# TIME DECAY OF SCALING CRITICAL ELECTROMAGNETIC SCHRÖDINGER FLOWS

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Abstract. We obtain a representation formula for solutions to Schrödinger equations with a class of homogeneous, scaling-critical electromagnetic potentials. As a consequence, we prove the sharp $L^1 \to L^\infty$ time decay estimate for the 3D-inverse square and the 2D-Aharonov-Bohm potentials.

## 1. Introduction and statements of the results

This work is concerned with the dispersive property of the following class of Schrödinger equations with singular homogeneous electromagnetic potentials

$$iu_t = \left(-i\nabla + \frac{\mathbf{A}\left(\frac{x}{|x|}\right)}{|x|}\right)^2 u + \frac{a\left(\frac{x}{|x|}\right)}{|x|^2}\, u; \tag{1.1}$$

here $u = u(x,t) : \mathbb{R}^{N+1} \to \mathbb{C}$, $N \geqslant 2$, $a \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$, $\mathbb{S}^{N-1}$ denotes the unit $(N-1)$-dimensional sphere, and $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$ satisfies the following transversality condition

$$\mathbf{A}(\theta) \cdot \theta = 0 \quad \text{for all } \theta \in \mathbb{S}^{N-1}. \tag{1.2}$$

We always denote by $r := |x|$, $\theta = x/|x|$, so that $x = r\theta$.

Equation (1.1) describes the dynamics of a (non relativistic) particle under the action of a fixed external electromagnetic field $(E, B)$ given by

$$E(x) := \nabla\left(\frac{a\left(\frac{x}{|x|}\right)}{|x|^2}\right), \qquad B(x) = d\left(\frac{\mathbf{A}\left(\frac{x}{|x|}\right)}{|x|}\right),$$

where $B$ is the differential of the linear 1-form associated to the vector field $\frac{\mathbf{A}(x/|x|)}{|x|}$. In dimension $N = 3$, due to the identification between 1-forms and 2-forms, the magnetic field $B$ is in fact determined by the vector field $\operatorname{curl} \frac{\mathbf{A}(x/|x|)}{|x|}$, in the sense that

$$B(x)v = \operatorname{curl}\left(\tfrac{\mathbf{A}(x/|x|)}{|x|}\right) \times v, \qquad N = 3,$$

for any vector $v \in \mathbb{R}^3$, the cross staying for the usual vectorial product.

Under the transversality condition (1.2), the hamiltonian

$$\mathcal{L}_{\mathbf{A},a} := \left(-i\,\nabla + \frac{\mathbf{A}\left(\frac{x}{|x|}\right)}{|x|}\right)^2 + \frac{a\left(\frac{x}{|x|}\right)}{|x|^2} \tag{1.3}$$

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formally acts on functions $f : \mathbb{R}^N \to \mathbb{C}$ as

$$\mathcal{L}_{\mathbf{A},a} f = -\Delta f + \frac{|\mathbf{A}\big(\frac{x}{|x|}\big)|^2 + a\big(\frac{x}{|x|}\big) - i\,\mathrm{div}_{\mathbb{S}^{N-1}} \mathbf{A}\big(\frac{x}{|x|}\big)}{|x|^2} f - 2i\,\frac{\mathbf{A}\big(\frac{x}{|x|}\big)}{|x|} \cdot \nabla f,$$

where $\mathrm{div}_{\mathbb{S}^{N-1}} \mathbf{A}$ denotes the Riemannian divergence of $\mathbf{A}$ on the unit sphere $\mathbb{S}^{N-1}$ endowed with the standard metric.

The free Schrödinger equation, i.e. (1.1) with $\mathbf{A} \equiv \mathbf{0}$ and $a \equiv 0$, can be somehow considered as the canonical example of dispersive equation. The unique solution $u \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^N))$ of the Cauchy problem

$$\begin{cases} iu_t = -\Delta u \\ u(x,0) = f(x) \in L^2(\mathbb{R}^N) \end{cases} \tag{1.4}$$

can be explicitly written as follows:

$$u(x,t) = e^{it\Delta} f(x) := \frac{1}{(4\pi i t)^{\frac{N}{2}}} e^{i\frac{|x|^2}{4t}} \int_{\mathbb{R}^N} e^{-i\frac{x\cdot y}{2t}} e^{i\frac{|y|^2}{4t}} f(y)\,dy. \tag{1.5}$$

This shows that, up to scalings and modulations, $u$ is the Fourier transform of the initial datum $f$. Formula (1.5) contains most of the relevant informations about the dispersion which arises along the evolution of the Schrödinger flow. In dimension $N = 1$, the evolution of an initial wave packet of the form

$$f_K(x) = e^{iKx} e^{-\frac{x^2}{2}}, \quad K \in \mathbb{N},$$

gives an important description of the phenomenon. Inserting $f = f_K$ in (1.5) gives in turn the solution

$$u_K(x,t) = e^{-\frac{K^2}{2}} u_0(x - iK, 2t); \quad u_0(z,t) = \frac{1}{\sqrt{it+1}} e^{-\frac{z^2}{2(it+1)}}.$$

This shows that each wave travels with a speed which is proportional to the frequency $K$, and describes both the phenomenon and the terminology of dispersion.

The above property can be quantified in terms of a priori estimates for solutions to (1.4). A first consequence of (1.5) is the time decay

$$\big\|e^{it\Delta} f(\cdot)\big\|_{L^p} \leqslant \frac{C}{|t|^{N\left(\frac{1}{2}-\frac{1}{p}\right)}} \|f\|_{L^{p'}}, \qquad p \geqslant 2; \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{1.6}$$

for some $C = C(p,N) > 0$ independent on $t$ and $f$. In the cases $p = \infty, p = 2$, (1.6) can be easily obtained by (1.5) and Plancherel; the rest of the range $2 < p < \infty$ follows by Riesz-Thorin interpolation. These inequalities play a fundamental role in many different fields, including scattering theory, harmonic analysis and nonlinear analysis. In particular, they standardly imply the following Strichartz estimates

$$\big\|e^{it\Delta} f\big\|_{L^p_t L^q_x} \leqslant C\|f\|_{L^2}, \tag{1.7}$$

for some $C > 0$, where $L^p_t L^q_x := L^p(\mathbb{R}; L^q(\mathbb{R}^N))$ and the couple $(p,q)$ satisfies the scaling condition

$$\frac{2}{p} + \frac{N}{q} = \frac{N}{2}, \qquad p \geqslant 2, \qquad (p,q,N) \neq (2,\infty,2). \tag{1.8}$$

The first result in this style has been obtained by Segal in [46] for the wave equation; then it was generalized by Strichartz in [49] in connection with the Restriction Theorem by Tomas in [50]. Later, Ginibre and Velo introduced in [23] (see also [24]) a different point of view which was extensively used by Yajima in [54] to prove a large amount of inequalities for the linear Schrödinger equation. Finally, Keel and Tao in [33] completed the picture of estimates (1.7), proving the difficult endpoint estimate $p = 2$, via bilinear techniques, for an abstract propagator verifying a time decay estimate in the spirit of (1.6).

Time decay and Strichartz estimates turn out to be a fundamental tool in the nonlinear applications, and consequently a large literature has been devoted, in the last years, to obtain them in more general situations, as for example perturbations of the Schrödinger equation with linear lower order terms, as in (1.1). In particular, since less regular terms usually arise in the physically relevant models, a deep effort has been spent in order to overcome the difficulty deriving from the fact that the Fourier transform does not fit well with differential operators with rough coefficients. Among these, electromagnetic Schrödinger hamiltonians have been object of study in several papers.

An electromagnetic Schrödinger equation has the form

$$iu_t = (-i\nabla + A(x))^2 u + V(x)u, \tag{1.9}$$

where $u = u(x,t) : \mathbb{R}^{N+1} \to \mathbb{C}$, $A : \mathbb{R}^N \to \mathbb{R}^N$, and $V : \mathbb{R}^N \to \mathbb{R}$. Homogeneous potentials like

$$|A| \sim \frac{1}{|x|}, \qquad |V| \sim \frac{1}{|x|^2} \tag{1.10}$$

represent a threshold for the validity of estimates (1.6) and (1.7), as shown by Goldberg, Vega and Visciglia in [26], when $A \equiv 0$ and later generalized by Fanelli and García in [18] for $A \neq 0$ (actually, the authors in [18, 26] disprove Strichartz estimates, and a byproduct of this fact is the failure of the usual time decay estimates). Notice that, for potentials as the ones in (1.10), equation (1.9) remains invariant under the usual scaling $u_\lambda(x,t) = u(x/\lambda, t/\lambda^2)$, $\lambda > 0$, and this is why we refer to it as the scaling-critical situation. We also recall that equation (1.9) is gauge invariant, namely if $u$ is a solution to (1.9), then $v = e^{i\phi(x)}u$ solves the same equation, with $A$ replaced by $A + \nabla\phi$ and the same magnetic field $B$.

In the purely electric case $A \equiv 0$, several authors studied the time dispersion when the potential $V$ is close to the scaling invariant case (1.10) (see [3, 4, 5, 15, 22, 25, 29, 30, 40, 43] and the references therein, both for Schrödinger and wave equations, and also the useful survey [45]). A typical perturbative approach consists in writing the action of the flow $e^{it(\Delta - V)}$ via spectral theorem, and then reducing matters of proving the desired estimate to perform a suitable analysis of the resolvent of $-\Delta + V$, in the Agmon-Hörmander style. We refer to the results by Goldberg-Schlag [25] and Rodnianski-Schlag [43], in which also time dependent potentials are treated, as standard examples of this technique for Schrödinger equations; in these papers, time decay estimates are obtained under integrability conditions on $V$ which are close to the scaling invariant case (1.10), but do not include the critical behavior $1/|x|^2$, due to the perturbation character of the strategy. Another possible approach consists in studying the mapping properties of the wave operators in $L^p$, and obtaining the time decay for the perturbed flow $e^{it(\Delta - V)}$ as a consequence of (1.6), via interwining properties. This point of view was introduced by Yajima in [55, 56, 57] and then followed by different authors (see e.g. [11, 52, 53]). Since it leads to a much stronger result, the integrability conditions which are needed for the potential $V$ are usually far from being optimal in the sense of (1.10). The unique situation in which, at our knowledge, the $L^p$-boundedness of the wave operators is proved under almost sharp assumptions on $V$ is the 1D-case, as it has been proven in [11]. About Strichartz estimates for $e^{it(\Delta - V)}$, the situation is quite clear, thanks to the results obtained by Burq, Planchon, Stalker and Tahvildar-Zadeh in [7, 8]; the authors can prove a suitable Morawetz-type estimate for the perturbed resolvent, by multiplier techniques, which implies, together with its free countepart and free Strichartz, the Strichartz estimates for a class of potentials $V$ which includes the ones which are critical in the sense of (1.10).

The situation in the electromagnetic case $A \neq 0, V \neq 0$ is quite more complicated and weaker results are available. Some additional difficulties, performing the above mentioned approach, come into play, due to the introduction of a first order term in the equation, which makes more complicate the analysis of the resolvent (see e.g. [12]). On the other hand, some results are available, both for estimates like (1.6) and (1.7), under suitable conditions on the potentials in (1.9), which as far as we know never permit to recover the critical cases as in (1.10) (see e.g. [9, 10, 16, 17, 12, 13, 14, 21, 38, 42, 47, 48] and the references therein).

In view of the above considerations, it should be quite interesting to produce a tool which might permit to prove the decay estimates (1.6) for equation (1.1), in which the potentials are scaling-critical. The main goal of this manuscript is to give an explicit representation formula for solutions to (1.1), which is in fact a generalization of (1.5). In the approach we follow in the sequel, the critical homogeneities and the transversality condition (1.2) play a fundamental role. We are now ready to prepare the setting of our main results.

A key role in the representation formula we are going to derive in section 4 is played by the spectrum of the angular component of the operator $\mathcal{L}_{\mathbf{A},a}$ on the unit $(N-1)$-dimensional sphere $\mathbb{S}^{N-1}$, i.e. of the operator

$$\begin{aligned} L_{\mathbf{A},a} &= \big(-i\,\nabla_{\mathbb{S}^{N-1}} + \mathbf{A}\big)^2 + a(\theta) \\ &= -\Delta_{\mathbb{S}^{N-1}} + \big(|\mathbf{A}|^2 + a(\theta) - i\ \mathrm{div}_{\mathbb{S}^{N-1}}\mathbf{A}\big) - 2i\,\mathbf{A}\cdot\nabla_{\mathbb{S}^{N-1}}. \end{aligned} \tag{1.11}$$

By classical spectral theory, $L_{\mathbf{A},a}$ admits a diverging sequence of real eigenvalues with finite multiplicity $\mu_1(\mathbf{A},a) \leqslant \mu_2(\mathbf{A},a) \leqslant \cdots \leqslant \mu_k(\mathbf{A},a) \leqslant \cdots$, see [19, Lemma A.5]. To each $k \in \mathbb{N}$, $k \geqslant 1$, we associate a $L^2\big(\mathbb{S}^{N-1},\mathbb{C}\big)$-normalized eigenfunction $\psi_k$ of the operator $L_{\mathbf{A},a}$ on $\mathbb{S}^{N-1}$ corresponding to the $k$-th eigenvalue $\mu_k(\mathbf{A},a)$, i.e. satisfying

$$\begin{cases} L_{\mathbf{A},a}\psi_k = \mu_k(\mathbf{A},a)\,\psi_k(\theta), & \text{in } \mathbb{S}^{N-1}, \\ \int_{\mathbb{S}^{N-1}} |\psi_k(\theta)|^2\,dS(\theta) = 1. \end{cases} \tag{1.12}$$

In the enumeration $\mu_1(\mathbf{A},a) \leqslant \mu_2(\mathbf{A},a) \leqslant \cdots \leqslant \mu_k(\mathbf{A},a) \leqslant \cdots$ we repeat each eigenvalue as many times as its multiplicity; thus exactly one eigenfunction $\psi_k$ corresponds to each index $k \in \mathbb{N}$. We can choose the functions $\psi_k$ in such a way that they form an orthonormal basis of $L^2(\mathbb{S}^{N-1},\mathbb{C})$. We also introduce the numbers

$$\alpha_k := \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k(\mathbf{A},a)}, \quad \beta_k := \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k(\mathbf{A},a)}, \tag{1.13}$$

so that $\beta_k = \frac{N-2}{2} - \alpha_k$, for $k = 1, 2, \ldots$, which will come into play in the sequel. Notice that $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \ldots$.

Under the condition

$$\mu_1(\mathbf{A},a) > -\left(\frac{N-2}{2}\right)^2 \tag{1.14}$$

the quadratic form associated to $\mathcal{L}_{\mathbf{A},a}$ is positive definite (see Section 2 below and the paper [19]), thus implying that the hamiltonian $\mathcal{L}_{\mathbf{A},a}$ is a symmetric semi-bounded operator on $L^2(\mathbb{R}^N;\mathbb{C})$ which then admits a self-adjoint extension (Friedrichs extension) with the natural form domain. As a consequence, under assumption (1.14) the unitary flow $e^{it\mathcal{L}_{\mathbf{A},a}}$ is well defined on the domain of $\mathcal{L}_{\mathbf{A},a}$ by Spectral Theorem; therefore, for every $u_0 \in L^2(\mathbb{R}^N;\mathbb{C})$, there exists a unique solution $u(\cdot,t) := e^{it\mathcal{L}_{\mathbf{A},a}}u_0 \in \mathcal{C}(\mathbb{R};L^2(\mathbb{R}^N))$ to (1.1) with $u(x,0) = u_0(x)$.

**Remark 1.1.** We notice that

$$u(\cdot,-s) := e^{-is\mathcal{L}_{\mathbf{A},a}}u_0 = \overline{e^{is\mathcal{L}_{-\mathbf{A},a}}\overline{u_0}};$$

henceforth, for the sake of simplify and without loss of generality, in the sequel we consider $u = u(x,t) : \mathbb{R}^N \times [0,+\infty) \to \mathbb{C}$.

The main theorem of the present paper provides a representation formula for such solution in terms of the following kernel

$$K(x,y) = \sum_{k=1}^{\infty} i^{-\beta_k} j_{-\alpha_k}(|x||y|)\psi_k\big(\tfrac{x}{|x|}\big)\overline{\psi_k\big(\tfrac{y}{|y|}\big)}, \tag{1.15}$$

where $\alpha_k, \beta_k$ are defined in (1.13) and, for every $\nu \in \mathbb{R}$,

$$j_\nu(r) := r^{-\frac{N-2}{2}} J_{\nu+\frac{N-2}{2}}(r)$$

with $J_\nu$ denoting the Bessel function of the first kind

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)}\left(\frac{t}{2}\right)^{2k}.$$

The following lemma provides uniform convergence on compacts of the queue of the series in (1.15).

**Lemma 1.2.** *Let $a \in L^\infty(\mathbb{S}^{N-1},\mathbb{R})$, $\mathbf{A} \in C^1(\mathbb{S}^{N-1},\mathbb{R}^N)$ such that (1.2) and (1.14) hold. Then there exists $k_0 \geqslant 1$ such that the series*

$$\sum_{k=k_0+1}^{\infty} i^{-\beta_k} j_{-\alpha_k}(|x||y|)\psi_k\big(\tfrac{x}{|x|}\big)\overline{\psi_k\big(\tfrac{y}{|y|}\big)}$$

*is uniformly convergent on compacts and*

$$K(x,y) - \sum_{k=1}^{k_0} i^{-\beta_k} j_{-\alpha_k}(|x||y|)\psi_k\big(\tfrac{x}{|x|}\big)\overline{\psi_k\big(\tfrac{y}{|y|}\big)} \in L^\infty_{\rm loc}(\mathbb{R}^{2N},\mathbb{C}).$$

We can now state the main result of this paper.

**Theorem 1.3** (Representation formula)**.** *Let $a \in L^\infty(\mathbb{S}^{N-1},\mathbb{R})$, $\mathbf{A} \in C^1(\mathbb{S}^{N-1},\mathbb{R}^N)$ such that (1.2) and (1.14) hold. Let $\mathcal{L}_{\mathbf{A},a}$ as in (1.3) and $K$ as in (1.15). If $u_0 \in L^2(\mathbb{R}^N)$ and $u(x,t) = e^{it\mathcal{L}_{\mathbf{A},a}}u_0(x)$, then, for all $t > 0$,*

$$u(x,t) = \frac{e^{\frac{i|x|^2}{4t}}}{i(2t)^{N/2}} \int_{\mathbb{R}^N} K\left(\frac{x}{\sqrt{2t}},\frac{y}{\sqrt{2t}}\right) e^{i\frac{|y|^2}{4t}} u_0(y)\,dy. \tag{1.16}$$

**Remark 1.4.** The integral at the right hand side of (1.16) is understood in the sense the improper multiple integrals, i.e.

$$u(x,t) = \frac{e^{\frac{i|x|^2}{4t}}}{i(2t)^{N/2}} \lim_{R\to+\infty} \int_{B_R} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) e^{i\frac{|y|^2}{4t}} u_0(y)\, dy,$$

where $B_R := \{y \in \mathbb{R}^N : |y| < R\}$.

**Remark 1.5.** Formula (1.16) is in fact a generalization of (1.5). Indeed, in the free case, i.e. $\mathbf{A} \equiv \mathbf{0}$ and $a \equiv 0$, the operator $L_{\mathbf{A},a}$ reduces to the Laplace Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$, whose eigenvalues are given by

$$\lambda_\ell = (N-2+\ell)\ell, \quad \ell = 0,1,2,\dots,$$

having the $\ell$-th eigenvalue $\lambda_\ell$ multiplicity

$$m_\ell = \frac{(N-3+\ell)!(N+2\ell-2)}{\ell!(N-2)!},$$

and whose eigenfunctions coincide with the usual spherical harmonics. For every $\ell \geqslant 0$, let $\{Y_{\ell,m}\}_{m=1,2,\dots,m_\ell}$ be a $L^2(\mathbb{S}^{N-1},\mathbb{C}^N)$-orthonormal basis of the eigenspace of $-\Delta_{\mathbb{S}^{N-1}}$ associated to $\lambda_\ell$ with $Y_{\ell,m}$ being spherical harmonics of degree $\ell$. Hence we have that

$$\text{if } k=1, \text{ then } \mu_1(\mathbf{0},0) = \lambda_0 = 0, \quad \alpha_1 = 0, \quad \beta_1 = \tfrac{N-2}{2},$$

$$\text{if } k>1 \text{ and } \textstyle\sum_{n=0}^{\ell-1} m_n < k \leqslant \sum_{n=0}^{\ell} m_n, \text{ then } \begin{cases} \mu_k(\mathbf{0},0) = \lambda_\ell \\ \alpha_k = -\ell \\ \beta_k = \frac{N-2}{2} + \ell \end{cases},$$

$$\{\psi_k\}_{k=1,2,\dots} = \{Y_{\ell,m}\}_{\substack{\ell=1,2,\dots \\ m=1,2,\dots,m_\ell}}.$$

The Jacobi-Anger expansion for plane waves combined with the Addition Theorem for spherical harmonics (see for example [28, formula (4.8.3), p. 116] and [6, Corollary 1]) yields

$$e^{ix\cdot y} = (2\pi)^{N/2}\big(|x||y|\big)^{-\frac{N-2}{2}} \sum_{\ell=0}^{\infty} i^\ell J_{\ell+\frac{N-2}{2}}\big(|x||y|\big)\left(\sum_{m=1}^{m_\ell} Y_{\ell,m}\big(\tfrac{x}{|x|}\big)\overline{Y_{\ell,m}\big(\tfrac{y}{|y|}\big)}\right)$$

for all $x, y \in \mathbb{R}^N$. Then in the free case $\mathbf{A} \equiv \mathbf{0}$, $a \equiv 0$, we have that

$$K(x,y) = \frac{e^{-ix\cdot y}}{(2\pi)^{\frac{N}{2}} i^{\frac{N-2}{2}}},$$

which, together with (1.16) and taking into account that if $t = -s$, $s > 0$, then $u(\cdot,-s) = \overline{e^{is(-\Delta)}\overline{u_0}}$, gives in turn (1.5).

**Remark 1.6.** We remark that, for every $y \in \mathbb{R}^N$ fixed, the function $K(\cdot,y)$ formally solves the equation

$$\mathcal{L}_{\mathbf{A},a} K(\cdot,y) = |y|^2 K(\cdot,y),$$

as one can easily check; in fact this fits with the free case, in which $K$ is the plane wave $e^{-ix\cdot y}$, up to constants.

Formula (1.16) is not present in the literature, at our knowledge; moreover, as far as we understand, it should provide a fundamental tool for several different applications. A first immediate consequence of the representation formula (1.16) is the following corollary.

**Corollary 1.7** (Time decay)**.** *Let $a \in L^\infty(\mathbb{S}^{N-1},\mathbb{R})$, $\mathbf{A} \in C^1(\mathbb{S}^{N-1},\mathbb{R}^N)$ such that* (1.2) *and* (1.14) *hold, and $\mathcal{L}_{\mathbf{A},a}$ as in* (1.3)*. If*

$$\sup_{x,y\in\mathbb{R}^N} |K(x,y)| < +\infty, \tag{1.17}$$

*with $K$ as in* (1.15)*, then the following estimate holds*

$$\left\|e^{it\mathcal{L}_{\mathbf{A},a}} f(\cdot)\right\|_{L^p} \leqslant \frac{C}{|t|^{N\left(\frac{1}{2}-\frac{1}{p}\right)}} \|f\|_{L^{p'}}, \quad p \in [2,+\infty], \quad \frac{1}{p}+\frac{1}{p'} = 1, \tag{1.18}$$

*for some $C = C(\mathbf{A},a,p) > 0$ which does not depend on $t$ and $f$.*

*Proof.* The proof is quite immediate. Formula (1.16), (1.17), and Remark 1.1 automatically yield (1.18) in the case $p = \infty$. The rest of the range $p \geqslant 2$ in (1.18) then follows by interpolation with the $L^2$ conservation. □

**Remark 1.8.** Once the matters to prove a time decay estimate are reduced to the study of the kernel $K$ in (1.15), the behavior of the spherical Bessel functions $j_{-\alpha_k}$ comes into play. The crucial fact to notice is that condition (1.17) is strictly related to the requirement $\alpha_1 \leqslant 0$ which in other words, by (1.13) means $\mu_1(A,a) \geqslant 0$. It is easy to verify that (1.17) implies that $\mu_1(A,a) \geqslant 0$, while arguing as in the proof of Lemma 1.2 we can easily check that $\mu_1(A,a) \geqslant 0$ implies that $K$ is locally bounded.

We are strongly motivated by the examples in the sequel to conjecture that in fact conditions (1.17) and $\mu_1(A,a) \geqslant 0$ are equivalent.

We now pass to give a couple of relevant examples in which the abstract assumption (1.17) can be checked by hands, and the optimal time decay can be obtained by working directly on the representation formula (1.16).

1.1. **Application 1: Aharonov-Bohm field.** We start with a 2D example of purely magnetic field, which is given by potentials associated to thin solenoids: if the radius of the solenoid tends to zero while the flux through it remains constant, then the particle is subject to a $\delta$-type magnetic field, which is called *Aharonov-Bohm* field. A vector potential associated to the Aharonov-Bohm magnetic field in $\mathbb{R}^2$ has the form

$$\boldsymbol{\mathcal{A}}(x_1,x_2) = \alpha\left(-\frac{x_2}{|x|^2},\frac{x_1}{|x|^2}\right), \quad (x_1,x_2) \in \mathbb{R}^2, \tag{1.19}$$

with $\alpha \in \mathbb{R}$ representing the circulation of $\boldsymbol{\mathcal{A}}$ around the solenoid. Notice that the potential in (1.19) is singular at $x = 0$, homogeneous of degree $-1$ and satisfies the transversality condition (1.2).

This situation corresponds to problem (1.1) with

$$N = 2, \quad \mathbf{A}(\theta) = \mathbf{A}(\cos t, \sin t) = \alpha(-\sin t, \cos t), \quad a(\theta) = 0,$$

so that equation (1.1) takes the form

$$iu_t = \left(-i\,\nabla + \alpha\left(-\frac{x_2}{|x|^2},\frac{x_1}{|x|^2}\right)\right)^2 u, \tag{1.20}$$

with $x = (x_1,x_2) \in \mathbb{R}^2$. In this case, an explicit calculation yields

$$\{\mu_k(\mathbf{A},0) : k \in \mathbb{N}\setminus\{0\}\} = \{(\alpha - j)^2 : j \in \mathbb{Z}\},$$

(see e.g. [34] for details), and in particular

$$\mu_1(\mathbf{A},0) = \big(\operatorname{dist}(\alpha,\mathbb{Z})\big)^2 \geqslant 0.$$

If $\operatorname{dist}(\alpha,\mathbb{Z}) \neq \frac{1}{2}$, then all the eigenvalues are simple and the eigenspace associated to the eigenvalue $(\alpha - j)^2$ is generated by $\psi(\cos t, \sin t) = e^{-ijt}$. If $\operatorname{dist}(\alpha,\mathbb{Z}) = \frac{1}{2}$, then all the eigenvalues have multiplicity 2. The following result is an interesting consequence of Theorem 1.3.

**Theorem 1.9** (Time decay for Aharonov-Bohm)**.** *Let $N = 2$, $\alpha \in \mathbb{R}$, and define*

$$\mathcal{L}_\alpha := \left(-i\,\nabla + \alpha\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)\right)^2.$$

*Then the following estimate holds*

$$\left\|e^{it\mathcal{L}_\alpha} f(\cdot)\right\|_{L^p} \leqslant \frac{C}{|t|^{2\left(\frac{1}{2}-\frac{1}{p}\right)}}\|f\|_{L^{p'}}, \quad p \in [2,+\infty], \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{1.21}$$

*for some constant $C = C(\alpha,p) > 0$ which does not depend on $t$ and $f$.*

**Remark 1.10.** Since $\operatorname{curl} A(x) \equiv 0$ if $x \neq 0$, the action of the magnetic field in the Aharonov-Bohm case is concentrated at the origin. However the potential $\mathbf{A}$ cannot be eliminated by gauge transformations; this is in fact the peculiar property of Aharonov-Bohm fields, which indeed describe an interesting difference between the classical and quantum version of the electromagnetic theory. Due to the above remark, estimates (1.21) are not trivial, as far as we understand, and at our knowledge they are not known. We finally stress that the algebraic structure of Aharonov-Bohm potentials is exactly the one which has been used in [18] in order to disprove the dispersion (in that case Strichartz inequalities) in the case of magnetic field which decay less than $|x|^{-1}$ at infinity, in dimension $N \geqslant 3$. In 2D, counterexamples as the ones in [18] are missing, and it is still unclear what might happen for potentials with less decay than the one in (1.20).

1.2. **Application 2: The inverse square potential.** We also present an application of formula (1.16) in the case of perturbation of the Laplace operator in dimension $N = 3$ with an inverse square electric potential; more precisely, we consider problem (1.1) with $\mathbf{A} = 0$ and $a(\theta) = a =$ constant, so that the hamiltonian (1.3) takes the form

$$\mathcal{L}_a := -\Delta + \frac{a}{|x|^2}, \quad \text{in } \mathbb{R}^3, \tag{1.22}$$

and condition (1.14) reads as $a > -\frac{1}{4}$. Since in this case the angular eigenvalue problem (1.11) becomes

$$\begin{cases} -\Delta_{\mathbb{S}^2}\psi_k = (\mu_k(\mathbf{0},a) - a)\psi_k, \text{ in } \mathbb{S}^2, \\ \|\psi_k\|_{L^2(\mathbb{S}^2)} = 1, \end{cases} \tag{1.23}$$

we have that $\{\psi_k\}_{k=1}^\infty$ are the well known spherical harmonics and

$$\mu_k(\mathbf{0},a) = a + \mu_k(\mathbf{0},0)\ .$$

As in Remark 1.5, for every $\ell \geqslant 0$, let $\{Y_{\ell,m}\}_{m=1,2,\dots,2\ell+1}$ be a $L^2(\mathbb{S}^{N-1},\mathbb{C}^N)$-orthonormal basis for the eigenspace of $-\Delta_{\mathbb{S}^2}$ associated to the $\ell$-th eigenvalue

$\lambda_\ell = \ell(\ell+1)$ of $-\Delta_{\mathbb{S}^2}$ (which has multiplicity $m_\ell = 2\ell+1$), with $Y_{\ell,m}$ being spherical harmonics of degree $\ell$. Hence we have that

$$\text{if } k=1, \text{ then } \mu_1(\mathbf{0},a) = a, \quad \alpha_1 = \tfrac{1}{2} - \sqrt{\tfrac{1}{4}+a}, \tag{1.24}$$

$$\text{if } k>1 \text{ and } \ell^2 < k \leqslant (\ell+1)^2, \text{ then } \begin{cases} \mu_k(\mathbf{0},a) = a + \ell(\ell+1) = \\ \alpha_k = \tfrac{1}{2} - \sqrt{\left(\ell+\tfrac{1}{2}\right)^2 + a} \end{cases}. \tag{1.25}$$

Notice that $\alpha_1 \leqslant 0$ if and only if $a \geqslant 0$. We define the well known zonal functions

$$Z^{(\ell)}_\theta(\theta') = \sum_{m=1}^{2\ell+1} Y_{\ell,m}(\theta)\overline{Y_{\ell,m}(\theta')}, \quad \theta,\theta' \in \mathbb{S}^2, \quad \ell = 0,1,2,\dots. \tag{1.26}$$

The study of the kernel in (1.15), which can be rewritten as

$$K(x,y) = \sum_{\ell=0}^{\infty} i^{-\sqrt{(\ell+1/2)^2+a}}\, j_{-\frac{1}{2}+\sqrt{(\ell+1/2)^2+a}}\big(|x||y|\big) Z^{(\ell)}_{\frac{x}{|x|}}\Big(\tfrac{y}{|y|}\Big), \tag{1.27}$$

permits to prove the following result.

**Theorem 1.11** (Time decay for inverse square potentials)**.** *Let $N=3$, $a > -\frac{1}{4}$, and define $\mathcal{L}_a$ by (1.22).*

i) *If $a \geqslant 0$, then the following estimates hold*

$$\left\| e^{it\mathcal{L}_a} f(\cdot)\right\|_{L^p} \leqslant \frac{C}{|t|^{3\left(\frac{1}{2}-\frac{1}{p}\right)}} \|f\|_{L^{p'}}, \quad p \in [2,+\infty], \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{1.28}$$

*for some constant $C = C(a,p) > 0$ which does not depend on $t$ and $f$.*

ii) *If $-\frac{1}{4} < a < 0$, let $\alpha_1$ as in (1.24), and define*

$$\|u\|_{p,\alpha_1} := \left(\int_{\mathbb{R}^3} (1+|x|^{-\alpha_1})^{2-p} |u(x)|^p \, dx\right)^{1/p}, \quad p \geqslant 1.$$

*Then the following estimates hold*

$$\left\| e^{it\mathcal{L}_a} f(\cdot)\right\|_{p,\alpha_1} \leqslant \frac{C(1+|t|^{\alpha_1})^{1-\frac{2}{p}}}{|t|^{3\left(\frac{1}{2}-\frac{1}{p}\right)}} \|f\|_{p',\alpha_1}, \quad p \geqslant 2, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{1.29}$$

*for some constant $C = C(a,p) > 0$ which does not depend on $t$ and $f$.*

**Remark 1.12.** As far as we know, the best dispersive results concerning this kind of operators are about Strichartz estimates, and have been obtained by Burq, Planchon, Stalker and Tahvildar-Zadeh in [7, 8]. As a fact, estimates (1.28) imply the ones obtained in [7], by the standard Ginibre-Velo and Keel-Tao techniques in [23, 33]. On the other hand, in [8] the authors can treat more general potentials with critical decay, including e.g. the cases in which $a = a(x/|x|)$ is a 0-degree homogeneous function; in addition, we think that the restriction $N=3$ in Theorem 1.11 is not in fact a relevant obstruction. We are motivated to claim that a deeper analysis of formula (1.16) should permit to prove the analog to Theorem 1.11 in the more general case $a = a(x/|x|)$, but this will not be treated in the present paper.

Moreover, notice that $\alpha_1 > 0$, in the range $-\frac{1}{4} < a < 0$, see (1.24), so that the decay in (1.29) is weaker than the usual one. We find it an interesting phenomenon, since on the other hand the usual Strichartz estimates are still true in this range, as proved in [7]. Estimates (1.29) are presumably sharp and, at our knowledge, new.

The rest of the paper is organized as follows. In Section 2, we describe the functional setting in which we work, in order to prepare the proof of the main result, Theorem 1.3; Section 3 is then devoted to the study of the spectral properties of a magnetic harmonic oscillator with inverse square potential, denoted by $T_{\mathbf{A},a}$ (see formula (2.11)), which comes into play when a suitable ansatz (formula (2.9)) is stated; finally, Section 4 is devoted to the proof of Theorem 1.3, while in the last Sections 5 and 6 we prove the applications, Theorems 1.9 and 1.11.

## 2. Functional setting

Let us define the following Hilbert spaces:

- the space $\mathcal{H}$ as the completion of $C^\infty_{\rm c}(\mathbb{R}^N\setminus\{0\},\mathbb{C})$ with respect to the norm
$$\|\phi\|_{\mathcal{H}}=\bigg(\int_{\mathbb{R}^N}\Big(|\nabla\phi(x)|^2+\Big(|x|^2+\frac{1}{|x|^2}\Big)|\phi(x)|^2\Big)\,dx\bigg)^{1/2};$$
- the space $H$ as the completion of $C^\infty_{\rm c}(\mathbb{R}^N,\mathbb{C})$ with respect to the norm
$$\|\phi\|_{H}=\bigg(\int_{\mathbb{R}^N}\Big(|\nabla\phi(x)|^2+\big(|x|^2+1\big)|\phi(x)|^2\Big)\,dx\bigg)^{1/2};$$
- the space $\mathcal{H}_{\mathbf{A}}$ as the completion of $C^\infty_{\rm c}(\mathbb{R}^N\setminus\{0\},\mathbb{C})$ with respect to the norm
$$\|\phi\|_{\mathcal{H}_{\mathbf{A}}}=\bigg(\int_{\mathbb{R}^N}\Big(|\nabla_{\mathbf{A}}\phi(x)|^2+\big(|x|^2+1\big)|\phi(x)|^2\Big)\,dx\bigg)^{1/2}$$
with $\nabla_{\mathbf{A}}\phi=\nabla\phi+i\,\frac{\mathbf{A}(x/|x|)}{|x|}\phi$.

From the above definition, it follows immediately that

$$\mathcal{H}\hookrightarrow H\quad\text{with continuous embedding.}\tag{2.1}$$

A further comparison between the above defined spaces can be derived from the well known diamagnetic inequality (see e.g. [36])

$$|\nabla|\phi|(x)|\leqslant\left|\nabla\phi(x)+i\frac{\mathbf{A}(x/|x|)}{|x|}\phi(x)\right|,\qquad N\geqslant 2,\tag{2.2}$$

which holds for a.e. $x\in\mathbb{R}^N$ and for all $\phi\in C^\infty_{\rm c}(\mathbb{R}^N\setminus\{0\},\mathbb{C})$, and the classical Hardy inequality (see e.g. [20, 27])

$$\int_{\mathbb{R}^N}|\nabla\phi(x)|^2\,dx\geqslant\Big(\frac{N-2}{2}\Big)^2\int_{\mathbb{R}^N}\frac{|\phi(x)|^2}{|x|^2}\,dx,\tag{2.3}$$

which holds for all $\phi\in\mathcal{C}^\infty_{\rm c}(\mathbb{R}^N,\mathbb{C})$ and $N\geqslant 3$. We notice that the presence of a vector potential satisfying a suitable non-degeneracy condition allows to recover a Hardy inequality even for $N=2$. Indeed, if $N=2$, (1.2) holds, and

$$\Phi_{\mathbf{A}}:=\frac{1}{2\pi}\int_0^{2\pi}\alpha(t)\,dt\notin\mathbb{Z},\quad\text{where }\alpha(t):=\mathbf{A}(\cos t,\sin t)\cdot(-\sin t,\cos t),\tag{2.4}$$

then functions in $C^\infty_{\rm c}(\mathbb{R}^N\setminus\{0\},\mathbb{C})$ satisfy the following Hardy inequality

$$\Big(\min_{k\in\mathbb{Z}}|k-\Phi_{\mathbf{A}}|\Big)^2\int_{\mathbb{R}^2}\frac{|u(x)|^2}{|x|^2}\,dx\leqslant\int_{\mathbb{R}^2}\left|\nabla u(x)+i\,\frac{\mathbf{A}\big(x/|x|\big)}{|x|}\,u(x)\right|^2dx\tag{2.5}$$

being $\big(\min_{k\in\mathbb{Z}}|k-\Phi_{\mathbf{A}}|\big)^2$ the best constant, as proved in [34].

Combining (2.3), (2.2), and (2.5), it is easy to verify that if $N \geqslant 3$, then $H = \mathcal{H} = \mathcal{H}_{\mathbf{A}}$, being the norms $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_{\mathbf{A}}}$ equivalent. If $N = 2$ then $\mathcal{H} \subsetneq H$; on the other hand, if $N = 2$ and (1.2), (2.4) hold, from (2.2) and (2.5) we deduce that $\mathcal{H} = \mathcal{H}_{\mathbf{A}}$, being the norms $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{\mathcal{H}_{\mathbf{A}}}$ equivalent.

From (2.1) and [32, Proposition 6.1], we also deduce that

$$\mathcal{H} \text{ is compactly embedded into } L^p(\mathbb{R}^N) \tag{2.6}$$

for all

$$2 \leqslant p < \begin{cases} 2^* = \frac{2N}{N-2}, & \text{if } N \geqslant 3, \\ +\infty, & \text{if } N = 2. \end{cases}$$

The quadratic form $Q_{\mathbf{A},a}$ associated to $\mathcal{L}_{\mathbf{A},a}$, i.e.

$$\begin{aligned} &Q_{\mathbf{A},a} : \mathcal{D}^{1,2}_*(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}, \\ &Q_{\mathbf{A},a}(\phi) := \int_{\mathbb{R}^N} \left[ \left|\nabla_{\mathbf{A}} \phi(x)\right|^2 - \frac{a\big(x/|x|\big)}{|x|^2} |\phi(x)|^2 \right] dx, \end{aligned} \tag{2.7}$$

with $\mathcal{D}^{1,2}_*(\mathbb{R}^N, \mathbb{C})$ being the completion of $C^\infty_{\rm c}(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})} := \left( \int_{\mathbb{R}^N} \left( \left|\nabla u(x)\right|^2 + \frac{|u(x)|^2}{|x|^2} \right) dx \right)^{1/2},$$

is positive definite if and only if (1.14) holds, see [19, Lemma 2.2]. In particular, assumption (1.14) ensures that the operator $\mathcal{L}_{\mathbf{A},a}$ is semibounded from below, self-adjoint on $L^2$ with the natural form domain, and that there exists some constant $C(N, \mathbf{A}, a) > 0$ such that

$$\int_{\mathbb{R}^N} \left[ \left|\nabla_{\mathbf{A}} \phi(x)\right|^2 - \frac{a\big(x/|x|\big)}{|x|^2} |\phi(x)|^2 + \frac{|x|^2}{4} |\phi(x)|^2 \right] dx \geqslant C(N, \mathbf{A}, a) \|\phi\|^2_{\mathcal{H}}, \tag{2.8}$$

for all $\phi \in \mathcal{H}$ (see [19]).

Up to a *pseudo conformal* change of variable, see [32], equation (1.1) can be rewritten in terms of a quantum harmonic oscillator with the singular electromagnetic potential, as stated in the following lemma.

**Lemma 2.1.** *Let (1.14) hold and $u \in C(\mathbb{R}; L^2(\mathbb{R}^N))$ be a solution to* (1.1). *Then*

$$\varphi(x,t) = (1+t^2)^{\frac{N}{4}} u\big(\sqrt{1+t^2}x, t\big) e^{-it\frac{|x|^2}{4}} \tag{2.9}$$

*satisfies*

$$\begin{aligned} &\varphi \in C(\mathbb{R}; L^2(\mathbb{R}^N)), \quad \varphi(x,0) = u(x,0), \\ &\|\varphi(\cdot,t)\|_{L^2(\mathbb{R}^N)} = \|u(\cdot,t)\|_{L^2(\mathbb{R}^N)} \textit{ for all } t \in \mathbb{R}, \end{aligned}$$

*and*

$$i\frac{d\varphi}{dt}(x,t) = \frac{1}{(1+t^2)} \left( \mathcal{L}_{\mathbf{A},a} \varphi(x,t) + \frac{1}{4}|x|^2 \varphi(x,t) \right). \tag{2.10}$$

A representation formula for solutions $u$ to (1.1) can be found by expanding the transformed solution $\varphi$ to (2.10) in Fourier series with respect to an orthonormal basis of $L^2(\mathbb{R}^N)$ consisting of eigenfunctions of the following quantum harmonic oscillator operator perturbed with singular homogeneous electromagnetic potentials

$$T_{\mathbf{A},a} : \mathcal{H} \to \mathcal{H}^\star, \quad T_{\mathbf{A},a} = \mathcal{L}_{\mathbf{A},a} + \frac{1}{4}|x|^2 \tag{2.11}$$

acting as

$$
\begin{aligned}
(2.12)\quad & {}_{\mathcal{H}^\star}\langle T_{\mathbf{A},a}v, w\rangle_{\mathcal{H}} \\
&= \int_{\mathbb{R}^N} \left( \nabla_{\mathbf{A}} v(x)\cdot \overline{\nabla_{\mathbf{A}} w(x)} - \frac{a(\frac{x}{|x|})}{|x|^2}\, v(x)\overline{w(x)} + \frac{|x|^2}{4} v(x)\overline{w(x)}\right) dx,
\end{aligned}
$$

for all $v, w \in \mathcal{H}$, where $\mathcal{H}^\star$ denotes the dual space of $\mathcal{H}$ and ${}_{\mathcal{H}^\star}\langle \cdot,\cdot\rangle_{\mathcal{H}}$ is the corresponding duality product.

## 3. The spectrum of $T_{\mathbf{A},a}$

From (2.6), (2.8), and classical spectral theory, we can easily deduce the following abstract description of the spectrum of $T_{\mathbf{A},a}$.

**Lemma 3.1.** *Let $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$ and $a \in L^\infty(\mathbb{S}^{N-1})$ such that (1.14) holds. Then the spectrum of the operator $T_{\mathbf{A},a}$ defined in (2.11–2.12) consists of a diverging sequence of real eigenvalues with finite multiplicity. Moreover, there exists an orthonormal basis of $L^2(\mathbb{R}^N)$ whose elements belong to $\mathcal{H}$ and are eigenfunctions of $T_{\mathbf{A},a}$.*

The following proposition gives a complete description of the spectrum of the operator $T_{\mathbf{A},a}$.

**Proposition 3.2.** *The set of the eigenvalues of the operator $T_{\mathbf{A},a}$ is*

$$\left\{\gamma_{m,k} : k, m \in \mathbb{N}, k \geqslant 1\right\}$$

*where*

$$
\gamma_{m,k} = 2m - \alpha_k + \frac{N}{2}, \quad \alpha_k = \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k(\mathbf{A}, a)}, \tag{3.1}
$$

*and $\mu_k(\mathbf{A}, a)$ is the $k$-th eigenvalue of the operator $L_{\mathbf{A},a}$ on the sphere $\mathbb{S}^{N-1}$. Each eigenvalue $\gamma_{m,k}$ has finite multiplicity equal to*

$$\#\left\{ j \in \mathbb{N}, j \geqslant 1 : \frac{\gamma_{m,k}}{2} + \frac{\alpha_j}{2} - \frac{N}{4} \in \mathbb{N}\right\}$$

*and a basis of the corresponding eigenspace is*

$$\left\{ V_{n,j} : j, n \in \mathbb{N}, j \geqslant 1, \gamma_{m,k} = 2n - \alpha_j + \frac{N}{2}\right\},$$

*where*

$$
V_{n,j}(x) = |x|^{-\alpha_j} e^{-\frac{|x|^2}{4}} P_{j,n}\Big(\frac{|x|^2}{2}\Big) \psi_j\Big(\frac{x}{|x|}\Big), \tag{3.2}
$$

*$\psi_j$ is an eigenfunction of the operator $L_{\mathbf{A},a}$ on the sphere $\mathbb{S}^{N-1}$ associated to the $j$-th eigenvalue $\mu_j(\mathbf{A}, a)$ as in (1.12), and $P_{j,n}$ is the polynomial of degree $n$ given by*

$$P_{j,n}(t) = \sum_{i=0}^{n} \frac{(-n)_i}{\left(\frac{N}{2} - \alpha_j\right)_i} \frac{t^i}{i!},$$

*denoting as $(s)_i$, for all $s \in \mathbb{R}$, the Pochhammer's symbol $(s)_i = \prod_{j=0}^{i-1}(s+j)$, $(s)_0 = 1$.*

*Proof.* Assume that $\gamma$ is an eigenvalue of $T_{\mathbf{A},a}$ and $g \in \mathcal{H} \setminus \{0\}$ is a corresponding eigenfunction, so that

$$\left(-i\,\nabla + \frac{\mathbf{A}\big(\frac{x}{|x|}\big)}{|x|}\right)^2 g(x) + \frac{a\big(\frac{x}{|x|}\big)}{|x|^2} g(x) + \frac{|x|^2}{4}\, g(x) = \gamma\, g(x) \tag{3.3}$$

in a weak $\mathcal{H}$-sense. From classical elliptic regularity theory, $g \in C^{1,\alpha}_{\rm loc}(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$. Hence $g$ can be expanded as

$$g(x) = g(r\theta) = \sum_{k=1}^{\infty} \phi_k(r)\psi_k(\theta) \quad \text{in } L^2(\mathbb{S}^{N-1}),$$

where $r = |x| \in (0, +\infty)$, $\theta = x/|x| \in \mathbb{S}^{N-1}$, and

$$\phi_k(r) = \int_{\mathbb{S}^{N-1}} g(r\theta)\overline{\psi_k(\theta)}\, dS(\theta).$$

Equations (1.12) and (3.3) imply that, for every $k$,

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(\gamma - \frac{\mu_k}{r^2} - \frac{r^2}{4}\right)\phi_k = 0 \quad \text{in } (0, +\infty). \tag{3.4}$$

Since $g \in \mathcal{H}$, we have that

$$\begin{aligned} \infty > \int_{\mathbb{R}^N} g^2(x)\, dx &= \int_0^{\infty} \left(\int_{\mathbb{S}^{N-1}} g^2(r\theta)\, dS(\theta)\right) r^{N-1}\, dr \\ &\geqslant \int_0^{\infty} r^{N-1}\phi_k^2(r)\, dr \end{aligned} \tag{3.5}$$

and

$$\infty > \int_{\mathbb{R}^N} \frac{g^2(x)}{|x|^2}\, dx \geqslant \int_0^{\infty} r^{N-3}\phi_k^2(r)\, dr. \tag{3.6}$$

For all $k = 1, 2, \ldots$ and $t > 0$, we define $w_k(t) = (2t)^{\frac{\alpha_k}{2}} e^{\frac{t}{2}} \phi_k(\sqrt{2t})$, with $\alpha_k$ as in (3.1). From (3.4), $w_k$ satisfies

$$t w_k''(t) + \left(\frac{N}{2} - \alpha_k - t\right) w_k'(t) - \left(\frac{N}{4} - \frac{\alpha_k}{2} - \frac{\gamma}{2}\right) w_k(t) = 0 \quad \text{in } (0, +\infty).$$

Therefore, $w_k$ is a solution of the well known Kummer Confluent Hypergeometric Equation (see [2] and [37]). Then there exist $A_k, B_k \in \mathbb{R}$ such that

$$w_k(t) = A_k M\Big(\frac{N}{4} - \frac{\alpha_k}{2} - \frac{\gamma}{2}, \frac{N}{2} - \alpha_k, t\Big) + B_k U\Big(\frac{N}{4} - \frac{\alpha_k}{2} - \frac{\gamma}{2}, \frac{N}{2} - \alpha_k, t\Big), \quad t \in (0, +\infty).$$

Here $M(c, b, t)$ and, respectively, $U(c, b, t)$ denote the Kummer function (or confluent hypergeometric function) and, respectively, the Tricomi function (or confluent hypergeometric function of the second kind); $M(c, b, t)$ and $U(c, b, t)$ are two linearly independent solutions to the Kummer Confluent Hypergeometric Equation

$$t w''(t) + (b - t) w'(t) - c w(t) = 0, \quad t \in (0, +\infty).$$

Since $\left(\frac{N}{2} - \alpha_k\right) > 1$, from the well-known asymptotics of $U$ at 0 (see e.g. [2]), we have that

$$U\Big(\frac{N}{4} - \frac{\alpha_k}{2} - \frac{\gamma}{2}, \frac{N}{2} - \alpha_k, t\Big) \sim \text{const}\, t^{1 - \frac{N}{2} + \alpha_k} \quad \text{as } t \to 0^+,$$

for some const $\neq 0$ depending only on $N, \gamma$, and $\alpha_k$. On the other hand, $M$ is the sum of the series

$$M(c,b,t) = \sum_{n=0}^{\infty} \frac{(c)_n}{(b)_n} \frac{t^n}{n!}.$$

We notice that $M$ has a finite limit at $0^+$, while its behavior at $\infty$ is singular and depends on the value $-c = -\frac{N}{4} + \frac{\alpha_k}{2} + \frac{\gamma}{2}$. If $-\frac{N}{4} + \frac{\alpha_k}{2} + \frac{\gamma}{2} = m \in \mathbb{N} = \{0, 1, 2, \cdots\}$, then $M\big(\frac{N}{4} - \frac{\alpha_k}{2} - \frac{\gamma}{2}, \frac{N}{2} - \alpha_k, t\big)$ is a polynomial of degree $m$ in $t$, which we will denote as $P_{k,m}$, i.e.,

$$P_{k,m}(t) = M\Big(-m, \tfrac{N}{2} - \alpha_k, t\Big) = \sum_{n=0}^{m} \frac{(-m)_n}{\big(\frac{N}{2} - \alpha_k\big)_n} \frac{t^n}{n!}.$$

If $\big(-\frac{N}{4} + \frac{\alpha_k}{2} + \frac{\gamma}{2}\big) \notin \mathbb{N}$, then from the well-known asymptotics of $M$ at $\infty$ (see e.g. [2]) we have that

$$M\Big(\frac{N}{4} - \frac{\alpha_k}{2} - \frac{\gamma}{2}, \frac{N}{2} - \alpha_k, t\Big) \sim \text{const}\, e^t t^{-\frac{N}{4} + \frac{\alpha_k}{2} - \frac{\gamma}{2}} \quad \text{as } t \to +\infty,$$

for some const $\neq 0$ depending only on $N, \gamma$, and $\alpha_k$.

Now, let us fix $k \in \mathbb{N}$, $k \geqslant 1$. From the above description, we have that

$$w_k(t) \sim \text{const}\, B_k t^{1 - \frac{N}{2} + \alpha_k} \quad \text{as } t \to 0^+,$$

for some const $\neq 0$, and hence

$$\phi_k(r) = r^{-\alpha_k} e^{-\frac{r^2}{4}} w_k\Big(\frac{r^2}{2}\Big) \sim \text{const}\, B_k r^{-(N-2)+\alpha_k} \quad \text{as } r \to 0^+,$$

for some const $\neq 0$. Therefore, condition (3.6) can be satisfied only for $B_k = 0$. If $\frac{\alpha_k}{2} + \frac{\gamma}{2} - \frac{N}{4} \notin \mathbb{N}$, then

$$w_k(t) \sim \text{const}\, A_k e^t t^{-\frac{N}{4} + \frac{\alpha_k}{2} - \frac{\gamma}{2}} \quad \text{as } t \to +\infty,$$

for some const $\neq 0$, and hence

$$\phi_k(r) = r^{-\alpha_k} e^{-\frac{r^2}{4}} w_k\Big(\frac{r^2}{4}\Big) \sim \text{const}\, A_k r^{-\frac{N}{2} - \gamma} e^{r^2/4} \quad \text{as } r \to +\infty,$$

for some const $\neq 0$. Therefore, condition (3.5) can be satisfied only for $A_k = 0$. If $\frac{\alpha_k}{2} + \frac{\gamma}{2} - \frac{N}{4} = m \in \mathbb{N}$, then $r^{-\alpha_k} e^{-\frac{r^2}{4}} P_{k,m}\big(\frac{r^2}{2}\big)$ solves (3.4); moreover the function

$$V_{m,k}(x) = |x|^{-\alpha_k} e^{-\frac{|x|^2}{4}} P_{k,m}\Big(\frac{|x|^2}{2}\Big) \psi_k\Big(\frac{x}{|x|}\Big)$$

belongs to $\mathcal{H}$, thus providing an eigenfunction of $L$. □

**Remark 3.3.** If $a(\theta) \equiv 0$, $\mathbf{A} \equiv \mathbf{0}$, the spectrum of $L_{\mathbf{0},0}$ is described in Remark 1.5, so that the spectrum of $T_{\mathbf{0},0}$ is $\frac{N}{2} + \mathbb{N}$. Hence, in this case we recover the eigenvalues of the Harmonic oscillator operator $-\Delta + \frac{|x|^2}{4}$ (see e.g. [32]).

**Remark 3.4.** It is easy to verify that

if $(m_1, k_1) \neq (m_2, k_2)$ then $V_{m_1,k_1}$ and $V_{m_2,k_2}$ are orthogonal in $L^2(\mathbb{R}^N)$.

By Lemma 3.1, it follows that

$$\Big\{ \widetilde{V}_{n,j} = \frac{V_{n,j}}{\|V_{n,j}\|_{L^2(\mathbb{R}^N)}} : j, n \in \mathbb{N}, j \geqslant 1 \Big\}$$

is an orthonormal basis of $L^2(\mathbb{R}^N)$.

**Remark 3.5.** Denoting by $L_m^\alpha(t)$ the generalized Laguerre polynomials

$$L_m^\alpha(t) = \sum_{n=0}^{m} (-1)^n \binom{m+\alpha}{m-n} \frac{t^n}{n!},$$

and $\beta_k = \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k(\mathbf{A}, a)}$ so that $\gamma_{m,k} = 2m + \beta_k + 1$, we can write

$$P_{k,m}\left(\frac{|x|^2}{2}\right) = M\Big(-\frac{\gamma_{m,k}}{2} + \frac{\beta_k}{2} + \frac{1}{2}, 1+\beta_k, \frac{|x|^2}{2}\Big) = \binom{m+\beta_k}{m}^{-1} L_m^{\beta_k}\Big(\frac{|x|^2}{2}\Big).$$

From the well known orthogonality relation

$$\int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(X)\, dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m},$$

where $\delta_{n,m}$ denotes the Kronecker delta, it is easy to check that

$$\|V_{m,k}\|^2_{L^2(\mathbb{R}^N)} = 2^{\beta_k}\Gamma(1+\beta_k)\binom{m+\beta_k}{m}^{-1}.$$

## 4. Proof of Theorem 1.3

The proof of Theorem 1.3 uses Lemma 1.2, which is proved below.

**Proof of Lemma 1.2.** From Theorem A.1 and Lemma A.2 in the Appendix, we deduce that there exist some $k_0 \in \mathbb{N}$, and $C_i > 0$, $i = 1, 2$, such that for every $k > k_0$,

$$-\alpha_k > C_1 k^{\delta_1}, \tag{4.1}$$

and

$$|\psi_k(\theta)| < C_2 k^{\delta_2} \quad \text{for all } \theta \in \mathbb{S}^{N-1}, \tag{4.2}$$

with $\delta_1 = \frac{1}{N-1}$ and $\delta_2 = \frac{2}{N-1}\lfloor\frac{N-1}{2}\rfloor$. From (4.1) and (4.2) it follows that, for all $k > k_0$, the $k$-th term of the series $K_k(x,y) = i^{-\beta_k} j_{-\alpha_k}(|x||y|)\psi_k\big(\frac{x}{|x|}\big)\overline{\psi_k\big(\frac{y}{|y|}\big)}$ belongs to $L^\infty_{\rm loc}(\mathbb{R}^{2N}, \mathbb{C})$. Furthermore, if we fix some compact set $\mathcal{K} \Subset \mathbb{R}^N$, there exists $k_0'$ such that $\Big|\frac{e|x||y|}{2(-\alpha_k+\frac{N}{2})}\Big| \leqslant \frac{1}{2}$ for every $x, y \in \mathcal{K}$ and $k > k_0'$. Therefore, for all $x, y \in \mathcal{K}$ and $k > k_0'$, we have that

$$\begin{aligned}
|K_k(x,y)| &\leqslant |j_{-\alpha_k}(|x||y|)| \Big|\psi_k\big(\tfrac{x}{|x|}\big)\Big| \Big|\psi_k\big(\tfrac{y}{|y|}\big)\Big| \\
&\leqslant Ck^{2\delta_2}\left(\frac{|x||y|}{2}\right)^{-\alpha_k} \sum_{m=0}^{\infty} \frac{1}{\Gamma(m+1)\Gamma(m-\alpha_k+\frac{N-2}{2}+1)}\left(\frac{|x||y|}{2}\right)^{2m} \\
&\leqslant Ck^{2\delta_2}\frac{\big(\frac{|x||y|}{2}\big)^{-\alpha_k}}{\Gamma(-\alpha_k+\frac{N}{2})} e^{\left(\frac{|x||y|}{2}\right)^2} \leqslant C' k^{2\delta_2}\left(\frac{e|x||y|}{2(-\alpha_k+\frac{N}{2})}\right)^{-\alpha_k} e^{\left(\frac{|x||y|}{2}\right)^2} \\
&\leqslant C'' k^{2\delta_2}\left(\frac{1}{2}\right)^{C_1 k^{\delta_1}} \equiv M_k
\end{aligned}$$

where $C''$ depends on $\mathcal{K}$ but not on $k$. Weierstrass M-test and convergence of $\sum_k M_k$ yields then the desired uniform convergence. $\square$

We are now ready to prove our main result, the representation formula given by Theorem 1.3.

**Proof of Theorem 1.3.** Let us expand the initial datum $u_0 = u(\cdot, 0) = \varphi(\cdot, 0)$ in Fourier series with respect to the orthonormal basis of $L^2(\mathbb{R}^N)$ introduced in Remark 3.4 as

$$u_0 = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} c_{m,k}\widetilde{V}_{m,k} \quad \text{in } L^2(\mathbb{R}^N), \quad \text{where } c_{m,k} = \int_{\mathbb{R}^N} u_0(x)\overline{\widetilde{V}_{m,k}(x)}\,dx, \tag{4.3}$$

and, for $t > 0$, the function $\varphi(\cdot, t)$ defined in (2.9) as

$$\varphi(\cdot, t) = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} \varphi_{m,k}(t)\widetilde{V}_{m,k} \quad \text{in } L^2(\mathbb{R}^N), \tag{4.4}$$

where

$$\varphi_{m,k}(t) = \int_{\mathbb{R}^N} \varphi(x,t)\overline{\widetilde{V}_{m,k}(x)}\,dx.$$

Since $\varphi(z, t)$ satisfies (2.10), we obtain that $\varphi_{m,k} \in C^1(\mathbb{R}, \mathbb{C})$ and

$$i\varphi'_{m,k}(t) = \frac{\gamma_{m,k}}{1+t^2}\varphi_{m,k}(t), \quad \varphi_{m,k}(0) = c_{m,k},$$

which by integration yields $\varphi_{m,k}(t) = c_{m,k}e^{-i\gamma_{m,k}\arctan t}$. Hence expansion (4.4) can be rewritten as

$$\varphi(z,t) = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} c_{m,k}e^{-i\gamma_{m,k}\arctan t}\widetilde{V}_{m,k}(z) \quad \text{in } L^2(\mathbb{R}^N), \qquad \text{for all } t > 0.$$

In view of (4.3), the above series can be written as

$$\varphi(z,t) = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} e^{-i\gamma_{m,k}\arctan t}\bigg(\int_{\mathbb{R}^N} u_0(y)\overline{\widetilde{V}_{m,k}(y)}\,dy\bigg)\widetilde{V}_{m,k}(z),$$

in the sense that, for all $t > 0$, the above series converges in $L^2(\mathbb{R}^N)$. Since $u_0(y)$ can be expanded as

$$u_0(y) = u_0\big(|y|\,\tfrac{y}{|y|}\big) = \sum_{j=1}^{\infty} u_{0,j}(|y|)\psi_j\big(\tfrac{y}{|y|}\big) \quad \text{in } L^2(\mathbb{S}^{N-1}),$$

where $u_{0,j}(|y|) = \int_{\mathbb{S}^{N-1}} u_0(|y|\theta)\overline{\psi_j(\theta)}\,dS(\theta)$, we conclude that

$$\begin{aligned}
\varphi(z,t) &= \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} \frac{e^{-i\gamma_{m,k}\arctan t}}{\|V_{m,k}\|_{L^2}^2}V_{m,k}(z)\bigg(\int_0^\infty u_{0,k}(r)r^{N-1-\alpha_k}P_{k,m}(\tfrac{r^2}{2})e^{-\frac{r^2}{4}}\,dr\bigg)\\
&= \sum_{k=1}^{\infty}\psi_k\big(\tfrac{z}{|z|}\big)\frac{e^{-i(\beta_k+1)\arctan t}}{2^{\beta_k}\Gamma(1+\beta_k)}\Bigg[\sum_{m=0}^{\infty}\frac{\binom{m+\beta_k}{m}}{e^{i2m\arctan t}}\times\\
&\quad\times\bigg(\int_0^\infty \frac{u_{0,k}(r)}{|rz|^{\alpha_k}}P_{k,m}\big(\tfrac{r^2}{2}\big)P_{k,m}\big(\tfrac{|z|^2}{2}\big)e^{-\frac{r^2+|z|^2}{4}}r^{N-1}dr\bigg)\Bigg].
\end{aligned}$$

By [2] we know that

$$P_{k,m}\Big(\frac{r^2}{2}\Big) = \frac{\Gamma(1+\beta_k)}{\Gamma(1+\beta_k+m)}e^{\frac{r^2}{2}}r^{-\beta_k}2^{\frac{\beta_k}{2}}\int_0^\infty e^{-t}t^{m+\frac{\beta_k}{2}}J_{\beta_k}(\sqrt{2}r\sqrt{t})\,dt,$$

where $J_{\beta_k}$ is the Bessel function of the first kind of order $\beta_k$. Therefore,

$$
\begin{aligned}
\varphi(z,t) = 4\sum_{k=1}^{\infty}\psi_k\big(\tfrac{z}{|z|}\big)\frac{\Gamma(1+\beta_k)}{e^{i(\beta_k+1)\arctan t}}\Bigg[\sum_{m=0}^{\infty}\binom{m+\beta_k}{m}\frac{e^{-i2m\arctan t}}{(\Gamma(1+\beta_k+m))^2}\times\\
\times\Big(\int_0^\infty\frac{u_{0,k}(r)}{|rz|^{\alpha_k+\beta_k}}e^{\frac{r^2+|z|^2}{4}}\Big(\int_0^\infty\!\!\int_0^\infty e^{-s^2-s'^2}(ss')^{2m+\beta_k+1}\times\\
\times J_{\beta_k}(\sqrt{2}rs)J_{\beta_k}(\sqrt{2}|z|s')\,ds\,ds'\Big)r^{N-1}dr\Big)\Bigg]\\
= 4\sum_{k=1}^{\infty}\psi_k\big(\tfrac{z}{|z|}\big)e^{-i(\beta_k+1)\arctan t}\Bigg[\int_0^\infty u_{0,k}(r)|rz|^{-\alpha_k-\beta_k}e^{\frac{r^2+|z|^2}{4}}r^{N-1}\times\\
\times e^{i(\arctan t+\frac{\pi}{2})\beta_k}\Bigg(\int_0^\infty\!\!\int_0^\infty\frac{ss'}{e^{s^2+s'^2}}\times\\
\times\Bigg(\sum_{m=0}^{\infty}\frac{(-1)^m e^{-i(\arctan t+\frac{\pi}{2})(2m+\beta_k)}}{\Gamma(1+m)\Gamma(1+\beta_k+m)}(ss')^{2m+\beta_k}\Bigg)\times\\
\times J_{\beta_k}(\sqrt{2}rs)J_{\beta_k}(\sqrt{2}|z|s')ds\,ds'\Bigg)dr\Bigg].
\end{aligned}
$$

Then, since $\sum\limits_{m=0}^{\infty}(-1)^m\frac{e^{-i(\arctan t+\frac{\pi}{2})(2m+\beta_k)}}{\Gamma(1+m)\Gamma(1+\beta_k+m)}(ss')^{2m+\beta_k} = J_{\beta_k}(2e^{-i(\arctan t+\frac{\pi}{2})}ss')$, we have

$$
\begin{aligned}
&\varphi(z,t)\\
&= 4\sum_{k=1}^{\infty}\psi_k\big(\tfrac{z}{|z|}\big)e^{i(\beta_k\frac{\pi}{2}-\arctan t)}\Bigg[\int_0^\infty u_{0,k}(r)|rz|^{-\frac{N-2}{2}}e^{\frac{r^2+|z|^2}{4}}\mathcal{I}_{k,t}(r,|z|)r^{N-1}dr\Bigg],
\end{aligned}
\tag{4.5}
$$

where

$$
\mathcal{I}_{k,t}(r,|z|) = \int_0^\infty\!\!\int_0^\infty ss'e^{-s^2-s'^2}J_{\beta_k}(2e^{-i(\arctan t+\frac{\pi}{2})}ss')J_{\beta_k}(\sqrt{2}rs)J_{\beta_k}(\sqrt{2}|z|s')\,ds\,ds'.
$$

From [51, formula (1), p. 395] (with $t=s'$, $p=1$, $a=\sqrt{2}|z|$, $b=2e^{-i(\arctan t+\frac{\pi}{2})}s$, $\nu=\beta_k$ which satisfy $\Re(\nu)>-1$ and $|\arg p|<\frac{\pi}{4}$), we know that

$$
\begin{aligned}
&\int_0^\infty s'e^{-s'^2}J_{\beta_k}(2e^{-i(\arctan t+\frac{\pi}{2})}ss')J_{\beta_k}(\sqrt{2}|z|s')\,ds'\\
&= \frac{1}{2}e^{-\frac{|z|^2+2e^{-i(2\arctan t+\pi)}s^2}{2}}I_{\beta_k}\Bigg(\frac{\sqrt{2}|z|s}{e^{i(\arctan t+\frac{\pi}{2})}}\Bigg),
\end{aligned}
$$

where $I_{\beta_k}$ denotes the modified Bessel function of order $\beta_k$. Hence

$$
\begin{aligned}
&\mathcal{I}_{k,t}(r,|z|)\\
&= \frac{1}{2}\int_0^\infty se^{-s^2}J_{\beta_k}(\sqrt{2}rs)e^{-\frac{|z|^2+2e^{-i(2\arctan t+\pi)}s^2}{2}}I_{\beta_k}(\sqrt{2}e^{-i(\arctan t+\frac{\pi}{2})}|z|s)\,ds\\
&= \frac{1}{4}\int_0^\infty se^{-\frac{s^2}{2}}J_{\beta_k}(rs)e^{-\frac{|z|^2+e^{-i(2\arctan t+\pi)}s^2}{2}}I_{\beta_k}(e^{-i(\arctan t+\frac{\pi}{2})}|z|s)\,ds.
\end{aligned}
$$

Since $I_\nu(x) = e^{-\frac{1}{2}\nu\pi i} J_\nu(xe^{\frac{\pi}{2}i})$ (see e.g. [2, 9.6.3, p. 375]), we obtain

$$\mathcal{I}_{k,t}(r,|z|) = \frac{1}{4} e^{-\frac{\beta_k}{2}\pi i} e^{-\frac{|z|^2}{2}} \int_0^\infty s e^{-\frac{s^2}{2}(e^{-i(2\arctan t+\pi)}+1)} J_{\beta_k}(rs) J_{\beta_k}(e^{-i\arctan t}|z|s)\, ds.$$

Applying [51, formula (1), p. 395] (with $t = s$, $p^2 = \frac{1+e^{-i(2\arctan t+\pi)}}{2}$, $a = r$, $b = e^{-i\arctan t}|z|$, $\nu = \beta_k$ which satisfy $\Re(\nu) > -1$ and $|\arg p| < \frac{\pi}{4}$) and [2, 9.6.3, p. 375], we obtain

$$\mathcal{I}_{k,t}(r,|z|) \tag{4.6}$$
$$= \frac{1}{4} e^{-\frac{\beta_k}{2}\pi i} e^{-\frac{|z|^2}{2}} \frac{1}{1+e^{-i(2\arctan t+\pi)}} e^{-\frac{r^2+|z|^2 e^{-2i\arctan t}}{2(1+e^{-i(2\arctan t+\pi)})}} I_{\beta_k}\left(\frac{r|z|e^{-i\arctan t}}{1+e^{-i(2\arctan t+\pi)}}\right)$$
$$= \frac{1}{4} e^{-\beta_k\pi i} e^{-\frac{|z|^2}{2}} \frac{1}{1+e^{-i(2\arctan t+\pi)}} e^{-\frac{r^2+|z|^2 e^{-2i\arctan t}}{2(1+e^{-i(2\arctan t+\pi)})}} J_{\beta_k}\left(i\frac{r|z|e^{-i\arctan t}}{1+e^{-i(2\arctan t+\pi)}}\right).$$

Noticing that

$$e^{-i\arctan t} = -\frac{i(t+i)}{\sqrt{1+t^2}}, \qquad \frac{1}{1+e^{-i(2\arctan t+\pi)}} = \frac{t-i}{2t},$$

from (4.5) and (4.6) we deduce

$$\varphi(z,t) \tag{4.7}$$
$$= \frac{e^{-i\arctan t}}{1+e^{-i(2\arctan t+\pi)}} \sum_{k=1}^\infty \psi_k\big(\tfrac{z}{|z|}\big) e^{-i\beta_k\frac{\pi}{2}} \Bigg[\int_0^\infty \frac{u_{0,k}(r)}{|rz|^{\frac{N-2}{2}}} e^{\frac{r^2}{2}\left(\frac{1}{2}-\frac{1}{1+e^{-i(2\arctan t+\pi)}}\right)} \times$$
$$\times\, e^{-\frac{|z|^2}{4}\left(1+\frac{2e^{-2i\arctan t}}{1+e^{-i(2\arctan t+\pi)}}\right)} J_{\beta_k}\left(i\frac{r|z|e^{-i\arctan t}}{1+e^{-i(2\arctan t+\pi)}}\right) r^{N-1}\, dr\Bigg]$$
$$= \frac{\sqrt{1+t^2}}{2ti} \sum_{k=1}^\infty \psi_k\big(\tfrac{z}{|z|}\big) e^{-i\beta_k\frac{\pi}{2}} \Bigg[\int_0^\infty \frac{u_{0,k}(r)}{|rz|^{\frac{N-2}{2}}} e^{-\frac{r^2}{4it}} e^{-\frac{|z|^2}{4it}} J_{\beta_k}\left(\frac{r|z|\sqrt{1+t^2}}{2t}\right) r^{N-1}\, dr\Bigg].$$

From (4.7) and (2.9) we get that for $t > 0$,

$$u(x,t) = (1+t^2)^{-\frac{N}{4}} e^{\frac{it|x|^2}{4(1+t^2)}} \varphi\left(\frac{x}{\sqrt{1+t^2}}, t\right) \tag{4.8}$$
$$= e^{-\frac{|x|^2}{4ti}} \frac{1}{2ti} |x|^{-\frac{N-2}{2}} \sum_{k=1}^\infty \psi_k\big(\tfrac{x}{|x|}\big) e^{-i\beta_k\frac{\pi}{2}} \left(\int_0^\infty u_{0,k}(r) e^{-\frac{r^2}{4ti}} J_{\beta_k}\left(\frac{r|x|}{2t}\right) r^{\frac{N}{2}}\, dr\right).$$

Notice that, by replacing $\int_0^\infty$ by $\int_0^R$ in (4.8) one obtains the series representation of the solution $u_R(x,t)$ with initial data $u_{0,R}(x) \equiv \chi_R(x)u_0(x)$ with $\chi_R(x)$ the characteristic function of the ball of radius $R$ centered at the origin. Since the evolution by Schrödinger equation is an isometry in $L^2$, we have that for all $t \in \mathbb{R}$ $\|u - u_R\|_{L^2}(t) = \|u_0 - u_{0,R}\|_{L^2} \to 0$, as $R \to \infty$. Hence $u(\cdot,t) = \lim_{R\to\infty} u_R(\cdot,t)$ in $L^2(\mathbb{R}^N)$. Since

$$u_{0,k}(r) = \int_{\mathbb{S}^{N-1}} u_0(r\theta)\overline{\psi_k(\theta)}\, dS(\theta),$$

and, by hypothesis, the queue of the series

$$K(x,y) = \sum_{k=1}^\infty e^{-i\beta_k\frac{\pi}{2}} \psi_k\big(\tfrac{x}{|x|}\big)\overline{\psi_k(\theta)} J_{\beta_k}\left(\frac{r|x|}{2t}\right)$$

is uniformly convergent on compacts, we can exchange integral and sum and write

$$
\begin{aligned}
u_R(x,t) = \frac{e^{-\frac{|x|^2}{4ti}}}{2ti}|x|^{-\frac{N-2}{2}} \int_0^R \int_{\mathbb{S}^{N-1}} u_0(r\theta) r^{\frac{N}{2}} e^{-\frac{r^2}{4ti}} \times \\
\times \left[ \sum_{k=1}^{\infty} e^{-i\beta_k \frac{\pi}{2}} \psi_k\big(\tfrac{x}{|x|}\big)\overline{\psi_k(\theta)} J_{\beta_k}\left(\frac{r|x|}{2t}\right)\right] dr\, dS(\theta) \\
= \frac{e^{\frac{i|x|^2}{4t}}}{i(2t)^{N/2}} \int_{B_R} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) e^{i\frac{|y|^2}{4t}} u_0(y)\, dy.
\end{aligned}
$$

Letting $R \to \infty$, we obtain (1.16) thus completing the proof of Theorem 1.3. □

## 5. Proof of Theorem 1.9

In view of Remark 1.1, it is enough to prove the stated estimate for $t > 0$. Moreover, thanks to Corollary 1.7, it is sufficient to prove condition (1.17), namely, uniform boundedness of

$$
K(x,y) = \frac{1}{(2\pi)^2} \sum_{j\in\mathbb{Z}} e^{-i|\alpha-j|\frac{\pi}{2}} e^{-ij \arctan\frac{x_2}{x_1}} e^{ij\arctan\frac{y_2}{y_1}} J_{|\alpha-j|}(|x||y|)
$$

which can be written as $K(x,y) = \frac{1}{(2\pi)^2} W\big(\arctan\frac{x_2}{x_1} - \arctan\frac{y_2}{y_1}, |x||y|\big)$ where

$$
W(z,s) = \sum_{j\in\mathbb{Z}} e^{-i|\alpha-j|\frac{\pi}{2}} e^{-ijs} J_{|\alpha-j|}(z).
$$

Notice that

$$
|\alpha - j| = \begin{cases} \alpha - j, & \text{if } j < \alpha, \\ j-\alpha, & \text{if } j > [\alpha] \equiv j_0, \end{cases}
$$

so that we can write

$$
\begin{aligned}
W(z,s) &= i^{-\alpha} \sum_{j<-|j_0|+1} i^j e^{-ijs} J_{\alpha-j}(z) \\
&\quad + \sum_{-|j_0|+1}^{|j_0|+1} e^{-i|\alpha-j|\frac{\pi}{2}} e^{-ijs} J_{|\alpha-j|}(z) + i^{\alpha} \sum_{j>|j_0|+1} i^{-j} e^{-ijs} J_{j-\alpha}(z) \\
&\equiv i^{-\alpha} S_1(z,s) + S_2(z,s) + i^{\alpha} S_3(z,s).
\end{aligned}
$$

$S_2(z,s)$ is clearly bounded. By using identity 9.1.27 in [2],

$$
J_\nu'(z) = \frac{1}{2}(J_{\nu-1}(z) - J_{\nu+1}(z)),
$$

we can compute

$$
\begin{aligned}
(5.1)\qquad \frac{d}{dz} S_1(z,s) &= \frac{1}{2} \sum_{j>|j_0|+1} i^{-j} e^{ijs} (J_{\alpha+j-1}(z) - J_{\alpha+j+1}(z)) \\
&= \frac{1}{2} \sum_{j>|j_0|} i^{-(j+1)} e^{i(j+1)s} J_{\alpha+j}(z) - \frac{1}{2} \sum_{j>|j_0|+2} i^{-(j-1)} e^{i(j-1)s} J_{\alpha+j}(z) \\
&= \frac{1}{2}\left(i^{-1}e^{is} J_{\alpha+|j_0|+1}(z) + J_{\alpha+|j_0|+2}(z)\right) i^{-|j_0|} e^{i|j_0|s} - (i\cos s)\, S_1(z,s),
\end{aligned}
$$

$$\begin{aligned}(5.2)\quad \frac{d}{dz}S_3(z,s) &= \frac{1}{2}\sum_{j>|j_0|+1} i^{-j}e^{-ijs}(J_{-\alpha+j-1}(z)-J_{-\alpha+j+1}(z))\\ &= \frac{1}{2}\sum_{j>|j_0|} i^{-(j+1)}e^{-i(j+1)s}J_{-\alpha+j}(z) - \frac{1}{2}\sum_{j>|j_0|+2} i^{-(j-1)}e^{-i(j-1)s}J_{-\alpha+j}(z)\\ &= \frac{1}{2}\left(i^{-1}e^{-is}J_{-\alpha+|j_0|+1}(z)+J_{-\alpha+|j_0|+2}(z)\right)i^{-|j_0|}e^{-i|j_0|s} - (i\cos s)\,S_3(z,s).\end{aligned}$$

Defining

$$F(z,s) = i^{-\alpha}S_1(z,s) + i^{\alpha}S_3(z,s),$$

we deduce from (5.1) and (5.2) that, for every $s$, $F(\cdot,s)$ satisfies the differential equation

$$\frac{d}{dz}F(z,s) + (i\cos s)\,F(z,s) = g(z,s), \tag{5.3}$$

with

$$\begin{aligned} g(z,s) =& \frac{i^{-\alpha}}{2}\left(i^{-1}e^{is}J_{\alpha+|j_0|+1}(z)+J_{\alpha+|j_0|+2}(z)\right)i^{-|j_0|-1}e^{i(|j_0|+1)s}\\ &+\frac{i^{\alpha}}{2}\left(i^{-1}e^{-is}J_{-\alpha+|j_0|+1}(z)+J_{-\alpha+|j_0|+2}(z)\right)i^{-|j_0|-1}e^{-i(|j_0|+1)s}.\end{aligned}$$

By integration of (5.3) we obtain that

$$F(z,s) = e^{-iz\cos s}\left(F(0,s)+\int_0^z e^{iz'\cos s}g(z',s)dz'\right).$$

Since $|j_0|+1\pm\alpha>0$, by the asymptotic behavior of Bessel functions close to the origin (see formula 9.1.7 in [2])

$$J_\nu(x) \simeq \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}$$

we conclude that $F(0,s)=0$, and hence

$$F(z,s) = \int_0^z e^{-i(z-z')\cos s}g(z',s)dz'.$$

Uniform (in $s$ and $z$) boundedness of $F(z,s)$ follows from uniform boundedness of the function

$$f(z,s) \equiv \int_0^z e^{iz'\cos s}\left(i^{-1}e^{is}J_\sigma(z')+J_{\sigma+1}(z')\right)dz'$$

for any $\sigma>0$. In order to prove it, we use the identity

$$J_\sigma(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\sigma\pi}{2}-\frac{\pi}{4}\right)+\xi_\sigma(z) \tag{5.4}$$

with

$$|\xi_\sigma(z)| \leqslant \frac{C_\sigma}{z^{\frac{1}{2}}(1+z)}, \tag{5.5}$$

which is a simple consequence of the asymptotic behavior of Bessel functions at infinity (see formula 9.2.1 in [2]). Therefore,

$$f(z,s)=\int_0^z \sqrt{\frac{2}{\pi z'}}e^{iz'\cos s}\left(e^{-i\frac{\pi}{2}}e^{is}\cos\left(z'-\frac{\sigma\pi}{2}-\frac{\pi}{4}\right)+\sin\left(z'-\frac{\sigma\pi}{2}-\frac{\pi}{4}\right)\right)dz'$$
$$+\int_0^z e^{iz'\cos s}(i^{-1}e^{is}\xi_\sigma(z')+\xi_{\sigma+1}(z'))dz'\equiv I_1(z)+I_2(z).$$

By (5.5), $I_2(z)$ is uniformly bounded. We notice now that

$$\sqrt{\frac{2}{\pi z'}}e^{iz'\cos s}\left(e^{-i\frac{\pi}{2}}e^{is}\cos\left(z'-\frac{\sigma\pi}{2}-\frac{\pi}{4}\right)+\sin\left(z'-\frac{\sigma\pi}{2}-\frac{\pi}{4}\right)\right)$$
$$=\frac{1}{2i}\sqrt{\frac{2}{\pi z'}}e^{iz'\cos s}\left(\left(e^{is}+1\right)e^{i\left(z'-\frac{\sigma\pi}{2}-\frac{\pi}{4}\right)}+\left(e^{is}-1\right)e^{-i\left(z'-\frac{\sigma\pi}{2}-\frac{\pi}{4}\right)}\right)$$

and since

$$\begin{aligned}\int_0^z \frac{e^{iz'(\cos s+1)}}{\sqrt{z'}}dz' &= \frac{1}{\sqrt{\cos s+1}}\int_0^{z(\cos s+1)}\frac{e^{iy}}{\sqrt{y}}dy\\ \int_0^z \frac{e^{iz'(\cos s-1)}}{\sqrt{z'}}dz' &= \frac{1}{\sqrt{1-\cos s}}\int_0^{z(1-\cos s)}\frac{e^{-iy}}{\sqrt{y}}dy\end{aligned}$$

and

$$\left|\frac{\left(e^{is}+1\right)}{\sqrt{\cos s+1}}\right|\leqslant C,\qquad \left|\frac{\left(e^{is}-1\right)}{\sqrt{1-\cos s}}\right|\leqslant C,$$

we conclude that $I_1(z)$ is uniformly bounded.

Therefore, $K(x,y)$ is uniformly bounded and then inequality (1.21) follows by Corollary 1.7. □

## 6. Proof of Theorem 1.11

In view of Remark 1.1, it is sufficient to consider the case $t>0$. Let $N=3$, $a>-\frac{1}{4}$, and $K$ as in (1.27). The proof of the theorem will follow from the following estimates for $K$:

$$\text{if } \alpha_1<0, \quad \text{then} \quad \sup_{x,y\in\mathbb{R}^3}|K(x,y)|<+\infty, \tag{6.1}$$

$$\text{if } \alpha_1>0, \quad \text{then} \quad \sup_{x,y\in\mathbb{R}^3}\frac{|K(x,y)|}{1+(|x||y|)^{-\alpha_1}}<+\infty. \tag{6.2}$$

Before proving the above estimates, let us show how (6.1) and (6.2) imply estimates (1.28) and (1.29) respectively, thus proving Theorem 1.11.

We notice that if $\alpha_1=0$ then $a=0$ and there is nothing to prove since in this case the result reduces to classical decay estimates for the free Schrödinger equation.

If $\alpha_1<0$, in view of (6.1) estimate (1.28) directly follows from Corollary 1.7.

If $\alpha_1>0$, then (6.2) and (1.16) imply that, for some $C>0$ (independent of $x$ and $t$),

$$|u(x,t)|\leqslant\frac{C}{t^{\frac{3}{2}}}\int_{\mathbb{R}^3}\left(1+\frac{|x|^{-\alpha_1}|y|^{-\alpha_1}}{t^{-\alpha_1}}\right)|u_0(y)|\,dy$$
$$=\frac{C}{t^{\frac{3}{2}}}\|u_0\|_{L^1(\mathbb{R}^3)}+\frac{C}{t^{\frac{3}{2}-\alpha_1}}\frac{1}{|x|^{\alpha_1}}\int_{\mathbb{R}^3}\frac{|u_0(y)|}{|y|^{\alpha_1}}dy,$$

for a.e. $x \in \mathbb{R}^3$ and all $t \geqslant 0$, which implies

$$\frac{|x|^{\alpha_1}}{1+|x|^{\alpha_1}}\,|u(x,t)| \leqslant C\frac{1+t^{\alpha_1}}{t^{\frac{3}{2}}}\int_{\mathbb{R}}\frac{1+|y|^{\alpha_1}}{|y|^{\alpha_1}}\,|u_0(y)|\,dy. \tag{6.3}$$

Let us introduce the weight function $w(y) = \big(\frac{1+|y|^{\alpha_1}}{|y|^{\alpha_1}}\big)^2$ and the weighted $L^p$ norm

$$\|v\|_{L^p_w} \equiv \begin{cases} \big(\int_{\mathbb{R}^3}|v(y)|^p w(y)dy\big)^{1/p}, & \text{if } 1 \leqslant p < +\infty,\\ \operatorname{ess\,sup}_{y\in\mathbb{R}^3}|v(y)|, & \text{if } p = +\infty.\end{cases}$$

$L^2$conservation and (6.3) yield the estimates

$$\left\|\frac{u(\cdot,t)}{\sqrt{w}}\right\|_{L^2_w} = \left\|\frac{u_0}{\sqrt{w}}\right\|_{L^2_w}, \quad \left\|\frac{u(\cdot,t)}{\sqrt{w}}\right\|_{L^\infty_w} \leqslant C\,\frac{1+t^{\alpha_1}}{t^{\frac{3}{2}}}\left\|\frac{u_0}{\sqrt{w}}\right\|_{L^1_w}.$$

Then, letting, for all $p > 2$,

$$\theta_p = 1-\frac{2}{p}, \quad p' = \frac{p}{p-1},$$

so that

$$\theta_p \in (0,1), \quad \frac{1}{p'} = \frac{1-\theta_p}{2}+\frac{\theta}{1}, \quad \frac{1}{p} = \frac{1-\theta_p}{2}+\frac{\theta}{\infty},$$

the Riesz-Thorin interpolation theorem yields

$$\left\|\frac{u(\cdot,t)}{\sqrt{w}}\right\|_{L^p_w} \leqslant C^{\theta_p}\left(\frac{1+t^{\alpha_1}}{t^{\frac{3}{2}}}\right)^{\theta_p}\left\|\frac{u_0}{\sqrt{w}}\right\|_{L^{p'}_w}$$

i.e.

$$\left(\int_{\mathbb{R}^3}|u(y,t)|^p\big(\tfrac{1+|y|^{\alpha_1}}{|y|^{\alpha_1}}\big)^{2-p}dy\right)^{1/p} \leqslant C^{1-\frac{2}{p}}\left(\frac{1+t^{\alpha_1}}{t^{\frac{3}{2}}}\right)^{1-\frac{2}{p}}\left(\int_{\mathbb{R}^3}|u_0(y)|^{p'}\big(\tfrac{1+|y|^{\alpha_1}}{|y|^{\alpha_1}}\big)^{2-p'}dy\right)^{1/p'}.$$

Hence, inequality (1.29) in Theorem 1.11 follows. Therefore, in order to prove the theorem, it is sufficient to prove estimates (6.1) and (6.2).

It is well known that the link between plane waves and a combination of zonal functions is given by the Jacobi-Anger expansion, combined with the addition theorem for spherical harmonics (see for example [51], [39] and the references therein). For $N = 3$, we get that

$$e^{-ix\cdot y} = 4\pi\sqrt{\frac{\pi}{2}}\sum_{\ell=0}^{\infty} i^{-\ell} j_\ell(|x||y|) Z^{(\ell)}_{x/|x|}(y/|y|). \tag{6.4}$$

We need to estimate the kernel $K$ in (1.15) which, as observed in (1.27), can be written as

$$K(x,y) = S\Big(|x||y|, \tfrac{x}{|x|}, \tfrac{y}{|y|}\Big)$$

where

$$S(r,\theta,\theta') = \sum_{\ell=0}^{\infty} i^{-b_\ell}\, j_{-a_\ell}(r) Z^{(\ell)}_\theta(\theta'),$$

with $b_\ell = \sqrt{(\ell+1/2)^2+a}$, $a_\ell = \frac{1}{2} - \sqrt{(\ell+1/2)^2+a}$. We split the sum into two terms

$$S(r,\theta,\theta') = \sum_{\ell=0}^{\ell_0-1} i^{-b_\ell} j_{-a_\ell}(r) Z_\theta^{(\ell)}(\theta') + \sum_{\ell=\ell_0}^{\infty} i^{-b_\ell} j_{-a_\ell}(r) Z_\theta^{(\ell)}(\theta') = S_1(r,\theta,\theta') + S_2(r,\theta,\theta'), \tag{6.5}$$

with $\ell_0 \geqslant 0$ such that $a_\ell > 0$ for any $\ell < \ell_0$ and $a_\ell < 0$ for any $\ell \geqslant \ell_0$ ($S_1$ is meant to be zero if $\ell_0 = 0$). Our goal is to show that the singularities in the Jacobi-Anger expansion are described by the first finite sum $S_1$ at the right-hand side of (6.5) while the second term $S_2$ at the right-hand side is uniformly bounded. Such boundedness for $S_2$ follows from the arguments below. We have that

$$\begin{aligned} S_2 &= \sum_{\ell=\ell_0}^{\infty} i^{-b_l} j_{-a_l}(r) Z_\theta^{(\ell)}(\theta') \\ &= \sum_{\ell=\ell_0}^{\infty} i^{-(\ell+\frac{1}{2})} j_\ell(r) Z_\theta^{(\ell)}(\theta') + \sum_{\ell=\ell_0}^{\infty} (i^{-b_\ell} j_{-a_\ell}(r) - i^{-(\ell+\frac{1}{2})} j_\ell(r)) Z_\theta^{(\ell)}(\theta') \\ &= i^{-\frac{1}{2}} \left[ (2\pi)^{-\frac{3}{2}} e^{-ir\theta\cdot\theta'} - \sum_{\ell=0}^{\ell_0-1} i^{-\ell} j_\ell(r) Z_\theta^{(\ell)}(\theta') \right] \\ &\quad + \sum_{\ell=\ell_0}^{\infty} \left( i^{-b_l} j_{-a_\ell}(r) - i^{-(\ell+\frac{1}{2})} j_\ell(r) \right) Z_\theta^{(\ell)}(\theta') . \end{aligned} \tag{6.6}$$

The first term at the right hand side of (6.6) is clearly bounded, since it is the difference between a plane wave and the first $(\ell_0 - 1)$ terms of its Jacobi-Anger expansion.

We first notice that the second term at the right hand side of (6.6) is bounded for $r \leqslant \delta$ if $\delta > 0$ is sufficiently small. Indeed from the estimates

$$\begin{aligned} |J_\nu(t)| &\leqslant \frac{1}{\Gamma(1+\nu)} \left(\frac{t}{2}\right)^\nu e^{t^2/4}, \quad \text{for all } \nu > 0,\ t \geqslant 0, \\ |Z_\theta^{(\ell)}(\theta')| &\leqslant Z_\theta^{(\ell)}(\theta) = \frac{2\ell+1}{4\pi}, \quad \text{for all } \ell \geqslant 0,\ \theta,\theta' \in \mathbb{S}^2, \end{aligned} \tag{6.7}$$

see for example [39], it follows that, if $r \leqslant \delta$,

$$\begin{aligned} \left| \sum_{\ell=\ell_0}^{\infty} i^{-b_\ell} j_{-a_\ell}(r) Z_\theta^{(\ell)}(\eta) \right| &\leqslant \sum_{\ell=\ell_0}^{\infty} \frac{2\ell+1}{4\pi\Gamma(b_\ell+1)} \frac{(\frac{r}{2})^{b_\ell}}{r^{\frac{1}{2}}} e^{r^2/4} \\ &\leqslant \frac{e^{\delta^2/4}}{\sqrt{2}\,4\pi} \sum_{\ell=\ell_0}^{\infty} \frac{2\ell+1}{\Gamma(b_\ell+1)} \left(\frac{r}{2}\right)^{-a_\ell} \leqslant \frac{e^{\delta^2/4}}{\sqrt{2}\,4\pi} \left(\frac{r}{2}\right)^{-a_{\ell_0}} \sum_{\ell=\ell_0}^{\infty} \frac{2\ell+1}{\Gamma(b_\ell+1)} \leqslant C r^{-a_{\ell_0}} \end{aligned}$$

for some constant $C > 0$ dependent on $\delta$ and $\ell_0$ but independent of $r, \theta, \theta'$.

Next, for $r > \delta$, we write

$$\begin{aligned} &\sum_{\ell=\ell_0}^{\infty} (i^{-b_\ell} j_{-a_\ell}(r) - i^{-(\ell+\frac{1}{2})} j_\ell(r)) Z_\theta^{(\ell)}(\theta') \\ &\quad = \frac{1}{2\pi i r^{\frac{1}{2}}} \int_\gamma e^{\frac{r}{2}(z-\frac{1}{z})} \left( \sum_{\ell=\ell_0}^{\infty} \left[ (iz)^{\ell+\frac{1}{2}-b_\ell} - 1 \right] \frac{Z_\theta^{(\ell)}(\theta')}{(iz)^{\ell+\frac{1}{2}}} \right) \frac{dz}{z}, \end{aligned} \tag{6.8}$$

where we have used the following representation for Bessel functions

$$J_\nu(r) = \frac{1}{2\pi i}\int_\gamma e^{\frac{r}{2}\left(z-\frac{1}{z}\right)}\frac{dz}{z^{\nu+1}},$$

with $\gamma$ being the positive oriented contour represented in Figure 1 (see [35, 5.10.7]). We have also exchanged sum and integral, which is allowed for any $r, \theta, \theta'$, as we will see below.

For convenience, we split the integral along $\gamma$ into the integrals $I_1$, along the circumference of radius 1 (to be denoted as $\Gamma_1$), and the integral $I_2$, along the lines running between $z = -\infty$ and $z = -1$ (to be denoted as $\Gamma_2$):

$$\int_\gamma = \int_{\Gamma_1} + \int_{\Gamma_2} \equiv I_1 + I_2.$$

Notice that, by analyticity of the integrand outside $z = \mathbb{R}^- + i0^\pm$, we can write

$$\int_\gamma = \int_{\Gamma_1^\varepsilon} + \int_{\Gamma_2^\varepsilon} = \lim_{\varepsilon\to 0^+}\left(\int_{\Gamma_1^\varepsilon} + \int_{\Gamma_2^\varepsilon}\right)$$

where $\Gamma_1^\varepsilon$ is the circumference of radius $1+\varepsilon$ around the origin and $\Gamma_2^\varepsilon$ runs along $(-\infty, -1-\varepsilon) + i0^\pm$. Notice that, for the integral along $\Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon$ since $|z| > 1$, one has absolute convergence for any given $r, \theta, \theta'$ and hence the exchange of integral and sum performed in formula (6.8) is allowed by Fubini's Theorem.

We start estimating the integral along $\Gamma_1$. Taking into account that

$$b_\ell - \left(\ell + \tfrac{1}{2}\right) = \sqrt{\left(\ell + \tfrac{1}{2}\right)^2 + a} - \tfrac{1}{2} - \ell = \tfrac{a}{2\ell+1} + O(\ell^{-3}),$$

we have that

$$\left[(iz)^{(\ell+\frac{1}{2})-b_\ell} - 1\right] = -\frac{a}{2\ell+1}\log(iz) + \frac{a^2}{2}\frac{(\log(iz))^2}{(2\ell+1)^2} + \frac{O(1)}{\ell^3} \tag{6.9}$$
$$\equiv J_{1,1}(z,\ell) + J_{1,2}(z,\ell) + J_{1,3}(z,\ell)$$

as $\ell \to +\infty$ uniformly with respect to $z \in \Gamma_1$. Since $z^{-b_\ell}$ and $z^{(\ell+\frac{1}{2})}$ have a branch-cut at $z \in \mathbb{R}^-$, the function $\log(iz)$ will also have a branch-cut at $z \in \mathbb{R}^-$, as well as the function $(iz)^{\frac{1}{2}}$ that will appear below. From (6.9) the contribution of the right hand side of ( 6.8) on $\Gamma_1$ can be written

$$I_1 = \frac{1}{2\pi i r^{\frac{1}{2}}}\int_{\Gamma_1} e^{\frac{r}{2}\left(z-\frac{1}{z}\right)}\left(\sum_{\ell=\ell_0}^\infty \Big(J_{1,1}(z,\ell) + J_{1,2}(z,\ell) + J_{1,3}(z,\ell)\Big)\frac{Z_\theta^{(\ell)}(\theta')}{(iz)^{\ell+\frac{1}{2}}}\right)\frac{dz}{z}$$
$$\equiv \mathcal{J}_{1,1} + \mathcal{J}_{1,2} + \mathcal{J}_{1,3}\ ,$$

where every summand $\mathcal{J}_{1,i}$, $i = 1, 2, 3$, corresponds to the integrand with the corresponding $J_{1,i}$. Since on $\Gamma_1$ we have that $|z| = 1$ and then $\left|e^{\frac{r}{2}\left(z-\frac{1}{z}\right)}\right| = 1$, from the estimate $\left|Z_\theta^{(\ell)}(\theta')\right| \leqslant \frac{2\ell+1}{4\pi}$ we deduce that, if $r > \delta$,

$$|\mathcal{J}_{1,3}| \leqslant \text{const}\, r^{-1/2}\sum_{\ell=\ell_0}^\infty \frac{2\ell+1}{\ell^3} \leqslant \text{const}\, \delta^{-1/2},$$

and hence $|\mathcal{J}_{1,3}|$ is bounded. Concerning $\mathcal{J}_{1,1}$, we notice that

$$-a\log(iz)\sum_{\ell=0}^{\infty}\frac{Z_{\theta}^{(\ell)}(\theta')}{2\ell+1}(iz)^{-\ell-\frac{1}{2}}=-\frac{a\log(iz)}{4\pi}\sum_{\ell=0}^{\infty}P_{\ell}(\theta\cdot\theta')(iz)^{-\ell-\frac{1}{2}} \tag{6.10}$$
$$=-\frac{1}{4\pi}\frac{a(iz)^{-\frac{1}{2}}\log(iz)}{\sqrt{1+2\theta\cdot\theta'\frac{i}{z}-\frac{1}{z^2}}}=-\frac{1}{4\pi}\frac{a(-iz)^{\frac{1}{2}}\log(iz)}{\sqrt{z^2+2iz(\theta\cdot\theta')-1}}$$

where we have used the well-known identity (see for example [39])

$$4\pi\frac{Z_{\theta}^{(\ell)}(\theta')}{2\ell+1}=P_{\ell}(\theta\cdot\theta'), \tag{6.11}$$

with $P_\ell$ being the Legendre polynomial of index $\ell$, and the identity (see Formula 22.9.12 in [2])

$$\sum_{\ell=0}^{\infty}P_{\ell}(t)w^{\ell}=\frac{1}{\sqrt{1-2wt+w^2}}, \tag{6.12}$$

which is valid for $|w|<1$. Hence, identity (6.10) is valid for $|z|>1$. Therefore

$$\mathcal{J}_{1,1}\sim(\textit{Bounded terms})-\frac{a}{8\pi^2 ir^{\frac{1}{2}}}\int_{\Gamma_1^{\varepsilon}}e^{\frac{r}{2}\left(z-\frac{1}{z}\right)}\frac{\log(iz)(-iz)^{\frac{1}{2}}}{\sqrt{z^2+2iz(\theta\cdot\theta')-1}}\frac{dz}{z}, \tag{6.13}$$

where the first term at the right-hand side of (6.13) represents a finite sum of terms, that are needed to complete the series (6.10) from $\ell=0$ to $\ell=\ell_0-1$, and which are uniformly bounded. Since $|e^{\frac{r}{2}(z-1/z)}|=1$ for all $r>0$ and $z\in\Gamma_1$, if $1-(\theta\cdot\theta')^2$ does not approach zero, i.e. if $\theta\cdot\theta'$ stays far away from $\pm1$, the second term at the right hand side of (6.13) is uniformly bounded with respect to $\varepsilon\to0^+$, $r>\delta$, and $1-(\theta\cdot\theta')^2>\delta$, due to the integrability of the two square root singularities of the integrand at $z_{\pm}=-i(\theta\cdot\theta')\pm\sqrt{1-(\theta\cdot\theta')^2}$.

When $(\theta\cdot\theta')=\mp1$, the two square root singularities at $z_\pm$ collapse into a stronger singularity at $z=\pm i$. Let us discuss e.g. the case $(\theta\cdot\theta')=-1$ (the case $(\theta\cdot\theta')=1$ can be treated similarly); then

$$\lim_{\varepsilon\to0^+}\int_{\Gamma_1^{\varepsilon}}e^{\frac{r}{2}\left(z-\frac{1}{z}\right)}\frac{(-iz)^{\frac{1}{2}}\log(iz)}{z-i}\frac{dz}{z} \tag{6.14}$$
$$=\pi^2 ie^{ir}+PV\int_{\Gamma_1}e^{\frac{r}{2}\left(z-\frac{1}{z}\right)}\frac{\log(iz)}{z-i}(-iz)^{\frac{1}{2}}\frac{dz}{z}.$$

Equation (6.14) is simply the Plemelj-Sokhotskyi formula (see for instance [1]) for the limit of Cauchy integrals when approaching a singular point. The first term at the right hand side of (6.14) is clearly bounded. The second term at the right hand side of (6.14) is a singular integral of the function $e^{\frac{r}{2}\left(z-\frac{1}{z}\right)}\frac{\log(iz)(-iz)^{1/2}}{z}$ which is differentiable for $z=e^{i\theta}$ with $\theta$ in the neighborhood of $\frac{\pi}{2}$ (remind that the discontinuity of the argument of $z$ is along the negative real line).

Hence, since the principal value of a Cauchy integral of a differentiable function is bounded (cf. [1]), we conclude the boundedness of $\mathcal{J}_{1,1}$ for any $r>\delta$. The fact that the principal value integral is bounded for any $r$ does not exclude the possibility of its diverging as $r\to\infty$. In order to exclude this possibility, we consider a neighborhood in $\Gamma_1$ of $z = i$:

$$\Gamma_1^{s_0} = \left\{ z = e^{i\left(\frac{\pi}{2}+s\right)},\ |s| < s_0 \ll 1 \right\}$$

and we integrate there for any $r \gg 1$ having into account that

$$\frac{(\pi + s)e^{i\frac{s}{2}}}{i(e^{is} - 1)} = -\frac{\pi}{s} + O(1) \quad \text{for } s \sim 0.$$

Hence,

$$\begin{aligned} PV \int_{-s_0}^{s_0} e^{ir\cos s} \frac{(\pi + s)e^{i\frac{s}{2}}}{i(e^{is} - 1)} ds &= PV \int_{-s_0}^{s_0} e^{ir\cos s}\big( -\frac{\pi}{s} + O(1)\big)\, ds \\ &= -PV \int_{-s_0}^{s_0} \pi \frac{e^{ir\cos s}}{s}\, ds + PV \int_{-s_0}^{s_0} O(1) e^{ir\cos s}\, ds. \end{aligned}$$

Since $PV \int_{-s_0}^{s_0} \pi \frac{e^{ir\cos s}}{s}\, ds = 0$, it follows that

$$PV \int_{-s_0}^{s_0} e^{ir\cos s} \frac{(\pi + s)e^{i\frac{s}{2}}}{i(e^{is} - 1)} ds = O(1).$$

Hence, the integral along $\Gamma_1^{s_0}$ is uniformly bounded for any $r$. The integral over $\Gamma_1 \backslash \Gamma_1^{s_0}$ is also uniformly bounded since $\left| e^{\frac{r}{2}\left(z - \frac{1}{z}\right)} \right| = 1$. Hence, the principal value integral over $\Gamma_1$ is uniformly bounded.

If one considers the two singularities $z_\pm$ sufficiently close, then, similarly to (6.14), the integral over $\Gamma_1$ can be written as

$$\int_{\Gamma_1} = \int_{Arc(z_-,z_+)} + \int_{\Gamma_1 \backslash Arc(z_-,z_+)} \tag{6.15}$$

where $Arc(z_-, z_+)$ is the small arc of $\Gamma_1$ between $z_+$ and $z_-$. The second integral at the right hand side of (6.15) can be easily estimated just like the principal value above and yields the same estimates uniformly in $\theta, \theta'$. The first term at the right hand side of (6.15) is, after writing

$$z_\pm = e^{i\left(\frac{\pi}{2} \pm \overline{s}\right)} = -i(\theta \cdot \theta') \pm \sqrt{1 - (\theta \cdot \theta')^2},$$

the integral of

$$\begin{aligned} &\frac{e^{ir\cos s}(s+\pi)e^{i\frac{s}{2}}}{\sqrt{\left(e^{i\left(s+\frac{\pi}{2}\right)} - e^{i\left(\frac{\pi}{2} - \overline{s}\right)}\right)\left(e^{i\left(s+\frac{\pi}{2}\right)} - e^{i\left(\overline{s}+\frac{\pi}{2}\right)}\right)}} \\ &= \frac{\pi e^{ir\cos s}}{\sqrt{|s+\overline{s}||s-\overline{s}|}}(1 + O(s - \overline{s}) + O(s + \overline{s})) \times \begin{cases} -1, & \text{if } s < -\overline{s} \\ +1 & \text{if } s > \overline{s} \\ -i & \text{if } -\overline{s} < s < \overline{s} \end{cases} \\ &= \nu(s) \frac{\pi e^{ir\cos s}}{\sqrt{|s+\overline{s}||s-\overline{s}|}} + O\left(\frac{1}{\sqrt{|s-\overline{s}|}}\right) + O\left(\frac{1}{\sqrt{|s+\overline{s}|}}\right), \end{aligned}$$

where $|\nu(s)| = 1$. As a consequence, we can estimate

$$\begin{aligned}\left|\int_{Arc(z_-,z_+)}\right| &= \left|\int_{-\bar{s}}^{\bar{s}} \frac{\pi e^{ir\cos s}}{\sqrt{\bar{s}^2 - s^2}} ds + O(1)\right| \\ &= \left|\int_{-1}^{1} \frac{\pi e^{ir\cos(\bar{s}t)}}{\sqrt{1-t^2}} dt + O(1)\right| \\ &\leqslant \text{const},\end{aligned}$$

uniformly with respect to $r$ and $\bar{s}$. Therefore, we conclude that the integral on $\Gamma_1$ is uniformly bounded both in $\delta$ and $r$.

Finally, the term $J_{1,2}$ in (6.9), inserted at the right hand side of (6.8), produces

$$\frac{1}{2\pi i r^{\frac{1}{2}}} \int_{\Gamma_1} e^{\frac{r}{2}(z-\frac{1}{z})} \frac{a^2(\log(iz))^2}{2} \left(\sum_{\ell=\ell_0}^{\infty} \frac{Z_\theta^{(\ell)}(\theta')}{(2\ell+1)^2} (iz)^{-\ell-\frac{1}{2}}\right) \frac{dz}{z}$$

where the series

$$F(\theta,\theta',z) = \sum_{\ell=0}^{\infty} \frac{Z_\theta^{(\ell)}(\theta')}{(2\ell+1)^2} (iz)^{-\ell-\frac{1}{2}} = \int g(\theta,\theta',z)\,dz$$

is the primitive in $z$ of the series

$$g(\theta,\theta',z) = -\frac{i}{2}\sum_{\ell=0}^{\infty} \frac{Z_\theta^{(\ell)}(\theta')}{2\ell+1} (iz)^{-\ell-\frac{3}{2}}.$$

Thus, using (6.11) and (6.12), we conclude that $F(\theta,\theta',z)$ is the primitive of

$$g(\theta,\theta',z) = -\frac{i}{8\pi} \frac{(iz)^{-\frac{3}{2}}}{\sqrt{1 + 2(\theta\cdot\theta')\frac{i}{z} - \frac{1}{z^2}}}$$

and hence, since $g(\theta,\theta',z)$ presents a square root singularity if $\theta\cdot\theta' \neq -1$ or a $1/(z-i)$ singularity at $z = i$ if $\theta\cdot\theta' = -1$, we conclude that $F(\theta,\theta',z)$ presents at most a log-type singularity, which is integrable, and consequently the integral yields a uniformly bounded contribution $\mathcal{J}_{1,2}$. Therefore, we conclude that $I_1$ is uniformly bounded.

We continue estimating $I_2$,

$$I_2 = \frac{1}{2\pi i r^{\frac{1}{2}}} \int_{\Gamma_2} e^{\frac{r}{2}(z-\frac{1}{z})} \left(\sum_{\ell=\ell_0}^{\infty} \left[(iz)^{\ell+\frac{1}{2}-b_\ell} - 1\right] \frac{Z_\theta^{(\ell)}(\theta')}{(iz)^{\ell+\frac{1}{2}}}\right) \frac{dz}{z}.$$

Introducing the changes of variables, $z = e^{\pm\pi i}e^t$, exchanging sum and integral (arguing as above) and rearranging terms, we rewrite it in the form

$$I_2 = \frac{1}{2\pi i r^{\frac{1}{2}}} \sum_{\ell=\ell_0}^{\infty} Z_\theta^{(\ell)}(\theta')\left(A_\ell(r) + B_\ell(r)\right) \equiv \frac{1}{2\pi i r^{\frac{1}{2}}}\left(\mathcal{J}_{2,1} + \mathcal{J}_{2,2}\right)$$

where

$$A_\ell(r) = -\frac{2\sin(\pi b_\ell)}{i^{b_\ell - 1}} \int_0^\infty e^{-r\sinh t}(e^{-b_\ell t} - e^{-(\ell+\frac{1}{2})t})dt$$

and

$$B_\ell(r) = -2i\int_0^\infty e^{-r\sinh t} e^{-(\ell+\frac{1}{2})t}\left(\frac{\sin\pi b_\ell}{i^{b_\ell}} - \frac{\sin\pi(\ell+\frac{1}{2})}{i^{\ell+\frac{1}{2}}}\right) dt.$$

We estimate $A_\ell$ by using again that $|Z_\theta^{(\ell)}(\theta')| \leqslant \frac{2\ell+1}{4\pi}$,

$$|\mathcal{J}_{2,1}| = \left|\sum_{\ell=\ell_0}^{\infty} Z_\theta^{(\ell)}(\theta')\, A_\ell(r)\right| \leqslant C \sum_{\ell=\ell_0}^{\infty}(2\ell+1)\int_0^\infty e^{-r\sinh t}\left|e^{-b_\ell t} - e^{-(\ell+\frac{1}{2})t}\right| dt$$

$$\leqslant C\sum_{\ell_0}^{\infty}(2\ell+1)\left|\frac{1}{b_\ell} - \frac{1}{\ell+\frac{1}{2}}\right| \leqslant C.$$

In order to estimate $B_\ell$, notice that

$$\left(\frac{\sin\pi b_\ell}{i^{b_\ell}} - \frac{\sin\pi(\ell+\frac{1}{2})}{i^{\ell+\frac{1}{2}}}\right)$$

$$= \frac{1}{i^{\ell+\frac{1}{2}}}\left[\sin\pi b_\ell - \sin\pi(\ell+\frac{1}{2})\right] - \sin\pi b_\ell\left[\frac{1}{i^{\ell+\frac{1}{2}}} - \frac{1}{i^{b_\ell}}\right]$$

$$= -\frac{(-1)^\ell}{i^{\ell+\frac{1}{2}}}\frac{1}{2}\left(\frac{a\pi}{2\ell+1}\right)^2 - \frac{(-1)^\ell}{i^{\ell+\frac{1}{2}}}\left(\frac{a\pi}{2(2\ell+1)}i + \frac{a^2\pi^2}{8(2\ell+1)^2}\right) + O(\ell^{-3})$$

$$= -\frac{i^{\ell+\frac{1}{2}}}{2\ell+1}\frac{a\pi}{2} - \frac{5i^{\ell-\frac{1}{2}}}{8}\left(\frac{a\pi}{(2\ell+1)}\right)^2 + O(\ell^{-3}).$$

By using formula (6.12), it readily follows

$$|\mathcal{J}_{2,2}| = \left|\sum_{\ell=\ell_0}^{\infty} Z_\theta^{(\ell)}(\theta')\, B_\ell(r)\right| \leqslant \Bigg|\int_0^\infty e^{-r\sinh t}e^{-\frac{t}{2}} \times$$

$$\times\Bigg[K_{\ell_0}(r,\theta,\theta',t) - i^{\frac{1}{2}}\frac{a\pi}{4}\frac{1}{\sqrt{1-ie^{-t}(\theta\cdot\theta') - e^{-2t}}}$$

$$-\sum_{\ell=\ell_0}^{\infty}\frac{5a^2\pi^2 i^{\ell-\frac{1}{2}}e^{-\ell t}Z_\theta^{(\ell)}(\theta')}{16}\left(\frac{1}{(2\ell+1)^2} + O(\ell^{-3})\right)\Bigg]\Bigg|,$$

where $K_{\ell_0}$ accounts for the terms that need to be added in order to use (6.12) and which is uniformly bounded.

Since $\sqrt{|1-ie^{-t}(\theta\cdot\theta') - e^{-2t}|} \geqslant \frac{\text{const}\sqrt{t}}{1+\sqrt{t}}$ for some const $>0$ and, using again that $|Z_\theta^{(\ell)}(\theta')| \leqslant \frac{2\ell+1}{4\pi}$,

$$\left|\sum_{\ell=\ell_0}^{\infty} i^\ell e^{-\ell t}Z_\theta^{(\ell)}(\theta')\left(\frac{1}{(2\ell+1)^2} + O(\ell^{-3})\right)\right| \leqslant C\sum_{\ell=\ell_0}^{\infty}\frac{e^{-\ell t}}{\ell} \leqslant 2C\log|t|$$

for some constat $C$ and any $\theta,\theta'$, we conclude the existence of other constants $C', C''$ such that

$$|\mathcal{J}_{2,2}| \leqslant \frac{C'}{4}\int_0^\infty e^{-r\sinh t}e^{-\frac{t}{2}}\left[\frac{1+\sqrt{t}}{\sqrt{t}} + \log|t|\right] \leqslant C''.$$

Hence the uniform boundedness of $\mathcal{J}_{2,2}$ follows. We conclude then

$$\sup_{\substack{r\geqslant 0\\ \theta,\theta'\in\mathbb{S}^{N-1}}} |S_2(r,\theta,\theta')| < +\infty. \tag{6.16}$$

If $\alpha_1 = a_0 < 0$, then $\ell_0 = 0$. Hence $S_1(r,\theta,\theta') = 0$ and (6.1) is proved.

If $\alpha_1 = a_0 > 0$, then $\ell_0 > 0$. From (6.7) and the fact that $a_\ell \leqslant a_0$ for all $\ell \in \mathbb{N}$, we deduce that

$$|S_1(r,\theta,\theta')| \leqslant \text{const}\, r^{-a_0} = \text{const}\, r^{-\alpha_1} \quad \text{for all } r \leqslant 1,\ \theta,\theta' \in \mathbb{S}^{N-1}. \tag{6.17}$$

On the other hand, from (5.4) and (5.5) we easily deduce that

$$|S_1(r,\theta,\theta')| \leqslant \text{const} \quad \text{for all } r \geqslant 1,\ \theta,\theta' \in \mathbb{S}^{N-1}. \tag{6.18}$$

Estimate (6.2) follows from (6.5), (6.16), (6.17), and (6.18). □

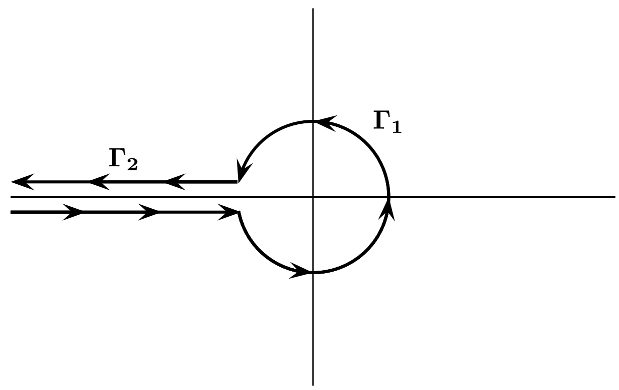


FIGURE 1. Integration oriented domain $\gamma$.

## APPENDIX

The asymptotic behavior of eigenvalues $\mu_k(\mathbf{A},a)$ as $k \to +\infty$ is described by Weyl's law, which is recalled in the theorem below. We refer to [41, 44] for a proof.

**Theorem A.1 (Weyl's law).** *For $a \in L^\infty(\mathbb{S}^{N-1},\mathbb{R})$ and $\mathbf{A} \in C^1(\mathbb{S}^{N-1},\mathbb{R}^N)$, let $\{\mu_k(\mathbf{A},a)\}_{k\geqslant 1}$ be the eigenvalues of the operator $L_{\mathbf{A},a} = \big(-i\,\nabla_{\mathbb{S}^{N-1}} + \mathbf{A}\big)^2 + a(\theta)$. Then*

$$\mu_k(\mathbf{A},a) = C(N,\mathbf{A},a)k^{2/(N-1)}\big(1+o(1)\big) \quad \text{as } k \to +\infty, \tag{A.19}$$

*for some positive constant $C(N,\mathbf{A},a)$ depending only on $N$, $\mathbf{A}$, and $a$.*

The following lemma provides an estimate of the $L^\infty$-norm of eigenfunctions of the operator $L_{\mathbf{A},a}$ in terms of the corresponding eigenvalues.

**Lemma A.2.** *For $a \in L^\infty(\mathbb{S}^{N-1},\mathbb{R})$, $\mathbf{A} \in C^1(\mathbb{S}^{N-1},\mathbb{R}^N)$, and $k \in \mathbb{N}\setminus\{0\}$, let $\psi_k$ be a $L^2$-normalized eigenfunction of the Schrödinger operator $L_{\mathbf{A},a}$ on the sphere associated to the $k$-th eigenvalue $\mu_k(\mathbf{A},a)$, i.e. satisfying (1.12). Then, there exists a constant $\widetilde{C}$ depending only on $N$, $\mathbf{A}$, and $a$ such that*

$$|\psi_k(\theta)| \leqslant \widetilde{C}\,|\mu_k|^{\lfloor (N-1)/2 \rfloor},$$

*where $\lfloor\cdot\rfloor$ denotes the floor function, i.e. $\lfloor x \rfloor := \max\{j \in \mathbb{Z} :\ j \leqslant x\}$.*

PROOF. Using classical elliptic regularity theory and bootstrap methods, we can easily prove that for any $j \in \mathbb{N}$ there exists a constant $C(N,\mathbf{A},a,j)$, depending only on $j$, $\mathbf{A}$, $a$, and $N$ but independent of $k$, such that, for large $k$,

$$\|\psi_k\|_{W^{2,\frac{2(N-1)}{(N-1)-2(j-1)}}(\mathbb{S}^{N-1})} \leqslant C(N,\mathbf{A},a,j)\big(\mu_k(\mathbf{A},a)\big)^j.$$

Choosing $j = \lfloor \frac{N-1}{2} \rfloor$, by Sobolev's inclusions we deduce that

$$W^{2,\frac{2(N-1)}{(N-1)-2(j-1)}}(\mathbb{S}^{N-1}) \hookrightarrow C^{0,\alpha}(\mathbb{S}^{N-1}) \hookrightarrow L^\infty(\mathbb{S}^{N-1}),$$

for any $0 < \alpha < 1 - \frac{N-1}{2} + \lfloor \frac{N-1}{2} \rfloor$, thus implying the required estimate. □


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