A Network Approach for Opinion Dynamics and Price Formation

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Abstract

If men define situations as real, they are real in their consequences.

W.I. Thomas and D.S. Thomas

In this work, we investigate the intertwined role of network interaction, opinion dynamics and price formation in a financial system. We propose a dynamical multi - agent framework where the interaction network and its topology, opinions and prices depend on one another, co - evolving in time. At first, we introduce some useful concepts in network theory and opinion dynamics. A method for classifying agents according to their topological role in the network is proposed. Second, we build on the existing literature on hetereogenous beliefs and evolutionary systems and provide a model with a specific update rule that leads to an evolving topology. The model is apt at describing social and behavioural phenomena that have recently received particular attention in the financial literature, such as hetereogeneous beliefs on market scenarios and the effects of the topology of interactions. We illustrate such dynamics via simulations, discussing the stylized facts that the model might be able to capture and we will discuss the use of social network data in order to calibrate the model. Third, we propose a model for formation of relative prices in a closed economy when agents have limited attention about a certain asset/sector.

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Chapter 1

Introduction

The main research question of this work is to investigate the intertwined role of network interaction, opinion patterns and price formation in a financial system. In particular, the motivation for this reseach came from the understanding that financial markets and, more generally, social systems, are endogenously driven by how the agents perceive and interpret the system itself. By quoting the so – called Thomas theorem in the abstract *'If men define situations as real, they are real in their consequences'* [1] we refer to a situation where opinions and reality mutually shape in a reciprocal way. In a social system, interpreting a situation implies and causes a subsequent action, but the interpretation is almost never *objective* and carries with itself a *certain degree of subjectivity* and this is what drives individual choices and behaviours.

In this light, *self – fulfilling prophecies*, i.e. statements or predictions that cause themselves to be true, might occur[2]. This concept is ubiquitous in social sciences but it has recently become a predominant paradigm in finance as well, where this concept is often referred to as *reflexivity.* This idea has been put forward, for example, by George Soros in his 1987 book *The Alchemy of Finance* on the basis on Karl Popper's view on social systems.

It is beyond the scope of this work to provide the reader with an extensive explanation of such a concept and the ways it has been declined. Hence, we will focus on the relevant ideas that will be useful to assess the research question, providing a brief

[1] For further details, see Robert K. Merton's historical account of the Thomas theorem, published in 1995.

[2] For an extensive account on this, we refer the reader to Merton's *Social Theory and Social Structure.*

summary of the three main streams of research used in the approach we propose in this work: i) network and graph theory, ii) opinion dynamics and iii) behavioural finance and Agent Based Models (ABMs). A more detailed literature review can be found, respectively, in Chapter 2, 3, and 4.

Structure of this work

Chapter 2 reports the mathematical concepts related to graph and network theory that will be used in Chapters 3 and 4. We will introduce basic definitions and results in graph and network theory, and more advanced concepts in network connectivity and graph topology, centrality measures and Perron – Frobenius theory for nonnegative matrices. In particular, we will define the Gantmacher form, which is related to a way to classify agents according to their role in opinion dynamics and we will propose an algorithm to obtain this classification.

Chapter 3 deals with the mathematical modelling of opinion dynamics where opinions are pooled in a social network. Starting from the work of Galton (1907) on the so called *wisdom of the crowds*, we will stress the relationship between opinions and the object of the opinion. We will then continue reporting models related to opinion dynamics in a social network setting: starting from DeGroot and French's early contributions to Berger's subsequent refinement, we will then continue to Friedkin's interpretation in terms of centrality measure and end with a later series of work by Krause and Lorenz which extend previous works in a non – linear setting. Contributions from other fields (physics, biology, robotics) will also be mentioned, although not in details. Analytical results on the behaviour of the system will be provided in terms of convergence theorems and ergodicity.

Chapter 4 will introduce two models. The first model relates to the recent series of work on behavioural finance and Agent Based Models (ABMs) in an evolutionary dynamical system when agents have heterogeneous adapetive beliefs (in particular we refer to the contributions original from Brock and Hommes's 1997 influential paper). We will show by computer simulations that opinion dynamics and an associated dynamical network process can replicate some well known stylized facts in financial systems.

From an empirical standpoint, we will suggest a way to calibrate the model by using so called *Big Data* and two examples will be given. The second model will deal with *limited attention* as a behavioural limitation of agents even when full information is available.

In Chapter 5 we will draw the conclusions and illustrate some open problems that will constitute the basis of future research work.

Big data and finance

The concept of *Big Data* refers to data sets that are extremely large, complex and from different sources. Availability of such data has been growing in the last few years with the advent of the Internet, social networks and increasing data storing capacity. Comparing data sets from different sources has proven to be a challenging research endeavour that is still in a incipient, perhaps undeveloped, state. Furthermore, *Big Data* represent a unique opportunity for cross fertilisation among different research fields.

Huge datasets usually represent a limitation for scientists in the sense that the need to provide quick and "online" answers to problems of different nature proves to be hard when one needs to analyse, for example, exabytes of data. In addition to this, more data do not necessarily imply better quality processing and prediction, so there will always be a certain trade – off between the amount of data that can be analysed and processed by a researcher and the quality of the research. Even the usage of massive parallel computing does not represent a definitive answer as, in turn, results need to be validated and interpreted, which is not always an easy task in this setting.

The explosion in data availability has not spared finance, which has always been one of the disciplines where data availability represented a clear advantage for researchers. In particular, the last five years have seen a dramatic increase in the availability of data that were not even contemplated in the past. Institutions, practitioners, and regulators call for more data availability and quality for such data. Former ECB president, Jean Claude Trichet, wrote[3]:

[3] A strategic vision for Statistics challenges for the next 10 years. `http://www.ecb.europa.eu/pub/pdf/other/strategicvisionstatistics2008en.pdf`

The ongoing financial turbulence has also led to a call for more transparency, inter alia, by developing relevant new statistics.

This process, however, does not come at zero cost. Not only would this collecting, validating and processing an enormous amount of data, but also to understand whether pattern detection and, more generally, statistical analysis can actually provide with the right answers to the right questions. Finance represents an obvious and interesting 'data playground' given to both availability and timeliness with which data are released. Even the Wall Street Journal stressed the need to reconsider the role of Big Data in the financial world[4]:

We will not treat in a detailed way the implications that this will have in the financial world, and the way financial models will develop in the next years. Our view is that, as every evolution in technology, Big Data analytics might represent a double – edged sword: on the one hand, increase in data availability could help in detecting patterns, prediction, validation, etc but, on the other hand, it could represent a problem for three main reasons. The first is that models that try to make sense out of pattern discovery that involves huge computations need to be relatively simple and a whole unifying methodology is yet to come. As such, interpretation of the results might become extremely difficult. The second reason lies in the validation of results: preprocessing and analysis is extremely costly and so is their replication. The third reason is overconfidence: more data availability does not necessarily imply a better analysis or prediction and could lead the research to draw conclusions that are not analytically sound. This work tries tackle the problem by proposing a dynamical model with some analytical results that could be used for testing, pattern discovery and prediction.

In the course of the empirical validation proposed in Chapter 4, we will make use of data retrieved from the micro – blogging social network Twitter. Twitter (www.twitter.com) is a massive social networking website where a number of 'tweets' (around 500 millions) are posted on a daily basis. Tweets cannot exceed 140 characters, meaning that users have to necessarily be coincise in their messages. Twitter has

[4]Why Every Finance Professional Needs a Degree in Big Data, available at `http://blogs.wsj.com/cfo/2013/07/25/why-every-finance-professional-needs-a-degree-in-big-data/`

played an important role in the Arab Spring, the Occupy Wall Street movement, in reporting natural disasters, etc. (see Kumar *et al.*, 2013).

People can tweet about different topics, including their opinion on stocks, by using the *hashtag* $. As such, they are offered a quick and easy way to exchange *opinions* about the value of a certain stock. The financial data provider Bloomberg offers an integrated service that extracts relevant tweets for trading sessions[5].

Data are retrieved via the Twitter *Application Programming Interface* (or API, available at https://dev.twitter.com). The problems connected with data retrieval, pre – processing and storage are not discussed here and we refer the reader to Kumar *et al.* (2013).

The study of the interaction of such social network website and the availability of the *actual* price dynamics represents an open problem. In particular, it represents an incredibly vast and unexplored dataset for assessing the impact opinions can have on financial markets and vice versa.

Motivating example

On April 23, 2013, at 1:07 PM the US stock crashed by about 1% in a couple of minutes, recovering afterwards in about the same time. Fig. 1.1 shows the DJ 30 industrial index time series during those minutes. We refer, in particular, to the sudden and dramatic drop *in the index* visibile at the end of the 1–minute intraday time series. What happened during those *few* minutes?

At 1:07 p.m., the Associated Press tweeted that the US president had been reported injured in an explosion at the White Hous (see Fig. 1.2). The tweet was fake, probably due to a hacker that accessed AP's account. The tweet was immediately re–tweeted a number of times (about 4000 in less than a minute, and even more in the next minutes).

American Press immediately noticed the hack, warning its followers and suspending the account in order to investigate the issue (see Fig. 1.3). Notwithstanding the very quick correction, the stock market plummeted, as previously described[6].

[5]Bloomberg Integrates Live Twitter Feeds With Financial Platform, `http://www.bloomberg.com/now/press-releases/bloomberg-integrates-live-twitter-feeds-with-financial-platform/`

[6]See `http://business.time.com/2013/04/24/how-does-one-fake-tweet-cause-a-stock-market-crash/`

Figure 1.1: April 23 DJ 30 Industrial crash – 1 minute intraday.

Figure 1.2: AP tweet reporting explosions at the White House.

Figure 1.3: AP's account suspension.

This two – minute storm that led to a $\sim 1\%$ drop and a loss of $130 billion in stock value cannot be explained by only *one* tweet: indirect (network) effects have to be taken into account, probably also due to the increase in automatic trading strategies.

We believe that, in addition to this, AP's authority as an independent and reliable agency, coupled with the "check mark" that certifies a *verified* Twitter account, led investors (indepedently from their strategy) to immediately *trust the consequences* of such a tweet. To quote the Thomas theorem, *the consequences were real*, although the actual "situation" was not.

Software and data

All data processing and simulations have been done in Matlab. Tri – dimensional plots have been exported to LaTeXvia `matlab2tikz`[7].

Twitter (`www.twitter.com`) data have been retrieved and pre – processed in Python with Python Twitter (a Python wrapper around the Twitter API (`https://dev.twitter.com/`) and checked via Twitty for Matlab.

All network visualizations are done with Gephi (`https://gephi.org/`).

#ixzz2jIV7iBU7 and http://www.ft.com/cms/s/0/33685e56-ac3d-11e2-a063-00144feabdc0.html#axzz2jIaJPrV5.

[7] http://www.mathworks.de/matlabcentral/fileexchange/22022-matlab2tikz/all_files

Highlights

- We will introduce concepts in network theory and opinion dynamics, providing an algorithm to classify agents in the network according to their topological relevance (essential and inessential) in opinion formation.
- We will propose an ABM (Agent Based Model) for price formation based on heterogeneous beliefs/opinions including a network interaction. The main peculiarity of this model is to find a mutual feedback relationship among opinions, topology and prices.
- We will use this model to replicate (by computer simulations) some stylized facts about financial markets.
- We will also illustrate a way to empirically validate such model by using "Big Data" from online social networks.
- We will introduce a model on limited attention and relative price formation.

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Last, but not least, I would like to express my love and gratitude to my family and all my friends.

Thank you!

Chapter 2

Graphs and networks

2.1 Introduction

In order to understand how social interaction in a network setting can influence the formation of opinions in a social system, we need to introduce mathematical tools apt at capturing the interaction occurring among agents. Hence, we will dedicate this Chapter to recall some useful definitions and concepts from graph and network theory.

The concept of *network* is quite broad in real life: from social networks to food webs, from the World Wide Web to networks of citation, a number of different problems have been tackled from a network perspective. From both a methodological and an empirical standpoint, networks have received growing attention among researchers in the last decades (Newman, 2003). Different disciplines - from physics to sociology, from biology to economics - have used network theory to approach both theoretical and empirical problems. For a general review on this topic see the aformentioned work of Newman (2003), Wasserman and Faust (1994) for a in-depth analysis social networks analysis and Jackson (2010) for a more recent textbook review on social and economic networks.

The mathematical background when analyzing networks is graph theory. Altough network theory *per se* has later evolved autonomously (especially for what concerns large complex networks), the main contributions from graph theory still remain the backbone for the quantitative treatment of network – related problems.

Network theory also represent an interesting case where an ubiquitous concept can lead to cross – fertilization between research fields that are apparently distant. As an example, consider centrality measures. Such measures were born in a sociological context, where the main problem was assessing the importance of an actor in a social network according to specific criteria. However, they proved to be very useful in different context, e.g. biological systems, financial systems, ...

The aim of this Chapter is twofold:

1. introduce some fundamental concepts in matrix and graph theory that will be used throughout the rest of the work;
2. providing with a way to efficiently classify agents (essential and inessential) in a network.

2.2 Graph theory: basic definitions and notations

The model we analyse in the next sections is based on a network-like structure, thus requiring a graph – theoretical approach. Since it is not the aim of this work to provide an extensive explanation of graph and network theory, we refer the reader to Harary (1969) and Newman (2010) for more details. Two relevant references for matrix theory are Gantmacher (1959) and Horn & Johnson (1990).

A graph $G = (V, E)$ is a pair of sets (V, E), where V is the set of n vertices and E is the set of m pairs of vertices of V. The pair (i, j) belonging to E is called an *edge* of G and i and j are called *adjacent*. A *directed graph* (digraph) is a graph in which each edge is an ordered pair (i, j) of vertices. G is simple if there one edge between two adjacent vertices. A weight w_{ij} can be associated to each edge (i, j) and, in this case, we will have a *weighted* or *valued* graph. A *path* is a sequence of distinct adjacent vertices and a $i - j$ path is a path from i to j. For a pair of vertices (i, j), i and j *communicate* if it holds $i - j$ and $j - i$. In case $w_{ii} > 0\ \forall i$, it always holds $i - i$, and we call i *self communicating*. A vertex i is *essential* if, $\forall j \in V$, with $i - j$, it holds $j - i$ and we denote such property with $i \sim j$. A vertex i is *inessential* if it is not essential.

V can be partitioned into self–communicating equivalence classes of vertices. In fact it easy to see that if i is self communicating and (i, j) communicate, then $i \sim i$, $i \sim j \Rightarrow j \sim i$ and, finally, if $i \sim j$ and $j \sim k \Rightarrow i \sim k$, i.e. a path from i to k and vice versa can always be found passing through j. Hence, $\sim$ is an equivalence relation. All the vertices in each equivalence class are essential and the class itself is called essential. A self communicating inessential vertex constitutes an equivalence class on its own. In an essential class, all vertices communicate to each other and do not communicate with other vertices belonging to other classes. On the other hand, inessential vertices are linked to either essential and inessential vertices.

A particular case arises when all vertices belong to one essential class, i.e. $\forall(i,j)$ $i \sim j$. In this case, the graph is called *strongly connected*.

A nonnegative $n-$square matrix $\mathbf{W}$ representing the adjacency relationships, possibly weighted and directed, between vertices of G is said to be the *adjacency matrix* of G. A nonnegative adjacency matrix $\mathbf{W}$ is said to be *diagonal dominant* if $w_{ii} \geq \sum_{j\neq i} w_{ij}$ $\forall i$. $\mathbf{W}$ is said to be *irreducible* if for some permutation matrix $\mathbf{P}$, the (permuted) matrix $\mathbf{PWP}^T$ is *not* block upper triangular. A matrix that is not irreducible is said to be *reducible*. The adjacency matrix of a strongly connected graph is irreducible. If there exist a $t \in \mathbb{N}$ such that $\mathbf{W}^t$ is (strictly) positive, then $\mathbf{W}$ is said to be *primitive*.

An n–vector $\mathbf{x}$ is said to be a *right eigenvector* of $\mathbf{W}$, with associated *eigenvalue* λ if $\mathbf{Wx} = \lambda\mathbf{x}$. For any $n \times n$ matrix, there exist n eigenvalues $\lambda_1, \lambda_1, \ldots, \lambda_n$.

A nonnegative n–square matrix $\mathbf{W} \in \mathbb{R}^{n\times n}$ is row – stochastic if the elements in each row sum up to 1, i.e. $\sum_{j=1}^{n} w_{ij} = 1$ $\forall i$ or, in matrix notation, $\mathbf{Wu} = \mathbf{u}$, where $\mathbf{u}$ is the n–th unit vector. It is easy to show that a row stochastic matrix has dominant eigenvalue $\lambda_1 = 1$ with $\mathbf{u}$ its associated eigenvector.

Gantmacher form Any square matrix $\mathbf{W}$ can be brought in its *Gantmacher form*:

$$\mathbf{W}_G = \mathbf{PWP}' = \begin{bmatrix} \mathbf{W}_{1,1} & & & & & \mathbf{0} \\ & \ddots & & & & \\ \mathbf{0} & & \mathbf{W}_{g,g} & & & \\ \mathbf{W}_{g+1,1} & \ldots & \mathbf{W}_{g+1,g} & \mathbf{W}_{g+1,g+1} & & \\ \vdots & & \vdots & \vdots & \ddots & \\ \mathbf{W}_{p,1} & \ldots & \mathbf{W}_{p,g} & \mathbf{W}_{p,g+1} & \ldots & \mathbf{W}_{p,p} \end{bmatrix}$$

where $\mathbf{P}$ is a suitable permutation matrix. The diagonal blocks $\mathbf{W}_{1,1}, \ldots, \mathbf{W}_{p,p}$ are square and irreducible (Gantmacher 1959, Lorenz 2007). For the non diagonal

Gantmacher blocks $\mathbf{W}_{kl}$ with $k = g+1, \dots, p$ and $l = 1, \dots, k-1$. Then, it holds that for every $k = g+1, \dots, p$ at least one block of $\mathbf{W}_{k,1}, \dots, \mathbf{W}_{k,k-1}$ contains at least one positive entry. It turns that $\mathbf{W}_{kk}$, with $k = 1, \dots, g$ are the sub matrices associated to essential classes. Two non – negative matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be of the same *type*, $\mathbf{A} \sim \mathbf{B}$ if they share the same non negative pattern, i.e. $a_{ij} > 0 \Leftrightarrow b_{ij} > 0$.

Perron – Frobenius theory An important theorem connected to these definition that will be fundamental in the next part of this work is the Perron – Frobenius theorem, an important result for non – negative matrices which is key to eigenvector centrality. It has important application in Markov chains (relevant for Chapter 3), graph theory, numerical analysis, etc. See Meyer (2000) for a review of the applications and Chang *et al.* (2008) for an extension to multilinear forms.

Theorem 1 (Perron – Frobenius (Weak form))**.** *Given* $\mathbf{A} \in \mathbb{R}^{n\times n}_{\geq 0}$*, then:*

- *the spectral radius of* $\mathbf{A}$*,* $\rho(\mathbf{A})$*, is also an eigenvalue;*
- $\exists\ \mathbf{x} \in \mathbb{R}^n$ *s.t.* $\mathbf{Ax} = \rho\mathbf{x}$

Theorem 2 (Perron – Frobenius (Strong form))**.** *Let* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *be an irreducible nonnegative matrix, then:*

- $\rho(\mathbf{A})$ *is an eigenvalue;*
- $\exists\ \mathbf{x} > 0$ *(strictly positive) s.t.* $\mathbf{Ax} = \rho\mathbf{x}$
- *(uniqueness) if* λ *is an eigenvalue of* $\mathbf{A}$*, with associated nonnegative eigenvector, then* $\lambda = \rho(\mathbf{A})$
- $\rho(\mathbf{A})$ *is a simple eigenvalue of* $\mathbf{A}$ *and;*
- *if* λ *is an eigenvalue of* $\mathbf{A}$*, then* $|\lambda| \leq \rho(\mathbf{A})$

2.3 Connectivity

The notion of *connectivity* in a graph (and, more generally, in a digraph) is crucial in our analysis and thus deserves a thorough explanation. When a pairwise interaction among a set of agents occurs, it is natural to wonder whether agents interact with each other not only via direct links but also via indirect links (e.g. a path). Answering this question will provide with useful tools in order to analyse the dynamic processes occurring on the network that we will outline in the next chapters.

Given an adjacency matrix $\mathbf{A}$ for a graph G, we say that vertex j is *reachable* from node i if and only if there exists a path from i to j.

A digraph $G = (V, E)$ is said to be *strongly connected* if, for any pair $(i, j) \in V$ there is a path from each i in the graph to every other j. A digraph is said to be *weakly connected* when, replacing all of its directed edges with undirected edges, the resulting graph is connected.

2.3.1 Condensation digraph

An important results of graph theory (see Harary, 1969) is that a non-connected graph G can be *uniquely* partitioned into separate strongly connected components (SCC). When each one of these strongly connected component is contracted to a single vertex, the resulting graph is a directed acyclic graph (DAG), the so called *condensation* of G. A DAG is a digraph with no directed cycles, i.e. there exists no path connecting a node i to itself.

Given a graph $G = (V, E)$, its condensation digraph $C(C) = (V_C, E_C)$ is a graph where the set of nodes V_C is represented by the strongly connected components of G. Accordingly, the condensation digraph of G, denoted $C(G)$, is defined as follows: the nodes of $C(G)$ are the SCCs of G, and there exists a directed edge in $C(G)$ from node H_1 to node H_2 if and only if there exists a directed edge in G from a node of H_1 to a node of H_2.

2.3.2 Eigenvector centrality

In order to understand the role of the influence of a node in a network, Bonacich (1987) introduced a measure to quantitatively assess the influence of agents taking into account also inderect connections: the basic idea is that the centrality scores of each agent is a linear combination of the scores of its neighbours. Related measures are the well – known Google Page rank and Katz centrality (see Newman, 2010)

Given a (weighted) adjacency matrix $\mathbf{A}$ for a strongly connected graph $G = (V, E)$. The eigenvector centrality $x_i > 0$ of a node i is a scalar value obtained as a linear combination of the eigenvector centralities of the adjacent nodes of i (Bonacich, 1987 and 1991). Formally:

$$x_i = \frac{1}{\lambda} \sum_{j=1}^{n} a_{ij} x_j \quad (i, j) \in V$$

which, in matrix notation, means to solve a system of n linear equations $\mathbf{Ax} = \lambda \mathbf{x}$.

Since, in general, there will be several eigenvalues satisfying the relationship, the fact that the centrality scores x_i must be strictly positive implies, by the Perron – Frobenius theorem (strong form) that only the dominant eigenvalue is to be chosen and the $i - th$ component of the associated eigenvector is the centrality score for node i.

2.3.3 Determining the Gantmacher form

If one knows in advance the topological structure of the matrix in terms of essential and inessential agents, it is easy to find the permutation matrix that leads to the Gantmacher normal form. However, especially from experimental data and simulations, this is not always the case and it is necessary to find an algorithm apt at finding the proper permutation matrix. In particular, in Chapter 4, in the part dedicated to the empirical validation of the models we propose, a number of large matrices would need to be brought into their Gantmacher form. As such, an easy and quite efficient method needs to be developed. We hereby propose a heuristics[1] in order to solve the problem. The algorithm outline is as follows:

1. find the Strongly Connected Components of G, e.g. via Tarjan's algorithm, of Depth First Search based algorithms ($O(|V| + |E|)$), see Knuth (1997);

2. find the condensation digraph $C(G)$ of G;

3. for each node k of $C(G)$ check links for nodes;

[1] This algorithm is inspired by Fig. 2 in Mirtabadei & Bullo (2011). We thank Jan Lorenz (private communication) for providing useful hints.

4. if node k has only in–coming links (i.e. sinks), then it is associated to an essential class;

5. if node k has only out – going links, then is is associated to an innessential class.

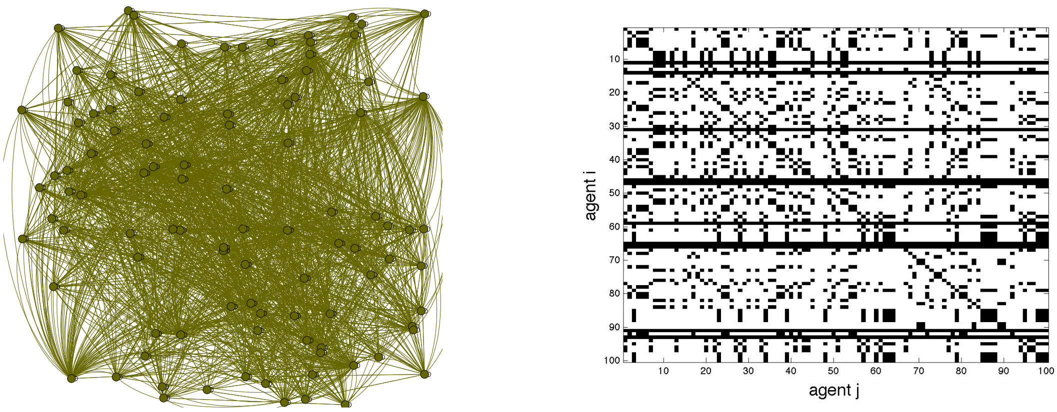
6. order nodes in $C(G)$ in order to find permutation matrix for $\mathbf{A}$.

The algorithm proves to be quite fast for the applications in Chapter 4 (Empirical validation). However, better solutions could be found that do not imply evaluating a node of G multiple times.

On the basis of such classification, a further refinement of Lorenz's classification has been proposed by Mirtabadei and Bullo (2011). See Table 2.3.3 for a comparison of the two classifications.

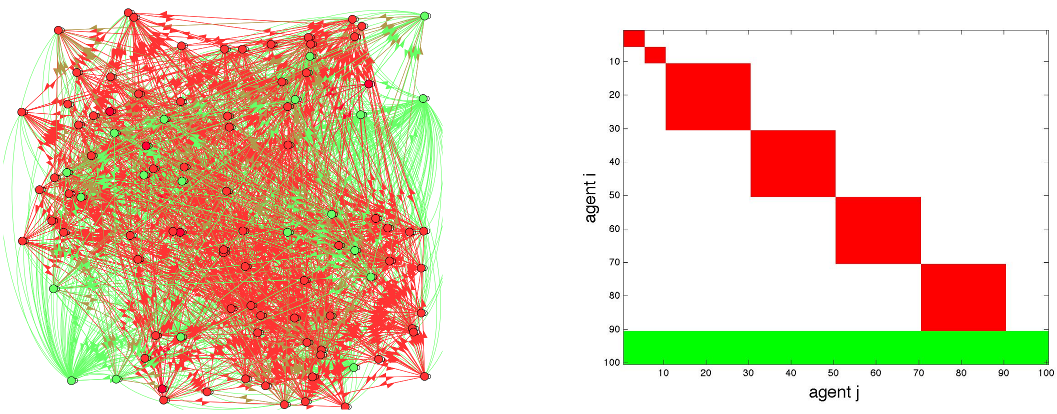
Agents classification (via SCCs)			
Lorenz (2006c)	Mirt. Bullo (2011)	SCCs condensation	SCC Subgraph
essential	closed minded	sink	complete
	moderate minded	not a sink	non – complete
inessential	open minded	not a sink	either

Figure 2.1a shows a directed network extracted from Twitter data when agents are not classified and the corresponding adjacency matrix (Figure and 2.1b) is not in its Gantmacher normal form. Figure 2.2a shows the result of the algorithm proposed above: the green agents (nodes) are the union of all inessential classes and the red ones are associated to the essential classes (along the main diagonal in the adjacency matrix, Fig. 2.2b).



(a) (b)

Figure 2.1: Agents not classified



(a) (b)

Figure 2.2: Agents classified

Chapter 3

Opinion dynamics

3.1 Introduction

3.1.1 Social opinions and reality

In this Chapter we introduce a general overview of the problems related to *opinion dynamics*, which includes a number of models in which agents interact and revise opinions in time. The basic concept that we want to address in this Chapter is that individual opinions on different subjects or, more specifically, *variables* are the result of an interaction among agents. In this light, we illustrate some models that relate with this problem, in order to explain in details a possible way opinions are formed in a social setting. Both the conceptual and analytical aspects that are explained in this Chapter form the basis for deriving our model of price formation in a financial market.

The fundamental research questions we want to address in this Chapter are the following ones:

- how can one model opinion dynamics with interacting agents?
- what is the role of the interaction topology in determining opinion patterns?
- does a feedback effect between opinions and the topology exist?

A fourth question, not fully explored in this work is whether, in general terms, the interaction of many agents in a social network will provide a better assessment of the *real* value of a variable. The adjective *real* clearly refers to the actual value the variable will assume. More precise or biased opinion patterns will have a different impact on whether the interaction provides with a more accurate way to gauge the actual value. This idea is often referred to as the *wisdom of the crowds* problem: is collective thinking more accurate than individual one? The answer to this question is not trivial. First, one needs to understand the process according to which different individual ideas and opinions upon a variable are merged into a collective one. Second, it is fundamental to understand whether either the individual or the collective opinion can have an influence on the actual value of the variable. In other words, one needs to model a system where a feedback relationship between opinions and the variable exists.

The concept of *wisdom of the crowds* has been thoruoghly explored in different streams of literature. However, it is with Galton's 1907 paper (**cite**) *Vox Populi* that the problem received one of the first quantitative interpretations. Galton describes a situation where a number of agents compete in gauging the weight of an ox, finding that the median guess is an accurate estimate of the actual weight. However, in ox weighting competitions, agents are asked to cast a prediction and no *interaction* occurs. Moreover, it is quite obvious that the guesses will have no influence on the ox's weight. In the end, the *aggregation* of the opinions is done by considering a simple statistics of the whole opinion distribution, which does not affect the variable (i.e., the weight).

In this light, what we want to quantitatively assess in this work is a more complex situation where:

1. interaction occurs;
2. opinions are a function of the actual value and, viceversa, the actual value is a function of the opinions.

Another issue that will hereby addressed is whether a feedback relationship between the interaction topology and the opinion structure occurs: in particular, we will use the Hegselmann & Krause model, a very well known non – linear model that aptly captures such feedback.

3.1.2 Introducing interaction among agents

In the modeling framework we hereby analyse, we consider a set of *interacting* agents and a certain dynamical process leading to the formation of a set of opinions, represented by a scalar value. The set of agents can be of a broad kind: from experts discussing a value connected to a particular issue to agents in financial markets. The interaction topology will clearly provide with both a way to model the opinion formation process and the influence that an agent has on others in such a process.

Some interesting questions naturally arise. Will all agents reach the same opinion? Will there be a consolidation of different opinions into fewer ones? We will refer to the first case as opinion *consensus* and to the latter as opinion *fragmentation* (see, for example, Hegselmann & Krause, 2005).

In order to analyse a dynamics of this kind, a specific mathematical formulation needs to be introduced. Probably, we can find one if the first mathematical formulations for opinion dynamics in the work of DeGroot (1974), where the author introduces an opinion (weighted) averaging scheme that will remain the baseline model in later works. The convex weighted average scheme clearly stems from the mathematical properties related to the product of stochastic matrices and, hence, Markov chains. DeGroot clearly refers to an opinion as a *subjective probability distribution*, which each agent assigns to a certain parameter. Such *a priori* opinions are then "pooled", to obtain an *a posteriori* set of opinions, thus implying a Bayesian framework. Such a model is suitable also when the opinions are formed upon a certain estimate of the parameter (not necessarily a distribution). A very important feature of DeGroot's model is that, in this setting, no external information, or data, about the parameter is available to agents. The only available information is the opinion of every other agent. In mathematical terms, DeGroot's model stems from the theory of stochastic matrices, which is a well known analytical tool in Markov Chain models. In particular, one of the main assumptions of this model is that agents fix the weight they give to other agents' opinions at the beginning, thus leading to a mathematical treatment analogous to the one of homogenous Markov chains. Previous formulations of this problem include the ones by J. French (1956), Harary (1959) and later contributions include the ones (among others) by Chatterjee and Seneta (1977) and Cohen *et al.* (1986). These works focus on reaching a common opinion, a *consensus*.

In this light, another interesting work is the one of Berger (1981), in which sufficient conditions for consensus are discussed in more details and an implicit relationships, later analyzed in the literature on economic networks (see Jackson, 2008), between *consensus* and eigenvector centrality is provided. Berger's paper shows also how a common opinion might be achieved only within sub – groups of agents, hence implying a certain degree of opinion fragmentation within the opinion structure between sub – groups (i.e. *dissent* as opposed to *consensus*).

Another approach to the opinion dynamics problem is the one analysed in Friedkin (1991), where the general consensus/opinion dynamics problem is seen under a graph/network theorical framework[1], relating to the conceptual and mathematical foundations for centrality measures in social networks. The author introduces a simple dynamical model showing opinion *polarization* in the sub – groups and an individual weighted averaging of other agents' opinion, called *compromise*. The analysis of the network of influence is key to understanding the role of agents in determining the opinion patterns, thus allowing to use the concept of *social influence* to attribute a measure of influence that takes into account also *interpersonal effects* to every agent.

The literature so far reviewed is based on models of interaction that are *linear* in their nature. Both DeGroot's and Friedkin's models, for example, feature a linear dynamical system whose asymptotic properties are key for the interpretation. As already mentioned, they make use of very well known properties of linear systems and, in particular, of homogeneous Markov chain. Berger's work implicitly introduces a graph – theoretical argument based on connectivity, and Friedkin's approach is entirely graph – based.

Their limitation lies in that such models do not capture the evolution in time of the interaction network and the fact that the interaction topology might be influenced by the opinions themselves. An interesting way to cope with both these problems is the approach proposed by Krause in a series of works (1997, 2000 and, with Hegselmann, in 2005). In this model, the non – linearity arises because the topology of the influence

[1] Previous works on this interpretation include Friedkin (1986) and Friedkin & Johnsen (1990).

network changes in time according to the opinion structure and, viceversa, the new opinion structure is derived by the network interaction. This feedback relationship does not allow to easily derive analytical results. A combination of matrix and graph – theoretical arguments can be used in order to partially overcome this problem, but most of the interpretation must rely on computer simulations. In fact, although it is possible to derive some sufficient conditions for the convergence of a system, it is still an open challenge to fully characterise in an analytical way the dynamical behaviour of such a system.

In their 2005 paper, Hegselmann & Krause review some of the above mentioned approaches, recalling both their mathematical background and their interpretation. Moreover, they detail the non – linear approach based on the so – called *bounded confidence*, i.e. when agents put a positive weight to another agent (thus *trusting* in her opinion) if the absolute value of the difference of the opinions does not exceed a certain threshold. According to such a threshold, this non – linear model can capture effects of opinion fragmentation or polarization as well as convergence to a consensus.

We define a stable opinion pattern in the dynamical system as that situation where agents stabilize their opinion and no further change occurs. Once such a stable pattern is achieved, we can, in fact, have the two following situations:

1. *consensus*, i.e. all agents have the same opinion;

2. *dissent*, i.e. agents will have different opinions that do not change over time. In particular, we can divide *dissent* into *polarization* (i.e. agents can have only two opinions) and *fragmentation* (when the number of possible opinions is higher than two).

Sufficient conditions for these models are found in the above mentioned works. In addition to this, we recall the work of Lorenz (2005, 2006, 2007) and, independently, Moreau (2005) who provided with sufficient conditions for the convergence of a product of stochastic matrices. We will observe, in particular, that the setup suggested by Lorenz provides with useful insights on the topological classification of agents. Further insights and some more technical results on the stabilisation patterns in the classification of agents can be found in Mirtabadei and Bullo (2011).

The goals of this Chapter are the following:

- illustrate the mathematical background connected to the opinion dynamics problem (this approach will constitute the basis for the models provided in Chapter 4);
- provide a classification for the agents via their topological interaction, in particular with reference to the the algorithm proposed in Chapter 2, which will allow to assess the role agents play within the stabilization of the opinion pattern (if it occurs);
- provide further insights of this problem by means of computer simulations.

3.2 The mathematical background

As mentioned in the Introduction to this Chapter, probably the most well – known approach to opinion dynamics is to be found in a series of papers (De Groot 1974, Friedkin & Johnsen, Friedkin 1991, Krause 2000, Hegselmann & Krause 2005, Lorenz 2006) where agents form their opinion upon a variable and, via a discrete – time dynamical model, they *revise* their opinions by *averaging* the opinions of other agents. The models we will use in this work deal with an averaging structure, i.e. a linear *convex* combination where the coefficients sum up to 1.

The mathematical concepts behind the results reported in this Chapter lie in the theory of non – negative matrices (Gantmacher, 1959; Seneta, 1981), Perron – Frobenius theory (Frobenius 1908, 1909, 1912; Horn & Johnson, 1990), Markov chains theory (Isaacson and Madsen 1976).

3.2.1 The homogeneous opinion dynamics process

Consider a model with n agents, and an *opinion vector* $\mathbf{x}(t) \in \mathbb{R}^n$ with $t \in [0, \ldots, +\infty)$ representing the time index. The element $x_i(t)$ represents the opinion of agent i at time t. The n agents can be seen as vertices on a network, discussing a particular issue or trying to obtain better guesses upon the value of a variable at each time step.

The general *homogeneous* consensus dynamical model can be written as (DeGroot, 1974):

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) \tag{3.1}$$

Where $\mathbf{A}$ is a row – stochastic matrix, i.e. $\sum_j a_{ij} = 1$, $\forall i$. We will refer to this matrix as the *influence* matrix[2]. Since the matrix $\mathbf{A}$ does not change over time, i.e. the influence weights are fixed, the process is said to be *homogeneus*, and it is easy to show that:

$$\mathbf{x}(t+1) = \mathbf{A}^{t+1}\mathbf{x}(0) \tag{3.2}$$

where $\mathbf{A}^{t+1}$ is the $(t+1)$–th power of matrix $\mathbf{A}$.

Within this framework, some questions naturally arise. Will there be, for $t \to \infty$, an asymptotic opinion vector? Should it exist, will such vector represent a common opinion, i.e. consensus, or will there be fragmentation in the opinion patterns? In addition: will the opinion pattern depend on the initial configuration?

The answer to these questions depends on the topology of the graph underlying the influence matrix. Moreover, to find the answer, we need to analyze only the powers of the influence matrix, in the sense that we want to understand whether the configuration arising for $t \to \infty$ is possibly stable or not.

We say that the dynamical model described in Eq. 3.2 reaches a *consensus* if $\forall \mathbf{x}(0) \in \mathbb{R}^n$, $\exists c$ s.t. $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{A}^{\infty}\mathbf{x}(0) = \mathbf{c}$, where $\mathbf{c}$ is a real – valued vector with all elements equal.

Theorem 3 (Conditions for convergence to consensus)**.** *A* consensus *for system 3.2 is achieved:*

- *if* $\forall (i,j) \in E$, $\exists k \in V$ *such that* $a_{ik} > 0$ *and* $a_{jk} > 0$ *[see DeGroot (1974) for the proof];*
- *if and only if* $\exists\ t_0 \in T$ *s.t. the matrix* $\mathbf{A}^{t_0}$ *contains* at least *one strictly positive column [see Berger (1981) for the proof].*

[2]We will also make use, according to the specific context, of the expressions *confidence* and *attention* matrix.

Theorem 4 (Conditions for convergence)**.** *Let* $\mathbf{A}$ *be in its Gantmacher form with diagonal Gantmacher blocks* $\mathbf{A}_k$*,* $k = 1, \ldots, s$ *and* $g = 0, \ldots s$*, then*

- $\lim_{t \to \infty} \mathbf{x}(t)$ *exists* $\forall \mathbf{x}(0) \in \mathbb{R}^n$ if and only if *the Gantmacher blocks are all primitive;*

- *the system reaches a consensus* if and only if $g = 1$ *(i.e. there exists only one essential class and no inessential classes).*

See Gantmacher (1959) for a proof of both parts of the theorem.

Notice that a stable opinion pattern (including a possible consensus) will be reached no earlier than the time when all the blocks are simultaneously strictly positive, i.e. at least at the maximum of the primitivity indexes.

This first general theorem states that a sufficient condition for consensus is found when any two agents have a positive weight on a same third agent. The value c of the consensus depends, clearly, on the initial opinion profile $\mathbf{x}(0)$. In order to understand this condition, one needs to recall the definition of a primitive matrix, i.e. a matrix for which there exists a strictly positive power. This represents a sufficient condition.

The second theorem employs the so – called Gantmacher form. Intuitively speaking, as explained in the previous Chapter, we can split agents into essential (i.e. agents that have positive weights only on agents belonging to the same group) and inessential agents. If each of the sub – groups of essential agents has a primitive structure, then a stable opinion configuration is achieved where only the opinions of the essential agents matter in determining the asymptotic opinion vector. More over, we have a necessary and sufficient condition for consensus when the sub – group of essential agents is only one ($g = 1$, i.e. only one essential class).

We define a *consensus matrix* $\tilde{\mathbf{A}}$ the rank – one strictly positive row – stochastic matrix whose rows are all equal. It is immediate to see that $\tilde{\mathbf{A}}\mathbf{y}$ always gives a consensus.

Homegeneous opinion dynamics and eigencentrality

An interesting interpretation of the DeGroot consensus problem can be found in its relationship with the eigenvector centrality. This part is based on Berger (1981) and

Golub & Jackson (2010).

Consider a row – stochastic, aperiodic and irreducible $n \times n$ matrix $\mathbf{A}$ (i.e. with $g = 1$ essential class), and an initial nonnegative opinion vector $\mathbf{x}(0)$. In this case, for $t \to \infty$, a consensus $\mathbf{c}$ is achieved (i.e. $c_i = c$, $\forall i \in V$). We now want to give a quantitative assesment of the level of influence that each agent has on the the consensus value.

Theorem 5. *Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an irreducible row – stochastic matrix, and let $\mathbf{x}(0) \in \mathbb{R}^n$ be any initial opinion profile, then there exists a unique lef – eigenvector $\mathbf{v} \in \mathbb{R}^n$ of $\mathbf{A}$ with associated eigenvalue 1 and $\sum_{i=1}^n v_i = 1$ such that:*

$$\lim_{t \to \infty} \mathbf{A}^t \mathbf{x}(0) = \mathbf{v}^\top \mathbf{x}(0) = \mathbf{c}$$

Proof. We need to find a nonnegative vector, with entries normalised in order to sum up to one such that for any initial opinion profile $\mathbf{x}(0)$ it must hold that:

$$\mathbf{c} = \lim_{t \to \infty} \mathbf{A}^t \mathbf{x}(0) = (\mathbf{v}^\top \mathbf{x}(0))\ \mathbf{u} \tag{3.3}$$

where $\mathbf{u}$ is the n–unit vector. But since:

$$\lim_{t \to \infty} \mathbf{A}^t \mathbf{x}(0) = \lim_{t \to \infty} \mathbf{A}^t (\mathbf{A}\mathbf{x}(0))$$

then it must hold:

$$\mathbf{v}^\top \mathbf{x}(0) = \mathbf{v}^\top \mathbf{A}\mathbf{x}(0)$$

implying that $\mathbf{v}^\top = \mathbf{v}^\top \mathbf{A}$, i.e. $\mathbf{v}$ is a left eigenvector associated with the eigenvalue 1 and, via the Perron – Frobenius theorem, has strictly positive entries.

□

We refer to v_i as the *influence weight* of agent i and it is a measure of social influence in reaching the consensus value, in the sense that the initial (and subsequent) opinion of agent i are weighted more or less in achieving the final consensus vector $\mathbf{c}$. In terms of a Markov chains, the vector $\mathbf{v}$ represents the asymptotic probability distribution for the transition matrix $\mathbf{A}^\top$.

3.2.2 Inhomogeneous opinion dynamics process

A more general approach is obtained when $\mathbf{A}$ is time dependent:

$$\mathbf{x}(t) = \mathbf{A}(t) \cdots \mathbf{A}(0)\mathbf{x}(0) \tag{3.4}$$

We refer to the process described by Eq. 3.4 as either *inhomogenous* (using the Markov chains expression) or *time – variant* (as opposed to the *time – invariant* process described in Eq. 3.2. Quite clearly, the "inhomogenous" or "time – variant" part of the model stems from changes in the agents' interaction topology and not directly from the opinion patterns in time. However, we will see that, in the case when the weight (and hence the interaction topology) depends on the opinions themselves, the feedback relationships between the opinions and the network topology will show non – trivial behaviours.

The definitions we previously gave in the homogeneous case naturally extend to the inhomogeneous case. The $n \times n$ non – negative matrix $\mathbf{A}(t)$ in our model is said to be an *influence* or *confidence* matrix if and only if:

1. it is row – stochastic, i.e. $\sum_j a_{ij}(t) = 1$, $\forall t$;
2. its diagonal is strictly positive, i.e. $a_{ii}(t) > 0$, $\forall i \in V$ and $\forall t$;

The objective of our analysis will be thus the sequence of matrices $(\mathbf{A}_0, \dots \mathbf{A}_t)$ and, in particular, their right – product $\mathbf{A}(t) \dots \mathbf{A}(0)$[3] and its asymptotic behaviour for $t \to \infty$.

The process described by Eq. 3.4 is clearly more difficult to treat than the homogeneous case and analytical results are harder to obtain. As usual, we are interested in determining sufficient and necessary conditions for convergence of the system. The structure of the sequence of matrices on $\mathbf{A}(t)$ will determine the asymptotic behaviour of the system. We will observe that, analogously to what happens in the homogenous case, consensus or, more generally, convergence to consensus in the sub – groups will be

[3] As opposed to the Markov chain process, in which case, we are interested in the left product of the elements of the sequence: $\mathbf{A}(0) \dots \mathbf{A}(t)$.

obtained as long as the weights keep being positive to a certain degree and/or sufficient time passes by.

Several interesting results have been proposed in the literature. Among these, we recall Moreau (2005), who describes a model of dynamic network interaction based on synchronization theory, providing results based on both graph theory and system theory; Fazeli and Ali Jadbabaie (2010), who use a Polya urn argument to show certain asymptotic properties and Fazeli & Jadbabaie (2012), who propose some analytical results on convergence where the underlying network process is a martingale.

Motivation for the averaging scheme

There are several reasons why a convex averaging scheme is useful in the opinion dynamics setting (see Krause (2000)); however, we will hereby illustrate only those connected to the simple property of a weighted mean. *Although quite simple in their mathematical formulation*, such properties are key to understand the rationale behind the model. Denote with $M(\mathbf{x})$ the (linear) operator "weighted average" over a vector of values $\mathbf{x} \in \mathbb{R}^n$, with weights that sum up to one, assume $c, k \in \mathbb{R}$. Then some properties hold true. Some of these properties, that hold in a simple linear setting, are of use in the context of the non − linear bounded confidence model described in the next section.

1. *Fixed point property* $M(c\mathbf{u}) = c$, with $c \in R$. In our setting, it means that, once a global consensus is reached, any further averaging will not change the consensus value. Moreover, in case of fragmentation, with no further change in the influence matrix, the opinion pattern will not change either..

2. *Invariance to scale transformations* $M(k\mathbf{x}) = kM(\mathbf{x})$. It implies that any opinion vector can be appropriately scaled by a scalar k. A possible application of this property is to understand the opinion magnitude change in the non − linear bounded confidence model (see the next subsection).

3. *Monotonic property* $\mathbf{x} \leq \mathbf{y} \Rightarrow M(\mathbf{x}) \leq M(\mathbf{y})$, which means that the operator is non − decreasing monotone with respect the consensus subgroups.

4. *Boundedness* $\min(\mathbf{x}) \leq M(\mathbf{x}) \leq \max(\mathbf{x})$. It is probably the most important one: it means that the operator M always lies between the extrema of the vector. In

our setting, this implies that *any* opinion profile at *any* t will always be bounded between the maximum and minimum value at time $t = 0$. Moreover, since the opinions $\mathbf{x}(t)$ is weighted average of the previous opinions $\mathbf{x}(t-1)$, because of the boundedness property, the maximum and the minimum at each t will be also bounded by the (previous) maximum and minimum at $t-1$. Hence, even in the non – linear bounded confidence model, we can always find well defined bounds for the extrema of the opinion distribution.

Using a simple arithmetic average scheme, however, is not the only way to introduce an averaging scheme: in fact, other types of means could be adopted in this framework, such as the geometric, harmonic or power mean (see Krause's 2008 presentation for a short summary).

Relationship with Markov chains The sequence of row – stochastic confidence matrices, multiplied from the right by an opinion vector, can be seen as a sequence of *transition* matrices in a Markov chain (when a vector of probabilities is multiplied from the left), where the element $a_{ij}(t)$ represents the probability of transition from state i to j at time t. Thus, there are two main processes associated to a sequence of stochastic matrices: an (inhomogeneous) opinion dynamics process and a (inhomogenous) Markov chain. Lorenz (2007) shows how the two processes are related in terms of matrix algebra. In particular, the author assumes a *positive diagonal*, which leads to an acyclic matrix (see Chapter 2), meaning that each agent i in the opinion dynamics process has always strictly positive self – confidence or that, in the Markov process, there is always a strictly positive probability to stay in the own state.

3.2.3 Speed of convergence

Once sufficient and necessary conditions for convergence have been established, the subsequent step in both the time – invariant and time – variant model is to find the number of time steps required to achieve a stable opinion configuration. This is a classical problem in Markov chain theory and will be only sketched in this work. For more references, see Isaacson & Madsen (1976).

A Markov chain is (in very informal terms) *ergodic* if the matrix product converges to a rank one stochastic matrix. A consensus matrix is hence a special case where all elements are strictly positive. *Ergodicity*, as a very broad concept, refers to the behaviour of the product of row – stochastic matrices. If convergence occurs, different *ergodicity coefficients* have been proposed as an estimate of time steps required to reach it.

In the homogenous case, this problem reduces to analyse the powers of the matrix $\mathbf{A}$. Consider the eigenvalues of $\mathbf{A}$: the Perron – Frobenius theorem ensures (see Chapter 2) that the first (dominant) eigenvalue is real (and moreover, equal to 1). As such, we can order the eigenvalues in decreasing order of magnitude: $1 = \lambda_1(\mathbf{A}) \geq |\lambda_2(\mathbf{A})| \geq \ldots \geq |\lambda_n(\mathbf{A})|$. If $|\lambda_2(\mathbf{A})| < 1$ then, for $t \to \infty$, it holds that $|\lambda_2(\mathbf{A}^t)| = |\lambda_2(\mathbf{A})|^t \to 0$. This implies that the power $\mathbf{A}^t$ converge to a rank – one matrix and $|\lambda_2(\mathbf{A})|$ can be used as an estimate of the speed of convergence (see Gross & Rothblum, 1993; Rothblum & Tan 1995).

In the inhomogeneous (time – variant) model, understanding whether the product $\mathbf{A}(t) \cdots \mathbf{A}(0)$ converges to a rank – one matrix is non – trivial and clearly the magnitude of second eigenvalue cannot be used since, in general, $\lambda_2[\mathbf{A}(t) \cdots \mathbf{A}(0)] \neq \lambda_2(\mathbf{A}(t)) \cdot \lambda_2(\mathbf{A}(0))$, i.e. the second eigenvalue has no *multiplicative* property with respect to the matrix product.

Several attempts to identify appropriate ergodicity coefficients have been proposed in the literature (see, e.g., Seneta 1984). The discussion of this problem falls outside the aims of this work. See also Ipsen & Selee (2011) for a vector – norm based approach to the coefficient of ergodicity that also explores concepts in network centrality.

Hence, for simplicity, we will focus on a particular ergodicity coefficient (Dobrushin, 1956), defined as follows:

$$\tau(A) := 1 - \min_{i,j \in V} \sum_{k=1}^{n} \min\{a_{ik}, a_{jk}\} \tag{3.5}$$

It is possible to prove that (see Isaacson & Madsen (1976), pag. 145, Lemma V.2.3):

$$\tau(\mathbf{A}(t) \cdots \mathbf{A}(0)) \leq \tau(\mathbf{A}(t)) \cdots \tau(\mathbf{A}(0))$$

i.e. that the coefficient $\tau(\mathbf{A})$ is *submultiplicative*.

A benchmark case: diagonal dominance We will now illustrate a simple lemma in order to clarify the concept of coefficient of ergodicity and the speed of convergence to a limiting opinion vector by analysising a benchmark case: diagonal dominance (i.e. the elements $a_{ii} \geq \sum_{j \neq i} a_{ij}$, $\forall i \in V$), when all the rest of the weights are uniformly assigned to the remaining $n-1$ vertexes.

We decompose the matrix A as the sum of two matrices: the diagonal matrix D, such that $d_{ii} = a_{ii}$, $\forall i \in V$ and the off–diagonal matrix B, such that $b_{ij} = a_{ij}$, for $i \neq j$ and 0 otherwise. We also assume that for each agent the attention paid to other agents is uniformly distributed, i.e. $b_{ij} = \frac{1-d_{ii}}{n-1}$.

Theorem 6. *For a diagonal dominant matrix A with uniformly distributed attention to other agents, the ergodicity coefficient is given by:*

$$\tau(A) = 1 - \frac{n}{n-1}\left(1 - \max_{i,j \in V} a_{ij}\right) \tag{3.6}$$

Proof.

$$\begin{aligned}
\tau(A) &= 1 - \min_{i,j \in V} \sum_{k=1}^{n} \min(a_{ij}, a_{jk}) = \\
&= 1 - \min_{i,j \in V} \sum_{k=1}^{n} \min(b_{ij}, b_{jk}) \\
&= 1 - \min_{i,j \in V} \sum_{k=1}^{n} \min\left(\frac{1-d_{ii}}{n-1}, \frac{1-d_{jj}}{n-1}\right) = \\
&= 1 - \min_{i,j \in V} \sum_{k=1}^{n} \frac{1}{n-1} \min\left(1-d_{ii}, 1-d_{jj}\right) = \\
&= 1 - \min_{i,j \in V} \frac{n}{n-1}\left(1 - \max(1-d_{ii}, 1-d_{jj})\right) = \\
&= 1 - \frac{n}{n-1}\left(1 - d^{MAX}\right) = \\
&= 1 - \frac{n}{n-1}\left(1 - \max_{ij, \in V}(a_{ij})\right) \qquad (3.7)
\end{aligned}$$

□

Corollary: given the assumptions above: $\lim_{n \to \infty} \tau(A) = \max(a_{ij})$

3.2.4 Opinion dynamics under bounded confidence

The inhomogenous case is often referred to as a *time – variant* model, in the sense that the weights each agent puts onto other agents changes over time. However, a rule for the evolution of the network topology must be established in order to analyze the system dynamics.

The main question in this case is whether the update rule will influence the reaching of a stable opinion pattern or fragmentation (either consensus within subgroups or dissent across different subgroups) will occur or even no convergence at all will be achieved.

Within this framework, "the most difficult type of model occurs if the weights depends on opinions itself because then the model turns from a linear one to a *non–linear* one" (cit. Hegselmann & Krause, 2005).

In particular, we hereby examine a system with *bounded confidence*, in which agents update their network interaction structure by putting weight on agents that have a similar opinion. The system becomes *non – linear* in the sense that the averaging process can automatically include or discard certain agents, hence underweighting certain opinions or overweighting other opinions if they are similar.

We define a *bounded confidence* opinion dynamics process, as the system with initial opinion vector $\mathbf{x}(0)$ and dynamics:

$$\mathbf{x}(t+1) = \mathbf{A}(t+1)\mathbf{x}(t) = \mathbf{A}(x(t), \epsilon)\mathbf{x}(t)$$

$$a_{ij}(t+1) := \begin{cases} \frac{1}{\#I(i,\mathbf{x}(t))} & \text{if } j \in I(i, \mathbf{x}(t)) \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

$$I(i, \mathbf{x}(t)) = \{j \text{ s.t. } |x_i(t) - x_j(t)| \leq \epsilon_i\}$$

This structure implies that, at each time t, every agent compares her opinion with that of others and finds a subset of agents whose opinion do not differ too much from theirs and puts *equal* weight on these agents. The assumption of equal weights can be relaxed, but it will not be analyzed in this work. It is also important to note that, in the set I lies agent i herself, thus leading to have a positive diagonal at each time t.

The non – linearity in the model is due to the fact that the update rule enhances the selection of similar opinions and implies discarding distant opinions. This approach naturally leads to *fragmentation* and *polarisation*, meant as a *hardening* of opinion patterns. Hegselmann & Krause (2005) describe, by simple examples, the possibility of splits between subgroups that become essential and that reach consensus within the agents belonging to the subgroup.

3.3 Convergence in the inhomogenous case

Finding sufficient and necessary conditions for convergence of an opinion dynamics scheme has been discussed throuroghly in the literature in the time – invariant case (e.g. DeGroot 1974, Berger 1981) in the light of homogeneous Markov chain and some results have been illustrated in the previous sections.

More recently, Moreau (2005), Hegselmann and Krause (2005), Lorenz (2007), Mirtabadei and Bullo (2011) have provided convergence results for the inhomogeneous case. In the previous section, we have reported some convergence results for the homogenous process.

The basic intution behind the convergence in the inhomogeneous case is that a time – variant model will converge provided that, as previously noted, both the weights keep beeing sufficiently positive and/or that enough time passes. The positive diagonal assumption will also play a significant role in ensuring this. Clearly, should the weights go to zero too fast, convergence (let alone consensus) might not be achieved (see Cohen 1986 for an example).

Krause (2000) proposes an interesting theorem for convergence (not limited to arithmetic means) and later, Hegselmann & Krause provide more general approach based on the arithmetic average setting based on the *accumulated weights*. The intuition behind this approach is to keep track of the weights of the chain of products.

Another way to see this problem is to exploit the positive diagonal. In fact, given a non – negative matrix $\mathbf{B}$ with positive diagonal and a non – negative matrix $\mathbf{C}$, then the product $\mathbf{BC}$ has at least the same positive entries of $\mathbf{C}$. Based on the above reported results, Lorenz (2005 and 2007) has proposed an convergence theorem based on the Gantmacher form.

Theorem 7. *Given* $\mathbf{A}(t)$*,* $t = 0, 1, 2, \ldots,$ *non – negative, row – stochastic with strictly positive diagonal, and* $\min^{+} \mathbf{A}(t+1)\mathbf{A}(t) \geq \delta_t$ *and* $\sum_{t=1}^{\infty} \delta_t = \infty$ *Gantmacher blocks converge to:*

$$\lim_{t\to\infty} (\mathbf{A}(t) \cdots \mathbf{A}(0)) = \left[\begin{array}{ccc|c} \mathbf{G}_1 & & & \mathbf{0} \\ & \ddots & & \\ \mathbf{0} & & \mathbf{G}_g & \mathbf{0} \\ \hline n.c & \ldots & n.c. & \mathbf{0} \end{array}\right]$$

where g *is the number of essential classes,* n_h *is the cardinality of essential class* h *and* $\mathbf{G}_h$ *are* $n_h \times n_h$ *row – stochastic consensus matrices, i.e. strictly positive stochastic matrices with all rows equal:*

$$\mathbf{G}_h = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \\ \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & \ldots & c_n \end{bmatrix} = \left[\begin{array}{c|c|c|c} \mathbf{c}_1 & \mathbf{c}_2 & \ldots & \mathbf{c}_n \end{array}\right]$$

3.4 Simulations

3.4.1 Bounded confidence model

The models we have analyzed in the previous parts of this Chapter are multi – faceted and computer simulations can give a better insight in understanding both the potential and the analytical characteristics. Since the homogeneous case is of little interest in this work, and it has been reported for completeness, we refer to the non – linear bounded confidence inhomogenous model (Eq. 3.8) in the following simulations.

All simulations are for $n = 100$ agents. The number of simulations, unless otherwise stated, is 1000. In the first part, the initial opinion vector $\mathbf{x}(0)$ is drawn from a uniform continous random distribution with support $[0, 1]$. Figures 3.1 and 3.2 report a simple one – run simulation with different values of ϵ (assumed to be equal for all agents). It is immediate to see that the higher the ϵ, the higher the chances to achieve consensus. However, polarized situations ($\epsilon = 0.15$) can be obtained, which show “hardening” in the opinions. When $\epsilon = 0.35$ a consensus is achieved in a very limited number of time steps.

Figure 3.3 shows a tri – dimensional plot of the opinion space with respect to different values of ϵ *once a stable configuration is achieved*, the y axis is the average relative frequence over 1000 simulations. One can see how the higher the ϵ, the less fragmented the opinion pattern is.

Figures 3.4, 3.5, 3.6, 3.7 report an analogous graph in time. The x axis refers to the opinion space, the z axis to the time steps and the y axis, again, to the average relative frequences over 1000 simulations for different values of ϵ. One can see how, with the increase of the confidence level, opinions tend to reach less fragmented, then polarized structures and, eventually, consesus. The plots show how both confidence levels and time contribute to the opinion patterns.

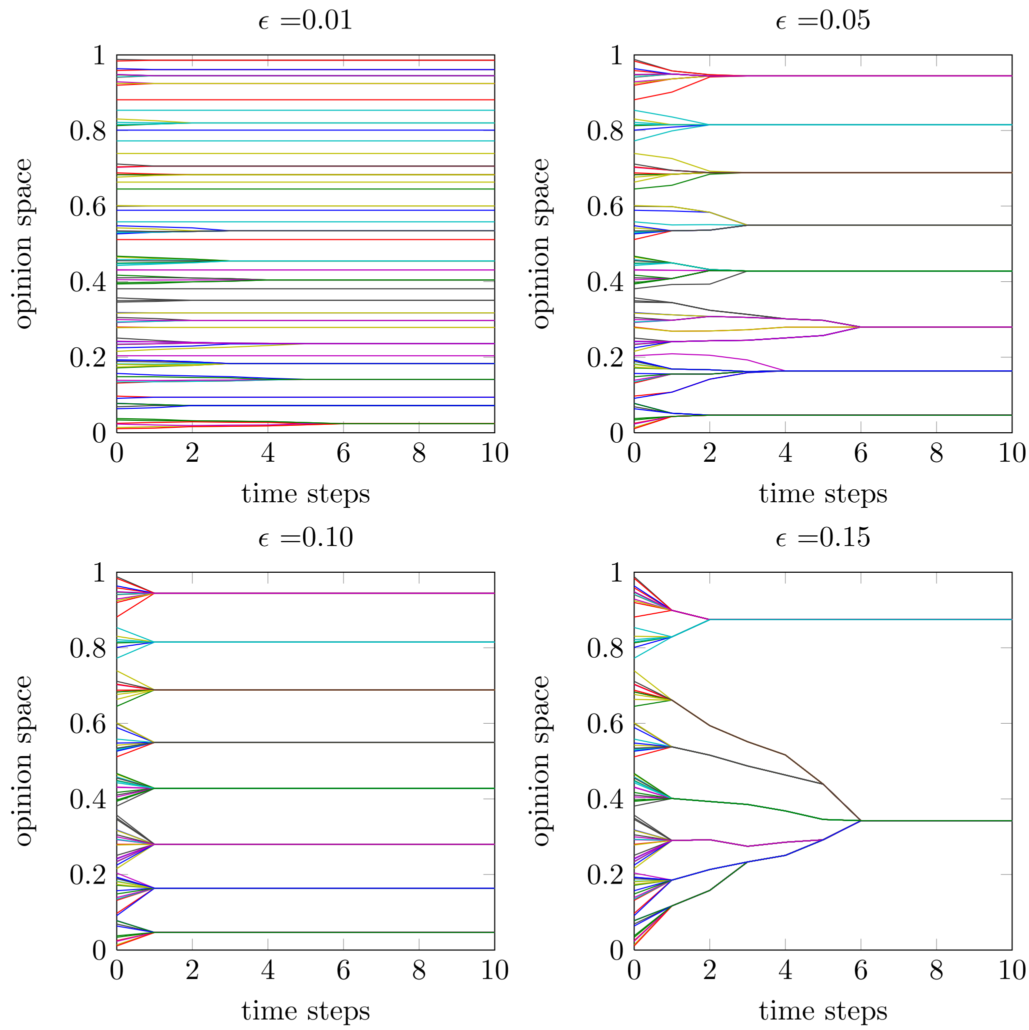


Figure 3.1: Single - run simulations

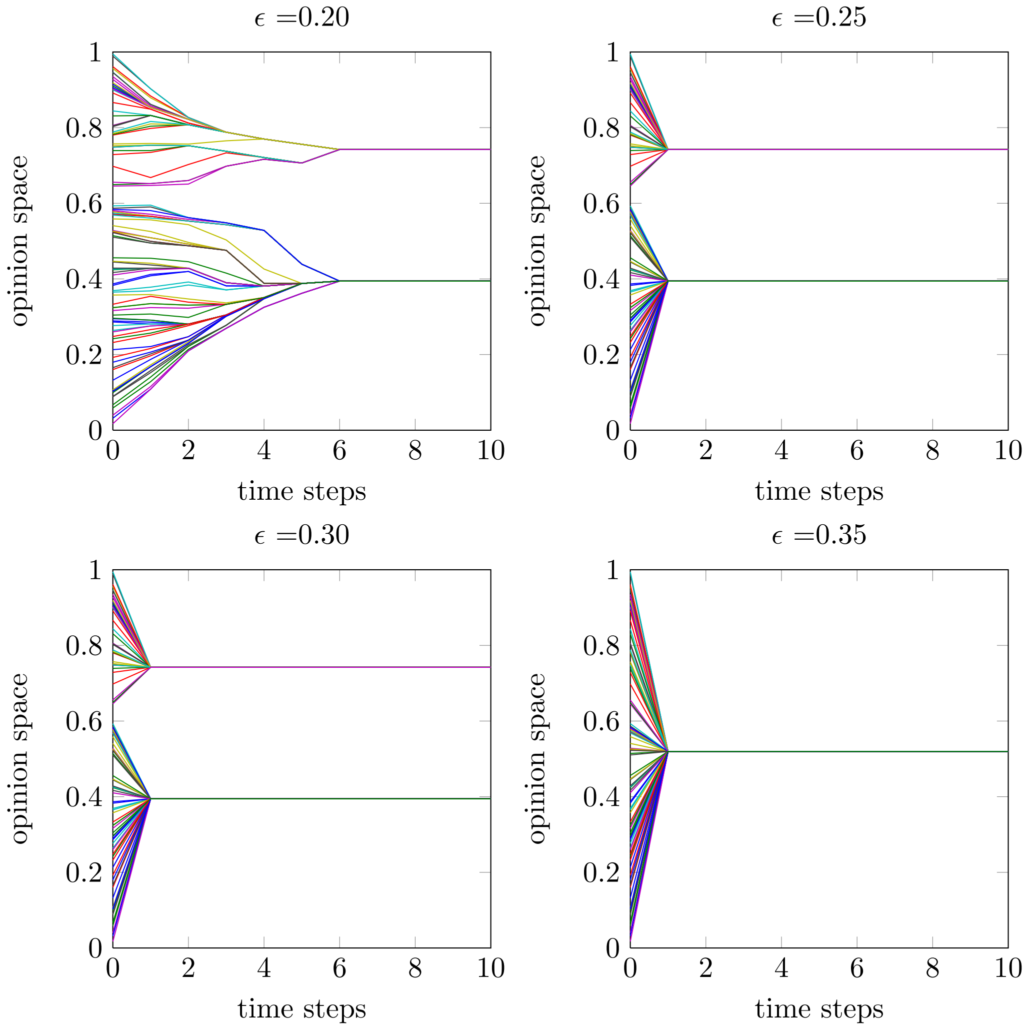


Figure 3.2: Single - run simulations

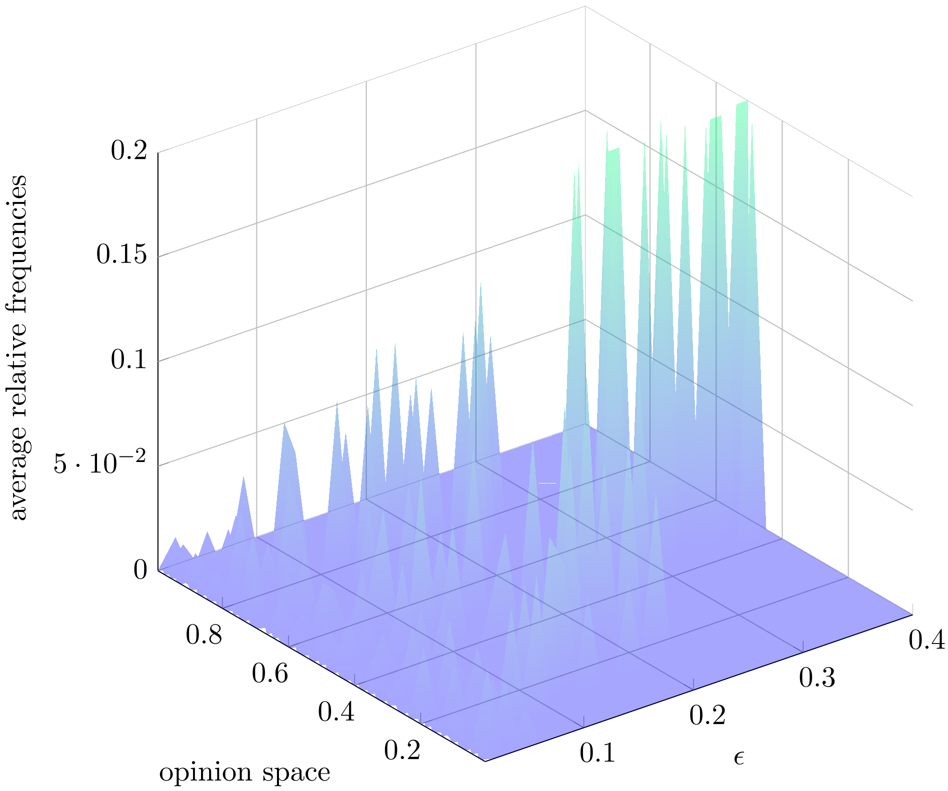


Figure 3.3: Average relative frequencies of $x \in [0, 1]$, $n = 100$ opinions (after stabilisation) for different values of ϵ (1000 simulations).

3.4.2 Distribution of the initial opinion profile

Since in the non – linear bounded confidence model the initial opinion profile can determine the degree of fragmentation of the opinion patterns, it is interesting to conclude this Chapter by modifying the distribution of the initial opinion profile $\mathbf{x}(0)$. Quite obviously, if the initial opinion profile is too dispersed (with respect to the confidence level ϵ), the initial fragmentation pattern will never converge to a consensus or to a less fragmented pattern.

While still keeping the opinion range $\mathbf{x}(0) \in [0, 1]$, we will use an initial opinion profile from random numbers drawn from a Beta distribution[4]. Figure 3.8 shows the distribution of the initial opinion profile for different values of the shape parameters a and b.

[4]The pdf of a Beta distribution is given by

$$f(x; a, b) = \frac{1}{B(a, b)} x^{a-1}(1 - x)^{b-1},$$

where $B(a, b)$ is the Beta function and a and b are the shape parameters of the distribution.

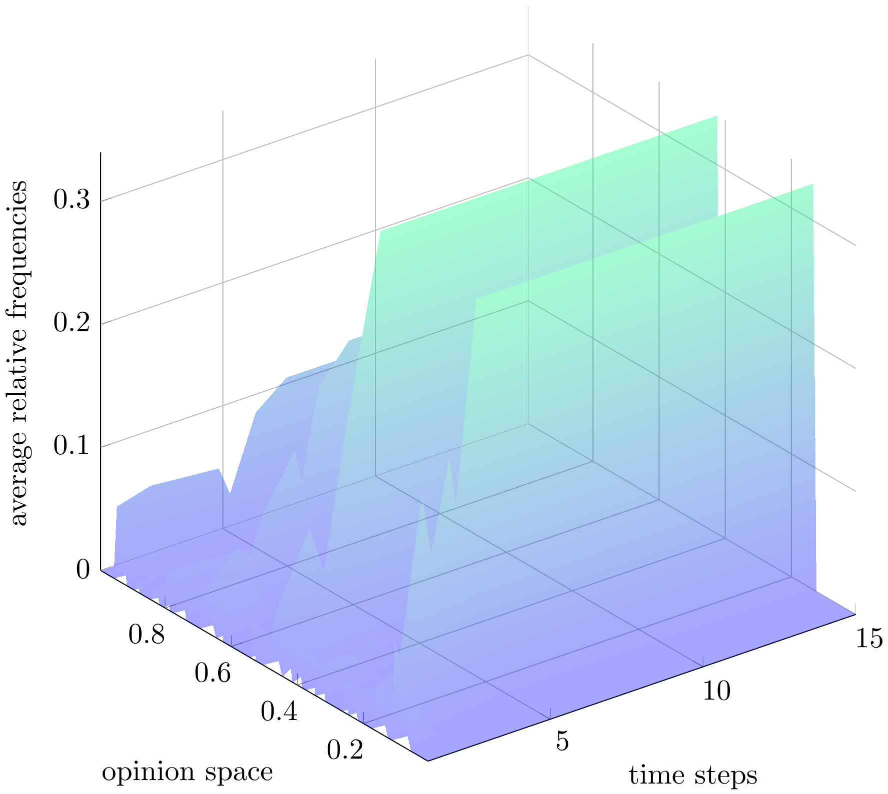


Figure 3.4: Dynamics (average relative frequencies) of $x \in [0, 1]$, $n = 100$ opinions with $\epsilon = 0.1$ (1000 simulations).

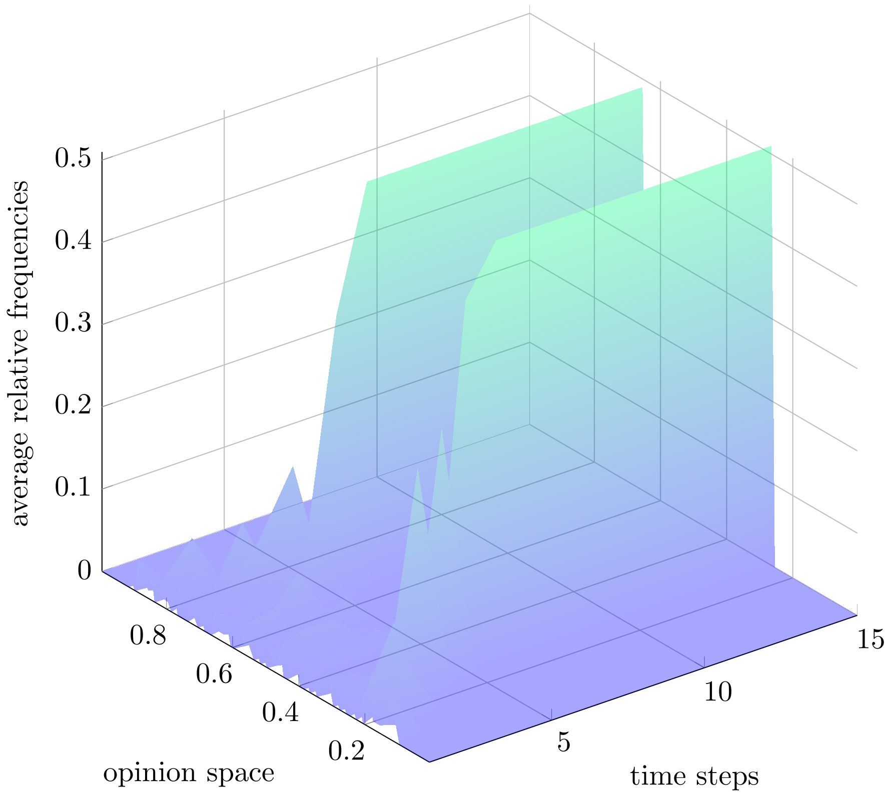


Figure 3.5: Dynamics (average relative frequencies) of $x \in [0, 1]$, $n = 100$ opinions with $\epsilon = 0.2$ (1000 simulations).

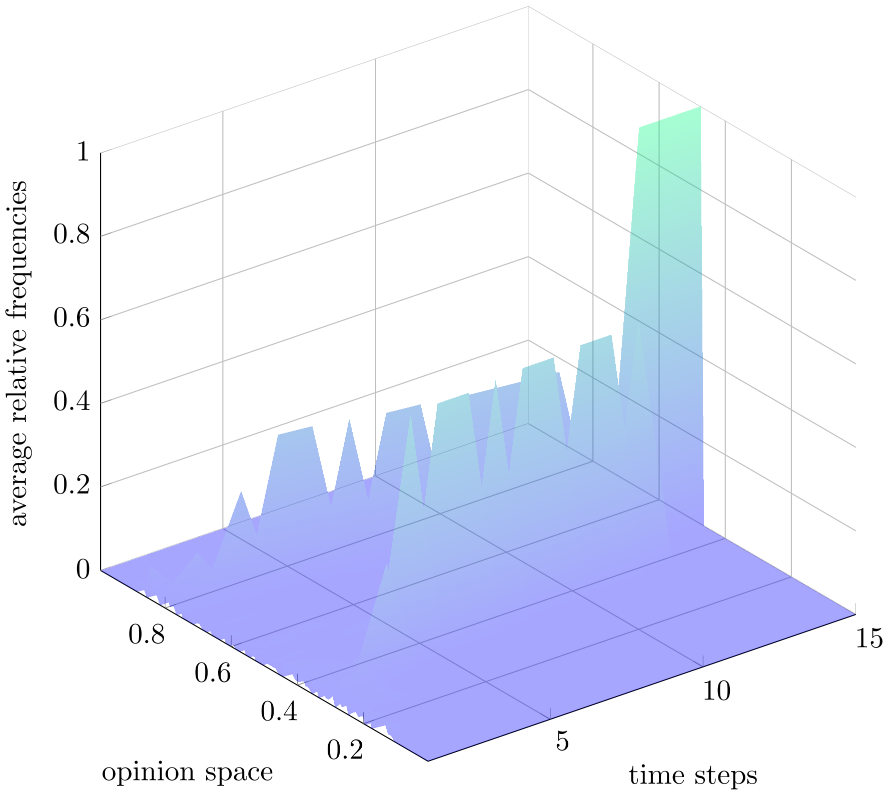


Figure 3.6: Dynamics (average relative frequencies) of $x \in [0, 1]$, $n = 100$ opinions with $\epsilon = 0.25$ (1000 simulations).

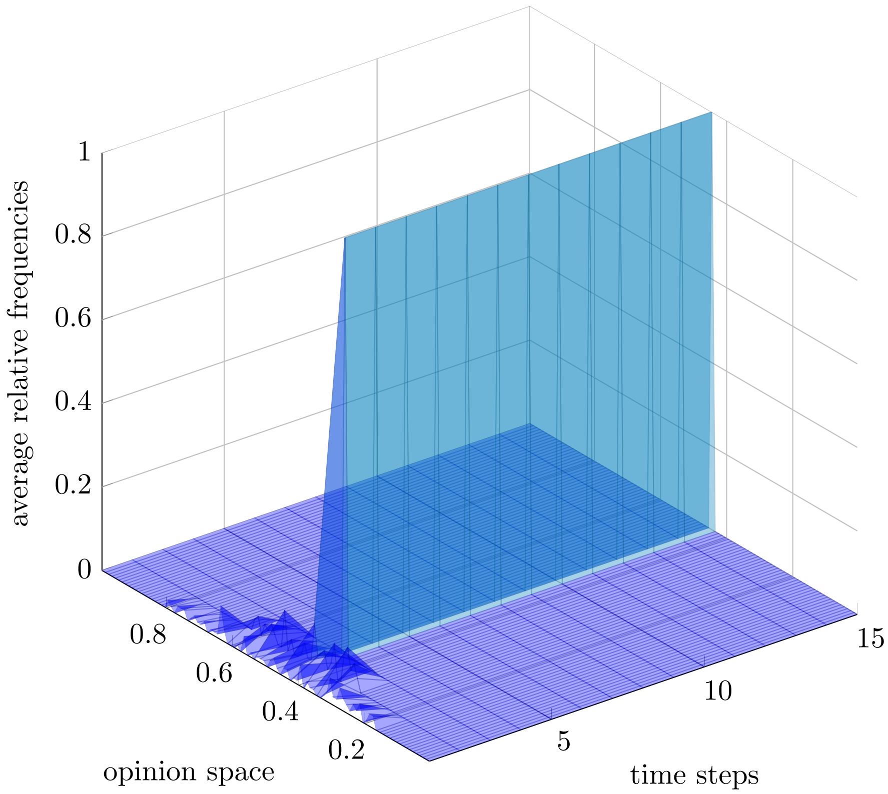


Figure 3.7: Dynamics (average relative frequencies) of $x \in [0, 1]$, $n = 100$ opinions with $\epsilon = 0.35$ (1000 simulations).

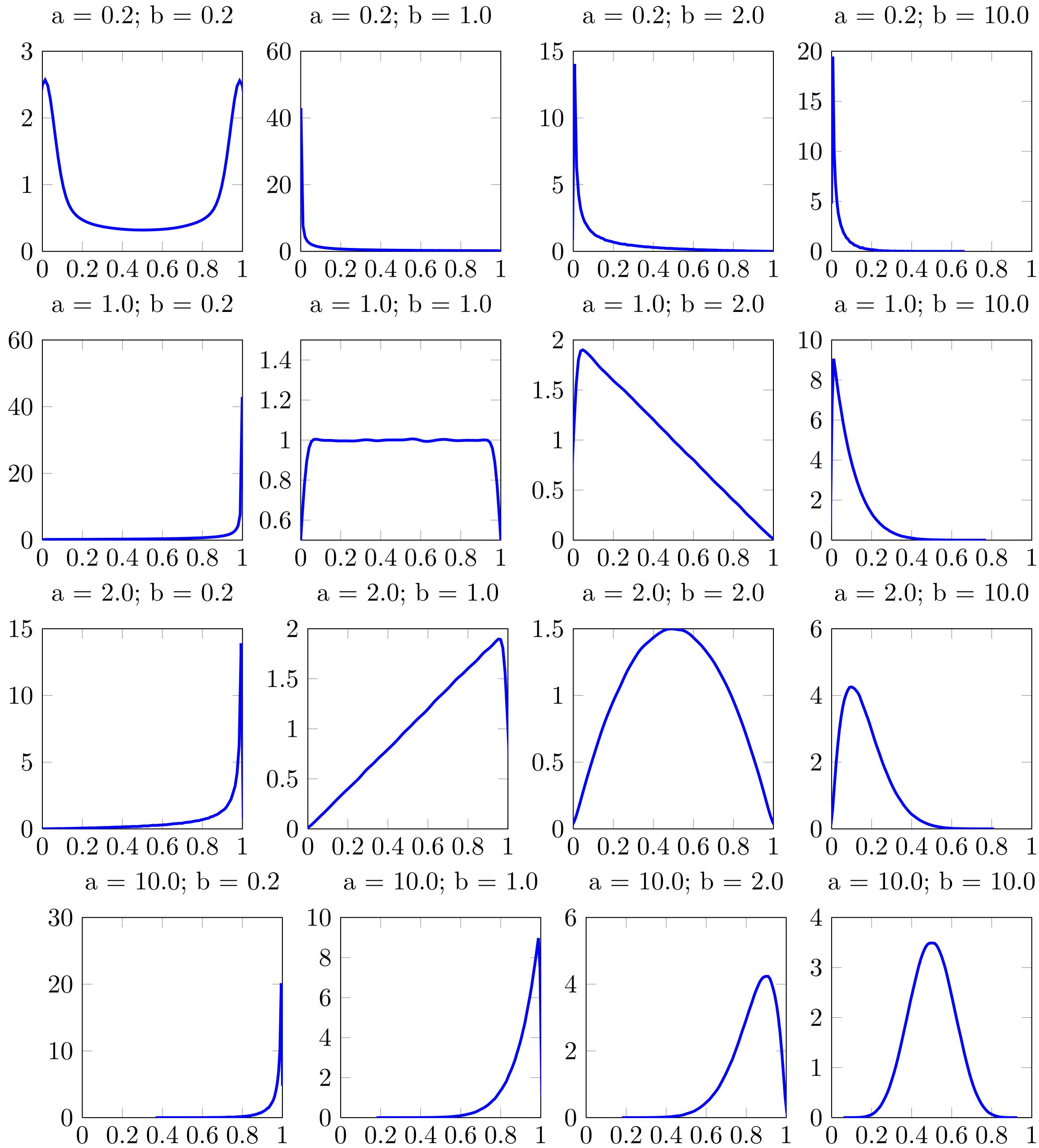


Figure 3.8: Distribution of the different initial profiles $\mathbf{x}(0)$ drawn from a Beta r.v. for different parameters a and b.

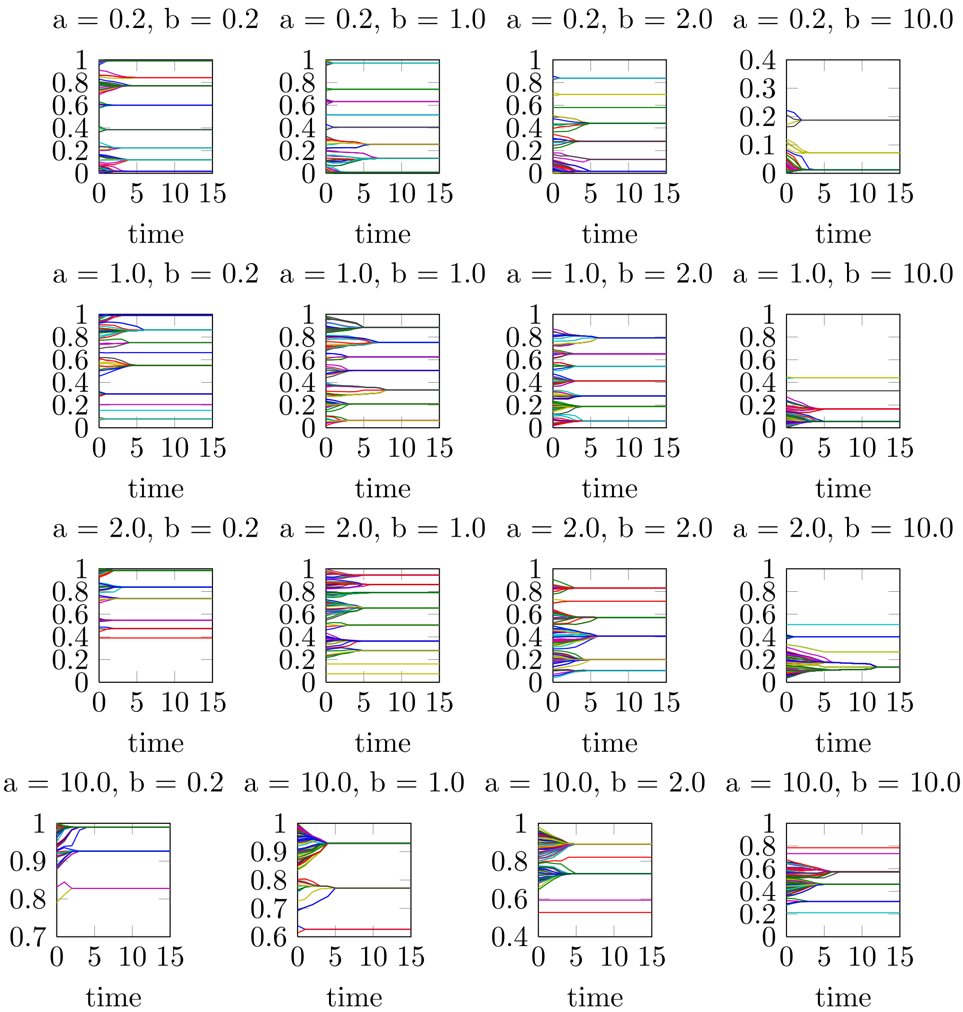


Figure 3.9: Opinion dynamics for $\epsilon = 0.05$ and different initial opinion profiles (see Fig. 3.8) drawn from a Beta distribution.

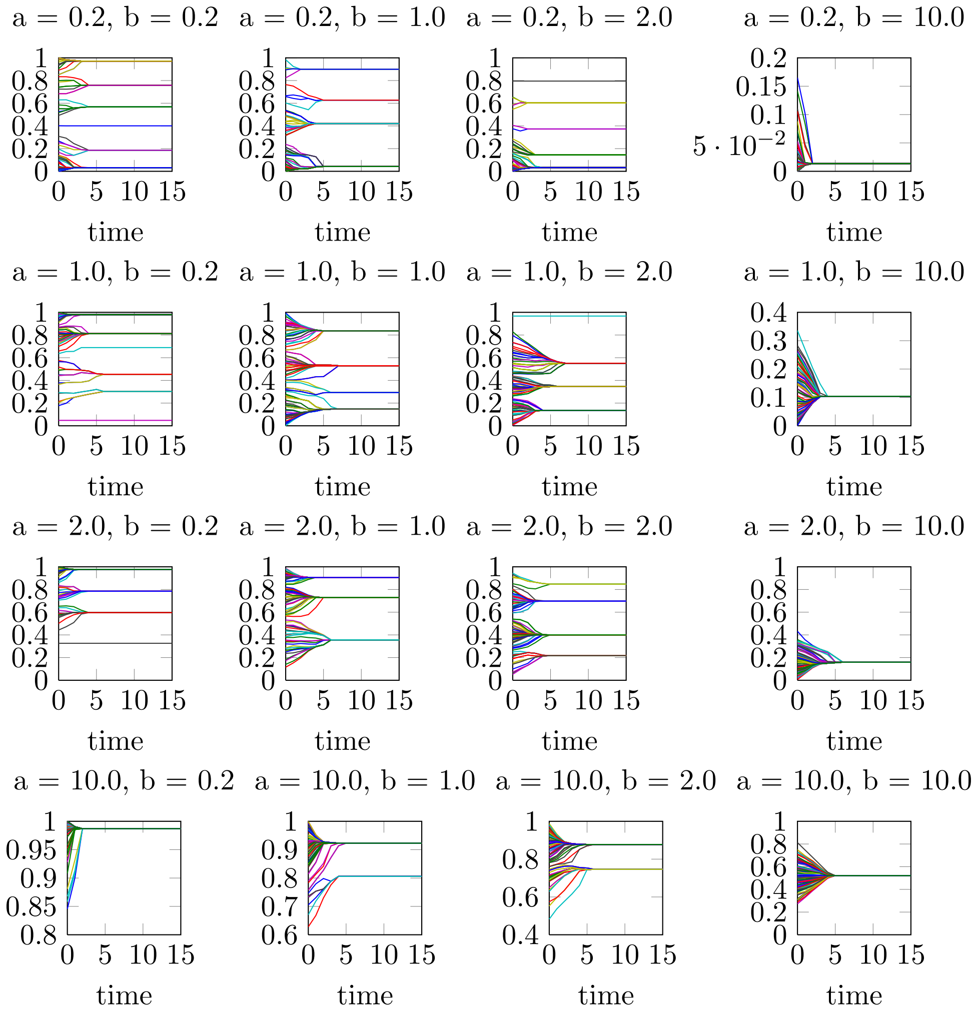


Figure 3.10: Opinion dynamics for $\epsilon = 0.2$ and different initial opinion profiles (see Fig. 3.8) drawn from a Beta distribution.

Chapter 4

Opinion dynamics and price formation

4.1 Introduction

The basic financial paradigm is built upon the assumption of *rationality* of agents and markets that are perfectly efficient (see Fama 1970). In order to overcome this limitation, behavioural approaches to finance have been proposed in the course of the last decades, where agents are *boundedly rational* and the marketplace implies a certain level of interaction among agents. Moreover, the idea that investors' psychology plays a fundamental role in determining price behaviour has led to a new stream of literature called *behavioural finance*, where the behaviour of agents in a financial system depart from the rationality assumption. Agents can be boundedly rational, thus leading to different behavioral aspects (e.g. momentum trading, trend extrapolation, noise trading, overreaction, overshooting, contrarian strategies). Some useful references include DeBondt and Thaler (1985), Hong and Stein (1999, 2003), and review papers such as Hirschleifer (2001), and Barberis and Thaler (2003).

One of the main emergent streams in the mathematical finance literature is related to models where heterogenous agents interact and determine asset dynamics. Among the several contributions in this field, those of Brock and Hommes (1998), Lux (1998), Gaunersdorfer and Hommes (2007) and Chiarella *et al.* (2007) are of particular interest. The basic idea of *heterogeneous* agents finds its root in overcoming the typical limitations embedded in models with representative agents: as a matter of fact, in the real world, agents naturally have heterogeneous beliefs about asset dynamics.

These models can be thought as *Agent Based Models* and often feature an adaptive and evolutionary dynamics (adaptive beliefs) based on the *success* of previous strategies (either in simple terms of price forecast or in terms of accumulated profits) where the *interaction* in these models stems from the agents' capability of observing other agents' strategies and imitating the most successful ones, by switching strategy.

This class of models seems to be more apt at providing a plausible explanation for phenomena such as clustered volatility and so called fat tails in the returns distribution. Moreover, such models provide with an interesting way of dealing with the *evolutionary* nature of agents' behaviour.

The concept of *agents interaction* naturally leads to interpreting the change in the belief systems via a network approach and, in this work, we make use of the opinion dynamics process illustrated in Chapter 3 to do so. In this Chapter we will propose two models:

- an agent based model of price formation in an open economy with heterogeneous opinions on the price, where the opinion dynamics is of the kind described in Chapter 3;
- a model of relative price formation in a closed economy when the attention matrix represents a confidence structure in other agents due to limited attention.

In the first model the connection among agents is captured along two dimensions: i) a network structure and ii) a more classical interaction via prices. In the second model, the holding pattern will be key in determining how agents will pay attention to one another. The strategies are then determined *both* by the network topology *and* the success of a certain strategy.

4.2 Heterogenous beliefs on prices - the model

Consider a set V of n agents (and denote each agent with an index i), one risky asset and one risk – free asset (with interest rate r). The price dynamics is denoted by

$(p(t))_{t\in\mathbb{N}}$ and will be outlined below. Let $z_i(t)$ be the quantity (number of shares) of the risky asset purchased by agent i at time t. Finally, let $(y(t))_{t\in\mathbb{N}}$ be the dividend dynamics (which will be assumed to be i.i.d. throughout this chapter). The wealth dynamical process for each agent can be described by the following equation in vector form:

$$\mathbf{w}(t+1) = (1+r)\mathbf{w}(t) + (p(t+1) + y(t+1) - (1+r)p(t))\,\mathbf{z}(t) \tag{4.1}$$

In order to capture the belief/opinion dynamics in terms of the models described in the previous chapter, we will denote the *opinion of agent i about the price and the dividend innovation* with the following expected value:

$$x_i(t) = \mathbb{E}_i(p(t)) + \mathbb{E}_i(y(t)) = \mathbb{E}_i(p(t) + y(t))$$

An important remark is that the opinion at time t *is formed* with the information set available at time $t-1$. Again, it is important to underline that opinions are heterogenous for what regards the first moment of the distribution. Such an assumption can be extended to higher moments of the distribution but this case is not be explored here. We assume, in fact, that the *opinion about the variance* of wealth is constant and equal for all agents, as it has been seen in a number of contributions (see, for example, Hommes and Wagener 2005 for a review paper; Chiarella *et al.*. 2007). The *opinion/belief* about wealth of agent i can be then described as follows:

$$\begin{aligned}
\mathbb{E}_i(w_i(t+1)) &= (1+r)w_i(t) + x_i(t+1)z_i(t) - (1+r)p(t)z_i(t) \\
V_i(w_i(t+1)) &= \sigma^2 \qquad \forall i \in V, \forall t \in \mathbb{N}
\end{aligned}$$

We now need to determine the optimal demand of the risky asset for each agent i; assuming that agents are optimizing their wealth in mean/variance, we will conveniently use a utility function $u_i(w(t)) = -\exp(-a_i W(t))$, where a_i is the CARA (Constant Absolut Risk Aversion) coefficient. This approach is consisent with the literature we refer to (Hommes & Wagener, Chiarella *et al.*).

The maximisation problem can be then expressed as follows:

$$\max_{z_i(t)} \quad \left\{\mathbb{E}_i(w(t+1),t) - \tfrac{a}{2}\sigma^2\right\} \tag{4.2}$$

which leads to the following optimal demand for agent i at time t:

$$z_i(t) = \frac{1}{a\sigma^2}\left(x_i(t+1) - (1+r)p(t)\right) \tag{4.3}$$

Moreover, we assume that there exists a limited *outside supply per agent*, that will be denoted by z^s (with nz^s obviously being the overal outside supply). In this way, we have a convenient way to obtain the equilibrium price of the risky asset in equilibrium via the following equilibrium equation:

$$\frac{1}{n}\sum_{i=1}^{N}\frac{1}{a\sigma^2}\left(x_i(t+1) - (1+r)p(t)\right) = z^s \tag{4.4}$$

which finally leads to the following *pricing equation*:

$$(1+r)p(t) = \underbrace{\frac{1}{n}\sum_{i=1}^{N} x_i(t+1)}_{\text{average opinion}} - \underbrace{a\sigma^2 z^s}_{\text{risk premium}} \tag{4.5}$$

As previously mentioned, we will assume that the dividend sequence $(y(t))_{t=1,2...}$ is i.i.d., with variance σ^2 and will be the only source of randomness in the model.

4.2.1 Network interaction for opinion dynamics

We now want to describe the opinion dynamics $\mathbf{x}(t)$ and the price dynamics $p(t)$ for the risky asset. Hommes and Wagener (2005) and, in more detail, Gaunersdorfer and Hommes (2007) assume an interaction based on the success of the trading strategy basing it on a *fitness function.*

Our contribution in this work lies in adopting a *different* interaction strategy, which is based on a precise update rule that modifies the underlying interaction topology among agents. Hence, we will assume a *pairwise interaction* (i.e, a network) with a dynamics of the kind described in the previous Chapter, i.e. we will consider a sequence of *influence matrices* $\mathbf{A}(t)_{t\in\mathbb{N}}$ of the type describe in Chapter 3.

If the number of essential classes is g, then there will be different *asymptotic* opinions for all agents belonging to a class, leading to a belief for each essential class.

The process that we will assume in this model involves a change in the influence topology at each t, according to the following rule.

$$a_{ij}(t) = (1-\alpha_i)\ a_{ij}(t-1) + \alpha_i\ c_{ij}(t) \tag{4.6}$$

$$c_{ij}(t) := \begin{cases} \frac{1}{\#I(i,\mathbf{x}(t-1))} & \text{if } j \in I(i,\mathbf{x}(t-1),p(t-1)) \\ 0 & \text{otherwise} \end{cases}$$

where $c_{ij}(t)$ represents the adaptive update of the influence matrix at time t ($\mathbf{C}(t)$ is row stochastic, see details below) and $\alpha \in [0,1]$ is an *update propensity* parameter. It is easy to show that since both $\mathbf{A}(t-1)$ and $\mathbf{C}(t)$ are row – stochastic and $\alpha \in [0,1]$, then also $\mathbf{A}(t)$ is row – stochastic.

The choice of the set $I(i,\mathbf{x}(t-1))$ for each i will lead to different types of models and will be now detailed.

Bounded confidence model

In the first case, we will follow an approach analogous to the one in Chapter 2 (bounded confidence, BC, Equation 3.8). In particular, we will consider an interaction topology for the adaptive update $\mathbf{C}(t)$, such that:

$$I(i,\mathbf{x}(t-1),p(t-1)) = \{j \text{ s.t. } |x_i(t-1) - x_j(t-1)| \le \epsilon_i\}$$

In this scenario, there is no need to look at *fundamentals* in order to understand the price dynamics. Prices will be driven only by the opinions and the opinions will be pooled according the bounded confidence non – linear model. The non – linearity in the model will lead to opinion polarization and fragmentation, thus the average in Eq. 4.5 can increase or decrease dramatically according to such level of polarisation. It is immediate to notice that $\mathbf{C}(t)$ is a rank – one matrix with positive diagonal.

Price adaptive strategy

The idea in this setting is that those agents whose opinion is the closest to the actual price at time t (price adaptive, PA) will be followed, in the next time step, by other agents . In an 'evolutionary' or 'adaptive' perspective, these agents are thus those capable of understanding *both* the impact of the opinions on price and the impact of the dividend innovation.

In such an adaptive setting, agents revise the influence matrix as in Eq. 4.6, but the set I is now given by:

$$I(i, \mathbf{x}(t-1), p(t-1)) = \{i\} \cup \{j \text{ s.t. } |1 - x_i(t-1)/p(t-1)| \leq \epsilon_i\} \tag{4.7}$$

In other words, in this case, $c_{ij}(t)$ represents an adaptive strategy where agents tend to follow more agents that had close guesses (opinions) to the actual realized price. $\mathbf{C}(t)$ is clearly a rank one matrix at each t. Since the price is driven both by the opinion dynamics and an exogenous divided innovation, this model reflects a situation where agents want to follow other agents that have proven to be successful in the past.

The reason why we insert the set $\{i\}$ in the set I in 4.7 is because we want to allow the matrix $\mathbf{A}(t)$ to have positive diagonal (also in the case $\alpha = 1$).

Fundamental benchmark

When all agents have identical long – term expectations (opinions) on the price, i.e. $x_i(t) = c, \forall i \in V$, we can assume $\mathbf{x}(t)$ to be the analogous of a consensus limit vector where all agents belong to the same essential class. The pricing equation becomes:

$$(1 + r)p(t) = c - a\sigma^2 z^s$$

We can also define a *fundamental price* given by the discounted sum of all expected future dividends:

$$p^*(t) = \sum_{k=1}^{\infty} \frac{\mathbb{E}(y_{t+k}, t) - a\sigma^2 z^2}{(1 + r)^k}$$

which can be set in an analogous way of a perpetuity with interest r. We assume that, at a specific time t, should influence matrices be fixed, there are g different asymptotic opinions, each of which can be associated to one **essential class** of the kind described in the previous chapter. When $(y(t))_{t=1,2,\ldots}$, then $\mathbb{E}[y_t] = \bar{y}$ and the fundamental price is constant over t:

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y} - a\sigma^2 z^s}{(1+r)^k} = \frac{\bar{y} - a\sigma^2 z^s}{r}$$

We will show, by computer simulations, that in this model based on fundamentals (FB), changes in opinion in the essential classes will lead to significant changes in prices.

In this case, we assume the set I to be determined as follows:

$$I(i, \mathbf{x}(t-1), p(t-1)) = \{i\} \cup \{j \text{ s.t. } |1 - x_i(t-1)/p^*| \leq \epsilon_i\}$$

This implies that all agents will see deviations from the fundamental as temporary and will tend to trust agents whose opinion does not deviate from the fundamental.

Reaching the fundamental There are possible two cases where the fundamental price is achieved in "aggregate" at time t:

1. in general, when the *average opinion* on the price is equal to the fundamental, or $\frac{1}{n}\mathbf{u}^\top \mathbf{x}(t) = p^*(t)$;
2. as a special case, when, at t, all agents have opinion on the price equal to the fundamental at time t.

The first, more general, case could occur in a number of situations. For example, even in presence of high variability in the opinions, the system can reach the fundamental anyway, if the average opinion coincides with it.

4.2.2 Simulations

In this section we will illustrate some results on the dynamics described by Equations 4.5, 4.6 according to the three models (BC, PA, FB). Given the number of parameters involved in this model, we will make the following assumptions:

1. The variance on the dividend innovations is constant $\forall t$ and $\forall i$ (i.e. on the innovation opinions) and equal to σ;

2. $\mathbb{E}_{it}(t_{t+1}) = \bar{y}$, $\forall i, t$;

3. in any case, $\epsilon_i = \epsilon$, $\forall i$;

All figures hereby reported are a representation of the price $p(t)$ in time, obtained from Eq. 4.5 by averaging out of 1000 realizations of the process in Eq. 4.6 from the *same* starting opinion profile $\mathbf{x}(0)$, that it is distributed as a log − normal with mean 3. The number of agents is $n = 100$.

Figures 4.1, 4.3, 4.2 show simulation price dynamics for the models BC, FB and PA. Statistics are shown in Tables 4.2a − 4.2f for the returns. These statistics and the price dynamics reflect some well–known stylized facts in financial markets, such as sudden drops, clustered volatility and fat tails. In particular, although the return distributions are slightly asymmetric, we used the excess kurtosis as a way to compare tail distribution with those of a normal distribution one can see that fat tails arise in the BC non − linear model (where polarisation occurs) when the update propensity parameter α is very high. This means that the quicker the fragmentation, the quicker non − linear phenomena arise and fat tails are produce. The PA model, on the contrary, produces no fat − tails, as the result of the reversion towards the fundamental.

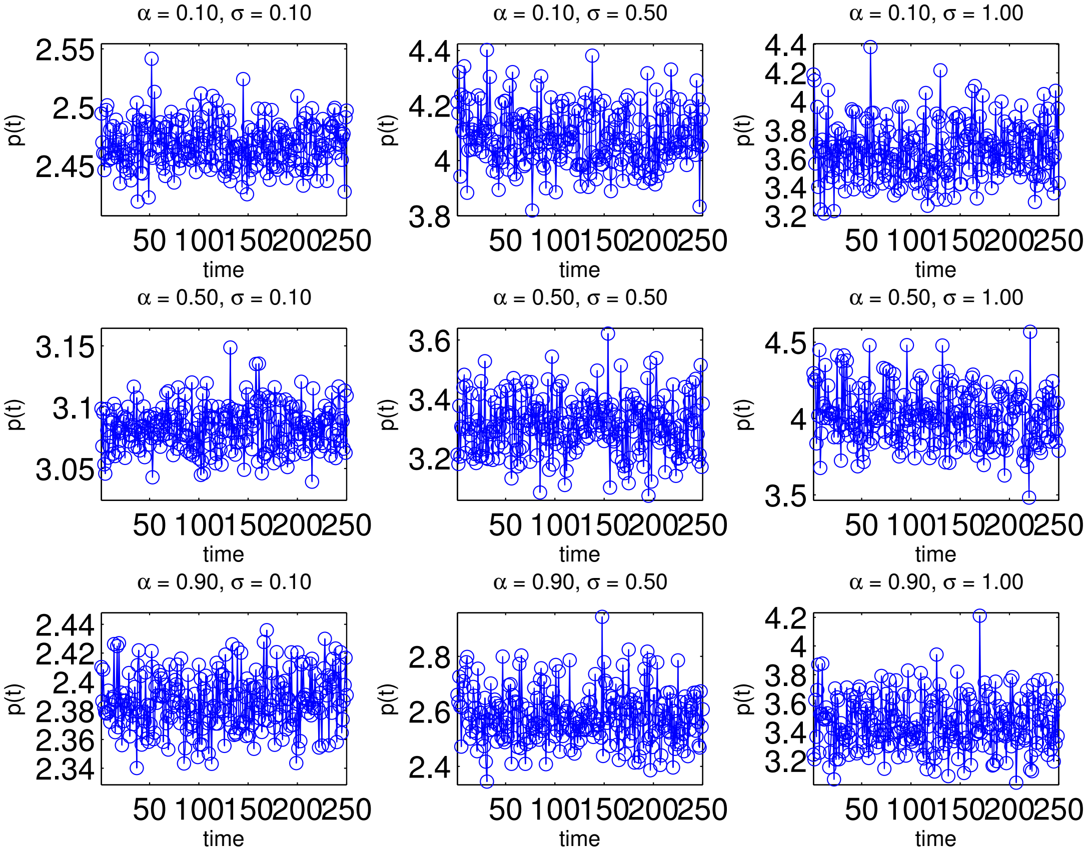


Figure 4.1: BC model for different values of α and σ.

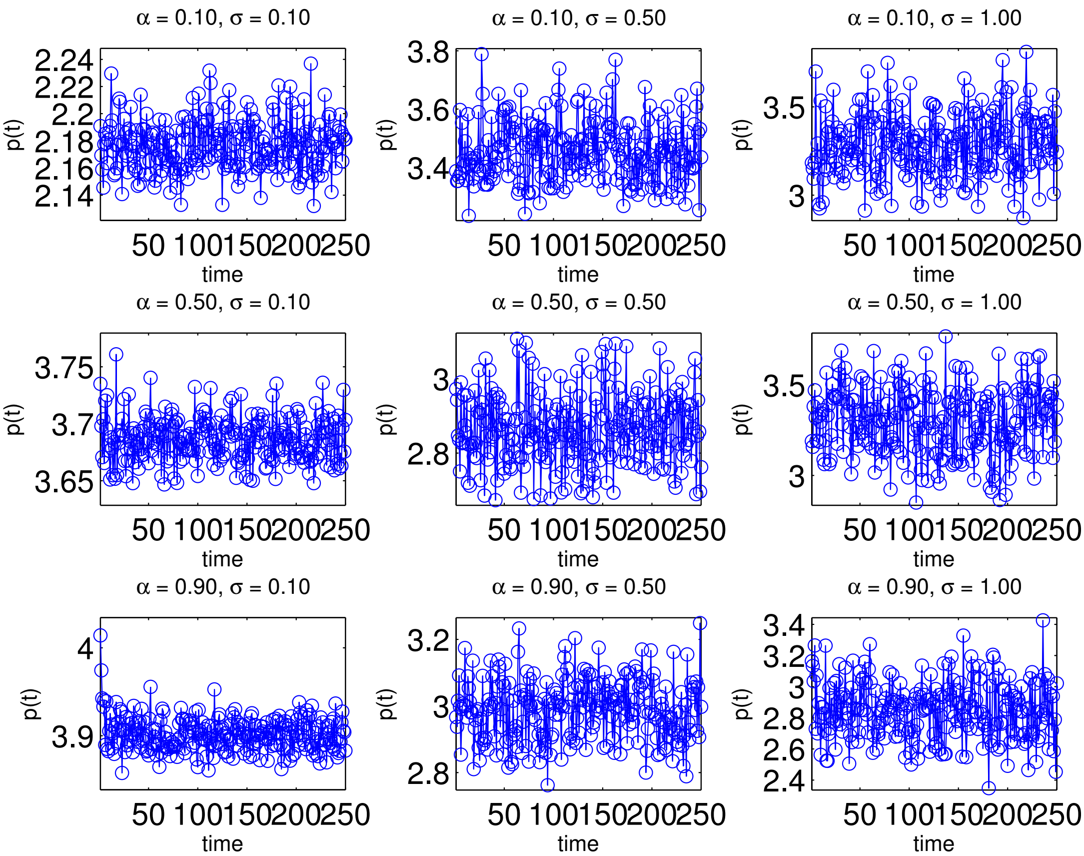


Figure 4.2: PA model for different values of α and σ.

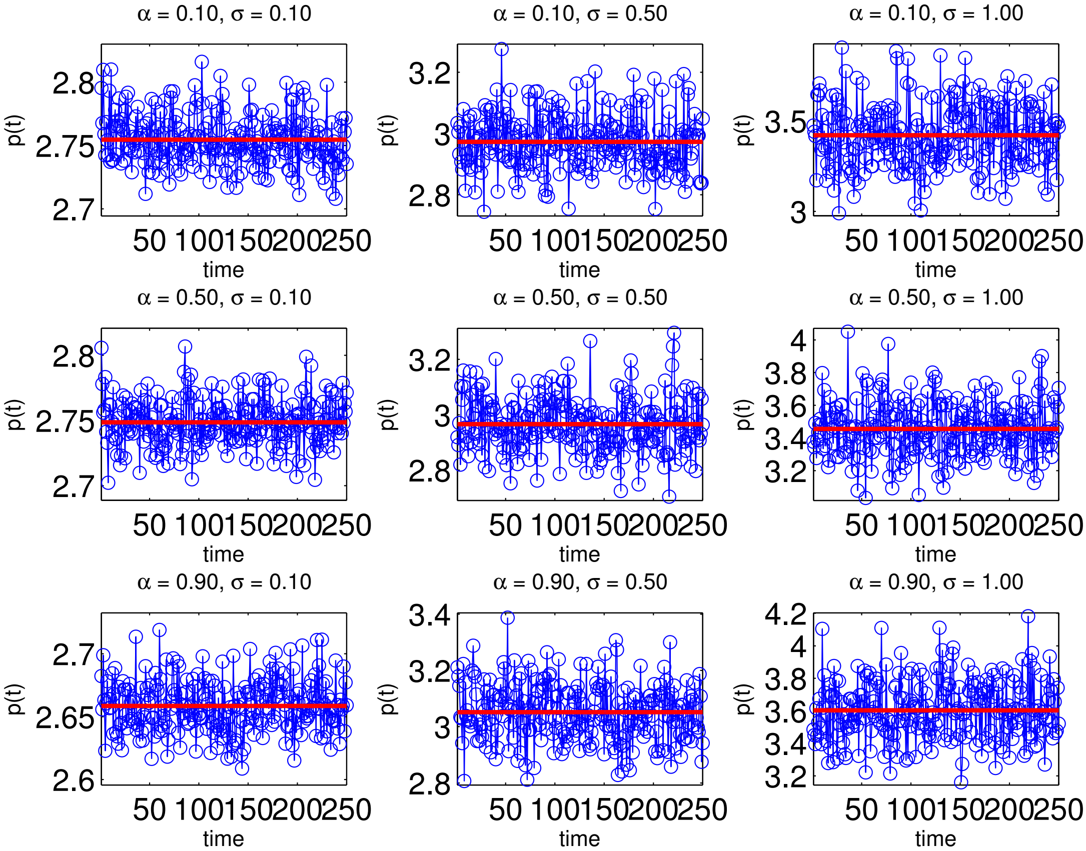


Figure 4.3: FB model for different values of α and σ. Fundamental value is in red.

Table 4.1: Statistics over 1000 simulations for the three models and different values of the update propensity parameter α and σ.

	$\sigma = 0.1$	$\sigma = 0.5$	$\sigma = 1$
$\alpha = 0.1$	0.20	-0.03	0.02
$\alpha = 0.5$	-0.21	-0.16	0.25
$\alpha = 0.9$	0.12	0.05	-0.02

(a) BC model – skewness

	$\sigma = 0.1$	$\sigma = 0.5$	$\sigma = 1$
$\alpha = 0.1$	-0.38	-0.45	-0.18
$\alpha = 0.5$	-0.00	0.42	0.00
$\alpha = 0.9$	0.29	-0.08	0.59

(b) BC – (excess) kurtosis

	$\sigma = 0.1$	$\sigma = 0.5$	$\sigma = 1$
$\alpha = 0.1$	0.14	-0.02	-0.09
$\alpha = 0.5$	0.09	-0.16	-0.13
$\alpha = 0.9$	0.08	0.11	0.12

(c) PA model - skewness

	$\sigma = 0.1$	$\sigma = 0.5$	$\sigma = 1$
$\alpha = 0.1$	0.10	-0.34	-0.14
$\alpha = 0.5$	0.66	-0.22	-0.01
$\alpha = 0.9$	-0.14	-0.12	-0.26

(d) PA model – (excess) kurtosis

	$\sigma = 0.1$	$\sigma = 0.5$	$\sigma = 1$
$\alpha = 0.1$	0.05	-0.18	0.12
$\alpha = 0.5$	-0.02	0.19	-0.17
$\alpha = 0.9$	0.18	-0.00	0.08

(e) FB model - skewness

	$\sigma = 0.1$	$\sigma = 0.5$	$\sigma = 1$
$\alpha = 0.1$	-0.09	0.23	-0.04
$\alpha = 0.5$	-0.21	0.40	0.20
$\alpha = 0.9$	-0.22	0.59	-0.39

(f) FB model - (excess) kurtosis

Opinion shifts

The simulation setup will aim at explaining sudden drops in price followed by recovery where herding behaviour may play a role. In particular, we will try to assess whethet the 'drop' in the DJ30 following the fake tweet by AP can be explained in terms of network effects. To do so, we assume a "sudden" exogenous change at $t = 50$ for either some essential classes or the union set of inessential classes. The basic idea is to assess whether changes in the opinion profiles of essential classes might have a different impact with respect to change in the opinion profiles of inessential classes. Although the results presented in the previous Chapter already provide a positive answer to such a question, it still remains to assess the impact of such changes. In particular, we will see that changes in inessential classes do have an impact on price but it will persist less.

Figure 4.4 shows that, in the BC model, when drops occur in the essential classe, they are kept for low values of σ: in fact, the *bounded* nature of the model will provide a polarisation versus a low market scenario rather than a recovery. This does not happen when the drop occurs in the inessential classes (Fig. 4.5). Figures 4.6 and 4.7 shows analogous findings for the PA model. The most interesting case is the FB model, where it is important to understand whether a return to the fundamental benchmark will occur and the time to recovery.

Figure 4.8 and 4.9 show an important fact: drops happen and persist when opinion shifts occur in both essential and inessential classes, then recovering quite rapidly. However, the drops are much more pronounced in the essential class case and for *lower* values of σ, thus showing that a higher variability in dividend innovations might lead to a more stable price dynamics w.r.t. suddend opinion shifts. The role of the update propensity parameter α is important for higher values of σ in that lower propensity implies a lower degree of abandoment of previously formed opinions on the deviations from the fundamental, hence making recovery faster.

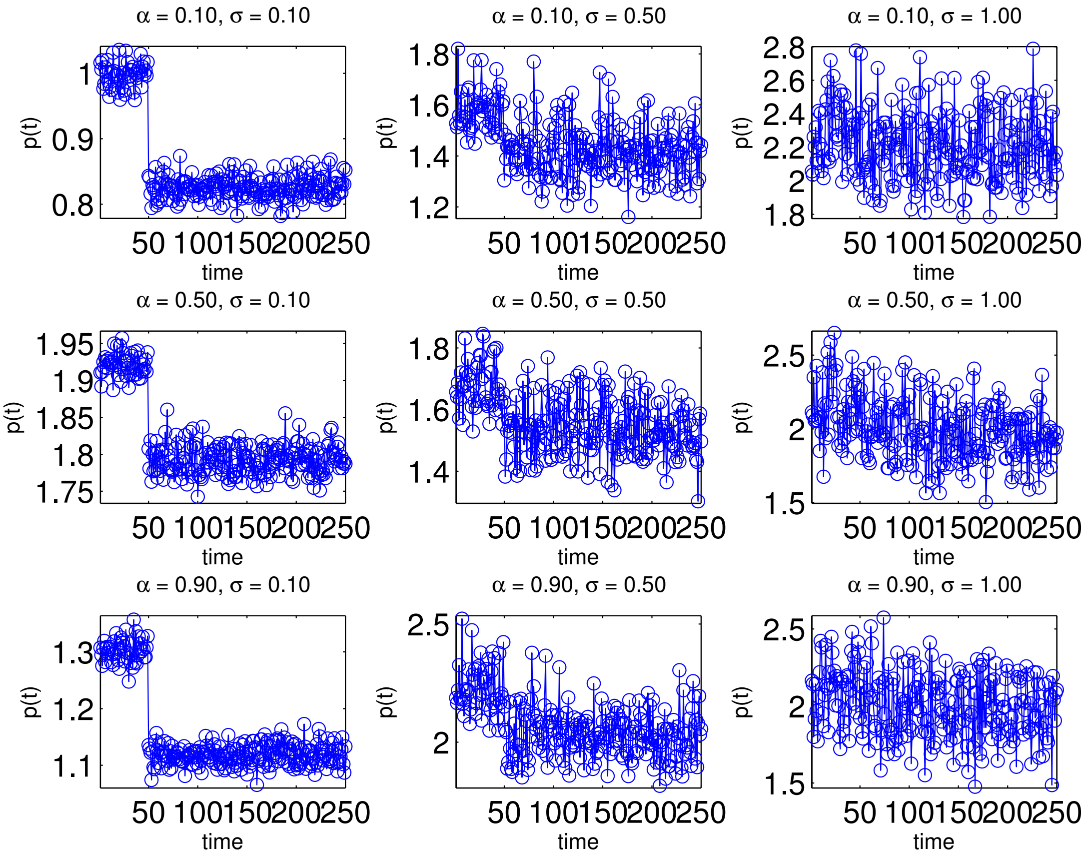


Figure 4.4: BC, drop in essential classes

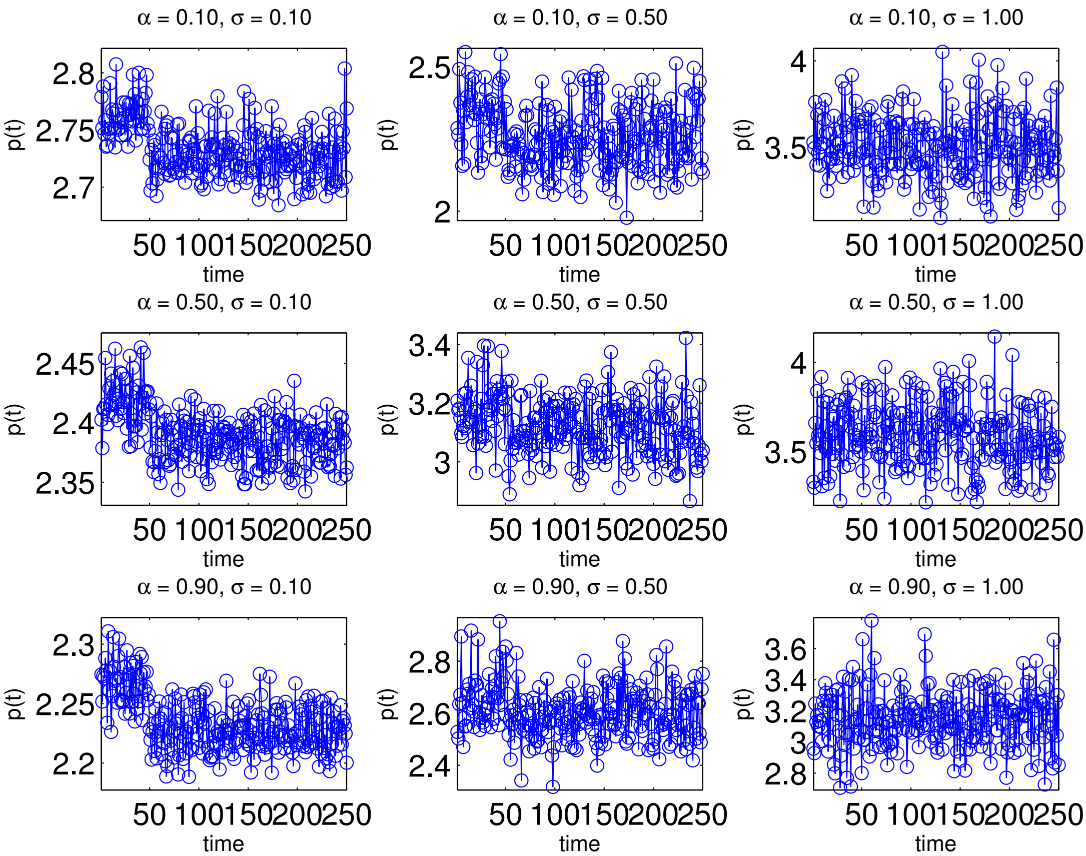


Figure 4.5: BC, drop in inessential classes

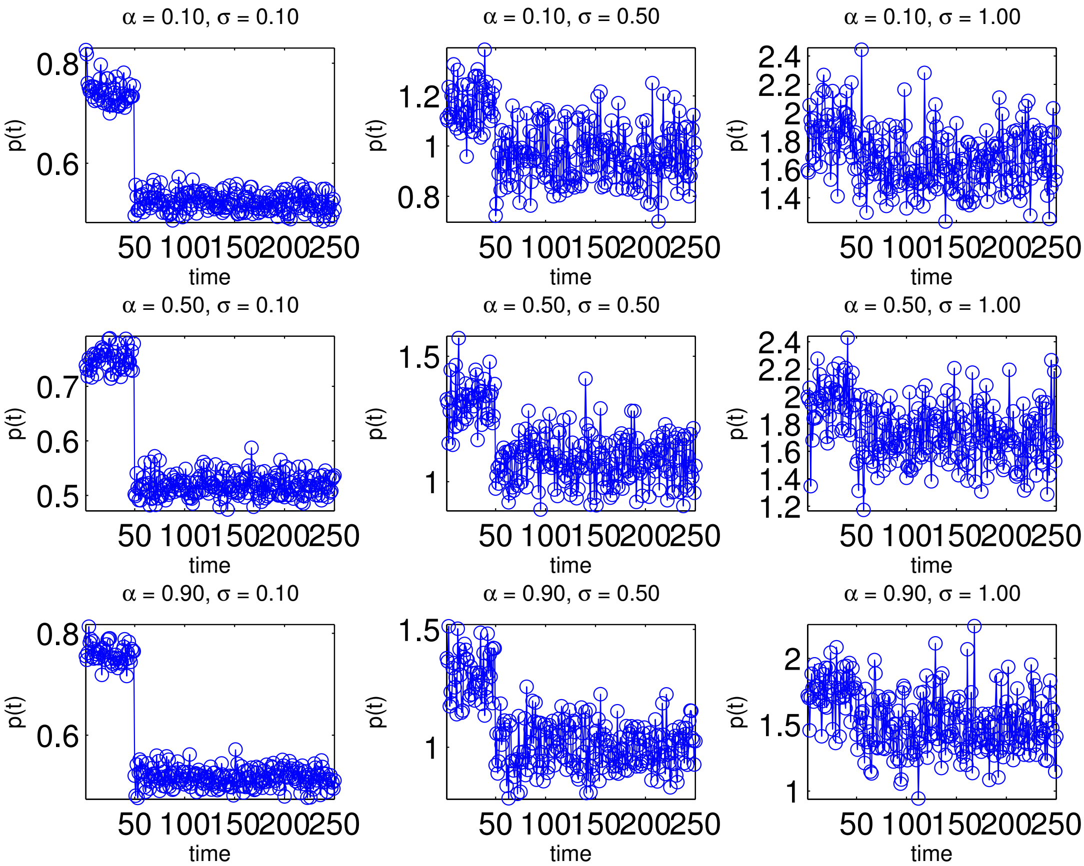


Figure 4.6: PA, drop in essential classes

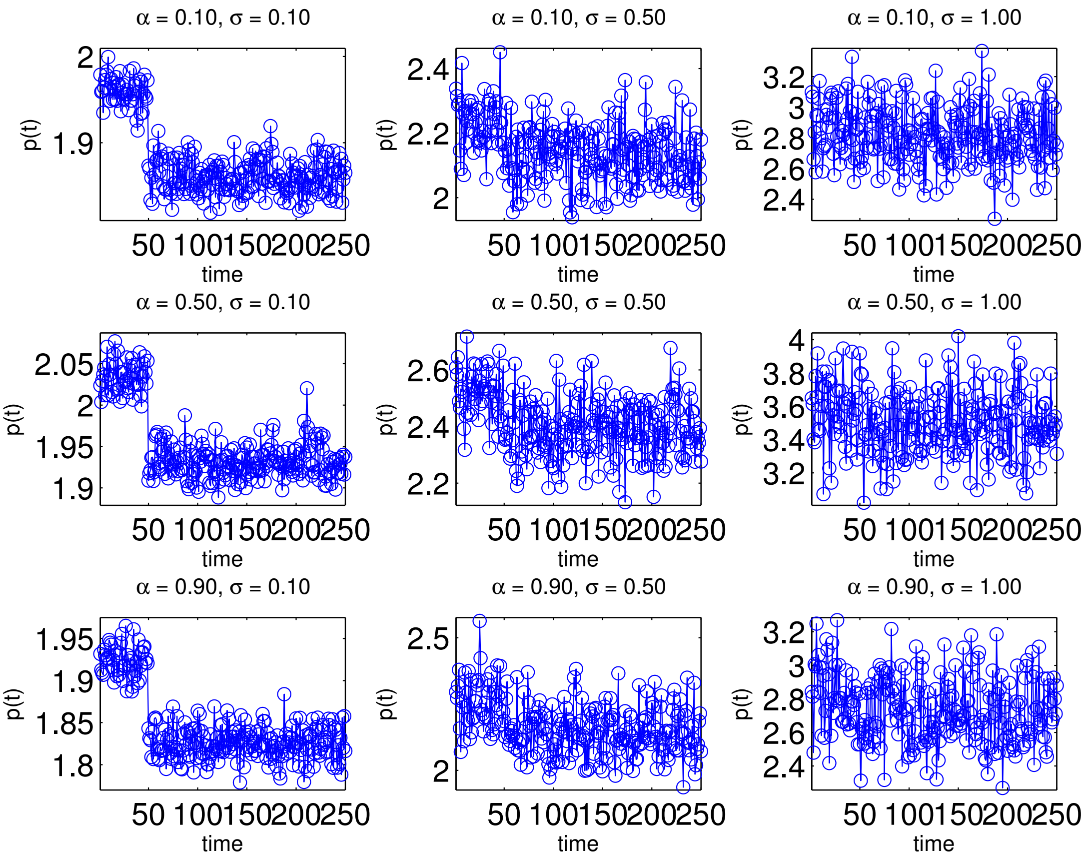


Figure 4.7: PA, drop in inessential classes

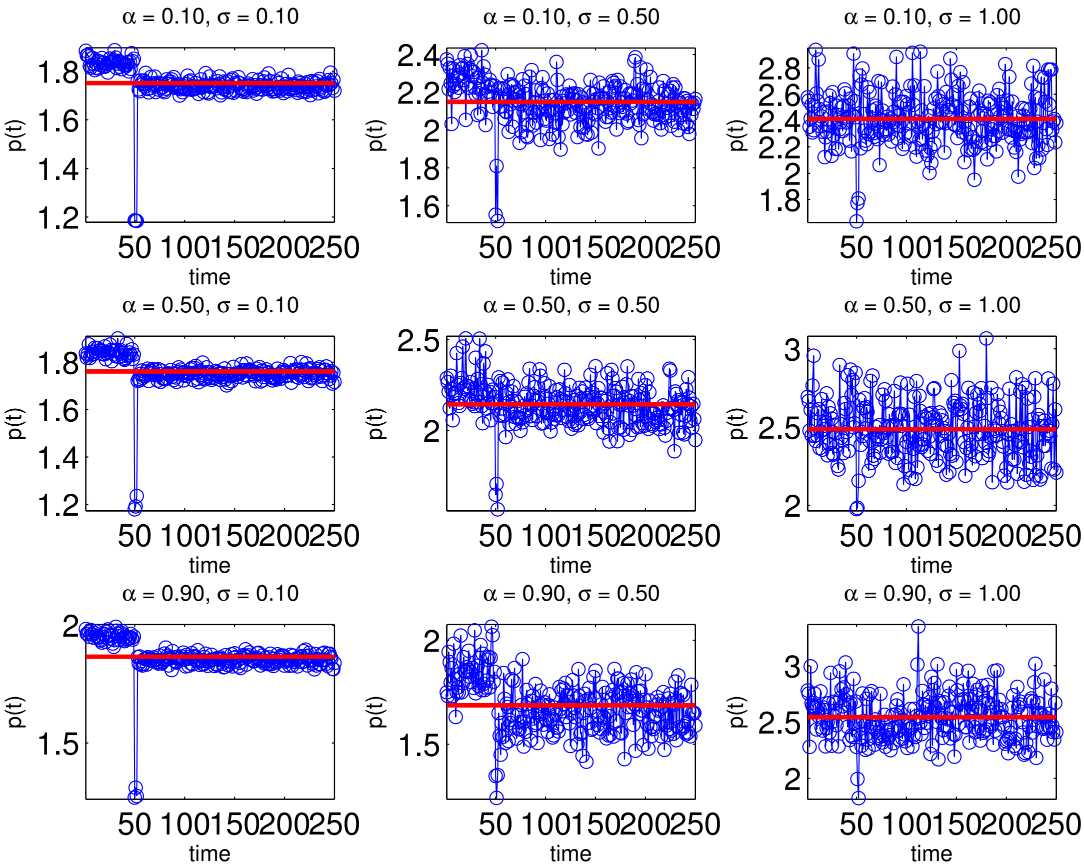


Figure 4.8: FB, drop in essential classes. The fundamental value is in red.

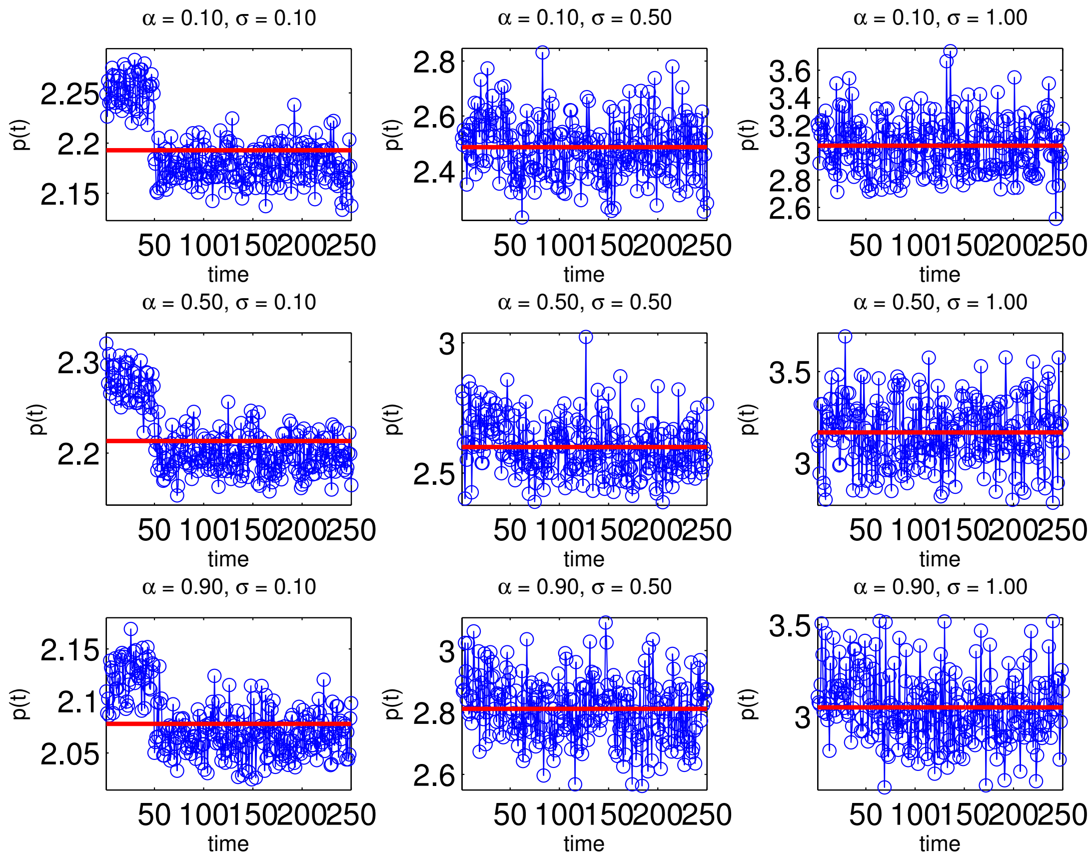
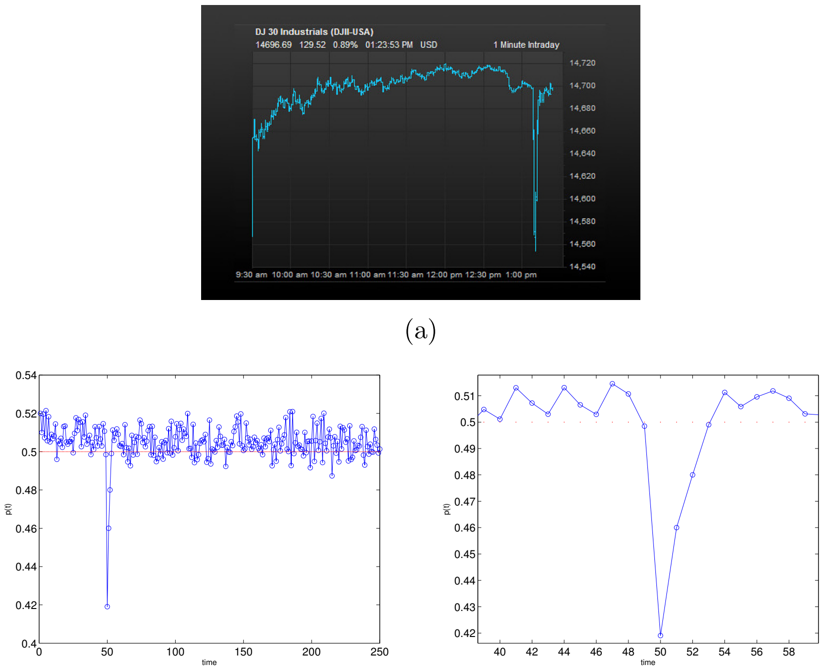


Figure 4.9: FB, drop in inessential classes. The fundamental value is in red.



(a)

(b) (c)

Figure 4.10: Possible explanation of the reaction to the fake tweet: change in opinion (drop) of 2 essential classes leads to drop in price (blue line) and return to fundamental after influence weights have been (red dotted line) due to the update to the fundamental. Right figure is a zoom in the periods where the drop happens.

In order to understand the DJ30 drop occurred in the early afternoon of April 23, 2013 described in the Introduction of this work (and reported again in Fig. 4.10a) we focused on a one – run simulation reported in Fig. 4.10b and, in more detail, in 4.10c. Here, we simulated a situation where drops in opinion occurred in only *two* essential classes for a total number of 10 agents shifting opinion. The drop is dramatic and shows the importance of the topological classification of agents.

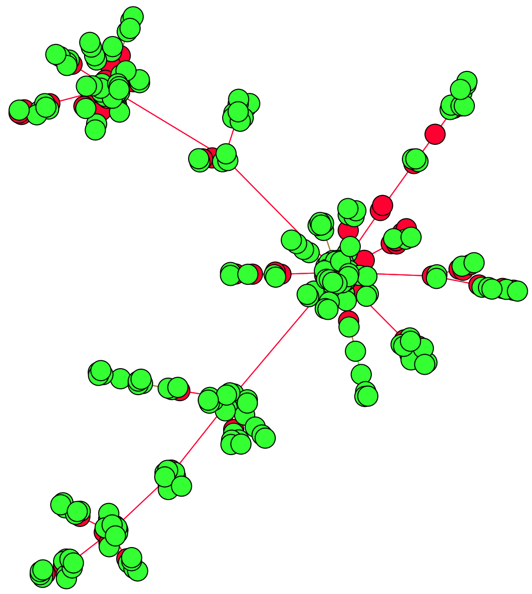
4.3 Empirical validation

The empirical validation of such a model is not an easy task. Although some stylized facts like clustered volatility, drops in prices that lead to fat tails in the returns distribution can be explained by simulations, the calibrations of such model is an open problem. Time series for prices are easily available but determining and calibrating the interaction network and the update propensity parameter seems to be the hardest task. However, we can use the wide availability of *Big Data* (i.e. Twitter microblogging) to have a proxy for such interaction. Applying Twitter data (and, in general, big data) in the context of financial markets is still a research endeavour (see Bollen, 2010). However, the recent usage of Twitter data in giving signals about markets by the famous data service provider Bloomberg, clearly shows a growing interest in this approach. We can then give an outline of the future research that will need to be done to accomplish this task:

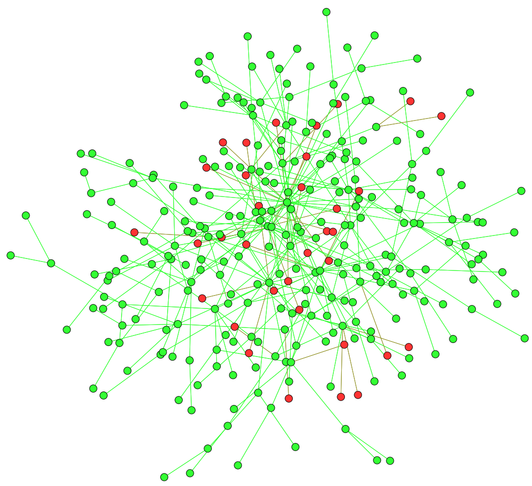
- Twitter and, more generally, social network provide with a huge amount of data, in the order of terabytes per minute. As such, algorithms must be fast and efficient, especially in the classification of agents.
- At each t, agents needs to be classified according to the their topological importance (this includes, but it is not limited to, the separation into essential and inessential classes).
- Social network data naturally embed a *multiple network structure*, i.e. a structure where several interactions might occur. A possible way to cope with this problem is to use a tensor sequence as a means to treat multiple networks (see Kolda, 2005) $(\boldsymbol{\mathcal{A}}(t))_{t\in\mathbb{N}}$

$$a_{ijk}(t) = \begin{cases} \in (0,1] \text{ if } j \text{ retweets } i\text{'s tweet about stock } \$k \text{ at } t \\ 0 \text{ otherwise} \end{cases}$$

Figures 4.11a and 4.11b shows the network stemming from the tensor with two fixed dimensions (time and stock), referring in particular to Google and Apple stock. Identification of essential (red) and inessential (green) classes is stressed.



(a) $GOOG 21-Sep-2013, data retrieved at 16:03:21 − 1 hour retweet time, $n = 150$, $m = 210$



(b) $AAPL 21-Sep-2013 17:03:21, data retrieved at − 1 hour retweet time, $n = 300$, $m = 598$

4.4 A model for limited attention

In this section, we will illustrate a second theoretical contribution concerning agents interaction in forming opinions in a financial market. However, there will be substantial differences with the model proposed in the previous section:

- we deal with a multiple – asset, closed financial systems and we will be interested in dealing with *relative* prices;
- agents put trust in other agents because they have *limited attention*, in the sense that, even though information is readily available to all agents at all times, they can pay attention (and process) only a limited amount of it, as such they find convenient to look at what other agents do.

Our model will allow to classify a financial system in terms of relative price formation according to whether stable or unstable equilibria in asset holding configurations are reached. Moreover, we will be able to characterise the system in terms of whether such configurations are uniform for all agents or certain phenomena of asset holding configuration *polarisation* or *fragmentation* occur. In particular, we refer to the concept of fragmentation as a more general case of the concept of polarisation (Hegselmann & Krause, 2005). Given the structure of the model, sufficient conditions to achieve such equilibria will be given.

4.4.1 The model

The general setting

We describe financial markets with an $n \times n$ non – negative matrix $\mathbf{S}(t)$, whose rows represent the agents and whose columns represent the sectors. The entries of the matrix $s_{ij}(t)$ represent the asset holdings of agent i in sector j at time t. We assume that the matrix of the initial holdings $\mathbf{S}(0)$ is row – stochastic, i.e. $\sum_j s_{ij}(0) = 1$, $\forall i$. The total sum of entries $\sum_{ij} s_{ij} = n$ is constant and does not change over time, i.e. the financial system is *closed* and n can be used as a numéraire. Because of this closure property of our model, it is possible to express the *relative prices of each asset* in the in the financial system as the sum of the relative asset holdings of all agents in each sector, i.e. the column totals $\mathbf{p}(t)$ divided by the numéraire n give the relative prices of the sectors in the financial markets at each time t, i.e.:

$$p_j(t) = \frac{1}{n} \sum_{i=1}^{n} s_{ij}(t) \quad \text{or, equivalently:} \quad \mathbf{p}(t) = \frac{1}{n} \mathbf{S}^\top(t)\mathbf{u}$$

where $p_j(t)$ is the relative price of asset/sector j at time t and $\mathbf{u}$ is the n – unit vector.

We can describe the attention at time t by an $n \times n$ non–negative matrix $\mathbf{A}(t)$ whose columns represent the attention each agent pays to *agents that specialise* in all other sectors. Assuming that each agent has 24 hours a day, their total attention is limited and sums up to 1, i.e. $\sum_j a_{ij} = 1$, i.e. $\mathbf{A}(t)$ is row stochastic $\forall t$. Agents are identical, except for their attention structure to public information.

Furthermore, we assume each agent is specialised in one sector, as a realistic assumption that is easily observed in financial markets. For example, if an agent is a trader in the US stock market, she spends most of her time on processing the information on US stocks and spends limited time on the subprime loan markets in the US or stock markets in Italy and Turkey. This assumption translates in our model by imposing a positive diagonal to the attention matrix, $a_{ii}(t) > 0$, $\forall i, t$, as described above, i.e. all agents are *self – communicating*. Note that $\mathbf{A}(t)$ is not necessarily symmetric since the amount of attention given by an agent to another is not necessarily mutual.

We assume a discrete dynamic process, where agents *learn* both the relative prices at time t and the asset holdings of other agents $s_{ij}(t)$. This information is readily available at all times t to all agents $i = 1, \ldots, n$.

We do not make any behavioural assumptions about the agents so agents are not necessarily aware of the attention other agents pay to the different sectors of the economy. However, since the holdings are available, they infer the behaviour of other agents from the relative prices in the economy.

We assume a dynamical discrete process in which agents revise their holding configurations in each asset. The new configuration is obtained by averaging at each time t their holdings and the holdings of other agents with weights given by the attention to themselves and the other agents. The new configuration is a convex linear combination since its weights sum up to one [1]. Therefore, the dynamics of the system can be described as follows:

$$\begin{aligned}\mathbf{S}(t) &= \mathbf{A}(t)\mathbf{S}(t-1) \\ &= \mathbf{A}(t)\mathbf{A}(t-1)\cdots\mathbf{A}(1)\mathbf{S}(0), \quad t = 1, 2, \ldots\end{aligned} \tag{4.8}$$

It is easy to note that, given $\mathbf{A}(t)$ row – stochastic $\forall t$, then $\mathbf{S}(t)$ is row – stochastic $\forall t$ as well, since it is row – stochastic at $t = 0$. In fact, denoting with $\mathbf{u}$, the n–th unit vector, it holds that:

$$\begin{aligned}\mathbf{S}(t)\mathbf{u} &= \mathbf{A}(t)\mathbf{S}(t-1)\mathbf{u} \\ &= \mathbf{A}(t)\cdots\mathbf{A}(2)\mathbf{A}(1)\mathbf{S}(0)\mathbf{u} \\ &= (\mathbf{A}(t)\cdots\mathbf{A}(2))\,\mathbf{A}(1)\mathbf{u} = \mathbf{u} \ .\end{aligned}$$

We will refer to as a *homogenous dynamics system* the special case when the attention structure does not change over time, i.e. $\mathbf{A}(t) = \mathbf{A}\ \forall t$. In this case, it holds that:

$$\mathbf{S}(t) = \mathbf{A}\mathbf{S}(t-1) = \mathbf{A}^t\mathbf{S}(0) \tag{4.9}$$

We will illustrate that, within this model, the dynamical process may lead to a stable holding configuration for all agents or, more generally, to a fragmentation of the holding patterns.

According to the topology of the attention matrix, an equilibrium will be either reached or not and, in case it will be reached, it can be *uniform*, i.e. all agents have identical asset holding configurations, or *non – uniform*, i.e. agents have sub sets of different identical holding configurations. We will call the latter case a *segmented* equilibrium.

[1] This approach is analogous to the one detailed in Hegselmann and Krause (2005) that models opinion dynamics.

Uniform non stable holding configuration A *uniform non stable holding configuration* when there exists a t such that the matrix $\mathbf{S}(t)$ is a row – stochastic matrix with equal rows, i.e. a rank – one matrix, or in other words, a matrix with constant column values. This configuration represents a situation where all agents have identical holding profiles in all sectors. A sufficient condition for the uniform unstable holding configuration is that the associated matrix $\mathbf{A}(t)$ is irreducible, i.e. all agents belong to an essential class, i.e. they communicate to each other. This does not necessarily imply convergence to a stable configuration.

Once a stable configuration has been reached for certain t, such a configuration holds also at time $t+1$. A *uniform stable holding configuration* for the system occurs when the matrix $\mathbf{S}$ converges to a row stochastic matrix with equal rows, i.e. a rank – one matrix, or in other words, a matrix with constant column values. This configuration represents a situation where all agents have identical holding profiles in all sectors. Therefore, a stable price is reached for each sector, obtained (as previously defined) as the column sum of the uniform holding matrix.

Uniform stable holding configuration We can identify g essential classes with identical asset holdings while the inessential classes may converge to a stable holding configuration which, in case it is achieved, will be a convex linear combination of the asset holdings achieved in the essential classes. For a a formal proof, see on the sufficient condition, Lorenz 2006 Theorem 1.

Homogenous dynamics

Although we will define update rules for the attention matrix $\mathbf{A}(t)$ at time t, as an illustrative example, we will now describe the particular case where the attention matrix does not change over time, i.e the system described in Equation 4.9. Some sufficient conditions for a stable configuration hold, in particular a stable configuration will be approached for any initial holding profile if there exists a $t \in \mathbb{N}$ such that any two agents put *jointly* a positive attention to a third agent/sector or that any pair of agents communicate to each other. In mathematical terms, this implies the irreducibility and primitivity of the attention matrix.

A generalisation of this result can be found using the Gantmacher form for the attention matrix $\mathbf{A}$. It can be shown that, for any given initial holding profile $\mathbf{S}(0)$ a stable non − uniform configuration is reached if the sub matrices associated to the essential classes are primitive. In the process of achieving a uniform holding profile, only the opinions of the essential classes play a role (Hegselmann & Krause, 2005). On the contrary, the *inessential* classes do not provide any contribution to the uniform holding configuration.

If the initial configuration $\mathbf{S}(0)$ is a matrix with equal rows, it is easy to show that such initial configuration is already uniform for any stochastic attention matrix $\mathbf{A}$ and the uniform holding configuration is given by $\mathbf{S}(0)$ (Berger, 1981).

Chapter 5

Conclusions and future research

In this work, we have proposed an analytical framework to model the role of opinion dynamics in a financial system. We have stressed the intertwined relationship among opinions, the actual price dynamics and the interaction network among agents. Different cases have been explored and we showed, by simulations, that some stylized facts can be reproduced and explained in this way. Moreover, we proposed an empirical validation scheme for the model.

This work stems from the need to interpret the role of opinion dynamics in an increasingly interconnected world, where network effects play an important role in determining the evolution of social systems.

Particular emphasis has been given to the topology of the interaction network and to the classification of agents according to their role in the dynamical process of opinion formation. We have explored the different models proposed in the literature and we have built on a well – known nonlinear model where opinion fragmentation might occur. In this case, we noticed that such dynamics reflects on the price formation process in a feedback loop in the sense that agents will tend to shape their opinion patterns according to the topology and vice – versa, the topology is determined by the opinion patterns.

Future research

The approach proposed in this work provides a framework that can be extended by future research on these topics.

For what concerns the network interaction topology, a detailed analysis on the properties of network dynamics when the network has a specific topology should be done. We refer to, e.g. random, small – world, scale – free or core – periphery (see Newman, 2010 for details on such network topologies) topology structures. Finding analytical results for specific topologies would be useful for empirical comparison and analysis of the dynamical system. In particular, we think of a scheme where the topology is imposed to the essential classes: this would drive different polarisation/fragmentation patterns that would then need to be compared with the actual empirical data.

Moreover, we would like to extend the model on price formation to more than one risky asset (see Chiarella *et al.* 2007 for this problem) by assuming different influence interaction networks for each asset. This problem is also known as *multidimensional opinion dynamics* and we propose a *multiple network* approach to this problem, by using a *tensor* representation (see Kolda and Bader 2009), as follows (see Fig. 5.1 for a graphical representation):

$$\boldsymbol{\mathcal{A}}(t) \in \mathbb{R}^{n \times n \times m} = [a_{ijs}(t)]$$

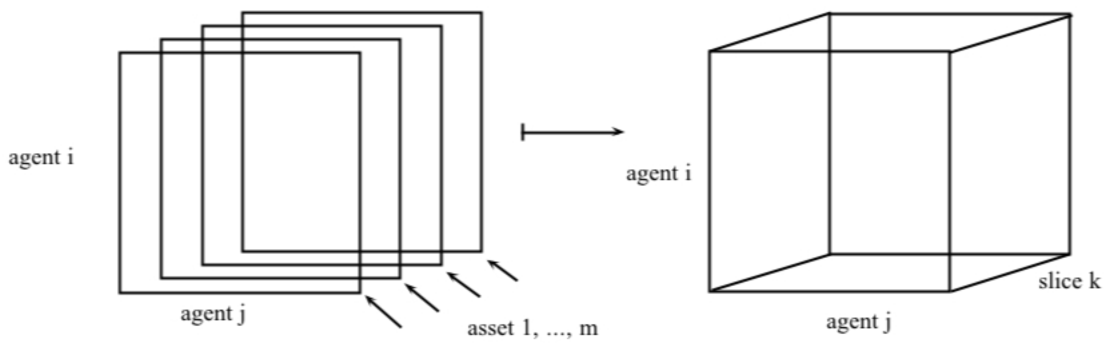


Figure 5.1: Tensor multiple network approach for opinion dynamics.

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