \in \mathbb{S}^{N-1}\}.$$

Then for any t such that $\overline{\mathcal{C}}_t \subset \mathcal{C}_\Omega$, the function $v := Tu \in H_\mu(\mathcal{C}_t)$ satisfies

$$\int_{\mathcal{C}_t} \nabla_{\mathcal{C}} v \cdot \nabla_{\mathcal{C}} w \, d\mu = -\int_{\Gamma_t} \frac{\partial v}{\partial s} w \, dS + \int_{\mathcal{C}_t} e^{-2s} \big(\widetilde{h}(s, \theta) v(s, \theta) + \widetilde{f}(s, \theta, v(s, \theta))\big) w(s, \theta) \, d\mu \tag{27}$$

for any $w \in H_\mu(\mathcal{C}_t)$.

In order to study solutions to (23), the properties of space H_μ have to be investigated. The next results go in this direction.

Lemma 4.3. *For every $t \in \mathbb{R}$, $H_\mu(\mathcal{C}_t) \hookrightarrow L^2(\Gamma_t)$ with compact embedding. Furthermore,*

$$v \mapsto \left(\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2 d\mu + \int_{\Gamma_t} v^2 dS\right)^{1/2} \tag{28}$$

is an equivalent norm in $H_\mu(\mathcal{C}_t)$; more precisely, there exists a constant $C > 0$ such that, for all $t \in \mathbb{R}$ and $v \in H_\mu(\mathcal{C}_t)$,

$$\begin{aligned} \frac{1}{C}\bigg(\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2\, d\mu + e^{2t}\int_{\mathcal{C}_t} e^{-2s} v^2\, d\mu\bigg) &\leqslant \int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2 d\mu + \int_{\Gamma_t} v^2 dS \\ &\leqslant C\bigg(\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2\, d\mu + e^{2t}\int_{\mathcal{C}_t} e^{-2s} v^2\, d\mu\bigg). \end{aligned} \tag{29}$$

Proof. The embedding $H_\mu(\mathcal{C}_t) \hookrightarrow L^2(\Gamma_t)$ and its compactness are just a consequence of the fact that $T : \mathcal{H}(\omega) \to H_\mu(\mathcal{C}_\omega)$ is an isometric isomorphism combined with Proposition 3.2 and compactness of classical Sobolev trace embeddings. To show that the quadratic form in (28) is an equivalent norm in $H_\mu(\mathcal{C}_t)$, we notice that, for all $v \in C^\infty_{\rm c}(\overline{\mathcal{C}}_t)$, integration by parts yields

$$\begin{aligned} \int_{\mathcal{C}_t} e^{-2s} v^2(s,\theta)\, d\mu(s,\theta) &= \int_{\mathbb{S}^{N-1}} \bigg(\int_t^{+\infty} e^{-2s} v^2(s,\theta) ds\bigg) dS(\theta) \\ &= \int_{\mathbb{S}^{N-1}} \bigg(\Big[-\frac{1}{2} e^{-2s} v^2(s,\theta)\Big]_{s=t}^{s=+\infty} + \int_t^{+\infty} e^{-2s}\frac{dv}{ds}(s,\theta) v(s,\theta) ds\bigg) dS(\theta) \\ &= \frac{1}{2} e^{-2t}\int_{\Gamma_t} v^2 dS + \int_{\mathcal{C}_t} e^{-2s} v \frac{dv}{ds}\, d\mu, \end{aligned} \tag{30}$$

which implies

$$\int_{\mathcal{C}_t} e^{-2s} v^2\, d\mu \leqslant \frac{1}{2} e^{-2t}\int_{\Gamma_t} v^2 dS + \frac{1}{2}\int_{\mathcal{C}_t} e^{-2s} v^2\, d\mu + \frac{e^{-2t}}{2}\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2\, d\mu$$

and hence

$$\int_{\mathcal{C}_t} e^{-2s} v^2\, d\mu \leqslant e^{-2t}\bigg(\int_{\Gamma_t} v^2 dS + \int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2\, d\mu\bigg).$$

On the other hand, (30) also implies

$$e^{-2t}\int_{\Gamma_t} v^2 dS \leqslant 3\int_{\mathcal{C}_t} e^{-2s} v^2\, d\mu + e^{-2t}\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2\, d\mu.$$

The conclusion then follows by density. □

The following lemma provides a Hardy type inequality with boundary terms.

Lemma 4.4. *For every $\sigma > 0$ and $t \in \mathbb{R}$, $H_\mu(\mathcal{C}_t) \subset L^2(\mathcal{C}_t, e^{-\sigma s} d\mu)$. Furthermore, for every $\sigma > 0$ there exists $\widetilde{C}_\sigma > 0$ such that, for all $t \in \mathbb{R}$ and $v \in H_\mu(\mathcal{C}_t)$,*

$$\int_{\mathcal{C}_t} e^{-\sigma s} v^2(s,\theta) d\mu(s,\theta) \leqslant \widetilde{C}_\sigma e^{-\sigma t}\bigg(\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2 d\mu + \int_{\Gamma_t} v^2 dS\bigg).$$

Proof. For all $v \in C^\infty_c(\overline{\mathcal{C}}_t)$, integration by parts yields

$$\begin{aligned}\int_{\mathcal{C}_t} e^{-\sigma s} v^2(s,\theta)\, d\mu(s,\theta) &= \int_{\mathbb{S}^{N-1}} \left(\int_t^{+\infty} e^{-\sigma s} v^2(s,\theta) ds \right) dS(\theta) \\ &= \int_{\mathbb{S}^{N-1}} \left(\left[-\frac{1}{\sigma} e^{-\sigma s} v^2(s,\theta) \right]_{s=t}^{s=+\infty} + \frac{2}{\sigma} \int_t^{+\infty} e^{-\sigma s} \frac{dv}{ds}(s,\theta) v(s,\theta) ds \right) dS(\theta) \\ &= \frac{1}{\sigma} e^{-\sigma t} \int_{\Gamma_t} v^2 dS + \frac{2}{\sigma} \int_{\mathcal{C}_t} e^{-\sigma s} v \frac{dv}{ds}\, d\mu,\end{aligned}$$

which implies

$$\int_{\mathcal{C}_t} e^{-\sigma s} v^2\, d\mu \leqslant \frac{1}{\sigma} e^{-\sigma t} \int_{\Gamma_t} v^2 dS + \frac{1}{2} \int_{\mathcal{C}_t} e^{-\sigma s} v^2\, d\mu + \frac{2 e^{-\sigma t}}{\sigma^2} \int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2\, d\mu$$

and hence

$$\int_{\mathcal{C}_t} e^{-\sigma s} v^2\, d\mu \leqslant e^{-\sigma t} \left(\frac{2}{\sigma} \int_{\Gamma_t} v^2 dS + \frac{4}{\sigma^2} \int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2\, d\mu \right).$$

The conclusion thereby follows with $\widetilde{C}_\sigma = \max\{2/\sigma, 4/\sigma^2\}$. □

The following Hardy-Sobolev type inequality holds.

Lemma 4.5. *For every $q \in \left[1, \frac{2N}{N-2}\right)$, there exists $C_{N,q} > 0$ such that, for all $t \in \mathbb{R}$ and $v \in H_\mu(\mathcal{C}_t)$,*

$$\left(\int_{\mathcal{C}_t} e^{(-N + \frac{N-2}{2} q)s} |v(s,\theta)|^q d\mu(s,\theta) \right)^{2/q} \leqslant C_{N,q}\, e^{(-\frac{2N}{q} + N - 2)t} \left(\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2 d\mu + \int_{\Gamma_t} v^2 dS \right).$$

Proof. From Proposition 3.5, there exists $c_{N,q} > 0$ such that

$$\left(\int_{B_1} |u(x)|^q dx \right)^{1/q} \leqslant c_{N,q} \|u\|_{\mathcal{H}(B_1)}$$

for all $u \in \mathcal{H}(B_1)$. Performing the change of variable $v(s,\theta) = Tu(s-t,\theta)$ in the above inequality for all $t \in \mathbb{R}$ and taking into account (29), we obtain the stated inequality with $C_{N,q} = c_{N,q}^2 C$. □

5. The Almgren frequency function

In this section, our purpose would be to construct an appropriate Almgren-type frequency function for the solution to problem (5). Since for a general function $u \in \mathcal{H}(\omega)$, the norm $\|u\|_{\mathcal{H}(\omega)}$ cannot be expressed in an integral form, we prefer to look for an Algrem-type function associated with the function $v := Tu$.

In a domain $\Omega \subset \mathbb{R}^N$, let $u \in \mathcal{H}_{\mathrm{loc}}(\Omega)$ be a solution of (5). Let $R > 0$ be such that $\overline{B}_R \subset \Omega$. According with [2, 21] (see also [14, 15, 16, 17]), for $t > -\log R$, we define the functions

$$D(t) := \int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2 d\mu - \int_{\mathcal{C}_t} e^{-2s} \widetilde{h} v^2 d\mu - \int_{\mathcal{C}_t} e^{-2s} \widetilde{f}(s,\theta,v) v\, d\mu \tag{31}$$

and

$$H(t) := \int_{\Gamma_t} v^2 dS, \tag{32}$$

where $v := Tu$ and T is defined in (8).

Lemma 5.1. *Let Ω be a domain in $\mathbb{R}^N$ and $u \in \mathcal{H}_{\rm loc}(\Omega)$ be a solution of (5), $u \not\equiv 0$, with h satisfying* (**H**) *and f satisfying* (**F**). *Let $H = H(t)$ be the function defined in (32). Then there exists $\bar t > 0$ such that $H(t) > 0$ for any $t > \bar t$.*

Proof. Let us argue by contradiction and assume that there exists $t_n \to +\infty$ such that $H(t_n) = 0$; in particular $v = 0$ on Γ_{t_n} and $v \in H_{\mu,0}(\mathcal{C}_{t_n})$. From (27), (**H**), (**F**), and Lemmas 4.4 and 4.5

$$\begin{aligned}
0 &= \int_{\mathcal{C}_{t_n}} |\nabla_{\mathcal{C}} v|^2 d\mu - \int_{\mathcal{C}_{t_n}} e^{-2s}\widetilde{h} v^2 d\mu - \int_{\mathcal{C}_{t_n}} e^{-2s}\widetilde{f}(s,\theta,v) v\, d\mu \\
&\geqslant \int_{\mathcal{C}_{t_n}} |\nabla_{\mathcal{C}} v|^2 d\mu - C_h \int_{\mathcal{C}_{t_n}} e^{-\varepsilon s} v^2 d\mu - C_f \int_{\mathcal{C}_{t_n}} e^{-2s} v^2 d\mu - C_f \int_{\mathcal{C}_{t_n}} e^{\left(-N+\frac{N-2}{2}p\right)s} |v|^p d\mu \\
&\geqslant \left(1 - C_h \widetilde{C}_\varepsilon e^{-\varepsilon t_n} - C_f \widetilde{C}_2 e^{-2t_n} - C_f C_{N,p}^{p/2} e^{(-N+\frac{N-2}{2}p)t_n} \big(\textstyle\int_{\mathcal{C}_{-\log R}} |\nabla_{\mathcal{C}} v|^2 d\mu\big)^{\frac{p-2}{2}}\right) \displaystyle\int_{\mathcal{C}_{t_n}} |\nabla_{\mathcal{C}} v|^2 d\mu \\
&= (1+o(1)) \int_{\mathcal{C}_{t_n}} |\nabla_{\mathcal{C}} v|^2 d\mu
\end{aligned}$$

which implies that $v \equiv 0$ in $\mathcal{C}_{t_n}$ for n sufficiently large. Hence $u \equiv 0$ in a neighborhood of the origin and, by classical unique continuation principles for second order elliptic equations with locally bounded coefficients (see e.g. [31]) we conclude that $u = 0$ a.e. in Ω, a contradiction. □

By virtue of Lemma 5.1, the *Almgren-type frequency function*

$$\mathcal{N}(t) = \frac{D(t)}{H(t)} \tag{33}$$

is well defined in $(\bar t, +\infty)$.

In order to obtain a suitable representation for the derivative of D we need the following Pohozaev-type identity.

Proposition 5.2. *Let h satisfy* (**H**), *f satisfy* (**F**), *and $u \in \mathcal{H}_{\rm loc}(\Omega)$ be a solution to (5). Let $R > 0$ be such that $\overline{B}_R \subset \Omega$. Then for every $t \in (-\log R, +\infty)$ the function $v := Tu \in H_\mu(\mathcal{C}_{-\log R})$ satisfies*

$$\begin{aligned}
\frac{1}{2}\int_{\Gamma_t} |\nabla v|^2 dS &= \int_{\Gamma_t} \left|\frac{\partial v}{\partial s}\right|^2 dS - \int_{\mathcal{C}_t} e^{-2s}\,\widetilde{h} v \frac{\partial v}{\partial s}\, d\mu + \frac{N-2}{2}\int_{\mathcal{C}_t} e^{-2s}\widetilde{f}(s,\theta,v) v\, d\mu \\
&- \int_{\mathcal{C}_t} e^{-(N+1)s} \nabla_x F(e^{-s}\theta, e^{\frac{N-2}{2}s} v(s,\theta)) \cdot \theta\, d\mu - N \int_{\mathcal{C}_t} e^{-Ns} F(e^{-s}\theta, e^{\frac{N-2}{2}s} v(s,\theta))\, d\mu \\
&+ \int_{\Gamma_t} e^{-Nt} F(e^{-t}\theta, e^{\frac{N-2}{2}t} v(t,\theta))\, dS.
\end{aligned} \tag{34}$$

Proof. Since $v \in H_\mu(\mathcal{C}_t)$ for any $t > -\log R$, then

$$\int_t^{+\infty} \left(\int_{\mathbb{S}^{N-1}} |\nabla_{\mathcal{C}} v(s,\theta)|^2 dS(\theta)\right) ds = \int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2 d\mu < +\infty,$$

which implies that the map $s \mapsto \int_{\mathbb{S}^{N-1}} |\nabla_{\mathcal{C}} v(s,\theta)|^2 dS(\theta)$ is integrable in $(t, +\infty)$ and hence

$$\liminf_{s\to+\infty} \int_{\mathbb{S}^{N-1}} |\nabla_{\mathcal{C}} v(s,\theta)|^2 dS(\theta) = 0\,.$$

By Lemmas 4.4, 4.5, we also have

$$\int_t^{+\infty} I_1(s)ds + \int_t^{+\infty} I_2(s)ds = \int_{\mathcal{C}_t} e^{-2s}v^2(s,\theta)\,d\mu + \int_{\mathcal{C}_t} e^{\left(-N+\frac{N-2}{2}p\right)s}|v|^p d\mu < +\infty$$

where

$$I_1(s) = \int_{\mathbb{S}^{N-1}} e^{-2s}v^2(s,\theta)dS(\theta), \quad I_2(s) = \int_{\mathbb{S}^{N-1}} e^{\left(-N+\frac{N-2}{2}p\right)s}|v(s,\theta)|^p dS(\theta),$$

so that $I_1, I_2 \in L^1(0,+\infty)$ and hence

$$\liminf_{s\to+\infty}(I_1(s)+I_2(s)) = 0.$$

Let $\{s_k\} \subset \mathbb{R}$ be an increasing sequence such that $s_k \to +\infty$ and

$$\lim_{k\to+\infty}\left(\int_{\mathbb{S}^{N-1}} |\nabla_{\mathcal{C}} v(s_k,\theta)|^2 dS(\theta)\right) + I_1(s_k) + I_2(s_k) = 0\,. \tag{35}$$

From Remark 3.6, $u \in C^1(\Omega\setminus\{0\})$ and hence $v \in C^1(\mathcal{C}_\Omega)$. Since v is a weak solution of (23) in $\mathcal{C}_\omega$ for any bounded domain ω satisfying $\partial\omega \in C^1$ and $\overline{\omega} \subset \Omega$, testing (23) with $\frac{\partial v}{\partial s}$ (we recall that $\frac{\partial v}{\partial s} \in H^1_\mu(\mathcal{C}_t \setminus \overline{\mathcal{C}_{s_k}})$ in view of Remark 3.6) and using (26) we obtain

$$\begin{aligned}
\int_{\mathcal{C}_t\setminus\overline{\mathcal{C}}_{s_k}} e^{-2s}\left(\widetilde{h}v + \widetilde{f}(s,\theta,v)\right)\frac{\partial v}{\partial s}\,d\mu &= \int_{\mathcal{C}_t\setminus\overline{\mathcal{C}}_{s_k}} \nabla_{\mathcal{C}} v\cdot\nabla_{\mathcal{C}}\left(\tfrac{\partial v}{\partial s}\right)d\mu + \int_{\Gamma_t}\left|\frac{\partial v}{\partial s}\right|^2 dS - \int_{\Gamma_{s_k}}\left|\frac{\partial v}{\partial s}\right|^2 dS \\
&= \frac{1}{2}\int_{\mathcal{C}_t\setminus\overline{\mathcal{C}}_{s_k}} \frac{\partial}{\partial s}(|\nabla_{\mathcal{C}} v|^2)\,d\mu + \int_{\Gamma_t}\left|\frac{\partial v}{\partial s}\right|^2 dS - \int_{\Gamma_{s_k}}\left|\frac{\partial v}{\partial s}\right|^2 dS \\
&= \frac{1}{2}\int_t^{s_k}\left(\frac{\partial}{\partial s}\int_{\mathbb{S}^{N-1}}|\nabla_{\mathcal{C}} v(s,\theta)|^2 dS(\theta)\right)ds + \int_{\Gamma_t}\left|\frac{\partial v}{\partial s}\right|^2 dS - \int_{\Gamma_{s_k}}\left|\frac{\partial v}{\partial s}\right|^2 dS \\
&= \frac{1}{2}\int_{\mathbb{S}^{N-1}}|\nabla_{\mathcal{C}} v(s_k,\theta)|^2 dS(\theta) - \frac{1}{2}\int_{\mathbb{S}^{N-1}}|\nabla_{\mathcal{C}} v(t,\theta)|^2 dS(\theta) + \int_{\Gamma_t}\left|\frac{\partial v}{\partial s}\right|^2 dS - \int_{\Gamma_{s_k}}\left|\frac{\partial v}{\partial s}\right|^2 dS\,.
\end{aligned} \tag{36}$$

By (35) we infer that

$$\lim_{k\to+\infty}\int_{\Gamma_{s_k}}\left|\frac{\partial v}{\partial s}\right|^2 dS = \lim_{k\to+\infty}\int_{\mathbb{S}^{N-1}}\left|\frac{\partial v}{\partial s}(s_k,\theta)\right|^2 dS(\theta) \leqslant \lim_{k\to+\infty}\int_{\mathbb{S}^{N-1}}|\nabla_{\mathcal{C}} v(s_k,\theta)|^2 dS(\theta) = 0. \tag{37}$$

Moreover an integration by parts in the left hand side of (36) yields

$$\begin{aligned}
\int_{\mathcal{C}_t\setminus\overline{\mathcal{C}}_{s_k}} e^{-2s}\,\widetilde{f}(s,\theta,v)\frac{\partial v}{\partial s}\,d\mu &= -\frac{N-2}{2}\int_{\mathcal{C}_t\setminus\overline{\mathcal{C}}_{s_k}} e^{-Ns}f(e^{-s}\theta,u(e^{-s}\theta))u(e^{-s}\theta)\,d\mu \\
&+ \int_{\mathcal{C}_t\setminus\overline{\mathcal{C}}_{s_k}} e^{-(N+1)s}\nabla_x F(e^{-s}\theta,u(e^{-s}\theta))\cdot\theta\,d\mu + N\int_{\mathcal{C}_t\setminus\overline{\mathcal{C}}_{s_k}} e^{-Ns}F(e^{-s}\theta,u(e^{-s}\theta))\,d\mu \\
&- \int_{\Gamma_t} e^{-Ns}F(e^{-s}\theta,u(e^{-s}\theta))\,dS + \int_{\Gamma_{s_k}} e^{-Ns}F(e^{-s}\theta,u(e^{-s}\theta))\,dS\,.
\end{aligned}$$

By (**F**) and (35) we have

$$\lim_{k\to+\infty}\left|\int_{\Gamma_{s_k}} e^{-Ns}F(e^{-s}\theta,u(e^{-s}\theta))\,dS\right| \leqslant \text{const}\lim_{k\to+\infty}(I_1(s_k)+I_2(s_k)) = 0. \tag{38}$$

Passing to the limit as $k \to +\infty$ in (36), by (**H**), (**F**), (35), (37) and (38) we arrive to the conclusion. □

In the next lemma we provide a useful representation for the derivative of D.

Lemma 5.3. *Under the same assumptions of Proposition 5.2, the function D defined in (31) belongs to $W^{1,1}_{\rm loc}(-\log R, +\infty)$ and*

$$\begin{aligned} D'(t) = &-2\int_{\Gamma_t}\left|\frac{\partial v}{\partial s}\right|^2 dS + 2\int_{\mathcal{C}_t} e^{-2s}\widetilde{h}v\frac{\partial v}{\partial s}\,d\mu - (N-2)\int_{\mathcal{C}_t} e^{-2s}\widetilde{f}(s,\theta,v)v\,d\mu \\ &+ 2\int_{\mathcal{C}_t} e^{-(N+1)s}\nabla_x F(e^{-s}\theta, e^{\frac{N-2}{2}s}v(s,\theta))\cdot\theta\,d\mu + 2N\int_{\mathcal{C}_t} e^{-Ns}F(e^{-s}\theta, e^{\frac{N-2}{2}s}v(s,\theta))\,d\mu \\ &- 2\int_{\Gamma_t} e^{-Nt}F(e^{-t}\theta, e^{\frac{N-2}{2}t}v(t,\theta))\,dS + e^{-2t}\int_{\Gamma_t}\big(\widetilde{h}v^2 + \widetilde{f}(t,\theta,v)v\big)dS \end{aligned}$$

in a distributional sense and for a.e. $t \in (-\log R, +\infty)$.

Proof. Since

$$D'(t) = -\int_{\Gamma_t}|\nabla_{\mathcal{C}}v|^2 dS + e^{-2t}\int_{\Gamma_t}\big(\widetilde{h}v^2 + \widetilde{f}(t,\theta,v)v\big)dS,$$

the proof directly follows from (34). □

The derivative of H is computed in the next lemma.

Lemma 5.4. *Under the same assumptions of Proposition 5.2 let H be as in* (32). *Then H is differentiable in $(-\log R, +\infty)$ and*

$$H'(t) = 2\int_{\Gamma_t} v\frac{\partial v}{\partial s}\,dS = -2D(t)$$

for any $t \in (\log R, +\infty)$.

Proof. By Remark 3.6 $v \in C^1(\mathcal{C}_\Omega)$. Moreover $H(t) = \int_{\mathbb{S}^{N-1}} v^2(t,\theta)\,dS(\theta)$ and hence

$$H'(t) = \int_{\mathbb{S}^{N-1}} 2v(t,\theta)\frac{\partial v}{\partial t}\,dS(\theta) = \int_{\Gamma_t} 2v\frac{\partial v}{\partial s}\,dS,$$

which, together with the identity

$$\int_{\mathcal{C}_t}|\nabla_{\mathcal{C}}v|^2 d\mu + \int_{\Gamma_t} v\frac{\partial v}{\partial s}\,dS = \int_{\mathcal{C}_t} e^{-2s}\big(\widetilde{h}v^2 + \widetilde{f}(s,\theta,v)v\big)d\mu$$

obtained by taking $w = v$ in (26), completes the proof of the lemma. □

Let us now compute the derivative of $\mathcal{N}$.

Lemma 5.5. *Let h satisfy* (**H**)*, f satisfy* (**F**)*, $u \in \mathcal{H}_{\rm loc}(\Omega)$ be a nontrivial solution of (5). Let $\mathcal{N}$ be the Almgren-type function defined in* (33)*. Then $\mathcal{N} \in W^{1,1}_{\rm loc}(\bar{t}, +\infty)$ and*

$$\mathcal{N}'(t) = \nu_1(t) + \nu_2(t) \tag{39}$$

in a distributional sense and for a.e. $t \in (\bar{t}, +\infty)$*, where*

$$\nu_1(t) := -2\frac{\left(\int_{\Gamma_t} \left|\frac{\partial v}{\partial s}\right|^2 dS\right)\left(\int_{\Gamma_t} v^2 dS\right) - \left(\int_{\Gamma_t} v\frac{\partial v}{\partial s} dS\right)^2}{\left(\int_{\Gamma_t} v^2 dS\right)^2}$$

and

$$\begin{aligned}\nu_2(t) = &\frac{2\int_{\mathcal{C}_t} e^{-2s}\widetilde{h}(s,\theta)v(s,\theta)\frac{\partial v}{\partial s}(s,\theta)\,d\mu + e^{-2t}\int_{\Gamma_t}\widetilde{h}v^2\,dS}{\int_{\Gamma_t} v^2\,dS}\\ &+ \frac{2\int_{\mathcal{C}_t} e^{-(N+1)s}\nabla_x F(e^{-s}\theta, e^{\frac{N-2}{2}s}v(s,\theta))\cdot\theta\,d\mu}{\int_{\Gamma_t} v^2\,dS}\\ &+ \frac{2N\int_{\mathcal{C}_t} e^{-Ns}F(e^{-s}\theta, e^{\frac{N-2}{2}s}v(s,\theta))\,d\mu - (N-2)\int_{\mathcal{C}_t} e^{-2s}\widetilde{f}(s,\theta,v(s,\theta))v(s,\theta)\,d\mu}{\int_{\Gamma_t} v^2\,dS}\\ &+ \frac{e^{-2t}\int_{\mathbb{S}^{N-1}}\widetilde{f}(t,\theta,v(t,\theta))v(t,\theta)\,dS(\theta) - 2e^{-Nt}\int_{\mathbb{S}^{N-1}} F(e^{-t}\theta, e^{\frac{N-2}{2}t}v(t,\theta))\,dS(\theta)}{\int_{\Gamma_t} v^2\,dS}.\end{aligned}$$

Proof. It follows from (33), Lemmas 5.3, 5.4. □

In order to show that the Almgren function $\mathcal{N}$ admits a finite limit as $t \to +\infty$ we need some preliminary estimates which will be proved in the next lemmas.

Lemma 5.6. *Under the same assumptions as in Lemma 5.5, let* $\mathcal{N}$ *be as in (33) and* $\bar{t}$ *as in Lemma 5.1. Then, up to choose a larger* $\bar{t}$*, we have*

$$\mathcal{N}(t) \geqslant -\overline{C}e^{-Mt}$$

and

$$D(t) + H(t) \geqslant \frac{1}{2}\left(\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2 d\mu + \int_{\Gamma_t} v^2 dS\right) \tag{40}$$

for any $t > \bar{t}$*, where* $\overline{C}$ *is a constant depending only on* $N, \varepsilon, p, u, h, f$ *and* $M = \min\left\{\varepsilon, \frac{2N}{p} - N + 2\right\}$.

Proof. Combining assumptions (**H**), (**F**) with Lemma 4.4 and Lemma 4.5, we obtain that, for $t > \bar{t}$ with $\bar{t}$ large,

$$\begin{aligned}D(t) &= \int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2 d\mu - \int_{\mathcal{C}_t} e^{-2s}\widetilde{h}v^2 d\mu - \int_{\mathcal{C}_t} e^{-2s}\widetilde{f}(s,\theta,v)v\,d\mu\\ &\geqslant \left(1 - C_h\widetilde{C}_\varepsilon e^{-\varepsilon t} - C_f\widetilde{C}_2 e^{-2t} - C_f C_{N,p} e^{\left(-\frac{2N}{p}+N-2\right)t}\left(\int_{\mathcal{C}_t} e^{\left(-N+\frac{N-2}{2}p\right)s}|v|^p d\mu\right)^{\frac{p-2}{p}}\right)\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} v|^2 d\mu\\ &\quad - \left(C_h\widetilde{C}_\varepsilon e^{-\varepsilon t} + C_f\widetilde{C}_2 e^{-2t} + C_f C_{N,p} e^{\left(-\frac{2N}{p}+N-2\right)t}\left(\int_{\mathcal{C}_t} e^{\left(-N+\frac{N-2}{2}p\right)s}|v|^p d\mu\right)^{\frac{p-2}{p}}\right)\int_{\Gamma_t} v^2 dS\\ &\geqslant -\left(C_h\widetilde{C}_\varepsilon e^{-\varepsilon t} + C_f\widetilde{C}_2 e^{-2t} + C_f C_{N,p} e^{\left(-\frac{2N}{p}+N-2\right)t}\left(\int_{\mathcal{C}_t} e^{\left(-N+\frac{N-2}{2}p\right)s}|v|^p d\mu\right)^{\frac{p-2}{p}}\right)\int_{\Gamma_t} v^2 dS\end{aligned}$$

for which yields the conclusion if $\bar{t}$ is chosen sufficiently large. □

Next we provide an estimate on the function ν_2 introduced in Lemma 5.5.

Lemma 5.7. *Under the same assumptions as in Lemma 5.5 we have*

$$|\nu_2(t)| \leqslant \overline{C}_1(e^{-\alpha t}+g(t))(\mathcal{N}(t)+1)+\overline{C}_2 e^{-2t} \qquad \text{for any } t>\overline{t}\,,$$

where $\overline{C}_1$ *and* $\overline{C}_2$ *are two positive constant depending only on* N,h,f,u *but independent of* t, $\alpha := \min\{\varepsilon, 2, \frac{2N}{p}-N+2\}$, *and* $g\in L^1(\overline{t},+\infty)$, $g\geqslant 0$ *a.e., satisfies*

$$\int_t^{+\infty} g(s)\,ds \leqslant \frac{p}{p-2}\left(\int_{\mathcal{C}_{\overline{t}}} e^{\left(-N+\frac{N-2}{2}p\right)s}|v(s,\theta)|^p\,d\mu\right)^{\frac{p-2}{p}} e^{\left(-\frac{2N}{p}+N-2\right)t} \qquad \text{for any } t>\overline{t}\,.$$

Proof. From (**H**) and (**F**) it follows that

$$(41)\qquad \begin{aligned} |\nu_2(t)| &\leqslant \frac{C_h\int_{\mathcal{C}_t} e^{-\varepsilon s}v^2(s,\theta)\,d\mu + C_h e^{-\varepsilon t}\int_{\mathcal{C}_t}|\nabla_{\mathcal{C}}v|^2 d\mu + C_h e^{-\varepsilon t}\int_{\Gamma_t} v^2 dS}{\int_{\Gamma_t} v^2\,dS}\\ &\quad+\frac{3NC_f\int_{\mathcal{C}_t} e^{-Ns}[u^2(e^{-s}\theta)+|u(e^{-s}\theta)|^p]\,d\mu}{\int_{\Gamma_t} v^2\,dS}\\ &\quad+\frac{3C_f e^{-Nt}\int_{\mathbb{S}^{N-1}}[u^2(e^{-t}\theta)+|u(e^{-t}\theta)|^p]\,dS(\theta)}{\int_{\Gamma_t} v^2\,dS}\\ &=\frac{C_h\int_{\mathcal{C}_t} e^{-\varepsilon s}v^2(s,\theta)\,d\mu + C_h e^{-\varepsilon t}\int_{\mathcal{C}_t}|\nabla_{\mathcal{C}}v|^2 d\mu + C_h e^{-\varepsilon t}\int_{\Gamma_t} v^2 dS}{\int_{\Gamma_t} v^2\,dS}\\ &\quad+\frac{3NC_f\left[\int_{\mathcal{C}_t} e^{-2s}v^2(s,\theta)\,d\mu+\int_{\mathcal{C}_t} e^{\left(-N+\frac{N-2}{2}p\right)s}|v(s,\theta)|^p\,d\mu\right]}{\int_{\Gamma_t} v^2\,dS}\\ &\quad+\frac{3C_f e^{-2t}\int_{\mathbb{S}^{N-1}} v^2(t,\theta)\,dS(\theta)}{\int_{\Gamma_t} v^2\,dS}+\frac{3C_f e^{\left(-N+\frac{N-2}{2}p\right)t}\int_{\mathbb{S}^{N-1}}|v(t,\theta)|^p\,dS(\theta)}{\int_{\Gamma_t} v^2\,dS}. \end{aligned}$$

By Lemma 4.5 and (40) we obtain for any $t>\overline{t}$

$$\begin{aligned} &\left(\int_{\mathcal{C}_t} e^{\left(-N+\frac{N-2}{2}p\right)s}|v(s,\theta)|^p d\mu\right)^{2/p} \leqslant C_{N,p}e^{\left(-\frac{2N}{p}+N-2\right)t}\left(\int_{\mathcal{C}_t}|\nabla_{\mathcal{C}}v|^2 d\mu+\int_{\Gamma_t} v^2 dS\right)\\ &\leqslant 2\,C_{N,p}e^{\left(-\frac{2N}{p}+N-2\right)t}(D(t)+H(t)) = 2\,C_{N,p}\,e^{\left(-\frac{2N}{p}+N-2\right)t}(\mathcal{N}(t)+1)\int_{\Gamma_t} v^2 dS \end{aligned}$$

and hence

$$(42)\qquad \begin{aligned} &\frac{3\,C_f e^{\left(-N+\frac{N-2}{2}p\right)t}\int_{\mathbb{S}^{N-1}}|v(t,\theta)|^p\,dS(\theta)}{\int_{\Gamma_t} v^2\,dS}\\ &\qquad\leqslant 6C_fC_{N,p}\,e^{\left(-\frac{2N}{p}+N-2\right)t}\;\frac{\int_{\Gamma_t} e^{\left(-N+\frac{N-2}{2}p\right)t}|v(t,\theta)|^p\,dS}{\left(\int_{\mathcal{C}_t} e^{\left(-N+\frac{N-2}{2}p\right)s}|v(s,\theta)|^p\,d\mu\right)^{2/p}}\,(\mathcal{N}(t)+1)\,. \end{aligned}$$

We also have

$$0 \leqslant g(t) := e^{\left(-\frac{2N}{p}+N-2\right)t} \frac{\int_{\Gamma_t} e^{\left(-N+\frac{N-2}{2}p\right)t}|v(t,\theta)|^p\,dS}{\left(\int_{\mathcal{C}_t} e^{\left(-N+\frac{N-2}{2}p\right)s}|v(s,\theta)|^p\,d\mu\right)^{2/p}} \tag{43}$$

$$= -\frac{p}{p-2}\Bigg\{\frac{d}{dt}\left[e^{\left(-\frac{2N}{p}+N-2\right)t}\left(\int_{\mathcal{C}_t} e^{\left(-N+\frac{N-2}{2}p\right)s}|v(s,\theta)|^p\,d\mu\right)^{\frac{p-2}{p}}\right]$$

$$-\left(-\tfrac{2N}{p}+N-2\right)e^{\left(-\frac{2N}{p}+N-2\right)t}\left(\int_{\mathcal{C}_t} e^{\left(-N+\frac{N-2}{2}p\right)s}|v(s,\theta)|^p\,d\mu\right)^{\frac{p-2}{p}}\Bigg\}$$

$$\leqslant -\frac{p}{p-2}\frac{d}{dt}\left[e^{\left(-\frac{2N}{p}+N-2\right)t}\left(\int_{\mathcal{C}_t} e^{\left(-N+\frac{N-2}{2}p\right)s}|v(s,\theta)|^p\,d\mu\right)^{\frac{p-2}{p}}\right]$$

in the distributional sense for almost every $t > \bar{t}$. But the right hand side of (43) is integrable in $(\bar{t}, +\infty)$ since

$$\lim_{t\to+\infty} e^{\left(-\frac{2N}{p}+N-2\right)t}\left(\int_{\mathcal{C}_t} e^{\left(-N+\frac{N-2}{2}p\right)s}|v(s,\theta)|^p\,d\mu\right)^{\frac{p-2}{p}} = 0$$

and hence we also have $g \in L^1(\bar{t}, +\infty)$.

Combining (41), (42) with Lemma 4.4, Lemma 4.5 and (40) we obtain

$$|\nu_2(t)| \leqslant \Bigg[6NC_fC_{N,p}\, e^{\left(-\frac{2N}{p}+N-2\right)t}\left(\int_{\mathcal{C}_{\bar{t}}} e^{\left(-N+\frac{N-2}{2}p\right)s}|v(s,\theta)|^p\,d\mu\right)^{1-\frac{2}{p}}$$

$$+ 2C_h(\widetilde{C}_\varepsilon+1)e^{-\varepsilon t} + 6NC_fC_{N,2}e^{-2t} + 6C_fC_{N,p}\, g(t)\Bigg](\mathcal{N}(t)+1) + 3C_fe^{-2t}\,.$$

The statements of the lemma follow from this last estimate and the definition of g. □

We can now prove that the function $\mathcal{N}$ admits a finite limit as $t \to +\infty$.

Lemma 5.8. *Under the same assumptions as in Lemma 5.5, the limit $\gamma := \lim_{t\to+\infty}\mathcal{N}(t)$ exists and is finite. Moreover $\gamma \geqslant 0$.*

Proof. From Lemma 5.6 we have that

$$\liminf_{t\to+\infty}\mathcal{N}(t) \geqslant 0\,. \tag{44}$$

On the other hand, by Lemma 5.5, Schwarz inequality, and Lemma 5.7, we have

$$(\mathcal{N}(t)+1)' = \mathcal{N}'(t) = \nu_1(t) + \nu_2(t) \leqslant \nu_2(t) \leqslant \overline{C}_1(e^{-\alpha t} + g(t))(\mathcal{N}(t)+1) + \overline{C}_2e^{-2t} \tag{45}$$

and in turn

$$\frac{d}{dt}\left[e^{\overline{C}_1\int_t^{+\infty}(e^{-\alpha s}+g(s))\,ds}(\mathcal{N}(t)+1)\right] \leqslant \overline{C}_2e^{-2t+\overline{C}_1\int_t^{+\infty}(e^{-\alpha s}+g(s))\,ds}\,.$$

Since the right hand side in the above line belongs to $L^1(\bar{t}, +\infty)$, after integration we deduce that $\mathcal{N}$ is bounded from above and hence, by (45) and Lemma 5.7, it follows that $\mathcal{N}'$ is the sum of the nonpositive function ν_1 and of the integrable function ν_2. This implies that

$$\lim_{t\to+\infty}\mathcal{N}(t)=\mathcal{N}(\bar t)+\lim_{t\to+\infty}\int_{\bar t}^{t}\mathcal{N}'(s)\,ds$$

exists and it is necessarily finite since $\mathcal{N}$ is bounded. This limit is necessarily nonnegative in view of (44). □

As a consequence of the convergence of $\mathcal{N}$, the following estimates on H hold.

Lemma 5.9. *Suppose that all the assumptions of Lemma 5.5 are satisfied. Then there exists a constant $K_1>0$ such that*

$$H(\lambda)\leqslant K_1e^{-2\gamma\lambda}\quad \textit{for any } \lambda>\bar t, \tag{46}$$

with $\gamma=\lim_{t\to+\infty}\mathcal{N}(t)$ as in Lemma 5.8. Moreover, for any $\sigma>0$ there exists a constant $K_2(\sigma)$ such that

$$H(\lambda)\geqslant K_2(\sigma)e^{-(2\gamma+\sigma)\lambda}\qquad \textit{for any } \lambda>\bar t\,. \tag{47}$$

Proof. By Lemma 5.4 and Lemma 5.8, we have

$$\frac{H'(\lambda)}{H(\lambda)}=-2\mathcal{N}(\lambda)=-2\left[\gamma-\int_{\lambda}^{+\infty}\mathcal{N}'(s)\,dx\right]\leqslant-2\gamma+2\int_{\lambda}^{+\infty}\nu_2(s)\,ds\,.$$

By Lemma 5.7 we then obtain

$$\frac{H'(\lambda)}{H(\lambda)}\leqslant-2\gamma+Ce^{-\alpha\lambda}\qquad\text{for any }\lambda>\bar t$$

where C is a constant depending only on N,h,f,u,ε,N,p and α is as in Lemma 5.7. Estimate (46) follows after integration in the last inequality.

On the other hand, for any $\sigma>0$ there exists $\lambda(\sigma)>0$ such that

$$\frac{H'(\lambda)}{H(\lambda)}\geqslant-2\gamma-\sigma\qquad\text{for any }\lambda>\overline{\lambda}(\sigma)\,.$$

Estimate (47) follows after integration. □

6. A blow-up argument

Convergence of the frequency function $\mathcal{N}$ as $t\to+\infty$ is a fundamental tool in the following blow-up argument. Hereafter, we denote as $0=\mu_1<\mu_2\leqslant\ldots\leqslant\mu_n\leqslant\ldots$ the eigenvalues of $-\Delta_{\mathbb{S}^{N-1}}$ with the usual notation of repeating them as many times as their multiplicity. Hence we have that $\mu_1=\lambda_0=0$ and

$$\text{if } k>1 \text{ and } \textstyle\sum_{n=0}^{\ell-1}m_n<k\leqslant\sum_{n=0}^{\ell}m_n,\ \text{ then } \mu_k=\lambda_\ell,$$

where m_n is defined in (6).

Lemma 6.1. *Under the same assumptions as in Lemma 5.5, let us define the family of functions $\{w_\lambda\}_{\lambda>\bar t}$*

$$w_\lambda(t,\theta):=\frac{v(t+\lambda,\theta)}{\sqrt{H(\lambda)}}\qquad \textit{for any } t\geqslant0 \textit{ and } \theta\in\mathbb{S}^{N-1}\,.$$

Let γ be the limit introduced in Lemma 5.8. Then

(i) *there exists* $k_0 \in \mathbb{N} \setminus \{0\}$ *such that* $\gamma = \sqrt{\mu_{\kappa_0}}$ *;*
(ii) *for any sequence* $\lambda_n \to +\infty$ *there exists a subsequence* λ_{n_k} *and an eigenfunction* ψ *of* $-\Delta_{\mathbb{S}^{N-1}}$ *corresponding to the eigenvalue* μ_{k_0} *such that* $\|\psi\|_{L^2(\mathbb{S}^{N-1})} = 1$ *and* $\{w_{\lambda_{n_k}}\}$ *converges to the function* $w(s,\theta) = e^{-\sqrt{\mu_{k_0}}s}\psi(\theta)$ *weakly in* $H_\mu(\mathcal{C}_0)$*, strongly in* $H_\mu(\mathcal{C}_t)$ *for any* $t > 0$*, and strongly in* $C^{1,\alpha}_{\rm loc}(\mathcal{C}_0)$ *for any* $\alpha \in (0,1)$*.*

Proof. We divide the proof into several steps.

Step 1. We observe that $\{w_\lambda\}_{\lambda>\bar t}$ is bounded in $H_\mu(\mathcal{C}_0)$. Indeed, by (40) and the definition of w_λ we deduce that

$$\mathcal{N}(\lambda) + 1 \geqslant \frac{1}{2}\left(\int_{\mathcal{C}_0} |\nabla_{\mathcal{C}} w_\lambda|^2 d\mu + \int_{\Gamma_0} w_\lambda^2 \, dS\right)$$

and by Lemma 4.3 and Lemma 5.8 we conclude that $\{w_\lambda\}_{\lambda>\bar t}$ is bounded in $H_\mu(\mathcal{C}_0)$.

Step 2. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lambda_n \to +\infty$. From Step 1, it follows that there exists a subsequence λ_{n_k} and a function $w \in H_\mu(\mathcal{C}_0)$ such that $w_{\lambda_{n_k}} \rightharpoonup w$ in $H_\mu(\mathcal{C}_0)$. We claim that w is harmonic on $\mathcal{C}_0$.

Indeed, by direct computation, we have that w_λ weakly solves the equation

$$-\Delta_{\mathcal{C}} w_\lambda(t,\theta) = g_1(\lambda,t,\theta) + g_2(\lambda,t,\theta), \quad \text{in } \mathcal{C}_0, \tag{48}$$

where

$$g_1(\lambda,t,\theta) = e^{-2\lambda}e^{-2t}\widetilde{h}(t+\lambda,\theta)w_\lambda(t,\theta), \quad g_2(\lambda,t,\theta) = \frac{e^{-2\lambda}}{\sqrt{H(\lambda)}}\, e^{-2t}\widetilde{f}(t+\lambda,\theta,\sqrt{H(\lambda)}w_\lambda(t,\theta)),$$

and hence, for any $\phi \in H_{\mu,0}(\mathcal{C}_0)$,

$$\int_{\mathcal{C}_0} \nabla_{\mathcal{C}} w_\lambda \cdot \nabla_{\mathcal{C}}\phi \, d\mu = \int_{\mathcal{C}_0} g_1(\lambda,t,\theta)\phi(t,\theta)\, d\mu + \int_{\mathcal{C}_0} g_2(\lambda,t,\theta)\phi(t,\theta)\, d\mu. \tag{49}$$

We estimate the two terms in the right hand side of (49). By (**H**), Lemma 4.4, and boundedness of $\{w_\lambda\}$ in $H_\mu(\mathcal{C}_0)$, we have that

$$\begin{aligned}\left|\int_{\mathcal{C}_0} g_1(\lambda,t,\theta)\phi(t,\theta)\, d\mu\right| &\leqslant C_h e^{-\varepsilon\lambda}\int_{\mathcal{C}_0} e^{-\varepsilon t}|w_\lambda(t,\theta)|\,|\phi(t,\theta)|\, d\mu \\ &\leqslant C_h\widetilde{C}_\varepsilon e^{-\varepsilon\lambda}\left(\int_{\mathcal{C}_0}|\nabla_{\mathcal{C}} w_\lambda|^2\, d\mu + \int_{\Gamma_0} w_\lambda^2 dS\right)^{\frac12}\left(\int_{\mathcal{C}_0}|\nabla_{\mathcal{C}}\phi|^2\, d\mu + \int_{\Gamma_0}\phi^2 dS\right)^{\frac12} = o(1)\end{aligned} \tag{50}$$

as $\lambda \to +\infty$. On the other hand by (**F**) we have

$$\begin{aligned}\left|\int_{\mathcal{C}_0} g_2(\lambda,t,\theta)\phi(t,\theta)\, d\mu\right| &\leqslant C_f e^{-2\lambda}\int_{\mathcal{C}_0} e^{-2t}|w_\lambda(t,\theta|\,|\phi(t,\theta)|\, d\mu \\ &\quad + C_f e^{\frac{p(N-2)-2N}{2}\lambda}\int_{\mathcal{C}_0} e^{\frac{p(N-2)-2N}{2}t}|v(t+\lambda,\theta)|^{p-2}|w_\lambda(t,\theta)|\,|\phi(t,\theta)|\, d\mu.\end{aligned} \tag{51}$$

One can show that the first term at the right hand side of (51) tends to zero as $\lambda \to +\infty$ by proceeding as in (50). Let us prove that also the second term at right hand side of (51) tends to zero. Indeed, by Hölder inequality, Lemma 4.5 and boundedness of $\{w_\lambda\}$ in $H_\mu(\mathcal{C}_0)$ we obtain

$$
\begin{aligned}
(52)\qquad & e^{\frac{p(N-2)-2N}{2}\lambda}\int_{\mathcal{C}_0} e^{\frac{p(N-2)-2N}{2}t}|v(t+\lambda,\theta)|^{p-2}|w_\lambda(t,\theta)|\,|\phi(t,\theta)|\,d\mu\\
&\leqslant e^{\frac{p(N-2)-2N}{2}\lambda}\left(\int_{\mathcal{C}_0} e^{\frac{p(N-2)-2N}{2}t}|v(t+\lambda,\theta)|^{p}\,d\mu\right)^{\frac{p-2}{p}}\times\\
&\quad\times\left(\int_{\mathcal{C}_0} e^{\frac{p(N-2)-2N}{2}t}|w_\lambda(t,\theta)|^{p}\,d\mu\right)^{\frac{1}{p}}\left(\int_{\mathcal{C}_0} e^{\frac{p(N-2)-2N}{2}t}|\phi(t,\theta)|^{p}\,d\mu\right)^{\frac{1}{p}}\\
&\leqslant C_{N,p}e^{\frac{p(N-2)-2N}{2}\lambda}e^{\frac{p-2}{p}\,\frac{2N-p(N-2)}{2}\lambda}\left(\int_{\mathcal{C}_\lambda} e^{\frac{p(N-2)-2N}{2}t}|v|^{p}d\mu\right)^{\frac{p-2}{p}}\\
&\quad\times\left(\int_{\mathcal{C}_0}|\nabla_{\mathcal{C}}w_\lambda|^2d\mu+\int_{\Gamma_0}w_\lambda^2\,dS\right)^{\frac12}\left(\int_{\mathcal{C}_0}|\nabla_{\mathcal{C}}\phi|^2d\mu+\int_{\Gamma_0}\phi^2\,dS\right)^{\frac12}\to 0^+
\end{aligned}
$$

as $\lambda\to+\infty$ since $p<2^*$. Passing to the limit in (49) along the sequence $\{\lambda_{n_k}\}$ and using (50), (51), (52) we obtain

$$\int_{\mathcal{C}_0}\nabla_{\mathcal{C}}w\cdot\nabla_{\mathcal{C}}\phi\,d\mu=0\qquad\text{for any }\phi\in H_{\mu,0}(\mathcal{C}_0)\,.$$

This proves that w is harmonic in $\mathcal{C}_0$.

Step 3. We claim that $w_{\lambda_{n_k}}\to w$ strongly in $H_\mu(\mathcal{C}_t)$ for any $t>0$. We first observe that, testing equation $\Delta_{\mathcal{C}}w=0$ with a function $\phi\in H_\mu(\mathcal{C}_0)$, we have

$$\int_{\mathcal{C}_0}\nabla_{\mathcal{C}}w\cdot\nabla_{\mathcal{C}}\phi\,d\mu=-\int_{\Gamma_0}\frac{\partial w}{\partial s}\,\phi\,dS\,. \tag{53}$$

Moreover, since $\int_{\Gamma_0}w_\lambda^2\,dS=1$, by compactness of the trace map we also have that

$$\int_{\Gamma_0}w^2\,dS=1. \tag{54}$$

To prove strong $H_\mu(\mathcal{C}_t)$-convergence of $w_{\lambda_{n_k}}\to w$, we notice that, by direct computation, the function $u_\lambda=T^{-1}w_\lambda$ is actually a rescaling of the function u, i.e.

$$u_\lambda(x)=\frac{e^{-\frac{N-2}{2}\lambda}}{\sqrt{H(\lambda)}}\,u(e^{-\lambda}x)$$

so that it solves the equation

$$-\Delta u_\lambda-\left(\frac{N-2}{2}\right)^2\frac{u_\lambda}{|x|^2}=G_1(\lambda,x)+G_2(\lambda,x),\quad\text{in }B_1,$$

where

$$G_1(\lambda,x)=e^{-2\lambda}h(e^{-\lambda}x)u_\lambda(x),\quad G_2(\lambda,x)=\frac{e^{-\frac{N-2}{2}\lambda}\cdot e^{-2\lambda}}{\sqrt{H(\lambda)}}\,f(e^{-\lambda}x,\sqrt{H(\lambda)}e^{\frac{N-2}{2}\lambda}u_\lambda(x)).$$

By (**H**) we obtain

$$|G_1(\lambda,x)|\leqslant C_h e^{-\varepsilon\lambda}|x|^{-2+\varepsilon}|u_\lambda(x)| \tag{55}$$

and by (**F**) and (46)

$$|G_2(\lambda,x)| \leqslant C_f e^{-2\lambda}|u_\lambda(x)| + C_f (H(\lambda))^{\frac{p-2}{2}} e^{\left(-N+\frac{N-2}{2}p\right)\lambda}|u_\lambda(x)|^{p-1}\,. \tag{56}$$

Taking into account that the set $\{u_\lambda\}_{\lambda>\bar t}$ is bounded in $\mathcal{H}(B_1)$ (we recall that T is an isometry), by (11) we also have that $\{u_\lambda\}_{\lambda>\bar t}$ is also bounded in $H^1(A)$ for any open set $A \Subset B_1 \setminus \{0\}$.

Therefore by (55), (56), (46), the fact that $p < 2^*$, and a standard bootstrap argument, we deduce that u_λ is bounded in $C^{1,\alpha}_{\rm loc}(B_1\setminus\{0\})$ for any $\alpha\in(0,1)$; the same holds true for the set $\{w_\lambda\}$ in $C^{1,\alpha}_{\rm loc}(\mathcal{C}_0)$ for any $\alpha\in(0,1)$.

Moreover along the subsequence $\{\lambda_{n_k}\}$ we have that

$$w_{\lambda_{n_k}} \to w \qquad \text{in } C^{1,\alpha}_{\rm loc}(\mathcal{C}_0)$$

for any $\alpha\in(0,1)$ and in particular

$$\frac{\partial w_{\lambda_{n_k}}}{\partial s} \to \frac{\partial w}{\partial s} \qquad \text{in } C^{0,\alpha}_{\rm loc}(\mathcal{C}_0)\,. \tag{57}$$

Taking $\lambda=\lambda_{n_k}$ in (48), testing in $\mathcal{C}_t$ with the function $w_{\lambda_{n_k}} - w \in H_\mu(\mathcal{C}_t)$, for any $t>0$ we obtain

$$\begin{aligned}\int_{\mathcal{C}_t} \nabla_{\mathcal{C}} w_{\lambda_{n_k}} \cdot \nabla_{\mathcal{C}}(w_{\lambda_{n_k}} - w)\,d\mu &= -\int_{\Gamma_t} \frac{\partial w_{\lambda_{n_k}}}{\partial s}(w_{\lambda_{n_k}} - w)\,dS \\ &+ \int_{\mathcal{C}_t} g_1(\lambda_{n_k},s,\theta)(w_{\lambda_{n_k}}(s,\theta)-w(s,\theta))\,d\mu + \int_{\mathcal{C}_t} g_2(\lambda_{n_k},s,\theta)(w_{\lambda_{n_k}}(s,\theta)-w(s,\theta))\,d\mu.\end{aligned}$$

Using (53) with $\phi = w_{\lambda_{n_k}} - w$, the last identity then gives

$$\begin{aligned}\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}}(w_{\lambda_{n_k}} - w)|^2\,d\mu &= -\int_{\Gamma_t} \left(\frac{\partial w_{\lambda_{n_k}}}{\partial s} - \frac{\partial w}{\partial s}\right)(w - w_{\lambda_{n_k}})\,dS \\ &+ \int_{\mathcal{C}_t} g_1(\lambda_{n_k},s,\theta)(w_{\lambda_{n_k}}(s,\theta)-w(s,\theta))\,d\mu + \int_{\mathcal{C}_t} g_2(\lambda_{n_k},s,\theta)(w_{\lambda_{n_k}}(s,\theta)-w(s,\theta))\,d\mu.\end{aligned}$$

Passing to the limit as $k\to+\infty$, proceeding as in (50)-(52) and using (57) and the fact that $w_{\lambda_{n_k}} \to w$ in $L^2(\Gamma_t)$, we obtain $\nabla_{\mathcal{C}} w_{\lambda_{n_k}} \to \nabla_{\mathcal{C}} w$ in $L^2(\mathcal{C}_t)$ and in turn, thanks to Lemma 4.3, $w_{\lambda_{n_k}} \to w$ strongly in $H_\mu(\mathcal{C}_t)$ for all $t>0$.

Step 4. We claim that there exists $k_0\in\mathbb{N}\setminus\{0\}$ such that

$$w(s,\theta) = e^{-\sqrt{\mu_{k_0}}\,s}\psi(\theta),$$

where ψ is an eigenfunction of $-\Delta_{\mathbb{S}^{N-1}}$ associated to the eigenvalue μ_{k_0} such that $\int_{\mathbb{S}^{N-1}}\psi^2 dS = 1$.

To prove the claim, we study the frequency functions associated to w_λ and w. According with (33), it is reasonable to associate to every solution w_λ of (48) the Almgren-type frequency function

$$\mathcal{N}_\lambda(t) := \frac{D_\lambda(t)}{H_\lambda(t)} \quad \text{for any } t\geqslant 0 \text{ and } \lambda > \bar t,$$

where

$$D_\lambda(t) := \int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} w_\lambda|^2\,d\mu - \int_{\mathcal{C}_t} g_1(\lambda,t,\theta) w_\lambda(s,\theta)\,d\mu - \int_{\mathcal{C}_t} g_2(\lambda,t,\theta) w_\lambda(s,\theta)\,d\mu$$

and

$$H_\lambda(t) := \int_{\Gamma_t} w_\lambda^2 dS\,.$$

By direct computation it follows that

$$\mathcal{N}(t+\lambda) = \mathcal{N}_\lambda(t) \quad \text{for any } t>0 \text{ and } \lambda > \bar{t}. \tag{58}$$

Since $w_{\lambda_{n_k}} \to w$ strongly in $H_\mu(\mathcal{C}_t)$ for any $t>0$, passing to the limit as $k\to\infty$ and proceeding as in (50)-(52), we obtain that, for any $t>0$,

$$D_{\lambda_{n_k}}(t) \to \int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} w|^2\, d\mu \tag{59}$$

and

$$H_{\lambda_{n_k}}(t) \to \int_{\Gamma_t} w^2\, dS\,. \tag{60}$$

We notice that

$$\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} w|^2\, d\mu + \int_{\Gamma_t} w^2\, dS > 0 \qquad \text{for any } t>0\,. \tag{61}$$

Indeed, if there exists $t>0$ such that $\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} w|^2\, d\mu + \int_{\Gamma_t} w^2\, dS = 0$, then by a classical unique continuation property we deduce that w is identically zero in $\mathcal{C}_0$ in contradiction with (54). Moreover

$$\int_{\Gamma_t} w^2\, dS > 0 \tag{62}$$

for any $t>0$ since otherwise, if there exists $t>0$ such that $\int_{\Gamma_t} w^2\, dS = 0$, then by (59), (60), (61) we would have

$$\gamma = \lim_{k\to+\infty} \mathcal{N}(t+\lambda_{n_k}) = \lim_{k\to+\infty} \mathcal{N}_{\lambda_{n_k}}(t) = \lim_{k\to+\infty} \frac{D_{\lambda_{n_k}}(t)}{H_{\lambda_{n_k}}(t)} = +\infty,$$

a contradiction. Therefore

$$\mathcal{N}_{\lambda_{n_k}}(t) \to \mathcal{N}_w(t) := \frac{\int_{\mathcal{C}_t} |\nabla_{\mathcal{C}} w|^2\, d\mu}{\int_{\Gamma_t} w^2 dS}$$

for any $t>0$. Combining this with (58) and Lemma 5.8 we deduce that

$$\mathcal{N}_w(t) = \gamma \quad \text{for any } t>0. \tag{63}$$

This means that $\mathcal{N}_w$ is constant and in particular, for almost every $t>0$, by Lemma 5.5 we have

$$0 = \mathcal{N}_w'(t) = -2\frac{\left(\int_{\Gamma_t} \left|\frac{\partial w}{\partial s}\right|^2 dS\right)\left(\int_{\Gamma_t} w^2 dS\right) - \left(\int_{\Gamma_t} w\frac{\partial w}{\partial s} dS\right)^2}{\left(\int_{\Gamma_t} w^2 dS\right)^2}.$$

The condition $\left(\int_{\Gamma_t} w\frac{\partial w}{\partial s} dS\right)^2 = \left(\int_{\Gamma_t} \left|\frac{\partial w}{\partial s}\right|^2 dS\right)\left(\int_{\Gamma_t} w^2 dS\right)$ implies that, for almost every $t>0$, the functions $\theta \mapsto w(t,\theta)$ and $\theta \mapsto \frac{\partial w}{\partial s}(t,\theta)$ are parallel as vectors of $L^2(\mathbb{S}^{N-1})$ and hence there exists a function η depending only on t such that

$$\frac{\partial w}{\partial t}(t,\theta) = \eta(t) w(t,\theta) \qquad \text{for any } t>0. \tag{64}$$

Clearly the function $\eta(t) = (w(t,\theta))^{-1}\frac{\partial w}{\partial t}(t,\theta)$ is well defined and continuous for any $t > 0$ thanks to (62). After integration in (64) we deduce that w admits the representation

$$w(t,\theta) = \varphi(t)\psi(\theta).$$

It is not restrictive assuming that $\int_{\mathbb{S}^{N-1}} \psi^2 dS = 1$. Inserting the above representation of w into the equation $\Delta_{\mathcal{C}} w = 0$, it follows that there exist $k_0 \in \mathbb{N}\setminus\{0\}$ and $c_1, c_2 \in \mathbb{R}$ such that

$$-\Delta_{\mathbb{S}^{N-1}}\psi(\theta) = \mu_{k_0}\psi(\theta) \quad \text{and} \quad \varphi(t) = c_1 e^{\sqrt{\mu_{k_0}}\,t} + c_2 e^{-\sqrt{\mu_{k_0}}\,t}.$$

Since $w \in H_\mu(\mathcal{C}_0)$ and $\int_{\Gamma_0} w^2 dS = 1$, then necessarily $c_1 = 0$ and $c_2 = 1$, so that we may write

$$w(t,\theta) = e^{-\sqrt{\mu_{k_0}}\,t}\psi(\theta). \tag{65}$$

Step 5. To conclude the proof, we observe that, inserting (65) into (63), we obtain that $\gamma = \sqrt{\mu_{k_0}}$. □

The next lemma provides an upper bound for the function v.

Lemma 6.2. *Suppose that all the assumptions of Lemma 5.5 are satisfied. Then, up to enlarge $\bar{t}$, there exists a constant C independent of s such that*

$$\sup_{\Gamma_s} v^2 \leqslant CH(s) \quad \textit{for any } s > \bar{t} \tag{66}$$

and

$$\sup_{\Gamma_s} v^2 \leqslant CK_1 e^{-2\gamma s} \quad \textit{for any } s > \bar{t}. \tag{67}$$

Proof. Estimate (67) follows from (66) and (46). In order to prove (66) we proceed by contradiction and assume that there exists a sequence $s_n \to +\infty$ such that

$$\sup_{\theta\in\mathbb{S}^{N-1}} v^2(s_n,\theta) > n\int_{\mathbb{S}^{N-1}} v^2(s_n,\theta)\,dS(\theta)\,.$$

Putting $\lambda_n := s_n - 1$ and dividing both sides of the last inequality by $\sqrt{H(\lambda_n)}$ we infer

$$\sup_{\theta\in\mathbb{S}^{N-1}} w^2_{\lambda_n}(1,\theta) > n\int_{\mathbb{S}^{N-1}} w^2_{\lambda_n}(1,\theta)\,dS(\theta) \tag{68}$$

with w_{λ_n} as in Lemma 6.1. By Lemma 6.1, along a suitable subsequence $\{\lambda_{n_k}\}$ we have

$$\sup_{\theta\in\mathbb{S}^{N-1}} w^2_{\lambda_{n_k}}(1,\theta) \to \sup_{\theta\in\mathbb{S}^{N-1}} e^{-2\gamma}\psi^2(\theta)$$

and

$$\int_{\mathbb{S}^{N-1}} w^2_{\lambda_{n_k}}(1,\theta)\,dS(\theta) \to e^{-2\gamma}\int_{\mathbb{S}^{N-1}} \psi^2(\theta)\,dS(\theta) = e^{-2\gamma},$$

hence contradicting (68). □

We now describe the behavior of $H(t)$ as $t \to +\infty$.

Lemma 6.3. *Suppose that all the assumptions of Lemma 5.5 are satisfied and let γ be as in Lemma 5.8. Then the limit*

$$\lim_{t\to+\infty} e^{2\gamma t}H(t) \tag{69}$$

exists and belongs to $(0,+\infty)$.

Proof. By Lemma 5.4, Lemma 5.8, and direct computations we obtain

$$\frac{d}{dt}(e^{2\gamma t}H(t)) = 2\gamma e^{2\gamma t}H(t) + e^{2\gamma t}H'(t) = 2e^{2\gamma t}H(t)(\gamma - \mathcal{N}(t)) = 2e^{2\gamma t}H(t)\int_t^{+\infty}\mathcal{N}'(s)\,ds\,.$$

Integration in $(\bar t, t)$ then yields

$$e^{2\gamma t}H(t) - e^{2\gamma \bar t}H(\bar t) = 2\int_{\bar t}^{t} e^{2\gamma s}H(s)\left(\int_s^{+\infty}\nu_1(z)dz\right)ds + 2\int_{\bar t}^{t} e^{2\gamma s}H(s)\left(\int_s^{+\infty}\nu_2(z)\,dz\right)ds. \tag{70}$$

By Lemma 5.5 we deduce that the function

$$s \mapsto e^{2\gamma s}H(s)\int_s^{+\infty}\nu_1(z)\,dz$$

is non positive.

On the other hand combining Lemma 5.7 with (46) we infer that

$$s \mapsto e^{2\gamma s}H(s)\int_s^{+\infty}\nu_2(z)\,dz$$

is integrable in a neighborhood of infinity. This implies that the right hand side of (70) admits a limit as $t \to +\infty$. This proves that the limit in (69) exists; on the other hand by (46) it is necessarily finite. It remains to prove that it is strictly positive.

Let R be such that $\overline{B}_R \subset \Omega$ and let $T := -\log R$. For any $k \in \mathbb{N}$ let us denote by ψ_k an eigenfunction of $-\Delta_{\mathbb{S}^{N-1}}$ corresponding to the eigenvalue μ_k and suppose that the set $\{\psi_k\}_{k\geqslant 1}$ is an orthonormal basis of $L^2(\mathbb{S}^{N-1})$. For any $t \geqslant T$, we define the functions

$$\varphi_k(t) := \int_{\mathbb{S}^{N-1}} v(t,\theta)\psi_k(\theta)\,dS(\theta)$$

and

$$\zeta_k(t) := \int_{\mathbb{S}^{N-1}}\left[e^{-2t}\widetilde{h}(t,\theta)v(t,\theta) + e^{-2t}\widetilde{f}(t,\theta,v(t,\theta))\right]\psi_k(\theta)\,dS(\theta)\,.$$

Since v is a solution to (23) then, for any $k \geqslant 1$, φ_k solves the equation

$$-\varphi_k''(t) + \mu_k\varphi_k(t) = \zeta_k(t) \quad \text{in } [T,+\infty).$$

Integration of the above ordinary differential equation yields

$$\varphi_k(t) = \left(c_1^k - \int_T^t \frac{e^{-\sqrt{\mu_k}\,s}}{2\sqrt{\mu_k}}\,\zeta_k(s)\,ds\right)e^{\sqrt{\mu_k}\,t} + \left(c_2^k + \int_T^t \frac{e^{\sqrt{\mu_k}\,s}}{2\sqrt{\mu_k}}\,\zeta_k(s)\,ds\right)e^{-\sqrt{\mu_k}\,t}$$

for some $c_1^k, c_2^k \in \mathbb{R}$. Let $k_0 \geqslant 1$ be as in Lemma 6.1 so that

$$\gamma := \lim_{t\to+\infty}\mathcal{N}(t) = \sqrt{\mu_{k_0}}.$$

By definition of φ_k and the Parseval identity we have $H(t) = \sum_{k=1}^{+\infty}|\varphi_k(t)|^2$. In particular, by (46)

$$|\varphi_k(t)| \leqslant \sqrt{H(t)} \leqslant \sqrt{K_1}e^{-\sqrt{\mu_{k_0}}\lambda} \quad \text{for all } t > \bar t. \tag{71}$$

Let m be the multiplicity of the eigenvalue μ_{k_0} and let j_0 be such that

$$\mu_{j_0} = \ldots = \mu_{k_0} = \ldots = \mu_{j_0+m-1}.$$

Let us fix an index $i \in \{j_0, \dots, j_0+m-1\}$ and provide an estimate for the function ζ_i. From (**H**), (**F**), (46), and Lemma 6.2 we infer

$$\begin{aligned}(72)\qquad |\zeta_i(t)| &\leqslant (C_h e^{-\varepsilon t} + C_f e^{-2t})\sqrt{H(t)} + C_f C^{\frac{p-1}{2}} \sqrt{\omega_{N-1}} (H(t))^{\frac{p-1}{2}} e^{\left(-N+\frac{N-2}{2}p\right)t} \\ &\leqslant \sqrt{K_1}(C_h e^{-\varepsilon t} + C_f e^{-2t}) e^{-\sqrt{\mu_{k_0}}\,t} + C_f C^{\frac{p-1}{2}} \sqrt{\omega_{N-1}} K_1^{\frac{p-1}{2}} e^{\left(-N+\frac{N-2}{2}p\right)t} e^{-(p-1)\sqrt{\mu_{k_0}}\,t}.\end{aligned}$$

Since $p > 2$, the previous estimate gives

$$s \mapsto e^{\sqrt{\mu_{k_0}}\,s}\zeta_i(s) \in L^1(0,+\infty), \quad s \mapsto e^{-\sqrt{\mu_{k_0}}\,s}\zeta_i(s) \in L^1(0,+\infty).$$

This implies that

$$\left(c_2^i + \int_T^t \frac{e^{\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\,\zeta_i(s)\,ds\right) e^{-\sqrt{\mu_{k_0}}\,t} = O(e^{-\sqrt{\mu_{k_0}}\,t}) = o(e^{\sqrt{\mu_{k_0}}\,t}) \qquad \text{as } t \to +\infty$$

and hence

$$c_1^i - \int_T^{+\infty} \frac{e^{-\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\,\zeta_i(s)\,ds = 0$$

since otherwise we would have $\lim_{t\to+\infty} \varphi_i(t) e^{-\sqrt{\mu_{k_0}}\,t} \neq 0$, in contradiction with (71).

Therefore we may write

$$(73)\qquad \varphi_i(t) = \left(\int_t^{+\infty} \frac{e^{-\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\,\zeta_i(s)\,ds\right) e^{\sqrt{\mu_{k_0}}\,t} + \left(c_2^i + \int_T^t \frac{e^{\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\,\zeta_i(s)\,ds\right) e^{-\sqrt{\mu_{k_0}}\,t}$$

and so by (72) we infer

$$(74)\qquad \varphi_i(t) = \left(c_2^i + \int_T^t \frac{e^{\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\,\zeta_i(s)\,ds\right) e^{-\sqrt{\mu_{k_0}}\,t} + o(e^{-(\sqrt{\mu_{k_0}}+\delta)\,t}) \qquad \text{as } t \to +\infty$$

where $\delta = \min\{\varepsilon, 2, -N + \frac{N-2}{2}p\}$.

Suppose by contradiction that $\lim\limits_{t\to+\infty} e^{2\sqrt{\mu_{k_0}}\,t} H(t) = 0$, so that, by (71), for any $k \geqslant 1$ we have

$$(75)\qquad \lim_{t\to+\infty} e^{\sqrt{\mu_{k_0}}\,t} \varphi_k(t) = 0\,.$$

Multiplying both sides of (74) by $e^{\sqrt{\mu_{k_0}}\,t}$ and exploiting (75) we get

$$c_2^i + \int_T^{+\infty} \frac{e^{\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\,\zeta_i(s)\,ds = 0$$

and hence

$$\varphi_i(t) = -\left(\int_t^{+\infty} \frac{e^{\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\,\zeta_i(s)\,ds\right) e^{-\sqrt{\mu_{k_0}}\,t} + o(e^{-(\sqrt{\mu_{k_0}}+\delta)\,t}) \quad \text{as } t \to +\infty.$$

Using again (72), we finally obtain

$$(76)\qquad \varphi_i(t) = O(e^{-(\sqrt{\mu_{k_0}}+\delta)\,t}) \quad \text{as } t \to +\infty.$$

Therefore, by (47) with $\sigma < 2\delta$ and (76), we have

$$\int_{\mathbb{S}^{N-1}} w_\lambda(0,\theta)\psi_i(\theta)\,dS(\theta) = (H(\lambda))^{-\frac{1}{2}} \varphi_i(\lambda) = o(1) \quad \text{as } \lambda \to +\infty,$$

for any $i \in \{j_0, \dots, j_0+m-1\}$. Passing to the limit as $k \to +\infty$ along a subsequence as in Lemma 6.1, then yields

$$\int_{\mathbb{S}^{N-1}} \psi(\theta)\psi_i(\theta)\, dS(\theta) = 0 \quad \text{for any } i \in \{j_0, \dots, j_0+m-1\}$$

with ψ as in Lemma 6.1. This contradicts the fact that $\|\psi\|_{L^2(\mathbb{S}^{N-1})} = 1$ and that ψ belongs to the space generated by $\psi_{j_0}, \dots, \psi_{j_0+m-1}$. The proof is thereby complete. □

We are now ready to prove the main theorem.

Proof of Theorem 2.1. Let $\{\psi_i\}_{i\geqslant 1}$ be as in the proof of Lemma 6.3. By Lemma 6.1 and Lemma 6.3, for any sequence $\lambda_n \to +\infty$ there exists a subsequence $\{\lambda_{n_k}\}$ such that, for any $\alpha \in (0,1)$,

$$e^{\gamma\lambda_{n_k}} v(\lambda_{n_k}, \theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in } C^{1,\alpha}(\mathbb{S}^{N-1}) \text{ as } k \to +\infty, \tag{77}$$

$$e^{\gamma\lambda_{n_k}} \frac{\partial v}{\partial t}(\lambda_{n_k}, \theta) \to -\gamma \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in } C^{0,\alpha}(\mathbb{S}^{N-1}) \text{ as } k \to +\infty, \tag{78}$$

and

$$e^{\gamma\lambda_{n_k}} \nabla_{\mathbb{S}^{N-1}} v(\lambda_{n_k}, \theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i \nabla_{\mathbb{S}^{N-1}} \psi_i(\theta) \quad \text{in } C^{0,\alpha}(\mathbb{S}^{N-1}, T\mathbb{S}^{N-1}) \text{ as } k \to +\infty \tag{79}$$

for some $\beta_{j_0}, \dots, \beta_{j_0+m-1} \in \mathbb{R}$ such that $(\beta_{j_0}, \dots, \beta_{j_0+m-1}) \neq (0, \dots, 0)$.

Let us prove that the coefficients $\beta_{j_0}, \dots, \beta_{j_0+m-1} \in \mathbb{R}$ depend neither on the sequence $\{\lambda_n\}$ nor on its subsequence $\{\lambda_{n_k}\}$.

First of all, for any $i \in \{j_0, \dots, j_0+m-1\}$ we have

$$\lim_{k\to+\infty} e^{\gamma\lambda_{n_k}} \varphi_i(\lambda_{n_k}) = \lim_{k\to+\infty} e^{\gamma\lambda_{n_k}} \int_{\mathbb{S}^{N-1}} v(\lambda_{n_k}, \theta)\, \psi_i(\theta)\, dS(\theta) = \beta_i. \tag{80}$$

On the other hand, by (73) with $t = T := -\log R$ and R as in the statement of Theorem 2.1, we infer

$$c_2^i = e^{\sqrt{\mu_{k_0}}\,T} \varphi_i(T) - e^{2\sqrt{\mu_{k_0}}\,T} \int_T^{+\infty} \frac{e^{-\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\, \zeta_i(s)\, ds,$$

which inserted in (74) gives

$$\begin{aligned} \varphi_i(t) = {} & \left(e^{\sqrt{\mu_{k_0}}\,T} \varphi_i(T) - e^{2\sqrt{\mu_{k_0}}\,T} \int_T^{+\infty} \frac{e^{-\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\, \zeta_i(s)\, ds \right) e^{-\sqrt{\mu_{k_0}}\,t} \\ & + e^{-\sqrt{\mu_{k_0}}\,t} \int_T^t \frac{e^{\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\, \zeta_i(s)\, ds + o(e^{-(\sqrt{\mu_{k_0}}+\delta)\,t}) \qquad \text{as } t \to +\infty\,. \end{aligned}$$

Multiplying both sides of the last identity by $e^{\sqrt{\mu_{k_0}}t}$ and passing to the limit as $t \to +\infty$ we obtain

$$e^{\sqrt{\mu_{k_0}}\,t} \varphi_i(t) \to e^{\sqrt{\mu_{k_0}}\,T} \varphi_i(T) - e^{2\sqrt{\mu_{k_0}}\,T} \int_T^{+\infty} \frac{e^{-\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\, \zeta_i(s)\, ds + \int_T^{+\infty} \frac{e^{\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\, \zeta_i(s)\, ds\,.$$

This combined with (80) yields

$$\beta_i = e^{\sqrt{\mu_{k_0}}\,T}\varphi_i(T) - e^{2\sqrt{\mu_{k_0}}\,T}\int_T^{+\infty}\frac{e^{-\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\,\zeta_i(s)\,ds + \int_T^{+\infty}\frac{e^{\sqrt{\mu_{k_0}}\,s}}{2\sqrt{\mu_{k_0}}}\,\zeta_i(s)\,ds\,.$$

Therefore the coefficients $\beta_{j_0},\dots,\beta_{j_0+m-1}$ do depend neither on $\{\lambda_n\}$ nor on $\{\lambda_{n_k}\}$ and hence (77)–(79) also hold as $\lambda\to+\infty$ and not only along the sequence $\{\lambda_{n_k}\}$. The proof of Theorem 2.1 then follows from (8), the fact that $v = Tu$, and direct computations. □

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