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FIXED POINTS AND CHAOTIC DYNAMICS FOR EXPANSIVE-CONTRACTIVE MAPS IN EUCLIDEAN SPACES, WITH SOME APPLICATIONS

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Introduction

In this work we introduce a topological method for the search of fixed points and periodic points for continuous maps defined on generalized rectangles in finite dimensional Euclidean spaces. We name our technique "Stretching Along the Paths" method, since we deal with maps that expand the arcs along one direction. Such theory was developed in the planar case by Papini and Zanolin in [117, 118] and it has been extended to the N-dimensional framework by the author and Zanolin in [124]. In the bidimensional setting, elementary theorems from plane topology suffice, while in the higher dimension some results from degree theory are needed, leading to the study of the so-called "Cutting Surfaces" [124]. In the past decade a big effort has been addressed towards the search of fixed and periodic points. In this respect several authors [11, 12, 56, 108, 130, 148, 150, 155, 167, 172, 176] have studied maps expansive along some directions and compressive along other ones, obtaining results that are also significant from a dynamical point of view, in particular in relation to Markov partitions. These achievements usually require sophisticated algebraic tools, like the Conley index or the Lefschetz number. Our approach, even if confined to a special configuration, looks instead more elementary. The description of the Stretching Along the Paths method and suitable variants of it can be found in Chapter 1.

In Chapter 2 we discuss which are the chaotic features that can be obtained for a given map when our technique applies. In particular, we are able to prove the semi-conjugacy to the Bernoulli shift and thus the positivity of the topological entropy, the presence of topological transitivity and sensitivity with respect to initial conditions, the density of periodic points. Moreover we show the mutual relationships among various classical notions of chaos (such as that by Devaney, Li-Yorke, etc.). We also introduce an alternative geometrical framework related to the so-called "Linked Twist Maps", where it is possible to employ our method in order to detect complex dynamics. Chapter 2 is as self-contained as possible and hence accessible also to the reader who is not familiar with the concept of chaos. Due to the large quantity of definitions available in the literature, our exposition is thought as a brief survey, where a

comparison with the notion of chaos in the sense of coin-tossing is presented, too. Such characterization is indeed natural when considered in relation with the approach in Chapter 1.

The theoretical results obtained so far find an application to discrete- and continuous-time systems in Chapters 3 and 4. Namely, in Chapter 3 we deal with some one-dimensional and planar discrete economic models, both of the Overlapping Generations and of the Duopoly Games classes. The bidimensional models are taken from [104, 135] and [2], respectively. On the other hand, in Chapter 4 we analyze some nonlinear ODEs with periodic coefficients. In more details, we consider a modified version of the Volterra predator-prey model, in which a periodic harvesting is included, as well as a simplification of the Lazer-McKenna suspension bridges model [88, 89] from [120, 121]. When dealing with ODEs with periodic coefficients, our method is applied to the associated Poincaré map and thus we are led back to work with discrete dynamical systems.

The above summary is just meant to mention the main aspects of this work. Each chapter is indeed equipped with a more detailed introduction, including the corresponding bibliography.

The contents of the present thesis are based on the papers [105, 120, 124, 125, 126] and partially on [123], where maps expansive along several directions were considered.

Notations

For the reader's convenience, we introduce some basic notations that will be used throughout this work.

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of *natural*, *integer*, *rational*, *real* and *complex numbers*, respectively. In particular $\mathbb{N}$ is the set of nonnegative integers, while $\mathbb{N}_0$ denotes the set of the positive integers. The sets of nonnegative and positive real numbers will be indicated with $\mathbb{R}^+$ and $\mathbb{R}_0^+$. Accordingly, the first quadrant and the open first quadrant will be denoted by $(\mathbb{R}^+)^2$ and $(\mathbb{R}_0^+)^2$, respectively.

For a subset M of the topological space W, we denote by $\overline{M}$, $\text{Int}(M)$ and ∂M the *closure*, the *interior* and the *boundary* of M in W, respectively. The set $M \subseteq W$ is said to be *dense* in W if $\overline{M} = W$. If $M \subseteq W$, we indicate with $W \setminus M$ the *complement* of M in W. In the case that $A, B \subseteq W$, we denote by $A \setminus B$ the *relative complement* of B in A. By $|A|$ we mean the *cardinality* of the set A.

We denote by Id the *identity map* and by Id_W the identity on the space W, when we need to specify it. Given a function f, we sometimes indicate with D_f its *domain*. By $f \restriction_M$ we mean the *restriction* of f to a subset M of its domain. For a function $f : W \supseteq D_f \to Z$ between the topological spaces W and Z, we define the *preimage* of $z \in Z$ as $f^{-1}(z) := \{w \in D_f : f(w) = z\}$.

A function $f : W \to W$ is called a *self-map* of the space W. The *iterates* of f are defined recursively with the convention $f^0 = \text{Id}_W$, $f^1 = f$ and $f^n = f \circ f^{n-1}$, $\forall n \geq 2$. We say that $\overline{w} \in W$ is a *fixed point* for f if $f(\overline{w}) = \overline{w}$. We say that $\overline{w} \in W$ is a *periodic point* for f if there exists an integer $l \geq 1$ such that $f^l(\overline{w}) = \overline{w}$. The minimal l such that $f^l(\overline{w}) = \overline{w}$ is called *period* of $\overline{w}$.

The N-dimensional Euclidean space $\mathbb{R}^N$ is endowed with the usual scalar product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\|$ and distance $\text{dist}(\cdot, \cdot)$. In the case of $\mathbb{R}$, the norm $\|\cdot\|$ will be replaced with the *absolute value* $|\cdot|$. If X is a metric space different from $\mathbb{R}^N$, we denote by d_X the distance defined on it. The distance between M_1, $M_2 \subseteq X$ is indicated with $d_X(M_1, M_2) := \inf\{d_X(x,y) : x \in M_1, y \in M_2\}$.

In a normed space $(X, \|\cdot\|_X)$, we denote by $B(x_0, r)$ and $B[x_0, r]$ the *open* and *closed balls* centered in $x_0 \in X$ with radius $r > 0$, i.e. $B(x_0, r) := \{x \in X : \|x - x_0\|_X < r\}$ and $B[x_0, r] := \{x \in X : \|x - x_0\|_X \leq r\}$. For $M \subseteq X$, we set $B(M, r) := \{x \in X : \exists\, w \in M \text{ with } \|x - w\|_X < r\}$. The set $B[M, r]$ is defined accordingly. If we wish to specify the dimension m of the ball, when it is contained in $\mathbb{R}^m$, we write $B_m(x_0, r)$ and $B_m[x_0, r]$ in place of $B(x_0, r)$ and $B[x_0, r]$. We denote by S^{N-1} the $N - 1$-dimensional *unit sphere* embedded in $\mathbb{R}^N$, that is, $S^{N-1} := \partial B_N[0, 1]$.

Given an open bounded subset Ω of $\mathbb{R}^N$, an element p of $\mathbb{R}^N$ and a continuous function $f : \overline{\Omega} \to \mathbb{R}^N$, we indicate with $\deg(f, \Omega, p)$ the *topological degree* of the map f with respect to Ω and p. We say that $\deg(f, \Omega, p)$ is defined if $f(x) \neq p,\ \forall x \in \partial\Omega$ [96].

We denote by $\mathcal{C}(D, \mathbb{R}^N)$ the set of the continuous maps $f : D \to \mathbb{R}^N$ on a compact set D, endowed with the infinity norm $|f|_\infty := \max_{x \in D} \|f(x)\|$.

Given a compact interval $[a, b] \subset \mathbb{R}$, we set $L^1([a, b]) := \{f : [a, b] \to \mathbb{R} : f \text{ is Lebesgue measurable and } \int_a^b |f(t)|dt < +\infty\}$, where the integral is the Lebesgue integral. This space is endowed with the norm $\|f\|_1 := \int_a^b |f(t)|dt$.

Contents

Part I

Expansive-contractive maps in Euclidean spaces

Chapter 1

Search of fixed points for maps expansive along one direction

One of the main areas in general topology concerns the search of fixed points for continuous maps defined on arbitrary topological spaces. Given a topological space $W \supseteq A \neq \emptyset$ and a continuous map $f : A \to W$, a *fixed point* for f is simply a point $\bar{x} \in A$ that is not moved by the map, i.e. $f(\bar{x}) = \bar{x}$. In spite of the simplicity of the definition, the concept of fixed point turns out to be central in the study of dynamical systems, since, for instance, fixed points of suitable operators correspond to periodic solutions (cf. Chapter 4). A concept related to that of fixed point is the one of *periodic point*. Namely, the periodic points of a continuous map f are the fixed points of the iterates of the map, that is, $\bar{x}$ is a periodic point for f if there exists an integer $l \geq 1$ such that $f^l(\bar{x}) = \bar{x}$.

As often happens, the more general are the spaces we consider, the more sophisticated are the tools to be employed: the search of fixed points and periodic points can indeed require the use of advanced theories, like that of the Conley or fixed point index [108, 110, 138, 155, 167, 172], Lefschetz number [86, 147, 148, 150] and many other geometric or algebraic methods [15, 46, 151, 166]. In the past decades several efforts have been made in order to find elementary tools to deal with such a problem. For example Poincaré-Miranda Theorem or some other equivalent versions of Brouwer fixed point Theorem have turned out to be very useful when dealing with N-dimensional Euclidean spaces [10, 99].

It is among these not too sophisticated approaches that the theory of the Cutting Surfaces for the search of fixed point and periodic points, introduced by the author and Zanolin in [124], has to be placed. More precisely, it concerns continuous maps defined on generalized N-dimensional rectangles and having an expansive direction. Our main tools are a modified version of the

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classical Hurewicz-Wallman Intersection Lemma [48, p.72], [66, D), p.40] and the Fundamental Theorem of Leray-Schauder [90, Théorème Fondamental]. The former result (also referred to Eilenberg-Otto [47, Th.3], according to [45, Theorem on partitions, p.100]) is one of the basic lemmas in dimension theory and it is known to be one of the equivalent versions of Brouwer fixed point Theorem. It may be interesting to observe that extensions of the Intersection Lemma led to generalizations of Brouwer fixed point Theorem to some classes of possibly noncontinuous functions [161]. On the other hand, the Leray-Schauder Continuation Theorem concerns topological degree theory and has found important applications in nonlinear analysis and differential equations (see [100]). Actually, in Theorem 1.2.11 of Section 1.2, we will employ a more general version of such result due to Fitzpatrick, Massabó and Pejsachowicz [51]. However, we point out that for our applications to the study of periodic points and chaotic-like dynamics we rely on the classical Leray-Schauder Fundamental Theorem. Another particular feature of our approach consists in a combination of Poincaré-Miranda Theorem [81, 101], which is an N-dimensional version of the intermediate value theorem, with the properties of topological surfaces cutting the arcs between two given sets. The theory of Cutting Surfaces is explained in Section 1.2.

When we confine ourselves to the bidimensional setting, suitable results from plane topology can be applied [120], like the Crossing Lemma 1.1.17, and no topological degree arguments are needed. In this case we enter the context of the "Stretching Along the Paths" method developed by Papini and Zanolin in [117, 118]. The name comes from the fact that one deals with maps that expand the arcs, in a sense that will be clarified in Section 1.1, where the interested reader can find all the related details. We stress that, with respect to [117, 118], the terminology and notation are slightly different and some proofs are new.

In any case, our approach based on the Cutting Surfaces represents a possible extension of the planar theory in [117, 118] to the N-dimensional setting, for any $N \geq 2$. By the similarity between the concepts employed and the results obtained in the two frameworks, we will keep the "stretching" terminology also for the higher dimensional case, without distinguishing between them. Indeed, the main difference between the planar setting and the N-dimensional one resides in the tools to be employed in order to get some results, like for instance the fixed point Theorems 1.1.3 and 1.2.20, on which the two theories are respectively based. Namely, as already mentioned, in the plane elementary results from general topology suffices, while in the higher dimension, one needs more sophisticated tools, such as topological degree.

1.1 The planar case

1.1.1 Stretching along the paths and variants

Before introducing the main concepts of the "Stretching Along the Paths" method, let us recall some facts about paths, arcs and continua.

Let W be a topological space. By a *path* γ in W we mean a continuous map $\gamma : \mathbb{R} \supseteq [a, b] \to W$. Its range will be denoted by $\overline{\gamma}$, that is, $\overline{\gamma} := \gamma([a, b])$. A *subpath* ω of γ is the restriction of γ to a closed subinterval of its domain and hence it is defined as $\omega := \gamma \restriction_{[c,d]}$, for some $[c, d] \subseteq [a, b]$. If W, Z are topological spaces and $\psi : W \supseteq D_\psi \to Z$ is a map which is continuous on a set $\mathcal{M} \subseteq D_\psi$, then for any path γ in W with $\overline{\gamma} \subseteq \mathcal{M}$, it follows that $\psi \circ \gamma$ is a path in Z with range $\psi(\overline{\gamma})$. Notice that there is no loss of generality in assuming the paths to be defined on $[0, 1]$. Indeed, if $\theta_1 : [a_1, b_1] \to W$ and $\theta_2 : [a_2, b_2] \to W$, with $a_i < b_i$, $i = 1, 2$, are two paths in W, we define the equivalence relation " $\sim$ " between θ_1 and θ_2 by setting $\theta_1 \sim \theta_2$ if there is a homeomorphism h of $[a_1, b_1]$ onto $[a_2, b_2]$ such that $\theta_2(h(t)) = \theta_1(t)$, $\forall t \in [a_1, b_1]$. It is easy to check that if $\theta_1 \sim \theta_2$, then the ranges of θ_1 and θ_2 coincide. Hence, for any path γ there exists an equivalent path defined on $[0, 1]$. In view of this fact we will usually deal with paths defined on $[0, 1]$, but sometimes we will also consider paths defined on an arbitrary interval $[a, b]$, when this can help to simplify the exposition. A concept similar to the one of path is that of arc. More precisely an *arc* is the homeomorphic image of the compact interval $[0, 1]$, while an *open arc* is an arc without its end-points. A *continuum* of W is a compact connected subset of W and a *subcontinuum* is a subset of a continuum which is itself a continuum. Now we can start with our definitions. Given a metric space X, we call *generalized rectangle* any set $\mathcal{R} \subseteq X$ homeomorphic to the unit square $\mathcal{Q} := [0, 1]^2$ of $\mathbb{R}^2$. If $\mathcal{R}$ is a generalized rectangle and $h : \mathcal{Q} \to h(\mathcal{Q}) = \mathcal{R}$ is a homeomorphism defining it, we call *contour* $\vartheta\mathcal{R}$ of $\mathcal{R}$ the set

$$\vartheta\mathcal{R} := h(\partial\mathcal{Q}),$$

where $\partial\mathcal{Q}$ is the usual boundary of the unit square. Notice that the contour $\vartheta\mathcal{R}$ is well-defined as it does not depend on the choice of the homeomorphism h. In fact, $\vartheta\mathcal{R}$ is also a homeomorphic image of S^1, that is, a Jordan curve. By an *oriented rectangle* we mean a pair

$$\widetilde{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-),$$

where $\mathcal{R} \subseteq X$ is a generalized rectangle and

$$\mathcal{R}^- := \mathcal{R}_\ell^- \cup \mathcal{R}_r^-$$

is the union of two disjoint arcs $\mathcal{R}_\ell^-$, $\mathcal{R}_r^- \subseteq \vartheta\mathcal{R}$, that we call the *left* and the *right sides* of $\mathcal{R}^-$. Since $\vartheta\mathcal{R}$ is a Jordan curve it follows that $\vartheta\mathcal{R} \setminus (\mathcal{R}_\ell^- \cup \mathcal{R}_r^-)$ consists of two open arcs. We denote by $\mathcal{R}^+$ the closure of such open arcs, that we name $\mathcal{R}_d^+$ and $\mathcal{R}_u^+$ (the *down* and *up sides* of $\mathcal{R}^+$). It is important to notice that we can always label the arcs $\mathcal{R}_\ell^-$, $\mathcal{R}_d^+$, $\mathcal{R}_r^-$ and $\mathcal{R}_u^+$, following the cyclic order $\ell - d - r - u - \ell$, and take a homeomorphism $g : \mathcal{Q} \to g(\mathcal{Q}) = \mathcal{R}$ so that

$$
\begin{aligned}
g(\{0\} \times [0,1]) &= \mathcal{R}_\ell^- , & g(\{1\} \times [0,1]) &= \mathcal{R}_r^- , \\
g([0,1] \times \{0\}) &= \mathcal{R}_d^+ , & g([0,1] \times \{1\}) &= \mathcal{R}_u^+ .
\end{aligned}
\tag{1.1.1}
$$

Both the term "generalized rectangle" for $\mathcal{R}$ and the decomposition of the contour $\vartheta\mathcal{R}$ into $\mathcal{R}^-$ and $\mathcal{R}^+$ are somehow inspired to the construction of rectangular domains around hyperbolic sets arising in the theory of Markov partitions [59, p.291], as well as by the Conley-Ważewski theory [36, 149]. Roughly speaking, in such frameworks, the sets labeled as $[\cdot]^-$, or as $[\cdot]^+$, are made by those points which are moved by the flow outward, respectively inward, with respect to $\mathcal{R}$. As we shall see in the next definition of *stretching along the paths* property, also in our case the $[\cdot]^-$-set is loosely related to the expansive direction. Indeed we have:

Definition 1.1.1. Let X be a metric space and let $\psi : X \supseteq D_\psi \to X$ be a map defined on a set D_ψ. Assume that $\widetilde{\mathcal{A}} := (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} := (\mathcal{B}, \mathcal{B}^-)$ are oriented rectangles of X and let $\mathcal{K} \subseteq \mathcal{A} \cap D_\psi$ be a compact set. We say that $(\mathcal{K}, \psi)$ *stretches* $\widetilde{\mathcal{A}}$ *to* $\widetilde{\mathcal{B}}$ *along the paths* and write

$$
(\mathcal{K}, \psi) : \widetilde{\mathcal{A}} \mathbin{\Leftrightarrow\!\!\!-\!\!\!\rightarrow} \widetilde{\mathcal{B}},
\tag{1.1.2}
$$

if the following conditions hold:

- ψ is continuous on $\mathcal{K}$;

- For every path $\gamma : [0,1] \to \mathcal{A}$ such that $\gamma(0) \in \mathcal{A}_\ell^-$ and $\gamma(1) \in \mathcal{A}_r^-$ (or $\gamma(0) \in \mathcal{A}_r^-$ and $\gamma(1) \in \mathcal{A}_\ell^-$), there exists a subinterval $[t', t''] \subseteq [0,1]$ with

$$
\gamma(t) \in \mathcal{K}, \quad \psi(\gamma(t)) \in \mathcal{B}, \ \ \forall t \in [t', t'']
$$

  and, moreover, $\psi(\gamma(t'))$ and $\psi(\gamma(t''))$ belong to different sides of $\mathcal{B}^-$.

In the special case in which $\mathcal{K} = \mathcal{A}$, we simply write

$$
\psi : \widetilde{\mathcal{A}} \mathbin{\Leftrightarrow\!\!\!-\!\!\!\rightarrow} \widetilde{\mathcal{B}}.
$$

The role of the compact set $\mathcal{K}$ is crucial in the results which use Definition 1.1.1. For instance, if (1.1.2) is satisfied with $\widetilde{\mathcal{B}} = \widetilde{\mathcal{A}}$, that is,

$$(\mathcal{K}, \psi) : \widetilde{\mathcal{A}} \Longleftrightarrow \widetilde{\mathcal{A}}, \tag{1.1.3}$$

we are able to prove the existence of a fixed point for ψ in $\mathcal{K}$ (see Theorem 1.1.3).

Notice that when (1.1.2) holds, the stretching condition

$$(\mathcal{K}', \psi) : \widetilde{\mathcal{A}} \Longleftrightarrow \widetilde{\mathcal{B}}$$

is fulfilled for any compact set $\mathcal{K}'$, with $\mathcal{K} \subseteq \mathcal{K}' \subseteq \mathcal{A} \cap D_\psi$, on which ψ is continuous. On the other hand, since $\mathcal{K}$ provides a localization of the fixed point, it is convenient to choose it as small as possible.

We point out that in Definition 1.1.1 (as well as in its variant in Definition 1.1.2 below) we don't require $\psi(\mathcal{A}) \subseteq \mathcal{B}$. In fact, we will refer to $\mathcal{B}$ as to a "target set" and not as to a codomain.

The relation introduced in Definition 1.1.1 is fundamental in our "stretching along the paths" method. Indeed, such strategy consists in checking property (1.1.3), so that, in view of Theorem 1.1.3, there exists at least a fixed point for ψ in $\mathcal{K}$. If in particular (1.1.3) is satisfied with respect to two or more pairwise disjoint compact sets $\mathcal{K}_i$'s we get a multiplicity of fixed points. On the other hand, when (1.1.3) holds for some iterate of ψ, the existence of periodic points is ensured. Since the stretching property in (1.1.2) is preserved under composition of mappings (cf. Lemma 1.1.7), combining such arguments, also the presence of chaotic dynamics follows. A detailed treatment of such topic can be found in Section 2.1.

Definition 1.1.2. Let X be a metric space and let $\psi : X \supseteq D_\psi \to X$ be a map defined on a set D_ψ. Assume that $\widetilde{\mathcal{A}} := (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} := (\mathcal{B}, \mathcal{B}^-)$ are oriented rectangles of X and let $\mathcal{K} \subseteq \mathcal{A} \cap D_\psi$ be a compact set. Let also $m \geq 2$ be an integer. We say that $(\mathcal{D}, \psi)$ *stretches $\widetilde{\mathcal{A}}$ to $\widetilde{\mathcal{B}}$ along the paths with crossing number m* and write

$$(\mathcal{D}, \psi) : \widetilde{\mathcal{A}} \overset{m}{\Longleftrightarrow} \widetilde{\mathcal{B}},$$

if there exist m pairwise disjoint compact sets

$$\mathcal{K}_0, \ldots, \mathcal{K}_{m-1} \subseteq \mathcal{D}$$

such that

$$(\mathcal{K}_i, \psi) : \widetilde{\mathcal{A}} \Longleftrightarrow \widetilde{\mathcal{B}}, \quad i = 0, \ldots, m-1.$$

When $\mathcal{D} = \mathcal{A}$, we simply write

$$\psi : \widetilde{\mathcal{A}} \overset{m}{\Longleftrightarrow} \widetilde{\mathcal{B}}.$$
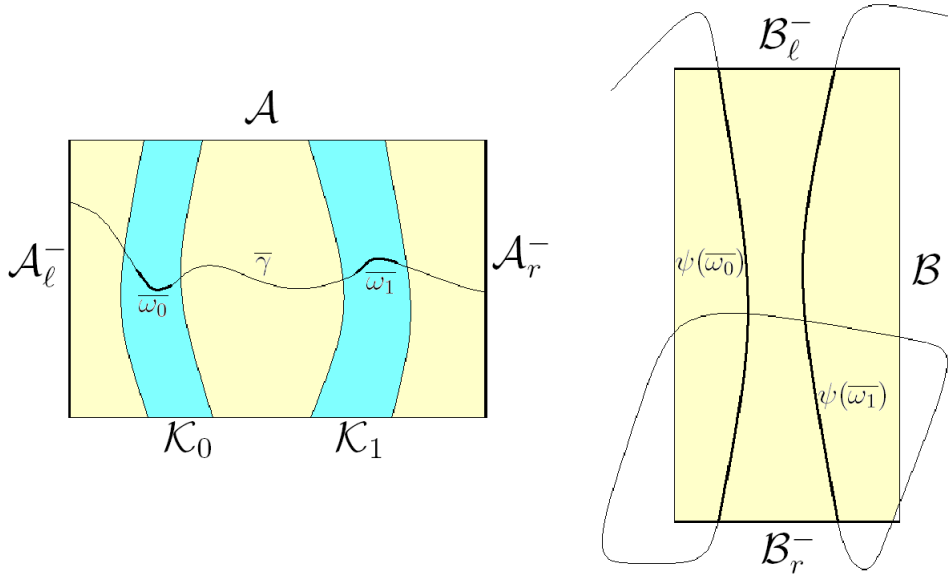
Figure 1.1: A pictorial comment to Definitions 1.1.1 and 1.1.2. The rectangles $\mathcal{A}$ and $\mathcal{B}$ have been oriented by selecting the sets $\mathcal{A}^-$ and $\mathcal{B}^-$ (drawn with thicker lines), respectively. We represent a case in which the relation $(\mathcal{K}_i, \psi) : \widetilde{\mathcal{A}} \Longleftrightarrow \widetilde{\mathcal{B}}$, $i = 0, 1$, is satisfied for a map $\psi : \mathbb{R}^2 \supseteq \mathcal{A} \to \mathbb{R}^2$ and for the two darker compact subsets $\mathcal{K}_0$ and $\mathcal{K}_1$ of $\mathcal{A}$, on which ψ is continuous. For a generic path $\gamma : [0, 1] \to \mathcal{A}$ with $\gamma(0)$ and $\gamma(1)$ belonging to different components of $\mathcal{A}^-$, we have highlighted two subpaths ω_0 and ω_1 with range in $\mathcal{K}_0$ and $\mathcal{K}_1$, respectively, such that their composition with ψ determines two new paths (drawn by bolder vertical lines) with values in $\mathcal{B}$ and joining the two sides of $\mathcal{B}^-$. In this framework, according to Definition 1.1.2, we could also write $\psi : \widetilde{\mathcal{A}} \overset{2}{\Longleftrightarrow} \widetilde{\mathcal{B}}$.

Definition 1.1.2 is an extension of Definition 1.1.1, as they are coincident when $m = 1$. The concept of "crossing number" is borrowed from Kennedy and Yorke [71] and adapted to our framework. Indeed in [71] the authors deal with a very general setting, that looks closely related to ours when restricted to the planar case. More precisely, in [71, horseshoe hypotheses Ω] a locally connected and compact subset Q of the separable metric space X is considered, on which two disjoint and compact sets end_0, $end_1 \subseteq Q$ are selected, so that any component of Q intersects both of them. On Q a continuous map $f : Q \to X$ is defined in such a way that every continuum $\Gamma \subseteq Q$ joining end_0 and end_1 (i.e. a *connection* according to [71]) admits at least $m \geq 2$ pairwise disjoint compact and connected subsets, whose images under f are again connections. Such subcontinua are named *preconnections* and m is the so-called *crossing number*.

Clearly, the arcs $\mathcal{R}_\ell^-$ and $\mathcal{R}_r^-$ in our definition of oriented rectangle are a particular case of the sets end_0 and end_1 considered by Kennedy and Yorke. Moreover, any path γ with values in $\mathcal{R}$ and joining $\mathcal{R}_\ell^-$ with $\mathcal{R}_r^-$ determines a connection (according to [71]) via its image $\overline{\gamma}$. Similarly, any subpath ω of γ, with $\omega = \gamma \!\restriction_{[t',t'']}$ like in Definition 1.1.1, makes $\overline{\omega}$ a preconnection according to Kennedy and Yorke. In fact, the idea of considering paths to detect a sort of expansion was independently developed by such authors in [72, 73], where a terminology analogous to ours was employed. A special request in our approach, not explicitly assumed in [71], is the one concerning the compact sets like $\mathcal{K}$. Indeed, in [71] there are no sets playing the role of the $\mathcal{K}_i$'s in Definition 1.1.2.

On the other hand, our stretching condition is strong enough to ensure the existence of fixed points and periodic points for the map ψ inside the $\mathcal{K}_i$'s, as we shall see in Theorems 1.1.3 and 1.1.8 below, while in [71, Example 10] a fixed point free map defined on $\mathbb{R}^2 \times S^1$ and satisfying the horseshoe hypotheses Ω is presented. Indeed, we point out that Theorem 1.1.3 allows not only to infer the existence of fixed points, but also to localize them. This turns out to be of particular importance when multiple coverings in the sense of Definition 1.1.2 occur.

In the proof of Theorems 1.1.3 and 1.1.8, we employ a classical result from plane topology (cf. Crossing Lemma 1.1.17), that we recall in Subsection 1.1.2 for the reader's convenience. See also [117, 118].

Theorem 1.1.3. *Let X be a metric space and let $\psi : X \supseteq D_\psi \to X$ be a map defined on a set D_ψ. Assume that $\widetilde{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$ is an oriented rectangle of X. If $\mathcal{K} \subseteq \mathcal{R} \cap D_\psi$ is a compact set for which it holds that*

$$(\mathcal{K}, \psi) : \widetilde{\mathcal{R}} \Longleftrightarrow \widetilde{\mathcal{R}}, \tag{1.1.4}$$

then there exists at least one point $z \in \mathcal{K}$ with $\psi(z) = z$.

Proof. By the definition of oriented rectangle, there exists a homeomorphism $h : \mathbb{R}^2 \supseteq \mathcal{Q} \to h(\mathcal{Q}) = \mathcal{R} \subseteq X$, mapping in a correct way (i.e. as in (1.1.1)) the sides of $\mathcal{Q} = [0,1]^2$ into the arcs that compose the sets $\mathcal{R}^-$ and $\mathcal{R}^+$. Then, passing to the planar map $\phi := h^{-1} \circ \psi \circ h$ defined on $D_\phi := h^{-1}(D_\psi) \subseteq \mathcal{Q}$, we can confine ourselves to the search of a fixed point for ϕ in the compact set $\mathcal{H} := h^{-1}(\mathcal{K}) \subseteq \mathcal{Q}$. The stretching assumption on ψ is now translated to

$$(\mathcal{H}, \phi) : \widetilde{\mathcal{Q}} \mathrel{\text{\reflectbox{$\looparrowright$}}} \widetilde{\mathcal{Q}}.$$

On $\widetilde{\mathcal{Q}}$ we consider the natural "left-right" orientation, choosing

$$\mathcal{Q}^- = (\{0\} \times [0,1]) \cup (\{1\} \times [0,1]).$$

A fixed point for ϕ in $\mathcal{H}$ corresponds to a fixed point for ψ in $\mathcal{K}$.
For $\phi = (\phi_1, \phi_2)$ and $x = (x_1, x_2)$, we define the compact set

$$V := \{x \in \mathcal{H} : 0 \le \phi_2(x) \le 1, \ x_1 - \phi_1(x) = 0\}.$$

The proof consists in showing that V contains a continuum $\mathcal{C}$ which joins in $\mathcal{Q}$ the lower side $[0,1] \times \{0\}$ to the upper side $[0,1] \times \{1\}$. To this end, in view of Lemma 1.1.17, it is sufficient to prove that V acts as a "cutting surface" (in the sense of Definition 1.2.1) between the left and the right sides of $\mathcal{Q}$, that is, V intersects any path in $\mathcal{Q}$ joining the left side $\{0\} \times [0,1]$ to the right side $\{1\} \times [0,1]$. Such cutting property can be checked via the intermediate value theorem by observing that if $\gamma = (\gamma_1, \gamma_2) : [0,1] \to \mathcal{Q}$ is a continuous map with $\gamma(0) \in \{0\} \times [0,1]$ and $\gamma(1) \in \{1\} \times [0,1]$ then, the stretching hypothesis $(\mathcal{H}, \phi) : \widetilde{\mathcal{Q}} \mathrel{\text{\reflectbox{$\looparrowright$}}} \widetilde{\mathcal{Q}}$ implies that there exists an interval $[t', t''] \subseteq [0,1]$ such that $\gamma(t) \in \mathcal{H}$, $\phi(\gamma(t)) \in \mathcal{Q}$, $\forall t \in [t', t'']$ and $\gamma_1(t') - \phi_1(\gamma(t')) \ge 0 \ge \gamma_1(t'') - \phi_1(\gamma(t''))$ or $\gamma_1(t') - \phi_1(\gamma(t')) \le 0 \le \gamma_1(t'') - \phi_1(\gamma(t''))$. Notice that, by the definition of V it follows that $\phi_2(z) \in [0,1], \forall z \in \mathcal{C}$. Hence, for every point $p = (p_1, p_2) \in \mathcal{C} \cap ([0,1] \times \{0\})$ we have $p_2 - \phi_2(p) \le 0$ and, similarly, $p_2 - \phi_2(p) \ge 0$ for every $p = (p_1, p_2) \in \mathcal{C} \cap ([0,1] \times \{1\})$. Applying Bolzano Theorem we obtain the existence of at least a point $v = (v_1, v_2) \in \mathcal{C} \subseteq V \subseteq \mathcal{H}$ such that $v_2 - \phi_2(v) = 0$. Hence v is a fixed point of ϕ in $\mathcal{H}$ and $z := h(v)$ is a fixed point for ψ in $\mathcal{K} \subseteq \mathcal{R}$. $\square$

Remark 1.1.4. Notice that, for the validity of Theorem 1.1.3, it is fundamental that the orientation of the generalized rectangle $\mathcal{R}$ in (1.1.4) remains the same for $\mathcal{R}$ considered as "starting set" and "target set" of the map ψ. Indeed, if one chooses two different orientations for $\mathcal{R}$, in general the above result does not hold anymore and the existence of fixed points for ψ is no longer ensured,

not only in $\mathcal{K}$, but even in $\mathcal{R}$, as shown by the example depicted in Figure 1.2.
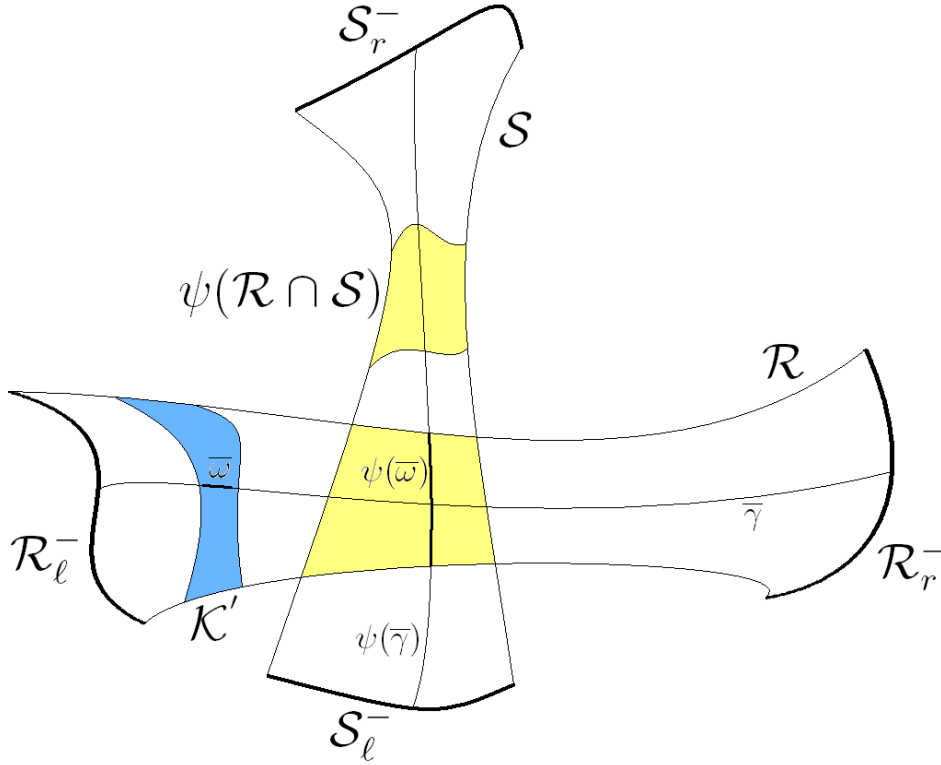
$\triangleleft$

Figure 1.2: The generalized rectangle $\mathcal{R}$ is transformed by a continuous planar map ψ onto the generalized rectangle $\mathcal{S} = \psi(\mathcal{R})$, so that, in particular, $\mathcal{S}_\ell^- = \psi(\mathcal{R}_\ell^-)$ and $\mathcal{S}_r^- = \psi(\mathcal{R}_r^-)$. The boundary sets $\mathcal{R}^- = \mathcal{R}_\ell^- \cup \mathcal{R}_r^-$ and $\mathcal{S}^- = \mathcal{S}_\ell^- \cup \mathcal{S}_r^-$ are drawn with thicker lines. As it is immediate to verify, for $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ and $\widetilde{\mathcal{S}} = (\mathcal{S}, \mathcal{S}^-)$, it holds that $\psi : \widetilde{\mathcal{R}} \Leftrightarrow \widetilde{\mathcal{S}}$. On the other hand, calling $\widetilde{\widetilde{\mathcal{R}}}$ the generalized rectangle $\mathcal{R}$ oriented by choosing $\overline{\vartheta \mathcal{R} \setminus \mathcal{R}^-}$ as $[\cdot]^-$-set, it also holds that $(\mathcal{K}', \psi) : \widetilde{\mathcal{R}} \Leftrightarrow \widetilde{\widetilde{\mathcal{R}}}$, where $\mathcal{K}'$ is the subset of $\mathcal{R}$ depicted with a darker color. However, since $\mathcal{R} \cap \mathcal{S}$ is mapped by ψ outside $\mathcal{R}$ (both $\mathcal{R} \cap \mathcal{S}$ and $\psi(\mathcal{R} \cap \mathcal{S})$ are drawn with the same light color), there cannot exist fixed points for ψ in $\mathcal{R}$ and, a fortiori, neither in $\mathcal{K}'$. Notice that Theorem 1.1.3 does not apply because we have taken two different orientations for $\mathcal{R}$. A similar geometrical framework was already considered in [69, Figure 8] and [123, Figure 4].

As a comment to Theorem 1.1.3, we discuss here its main differences with respect to the more classical Brouwer fixed point Theorem (recalled in the two-dimensional case as Theorem 1.1.14 in Subsection 1.1.2). It is a well-

known fact that the fixed point property for continuous maps is preserved by homeomorphisms. Therefore, it is straightforward to prove the existence of a fixed point for a continuous map ψ if $\psi(\mathcal{R}) \subseteq \mathcal{R}$, with $\mathcal{R}$ a generalized rectangle of a metric space X. The framework depicted in Theorem 1.1.3 is quite different. Indeed, first of all, the stretching assumption $(\mathcal{K}, \psi) : \widetilde{\mathcal{R}} \Leftrightarrow \widetilde{\mathcal{R}}$ does not imply $\psi(\mathcal{R}) \subseteq \mathcal{R}$ and, secondly, we need ψ to be continuous only on $\mathcal{K}$ and not on the whole set $\mathcal{R}$. Finally, as already noticed, we stress that our result also localizes the presence of a fixed point in the subset $\mathcal{K}$. From the point of view of the applications, this means that we are able to obtain a multiplicity of fixed points provided that the stretching property is fulfilled with respect to pairwise disjoint compact subsets of $\mathcal{R}$. Indeed, if the condition $(\mathcal{D}, \psi) : \widetilde{\mathcal{R}} \overset{m}{\Leftrightarrow} \widetilde{\mathcal{R}}$ is satisfied with a crossing number $m \geq 2$, in view of Theorem 1.1.3, the map ψ has at least a fixed point in each of the compact sets $\mathcal{K}_i$'s, $i = 0, \ldots, m-1$, from Definition 1.1.2 and therefore there are at least m fixed points for ψ in $\mathcal{R}$ (see Theorem 1.1.8).

A different case, related to the stretching relation from Definition 1.1.1, in which it is possible to find fixed points is when the special geometric configuration in Definition 1.1.5 gets realized, as stated in Theorem 1.1.6 below.

Definition 1.1.5. Let $\widetilde{\mathcal{A}} := (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} := (\mathcal{B}, \mathcal{B}^-)$ be two oriented rectangles of a metric space X. We say that $\widetilde{\mathcal{A}}$ is a *horizontal slab* of $\widetilde{\mathcal{B}}$ and write

$$\widetilde{\mathcal{A}} \subseteq_h \widetilde{\mathcal{B}}$$

if $\mathcal{A} \subseteq \mathcal{B}$ and, either

$$\mathcal{A}_\ell^- \subseteq \mathcal{B}_\ell^- \quad \text{and} \quad \mathcal{A}_r^- \subseteq \mathcal{B}_r^- \,,$$

or

$$\mathcal{A}_\ell^- \subseteq \mathcal{B}_r^- \quad \text{and} \quad \mathcal{A}_r^- \subseteq \mathcal{B}_\ell^- \,,$$

so that any path in $\mathcal{A}$ joining the two sides of $\mathcal{A}^-$ is also a path in $\mathcal{B}$ and joins the two opposite sides of $\mathcal{B}^-$.
We say that $\widetilde{\mathcal{A}}$ is a *vertical slab* of $\widetilde{\mathcal{B}}$ and write

$$\widetilde{\mathcal{A}} \subseteq_v \widetilde{\mathcal{B}}$$

if $\mathcal{A} \subseteq \mathcal{B}$ and every path in $\mathcal{B}$ joining the two sides of $\mathcal{B}^-$ admits a subpath in $\mathcal{A}$ that joins the two opposite sides of $\mathcal{A}^-$.
Given three oriented rectangles $\widetilde{\mathcal{A}} := (\mathcal{A}, \mathcal{A}^-)$, $\widetilde{\mathcal{B}} := (\mathcal{B}, \mathcal{B}^-)$ and $\widetilde{\mathcal{E}} := (\mathcal{E}, \mathcal{E}^-)$ of the metric space X, with $\mathcal{E} \subseteq \mathcal{A} \cap \mathcal{B}$, we say that $\widetilde{\mathcal{B}}$ *crosses* $\widetilde{\mathcal{A}}$ *in* $\widetilde{\mathcal{E}}$ and write

$$\widetilde{\mathcal{E}} \in \{\widetilde{\mathcal{A}} \pitchfork \widetilde{\mathcal{B}}\},$$

if

$$\widetilde{\mathcal{E}} \subseteq_h \widetilde{\mathcal{A}} \quad \text{and} \quad \widetilde{\mathcal{E}} \subseteq_v \widetilde{\mathcal{B}}.$$

The above definitions, which are adapted from the concept of "slice" in [118, 119] and imitate the classical terminology in [162, Ch.2.3], are topological in nature and therefore do not necessitate any metric assumption (like smoothness, lipschitzeanity, or similar properties often required in the literature). We also notice that the terms "horizontal" and "vertical" are employed in a purely conventional manner, as it looks clear from Figure 1.3: the horizontal is the expansive direction and the vertical is the contractive one (in a quite broad sense). For instance, in [124] the terms "vertical" and "horizontal" were interchanged in regard to the N-dimensional setting, but this did not make any difference with respect to the meaning of the results obtained.

The next theorem depicts a situation where the "starting set" and the "target set" of the mapping ψ are two intersecting oriented rectangles. A graphical illustration of it can be found in Figure 1.3.

Theorem 1.1.6. *Let X be a metric space and let $\psi : X \supseteq D_\psi \to X$ be a map defined on a set D_ψ. Assume that $\widetilde{\mathcal{A}} := (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} := (\mathcal{B}, \mathcal{B}^-)$ are oriented rectangles of X and let $\mathcal{K} \subseteq \mathcal{A} \cap D_\psi$ be a compact set such that*

$$(\mathcal{K}, \psi) : \widetilde{\mathcal{A}} \Leftrightarrow \widetilde{\mathcal{B}}. \tag{1.1.5}$$

If there exists an oriented rectangle $\widetilde{\mathcal{E}} := (\mathcal{E}, \mathcal{E}^-)$ with $\widetilde{\mathcal{E}} \in \{\widetilde{\mathcal{A}} \pitchfork \widetilde{\mathcal{B}}\}$, then ψ has at least a fixed point in $\mathcal{K} \cap \mathcal{E}$.

Proof. In order to achieve the thesis, we show that

$$(\mathcal{K} \cap \mathcal{E}, \psi) : \widetilde{\mathcal{B}} \Leftrightarrow \widetilde{\mathcal{B}}. \tag{1.1.6}$$

Indeed, let γ be a path with $\overline{\gamma} \subseteq \mathcal{B}$ and $\overline{\gamma} \cap \mathcal{B}_\ell^- \neq \emptyset$, $\overline{\gamma} \cap \mathcal{B}_r^- \neq \emptyset$. Then, since $\widetilde{\mathcal{E}} \subseteq_v \widetilde{\mathcal{B}}$, there exists a subpath ω of γ such that $\overline{\omega} \subseteq \mathcal{E}$ and $\overline{\omega} \cap \mathcal{E}_\ell^- \neq \emptyset$, $\overline{\omega} \cap \mathcal{E}_r^- \neq \emptyset$. Recalling now that $\widetilde{\mathcal{E}} \subseteq_h \widetilde{\mathcal{A}}$, it holds that $\overline{\omega} \subseteq \mathcal{E} \subseteq \mathcal{A}$ and $\overline{\omega} \cap \mathcal{A}_\ell^- \neq \emptyset$, $\overline{\omega} \cap \mathcal{A}_r^- \neq \emptyset$. Finally, since $(\mathcal{K}, \psi) : \widetilde{\mathcal{A}} \Leftrightarrow \widetilde{\mathcal{B}}$, there is a subpath η of ω such that $\overline{\eta} \subseteq \mathcal{K} \cap \mathcal{E}$, $\psi(\overline{\eta}) \subseteq \mathcal{B}$, with $\psi(\overline{\eta}) \cap \mathcal{B}_\ell^- \neq \emptyset$, $\psi(\overline{\eta}) \cap \mathcal{B}_r^- \neq \emptyset$. In this way we have proved that any path γ with $\overline{\gamma} \subseteq \mathcal{B}$ and $\overline{\gamma} \cap \mathcal{B}_\ell^- \neq \emptyset$, $\overline{\gamma} \cap \mathcal{B}_r^- \neq \emptyset$ admits a subpath η such that $\overline{\eta} \subseteq \mathcal{K} \cap \mathcal{E}$ and $\psi(\overline{\eta}) \subseteq \mathcal{B}$ with $\psi(\overline{\eta}) \cap \mathcal{B}_\ell^- \neq \emptyset$, $\psi(\overline{\eta}) \cap \mathcal{B}_r^- \neq \emptyset$. Therefore condition (1.1.6) has been checked and the existence of at least a fixed point for ψ in $\mathcal{K} \cap \mathcal{E}$ follows from Theorem 1.1.3. Notice that ψ is continuous on $\mathcal{K} \cap \mathcal{E}$ as, by (1.1.5), it is continuous on $\mathcal{K}$. $\square$
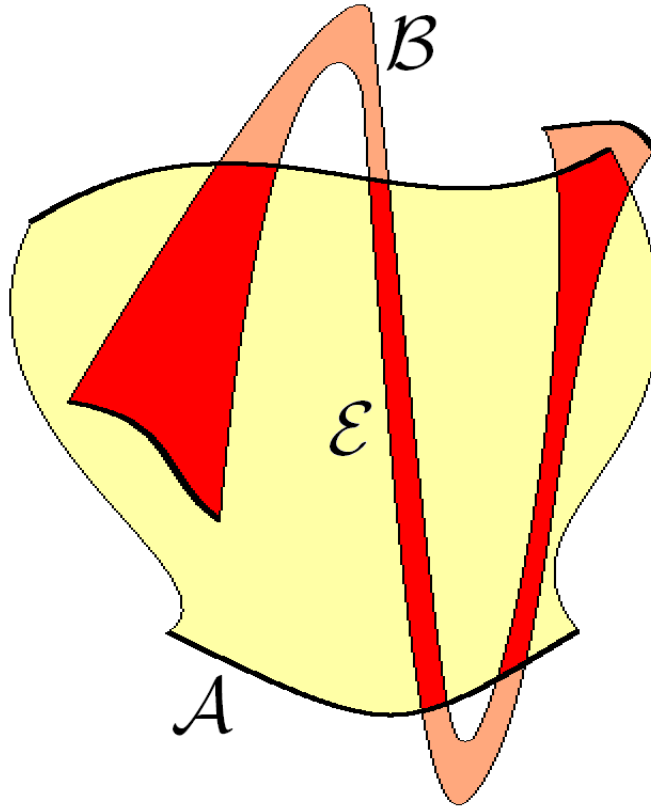
Figure 1.3: The continuous planar map ψ transforms the generalized rectangle $\mathcal{A}$ onto the snake-like generalized rectangle $\mathcal{B} \supseteq \psi(\mathcal{A})$, so that $\mathcal{B}_\ell^- \supseteq \psi(\mathcal{A}_\ell^-)$ and $\mathcal{B}_r^- \supseteq \psi(\mathcal{A}_r^-)$. The boundary sets $\mathcal{A}^- = \mathcal{A}_\ell^- \cup \mathcal{A}_r^-$ and $\mathcal{B}^- = \mathcal{B}_\ell^- \cup \mathcal{B}_r^-$ have been drawn with thicker lines. Clearly, for $\widetilde{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$, it holds that $\psi : \widetilde{\mathcal{A}} \Longleftrightarrow \widetilde{\mathcal{B}}$. Notice that we do not require that the end sets $\mathcal{B}_\ell$ and $\mathcal{B}_r$ of $\mathcal{B}$ lie outside $\mathcal{A}$, differently from the approaches based on degree theory, as discussed in [125]. Among the three intersections between $\mathcal{A}$ and $\mathcal{B}$, only the central one, that we call $\mathcal{E}$, corresponds to a crossing in the sense of Definition 1.1.5, since $\widetilde{\mathcal{E}} \subseteq_h \widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{E}} \subseteq_v \widetilde{\mathcal{B}}$. Therefore, Theorem 1.1.6 ensures the existence of at least a fixed point for ψ in $\mathcal{E}$. Even if the intersection between $\mathcal{A}$ and $\mathcal{B}$ on the right is "not far" from belonging to $\{\widetilde{\mathcal{A}} \pitchfork \widetilde{\mathcal{B}}\}$, it is possible to define ψ so that it has no fixed points therein. This shows that our fixed point theorems are, in some sense, "sharp". See [125] for more details and corresponding examples.

In regard to Theorem 1.1.6, we stress that if (1.1.5) holds and there exist $m \geq 2$ pairwise disjoint oriented rectangles

$$\widetilde{\mathcal{E}}_0, \ldots, \widetilde{\mathcal{E}}_{m-1} \in \{\, \widetilde{\mathcal{A}} \pitchfork \widetilde{\mathcal{B}} \,\}, \tag{1.1.7}$$

then the map ψ has at least a fixed point in $\mathcal{K} \cap \mathcal{E}_i$, $\forall\, i = 0, \ldots, m-1$. Hence, similarly to Definition 1.1.2, also in this framework it is possible to find a multiplicity of fixed points, obtaining conclusions analogous to the ones in Theorem 1.1.8 below.

If in place of fixed points we are concerned with the search of periodic points of any period (as in the following Theorems 1.1.8–1.1.10), then, in order to apply our stretching along the paths method, we need to check the stretching relation to be preserved under composition of maps. This fact can be easily proved by induction: the basic step is the content of the next lemma.

Lemma 1.1.7. *Let X be a metric space and let $\varphi : X \supseteq D_\varphi \to X$ and $\psi : X \supseteq D_\psi \to X$ be maps defined on the sets D_φ and D_ψ, respectively. Assume that $\widetilde{\mathcal{A}} := (\mathcal{A}, \mathcal{A}^-)$, $\widetilde{\mathcal{B}} := (\mathcal{B}, \mathcal{B}^-)$ and $\widetilde{\mathcal{C}} := (\mathcal{C}, \mathcal{C}^-)$ are oriented rectangles of X. If $\mathcal{H} \subseteq \mathcal{A} \cap D_\varphi$ and $\mathcal{K} \subseteq \mathcal{B} \cap D_\psi$ are compact sets such that*

$$(\mathcal{H}, \varphi) : \widetilde{\mathcal{A}} \Leftrightarrow \widetilde{\mathcal{B}} \quad and \quad (\mathcal{K}, \psi) : \widetilde{\mathcal{B}} \Leftrightarrow \widetilde{\mathcal{C}},$$

then it follows that

$$\big(\mathcal{H} \cap \varphi^{-1}(\mathcal{K}), \psi \circ \varphi\big) : \widetilde{\mathcal{A}} \Leftrightarrow \widetilde{\mathcal{C}}.$$

Proof. Let $\gamma : [0,1] \to \mathcal{A}$ be a path such that $\gamma(0)$ and $\gamma(1)$ belong to the different sides of $\mathcal{A}^-$. Then, since $(\mathcal{H}, \varphi) : \widetilde{\mathcal{A}} \Leftrightarrow \widetilde{\mathcal{B}}$, there exists a subinterval $[t', t''] \subseteq [0,1]$ such that

$$\gamma(t) \in \mathcal{H}, \quad \varphi(\gamma(t)) \in \mathcal{B}, \quad \forall\, t \in [t', t'']$$

and, moreover, $\varphi(\gamma(t'))$ and $\varphi(\gamma(t''))$ belong to different components of $\mathcal{B}^-$. Let us call ω the restriction of γ to $[t', t'']$ and define $\nu : [t', t''] \to \mathcal{B}$ as $\nu := \varphi \circ \omega$. Notice that $\nu(t')$ and $\nu(t'')$ belong to the different sides of $\mathcal{B}^-$ and so, by the stretching hypothesis $(\mathcal{K}, \psi) : \widetilde{\mathcal{B}} \Leftrightarrow \widetilde{\mathcal{C}}$, there is a subinterval $[s', s''] \subseteq [t', t'']$ such that

$$\nu(t) \in \mathcal{K}, \quad \psi(\nu(t)) \in \mathcal{C}, \quad \forall\, t \in [s', s'']$$

with $\psi(\nu(s'))$ and $\psi(\nu(s''))$ belonging to different components of $\mathcal{C}^-$. Rewriting all in terms of γ, this means that we have found a subinterval $[s', s''] \subseteq [0,1]$ such that

$$\gamma(t) \in \mathcal{H} \cap \varphi^{-1}(\mathcal{K}), \quad \psi(\varphi(\gamma(t))) \in \mathcal{C}, \quad \forall\, t \in [s', s'']$$

and $\psi(\varphi(\gamma(s')))$ and $\psi(\varphi(\gamma(s'')))$ belong to the different sides of $\mathcal{C}^-$. By the arbitrariness of the path γ, the stretching property

$$\left(\mathcal{H} \cap \varphi^{-1}(\mathcal{K}), \psi \circ \varphi\right) : \widetilde{\mathcal{A}} \looparrowright \widetilde{\mathcal{C}}$$

is thus fulfilled. We just point out that the continuity of the composite mapping $\psi \circ \varphi$ on the compact set $\mathcal{H} \cap \varphi^{-1}(\mathcal{K})$ follows from the continuity of φ on $\mathcal{H}$ and of ψ on $\mathcal{K}$, respectively. $\square$

Theorem 1.1.8. *Let X be a metric space and $\psi : X \supseteq D_\psi \to X$ be a map defined on a set D_ψ. Assume that $\widetilde{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$ is an oriented rectangle X. If $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}$ are $m \geq 2$ pairwise disjoint compact subsets of $\mathcal{R} \cap D_\psi$ and*

$$(\mathcal{K}_i, \psi) : \widetilde{\mathcal{R}} \looparrowright \widetilde{\mathcal{R}}, \ \text{for } i = 0, \ldots, m - 1,$$

then the following conclusions hold:

- *The map ψ has at least a fixed point in $\mathcal{K}_i$, $i = 0, \ldots, m - 1$;*

- *For each two-sided sequence $(s_h)_{h \in \mathbb{Z}} \in \{0, \ldots, m - 1\}^{\mathbb{Z}}$, there exists a sequence of points $(x_h)_{h \in \mathbb{Z}}$ such that $\psi(x_{h-1}) = x_h \in \mathcal{K}_{s_h}$, $\forall h \in \mathbb{Z}$;*

- *For each sequence $\boldsymbol{s} = (s_n)_n \in \{0, \ldots, m - 1\}^{\mathbb{N}}$, there exists a compact connected set $\mathcal{C}_{\boldsymbol{s}} \subseteq \mathcal{K}_{s_0}$ satisfying*

$$\mathcal{C}_{\boldsymbol{s}} \cap \mathcal{R}_d^+ \neq \emptyset, \quad \mathcal{C}_{\boldsymbol{s}} \cap \mathcal{R}_u^+ \neq \emptyset$$

  and such that $\psi^i(x) \in \mathcal{K}_{s_i}$, $\forall i \geq 1$, $\forall x \in \mathcal{C}_{\boldsymbol{s}}$;

- *Given an integer $j \geq 2$ and a $j + 1$-uple $(s_0, \ldots, s_j)$, $s_i \in \{0, \ldots, m - 1\}$, for $i = 0, \ldots, j$, and $s_0 = s_j$, then there exists a point $w \in \mathcal{K}_{s_0}$ such that*

$$\psi^i(w) \in \mathcal{K}_{s_i}, \forall i = 1, \ldots, j \quad \text{and} \quad \psi^j(w) = w.$$

As we shall see in Section 2.1, the previous result turns out to be our fundamental tool in the proof of Theorem 2.1.3 about chaotic dynamics. On the other hand, Theorem 1.1.8 can be viewed as a particular case of Theorem 1.1.10 below. The proof of the former result is thus postponed since it comes as a corollary of the latter more general theorem.

Before stating Theorem 1.1.10, we just make an observation that will reveal its significance in Section 2.2 when dealing with symbolic dynamics.

Remark 1.1.9. We observe that in the hypotheses of Theorem 1.1.8, or equivalently when we enter the framework of Definition 1.1.2 with $\widetilde{\mathcal{A}} = \widetilde{\mathcal{B}}$, i.e., when there exist $m \geq 2$ pairwise disjoint compact subsets $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}$ of an oriented rectangle $\mathcal{A} \subseteq X$, for which $(\mathcal{K}_i, \psi) : \widetilde{\mathcal{A}} \Longleftrightarrow \widetilde{\mathcal{A}}$ holds, then it is possible to find m pairwise disjoint vertical slabs $\widetilde{\mathcal{R}}_i$ of $\widetilde{\mathcal{A}}$ such that $\mathcal{R}_i \supseteq \mathcal{K}_i$, for $i = 0, \ldots, m-1$. Indeed, by Theorem 1.1.8, we know that any $\mathcal{K}_i$ contains a compact connected set $\mathcal{C}_i$ (actually, infinitely many) joining $\mathcal{A}_d^+$ and $\mathcal{A}_u^+$. Thus, the idea is to "fill" and "fatten" each $\mathcal{C}_i$ up, in order to obtain a compact set $\mathcal{R}_i$ that still joins $\mathcal{A}_d^+$ and $\mathcal{A}_u^+$, but homeomorphic to the unit square of $\mathbb{R}^2$ and containing only $\mathcal{K}_i$ among all the $\mathcal{K}_j$'s. Moreover we require $\mathcal{A}_d^+ \cap \mathcal{R}_i$ and $\mathcal{A}_d^+ \cap \mathcal{R}_i$ to be arcs. Notice that this is possible because the $\mathcal{K}_i$'s are compact and disjoint. To such generalized rectangles $\mathcal{R}_i$'s we give the orientation "inherited" from $\mathcal{A}$, that is, we set $\mathcal{R}_d^i := \mathcal{A}_d^+ \cap \mathcal{R}_i$ and $\mathcal{R}_u^i := \mathcal{A}_u^+ \cap \mathcal{R}_i$, for $i = 0, \ldots, m-1$, where we have denoted by $\mathcal{R}_d^i$ and $\mathcal{R}_u^i$ the two sides of $\mathcal{R}_i^+$. Indicating with $\mathcal{R}_\ell^i$ and $\mathcal{R}_r^i$ the two parts of $\mathcal{R}_i^-$, we find that they coincide with the two components of $\overline{\vartheta \mathcal{R}_i \setminus (\mathcal{R}_d^i \cup \mathcal{R}_u^i)}$. In particular, we can name such sets following the cyclic order $\ell - d - r - u - \ell$. As usual we put $\widetilde{\mathcal{R}}_i = (\mathcal{R}_i, \mathcal{R}_i^-)$ and, by construction, these are the desired vertical slabs of $\widetilde{\mathcal{A}}$. See Figure 1.4 for a graphical illustration.

If in addition the map ψ is continuous on $\mathcal{A}$, we claim that

$$\psi : \widetilde{\mathcal{R}}_i \Longleftrightarrow \widetilde{\mathcal{R}}_j, \, \forall \, i, j \in \{0, \ldots, m-1\}.$$

At first we notice that ψ is continuous on each $\mathcal{R}_i$ because $\bigcup_{i=0}^{m-1} \mathcal{R}_i \subseteq \mathcal{A}$. Moreover, for any $i = 0, \ldots, m-1$ and for every path $\gamma : [a, b] \to \mathcal{R}_i$, with $\gamma(a)$ and $\gamma(b)$ belonging to different components of $\mathcal{R}_i^-$, there exists a subinterval $[t', t''] \subseteq [a, b]$ such that $\gamma(t) \in \mathcal{K}_i$ and $\psi(\gamma(t)) \in \mathcal{A}, \, \forall \, t \in [t', t'']$, with $\psi(\gamma(t'))$ and $\psi(\gamma(t''))$ belonging to different sides of $\mathcal{A}^-$. This follows from the fact that γ can be extended to a path $\gamma^* : [a', b'] \to \mathcal{A}$, with $[a', b'] \supseteq [a, b]$, such that $\gamma^* \restriction_{[a,b]} = \gamma$ and $\gamma^*(a'), \gamma^*(b')$ belong to different sides of $\mathcal{A}^-$, and by recalling that $(\mathcal{K}_i, \psi) : \widetilde{\mathcal{A}} \Longleftrightarrow \widetilde{\mathcal{A}}$. Then, since $\bigcup_{i=0}^{m-1} \mathcal{R}_i \subseteq \mathcal{A}$, for any fixed $j = 0, \ldots, m-1$, there exists a subinterval $[s', s''] \subseteq [t', t'']$ such that $\psi(\gamma(t)) \in \mathcal{R}_j, \, \forall \, t \in [s', s'']$, with $\psi(\gamma(s'))$ and $\psi(\gamma(s''))$ belonging to different sides of $\mathcal{R}_j^-$. But this means that $\psi : \widetilde{\mathcal{R}}_i \Longleftrightarrow \widetilde{\mathcal{R}}_j$. By the arbitrariness of $i, j \in \{0, \ldots, m-1\}$, the proof of our claim is complete.
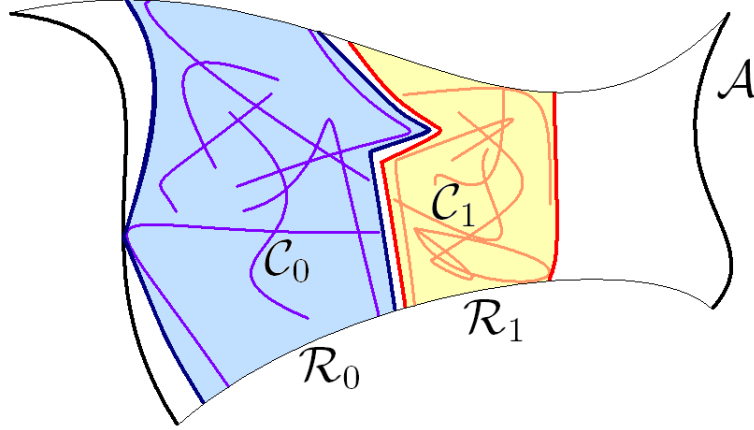
$\triangleleft$

Figure 1.4: With reference to the situation described in Remark 1.1.9 for $m = 2$, we have depicted the generalized rectangle $\mathcal{A}$, that we orientate by choosing as $\mathcal{A}^-$ the boundary set drawn with thicker lines. The continua $\mathcal{C}_0$ and $\mathcal{C}_1$, contained respectively in the disjoint compact sets $\mathcal{K}_0$ and $\mathcal{K}_1$ (not represented in the picture, in order not to overburden it) and joining the two sides of $\mathcal{A}^+$, are embedded (together with the $\mathcal{K}_i$'s) in the disjoint generalized rectangles $\mathcal{R}_0$ and $\mathcal{R}_1$, whose $[\cdot]^-$-sets have been indicated with thicker lines. With this choice, $\widetilde{\mathcal{R}}_0$ and $\widetilde{\mathcal{R}}_1$ are vertical slabs of $\widetilde{\mathcal{A}}$.

Theorem 1.1.10. *Assume there is a double sequence of oriented rectangles* $(\widetilde{\mathcal{R}}_i)_{i \in \mathbb{Z}}$ *(with* $\widetilde{\mathcal{R}}_i = (\mathcal{R}_i, \mathcal{R}_i^-)$ *) of a metric space X and a sequence* $((\mathcal{K}_i, \psi_i))_{i \in \mathbb{Z}}$, *with* $\mathcal{K}_i \subseteq \mathcal{R}_i$ *compact sets, such that*

$$(\mathcal{K}_i, \psi_i) : \widetilde{\mathcal{R}}_i \mathrel{\Leftarrow\!\!\!\Rightarrow} \widetilde{\mathcal{R}}_{i+1}, \quad \forall\, i \in \mathbb{Z}.$$

Let us denote by $\mathcal{R}_\ell^i$ and $\mathcal{R}_r^i$ the two components of $\mathcal{R}_i^-$ and by $\mathcal{R}_d^i$ and $\mathcal{R}_u^i$ the two components of $\mathcal{R}_i^+$. Then the following conclusions hold:

- *There is a sequence $(w_k)_{k \in \mathbb{Z}}$ such that $w_k \in \mathcal{K}_k$ and $\psi_k(w_k) = w_{k+1}$, for all $k \in \mathbb{Z}$;*

- *For each $j \in \mathbb{Z}$ there exists a compact connected set $\mathcal{C}_j \subseteq \mathcal{K}_j$ satisfying*

$$\mathcal{C}_j \cap \mathcal{R}_d^j \neq \emptyset, \quad \mathcal{C}_j \cap \mathcal{R}_u^j \neq \emptyset$$

  and such that, for every $w \in \mathcal{C}_j$, there is a sequence $(y_i)_{i \geq j}$ with $y_j = w$ and

$$y_i \in \mathcal{K}_i, \quad \psi_i(y_i) = y_{i+1}, \ \forall\, i \geq j;$$

- *If there are integers h and l, with $h < l$, such that $\widetilde{\mathcal{R}}_h = \widetilde{\mathcal{R}}_l$, then there exists a finite sequence $(z_i)_{h \leq i \leq l-1}$, with $z_i \in \mathcal{K}_i$ and $\psi_i(z_i) = z_{i+1}$ for each $i = h, \ldots, l-1$, such that $z_l = z_h$, that is, z_h is a fixed point of $\psi_{l-1} \circ \cdots \circ \psi_h$ in $\mathcal{K}_h$.*

Proof. We prove the conclusions of the theorem in the reverse order. So, let us start with the verification of the last assertion. By the assumptions and by Definition 1.1.1, it is easy to check that

$$(\mathcal{H}, \psi_{l-1} \circ \cdots \circ \psi_h) : \widetilde{\mathcal{R}}_h \Longleftrightarrow \widetilde{\mathcal{R}}_l, \tag{1.1.8}$$

where

$$\mathcal{H} := \{z \in \mathcal{K}_h : \psi_i \circ \cdots \circ \psi_h(z) \in \mathcal{K}_{i+1}, \forall i = h, \dots, l-1\}.$$

With the positions $\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}_h = \widetilde{\mathcal{R}}_l$ and $\phi = \psi_{l-1} \circ \cdots \circ \psi_h$, we read condition (1.1.8) as $(\mathcal{H}, \phi) : \widetilde{\mathcal{R}} \Longleftrightarrow \widetilde{\mathcal{R}}$ and therefore the thesis follows immediately by Theorem 1.1.3.

As regards the second conclusion, without loss of generality, we can assume $j = 0$. Let us define the closed set

$$\mathcal{S} := \{z \in \mathcal{K}_0 : \psi_j \circ \cdots \circ \psi_0(z) \in \mathcal{K}_{j+1}, \forall j \geq 0\}$$

and fix a path $\gamma_0 : [0, 1] \to \mathcal{R}_0$ such that $\gamma_0(0)$ and $\gamma_0(1)$ belong to the different components of $\mathcal{R}_0^-$. Then, since $(\mathcal{K}_0, \psi_0) : \widetilde{\mathcal{R}}_0 \Longleftrightarrow \widetilde{\mathcal{R}}_1$, there exists a subinterval

$$[t_1', t_1''] \subseteq [t_0', t_0''] := [0, 1]$$

such that

$$\gamma_0(t) \in \mathcal{K}_0 \quad \text{and} \quad \gamma_1(t) := \psi_0(\gamma_0(t)) \in \mathcal{R}_1, \quad \forall t \in [t_1', t_1''].$$

By the same assumption, we also have that $\psi_0(\gamma_0(t_1'))$ and $\psi_0(\gamma_0(t_1''))$ belong to different components of $\mathcal{R}_1^-$. Similarly, there exists a subinterval

$$[t_2', t_2''] \subseteq [t_1', t_1'']$$

such that

$$\gamma_1(t) \in \mathcal{K}_1 \quad \text{and} \quad \gamma_2(t) := \psi_1(\gamma_1(t)) \in \mathcal{R}_2, \quad \forall t \in [t_2', t_2''],$$

with $\psi_1(\gamma_1(t_2'))$ and $\psi_1(\gamma_1(t_2''))$ belonging to the different components of $\mathcal{R}_2^-$. Defining

$$\Gamma_2 := \{x \in \gamma_0([t_1', t_1'']) : \psi_0(x) \in \gamma_1([t_2', t_2''])\} \subseteq \{z \in \mathcal{K}_0 : \psi_0(z) \in \mathcal{K}_1\}$$

and proceeding by induction, we can find a decreasing sequence of nonempty compact sets

$$\Gamma_0 := \gamma_0([t_0', t_0'']) \supseteq \Gamma_1 := \gamma_0([t_1', t_1'']) \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_n \supseteq \Gamma_{n+1} \supseteq \dots$$

such that $\psi_j \circ \cdots \circ \psi_0(\Gamma_{j+1}) \subseteq \mathcal{R}_{j+1}$, with $\psi_j \circ \cdots \circ \psi_0(\Gamma_{j+1}) \cap \mathcal{R}_\ell^{j+1} \neq \emptyset$ and $\psi_j \circ \cdots \circ \psi_0(\Gamma_{j+1}) \cap \mathcal{R}_r^{j+1} \neq \emptyset$, for $j \geq 0$. Moreover, for every $i \geq 1$, we have that

$$\Gamma_{i+1} \subseteq \{z \in \mathcal{K}_0 : \psi_{j-1} \circ \cdots \circ \psi_0(z) \in \mathcal{K}_j, \forall j = 1, \ldots, i\}.$$

As it is straightforward to see, $\cap_{j=0}^{+\infty} \Gamma_j \neq \emptyset$ and for any $z \in \cap_{j=0}^{+\infty} \Gamma_j$ it holds that $\psi_n \circ \cdots \circ \psi_0(z) \in \mathcal{K}_{n+1}$, $\forall n \in \mathbb{N}$. In this way we have shown that any path γ_0 joining in $\mathcal{R}_0$ the two sides of $\mathcal{R}_0^-$ intersects $\mathcal{S}$. The existence of the connected compact set $\mathcal{C}_0 \subseteq \mathcal{S} \subseteq \mathcal{K}_0$ joining the two components of $\mathcal{R}_0^+$ comes from Lemma 1.1.17. By the definition of $\mathcal{S}$ it is obvious that any point of $\mathcal{C}_0 \subseteq \mathcal{S}$ generates a sequence as required in the statement of the theorem.

The first conclusion follows now by a standard diagonal argument (see, e.g., [69, Proposition 5] and [117, Theorem 2.2]), which allows to extend the result to bi-infinite sequences once it has been proved for one-sided sequences. $\qquad \square$

Proof of Theorem 1.1.8. The first conclusion easily follows from Theorem 1.1.3. As regards the remaining ones, given a two-sided sequence $(s_h)_{h \in \mathbb{Z}} \in \{0, \ldots, m-1\}^{\mathbb{Z}}$, they can be achieved by applying Theorem 1.1.10 with the positions $\mathcal{R}_i = \mathcal{R}$, $\psi_i = \psi$ and $\mathcal{K}_i = \mathcal{K}_{s_i}$, $\forall i \in \mathbb{Z}$. The details are omitted since they are of straightforward verification. $\qquad \square$

We end this subsection with the presentation and discussion of some stretching relations alternative to the one in Definition 1.1.1. In particular we will ask ourselves if the results on the existence and localization of fixed points are still valid with respect to these new concepts.

For all the next definitions, the basic setting concerns the following framework: *Let X be a metric space. Assume $\psi : X \supseteq D_\psi \to X$ is a map defined on a set D_ψ and let $\widetilde{\mathcal{A}} := (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} := (\mathcal{B}, \mathcal{B}^-)$ be oriented rectangles of X. Let also $\mathcal{K} \subseteq \mathcal{A} \cap D_\psi$ be a compact set.*

Definition 1.1.11. We say that $(\mathcal{K}, \psi)$ *stretches* $\widetilde{\mathcal{A}}$ *to* $\widetilde{\mathcal{B}}$ *along the continua* and write

$$(\mathcal{K}, \psi) : \widetilde{\mathcal{A}} \stackrel{\frown}{\leftrightsquigarrow} \widetilde{\mathcal{B}},$$

if the following conditions hold:

- ψ is continuous on $\mathcal{K}$;

- For every continuum $\Gamma \subseteq \mathcal{A}$ with $\Gamma \cap \mathcal{A}_\ell^- \neq \emptyset$ and $\Gamma \cap \mathcal{A}_r^- \neq \emptyset$, there exists a continuum $\Gamma' \subseteq \Gamma \cap \mathcal{K}$ such that $\psi(\Gamma') \subseteq \mathcal{B}$ and

$$\psi(\Gamma') \cap \mathcal{B}_\ell^- \neq \emptyset, \quad \psi(\Gamma') \cap \mathcal{B}_r^- \neq \emptyset.$$

While the above definition is based on the one given by Kennedy and Yorke in [71], the following bears some resemblances to that of "family of expanders" considered in [69].

Definition 1.1.12. We say that $(\mathcal{K}, \psi)$ *expands* $\widetilde{\mathcal{A}}$ *across* $\widetilde{\mathcal{B}}$ and write

$$(\mathcal{K}, \psi) : \widetilde{\mathcal{A}} \; \overset{\frown}{\underset{\smile}{\longrightarrow}} \; \widetilde{\mathcal{B}},$$

if the following conditions hold:

- ψ is continuous on $\mathcal{K}$;

- For every continuum $\Gamma \subseteq \mathcal{A}$ with $\Gamma \cap \mathcal{A}_\ell^- \neq \emptyset$ and $\Gamma \cap \mathcal{A}_r^- \neq \emptyset$, there exists a nonempty compact set $P \subseteq \Gamma \cap \mathcal{K}$ such that $\psi(P)$ is a continuum contained in $\mathcal{B}$ and

$$\psi(P) \cap \mathcal{B}_\ell^- \neq \emptyset, \quad \psi(P) \cap \mathcal{B}_r^- \neq \emptyset.$$

When it is possible to take $\mathcal{K} = \mathcal{A}$, we simply write $\psi : \widetilde{\mathcal{A}} \; \overset{\frown}{\underset{\smile}{\longrightarrow}} \; \widetilde{\mathcal{B}}$ and $\psi : \widetilde{\mathcal{A}} \; \overset{\frown}{\underset{\smile}{\longrightarrow}} \; \widetilde{\mathcal{B}}$ in place of $(\mathcal{A}, \psi) : \widetilde{\mathcal{A}} \overset{\frown}{\underset{\smile}{\longrightarrow}} \widetilde{\mathcal{B}}$ and $(\mathcal{A}, \psi) : \widetilde{\mathcal{A}} \; \overset{\frown}{\underset{\smile}{\longrightarrow}} \; \widetilde{\mathcal{B}}$, respectively. In analogy to Definition 1.1.2, one can define the variants $\overset{\frown}{\underset{\smile}{\longrightarrow}}^m$ and $\overset{\frown}{\underset{\smile}{\longrightarrow}}^m$, when multiple coverings occur.

As we shall see in a moment, the above relations behave in a different way with respect to the possibility of detecting fixed points. Indeed, since the ranges of paths are a particular kind of continua, one can follow the same steps as in the proof of Theorem 1.1.3 and show that an analogous result still holds when the property of stretching along the paths is replaced with the one of stretching along the continua (cf. [125, Theorem 2.10]). On the other hand, this is no more true when the relation in Definition 1.1.12 is fulfilled, since in general it guarantees neither the existence of fixed points nor their localization (in case that fixed points do exist).

Concerning the existence of fixed points, a possible counterexample is described in Figure 1.5 and is inspired to the bulging horseshoe in [69, Fig. 4]. For sake of conciseness, we prefer to present it by means of a series of graphical illustrations in Figures 1.5–1.8: we point out, however, that it is based on a concrete definition of a planar map (whose form, although complicated, can be explicitly given in analytical terms).

With respect to the localization of fixed points, a counterexample can instead be obtained by suitably adapting a one-dimensional map to the planar case. Indeed, if $f : \mathbb{R} \supseteq [0, 1] \to \mathbb{R}$ is any continuous function, we can set

$$\psi(x_1, x_2) := (f(x_1), x_2) \tag{1.1.9}$$

and have a continuous planar map defined on the unit square $[0,1]^2$ of $\mathbb{R}^2$, inheriting all the interesting properties of f. Notice that, in this special case, any fixed point x^* for f generates a vertical line (x^*, s) (with $s \in [0,1]$) of fixed points for ψ. The more general framework of a map ψ defined as

$$\psi(x_1, x_2) := (f(x_1), g(x_2)),$$

for $g : [0,1] \to [0,1]$ a continuous function, could be considered as well.
In view of the above discussion, we define a continuous map $f : [0,1] \to [0,1]$ of the form

$$f(s) := \begin{cases} \frac{1-c}{a}s + c & 0 \leq s < a, \\[2mm] \frac{1}{a-b}(s-b) & a \leq s \leq b, \\[2mm] \frac{d}{1-b}(s-b) & b < s \leq 1, \end{cases} \qquad (1.1.10)$$

where a, b, c, d are fixed constants such that $0 < a < b < 1$ and $0 < c < d < 1$. For $\widetilde{\mathcal{A}}$ the unit square oriented in the standard left-right manner and ψ as in (1.1.9), it holds that $\psi : \widetilde{\mathcal{A}} \; \overset{\frown}{\leadsto} \; \widetilde{\mathcal{A}}$. In particular, we can write both

$$(\mathcal{K}_0, \psi) : \widetilde{\mathcal{A}} \; \overset{\frown}{\leadsto} \; \widetilde{\mathcal{A}}, \quad \text{for } \mathcal{K}_0 = [a,b]$$

and

$$(\mathcal{K}_1, \psi) : \widetilde{\mathcal{A}} \; \overset{\frown}{\leadsto} \; \widetilde{\mathcal{A}}, \quad \text{for } \mathcal{K}_1 = [0,a] \cup [b,1].$$

In the former case, we also have

$$(\mathcal{K}_0, \psi) : \widetilde{\mathcal{A}} \Leftdoublearrow \widetilde{\mathcal{A}}$$

and therefore, consistently with Theorem 1.1.3, there exists at least a fixed point for ψ in $\mathcal{K}_0$. On the other hand, as it is clear from Figure 1.9, there are no fixed points for ψ in $\mathcal{K}_1$. Thus, the localization of the fixed points is not guaranteed when only the relation $\overset{\frown}{\leadsto}$ is satisfied. The same example could be slightly modified in order to have that $\psi : \widetilde{\mathcal{A}} \; \overset{\frown}{\leadsto}^m \; \widetilde{\mathcal{A}}$ for an arbitrary $m \geq 2$, but just one fixed point does exist.
We finally observe that, playing with the coefficients a, b, c, d and choosing a suitable compact set $\mathcal{K}_2 \subset [0,1]$, it is possible to have

$$(\mathcal{K}_2, \psi) : \widetilde{\mathcal{A}} \; \overset{\frown}{\leadsto} \; \widetilde{\mathcal{A}},$$

for a case in which neither ψ nor ψ^2 possess fixed points in $\mathcal{K}_2$. The set $\mathcal{K}_2$ will consist of the union of some compact subintervals of $\mathcal{K}_0$ and $\mathcal{K}_1$. A graphical representation of f and f^2 is given in Figure 1.9.
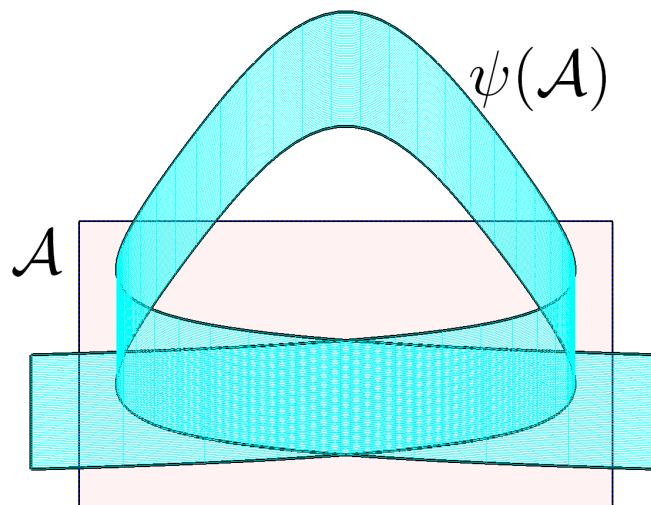
Figure 1.5: A rectangle $\mathcal{A}$, oriented by taking as $[\cdot]^-$-set the union of its left and right vertical segments, is deformed by a continuous planar map ψ onto $\psi(\mathcal{A})$, a ribbon bent across the rectangle itself. The left and right sides of the domain are homeomorphically transformed onto the two endings of the ribbon. Th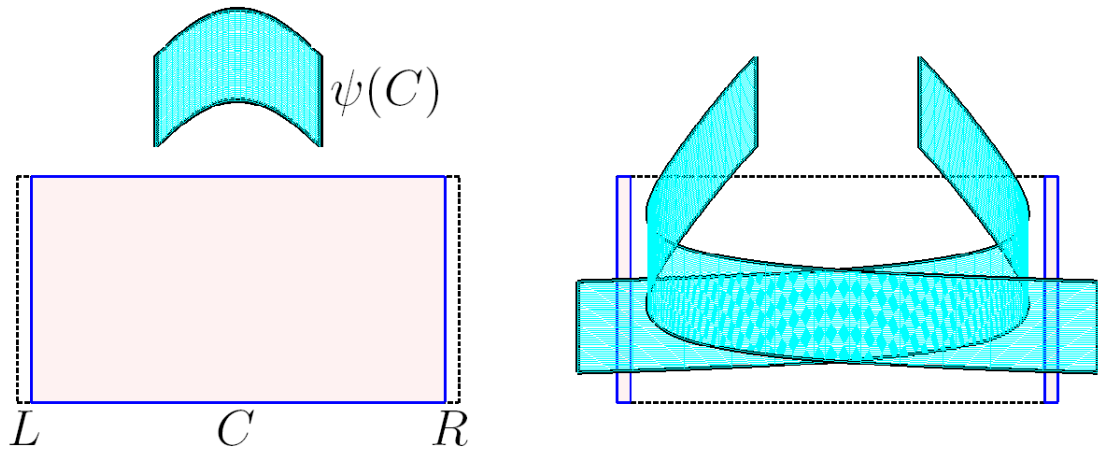is is an example of the case $\psi : \widetilde{\mathcal{A}} \; \overset{\frown}{\hookrightarrow} \widetilde{\mathcal{A}}$.

Figure 1.6: In regard to the example portrayed in the previous picture, we emphasize how some sub-rectangles of $\mathcal{A}$ are transformed by the map ψ. In particular, the main central part C of the rectangle is pushed out of $\mathcal{A}$ (picture at the left), while the two narrow rectangles L and R near the left and right sides of it, respectively, are stirred onto two overlapping bent strips (picture at the right). This can happen in two different ways, as shown in Figures 1.7–1.8. One can pass from a configuration to the other by considering, instead of the map ψ, the related map $(x_1, x_2) \mapsto \psi(-x_1, x_2)$.
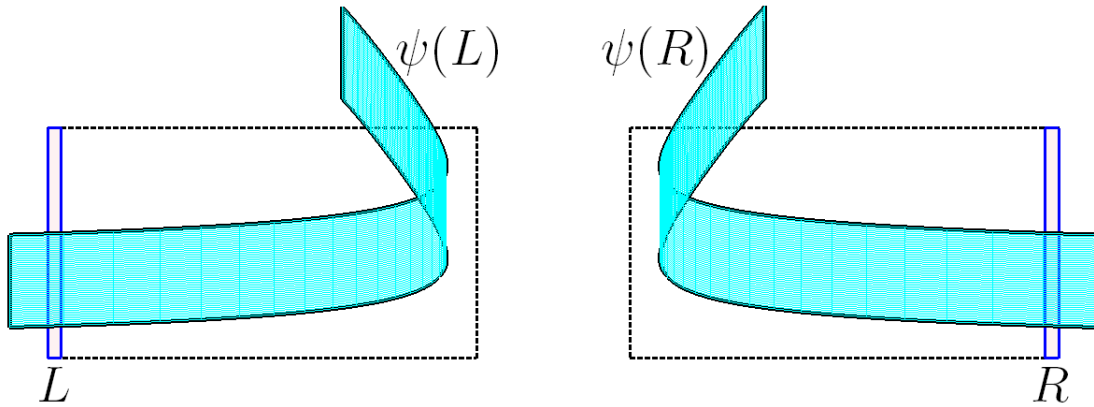

Figure 1.7: In this case, we enter the setting of Theorem 1.1.3. By applying our result to the narrow rectangles L and R (provided they are suitably oriented in an obvious left-right manner), we can prove the existence of at least two fixed points for the planar map ψ in $\mathcal{A}$ (one fixed point lies in L and the other in R).
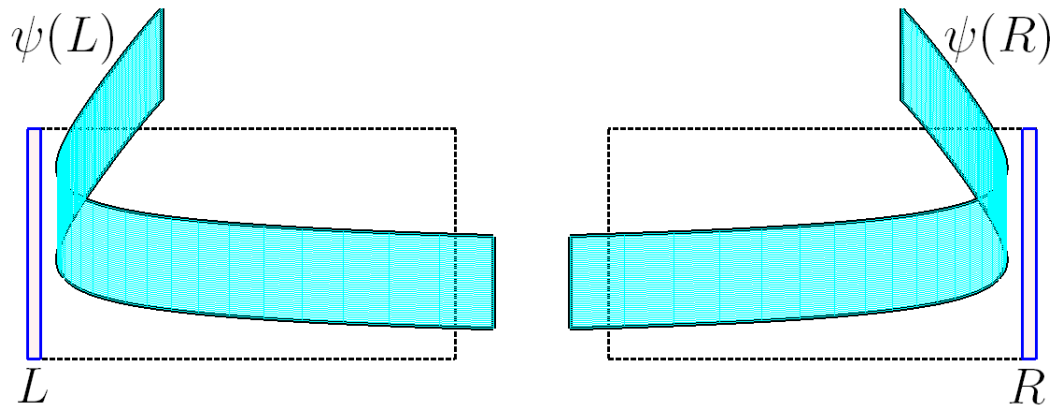
Figure 1.8: In the framework depicted above, we do not have any fixed point. In fact, the central main part C of the rectangle $\mathcal{A}$ is mapped by ψ onto a hat-like figure outside the domain (see the left of Figure 1.6) and, at the same time, each of the two narrow rectangles L and R, which constitute the remaining part of the domain, is mapped onto a set which is disjoint from itself.
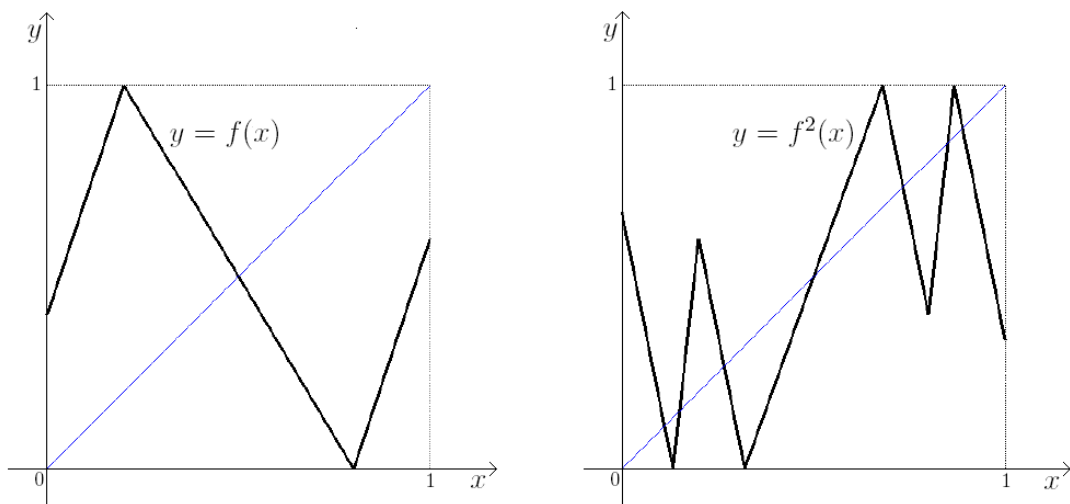

Figure 1.9: Example of the graph of a function f (left) and of its iterate f^2 (right), with f defined as in (1.1.10). We have drawn the line $y = x$ in order to show the fixed points and the points of period 2.

1.1.2 The Crossing Lemma

The fundamental result from plane topology that we have employed in the proof of the main theorems of Subsection 1.1.1 is the Crossing Lemma (cf. Lemma 1.1.17 below), which establishes the existence of a continuum $\mathcal{C}$ joining two opposite sides of a rectangle and contained in a given compact set $\mathcal{S}$. More precisely, the existence of $\mathcal{C}$ is guaranteed by the *cutting property* (CP) for $\mathcal{S}$ in Lemma 1.1.17, that is, a condition which asserts that $\mathcal{S}$ intersects any path contained in the rectangle and joining the other two sides. In the higher dimensional case the set $\mathcal{S}$ looks like a *cutting surface* (see Section 1.2).

In the next pages we furnish a verification of the Crossing Lemma, presenting a few topological facts that allow to achieve it. Even if, for the reader's convenience, we try to make the treatment as self-contained as possible, some theorems are stated below without proof.

The first among such results is Whyburn Lemma (see Lemma 1.1.13). It is a central tool in bifurcation theory and in the application of continuation methods to nonlinear equations [3, 100, 133]. We quote here one of its most used versions, as taken from [133, Lemma 1.33]. The interested reader can find a proof as well as further details in Chapter V of Kuratowski's book [84]. We refer also to [3] and [100] for interesting surveys about the role of connectedness in the study of fixed point theory and operator equations.

Lemma 1.1.13 (Whyburn Lemma)**.** *If A and B are (nonempty) disjoint closed subsets of a compact metric space K such that no connected components of K intersect both A and B, then $K = K_A \cup K_B$, where K_A and K_B are disjoint compact sets containing A and B, respectively.*

In the applications, Whyburn Lemma is often employed in the form of the following alternative:

Either there exists a continuum $\mathcal{C} \subseteq K$ with $\mathcal{C} \cap A \neq \emptyset$ and $\mathcal{C} \cap B \neq \emptyset$, or A and B are separated, in the sense that there exists K_A and K_B as in Lemma 1.1.13.

The other result that we state without proof is a two-dimensional version of the well-known Brouwer fixed point Theorem. In such a simplified form, it can be proved by using the winding number [35].

Theorem 1.1.14. *Let $\phi : D \to D$ be a continuous self-mapping of a set $D \subseteq \mathbb{R}^2$, where D is homeomorphic to a closed disc. Then ϕ has at least one fixed point in D.*

In the sequel, the set D will be the product of two nondegenerate compact intervals $[a, b]$ and $[c, d]$.

The next lemma is a particular case of the celebrated Leray-Schauder Continuation Principle [90, Théorème Fondamental]. Although its intuitiveness, the following result plays a useful role in various contexts: for a different application, see for instance [82], while the interested reader can find more corresponding details in [100, 171] and in the bibliography therein. By its significance in our approach, we furnish two proofs of Lemma 1.1.15: the first one is less direct, but elementary, while the second one is based on degree theory and thus it is immediately extendable to an higher dimensional setting [124]. Introducing the projections of the plane onto the coordinate axes

$$\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}, \quad \pi_1(x, y) := x, \ \pi_2(x, y) := y, \ \forall (x, y) \in \mathbb{R}^2 \,,$$

it can be stated as follows:

Lemma 1.1.15. *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *be a continuous map such that*

$$f(x, y) \leq 0, \quad \forall (x, y) \in [a, b] \times \{c\},$$

$$f(x, y) \geq 0, \quad \forall (x, y) \in [a, b] \times \{d\}.$$

Then there exists a compact connected set

$$\mathcal{S} \subseteq \{(x, y) \in [a, b] \times [c, d] : f(x, y) = 0\},$$

such that

$$\pi_1(\mathcal{S}) = [a, b].$$

Proof. Let

$$K := \{(x, y) \in [a, b] \times [c, d] : f(x, y) = 0\}$$

be the zero-set for the map f and let

$$A := K \cap (\{a\} \times [c, d]), \quad B := K \cap (\{b\} \times [c, d]).$$

By the intermediate value theorem, A and B are both nonempty. They are also disjoint compact subsets of the compact set K.
Assume, by contradiction, that there are no compact connected subsets of K intersecting both A and B. Then, according to Whyburn Lemma 1.1.13, there exist compact sets K_A and K_B with

$$K = K_A \cup K_B, \quad K_A \supseteq A, \ K_B \supseteq B, \ K_A \cap K_B = \emptyset.$$

We can thus define the continuous function (see [114, Lemma on Interpolation])

$$g(x, y) := \frac{\mathrm{dist}(p, K_A) - \mathrm{dist}(p, K_B)}{\mathrm{dist}(p, K_A) + \mathrm{dist}(p, K_B)} \quad \text{for} \ \ p = (x, y) \in [a, b] \times [c, d],$$

which has $[-1, 1]$ as range and attains the value -1 on K_A and the value 1 on K_B. Notice that the denominator never vanishes as the compact sets K_A and K_B are disjoint.

We also introduce the projections of the real line onto the intervals $[a, b]$ and $[c, d]$, respectively. For $s \in \mathbb{R}$ they are defined as

$$P_{[a,b]}(s) := \max\{a, \min\{s, b\}\}, \quad P_{[c,d]}(s) := \max\{c, \min\{s, d\}\}. \qquad (1.1.11)$$

Finally, we consider the continuous map

$$\phi = (\phi_1, \phi_2) : [a, b] \times [c, d] \to [a, b] \times [c, d],$$

with components

$$\phi_1(x, y) := P_{[a,b]}(x - g(x, y)), \quad \phi_2(x, y) := P_{[c,d]}(y - f(x, y)).$$

Since $D := [a, b] \times [c, d]$ is homeomorphic to the closed unit disc, Brouwer Theorem 1.1.14 ensures the existence of a fixed point $z = (z_1, z_2) \in D$ for the map ϕ, that is,

$$z_1 = P_{[a,b]}(z_1 - g(z_1, z_2)), \quad z_2 = P_{[c,d]}(z_2 - f(z_1, z_2)). \qquad (1.1.12)$$

We claim that

$$z_2 - f(z_1, z_2) \in [c, d]. \qquad (1.1.13)$$

Indeed, if $z_2 - f(z_1, z_2) \notin [c, d]$, then either $z_2 - f(z_1, z_2) < c$ or $z_2 - f(z_1, z_2) > d$. In the former case, $P_{[c,d]}(z_2 - f(z_1, z_2)) = c$ and thus, from the second relation in (1.1.12), we find $z_2 = c$. By the sign condition on $f(x, y)$ for $(x, y) \in [a, b] \times \{c\}$, it follows that $f(z) = f(z_1, c) \le 0$ and therefore $z_2 - f(z_1, z_2) = c - f(z_1, c) \ge c$. This contradicts the assumption $z_2 - f(z_1, z_2) < c$. On the other hand, if $z_2 - f(z_1, z_2) > d$, then $P_{[c,d]}(z_2 - f(z_1, z_2)) = d$ and thus the second relation in (1.1.12) implies $z_2 = d$. By the sign condition on $f(x, y)$ for $(x, y) \in [a, b] \times \{d\}$, it follows that $f(z) = f(z_1, d) \ge 0$ and therefore $z_2 - f(z_1, z_2) = d - f(z_1, d) \le d$. This contradicts the assumption $z_2 - f(z_1, z_2) > d$. Condition (1.1.13) is thus proved.

From (1.1.13) and the second relation in (1.1.12), we obtain $z_2 = P_{[c,d]}(z_2 - f(z_1, z_2)) = z_2 - f(z_1, z_2)$. Hence $f(z) = 0$, that is, $z \in K = K_A \cup K_B$ and, recalling the definition of the interpolation map g, we conclude that

$$g(z) = \pm 1. \qquad (1.1.14)$$

Arguing as in the proof of (1.1.13), we can also show that

$$z_1 - g(z_1, z_2) \in [a, b].$$

Then, the first relation in (1.1.12) implies that $z_1 = P_{[a,b]}(z_1 - g(z_1, z_2)) = z_1 - g(z_1, z_2)$. Hence,

$$g(z) = 0,$$

in contradiction with (1.1.14). Thus, by Whyburn Lemma 1.1.13, the existence of a continuum $\mathcal{S} \subseteq K = \{(x, y) \in [a, b] \times [c, d] : f(x, y) = 0\}$, with $\mathcal{S} \cap A \neq \emptyset$ and $\mathcal{S} \cap B \neq \emptyset$, follows. This means that the set $\mathcal{S}$ contains points of the form (a, u) and (b, v), with $u, v \in [c, d]$, and consequently the image of $\mathcal{S}$ under π_1 covers the interval $[a, b]$. The verification of the lemma is complete.

We present also an alternative proof via degree theory.
Consider the auxiliary function $\widehat{f} : [a, b] \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widehat{f}(x, y) := f(x, P_{[c,d]}(y)) + \min\{y - c, \max\{0, y - d\}\}.$$

The map $\widehat{f}$ is continuous and such that

$$\widehat{f}(x, y) = f(x, y), \quad \forall\, (x, y) \in [a, b] \times [c, d].$$

Moreover, it satisfies the following inequalities:

$$\widehat{f}(x, y) < 0, \quad \forall\, (x, y) \in [a, b] \times (-\infty, c\,[,$$
$$\widehat{f}(x, y) > 0, \quad \forall\, (x, y) \in [a, b] \times \,]\,d, +\infty).$$

If we treat the variable $x \in [a, b]$ as a parameter and consider the open set $\Omega := \,]c - 1, d + 1[$, we have that $\widehat{f}(x, y) \neq 0$, for every $x \in [a, b]$ and $y \in \partial\Omega$. The Brouwer degree $\deg(\widehat{f}(a, \cdot), \Omega, 0)$ is well-defined and nontrivial, since, by the sign condition $\widehat{f}(a, c - 1) < 0 < \widehat{f}(a, d + 1)$, it holds that

$$\deg\left(\widehat{f}(a, \cdot), \Omega, 0\right) = 1.$$

The Leray-Schauder Continuation Theorem (see [90, 100, 171]) ensures that the set of solution pairs

$$\widehat{K} := \left\{(x, y) \in [a, b] \times \Omega : \widehat{f}(x, y) = 0\right\}$$

contains a continuum along which x assumes all the values in $[a, b]$. By the definition of $\widehat{f}$, it is clear that $\widehat{K} = K$ and this gives another proof of the existence of $\mathcal{S}$. $\qquad\square$

The next result shows that the continuum $\mathcal{S}$ found in Lemma 1.1.15 can be ε-approximated by paths.

Lemma 1.1.16. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous map such that*

$$f(x, y) \leq -1, \qquad \forall \, (x, y) \in [a, b] \times \{c\},$$
$$f(x, y) \geq 1, \qquad \forall \, (x, y) \in [a, b] \times \{d\}.$$

Then, for each $\varepsilon > 0$, there exists a continuous map

$$\gamma = \gamma_\varepsilon : [0, 1] \to [a, b] \times [c, d],$$

such that

$$\gamma(0) \in \{a\} \times [c, d], \quad \gamma(1) \in \{b\} \times [c, d]$$

and

$$|f(\gamma(t))| < \varepsilon, \quad \forall \, t \in [0, 1].$$

Proof. Without loss of generality, we can assume $0 < \varepsilon < 1$.
First of all, we introduce the continuous function

$$\tilde{f} : \mathbb{R}^2 \to \mathbb{R}, \quad \tilde{f}(x, y) := f(P_{[a,b]}(x), P_{[c,d]}(y)),$$

which extends f to the whole plane by means of the projections in (1.1.11).
From the sign conditions on f, we easily see that

$$\tilde{f}(x, y) \leq -1, \qquad \forall \, (x, y) \in \mathbb{R}^2 : y \leq c \,,$$
$$\tilde{f}(x, y) \geq 1, \qquad \forall \, (x, y) \in \mathbb{R}^2 : y \geq d \,.$$

We also define the set

$$T_\varepsilon := \left\{ (x, y) \in \mathbb{R}^2 : |\tilde{f}(x, y)| < \varepsilon \right\},$$

which is an open subset of the strip $\mathbb{R} \times \,]c, d[$.
By Lemma 1.1.15, there exists a compact connected set

$$\mathcal{S} \subseteq \{(x, y) \in [a, b] \times [c, d] : f(x, y) = 0\},$$

such that $\pi_1(\mathcal{S}) = [a, b]$. Notice that

$$\mathcal{S} \subseteq T_\varepsilon \,.$$

For every $p \in \mathcal{S}$, there exists an open disc $B(p, \delta_p)$ with center in p and radius $\delta_p > 0$, such that $B(p, \delta_p) \subseteq T_\varepsilon$. By compactness, we can find a finite number of points $p_1, \ldots, p_n \in \mathcal{S}$ such that

$$\mathcal{S} \subseteq B := \bigcup_{i=1}^{n} B(p_i, \delta_i) \subseteq T_\varepsilon \,,$$

where we have set $\delta_i := \delta_{p_i}$. Without loss of generality (adding two further discs, if necessary), we can suppose that

$$p_1 \in (\{a\} \times [c,d]) \cap T_\varepsilon \quad \text{and} \quad p_n \in (\{b\} \times [c,d]) \cap T_\varepsilon .$$

The open set B is connected. Indeed, it is union of connected sets (the open discs $B(p_i, \delta_i)$) and each of such connected sets has nonempty intersection with the connected set $\mathcal{S}$.

Since every open connected set in the plane is also arcwise connected, there exists a continuous map $\omega_\varepsilon : [s_0, s_1] \to B$ with $\omega_\varepsilon(s_0) = p_1$ and $\omega_\varepsilon(s_1) = p_n$. Then, we define

$$s' := \max\{s \in [s_0, s_1] : \omega_\varepsilon(s) \in \{a\} \times [c,d]\}$$

and

$$s'' := \min\{s \in [s', s_1] : \omega_\varepsilon(s) \in \{b\} \times [c,d]\}.$$

Hence, the path

$$\gamma_\varepsilon(t) := \omega_\varepsilon(s' + t \cdot (s'' - s')), \quad \text{for } t \in [0,1],$$

is continuous and has the desired properties. Indeed, $\gamma_\varepsilon(t) \in ([a,b] \times \mathbb{R}) \cap T_\varepsilon \subseteq [a,b] \times]c,d[$ and therefore $|\tilde{f}(\gamma_\varepsilon(t))| = |f(\gamma_\varepsilon(t))| < \varepsilon$, for all $t \in [0,1]$. Moreover, $\gamma_\varepsilon(0) = \omega_\varepsilon(s') \in \{a\} \times [c,d]$ and $\gamma_\varepsilon(1) = \omega_\varepsilon(s'') \in \{b\} \times [c,d]$. The proof is complete. $\square$

We are now in position to prove the following:

Lemma 1.1.17 (Crossing Lemma)**.** *Let $K \subseteq [0,1]^2$ be a compact set which satisfies the* cutting property[1]:

(CP)
$$K \cap \gamma([0,1]) \neq \emptyset,$$
for each continuous map $\gamma : [0,1] \to [0,1]^2$,
with $\gamma(0) \in \{0\} \times [0,1]$ and $\gamma(1) \in \{1\} \times [0,1]$.

Then there exists a continuum $\mathcal{C} \subseteq K$ with

$$\mathcal{C} \cap ([0,1] \times \{0\}) \neq \emptyset \quad \text{and} \quad \mathcal{C} \cap ([0,1] \times \{1\}) \neq \emptyset.$$

Proof. By the assumptions, the sets

$$A := K \cap ([0,1] \times \{0\}), \quad B := K \cap ([0,1] \times \{1\})$$

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[1]In view of Definition 1.2.1 in Section 1.2, we could also say that K *cuts the arcs between* $\{0\} \times [0,1]$ *and* $\{1\} \times [0,1]$.

are compact, nonempty and disjoint. The thesis is achieved if we prove that there exists a continuum $\mathcal{C} \subseteq K$ with $\mathcal{C} \cap A \neq \emptyset$ and $\mathcal{C} \cap B \neq \emptyset$. If, by contradiction, there is no continuum of this kind, Whyburn Lemma 1.1.13 implies that the set K can be decomposed as

$$K = K_A \cup K_B, \quad K_A \cap K_B = \emptyset, \quad K_A \supseteq A, \quad K_B \supseteq B,$$

with K_A and K_B compact sets. Let us then define the compact sets

$$K'_A := K_A \cup ([0,1] \times \{0\}), \quad K'_B := K_B \cup ([0,1] \times \{1\}).$$

We claim that they are disjoint. Indeed, since $K_A \cap K_B = \emptyset$, if there exists $\bar{x} \in K'_A \cap K'_B$, then $\bar{x} \in K_A \cap ([0,1] \times \{1\})$ or $\bar{x} \in K_B \cap ([0,1] \times \{0\})$. In the former case, recalling that $K_A \subseteq K$, we find that $\bar{x} \in B \subseteq K_B$ and thus $\bar{x} \in K_A \cap K_B = \emptyset$. The same contradiction can be achieved by assuming $\bar{x} \in K_B \cap ([0,1] \times \{0\})$. The claim on the disjointness of K'_A and K'_B is thus checked.

Arguing like in the proof of Lemma 1.1.15, we introduce the continuous interpolation function

$$f(x,y) := \frac{\operatorname{dist}(p, K'_A) - \operatorname{dist}(p, K'_B)}{\operatorname{dist}(p, K'_A) + \operatorname{dist}(p, K'_B)} \quad \text{for} \quad p = (x,y) \in [0,1]^2$$

which has $[-1,1]$ as range and attains the value -1 on K'_A and the value 1 on K'_B. By Lemma 1.1.16 there exists a continuous map $\gamma : [0,1] \to [0,1]^2$ with $\gamma(0) \in \{0\} \times [0,1]$ and $\gamma(1) \in \{1\} \times [0,1]$ such that

$$|f(\gamma(t))| < \frac{1}{2}, \quad \forall t \in [0,1]. \tag{1.1.15}$$

The cutting property (CP) ensures the existence of $t^* \in [0,1]$ such that

$$\gamma(t^*) \in K.$$

By the splitting of $K = K_A \cup K_B$ and the definition of f we find that

$$f(\gamma(t^*)) = \pm 1,$$

in contradiction with (1.1.15). This concludes the proof. $\qquad\square$

The above Crossing Lemma may be used to give a simple proof of Poincaré-Miranda Theorem in dimension two. We recall that Poincaré-Miranda Theorem asserts the existence of a zero for a continuous vector field $F = (F_1, F_2)$ defined on a rectangle $[a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$, such that $F_1(a_1, x_2) \cdot F_1(b_1, x_2) \leq 0$,

for every $x_2 \in [a_2, b_2]$ and $F_2(x_1, a_2) \cdot F_2(x_1, b_2) \leq 0$, for every $x_1 \in [a_1, b_1]$. The result holds also for the standard hypercube of $\mathbb{R}^N$ and for N-dimensional rectangles, as we shall see in Section 1.2 (cf. Theorem 1.2.7). Such theorem is usually referred to Carlo Miranda, who in 1940 noticed its equivalence to Brouwer fixed point Theorem [107]. On the other hand, as remarked in [101], Henry Poincaré in [127, 128] had already announced this result with a suggestion of a correct proof using the Kronecker's index. In [128], with regard to the two-dimensional case, Poincaré (assuming strict inequalities for the components of the vector field on the boundary of the rectangle) described also an heuristic proof as follows: the "curve" $F_2 = 0$ departs from a point of the side $x_1 = b_1$ and ends at some point of $x_1 = a_1$; in the same manner, the curve $F_1 = 0$, departing from a point of $x_2 = b_2$ and ending at some point of $x_2 = a_2$, must necessarily meet the first "curve" in the interior of the rectangle. Further information and historical remarks about Poincaré-Miranda Theorem can be found in [81, 101].

Following Poincaré's heuristic argument, one could adapt his proof in the following manner: if $\gamma(t)$ is a path contained in the rectangle $[a_1, b_1] \times [a_2, b_2]$ and joining the left and the right sides, by Bolzano Theorem there exists at least a zero of $F_1(\gamma(t))$ and this, in turns, means that any path as above meets the set $\mathcal{S} := F_1^{-1}(0)$. The Crossing Lemma then implies the existence of a compact connected set $\mathcal{C}_1 \subseteq F_1^{-1}(0)$ which intersects the lower and the upper sides of the rectangle. At this point one can easily achieve the conclusion in various different ways. For instance, one could just repeat the same argument on F_2 in order to obtain a compact connected set $\mathcal{C}_2 \subseteq F_2^{-1}(0)$ which intersects the left and the right sides of the rectangle and thus prove the existence of a zero of the vector field F using the fact that $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$. Alternatively, one could apply Bolzano Theorem and find a zero for F_2 restricted to $\mathcal{C}_1$ (see also [134] for a similar use of a variant of the Crossing Lemma and [124] for extensions to the N-dimensional setting). Conversely, it is possible to provide a proof of the Crossing Lemma via Poincaré-Miranda Theorem [118].

1.2 The N-dimensional setting

1.2.1 Cutting surfaces and topological results

In addition to the notions on paths introduced in Section 1.1, we need some further concepts in order to present a generalization to higher dimensional settings of the planar theory previously explained.

If $\gamma_1, \gamma_2 : [0, 1] \to W$ are paths in the topological space W with $\gamma_1(1) = \gamma_2(0)$,

we define the *gluing of γ_1 with γ_2* as the path $\gamma_1 \star \gamma_2 : [0,1] \to W$ such that

$$\gamma_1 \star \gamma_2(t) := \begin{cases} \gamma_1(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Moreover, given a path $\gamma : [0,1] \to W$, we denote by $\gamma^- : [0,1] \to W$ the path having $\overline{\gamma}$ as support, but run with reverse orientation with respect to γ, i.e. $\gamma^-(t) := \gamma(1-t)$, for all $t \in [0,1]$. We also recall that a topological space W is said *arcwise connected* if, for any couple of distinct points $p, q \in W$, there is a path $\gamma : [0,1] \to W$ such that $\gamma(0) = p$ and $\gamma(1) = q$. In the case of a Hausdorff topological space W, the range $\overline{\gamma}$ of γ turns out to be a locally connected metric continuum (a Peano space according to [64]). Thus, if W is a metric space, the above definition of arcwise connectedness is equivalent to the fact that, given any two points $p, q \in W$ with $p \ne q$, there exists an arc (i.e. the homeomorphic image of $[0,1]$) contained in W and having p and q as extreme points (see [64, pp.115-131]).

We start our explanation by presenting some topological lemmas concerning the relationship between particular surfaces and zero sets of continuous real valued functions. Analogous results can be found, often in a more implicit form, in different contexts. However, since for our applications we need a specific version of the statements, we give an independent proof with all the details.
At first, let us introduce the central concept of "cutting set".

Definition 1.2.1. Let X be an arcwise connected metric space and let A, B, C be closed nonempty subsets of X with $A \cap B = \emptyset$. We say that C *cuts the arcs between A and B* if for any path $\gamma : [0,1] \to X$, with $\overline{\gamma} \cap A \ne \emptyset$ and $\overline{\gamma} \cap B \ne \emptyset$, it follows that $\overline{\gamma} \cap C \ne \emptyset$. In the sequel, if X is a subspace of a larger metric space Z and we wish to stress that we consider only paths contained in X, we make more precise our definition by saying that C *cuts the arcs between A and B in X*.

Such definition is a modification of the classical one regarding the cutting of a space between two points in [84]. See [18] for a more general concept concerning a set C that intersects every connected set meeting two nonempty sets A and B. In the special case that A and B are the opposite faces of an N-dimensional cube, J. Kampen [67, p.512] says that C separates A and B. We prefer to use the "cutting" terminology in order to avoid any misunderstanding with other notions of separation which are more common in topology. In particular we remark that our definition agrees with the usual one of cut when A, B, C are pairwise disjoint [49].

In the sequel, even when not explicitly mentioned, we assume the basic space X to be arcwise connected. In some of the next results the local arcwise connectedness of X is required, too, i.e. for any $p \in X$ it holds that each neighborhood of p contains an arcwise connected neighborhood of p. With this respect, we recall that any connected and locally arcwise connected metric space is arcwise connected (see [84, Th.2, p.253]).

Lemma 1.2.2. *Let X be a connected and locally arcwise connected metric space and let $A, B, C \subseteq X$ be closed and nonempty sets with $A \cap B = \emptyset$. Then C cuts the arcs between A and B if and only if there exists a continuous function $f : X \to \mathbb{R}$ such that*

$$f(x) \le 0, \forall\, x \in A, \qquad f(x) \ge 0, \forall\, x \in B \qquad (1.2.1)$$

and

$$C = \{x \in X : f(x) = 0\}. \qquad (1.2.2)$$

Proof. Assume there exists a continuous function $f : X \to \mathbb{R}$ satisfying (1.2.1) and (1.2.2). Let $\gamma : [0,1] \to X$ be a path such that $\gamma(0) \in A$ and $\gamma(1) \in B$. We want to prove that $\overline{\gamma} \cap C \ne \emptyset$. Indeed, for the composite continuous function $\theta := f \circ \gamma : [0,1] \to \mathbb{R}$, we have that $\theta(0) \le 0 \le \theta(1)$ and so Bolzano Theorem ensures the existence of $t^* \in [0,1]$ with $\theta(t^*) = 0$. This means that $\gamma(t^*) \in C$ and therefore $\overline{\gamma} \cap C \ne \emptyset$. Hence, we have proved that C cuts the arcs between A and B.
Conversely, let us assume that C cuts the arcs between A and B. We introduce the auxiliary functions

$$\rho : X \to \mathbb{R}^+,$$

$$\rho(x) := \mathrm{dist}(x, C), \forall\, x \in X \qquad (1.2.3)$$

and

$$\mu : X \to \{-1, 0, 1\},$$

$$\mu(x) := \begin{cases} 0 & \text{if } x \in C, \\ -1 & \text{if } x \notin C \text{ and } \exists \text{ a path } \gamma_x : [0,1] \to X \setminus C \\ & \text{such that } \gamma_x(0) \in A \text{ and } \gamma_x(1) = x, \\ 1 & \text{elsewhere.} \end{cases} \qquad (1.2.4)$$

Observe that ρ is a continuous function with $\rho(x) = 0$ if and only if $x \in C$ and also $\mu(x) = 0$ if and only if $x \in C$. Moreover, μ is bounded.
Let $x_0 \notin C$. We claim that μ is continuous in x_0. Actually, μ is locally constant on $X \setminus C$. Indeed, since $x_0 \in X \setminus C$ (an open set) and X is locally arcwise connected, there is a neighborhood U_{x_0} of x_0 with $U_{x_0} \subseteq X \setminus C$, such that for each $x \in U_{x_0}$ there exists a path $\omega_{x_0,x}$ joining x_0 to x in U_{x_0}. Clearly, if

there is a path γ_{a,x_0} in $X \setminus C$ joining some point $a \in A$ with x_0, then the path $\gamma_{a,x_0} \star \omega_{x_0,x}$ connects a to x in $X \setminus C$. This proves that if $\mu(x_0) = -1$, then $\mu(x) = -1$ for every $x \in U_{x_0}$. On the other hand, if there is a path $\gamma_{a,x}$ in $X \setminus C$ which connects some point $a \in A$ to $x \in U_{x_0}$, then the path $\gamma_{a,x} \star \omega_{x_0,x}^-$ connects a to x_0 in $X \setminus C$. This shows that if $\mu(x_0) = 1$ (that is, it is not possible to connect x_0 to any point of A in $X \setminus C$ using a path), then $\mu(x) = 1$ for every $x \in U_{x_0}$ (that is, it is not possible to connect any point $x \in U_{x_0}$ to any point of A in $X \setminus C$ using a path).
We can now define

$$f : X \to \mathbb{R}, \qquad f(x) := \rho(x)\mu(x). \tag{1.2.5}$$

Clearly, $f(x) = 0$ if and only if $x \in C$ and, moreover, f is continuous. Indeed, if $x_0 \notin C$, we have that f is continuous in x_0 because both ρ and μ are continuous in x_0. If $x_0 \in C$ and $x_n \to x_0$ (as $n \to \infty$), then $\rho(x_n) \to 0$ and $|\mu(x_n)| \leq 1$, so that $f(x_n) \to 0 = f(x_0)$. Finally, by the definition of μ in (1.2.4), it holds that $\mu(a) = -1$, for every $a \in A \setminus C$, and hence, for such an a, it holds that $f(a) < 0$. On the contrary, if we suppose that $b \in B \setminus C$, we must have $\mu(b) = 1$. In fact, by the cutting condition, there is no path connecting in $X \setminus C$ the point b to any point of A. Therefore, in this case we have $f(b) > 0$. The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Considering the functions μ and f as in (1.2.4)–(1.2.5), we notice that their definition, although adequate from the point of view of the proof of Lemma 1.2.2, perhaps does not represent an optimal choice. For instance, we would like the sign of f to coincide for all the points located "at the same side" of A (respectively of B) with respect to C (see Figure 1.10). Having such request in mind, we will propose a different definition for the function μ in (1.2.8) below. With this respect, we introduce the following sets that we call *the side of A in X* and *the side of B in X*, respectively:

$$\mathcal{S}(A) := \{x \in X : \overline{\gamma} \cap A \neq \emptyset, \, \forall \, \text{path} \, \gamma : [0,1] \to X, \text{with } \gamma(0) = x, \gamma(1) \in B\},$$

$$\mathcal{S}(B) := \{x \in X : \overline{\gamma} \cap B \neq \emptyset, \, \forall \, \text{path} \, \gamma : [0,1] \to X, \text{with } \gamma(0) = x, \gamma(1) \in A\},$$

that is, a point x belongs to the side of A (to the side of B) if, whenever we try to connect x to B (to A) by a path, we first meet A (we first meet B). By definition, it follows that

$$A \subseteq \mathcal{S}(A), \quad B \subseteq \mathcal{S}(B).$$

In addition it is possible to check that $\mathcal{S}(A)$ and $\mathcal{S}(B)$ are closed and disjoint. This is the content of the next lemma.

Lemma 1.2.3. *Let X be a connected and locally arcwise connected metric space and let $A, B \subseteq X$ be closed and nonempty sets with $A \cap B = \emptyset$. Then $\mathcal{S}(A)$ and $\mathcal{S}(B)$ are closed and, moreover,*

$$\mathcal{S}(A) \cap \mathcal{S}(B) = \emptyset.$$

Proof. First of all, we show that $\mathcal{S}(A)$ is closed by proving that if $w \notin \mathcal{S}(A)$ then there exists a neighborhood U_w of w with $U_w \subseteq X \setminus \mathcal{S}(A)$. Indeed, if $w \notin \mathcal{S}(A)$, there is a path $\gamma : [0,1] \to X$ with $\gamma(0) = w$ and $\gamma(1) = b \in B$ and such that $\gamma(t) \notin A$, for every $t \in [0,1]$. Since A is closed and X is locally arcwise connected, there exists an arcwise connected set V_w with $w \in V_w \subseteq X \setminus A$. Hence, for every $x \in V_w$, there is a path ω_x connecting x to w in V_w. As a consequence, we find that the path $\gamma_x := \omega_x \star \gamma$ connects $x \in V_w$ to $b \in B$ with $\overline{\gamma_x} \subseteq X \setminus A$. Clearly, the neighborhood $U_w := V_w$ satisfies our requirement and this proves that $X \setminus \mathcal{S}(A)$ is open. The same argument ensures that $\mathcal{S}(B)$ is closed.

It remains to show that $\mathcal{S}(A)$ and $\mathcal{S}(B)$ are disjoint. Since $A \cap B = \emptyset$, it follows immediately from the definition of side that

$$A \cap \mathcal{S}(B) = \emptyset, \quad B \cap \mathcal{S}(A) = \emptyset.$$

Assume by contradiction that there exists

$$x \in \mathcal{S}(A) \cap \mathcal{S}(B),$$

with $x \notin A \cup B$. Since X is arcwise connected, there is a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = b \in B$. The fact that $x \in \mathcal{S}(A) \setminus A$ implies that there is $t_1 \in]0,1[$ such that $\gamma(t_1) = a_1 \in A$. On the other hand, since $x \in \mathcal{S}(B) \setminus B$, there exists $s_1 \in]0, t_1[$ such that $\gamma(s_1) = b_1 \in B$. Proceeding by induction and using repeatedly the definition of $\mathcal{S}(A)$ and $\mathcal{S}(B)$, we obtain a sequence

$$t_1 > s_1 > t_2 > \cdots > t_j > s_j > t_{j+1} > \cdots > 0$$

with $\gamma(t_i) = a_i \in A$ and $\gamma(s_i) = b_i \in B$. For $t^* = \inf t_n = \inf s_n \in [0, 1[$, it holds that $\gamma(t^*) = \lim a_n = \lim b_n \in A \cap B$, a contradiction. This concludes the proof. $\qquad\square$

Thanks to Lemma 1.2.3, it is possible to consider the cutting property with the sets $\mathcal{S}(A)$ and $\mathcal{S}(B)$ playing the role of A and B in Definition 1.2.1. In particular, in connection with Lemma 1.2.2, the following results hold:

Lemma 1.2.4. *Let X be a connected and locally arcwise connected metric space and let $A, B, C \subseteq X$ be closed and nonempty sets with $A \cap B = \emptyset$. Then C cuts the arcs between A and B if and only if C cuts the arcs between $\mathcal{S}(A)$ and $\mathcal{S}(B)$.*

Proof. One direction of the inference is obvious. In fact, every path joining A to B is also a path joining $\mathcal{S}(A)$ to $\mathcal{S}(B)$. Thus, if C cuts the arcs between $\mathcal{S}(A)$ and $\mathcal{S}(B)$, then it also cuts the arcs between A and B.

Conversely, let us assume that C cuts the arcs between A and B. We want to prove that C cuts the arcs between $\mathcal{S}(A)$ and $\mathcal{S}(B)$. By the definition of $\mathcal{S}(A)$ and $\mathcal{S}(B)$, it is straightforward to check that C cuts the arcs between A and $\mathcal{S}(B)$, as well as that it cuts the arcs between $\mathcal{S}(A)$ and B. Suppose by contradiction that there exists a path

$$\gamma : \left[\tfrac{1}{2}, 1\right] \to X \setminus C$$

such that $\gamma(\tfrac{1}{2}) = a \in \mathcal{S}(A) \setminus A$ and $\gamma(1) = b \in \mathcal{S}(B) \setminus B$. We choose a point $a_0 \in A$ and connect it to $a \in \mathcal{S}(A)$ by a path $\omega : [0, \tfrac{1}{2}] \to X$ with $\omega(0) = a_0$ and $\omega(\tfrac{1}{2}) = a$. Introduce the new path $\gamma_0 := \omega \star \gamma : [0, 1] \to X$, with $\gamma_0(0) = a_0 \in A$ and $\gamma_0(1) = b \in \mathcal{S}(B)$. By the definition of $\mathcal{S}(B)$, there exists $s_1 \in]0, 1[$ such that $\gamma_0(s_1) = b_1 \in B$. Since $B \cap \mathcal{S}(A) = \emptyset$, it follows that $b_1 \notin \mathcal{S}(A)$. Notice also that $0 < s_1 < \tfrac{1}{2}$: indeed, if $\tfrac{1}{2} < s_1 < 1$, recalling that $\gamma_0(\tfrac{1}{2}) = \gamma(\tfrac{1}{2}) = a \in \mathcal{S}(A)$ and $\gamma_0(s_1) = \gamma(s_1) = b_1 \in B$, there must be a $\tilde{t} \in [\tfrac{1}{2}, s_1]$ such that $\gamma(\tilde{t}) \in C$, in contrast to the assumption on γ. The restriction of the path γ_0 to the interval $[s_1, \tfrac{1}{2}]$ defines a path joining $b_1 \in B$ to $a \in \mathcal{S}(A)$. Therefore there exists $t_1 \in]s_1, \tfrac{1}{2}[$ such that $\gamma_0(t_1) = a_1 \in A$. The restriction of the path γ_0 to the interval $[t_1, 1]$ defines a path joining $a_1 \in A$ to $b \in \mathcal{S}(B)$. Hence there exists $s_2 \in]t_1, 1[$ with $\gamma_0(s_2) = b_2 \in B$. As above, we can also observe that $b_2 \notin \mathcal{S}(A)$ and $t_1 < s_2 < \tfrac{1}{2}$. Proceeding by induction, we obtain a sequence

$$s_1 < t_1 < s_2 < \cdots < s_j < t_j < s_{j+1} < \frac{1}{2}$$

with $\gamma(t_i) = a_i \in A$ and $\gamma(s_i) = b_i \in B$. For $t^* = \sup t_n = \sup s_n \in]0, \tfrac{1}{2}]$, we have that $\gamma_0(t^*) = \omega(t^*) = \lim a_n = \lim b_n \in A \cap B$, a contradiction. The proof is complete. $\qquad\square$

Lemma 1.2.5. *Let X be a connected and locally arcwise connected metric space and let $A, B, C \subseteq X$ be closed and nonempty sets with $A \cap B = \emptyset$. Then C cuts the arcs between A and B if and only if there exists a continuous function $f : X \to \mathbb{R}$ such that*

$$f(x) \le 0, \, \forall \, x \in \mathcal{S}(A), \qquad f(x) \ge 0, \, \forall \, x \in \mathcal{S}(B) \qquad (1.2.6)$$

and

$$C = \{x \in X : f(x) = 0\}. \qquad (1.2.7)$$

Proof. Clearly, if there exists a continuous function $f : X \to \mathbb{R}$ satisfying (1.2.6) and (1.2.7), then (1.2.1) and (1.2.2) hold, too. Hence Lemma 1.2.2 implies that C cuts the arcs between A and B.

Conversely, if C cuts the arcs between A and B, by Lemma 1.2.4 it follows that C cuts the arcs between $\mathcal{S}(A)$ and $\mathcal{S}(B)$. Thus we can apply Lemma 1.2.2 with respect to the triple $(\mathcal{S}(A), \mathcal{S}(B), C)$. In particular, the function f is defined as in (1.2.5), with ρ like in (1.2.3) and $\mu : X \to \{-1, 0, 1\}$ as follows:

$$
\mu(x) := \begin{cases} 0 & \text{if } x \in C, \\ -1 & \text{if } x \notin C \text{ and } \exists \text{ a path } \gamma_x : [0,1] \to X \setminus C \\ & \text{such that } \gamma_x(0) \in \mathcal{S}(A) \text{ and } \gamma_x(1) = x, \\ 1 & \text{elsewhere.} \end{cases} \tag{1.2.8}
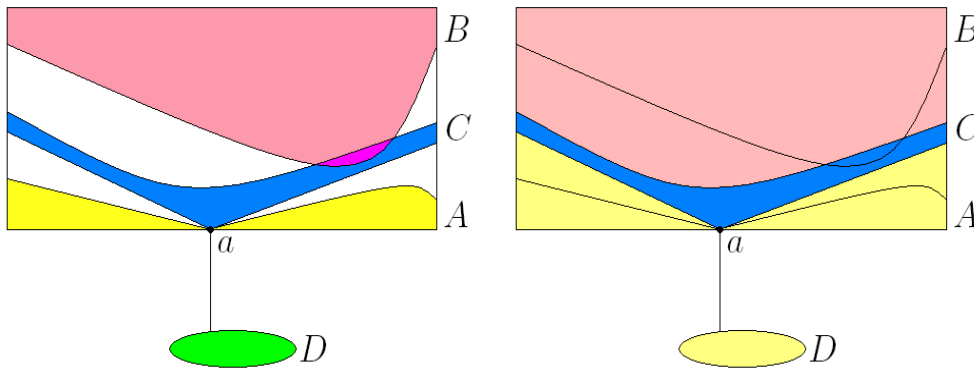$$

$\square$

Figure 1.10: The picture on the left-hand side gives an illustration of the framework described in Definition 1.2.1. The space X is the figure itself as a subset of the plane and the set C (darker region) cuts the arcs between A and B. We allow a nonempty intersection between B and C as well as between A and C (the singleton $\{a\}$). Notice that the only manner to connect with a path the points of A to the points of the "appendix" D is passing through the point $a \in A \cap C$. Therefore on the set D the map f in Lemma 1.2.2 with μ as in (1.2.4) would be positive, while it is negative on $A \setminus C$. The picture on the right-hand side provides an interpretation of Lemma 1.2.5. In regard to a function f having its factor μ defined like in (1.2.8), we have painted with a light color (yellow, in the colored version) the points where $f < 0$ and in grey (pink) the points where $f > 0$. Notice that the region D has been painted in light color, because $D \subseteq \mathcal{S}(A)$.

So far we have considered only the case of paths connecting two disjoint sets A and B. This choice is motivated by the foregoing examples for subsets of N-dimensional spaces. For the sake of completeness we end this subsection by discussing the framework in which A and B are joined by a continuum. We confine ourselves to the following lemma which will find an application in Theorem 1.2.20.

Lemma 1.2.6. *Let X be a connected and locally arcwise connected metric space and let $A, B \subseteq X$ be closed and nonempty sets with $A \cap B = \emptyset$. Let $\Gamma \subseteq X$ be a compact connected set such that*

$$\Gamma \cap A \neq \emptyset, \quad \Gamma \cap B \neq \emptyset.$$

Then, for every $\varepsilon > 0$, there exists a path $\gamma = \gamma_\varepsilon : [0,1] \to X$ with $\gamma(0) \in A$, $\gamma(1) \in B$ and

$$\overline{\gamma} \subseteq B(\Gamma, \varepsilon).$$

Moreover, if X is locally compact and $C \subseteq X$ is a closed set which cuts the arcs between A and B, then

$$\Gamma \cap C \neq \emptyset.$$

Proof. Let $\varepsilon > 0$ be fixed and consider, for every $p \in \Gamma$, a radius $\delta_p \in \,]0, \varepsilon[$ such that any two points in $B(p, \delta_p)$ can be joined by a path in $B(p, \varepsilon)$. Since Γ is compact, we can find a finite number of points

$$p_1, p_2, \ldots, p_k \in \Gamma,$$

such that

$$\Gamma \subseteq \mathcal{B} := \bigcup_{i=1}^{k} B(p_i, \delta_i) \subseteq B(\Gamma, \varepsilon), \quad \text{where} \quad \delta_i := \delta_{p_i}.$$

By the connectedness of Γ, for any partition of $\{1, \ldots, k\}$ into two nonempty disjoint subsets J_1 and J_2, there exist $i \in J_1$ and $j \in J_2$ such that $B(p_i, \delta_i) \cap B(p_j, \delta_j) \neq \emptyset$. This, in turns, implies that we can rearrange the p_i's (possibly changing their order in the labelling) so that

$$B(p_i, \delta_i) \cap B(p_{i+1}, \delta_{i+1}) \neq \emptyset, \quad \forall\, i = 1, \ldots, k-1.$$

Hence we can conclude that for any pair of points $w, z \in \mathcal{B}$, with $w \neq z$, there is a path $\gamma = \gamma_{w,z}$ joining w with z and such that $\overline{\gamma} \subseteq B(\Gamma, \varepsilon)$. In particular, taking $a \in A \cap \Gamma$ and $b \in B \cap \Gamma$, we have that there exists a path $\gamma = \gamma_\varepsilon : [0,1] \to B(\Gamma, \varepsilon)$, with $\gamma(0) = a$ and $\gamma(1) = b$ and this proves the first part of the statement.

Assume now that X is locally compact (i.e., for any $p \in X$ and $\eta > 0$, there exists $0 < \mu_p \leq \eta$ such that $\overline{B(p, \mu_p)}$ is compact). By the compactness of Γ we can find a finite number of points $q_1, q_2, \ldots, q_l \in \Gamma$ and corresponding radii $\mu_i := \mu_{q_i}$ such that

$$\Gamma \subseteq \mathcal{A} := \bigcup_{i=1}^{l} B(q_i, \mu_i)$$

and $\overline{\mathcal{A}} = \bigcup_{i=1}^{l} \overline{B(q_i, \mu_i)}$ is compact. Since $\mathcal{A}$ is an open neighborhood of the compact set Γ, there exists $\varepsilon_0 > 0$ such that $B(\Gamma, \varepsilon_0) \subseteq \mathcal{A}$. Hence, for each $0 < \varepsilon \leq \varepsilon_0$ we have that the set $\overline{B(\Gamma, \varepsilon)}$ is compact. Taking $\varepsilon = \frac{1}{n}$, we know that for every $n \in \mathbb{N}_0$ there exists a path $\gamma_n : [0, 1] \to X$, with $\gamma_n(0) \in A$, $\gamma_n(1) \in B$ and $\overline{\gamma_n} \subseteq B(\Gamma, \frac{1}{n})$. But, since C cuts the arcs between A and B, it follows that for every $n \in \mathbb{N}_0$, there is a point $c_n \in C \cap B(\Gamma, \frac{1}{n})$. For $n \geq \hat{n}$ large enough ($\hat{n} > 1/\varepsilon_0$), the sequence $(c_n)_{n \geq \hat{n}}$ is contained in the compact set $\overline{B(\Gamma, \varepsilon_0)}$ and therefore it admits a converging subsequence $c_{n_k} \to c^* \in \overline{B(\Gamma, \varepsilon_0)}$. Since $d_X(c_{n_k}, \Gamma) < \frac{1}{n_k}$ and the sets C, Γ are closed, the limit point $c^* \in \Gamma \cap C$. This concludes the proof. $\qquad\square$

1.2.2 The Intersection Lemma

We continue our exposition by presenting some applications of the topological lemmas just proved to the intersection of generalized surfaces which separate the opposite edges of an N-dimensional cube. Such generalized surfaces (cf. Definition 1.2.8) will be described as zero-sets of continuous scalar functions and therefore a nonempty intersection will be obtained as a zero of a suitably defined vector field. To this aim, we recall a classical result about the existence of zeros for continuous maps in $\mathbb{R}^N$, the Poincaré-Miranda Theorem.

Theorem 1.2.7. *Let $I^N := [0, 1]^N$ be the N-dimensional unit cube of $\mathbb{R}^N$, for which we denote by $[x_i = k] := \{x = (x_1, \ldots, x_N) \in I^N : x_i = k\}$. Let $F = (F_1, \ldots, F_N) : I^N \to \mathbb{R}^N$ be a continuous mapping such that, for each $i \in \{1, \ldots, N\}$,*

$$F_i(x) \leq 0, \ \forall x \in [x_i = 0] \ \text{ and } \ F_i(x) \geq 0, \ \forall x \in [x_i = 1]$$

or

$$F_i(x) \geq 0, \ \forall x \in [x_i = 0] \ \text{ and } \ F_i(x) \leq 0, \ \forall x \in [x_i = 1].$$

Then there exists $\bar{x} \in I^N$ such that $F(\bar{x}) = 0$.

Let us introduce now the spaces we are going to consider.

Definition 1.2.8. Let Z be a metric space and

$$h : \mathbb{R}^N \supseteq I^N \to X \subseteq Z$$

be a homeomorphism of $I^N := [0, 1]^N$ onto its image X. We call the pair

$$\widehat{X} := (X, h)$$

a *generalized N-dimensional rectangle of Z*. We also set

$$X_i^\ell := h([x_i = 0]), \quad X_i^r := h([x_i = 1])$$

and name them the *left* and the *right i-faces of X*.
Finally, we define

$$\vartheta X := h\left(\partial I^N\right)$$

and call it the *contour* of X.

As it is immediate to see, the generalized N-dimensional rectangles are a natural extension to the higher dimension of the generalized rectangles in Section 1.1. In a similar way also the concept of oriented rectangle will be transposed to the N-dimensional framework (cf. Definition 1.2.17).

Our main result on the intersection of generalized N-dimensional rectangles is Theorem 1.2.9 below, which can be considered as a variant of Hurewicz-Wallman Lemma about dimension [66]. The statements of the two results are in fact very similar, but the lemma in [66] concerns, instead of our concept of cutting, the stronger notion of separation and requires the sets A, B, C in Definition 1.2.1 to be pairwise disjoint (see [84]). Furthermore, with reference to Definition 1.2.8, in the statement of Hurewicz-Wallman Lemma only the very special case in which $Z = \mathbb{R}^N$, $X = I^N$ and $h = \mathrm{Id}_{\mathbb{R}^N}$ is considered. For such reasons we have chosen to provide all the details.

Theorem 1.2.9 (Intersection Lemma). *Let $\widehat{X} := (X, h)$ be a generalized N-dimensional rectangle of a metric space Z. Assume that, $\forall i \in \{1, \ldots, N\}$, there exists a compact set $\mathcal{S}_i \subseteq X$ that cuts the arcs between X_i^ℓ and X_i^r in X. Then*

$$\bigcap_{i=1}^N \mathcal{S}_i \neq \emptyset.$$

Proof. Through the inverse of the homeomorphism $h : \mathbb{R}^N \supseteq I^N \to X \subseteq Z$ we can define the compact sets

$$C_i := h^{-1}(\mathcal{S}_i),$$

which cut the arcs between $[x_i = 0]$ and $[x_i = 1]$ in I^N (for $i = 1, \ldots, N$). Clearly, it is sufficient to prove that

$$\bigcap_{i=1}^N C_i \neq \emptyset.$$

By Lemma 1.2.2, for every $i = 1, \ldots, N$, there exists a continuous function $f_i : I^N \to \mathbb{R}$ such that $f_i \leq 0$ on $[x_i = 0]$ and $f_i \geq 0$ on $[x_i = 1]$. Moreover,

$$C_i = \left\{ x \in I^N : f_i(x) = 0 \right\}.$$

The continuous vector field $\vec{f} := (f_1, \ldots, f_N) : I^N \to \mathbb{R}^N$ satisfies the assumptions of Poincaré-Miranda Theorem 1.2.7 and therefore there exists $\bar{x} \in I^N$ such that

$$f_i(\bar{x}) = 0, \quad \forall\, i = 1, \ldots, N.$$

Hence $\bar{x} \in \bigcap_{i=1}^N C_i$ and the proof is complete. $\qquad\square$

We recall that the previous result was applied in [124] in order to extend some recent theorems about fixed points and periodic points for continuous mappings in Euclidean spaces. In particular, in [124, Corollary 3.5] we generalized, via a simplified proof, a theorem by Kampen [67], while in [124, Theorem 3.8] we obtained an extension of a result by Zgliczyński [174] about periodic points associated to Markov partitions.

1.2.3   Zero sets of maps depending on parameters

As a next step we deal with the intersection of generalized surfaces which separate the opposite edges of an N-dimensional cube, in the case that the number of cutting surfaces is smaller than the dimension of the space. Our main tool is a result by Fitzpatrick, Massabó and Pejsachowicz (see [51, Theorem 1.1]) on the covering dimension of the zero set of an operator depending on parameters. For the reader's convenience, we recall the concept of covering dimension [48] in Definition 1.2.10 below, where by *order* of a family $\mathfrak{A}$ of subsets of the metric space Z we mean the largest integer n such that the family $\mathfrak{A}$ contains $n + 1$ sets with a non-empty intersection; if no such integer exists, the family $\mathfrak{A}$ has order infinity.

Definition 1.2.10. [48, p.54, p.208] Let Z be a metric space. We say that $\dim Z \leq n$ if every finite open cover of the space Z has a finite open [closed] refinement of order $\leq n$. The object $\dim Z \in \mathbb{N} \cup \{\infty\}$ is called the *covering dimension* or the *Čech-Lebesgue dimension* of the metric space Z. According to [51], if $z_0 \in Z$, we also say that $\dim Z \geq j$ *at z_0* if each neighborhood of z_0 has dimension at least j.

By a classical result from topology (cf. [48, The coincidence theorem]), in separable metric spaces the covering dimension coincides with the inductive dimension [48, p.3].

In view of the following results we also recall that, given an open bounded set $\mathcal{O} \subseteq \mathbb{R}^N$ and $n \in \{1, \ldots, N-1\}$, a continuous map $\pi : \overline{\mathcal{O}} \to \mathbb{R}^{N-n}$ is a *complement* for the continuous map $F : \overline{\mathcal{O}} \to \mathbb{R}^n$ if the topological degree $\deg((\pi, F), \mathcal{O}, 0)$ is defined and nonzero [51]. According to [64], a mapping f of a space X into a space Y is said to be *inessential* if f is homotopic to a constant; otherwise f is *essential*.

At last we introduce a further notation. Given an N-dimensional rectangle $\mathcal{R} := \prod_{i=1}^N [a_i, b_i]$, we denote its opposite i-faces by

$$\mathcal{R}_i^\ell := \{x \in \mathcal{R} : x_i = a_i\} \quad \text{and} \quad \mathcal{R}_i^r := \{x \in \mathcal{R} : x_i = b_i\}.$$

Theorem 1.2.11. *Let $\mathcal{R} := \prod_{i=1}^N [a_i, b_i]$ be an N-dimensional rectangle and let $P = (p_1, \ldots, p_N)$ be any interior point of $\mathcal{R}$. Let $n \in \{1, \ldots, N-1\}$ be fixed. Suppose that $F = (F_1, \ldots, F_n) : \mathcal{R} \to \mathbb{R}^n$ is a continuous mapping such that, for each $i \in \{1, \ldots, n\}$,*

$$F_i(x) < 0, \; \forall\, x \in \mathcal{R}_i^\ell \quad \text{and} \quad F_i(x) > 0, \; \forall\, x \in \mathcal{R}_i^r$$

or

$$F_i(x) > 0, \; \forall\, x \in \mathcal{R}_i^\ell \quad \text{and} \quad F_i(x) < 0, \; \forall\, x \in \mathcal{R}_i^r.$$

Define also the affine map

$$\pi : \mathbb{R}^N \to \mathbb{R}^{N-n}, \quad \pi_j(x_1, \ldots, x_N) := x_j - p_j, \; j = n+1, \ldots, N. \qquad (1.2.9)$$

Then there exists a connected subset $\mathcal{Z}$ of

$$F^{-1}(0) = \{x \in \mathcal{R} : F_i(x) = 0, \; \forall\, i = 1, \ldots, n\}$$

whose dimension at each point is at least $N - n$. Moreover,

$$\dim(\mathcal{Z} \cap \partial\mathcal{R}) \geq N - n - 1$$

and

$$\pi : \mathcal{Z} \cap \partial\mathcal{R} \to \mathbb{R}^{N-n} \setminus \{0\}$$

is essential.

Proof. We define the continuous mapping

$$H := (F, \pi) : \mathcal{R} \to \mathbb{R}^N.$$

By the assumptions on F and π we have

$$\deg(H, \mathrm{Int}(\mathcal{R}), 0) = (-1)^d \neq 0,$$

where d is the number of components $i \in \{1, \ldots, n\}$ such that $F_i(x) > 0$ for $x \in \mathcal{R}_i^\ell$ and $F_i(x) < 0$ for $x \in \mathcal{R}_i^r$. Hence π turns out to be a complementing map for F (according to [51]). A direct application of [51, Theorem 1.1] gives the thesis (observe that the dimension m in [51, Theorem 1.1] corresponds to our $N - n$). $\qquad\square$

Notice that if $a_i < 0 < b_i$, for $i = 1, \ldots, N$, then we can take $P = 0$, so that the complementing map is just the projection $\pi : \mathbb{R}^N \to \mathbb{R}^{N-n}$.

A more elementary version of Theorem 1.2.11 can be given for the zero set of a vector field with range in $\mathbb{R}^{N-1}$. In this case it is possible to achieve the thesis by a direct use of the classical Leray-Schauder Continuation Theorem [90], instead of the more sophisticated tools in [51]. Namely, we have:

Theorem 1.2.12. *Let $\mathcal{R} := \prod_{i=1}^{N}[a_i, b_i]$ be an N-dimensional rectangle and let $F = (F_1, \ldots, F_{N-1}) : \mathcal{R} \to \mathbb{R}^{N-1}$ be a continuous mapping such that, for each $i \in \{1, \ldots, N - 1\}$,*

$$F_i(x) < 0, \ \forall\, x \in \mathcal{R}_i^\ell \quad and \ \ F_i(x) > 0, \ \forall\, x \in \mathcal{R}_i^r$$

or

$$F_i(x) > 0, \ \forall\, x \in \mathcal{R}_i^\ell \quad and \ \ F_i(x) < 0, \ \forall\, x \in \mathcal{R}_i^r\,.$$

Then there exists a closed connected subset $\mathcal{Z}$ of

$$F^{-1}(0) = \{x \in \mathcal{R} : \ F_i(x) = 0, \ \forall\, i = 1, \ldots, N - 1\}$$

such that

$$\mathcal{Z} \cap \mathcal{R}_N^\ell \neq \emptyset, \quad \mathcal{Z} \cap \mathcal{R}_N^r \neq \emptyset.$$

Proof. We split $x = (x_1, \ldots, x_{N-1}, x_N) \in \mathcal{R} \subseteq \mathbb{R}^N$ as $x = (y, \lambda)$ with

$$y = (x_1, \ldots, x_{N-1}) \in \mathcal{M} := \prod_{i=1}^{N-1}[a_i, b_i], \quad \lambda = x_N \in [a_N, b_N]$$

and define

$$f = f(y, \lambda) : \mathcal{M} \times [a_N, b_N] \to \mathbb{R}^{N-1}, \quad f(y, \lambda) := F(x_1, \ldots, x_{N-1}, \lambda),$$

treating the variable $x_N = \lambda$ as a parameter for the $(N-1)$-dimensional vector field

$$f_\lambda(\cdot) = f(\cdot, \lambda).$$

By the assumptions on F we have

$$\deg(f_\lambda, \mathrm{Int}(\mathcal{M}), 0) = (-1)^d \neq 0, \quad \forall\, \lambda \in [a_N, b_N],$$

where d is the number of components $i \in \{1, \ldots, N-1\}$ such that $F_i(x) > 0$ for $x \in \mathcal{R}_i^\ell$ and $F_i(x) < 0$ for $x \in \mathcal{R}_i^r$. The Leray-Schauder Continuation Theorem [90, Théorème Fondamental] ensures the existence of a closed connected set

$$\mathcal{Z} \subseteq \left\{ (y, \lambda) \in \mathcal{M} \times [a_N, b_N] \ : \ f(y, \lambda) = 0 \in \mathbb{R}^{N-1} \right\},$$

whose projection onto the λ-component covers the interval $[a_N, b_N]$. By the above positions the thesis immediately follows. $\qquad\square$

For the interested reader, we mention that Theorem 1.2.12 can be found in [83] and that it was then applied in [82].

In the next lemma we take the unit cube $I^N := [0, 1]^N$ as N-dimensional rectangle and choose the interior point $P = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \right)$, in order to apply Theorem 1.2.11. Obviously, any other point interior to I^N could be chosen as well.

Lemma 1.2.13. *Let $n \in \{1, \ldots, N-1\}$ be fixed. Assume that, $\forall\, i \in \{1, \ldots, n\}$, there is a compact set $\mathcal{S}_i \subseteq I^N$ that cuts the arcs between $[x_i = 0]$ and $[x_i = 1]$ in I^N. Then there exists a connected subset $\mathcal{Z}$ of $\bigcap_{i=1}^{n} \mathcal{S}_i \neq \emptyset$, whose dimension at each point is at least $N - n$. Moreover,*

$$\dim \left(\mathcal{Z} \cap \partial I^N \right) \geq N - n - 1$$

and

$$\pi : \mathcal{Z} \cap \partial I^N \to \mathbb{R}^{N-n} \setminus \{0\}$$

is essential, where π is the affine map defined in (1.2.9).

Proof. For any fixed index $i^* \in \{1, \ldots, n\}$ we introduce the tunnel set

$$T_{i^*} := \prod_{i=1}^{i^*-1} [0, 1] \times \mathbb{R} \times \prod_{i=i^*+1}^{N} [0, 1].$$

It is immediate to check that $\mathcal{S}_{i^*}$ cuts the arcs between $[x_{i^*} = 0]$ and $[x_{i^*} = 1]$ in T_{i^*}.
By Lemma 1.2.5 there exists a continuous function $f_{i^*} : T_{i^*} \to \mathbb{R}$ such that

$$f_{i^*}(x) \leq 0\,, \ \forall\, x \in T_{i^*} \text{ with } x_{i^*} \leq 0 \quad \text{and} \quad f_{i^*}(x) \geq 0\,, \ \forall\, x \in T_{i^*} \text{ with } x_{i^*} \geq 1\,,$$

and moreover

$$\mathcal{S}_{i^*} = \{x \in T_{i^*} \ : \ f_{i^*}(x) = 0\}.$$

By this latter property and the fact that $\mathcal{S}_{i^*} \subseteq I^N$ it follows that

$$f_{i^*}(x) < 0, \ \forall\, x \in T_{i^*} \text{ with } x_{i^*} < 0 \quad \text{and} \quad f_{i^*}(x) > 0, \ \forall\, x \in T_{i^*} \text{ with } x_{i^*} > 1\,.$$

Now we define, for $x = (x_1, \ldots, x_{i^*-1}, x_{i^*}, x_{i^*+1}, \ldots, x_N) \in \mathbb{R}^N$, the continuous function

$$F_{i^*}(x) := f_{i^*}\big(\eta_{[0,1]}(x_1), \ldots, \eta_{[0,1]}(x_{i^*-1}), x_{i^*}, \eta_{[0,1]}(x_{i^*+1}), \ldots, \eta_{[0,1]}(x_N) \big),$$

where

$$\eta : \mathbb{R} \to [0,1], \ \ \eta_{[0,1]}(s) := \max\{0, \min\{s, 1\}\} \tag{1.2.10}$$

is the projection of $\mathbb{R}$ onto the interval $[0,1]$. As a consequence of the above positions we find that

$$F_{i^*}(x) < 0, \ \forall\, x \in \mathbb{R}^N \ : \ x_{i^*} < 0 \quad \text{and} \quad F_{i^*}(x) > 0, \ \forall\, x \in \mathbb{R}^N \ : \ x_{i^*} > 1\,.$$

We can thus apply Theorem 1.2.11 to the map $F = (F_1, \ldots, F_n)$ restricted to the N-dimensional rectangle

$$\mathcal{R} := \prod_{i=1}^{n} [-1, 2] \times \prod_{i=n+1}^{N} [0, 1].$$

Clearly,

$$\big(F \upharpoonright_{\mathcal{R}} \big)^{-1}(0) = \bigcap_{i=1}^{n} \mathcal{S}_i \subseteq I^N$$

and the proof is complete. $\qquad\square$

Remark 1.2.14. Both in Theorem 1.2.11 and in Lemma 1.2.13 the fact that we have privileged the first n components is purely conventional. It is evident that the results are valid for any finite sequence of indexes $i_1 < i_2 < \cdots < i_n$ in $\{1, \ldots, N\}$. The same observation applies systematically to all the other results (preceding and subsequent) in which some directions are conventionally chosen. Moreover, notice that Lemma 1.2.13 is invariant under homeomorphisms in a sense that is described in Theorem 1.2.15 below.

$\triangleleft$

Theorem 1.2.15. *Let $\widehat{X} := (X, h)$ be a generalized N-dimensional rectangle of a metric space Z. Let $i_1 < i_2 < \cdots < i_n$ be a finite sequence of $n \geq 1$ indexes in $\{1, \ldots, N\}$. Assume that, for each $j \in \{i_1, \ldots, i_n\}$, there is a compact set $\mathcal{S}_j \subseteq X$ that cuts the arcs between X_j^ℓ and X_j^r in X. Then there exists a compact*

connected subset $\mathscr{Z}$ of $\bigcap_{k=1}^{n} \mathcal{S}_{i_k} \neq \emptyset$, whose dimension at each point is at least $N - n$. Moreover,

$$\dim(\mathscr{Z} \cap \vartheta X) \geq N - n - 1$$

and

$$\pi : h^{-1}(\mathscr{Z}) \cap \partial I^N \to \mathbb{R}^{N-n} \setminus \{0\}$$

is essential, where π is defined as in (1.2.9) for $P = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right).$

Proof. The result easily follows by moving to the setting of Lemma 1.2.13 through the homeomorphism h^{-1} and repeating the arguments used therein. $\qquad \square$

At last we present a result (Corollary 1.2.16) which plays a crucial role in the subsequent proofs. It concerns the case $n = N - 1$ and could be obtained by suitably adapting the arguments employed in Lemma 1.2.13. However, due to its significance for our applications, we prefer to provide a detailed proof using Theorem 1.2.12 (which requires only the knowledge of the Leray-Schauder principle and therefore, in some sense, is more elementary). Corollary 1.2.16 extends to an arbitrary dimension some results in [134, Appendix] which were there proved only for $N = 2$ using [142].

Corollary 1.2.16. *Let $\widehat{X} := (X, h)$ be a generalized N-dimensional rectangle of a metric space Z. Let $k \in \{1, \ldots, N\}$ be fixed. Assume that, for each $j \in \{1, \ldots, N\}$ with $j \neq k$, there exists a compact set $\mathcal{S}_j \subseteq X$ that cuts the arcs between X_j^ℓ and X_j^r in X. Then there exists a compact connected subset $\mathcal{C}$ of $\bigcap_{\substack{i=1 \\ i \neq k}}^{N} \mathcal{S}_i \neq \emptyset$, such that*

$$\mathcal{C} \cap X_k^\ell \neq \emptyset, \quad \mathcal{C} \cap X_k^r \neq \emptyset.$$

Proof. Without loss of generality (if necessary, by a permutation of the coordinates), we assume $k = N$. In this manner, using the homeomorphism $h^{-1} : Z \supseteq X = h(I^N) \to I^N$, we can confine ourselves to the following framework:
For each $j \in \{1, \ldots, N - 1\}$, there exists a compact set

$$\mathcal{S}_j' := h^{-1}(\mathcal{S}_j) \subseteq I^N$$

that cuts the arcs between $[x_j = 0]$ and $[x_j = 1]$ in I^N.
Proceeding as in the proof of Lemma 1.2.13, for any fixed $i^* \in \{1, \ldots, N - 1\}$,

we introduce the tunnel set

$$T_{i^*} := \prod_{i=1}^{i^*-1} [0,1] \times \mathbb{R} \times \prod_{i=i^*+1}^{N} [0,1]$$

and find that $\mathcal{S}'_{i^*}$ cuts the arcs between $[x_{i^*} = 0]$ and $[x_{i^*} = 1]$ in T_{i^*}. Hence, by Lemma 1.2.5 there exists a continuous function $f_{i^*} : T_{i^*} \to \mathbb{R}$ such that

$$f_{i^*}(x) \leq 0, \ \forall x \in T_{i^*} \text{ with } x_{i^*} \leq 0 \quad \text{and} \quad f_{i^*}(x) \geq 0, \ \forall x \in T_{i^*} \text{ with } x_{i^*} \geq 1,$$

as well as

$$\mathcal{S}'_{i^*} = \{x \in T_{i^*} : f_{i^*}(x) = 0\}.$$

More precisely it holds that

$$f_{i^*}(x) < 0, \ \forall x \in T_{i^*} \text{ with } x_{i^*} < 0 \quad \text{and} \quad f_{i^*}(x) > 0, \ \forall x \in T_{i^*} \text{ with } x_{i^*} > 1.$$

We define, for $x = (x_1, \ldots, x_{i^*-1}, x_{i^*}, x_{i^*+1}, \ldots, x_N) \in \mathbb{R}^N$, the continuous function

$$F_{i^*}(x) := f_{i^*}\big(\eta_{[0,1]}(x_1), \ldots, \eta_{[0,1]}(x_{i^*-1}), x_{i^*}, \eta_{[0,1]}(x_{i^*+1}), \ldots, \eta_{[0,1]}(x_N)\big),$$

where $\eta_{[0,1]}$ is the projection of $\mathbb{R}$ onto the interval $[0,1]$ defined as in (1.2.10). Then we have

$$F_{i^*}(x) < 0, \ \forall x \in \mathbb{R}^N : x_{i^*} < 0 \quad \text{and} \quad F_{i^*}(x) > 0, \ \forall x \in \mathbb{R}^N : x_{i^*} > 1.$$

Now we consider the map $F = (F_1, \ldots, F_{N-1})$ restricted to the N-dimensional rectangle

$$\mathcal{R} := \prod_{i=1}^{N-1} [-\varepsilon, 1+\varepsilon] \times [0,1],$$

for any fixed $\varepsilon > 0$. Since

$$\big(F \!\restriction_{\mathcal{R}}\big)^{-1}(0) = \bigcap_{i=1}^{N-1} \mathcal{S}'_i \subseteq I^N,$$

we can set

$$\mathcal{C} := h(\mathcal{Z}),$$

where $\mathcal{Z} \subseteq \big(F \!\restriction_{\mathcal{R}}\big)^{-1}(0)$ comes from the statement of Theorem 1.2.12, and this concludes the proof. $\qquad\square$

As we shall see, the latter result turns out to be our main ingredient in the next subsection for the study of the dynamics of continuous maps defined on topological N-dimensional rectangles which possess, in a very broad sense, a one-dimensional expansive direction. Indeed, it allows to prove the crucial fixed point Theorem 1.2.20 (similar to Theorem 1.1.3) and subsequent results on the existence of multiple periodic points. Notice that the role of Corollary 1.2.16 in the proof of Theorem 1.2.20 is analogous to the one of the Crossing Lemma 1.1.17 in the verification of Theorem 1.1.3.

1.2.4 Stretching along the paths in the N-dimensional case

Finally, we are in position to provide an extension to N-dimensional spaces of the results obtained in Section 1.1 for the planar maps which expand the paths along a certain direction. To this aim, we reconsider Definition 1.2.8 in order to focus our attention on generalized N-dimensional rectangles in which we have fixed once for all the left and right sides. In the applications, these opposite sides give a sort of orientation to the generalized N-dimensional rectangles and in fact they are usually related to the expansive direction.

Definition 1.2.17. Let Z be a metric space and let $\widehat{X} := (X, h)$ be a generalized N-dimensional rectangle of Z. We set

$$X_\ell := h([x_N = 0]), \quad X_r := h([x_N = 1])$$

and

$$X^- := X_\ell \cup X_r \,.$$

The pair

$$\widetilde{X} := (X, X^-)$$

is called an *oriented N-dimensional rectangle of Z*.

Remark 1.2.18. A comparison between Definitions 1.2.8 and 1.2.17 shows that an oriented N-dimensional rectangle is just a generalized N-dimensional rectangle in which we privilege the two subsets of its contour which correspond to the opposite faces for some fixed component (namely, the x_N-component). In analogy with Remark 1.2.14, we point out that the choice of the N-th component is purely conventional. For example, in some other papers (see [56, 123, 176]) and also in the planar examples from Section 1.1, the first component was selected. Clearly, there is no substantial difference as the homeomorphism $h : \mathbb{R}^N \supseteq I^N \to X \subseteq Z$ could be composed with a permutation matrix, yielding to a new homeomorphism with the same image set. From

this point of view, our definition fits to the one of h-set of $(1, N-1)$-type, given by Zgliczyński and Gidea in [176] for a subset of $\mathbb{R}^N$ which is obtained as the inverse image of the unit cube through a homeomorphism of $\mathbb{R}^N$ onto itself. The similar concept of $(1, N-1)$-window is then considered by Gidea and Robinson in [56]: it is defined as a homeomorphic copy of the unit cube I^N of $\mathbb{R}^N$ through a homeomorphism whose domain is an open neighborhood of I^N.

$\triangleleft$

We now adapt to maps between oriented N-dimensional rectangles the concept of stretching along the paths, already considered in Definition 1.1.1 in regard to the planar case.

Definition 1.2.19. Let Z be a metric space and let $\psi : Z \supseteq D_\psi \to Z$ be a map defined on a set D_ψ. Assume that $\widetilde{X} := (X, X^-)$ and $\widetilde{Y} := (Y, Y^-)$ are oriented rectangles of Z and let $\mathcal{K} \subseteq X \cap D_\psi$ be a compact set. We say that $(\mathcal{K}, \psi)$ *stretches $\widetilde{X}$ to $\widetilde{Y}$ along the paths* and write

$$(\mathcal{K}, \psi) : \widetilde{X} \Leftrightarrow \widetilde{Y},$$

if the following conditions hold:

- ψ is continuous on $\mathcal{K}$;

- For every path $\gamma : [0, 1] \to X$ such that $\gamma(0) \in X_\ell^-$ and $\gamma(1) \in X_r^-$ (or $\gamma(0) \in X_r^-$ and $\gamma(1) \in X_\ell^-$), there exists a subinterval $[t', t''] \subseteq [0, 1]$ such that
$$\gamma(t) \in \mathcal{K}, \quad \psi(\gamma(t)) \in Y, \quad \forall t \in [t', t'']$$
and, moreover, $\psi(\gamma(t'))$ and $\psi(\gamma(t''))$ belong to different sides of Y^-.

By the similarity between Definitions 1.1.1 and 1.2.19, it is immediate to see that the remarks in Section 1.1 about the stretching along the path relation for planar maps remain valid also in the higher dimensional setting. In particular the comments on the relationship with the Brouwer fixed point Theorem or with the theory of the topological horseshoes in [69, 71] hold true. Suitable modifications of Definitions 1.1.2–1.1.12 could be presented as well. For the sake of conciseness we prefer to omit them. In view of its significance for the following treatment, we present however the next fixed point theorem, which shows that the set $\mathcal{K}$ still plays a crucial role.

Theorem 1.2.20. *Let Z be a metric space and let $\psi : Z \supseteq D_\psi \to Z$ be a map defined on a set D_ψ. Assume that $\widetilde{X} := (X, X^-)$ is an oriented N-dimensional rectangle of Z. If $\mathcal{K} \subseteq X \cap D_\psi$ is a compact set for which it holds that*

$$(\mathcal{K}, \psi) : \widetilde{X} \Longleftrightarrow \widetilde{X}, \tag{1.2.11}$$

then there exists at least one point $z \in \mathcal{K}$ with $\psi(z) = z$.

Proof. Let $h : I^N = [0,1]^N \to h(I^N) = X \subseteq Z$ be a homeomorphism such that $X_\ell = h([x_N = 0])$, $X_r = h([x_N = 1])$ and consider the compact subset of I^N

$$\mathcal{W} := h^{-1}(\mathcal{K} \cap \psi^{-1}(X))$$

as well as the continuous mapping $\phi = (\phi_1, \ldots, \phi_N) : \mathcal{W} \to I^N$ defined by

$$\phi(x) := h^{-1}\big(\psi(h(x))\big), \quad \forall\, x \in \mathcal{W}.$$

By Tietze-Urysohn Theorem [48, p.87] there exists a continuous map

$$\varphi = (\varphi_1, \ldots, \varphi_N) : I^N \to I^N, \quad \varphi \restriction_\mathcal{W} = \phi.$$

Let us introduce, for every $i = 1, \ldots, N-1$, the closed sets

$$\mathcal{S}_i := \big\{ x = (x_1, \ldots, x_{N-1}, x_N) \in I^N \,:\, x_i = \varphi_i(x) \big\} \subseteq I^N.$$

Since $\varphi(I^N) \subseteq I^N$, by the continuity of the φ_i's, it is straightforward to check that $\mathcal{S}_i$ cuts the arcs between $[x_i = 0]$ and $[x_i = 1]$ in I^N for $i = 1, \ldots, N-1$. Indeed, if $\gamma : [0,1] \to I^N$ is a path with $\gamma_i(0) = 0$ and $\gamma_i(1) = 1$, then, for the auxiliary function $g : [0,1] \ni t \mapsto \gamma_i(t) - \varphi_i(\gamma(t))$, we have $g(0) \leq 0 \leq g(1)$ and therefore Bolzano Theorem implies the existence of $s \in [0,1]$ such that $\gamma_i(s) = \varphi_i(\gamma(s))$, that is, $\overline{\gamma} \cap \mathcal{S}_i \neq \emptyset$. The cutting property is thus proved. Now Corollary 1.2.16 guarantees that there is a continuum

$$\mathcal{C} \subseteq \bigcap_{i=1}^{N-1} \mathcal{S}_i \tag{1.2.12}$$

such that

$$\mathcal{C} \cap [x_N = 0] \neq \emptyset, \quad \mathcal{C} \cap [x_N = 1] \neq \emptyset.$$

Lemma 1.2.6 implies that for every $\varepsilon > 0$ there exists a path $\gamma_\varepsilon : [0,1] \to I^N$ such that

$$\gamma_\varepsilon(0) \in [x_N = 0], \quad \gamma_\varepsilon(1) \in [x_N = 1] \quad \text{and} \quad \gamma_\varepsilon(t) \in B(\mathcal{C}, \varepsilon) \cap I^N, \ \forall\, t \in [0,1].$$

By the stretching assumption (1.2.11) and the definition of $\mathcal{W}$ and ϕ, there is a subpath ω_ε of γ_ε such that

$$\overline{\omega_\varepsilon} \subseteq \mathcal{W} \quad \text{and} \quad \phi(\overline{\omega_\varepsilon}) \subseteq I^N, \quad \text{with} \quad \phi(\overline{\omega_\varepsilon}) \cap [x_N = 0] \neq \emptyset, \quad \phi(\overline{\omega_\varepsilon}) \cap [x_N = 1] \neq \emptyset.$$

Bolzano Theorem applied to the continuous mapping $x \mapsto x_N - \varphi_N(x)$ on $\overline{\omega_\varepsilon}$ ensures the existence of a point

$$\tilde{x}^\varepsilon = (\tilde{x}_1^\varepsilon, \dots, \tilde{x}_N^\varepsilon) \in \overline{\omega_\varepsilon} \subseteq \mathcal{W}$$

such that

$$\tilde{x}_N^\varepsilon = \varphi_N(\tilde{x}^\varepsilon).$$

Taking $\varepsilon = \frac{1}{n}$ (for $n \in \mathbb{N}_0$) and letting $n \to \infty$, by a standard compactness argument we find a point

$$\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_N) \in \mathcal{C} \cap \mathcal{W}$$

such that

$$\tilde{x}_N = \varphi_N(\tilde{x}).$$

By (1.2.12), recalling also the definition of the $\mathcal{S}_i$'s, we get

$$\tilde{x} = \varphi(\tilde{x}) \in \mathcal{W}.$$

Then, since $\varphi \restriction_\mathcal{W} = \phi$, by the relation

$$h(\phi(x)) = \psi(h(x)), \quad \forall\, x \in \mathcal{W},$$

we have that $h(\tilde{x}) = \psi(h(\tilde{x})) \in h(\mathcal{W})$ and therefore

$$z := h(\tilde{x}) \in \mathcal{K} \cap \psi^{-1}(X)$$

is the desired fixed point for ψ. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Having proved Theorems 1.2.15 and 1.2.20, we have available the tools for extending to any finite dimension the results obtained in Section 1.1 for the planar case [117, 118]. For brevity's sake we focus our attention only on a few of them, that we present below in the more general setting. Notice that now we can complement the previous results adding some information on the dimension of the cutting surfaces.
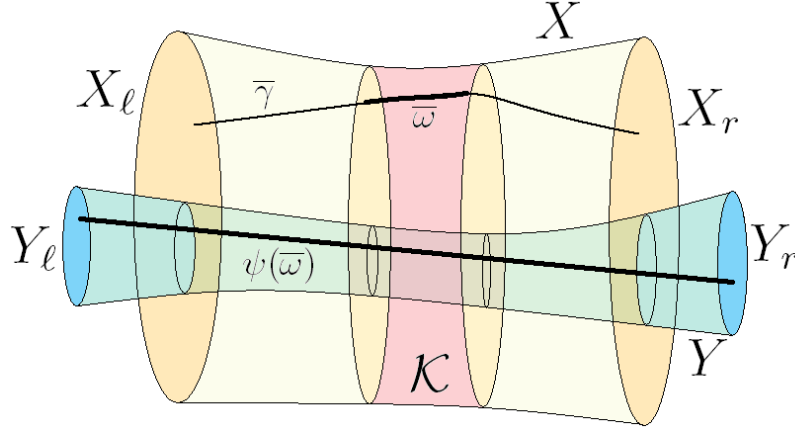
Figure 1.11: The tubular sets X and Y represent two generalized 3-dimensional rectangles, in which we have indicated the compact set $\mathcal{K}$ and the boundary sets X_ℓ and X_r, as well as Y_ℓ and Y_r. The function ψ is continuous on $\mathcal{K}$ and maps X onto $Y = \psi(X)$, so that $Y_\ell = \psi(X_\ell)$ and $Y_r = \psi(X_r)$. However, in this particular case the map ψ stretches the paths of X not only across Y, but also across X itself and therefore the existence of a fixed point for ψ inside $\mathcal{K}$ is ensured by Theorem 1.2.20.

Theorem 1.2.21. *Let Z be a metric space and $\psi : Z \supseteq D_\psi \to Z$ be a map defined on a set D_ψ. Assume that $\widetilde{X} := (X, X^-)$ is an oriented N-dimensional rectangle of Z. If $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}$ are $m \geq 2$ pairwise disjoint compact subsets of $X \cap D_\psi$ and*

$$(\mathcal{K}_i, \psi) : \widetilde{X} \Longleftrightarrow \widetilde{X}, \ \text{for } i = 0, \ldots, m-1,$$

then the following conclusions hold:

- *The map ψ has at least a fixed point in $\mathcal{K}_i$, $i = 0, \ldots, m-1$;*

- *For each two-sided sequence $(s_h)_{h \in \mathbb{Z}} \in \{0, \ldots, m-1\}^{\mathbb{Z}}$, there exists a sequence of points $(x_h)_{h \in \mathbb{Z}}$ such that $\psi(x_{h-1}) = x_h \in \mathcal{K}_{s_h}$, $\forall h \in \mathbb{Z}$;*

- *For each sequence of m symbols $\boldsymbol{s} = (s_n)_n \in \{0, 1, \ldots, m-1\}^{\mathbb{N}}$, there exists a compact connected set $\mathcal{C}_{\boldsymbol{s}} \subseteq \mathcal{K}_{s_0}$ which cuts the arcs between X_ℓ and X_r in X and such that, for every $w \in \mathcal{C}_{\boldsymbol{s}}$, there is a sequence $(y_n)_n$ with $y_0 = w$ and*

$$y_n \in \mathcal{K}_{s_n}, \ \ \psi(y_n) = y_{n+1}, \ \forall n \geq 0.$$

The dimension of $\mathcal{C}_{\boldsymbol{s}}$ at any point is at least $N-1$. Moreover, $\dim(\mathcal{C}_{\boldsymbol{s}} \cap \vartheta X) \geq N - 2$ and $\pi : h^{-1}(\mathcal{C}_{\boldsymbol{s}}) \cap \partial I^N \to \mathbb{R}^{N-1} \setminus \{0\}$ is essential (where π is defined as in (1.2.9) for $p_i = \frac{1}{2}$, $\forall i$ and h is the homeomorphism defining X);

- *Given an integer $j \geq 2$ and a $j+1$-uple $(s_0, \ldots, s_j)$, $s_i \in \{0, \ldots, m-1\}$, for $i = 0, \ldots, j$, and $s_0 = s_j$, then there exists a point $w \in \mathcal{K}_{s_0}$ such that*

$$\psi^i(w) \in \mathcal{K}_{s_i}, \ \forall i = 1, \ldots, j, \ \ and \ \ \psi^j(w) = w.$$

Like in the two-dimensional framework, such result can be easily obtained as a corollary of the more general:

Theorem 1.2.22. *Let $(\widetilde{X}_i)_{i \in \mathbb{Z}}$ (with $\widetilde{X}_i = (X_i, X_i^-)$) be a double sequence of oriented N-dimensional rectangles of a metric space Z and let $(\mathcal{K}_i, \psi_i)_{i \in \mathbb{Z}}$, with $\mathcal{K}_i \subseteq X_i$, be a sequence such that*

$$(\mathcal{K}_i, \psi_i) : \widetilde{X}_i \Longleftrightarrow \widetilde{X}_{i+1}, \quad \forall i \in \mathbb{Z}.$$

Let us denote by X_ℓ^i and X_r^i the two components of X_i^-. Then the following conclusions hold:

- *There is a sequence $(w_k)_{k \in \mathbb{Z}}$ such that $w_k \in \mathcal{K}_k$ and $\psi_k(w_k) = w_{k+1}$, for all $k \in \mathbb{Z}$;*

- *For each $j \in \mathbb{Z}$ there exists a compact connected set $\mathcal{C}_j \subseteq \mathcal{K}_j$ which cuts the arcs between X_ℓ^j and X_r^j in X_j and such that, for every $w \in \mathcal{C}_j$, there is a sequence $(y_i)_{i \geq j}$ with $y_j = w$ and*

$$y_i \in \mathcal{K}_i, \ \ \psi_i(y_i) = y_{i+1}, \ \forall i \geq j.$$

  The dimension of $\mathcal{C}_j$ at any point is at least $N-1$. Moreover, $\dim(\mathcal{C}_j \cap \vartheta X_j) \geq N-2$ and $\pi : h_j^{-1}(\mathcal{C}_j) \cap \partial I^N \to \mathbb{R}^{N-1} \setminus \{0\}$ is essential (where π is defined as in (1.2.9) for $p_i = \frac{1}{2}$, $\forall i$ and h_j is the homeomorphism defining X_j);

- *If there are integers h and l, with $h < l$, such that $\widetilde{X}_h = \widetilde{X}_l$, then there exists a finite sequence $(z_i)_{h \leq i \leq l-1}$, with $z_i \in \mathcal{K}_i$ and $\psi_i(z_i) = z_{i+1}$ for each $i = h, \ldots, l-1$, such that $z_l = z_h$, that is, z_h is a fixed point of $\psi_{l-1} \circ \cdots \circ \psi_h$.*

Proof. The thesis follows by steps analogous to the ones in the proof of Theorem 1.1.10, using Theorem 1.2.20 in place of Theorem 1.1.3 and recalling Definition 1.2.1. The estimates on the dimension of the continuum $\mathcal{C}_j$ come from Theorem 1.2.15. $\qquad\square$

As a final remark, we observe that, instead of maps expansive just along one direction, it is possible to deal with the more general case of functions defined on N-dimensional rectangles (or homeomorphic images of them) with u

unstable directions (along which an expansion occurs) and $s = N - u$ stable directions (along which the system is compressive). With this respect, many works are available in the literature, like, for instance, [11, 12, 56, 67, 130, 176]. In particular, we recall that a similar framework was analyzed in [123], where the author and Zanolin introduced a "deformation relation" among generalized N-dimensional rectangles with some similarities to the stretching property in Definition 1.2.19. Indeed also that relation could be viewed (cf. [123, Remark 3.4]) as an higher-dimensional counterpart of the planar stretching property in Definition 1.1.1 from [117, 118] and moreover it was suitable for the detection of fixed points, periodic points and chaotic dynamics, too. However, its definition required some assumptions on the topological degree of the map involved, while the conditions in Definition 1.2.19 do not need any sophisticated topological tool. Therefore, it is in this perspective of simplicity that the above results from [124] have to be considered.

Chapter 2

Chaotic dynamics for continuous maps

In the present chapter we show how to apply the method of "stretching along the paths" from Chapter 1 to prove the presence of chaotic dynamics in discrete-time dynamical systems. In order to do that, we first set our framework among more classical ones available in the literature, concerning the concepts of "covering" and "crossing". Indeed, this preliminary introductory discussion is meant to give an outline of the reasons that led various authors to deal with similar settings, as well as to quickly present the topics developed along Sections 2.1–2.4, where the undefined terms will be rigorously explained. Generally speaking, in the investigation of a dynamical system defined by a map f on a metric space, it can be difficult or impossible to find a direct proof of the presence of chaos, according to one or the other of the several, more or less equivalent, existing definitions. Hence, a canonical strategy is to establish a conjugacy or semi-conjugacy between f, perhaps restricted to a suitable (positively) invariant subset of its domain, and the shift map $\sigma : (s_i)_{i \in \mathbb{I}} \mapsto (s_{i+1})_{i \in \mathbb{I}}$ on $\{0, \ldots, m-1\}^{\mathbb{I}}$, for $\mathbb{I} = \mathbb{N}$ or $\mathbb{I} = \mathbb{Z}$, i.e. on the space of unilateral or bilateral sequences of $m \geq 2$ symbols, which displays many chaotic features, like transitivity, sensitivity, positive entropy, etc. In such indirect manner, one can conclude that f possesses all the properties of σ that are preserved by the conjugacy/semi-conjugacy relation. In particular, the known fact (cf. Section 2.2) that $h_{\text{top}}(f^n) = n h_{\text{top}}(f)$ for $n \geq 1$, where $h_{\text{top}}(f)$ denotes the topological entropy of f, guarantees that if a power of a map is conjugate or semi-conjugate to σ, then such map has positive entropy. Accordingly, to show the existence of chaos (positive entropy) of a map f it is sufficient to look at any of its iterates.

A crucial role in this strategy is played by the study of a topological property for maps which is called "crossing" and, broadly speaking, refers to the way

in which the image of suitable sets under the iterates of the maps intersect the original sets. In particular, the kind of crossing known as "horseshoe" has turned out to be a fundamental tool of analysis to prove the existence of chaotic dynamics in a well-defined manner. The name and the idea of horseshoe derive from the celebrated article by Smale [144] that provided a mathematically rigorous and geometrically suggestive proof of the existence of chaos for a special planar map.

More precisely (see [41, 112, 162] for the technical details), in the horseshoe model a square S is first shrunk uniformly in one direction and expanded in the other one. Subsequently, the elongated rectangle obtained in the previous step is bent along the original square in order to cross it twice. The resulting map F is a diffeomorphism with F and F^{-1} transforming the vertical and horizontal lines of the square into similar lines crossing the domain. In Smale's construction, the set $\mathcal{I}_S := \{q \in S : F^k(q) \in S, \ \forall\, k \in \mathbb{Z}\}$, consisting of the points which remain in the square under all the iterates of F in both forward and backward time, is a compact invariant set for F which contains as a dense subset the periodic points of F and such that F is sensitive on initial conditions and topologically transitive on $\mathcal{I}_S$. Actually, F acts on $\mathcal{I}_S$ like the shift map σ on two symbols since $F\!\restriction_{\mathcal{I}_S}$ and σ are conjugate. As mentioned above, the conjugacy, which is given by a homeomorphism $\pi : \mathcal{I}_S \to \{0,1\}^{\mathbb{Z}}$, with $\pi \circ F = \sigma \circ \pi$, allows to transfer to $F\!\restriction_{\mathcal{I}_S}$ the well-known dynamical properties of the Bernoulli shift.

In the applications to concrete dynamical systems, the presence of a complex behaviour for a given map F can be verified by proving the existence of a horseshoe structure either for the map itself or for one of its iterates. This led some authors (see [27]) to define a *horseshoe* like a set Λ that is invariant under F^{n_0}, for some $n_0 \geq 1$, with the property that $F^{n_0}\!\restriction_{\Lambda}$ is topologically conjugate to σ (or, more generally, to a nontrivial subshift of finite type). As pointed out by Burns and Weiss in [27], the difficulty in finding a horseshoe lies in showing that the map $\pi : \Lambda \to \{0,1\}^{\mathbb{Z}}$ is injective. Such step, in turns, often requires the verification of some assumptions on F, like being a diffeomorphism satisfying suitable hyperbolicity conditions, which either are not fulfilled or are hard to check for a given map. Hence, more general and less stringent definitions of horseshoe have been suggested to reproduce some geometrical features typical of the Smale horseshoe while discarding the hyperbolicity conditions associated with it, which are difficult or impossible to prove in practical cases. This led to the study of the so-called "topological horseshoes" [27, 71, 176].

The core of such field of research consists in providing an adequate notion of crossing property for higher dimensional dynamical systems, so that a map F (or one of its iterates) satisfying certain geometrical conditions is proved to be

semi-conjugate to a full shift on $m \geq 2$ symbols. This is enough to conclude that F displays chaotic dynamics in the sense that F has, for instance, positive topological entropy. More generally, one could show that F is semi-conjugate to a nontrivial subshift of finite type, by employing some tools from symbolic dynamics [75, 94], as explained in Section 2.2.

In regard to the one-dimensional case, a classical example of horseshoe-type crossing property relies on the definition of "covering". We recall that, according to [23], given a continuous map $f : \mathbb{R} \to \mathbb{R}$ and two intervals $I, J \subseteq \mathbb{R}$, we say that I f-*covers* J if there exists a subinterval $I_0 \subseteq I$ such that $f(I_0) = J$. We also say that I f-*covers* J m *times* if there exist $m \geq 2$ subintervals $I_0, \ldots, I_{m-1} \subseteq I$, with pairwise disjoint interiors, such that $f(I_k) = J$, for $k = 0, \ldots, m-1$. Special significance from the point of view of complex dynamics is assigned to the case in which I f-covers I m times. Different authors have investigated possible variants of the above definitions, dealing with the framework in which there are $m \geq 2$ intervals $I_0, \ldots, I_{m-1}$ such that I_i f-covers I_j, for each $i, j \in \{0, \ldots, m-1\}$, and some disjointness condition is imposed on them. For instance, the case $m = 2$ with I_0, I_1 open disjoint intervals was taken by Glendinning in [57] as a definition of "horseshoe" for f, while Block and Coppel in [21, 22] define "turbulence" for f the similar framework when I_0 and I_1 are compact intervals with disjoint interiors and "strict turbulence" when I_0, I_1 are compact and disjoint. In such settings one obtains the typical features associated to the concept of chaos (like, e.g., existence of periodic points of each period, semi-conjugacy to the Bernoulli shift for the map f or for some of its iterates and thus positive topological entropy, ergodicity with respect to some invariant measure, sensitivity to initial conditions, transitivity). Besides the works already quoted, see [7, 24, 87, 141].

An elementary introduction to these ideas is provided by the analysis of the logistic map $F : [0, 1] \to \mathbb{R}$, $F(x) = \mu x(1-x)$, with μ a positive real parameter. It is easy to see that for $\mu \geq 4$ the interval $[0, 1]$ F-covers $[0, 1]$ twice, whereas for $\mu < 4$ the existence of multiple coverings can be established for some iterate of F, as shown in Section 2.2. The concepts of covering and horseshoe can also be used to revisit some classical results in the theory of one-dimensional unimodal maps, such as Li-Yorke celebrated Theorem [93]. For instance, [57, Theorem 11.13] shows that, whenever a unimodal map f has a point of period three, then its second iterate f^2 has a horseshoe. However, it is in the higher-dimensional frameworks that the approach described above shows its full power.

The most natural way to extend the notion of multiple f-covering to the abstract setting of a continuous self-map f of a metric space X may be that of assuming the existence of $m \geq 2$ compact sets $K_0, \ldots, K_{m-1} \subseteq X$ such

that $f(K_i) \supseteq \bigcup_{j=0}^{m-1} K_j$, $\forall\, i = 0, \ldots, m-1$. Moreover, in order to avoid trivial conclusions, one usually requires some disjointness conditions on the sets K_i's, such as $K_i \cap K_j = \emptyset$, $\forall\, i \neq j$ $\left(\text{or, more generally, } \bigcap_{i=0}^{m-1} K_i = \emptyset\right)$. Blokh and Teoh [24] describe such framework by saying that $(f, K_0, \ldots, K_{m-1})$ form a m-horseshoe (or a weak m-horseshoe, respectively). Under these conditions it is possible to show the existence of some chaotic behaviour for the discrete dynamical system generated by f (see [87, 141]). In particular, in [24] it is proved that if a power of f admits a weak m-horseshoe, then the topological entropy of f is positive and there exists a set B and a power g of f such that B is g-invariant and $g \restriction_B$ is semi-conjugate to the Bernoulli shift on $\{0, \ldots, m-1\}^{\mathbb{N}}$. In spite of the generality of the setting in which such results can be obtained and their effectiveness for the one-dimensional framework, when one tries to apply the theory to specific higher dimensional mathematical models arising in applications, it may be expedient to follow Burns and Weiss' suggestion [27] and replace the previous covering relation with a weaker condition of the form $f(K_i)$ "goes across" $\bigcup_{j=0}^{m-1} K_j$, $\forall\, i = 0, \ldots, m-1$, which does not require the map f to be surjective on the K_i's.

In our approach the expression "goes across" (called also "Markov property" [59, p.291]) has to be understood in relation to the stretching along the paths effect of a map f from Definition 1.1.1 or Definition 1.2.19 [1]. More precisely, we assume that each path γ in our generalized (N-dimensional) rectangle $\mathcal{R}$, joining the two sides of $\mathcal{R}^-$, intercepts every K_i and is then expanded to a path $f \circ \gamma$ which crosses all the K_j's. Further details can be found in Section 2.2.

Different characterizations of the concept of "crossing" have been suggested by various authors in order to establish the presence of complex dynamics for continuous maps in higher dimensional spaces (see, for instance, [69, 71, 108, 148, 172, 174, 176] and the references therein). In this respect, the already discussed approach by Kennedy, Koçak and Yorke in [69] and Kennedy and Yorke in [71] is perhaps the most general, both with regard to the spaces considered and the restrictions on the maps involved. However, as we have seen, the great generality of such setting comes with a price, in the sense that the existence of periodic points is not ensured. Other authors (for example in [108, 148, 150, 167, 172, 174, 176]) have developed theories of topological horseshoes more focused on the search of fixed and periodic points for maps defined on subsets of the N-dimensional Euclidean space. The tools employed in these and related works range from the Conley index [108] to the Lefschetz

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[1] By the similarity between the results obtained in Chapter 1 for the planar and the N-dimensional settings, for the sake of generality, we will generally work in the higher dimensional framework, maybe not indicating the dimension, when no confusion may occur.

fixed point theory [148] and the topological degree [176].

Our framework may be looked at as an intermediate point of view between the theory of topological horseshoes developed by Kennedy and Yorke in [71] and the above-mentioned works based on some more or less sophisticated fixed point index theories. Indeed, compared to [71], the method of stretching along the paths, thanks to its specialized framework, allows one to obtain sharper results concerning the existence of periodic points. On the other hand, our approach, although mathematically rigorous, avoids the use of more advanced topological theories and it is relatively easy to apply to specific models arising in applications.

In more details, in Section 2.1 we describe the chaotic features that we are able to obtain with our method. Some tools from symbolic dynamics are introduced in Section 2.2, in order to collate our achievements on chaotic dynamics with other related results descending from classical approaches. Such discussion on the various notions of chaos is pursued further in Section 2.3. An alternative geometrical context for the applicability of the stretching along the paths method is presented in Section 2.4, where we deal with the so-called "Linked Twist Maps".

2.1 Definitions and main results

In the literature various different notions of complex dynamics can be found, so that one could say "as many authors, as many definitions of chaos" [20]. However, some common features are shared by several of these definitions, such as the unpredictability of the future behaviour of the system under consideration. In particular, one of the most natural notions of chaos is related to the possibility of realizing a generic coin-flipping experiment [145]. The definition of chaos that we choose is adapted from that considered by Kirchgraber and Stoffer in [74] under the name of *chaos in the coin-tossing sense*. In fact, exactly like in [74], our definition concerns the possibility of reproducing, via the iterates of a given map ψ, any coin-flipping sequence of two symbols. On the other hand, our definition, when compared to [74], is enhanced by the possibility of realizing periodic sequences of two symbols by means of periodic points of ψ. More formally, we have:

Definition 2.1.1. Let X be a metric space, $\psi : X \supseteq D_\psi \to X$ be a map and let $\mathcal{D} \subseteq D_\psi$. We say that ψ *induces chaotic dynamics on two symbols on the set* $\mathcal{D}$ if there exist two nonempty disjoint compact sets

$$\mathcal{K}_0, \, \mathcal{K}_1 \subseteq \mathcal{D},$$

such that, for each two-sided sequence $(s_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$, there exists a corresponding sequence $(w_i)_{i \in \mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ such that

$$w_i \in \mathcal{K}_{s_i} \quad \text{and} \quad w_{i+1} = \psi(w_i), \quad \forall\, i \in \mathbb{Z}, \tag{2.1.1}$$

and, whenever $(s_i)_{i \in \mathbb{Z}}$ is a k-periodic sequence (that is, $s_{i+k} = s_i$, $\forall i \in \mathbb{Z}$) for some $k \geq 1$, there exists a corresponding k-periodic sequence $(w_i)_{i \in \mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ satisfying (2.1.1). When we want to emphasize the role of the sets $\mathcal{K}_j$'s, we also say that ψ *induces chaotic dynamics on two symbols on the set* $\mathcal{D}$ *relatively to* $\mathcal{K}_0$ *and* $\mathcal{K}_1$.

To get a feel of such definition, we can imagine to associate the name " head " $= H$ to the set $\mathcal{K}_0$ and the name " tail " $= T$ to $\mathcal{K}_1$. If we consider any sequence of symbols

$$(s_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}} \cong \{H, T\}^{\mathbb{Z}}$$

so that, for each i, s_i is either " head " or " tail ", then we have the same itinerary of heads and tails realized through the map ψ. Namely, there exists a sequence $(w_i)_{i \in \mathbb{Z}}$ of points of the metric space X which is a full orbit for ψ, i.e.

$$w_{i+1} = \psi(w_i), \forall\, i \in \mathbb{Z},$$

and such that $w_i \in \mathcal{K}_0$ or $w_i \in \mathcal{K}_1$, according to the fact that the i-th term of the sequence $(s_i)_i$ is " head " or " tail ". Moreover, as already remarked, our definition extends that in [74] in the sense that any periodic sequence of heads and tails can be realized by suitable points which are periodic points for ψ. For instance, there exists a fixed point of ψ in the set $\mathcal{K}_1$ corresponding to the constant sequence of symbols $s_i = $ " tail ", $\forall\, i \in \mathbb{Z}$. There is also a point $w \in \mathcal{K}_0$ of period three with $\psi(w) \in \mathcal{K}_0$ and $\psi^2(w) \in \mathcal{K}_1$, corresponding to the periodic sequence $\ldots HHT\ HHT\ HHT \ldots$, and so on.

We stress that we have presented our definition of chaos with reference to just two symbols in order to make the connection with the coin-flipping process clearer. On the other hand, it is possible to define *chaotic dynamics on m symbols* for an arbitrary integer $m \geq 2$ in a completely analogous manner, when there are m pairwise disjoint compact sets $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}$ acting as $\mathcal{K}_0$ and $\mathcal{K}_1$ in Definition 2.1.1. More precisely we have:

Definition 2.1.2. Let X be a metric space, $\psi : X \supseteq D_\psi \to X$ be a map and let $\mathcal{D} \subseteq D_\psi$. Let also $m \geq 2$ be an integer. We say that ψ *induces chaotic dynamics on m symbols on the set* $\mathcal{D}$ if there exist m nonempty pairwise disjoint compact sets

$$\mathcal{K}_0, \ldots, \mathcal{K}_{m-1} \subseteq \mathcal{D},$$

such that, for each two-sided sequence $(s_i)_{i \in \mathbb{Z}} \in \{0, \ldots, m-1\}^{\mathbb{Z}}$, there exists a corresponding sequence $(w_i)_{i \in \mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ such that

$$w_i \in \mathcal{K}_{s_i} \quad \text{and} \quad w_{i+1} = \psi(w_i), \ \forall i \in \mathbb{Z} \tag{2.1.2}$$

and, whenever $(s_i)_{i \in \mathbb{Z}}$ is a k-periodic sequence (that is, $s_{i+k} = s_i, \forall i \in \mathbb{Z}$) for some $k \geq 1$, there exists a corresponding k-periodic sequence $(w_i)_{i \in \mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ satisfying (2.1.2). When we want to emphasize the role of the sets $\mathcal{K}_j$'s, we also say that ψ *induces chaotic dynamics on* m *symbols on the set* $\mathcal{D}$ *relatively to* $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}$.

Such variant of Definition 2.1.1 will be of particular importance when dealing with Linked Twist Maps (see Section 2.4). Moreover, the latter generalization agrees with the kind of chaotic behaviour detected in various papers, where the dynamical systems are investigated using some topological tools related to fixed point theory [150, 167]. Hence, in order to collate our notion of chaos to other ones available in the literature, from now on we will consider Definition 2.1.2 instead of Definition 2.1.1.

In regard to the results in [69, 71], we recall that the authors therein also deal with chaotic dynamics meant as the possibility of realizing any sequence of symbols. The difference, as it should be clear from the discussion in Section 1.1, resides in the possibility of realizing periodic coin-flipping sequences through periodic itineraries. Nonetheless, the kind of chaos detected in [69, 71] for a map f satisfying the "horseshoe hypotheses Ω" in [71] with crossing number $m \geq 2$ allows to prove the existence of a compact f-invariant set Q_I (i.e. $f(Q_I) = Q_I$), such that $f \restriction_{Q_I}$ is semi-conjugate to the one-sided shift on m symbols. Furthermore, in [69, Lemma 4] (Chaos Lemma), the existence of a smaller compact invariant set $Q_* \subseteq Q_I$ is obtained, on which the map f is sensitive and such that each forward itinerary on m symbols is realized by the f-itinerary generated by some point of Q_*.

Before showing in Theorem 2.1.4 what we are able to prove in this direction, let us see which is the link between the method of stretching along the paths from Chapter 1 and the notion of chaotic dynamics in the sense of Definition 2.1.2:

Theorem 2.1.3. *Let* $\widetilde{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$ *be an oriented (N-dimensional) rectangle of a metric space X and let* $\mathcal{D} \subseteq \mathcal{R} \cap D_\psi$, *with* D_ψ *the domain of a map* $\psi : X \supseteq D_\psi \to X$. *If* $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}$ *are* $m \geq 2$ *pairwise disjoint compact sets contained in* $\mathcal{D}$ *and*

$$(\mathcal{K}_i, \psi) : \widetilde{\mathcal{R}} \Longleftrightarrow \widetilde{\mathcal{R}}, \text{ for } i = 0, \ldots, m-1,$$

then ψ induces chaotic dynamics on m symbols on the set $\mathcal{D}$ relatively to $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}$.

Proof. Recalling Definition 2.1.2, the thesis is just a reformulation of the second and the forth conclusions in Theorem 1.1.8 (or in Theorem 1.2.21, in the case of an N-dimensional setting)[2]. □

The same conclusions on chaotic dynamics could be obtained in the framework of Theorem 1.1.6, when assuming condition (1.1.7). As discussed in [123], such remark looks useful in view of possible applications of our method to the detection of chaos via computer-assisted proofs.

When we enter the setting of Definition 2.1.2, many interesting properties for the map ψ can be proved. They are gathered in Theorem 2.1.4 below. The precise explanation of some concepts (like topological entropy, sensitivity, transitivity, etc.) will be given in Sections 2.2–2.3, where the reader can find a more detailed discussion about various classical notions of chaos considered in the literature.

Theorem 2.1.4. *Let ψ be a map inducing chaotic dynamics on $m \geq 2$ symbols on a set $\mathcal{D} \subseteq X$ and which is continuous on*

$$\mathcal{K} := \bigcup_{i=0}^{m-1} \mathcal{K}_i \subseteq \mathcal{D},$$

where $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}, \mathcal{D}$ and X are as in Definition 2.1.2. Setting

$$\mathcal{I}_\infty := \bigcap_{n=0}^{\infty} \psi^{-n}(\mathcal{K}), \qquad (2.1.3)$$

then there exists a nonempty compact set

$$\mathcal{I} \subseteq \mathcal{I}_\infty \subseteq \mathcal{K},$$

on which the following are fulfilled:

(i) $\mathcal{I}$ is invariant for ψ (that is, $\psi(\mathcal{I}) = \mathcal{I}$);

---

[2]Recalling also the third conclusion in Theorem 1.1.8 (or in Theorem 1.2.21), we could complement Theorem 2.1.3 with the further information about the existence, for any forward sequence on m symbols, of a corresponding continuum joining the lower and the upper sides of $\mathcal{R}$ and consisting of the points whose forward ψ-itinerary realizes the given sequence. However, for the sake of simplicity and since in the applications we don't use this fact, we have decided to omit it. The same observation also applies to the results obtained in Section 2.4 in relation to the framework of the Linked Twist Maps.

(ii) $\psi \upharpoonright_{\mathcal{I}}$ *is semi-conjugate to the one-sided Bernoulli shift on m symbols, i.e. there exists a continuous map π of $\mathcal{I}$ onto $\Sigma_m^+ := \{0, \ldots, m-1\}^{\mathbb{N}}$, endowed with the distance*

$$\hat{d}(\boldsymbol{s}', \boldsymbol{s}'') := \sum_{i \in \mathbb{N}} \frac{|s_i' - s_i''|}{m^{i+1}}, \qquad (2.1.4)$$

for $\boldsymbol{s}' = (s_i')_{i \in \mathbb{N}}$ and $\boldsymbol{s}'' = (s_i'')_{i \in \mathbb{N}} \in \Sigma_m^+$, such that the diagram

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\psi} & \mathcal{I} \\ \pi \downarrow & & \downarrow \pi \\ \Sigma_m^+ & \xrightarrow{\sigma} & \Sigma_m^+ \end{array} \qquad (2.1.5)$$

commutes, i.e $\pi \circ \psi = \sigma \circ \pi$, where $\sigma : \Sigma_m^+ \to \Sigma_m^+$ is the Bernoulli shift defined by $\sigma((s_i)_i) := (s_{i+1})_i, \forall i \in \mathbb{N}$;

(iii) *The set $\mathcal{P}$ of the periodic points of $\psi \upharpoonright_{\mathcal{I}_\infty}$ is dense in $\mathcal{I}$ and the preimage $\pi^{-1}(\boldsymbol{s}) \subseteq \mathcal{I}$ of every k-periodic sequence $\boldsymbol{s} = (s_i)_{i \in \mathbb{N}} \in \Sigma_m^+$ contains at least one k-periodic point.*

Furthermore, from conclusion (ii) it follows that:

(iv)

$$h_{\text{top}}(\psi) \geq h_{\text{top}}(\psi \upharpoonright_{\mathcal{I}}) \geq h_{\text{top}}(\sigma) = \log(m),$$

where h_{top} is the topological entropy;

(v) *There exists a compact invariant set $\Lambda \subseteq \mathcal{I}$ such that $\psi|_\Lambda$ is semi-conjugate to the one-sided Bernoulli shift on m symbols, topologically transitive and has sensitive dependence on initial conditions.*

Proof. Let us begin by checking that the set $\mathcal{I}_\infty$ in (2.1.3) is compact and nonempty. By the continuity of the map ψ on $\mathcal{K}$, it follows that $\mathcal{I}_\infty$ is closed and, being contained in the compact set $\mathcal{K}$, it is compact, too. The fact that $\mathcal{I}_\infty$ is nonempty follows from Definition 2.1.2 on chaotic dynamics, by observing that $z \in \mathcal{I}_\infty \Leftrightarrow \psi^n(z) \in \mathcal{K}, \forall n \geq 0$. This remark also implies that $\psi(\mathcal{I}_\infty) \subseteq \mathcal{I}_\infty$: indeed, it is straightforward to see that if $z \in \mathcal{I}_\infty$, then also $\psi(z) \in \mathcal{I}_\infty$.

Calling $\mathcal{P}$ the subset of $\mathcal{I}_\infty$ consisting of the periodic points of $\psi \upharpoonright_{\mathcal{I}_\infty}$, that is,

$$\mathcal{P} := \left\{ w \in \mathcal{I}_\infty : \exists k \in \mathbb{N}_0, \ \psi^k(w) = w \right\}, \qquad (2.1.6)$$

we claim that $\psi(\mathcal{P}) = \mathcal{P}$. Indeed, if $z \in \mathcal{P}$, then there exists $l \in \mathbb{N}_0$ such that $\psi^l(z) = z$. Hence, on the one hand, $\psi(z) = \psi(\psi^l(z)) = \psi^{l+1}(z) = \psi^l(\psi(z))$ and thus $\psi(z) \in \mathcal{P}$, too. This shows that $\psi(\mathcal{P}) \subseteq \mathcal{P}$. Notice that, repeating the same argument, it is possible to prove that if $z \in \mathcal{P}$, then $\psi^h(z) \in \mathcal{P}$, for any $h \geq 1$. On the other hand, if $\psi^l(z) = z$, for some $l \in \mathbb{N}_0$, then two possibilities can occur for l, that is, $l = 1$ or $l \geq 2$. In the former case we get $\psi(z) = z$ and so $z \in \psi(\mathcal{P})$, while in the latter we obtain $z = \psi^l(z) = \psi(\psi^{l-1}(z))$. Hence, since $\psi^{l-1}(z) \in \mathcal{P}$ whenever $z \in \mathcal{P}$, we find again $z \in \psi(\mathcal{P})$. In any case we have proved that, if $z \in \mathcal{P}$, then $z \in \psi(\mathcal{P})$, i.e. $\mathcal{P} \subseteq \psi(\mathcal{P})$. The claim is thus checked.

At this point we observe that, since $\mathcal{P}$ is contained in the compact set $\mathcal{I}_\infty$, also

$$\mathcal{I} := \overline{\mathcal{P}} \subseteq \mathcal{I}_\infty, \qquad (2.1.7)$$

and moreover $\mathcal{I}$ is compact, as it is closed in a compact set. From $\psi(\mathcal{P}) = \mathcal{P}$, it follows that

$$\psi(\mathcal{I}) = \psi(\overline{\mathcal{P}}) \supseteq \psi(\mathcal{P}) = \mathcal{P}.$$

But again, by the compactness of $\psi(\mathcal{I})$, it holds that

$$\psi(\mathcal{I}) \supseteq \overline{\mathcal{P}} = \mathcal{I}.$$

Let us show that the reverse inclusion is also fulfilled for $\mathcal{I}$, that is $\psi(\mathcal{I}) \subseteq \mathcal{I}$. Indeed, since ψ is continuous, we have

$$\psi(\mathcal{I}) = \psi(\overline{\mathcal{P}}) \subseteq \overline{\psi(\mathcal{P})} = \overline{\mathcal{P}} = \mathcal{I}.$$

Hence, the invariance of $\mathcal{I}$ is verified, in agreement with conclusion (i).

Let us consider now the diagram

$$
\begin{array}{ccc}
\mathcal{I}_\infty & \xrightarrow{\ \psi\ } & \mathcal{I}_\infty \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
\Sigma_m^+ & \xrightarrow{\ \sigma\ } & \Sigma_m^+
\end{array}
$$

with (Σ_m^+, σ) the Bernoulli system, and define the map $\pi : \mathcal{I}_\infty \to \Sigma_m^+$ by associating to any $w \in \mathcal{I}_\infty$ the sequence $(s_n)_{n \in \mathbb{N}} \in \Sigma_m^+$ such that $s_n = j$ if $\psi^n(w) \in \mathcal{K}_j$, for $j = 0, \ldots, m-1$. More formally, we notice that for any $w \in \mathcal{I}_\infty$, there exists a forward itinerary $(w_i)_{i \in \mathbb{N}}$ such that $w_0 = w$ and $\psi(w_i) = w_{i+1} \in \mathcal{K}$, for every $i \in \mathbb{N}$. Hence the function $g_1 : \mathcal{I}_\infty \to \mathcal{I}_\infty^{\mathbb{N}}$, which maps any $w \in \mathcal{I}_\infty$ into the one-sided sequence of points from the set $\mathcal{I}_\infty$

$$\mathbf{s}_w := (w_i)_{i \in \mathbb{N}}, \quad \text{where } w_i := \psi^i(w), \quad \forall\, i \in \mathbb{N},$$

with the usual convention $\psi^0 = \mathrm{Id}_{\mathcal{I}_\infty}$ and $\psi^1 = \psi$, is well-defined. Since the sets $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}$ are pairwise disjoint, for every term w_i of $\mathbf{s}_w$ there exists a unique index

$$s_i = s_i(w_i), \quad \text{with} \quad s_i \in \{0, \ldots, m-1\},$$

such that $w_i \in \mathcal{K}_{s_i}$. Therefore the map $g_2 : \mathcal{I}_\infty^{\mathbb{N}} \to \Sigma_m^+$,

$$g_2 : \mathbf{s}_w \mapsto (s_i)_{i \in \mathbb{N}} \in \Sigma_m^+$$

is also well-defined. Thus, by Definition 2.1.2 the map

$$\pi := g_2 \circ g_1 : \mathcal{I}_\infty \to \Sigma_m^+$$

is a surjection that makes the diagram (2.1.5) commute and the preimage through π of any k-periodic sequence in Σ_m^+ contains at least one k-periodic point of $\mathcal{I}_\infty$. To check that π is continuous, we prove the continuity in a generic $\bar{z} \in \mathcal{I}_\infty$, by showing that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\forall z \in \mathcal{I}_\infty$ with $d_X(z, \bar{z}) < \delta$, then $\hat{d}(\pi(z), \pi(\bar{z})) < \varepsilon$, where d_X is the distance on X and $\hat{d}$ is the metric defined in (2.1.4). Let us fix $\varepsilon > 0$ and let $n \in \mathbb{N}$ such that $0 < 1/m^n < \varepsilon$. We notice that it is sufficient to prove that $(\pi(z))_i = (\pi(\bar{z}))_i$, for any $i = 0, \ldots, n$. Indeed, if this is the case, by the definition of $\hat{d}$, it follows that $\hat{d}(\pi(z), \pi(\bar{z})) \leq 1/m^n < \varepsilon$.
Since $\bar{z} \in \mathcal{I}_\infty$, there exists a sequence $(s_0, \ldots, s_n) \in \{0, \ldots, m-1\}^{n+1}$ such that

$$\bar{z} \in \mathcal{K}_{s_0}, \, \psi(\bar{z}) \in \mathcal{K}_{s_1}, \ldots, \psi^n(\bar{z}) \in \mathcal{K}_{s_n}.$$

By the pairwise disjointness of the sets $\mathcal{K}_i$'s, it holds that

$$\eta := \min\{d_X(\mathcal{K}_i, \mathcal{K}_j) : i, j = 0, \ldots, m-1\} > 0.$$

Hence, for any $z \in \mathcal{I}_\infty$ with $d_X(z, \bar{z}) < \eta/2$, it follows that $z \in \mathcal{K}_{s_0}$, too. By the continuity of ψ in $\bar{z}$, there exists $\delta_1 > 0$ such that $\forall z \in \mathcal{I}_\infty$ with $d_X(z, \bar{z}) < \delta_1$, then $d_X(\psi(z), \psi(\bar{z})) < \eta/2$. But this means that $\psi(z) \in \mathcal{K}_{s_1}$. Analogously, by the continuity of ψ^2 in $\bar{z}$, there exists $\delta_2 > 0$ such that $\forall z \in \mathcal{I}_\infty$ with $d_X(z, \bar{z}) < \delta_2$, then $d_X(\psi^2(z), \psi^2(\bar{z})) < \eta/2$ and thus $\psi^2(z) \in \mathcal{K}_{s_2}$, for any such z. Proceeding in such way until the n-th iterate of ψ and setting

$$\delta := \min\left\{\frac{\eta}{2}, \delta_1, \ldots, \delta_n\right\},$$

we find that, for any $z \in \mathcal{I}_\infty$ with $d_X(z, \bar{z}) < \delta$, it holds that

$$z \in \mathcal{K}_{s_0}, \, \psi(z) \in \mathcal{K}_{s_1}, \ldots, \psi^n(z) \in \mathcal{K}_{s_n},$$

exactly as for $\bar{z}$. But this means that $(\pi(z))_i = (\pi(\bar{z}))_i$, for any $i = 0, \ldots, n$, and hence $\hat{d}(\pi(z), \pi(\bar{z})) \leq 1/m^n < \varepsilon$. The continuity of π is thus proved.

Considering in diagram (2.1.5) the restriction of ψ to $\mathcal{P} \subseteq \mathcal{I}_\infty$, we find the commutative diagram

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\psi} & \mathcal{P} \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
\mathcal{P}_m^+ & \xrightarrow{\sigma} & \mathcal{P}_m^+
\end{array}
$$

where $\mathcal{P}_m^+ \subseteq \Sigma_m^+$ is the set of the periodic sequences of m symbols. Notice that $\pi(\mathcal{P}) = \mathcal{P}_m^+$ thanks to Definition 2.1.2. Recalling the well-known fact that $\mathcal{P}_m^+$ is dense in Σ_m^+ [68, Proposition 1.9.1], by the continuity of π, it follows that

$$
\pi(\mathcal{I}) = \pi(\overline{\mathcal{P}}) \subseteq \overline{\mathcal{P}_m^+} = \Sigma_m^+.
$$

On the other hand $\pi(\mathcal{I})$ is a compact set containing $\pi(\mathcal{P}) = \mathcal{P}_m^+$, and hence

$$
\pi(\mathcal{I}) \supseteq \overline{\mathcal{P}_m^+} = \Sigma_m^+.
$$

Therefore, we can conclude that $\pi(\mathcal{I}) = \Sigma_m^+$ and the diagram

$$
\begin{array}{ccc}
\mathcal{I} & \xrightarrow{\psi} & \mathcal{I} \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
\Sigma_m^+ & \xrightarrow{\sigma} & \Sigma_m^+
\end{array}
$$

still commutes. Moreover, the preimage through π of any k-periodic sequence in Σ_m^+ contains at least one k-periodic point of $\mathcal{I}$, as $\mathcal{P} \subseteq \mathcal{I}$. Conclusions (ii) and (iii) are thus proved.

Assertion (iv), about the positive topological entropy, comes from property (ii) about the semi-conjugacy to the Bernoulli shift. For a proof, see [160, Theorem 7.12].

Finally, conclusion (v), about the existence of the compact invariant set $\Lambda \subseteq \mathcal{I}$, follows by applying Theorem 2.3.3 with the positions $(X, f) = (\mathcal{I}, \psi \restriction_\mathcal{I})$ and $(Y, g) = (\Sigma_m^+, \sigma)$. $\qquad\qquad\square$

We point out that results similar to Theorem 2.1.4 are almost known in the literature (see e.g. [148, Theorem 3] for a related statement), maybe with exception of conclusion (v). However, for the sake of completeness, we have decided to prove it in full details. Notice that, in frameworks more general than ours, the set $\mathcal{I}_\infty$ in (2.1.3), and a fortiori $\mathcal{P}$ in (2.1.6), can be empty and thus it is not possible to define $\mathcal{I}$ as in (2.1.7) in order to have an invariant

set. In our case we can do like that since, by Definition 2.1.2, the sequences of symbols are realized by points in $\mathcal{I}_\infty$ and the periodic sequences of symbols are reproduced by points in $\mathcal{P}$. We also remark that the density of the periodic points of ψ in $\mathcal{I}$ does not imply that the periodic points are dense in the smaller set Λ, on which ψ is transitive and sensitive. Therefore, we cannot conclude the presence of Devaney chaos (cf. Definition 2.3.2) for ψ on Λ.

Remark 2.1.5. If, in addition to the hypotheses of Theorem 2.1.4, the map ψ is also injective on $\mathcal{K}$, then one can deal with bi-infinite sequences of m symbols instead of forward ones and prove the stronger property for ψ of semi-conjugacy to the two-sided Bernoulli shift $\sigma : \Sigma_m := \{0, \ldots, m-1\}^{\mathbb{Z}} \to \Sigma_m$, defined as $\sigma((s_i)_i) := (s_{i+1})_i$, $\forall i \in \mathbb{Z}$. Indeed, it is sufficient to replace the set $\mathcal{I}_\infty$ in (2.1.3) with

$$\mathcal{I}_\infty := \bigcap_{n=-\infty}^{\infty} \psi^{-n}(\mathcal{K})$$

and consider $\mathcal{I}$ as in (2.1.7), in order to obtain, via similar steps, the analogue of the conclusions of Theorem 2.1.4 with respect to two-sided sequences, rather than one-sided sequences. The precise statement reads as follows:

Theorem 2.1.6. *Let ψ be a map inducing chaotic dynamics on $m \geq 2$ symbols on a set $\mathcal{D} \subseteq X$ and which is continuous and injective on*

$$\mathcal{K} := \bigcup_{i=0}^{m-1} \mathcal{K}_i \subseteq \mathcal{D},$$

where $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}, \mathcal{D}$ and X are as in Definition 2.1.2. Setting

$$\mathcal{I}_\infty := \bigcap_{n=-\infty}^{\infty} \psi^{-n}(\mathcal{K}),$$

then there exists a nonempty compact set

$$\mathcal{I} \subseteq \mathcal{I}_\infty \subseteq \mathcal{K},$$

on which the following are fulfilled:

(i) *$\mathcal{I}$ is invariant for ψ (that is, $\psi(\mathcal{I}) = \mathcal{I}$);*

(ii) *$\psi \restriction_{\mathcal{I}}$ is semi-conjugate to the two-sided Bernoulli shift on m symbols, i.e. there exists a continuous map π of $\mathcal{I}$ onto $\Sigma_m := \{0, \ldots, m-1\}^{\mathbb{Z}}$, endowed with the distance*

$$\hat{d}(\boldsymbol{s}', \boldsymbol{s}'') := \sum_{i \in \mathbb{Z}} \frac{|s_i' - s_i''|}{m^{|i|+1}},$$

for $\boldsymbol{s'} = (s_i')_{i\in\mathbb{Z}}$ and $\boldsymbol{s''} = (s_i'')_{i\in\mathbb{Z}} \in \Sigma_m$, such that the diagram

$$
\begin{array}{ccc}
\mathcal{I} & \stackrel{\psi}{\longrightarrow} & \mathcal{I} \\
\pi \downarrow & & \downarrow \pi \\
\Sigma_m & \stackrel{}{\underset{\sigma}{\longrightarrow}} & \Sigma_m
\end{array}
$$

commutes, i.e $\pi \circ \psi = \sigma \circ \pi$, where $\sigma : \Sigma_m \to \Sigma_m$ is the Bernoulli shift defined by $\sigma((s_i)_i) := (s_{i+1})_i$, $\forall i \in \mathbb{Z}$;

(iii) *The set $\mathcal{P}$ of the periodic points of $\psi \restriction_{\mathcal{I}_\infty}$ is dense in $\mathcal{I}$ and the preimage $\pi^{-1}(\boldsymbol{s}) \subseteq \mathcal{I}$ of every k-periodic sequence $\boldsymbol{s} = (s_i)_{i\in\mathbb{N}} \in \Sigma_m$ contains at least one k-periodic point.*

Furthermore, from conclusion (ii) it follows that:

(iv)
$$
h_{\mathrm{top}}(\psi) \geq h_{\mathrm{top}}(\psi \restriction_{\mathcal{I}}) \geq h_{\mathrm{top}}(\sigma) = \log(m),
$$

where h_{top} is the topological entropy;

(v) *There exists a compact invariant set $\Lambda \subseteq \mathcal{I}$ such that $\psi|_\Lambda$ is semi-conjugate to the two-sided Bernoulli shift on m symbols, topologically transitive and has sensitive dependence on initial conditions.*

For further details see [126], where some alternative approaches are expounded, such as that in [86].

The present observation applies, for instance, when dealing with the original Smale horseshoe map, introduced at the beginning of Chapter 2. Another field of applicability is that of the ODEs with periodic coefficients. Namely, in such a case, if one looks for periodic solutions, the corresponding map ψ is the homeomorphism called Poincaré map associated to the system. Some examples will be presented in Chapter 4.

$\lhd$

To conclude the discussion about the results in [69], we notice that, thanks to the fact that the kind of chaos in Definition 2.1.2 is stricter than the one detected in [69, 71], we are able to obtain a result (Theorem 2.1.4) that covers the Chaos Lemma ([69, Lemma 4]), adding further information. In fact, the set $\mathcal{I}$ in Theorem 2.1.4 plays the role of Q_I in [69] and the subsets $\Lambda \subseteq \mathcal{I}$ and $Q_* \subseteq Q_I$ share some common features, like the invariance and the sensitivity of

the maps defined on them, but we have in addition the density of the periodic points on $\mathcal{I}$, while, as already observed, in [69, 71] the existence of periodic points is not ensured at all. Also the transitivity is missing in the statement of the Chaos Lemma ([69, Lemma 4]), even if, after a look at its proof, it is evident that such property holds in that framework, too. Indeed the invariant set Q_* is the ω-limit set of a certain point x^* of Q_I and thus the orbit of x^* is dense in Q_*. On the other hand, by the invariance of Q_*, this is enough to infer the transitivity on Q_*. As we shall see in Section 2.3, the same argument is employed in the proof of Theorem 2.3.3, where the reader can find the missing definitions and details.

Finally, a natural question that can arise is whether our stretching condition in Theorem 2.1.3 is strong enough to imply a conjugacy to the Bernoulli shift, rather than just a semi-conjugacy as stated in Theorem 2.1.4. The answer in general is negative. However, it is possible to add further assumptions in order to get it. A first step in this direction is represented by some recent works by Zgliczyński and collaborators [34, 175], where they combine the theory of covering relations from [176] with certain cone conditions to establish the properties implied by hyperbolicity, such as the existence of stable and unstable manifolds. Moreover, in their framework, every periodic sequence of symbols is realized by the itinerary generated by a unique point: in symbols, using the notation introduced in the statement of Theorem 2.1.4, this means that the preimage $\pi^{-1}(\mathbf{s}) \subseteq \mathcal{I}$ of every k-periodic sequence $\mathbf{s} = (s_i)_{i \in \mathbb{N}} \in \Sigma_m^+$ contains exactly one k-periodic point.

2.2   Symbolic dynamics

In this section we furnish some tools from symbolic dynamics that will be useful to investigate more deeply the relationship between the notion of chaos in the sense of Definition 2.1.2 and other ones widely considered in the literature. In order to accomplish such task, we need to recall a few basic definitions: some of them have already been introduced in Section 2.1, but we prefer to gather them here for the sake of consistency.

Given an integer $m \geq 2$, we denote by $\Sigma_m := \{0, \ldots, m-1\}^{\mathbb{Z}}$ the set of the two-sided sequences of m symbols and by $\Sigma_m^+ := \{0, \ldots, m-1\}^{\mathbb{N}}$ the set of one-sided sequences of m symbols. These compact spaces are usually endowed with the distance

$$\hat{d}(\mathbf{s}', \mathbf{s}'') := \sum_{i \in \mathbb{I}} \frac{|s_i' - s_i''|}{m^{|i|+1}}, \quad \text{for } \mathbf{s}' = (s_i')_{i \in \mathbb{I}}, \ \mathbf{s}'' = (s_i'')_{i \in \mathbb{I}}, \quad (2.2.1)$$

where $\mathbb{I} = \mathbb{Z}$ or $\mathbb{I} = \mathbb{N}$, respectively. The metric in (2.2.1) could be replaced with

$$\tilde{d}(\mathbf{s}', \mathbf{s}'') := \sum_{i \in \mathbb{I}} \frac{d(s_i', s_i'')}{m^{|i|+1}}, \quad \text{for } \mathbf{s}' = (s_i')_{i \in \mathbb{I}}, \ \mathbf{s}'' = (s_i'')_{i \in \mathbb{I}},$$

where $d(\cdot, \cdot)$ is the discrete distance on $\{0, \ldots, m-1\}$, that is, $d(s_i', s_i'') = 0$ for $s_i' = s_i''$ and $d(s_i', s_i'') = 1$ for $s_i' \neq s_i''$. The significance of this second choice reveals when one needs to look at the elements from $\{0, \ldots, m-1\}$ as symbols instead of numbers.

On such spaces we define the *one-sided Bernoulli shift* $\sigma : \Sigma_m^+ \to \Sigma_m^+$ and the *two-sided Bernoulli shift* $\sigma : \Sigma_m \to \Sigma_m$ on m symbols as $\sigma((s_i)_i) := (s_{i+1})_i$, $\forall i \in \mathbb{I}$, for $\mathbb{I} = \mathbb{N}$ or $\mathbb{I} = \mathbb{Z}$, respectively. Both maps are continuous and the two-sided shift is a homeomorphism.

A precious tool for the detection of complex dynamics is the *topological entropy* and indeed its positivity is generally considered as one of the trademarks of chaos. Such object can be introduced for any continuous self-map f of a compact topological space X and we indicate it with the symbol $h_{\text{top}}(f)$. Its original definition due to Adler, Konheim and McAndrew [1] is based on the open coverings. More precisely, for an open cover α of X, we define the *entropy of α* as $H(\alpha) := \log N(\alpha)$, where $N(\alpha)$ is the minimal number of elements in a finite subcover of α. Given two open covers α and β of X, we define their *join* $\alpha \vee \beta$ as the open cover of X made by all sets of the form $A \cap B$, with $A \in \alpha$ and $B \in \beta$. Similarly one can define the join $\vee_{i=1}^n \alpha_i$ of any finite collection of open covers of X. If α is an open cover of X and $f : X \to X$ a continuous map, we denote by $f^{-1}\alpha$ the open cover consisting of all sets $f^{-1}(A)$, with $A \in \alpha$. By $\vee_{i=0}^{n-1} f^{-i}\alpha$ we mean $\alpha \vee f^{-1}\alpha \vee \cdots \vee f^{-n+1}\alpha$. Finally, we have:

$$h_{\text{top}}(f) := \sup_\alpha \left(\lim_{n \to \infty} \frac{1}{n} \left(H\left(\vee_{i=0}^{n-1} f^{-i}\alpha \right) \right) \right),$$

where α ranges over all open covers of X.

Among the several properties of the topological entropy, we recall just the ones that are useful in view of the subsequent discussion.

In regard to the (one-sided or two-sided) Bernoulli shift σ on m symbols, it holds that

$$h_{\text{top}}(\sigma) = \log(m).$$

Given a continuous self-map f of a compact topological space X and a *invariant* (resp. *positively invariant*) subset $\mathcal{I} \subseteq X$, i.e such that $f(\mathcal{I}) = \mathcal{I}$ (resp. $f(\mathcal{I}) \subseteq \mathcal{I}$), then

$$h_{\text{top}}(f) \geq h_{\text{top}}(f \upharpoonright_{\mathcal{I}}). \tag{2.2.2}$$

Denoting by f^n the n-th iterate of the continuous self-map f of a compact topological space X, we have

$$h_{\text{top}}(f^n) = nh_{\text{top}}(f), \ \forall n \geq 1. \tag{2.2.3}$$

Given two continuous self-maps $f : X \to X$ and $g : Y \to Y$ of the compact topological spaces X and Y and a continuous onto map $\phi : X \to Y$ that makes the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & X \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\
Y & \xrightarrow[\ g\ ]{} & Y
\end{array}
\tag{2.2.4}
$$

commute, i.e. such that $\phi \circ f = g \circ \phi$, then it holds that

$$h_{\text{top}}(f) \geq h_{\text{top}}(g).$$

If ϕ is also injective, the above inequality is indeed an equality.

When the diagram in (2.2.4) commutes, we say that f and g are *topologically semi-conjugate* and that ϕ is a *semi-conjugacy* between them. If ϕ is also one-to-one, then f and g are called *topologically conjugate* and ϕ is named *conjugacy*.

Thus, when for a continuous self-map f of a compact topological space X and a (positively) invariant subset $\mathcal{I} \subseteq X$ it holds that $f \restriction_{\mathcal{I}}$ is semi-conjugate to the (one-sided or two-sided) Bernoulli shift σ on m symbols, then

$$h_{\text{top}}(f) \geq h_{\text{top}}(f \restriction_{\mathcal{I}}) \geq h_{\text{top}}(\sigma) = \log(m). \tag{2.2.5}$$

If $f \restriction_{\mathcal{I}}$ is conjugate to σ, then the second inequality is indeed an equality.

We notice that, although the topological entropy can be defined for continuous self-maps of topological spaces, we confine ourselves to the case of metric spaces. More precisely, when dealing with chaotic dynamics, we will consider *dynamical systems*, i.e. couples (X, f), where X is a compact metric space and $f : X \to X$ is continuous and surjective.

For further features of h_{top} and additional details, see [1, 68, 160]. With reference to the case of compact metric spaces, alternative definitions of entropy can be found in [25, 42].

Let us begin our excursus on complex dynamics by returning on the concepts of "covering" and "crossing" introduced at the beginning of Chapter 2 and fundamental in the theory of symbolic dynamics.

For the reader's convenience, we recall that, given a continuous mapping $f : \mathbb{R} \to \mathbb{R}$ and two intervals $I, J \subseteq \mathbb{R}$, we say that I f-*covers* J if $f(I) \supseteq J$, or

equivalently, if there exists a subinterval $I_0 \subseteq I$ such that $f(I_0) = J$. We also say that *I f-covers J m times* if there are $m \geq 2$ subintervals $I_0, \ldots, I_{m-1} \subseteq I$, with pairwise disjoint interiors, such that $f(I_k) = J$, for $k = 0, \ldots, m-1$ (see [23]). In particular, we will focus on the case in which $I = J$ and there exist m compact intervals $I_0, \ldots, I_{m-1} \subseteq I$, with pairwise disjoint interiors, such that I_i f-covers I_j, for some (maybe all) $i, j \in \{0, \ldots, m-1\}$.

The natural extension of the concept of f-covering to the setting of a continuous self-map f of a generic metric space X can be obtained by replacing intervals with compact subsets of X in the previous definitions. Indeed, one can assume the existence of m pairwise disjoint compact sets $C_0, \ldots, C_{m-1} \subseteq X$ such that $f(C_i) \supseteq C_j$, for some $i, j \in \{0, \ldots, m-1\}$. On the other hand, as suggested in [27], instead of the above covering relation, one could deal with a weaker condition of the form $f(C_i)$ "goes across" C_j, which does not require the map f to be surjective on the C_j's. For example, in our approach, we interpret the expression "goes across" in terms of the stretching along the paths effect from Definition 1.1.1 or Definition 1.2.19, that is, $f(C_i)$ "goes across" C_j means for us that $f : \widetilde{C_i} \Leftrightarrow \rightarrow \widetilde{C_j}$, where $\widetilde{C_i}$ and $\widetilde{C_j}$ are (N-dimensional) oriented rectangles of X. In this respect, an interesting framework is the one described in Definition 1.1.2 for $\widetilde{A} = \widetilde{B} = \widetilde{\mathcal{R}}$, or in the statement of Theorem 2.1.3 with $X = \mathbb{R}^2$. [3] In such situation, there exist $m \geq 2$ pairwise disjoint compact subsets $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}$ of a generalized planar rectangle $\mathcal{R}$ for which $(\mathcal{K}_i, \psi) : \widetilde{\mathcal{R}} \Leftrightarrow \rightarrow \widetilde{\mathcal{R}}$ holds, where $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$. Then, in view of Remark 1.1.9, it is possible to find m pairwise disjoint vertical slabs $\widetilde{\mathcal{R}}_i$ of $\widetilde{\mathcal{R}}$ (cf. Definition 1.1.5), with $\mathcal{R}_i \supseteq \mathcal{K}_i$, for $i = 0, \ldots, m-1$, which satisfy the relation $\psi : \widetilde{\mathcal{R}}_i \Leftrightarrow \rightarrow \widetilde{\mathcal{R}}_j$, $\forall i, j \in \{0, \ldots, m-1\}$.

The pretext for considering the above covering and crossing relations comes from the previously observed fact that, whenever a map ψ induces chaotic dynamics in the sense of Definition 2.1.2, then its entropy is positive (cf. conclusion (iv) in Theorem 2.1.4) and this happens, for instance, when the stretching condition in Theorem 2.1.3 is fulfilled. Actually, we are going to show that a positive entropy can be obtained even under weaker assumptions.

For clarity's sake, we first explain the idea in the one-dimensional setting, in order to simplify the comprehension of the generic N-dimensional case (for $N \geq 2$). Hence, let us suppose that $f : I \to I$ is a continuous self-mapping of a compact interval $I \subset \mathbb{R}$ and assume there exists $a \in I$ such that, calling

---

[3]The case of sets homeomorphic to $[0, 1]^2$ but contained in a generic metric space X could be considered as well. However, since in the applications the space X is the Euclidean plane and the generalized rectangles are compact regions bounded by graphs of continuous functions, for simplicity we confine ourselves to the planar setting.

$b = f(a)$, $c = f(b) = f^2(a)$ and $d = f(c) = f^2(b) = f^3(a)$, it holds that

$$d \leq a < b < c \quad \text{or} \quad d \geq a > b > c. \tag{2.2.6}$$

In particular, if $d = a$ then the map f has in I a point of period three. This is in fact the case mentioned in the title of the well-known paper [93]. Notice that, setting

$$I_0 := [\min\{a, b\}, \max\{a, b\}] \quad \text{and} \quad I_1 := [\min\{b, c\}, \max\{b, c\}], \tag{2.2.7}$$

it follows that I_0 f-covers I_1 and I_1 f-covers I_0, as well as I_1 f-covers I_1. Under condition (2.2.6), the authors in [93] can prove the presence in I of periodic points of any period and also the existence of an uncountable scrambled set. We will return on this concept in Definition 2.3.1. Before that, we want to show the positivity of the topological entropy of f in the just described framework. To such end, we need some tools from the theory of symbolic dynamics [75, 94].

Given a continuous map $g : J \to J$ defined on a compact interval $J \subset \mathbb{R}$ and $n \geq 2$ closed subintervals $J_0, \ldots, J_{n-1} \subseteq J$, with pairwise disjoint interiors, we associate to the dynamical system (J, g) the $n \times n$ *transition matrix* $T = T(i, k)$, for $i, k = 0, \ldots, n-1$, defined as

$$T(i, k) = \begin{cases} 1 & \text{if } J_i \text{ } g\text{-covers } J_k, \\ 0 & \text{else.} \end{cases} \tag{2.2.8}$$

Moreover, let

$$\Sigma_T^+ := \{(s_i)_{i \in \mathbb{N}} \in \Sigma_n^+ : T(s_k, s_{k+1}) = 1, \forall k \in \mathbb{N}\} \tag{2.2.9}$$

be the *space of the admissible sequences for T* and

$$\sigma_T : \Sigma_T^+ \to \Sigma_T^+, \quad \sigma_T := \sigma \upharpoonright_{\Sigma_T^+} \tag{2.2.10}$$

be the *subshift of finite type for the matrix T*. In particular, according to [27], we say that such subshift of finite type is *nontrivial* if T is an *irreducible* matrix (that is, for every couple of integers i, $k \in \{0, \ldots, n-1\}$ there exists a positive integer l such that $T^l(i, k) > 0$), which is not a permutation on n symbols[4]. The space Σ_T^+ inherits the metric from Σ_n^+ and, with this choice, σ_T is continuous. Furthermore, it is possible to prove (cf. [75, Observation 1.4.2]) that, when T is irreducible, $h_{\text{top}}(\sigma_T) = \log(\overline{\lambda})$, where $\overline{\lambda}$ is the largest real eigenvalue of T in absolute value, also called *Perron eigenvalue of T*. Therefore, if there exists a

---

[4]Notice that, when T is the matrix whose entries are all 1, then $\Sigma_T^+ = \Sigma_n^+$ and σ_T coincides with the classical Bernoulli shift σ, also known as *full shift* [75].

(positively) invariant set $\mathcal{I}^* \subseteq J$ such that $g \restriction_{\mathcal{I}^*}$ is semi-conjugate to σ_T, it holds that

$$h_{\text{top}}(g) \geq h_{\text{top}}(g \restriction_{\mathcal{I}^*}) \geq h_{\text{top}}(\sigma_T) = \log(\overline{\lambda}). \tag{2.2.11}$$

In the case under investigation, we can take

$$\mathcal{I}^* := \bigcap_{i=0}^{+\infty} g^{-i}(J) \tag{2.2.12}$$

and define the semi-conjugacy $\pi : \mathcal{I}^* \to \Sigma_T^+$ between $g \restriction_{\mathcal{I}^*}$ and σ_T as

$$(\pi(x))_i = k \quad \text{iff} \quad g^i(x) \in J_k, \quad \text{for} \quad i \in \mathbb{N}. \tag{2.2.13}$$

We recall that, according to [27, Lemma 1.2], for the Perron eigenvalue it holds that $\overline{\lambda}$ is strictly greater than 1 unless T is a permutation matrix (otherwise, $\overline{\lambda} = 1$). Thus, the topological entropy of any nontrivial subshift of finite type is positive.

We have now available all the ingredients to deal with a map f with the properties expressed in (2.2.6). The transition matrix for f associated to the intervals I_0 and I_1 in (2.2.7) is

$$T_f = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2.2.14}$$

Defining the (positively) invariant set $\mathcal{I}^*$ for f as in (2.2.12), it is possible to obtain a semi-conjugacy as in (2.2.13) between $f \restriction_{\mathcal{I}^*}$ and the subshift of finite type σ_{T_f} for the matrix T_f. Thus, we can conclude that

$$h_{\text{top}}(f) \geq h_{\text{top}}(\sigma_{T_f}) = \log\left(\frac{1+\sqrt{5}}{2}\right), \tag{2.2.15}$$

since the Perron eigenvalue of T_f is $\overline{\lambda} = \frac{1+\sqrt{5}}{2}$. This quantity coincides with the *golden mean ratio* and the map σ_{T_f} is also known under the name of *golden mean shift*.

We stress that more general situations can be handled in a similar manner. For instance, a further case that can be considered is when for a continuous map $g : J \to J$, some multiple coverings among the subintervals $J_0, \ldots, J_{n-1}$ of J occur. In such a framework, the transition matrix T is replaced by the $n \times n$ *adjacency matrix* $A = A(i,k)$, for $i,k = 0, \ldots, n-1$, defined as

$$A(i,k) = \begin{cases} l & \text{if } J_i \ g\text{-covers } J_k \ l \text{ times}, \\ 0 & \text{if } J_i \text{ does not } g\text{-cover } J_k, \end{cases} \tag{2.2.16}$$

where l is a positive integer[5]. The entries of A are now nonnegative integers, possibly different from $0, 1$. Also in this case, one can associate a suitable subshift of finite type to A. To understand how to act, it is necessary to recall some notions from graph theory [94, 137].

A *graph G* consists of a finite set $\mathcal{V} = \mathcal{V}(G)$ of *vertices* and of a finite set of *edges* $\mathcal{E} = \mathcal{E}(G)$ among them. Every edge $e \in \mathcal{E}(G)$ starts at a vertex denoted by $i(e)$ (*initial state*) and ends at a vertex denoted by $t(e)$ (*terminal state*), which can possibly coincide with $i(e)$. Of course, there may be several edges between the same initial and terminal states. In the special case that $A = A(i, k)$ is an adjacency matrix, we call *graph of A* the graph $G = G(A)$, with vertex set $\mathcal{V}(G) = \{0, \ldots, n-1\}$ and with $A(i, k)$ edges with initial state in i and terminal state in k, for $i, k = 0, \ldots, n-1$. The cardinality of the set of the edges is $|\mathcal{E}(G)| = \sum_{i,k \in \{0,\ldots,n-1\}} A(i, k)$. We name N this quantity, so that $\mathcal{E}(G) = \{e_1, \ldots, e_N\}$ is the set of the edges.

The shift σ_T in (2.2.10) for the transition matrix T in (2.2.8) is called a *vertex subshift*. We want now to define an *edge subshift* for the adjacency matrix A in (2.2.16). In order to realize it, we construct the transition matrix $T' = T'(i, k)$ on $\mathcal{E}(G)$, that is, the $N \times N$ matrix with entries

$$T'(i, k) = \begin{cases} 1 & \text{if } t(e_i) = i(e_k), \\ 0 & \text{else.} \end{cases}$$

The desired edge subshift for A corresponds to the vertex subshift for T', $\sigma_{T'} : \Sigma_{T'}^+ \to \Sigma_{T'}^+$, $\sigma_{T'} := \sigma \restriction_{\Sigma_{T'}^+}$, where the definition of $\Sigma_{T'}^+$ is analogous to the one of Σ_T^+ in (2.2.9) [6].

The edge subshifts can thus be seen as a counterpart of the vertex subshifts in relation to matrices with nonnegative integer entries, possibly different from $0, 1$. An example of matrices of such kind is represented by the powers of a given transition matrix and it is particularly important to handle them when dealing with the iterates of a given function.

Notice that, both in the case of edge subshifts and in the case of vertex subshifts, we have confined ourselves to the one-sided sequences of symbols, since we do not assume g to be one-to-one on its domain. Analogous results hold true for bi-infinite sequences under the additional hypothesis of injectivity for g (cf. Remark 2.1.5).

---

[5]If $l = 1$, condition "J_i g-covers J_k l times" simply reads as "J_i g-covers J_k".

[6]Observe that, when the entries of A are already from $\{0, 1\}$, it is still possible to construct an associated transition matrix T' as described above, that will be possibly different from A (also the dimensions n and N won't coincide in general). On the other hand, it is not difficult to show that the vertex subshift σ_A and the corresponding edge subshift $\sigma_{T'}$ are conjugate. Indeed, the conjugacy $h : \Sigma_{T'}^+ \to \Sigma_A^+$ can be defined as $h((e_j)_{j \in \mathbb{N}}) = (v_j)_{j \in \mathbb{N}}$, where $v_j = i(e_j)$, $\forall j \in \mathbb{N}$, with $v_j \in \mathcal{V}$ and $e_j \in \mathcal{E}$.

If, in place of self-maps on intervals, one deals with functions $f : X \to X$ defined on a generic metric space X, and considers, instead of the one-dimensional covering relation for intervals, some covering or crossing relation among pairwise disjoint compact subsets (or at least compact subsets with pairwise disjoint interiors, in the case of Markov partitions [139]) of X, then, to such framework, it is possible to associate a transition matrix or an adjacency matrix exactly as in (2.2.8) and (2.2.16), respectively, with intervals replaced by compact sets.

In particular, when the hypotheses of Theorem 2.1.3 are fulfilled with $X = \mathbb{R}^2$ and $m = 2$, we have already observed that there exist two vertical slabs $\widetilde{\mathcal{R}}_0$ and $\widetilde{\mathcal{R}}_1$ of $\widetilde{\mathcal{R}}$, with $\mathcal{R}_0 \supseteq \mathcal{K}_0$ and $\mathcal{R}_1 \supseteq \mathcal{K}_1$, which satisfy $\psi : \widetilde{\mathcal{R}}_i \Leftrightarrow \to \widetilde{\mathcal{R}}_j$, for $i, j = 0, 1$. Thus, the transition matrix for ψ associated to $\widetilde{\mathcal{R}}_0$ and $\widetilde{\mathcal{R}}_1$ is

$$T_\psi = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2.2.17}$$

Analogous conclusions can be drawn when restricting our stretching along the paths relation to the one-dimensional setting. In such a case, a pair $\widetilde{I} = (I, I^-)$, where $I = [a, b]$ is a compact interval and $I^- = \{a, b\}$ is the set of its extreme points, may be seen as a degenerate kind of oriented rectangle. Accordingly, the stretching property $(\mathcal{K}, \psi) : \widetilde{I} \Leftrightarrow \to \widetilde{I}$ is equivalent to the fact that $\mathcal{K}$ contains a compact interval I_0 such that $\psi(I_0) = I$. This shows that, in the one-dimensional case, our stretching property reduces to the classical covering relation discussed above. Moreover, noticing that, in the just described framework, we have $(I_0, \psi) : \widetilde{I} \Leftrightarrow \to \widetilde{I}$, then, if I contains two disjoint compact subintervals I_0 and I_1 such that $\psi(I_i) = I$, for $i = 0, 1$, we enter the setting of Theorem 2.1.3, with $\widetilde{I}$ playing the role of $\widetilde{\mathcal{R}}$ and with $\mathcal{K}_i := I_i$.

A look at the matrices T_f in (2.2.14) and T_ψ in (2.2.17) shows that the assumptions of Theorem 2.1.3, when restricted to the one-dimensional setting, are stronger than the conditions in (2.2.6) [7]. Recalling (2.2.15), this verifies our claim that a positive topological entropy can be obtained under hypotheses weaker than the ones in Theorem 2.1.3. On the other hand, with respect to [93], the assumptions in Theorem 2.1.3 allow to obtain sharper consequences from a dynamical point of view (cf. Section 2.3).

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[7]At first sight, this conclusion may seem rather obvious, since a map ψ as in Theorem 2.1.3 has periodic points of all periods and thus, in particular, also a point of period three. However, things are not so simple. Indeed, as a consequence of the Šarkowskii Theorem [152], any continuous self-map f defined on a compact interval and possessing a point of period three has periodic points of each period. Moreover, the conditions in (2.2.6) do not necessarily imply the existence of a point of period three, being in fact more general. This is the reason that led us to introduce some elements from symbolic dynamics in order to compare our framework to the one in [93].

Before abandoning the one-dimensional setting, let us employ the just introduced tools from symbolic dynamics to study the behaviour of the logistic map

$$F : [0,1] \to \mathbb{R}, \ F(x) := \mu x (1-x), \qquad (2.2.18)$$

when the parameter μ ranges in $[0, +\infty]$.

It is easy to see that for $\mu > 4$ the interval $[0,1]$ F-covers $[0,1]$ twice. Indeed, the maximum of F is attained when $x = 1/2$ and $F(1/2) = \mu/4$, so that $F(1/2) \geq 1$ for $\mu \geq 4$. Since $F(0) = F(1) = 0$, setting $\alpha := F^{-1}(\{1\}) \cap ([0, 1/2])$ and $\beta := F^{-1}(\{1\}) \cap ([1/2, 1])$, we have $(I_i, F) : \widetilde{I} \Leftrightarrow \to \widetilde{I}$, $i = 0, 1$, where $\widetilde{I} := ([0,1], \{0,1\})$, $I_0 := [0, \alpha]$ and $I_1 := [\beta, 1]$. [8] Unfortunately, in such a case, F does not map $[0,1]$ into itself and, for almost every initial point, the iterates F^n of the map limit to $-\infty$ as $n \to \infty$. This generates problems with numerical simulations: indeed, although Theorems 2.1.3 and 2.1.4 ensure the existence of a chaotic invariant set in $[0,1]$, such region is not visible on the computer screen, because almost all points eventually leave the domain through the iterates of F.
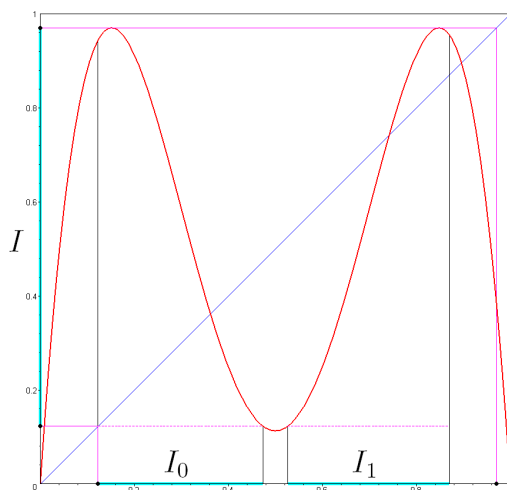
Figure 2.1: The graph of the second iterate of the logistic map F in (2.2.18) for $\mu = 3.88$.

When $\mu \leq 4$, the existence of multiple coverings can be established for some iterate of F through geometric arguments analogous to the ones employed for the case $\mu > 4$, that allow to conclude that the conditions of Theorem 2.1.3 are verified for sufficiently large values of the parameter $\mu \leq 4$. A corresponding example is illustrated in Figure 2.1, which depicts the second iterate of F for

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[8]Actually, $[0,1]$ F-covers $[0,1]$ twice also for the "watershed" value $\mu = 4$. However, in this case I_0 and I_1 are not disjoint and thus Theorem 2.1.3 does not apply directly.

$\mu = 3.88$. The interval $I \subseteq [0,1]$, drawn with a thicker line on the y-axis, is oriented by choosing as I^- its extreme points. For $\widetilde{I} = (I, I^-)$ and calling I_0 and I_1 the two subintervals of I highlighted on the x-axis, it is immediate to verify that $F^2(I_i) = I$, for $i = 0, 1$, and hence $(I_i, F^2) : \widetilde{I} \Leftrightarrow \rightarrow \widetilde{I}$, $i = 0, 1$. Therefore, the map F^2 induces chaotic dynamics on two symbols relatively to I_0, I_1 and possesses all the chaotic properties listed in Theorem 2.1.4. In particular, from this something can still be deduced for F. Indeed, by the semi-conjugacy between F^2 restricted to a suitable invariant set and the one-sided shift σ on two symbols (cf. Theorem 2.1.4, conclusions (ii) and (v)), it holds that

$$h_{\text{top}}\left(F^2\right) \geq \log(2)$$

and, since by (2.2.3) we have $h_{\text{top}}\left(F^2\right) = 2h_{\text{top}}(F)$, it follows

$$h_{\text{top}}(F) \geq \log(\sqrt{2}) > 0.$$

We conclude this section by reconsidering Definition 2.1.2 and Theorem 2.1.3 in the light of the above results from symbolic dynamics.

Indeed, when a map satisfies the hypotheses of Theorem 2.1.3 with $X = \mathbb{R}^2$ and for some $m \geq 2$, then the associated transition matrix has all elements equal to 1 (see (2.2.17) for the case $m = 2$). On the other hand, one could face the more general situation in which the transition matrix T associated to the system is irreducible, but possibly contains also some 0's [86]. The topological entropy can then be estimated as in (2.2.11) and one could say that the map ψ induces chaotic dynamics when the topological entropy of the corresponding subshift of finite type σ_T is positive. As already mentioned, this happens, for example, when σ_T is nontrivial [27].

According to [68, 75], if T is irreducible, the map σ_T is Devaney chaotic (cf. Definition 2.3.2) on Σ_T^+ and thus it displays transitivity and sensitivity with respect to initial conditions and the set of periodic points of σ_T is dense in Σ_T^+. However, in order to proceed as in the proof of Theorem 2.1.3 and transfer the chaotic features of σ_T to ψ (via Theorem 2.3.3), we need the periodic sequences of σ_T to be realized by periodic points of ψ. In view of Theorem 2.1.3, this happens, for instance, when our stretching relation is fulfilled with respect to some oriented rectangles, but maybe not all the possible coverings are realized, so that the conclusions of Theorem 2.1.4 hold with the full shift σ on Σ_m^+ replaced by σ_T on Σ_T^+.

At last we notice that a concrete framework in which it is possible to obtain chaotic dynamics in the sense of Definition 2.1.2 is when there exist $m \geq 2$ pairwise disjoint subsets C_i, $i = 0, \ldots, m-1$, of a metric space X, each with the

Fixed Point Property (FPP) [9] and such that there exists a homeomorphism φ defined on $\bigcup_{i=0}^{m-1} C_i$, with $\varphi(C_i) \supseteq C_j, \forall\, i, j = 0, \ldots, m-1$. Then, since $\varphi(C_i) \supseteq C_i$ is equivalent to $\varphi^{-1}(C_i) \subseteq C_i$, by the continuity of $\varphi^{-1} : C_i \to C_i$, it follows that there exists $x^* \in C_i$ with $\varphi^{-1}(x^*) = x^*$, i.e. $\varphi(x^*) = x^*$. Thus, the presence of at least a fixed point is ensured in each C_i, $i = 0, \ldots, m-1$. Working with the iterates of the map φ, the existence of periodic points in the C_i's can be shown in an analogous manner. To check that the periodic sequences on m symbols are realized by periodic itineraries, it is then sufficient to follow the same steps as in the proof of the second and the forth conclusions in Theorem 1.1.8. Due to the invertibility of φ, we observe that it is possible to deal with two-sided sequences (cf. Remark 2.1.5). In view of the previous discussion, one could also consider the more general case in which $\varphi(C_i) \supseteq C_j$, for *some* $i, j = 0, \ldots, m-1$, but the associated transition matrix T is still irreducible. Obviously, in this setting, only the admissible (one-sided or two-sided) sequences can be realized.

2.3 On various notions of chaos

Let us pursue further the discussion on chaotic dynamics started in Sections 2.1–2.2. In particular, we will try to show the mutual relationships among some of the most classical definitions of chaos (such as the ones by Li-Yorke, Devaney, etc.), considering also the notion of chaotic dynamics in the sense of Definition 2.1.2.

Firstly, we would like to conclude the comparison between our results in Section 2.1 and the achievements in [93]. As we have seen, the presence of a point of period three for a continuous self-map f defined on a compact interval, or more generally the conditions (2.2.6) from [93], ensure the positivity of the topological entropy for f. Thanks to some tools from symbolic dynamics, we have also realized that our assumptions in Theorem 2.1.3 are stronger than the ones in (2.2.6). Now we are going to show that the consequences of Theorem 2.1.3, listed in Theorem 2.1.4, are strictly sharper than the conclusions in [93]. The main result obtained by Li and Yorke in [93] is often recalled as a particular case of the Šarkowskii Theorem [152] but, actually, the authors in [93] proved much more than the presence of periodic points of any period for an interval map possessing a point of period three. Indeed, the existence of an uncountable *scrambled* set (cf. Definition 2.3.1) was obtained for such a map, which was called "chaotic" in [93] for the first time in the literature, even if the precise corresponding definition of chaos (now known as *Li-Yorke chaos*) was not given

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[9]We say that a topological space W has FPP if every continuous self-map of W has a fixed point in W.

there.

Definition 2.3.1. Let (X, d_X) be a metric space and $f : X \to X$ be a continuous map. We say that $S \subseteq X$ is a *scrambled set for f* if for any $x, y \in S$, with $x \neq y$, it holds that

$$\liminf_{n \to \infty} d_X(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n \to \infty} d_X(f^n(x), f^n(y)) > 0.$$

If the set S is uncountable, we say that f is *chaotic in the sense of Li-Yorke*.

We remark that according to [93] the scrambled set S should satisfy an extra assumption, i.e.

$$\limsup_{n \to \infty} d_X(f^n(x), f^n(p)) > 0,$$

for any $x \in S$ and for any periodic point $p \in X$. However, in [7] this condition has been proved to be redundant in any compact metric space and therefore it is usually omitted.

We also point out that the original framework in [93] was one-dimensional. The subsequent extension to generic metric spaces is due to different authors (see e.g. [7, 20, 65, 77]) that have collated the concept of chaos from [93] to other ones available in the literature. We will try to present some of these connections in the next pages.

We warn the reader that, in what follows, the term chaotic will be referred without distinction to a dynamical system, meant as a couple (X, f), where X is a compact metric space and $f : X \to X$ is continuous and surjective, as well as only to the map f defining it. In this respect, with a slight abuse of terminology, some properties of the map f (such as transitivity, sensitivity, etc.) will be transferred to the system (X, f). When we need to specify the distance d_X on X, we will also write (X, f, d_X) in place of (X, f). Since the map f has to be onto, if we are in the framework described in Theorem 2.1.4, the dynamical system we usually consider is given by $(\mathcal{I}, \psi \restriction_{\mathcal{I}})$, where $\mathcal{I}$ is the invariant set in (2.1.7).

In order to understand the relationship between the kind of chaos expressed in Definition 2.1.2 and the Li-Yorke chaos, a key role is played by the topological entropy. Indeed, as we have seen in Theorem 2.1.4, conclusion (iv), thanks to the semi-conjugacy with the Bernoulli shift, the topological entropy of ψ is positive in the setting described in Definition 2.1.2. On the other hand, in [20, Theorem 2.3] it is established that any dynamical system with positive topological entropy admits an uncountable scrambled set and therefore it is chaotic in the sense of Li-Yorke. Hence, we can conclude that our notion of chaos is stronger than the one in Definition 2.3.1, since any system chaotic according to Definition 2.1.2 is also Li-Yorke chaotic, while the vice versa does

not hold in general. In fact, there exist maps Li-Yorke chaotic but with zero topological entropy: for an example on the unit interval, see [146].

After the discussion on the concepts introduced in [93] and on the results obtained therein, the next feature related to the presence of chaos we take into consideration is the *sensitivity with respect to initial data*. This is one of the three requirements in Devaney's definition of chaos, together with the *topological transitivity* and the *density of periodic points*, and it is maybe the most intuitive among them. Indeed, it is pretty natural to associate the idea of chaos to a certain unpredictability of the forward behaviour of the system under consideration. The sensitivity with respect to initial data expresses exactly such concept: no matter how close two points start, there exists an instant in the future in which they are at a given positive distance. However, although its intuitiveness, the sensitivity on initial conditions has been proved to be redundant in Devaney's definition of chaos in any infinite metric space [13] [10].

In view of the subsequent treatment, we present the complete definition of *Devaney chaos* for the reader's convenience, even if at first we will focus only on the third condition.

Definition 2.3.2. Given a metric space (X, d_X) and a continuous function $f : X \to X$, we say that f is *chaotic in the sense of Devaney* if:

- f is *topologically transitive*, i.e. for any couple of nonempty open subsets $U, V \subseteq X$ there exists an integer $n \geq 1$ such that $U \cap f^n(V) \neq \emptyset$;

- The set of the periodic points for f is dense in X;

- f is *sensitive with respect to initial data* (or f displays *sensitive dependence on initial conditions*) *on* X, i.e. there exists $\delta > 0$ such that for any $x \in X$ there is a sequence $(x_i)_{i \in \mathbb{N}}$ of points in X such that $x_i \to x$ when $i \to \infty$ and for each $i \in \mathbb{N}$ there exists a positive integer m_i with $d_X(f^{m_i}(x_i), f^{m_i}(x)) \geq \delta$.

We stress that some authors call Devaney chaotic maps for which the three above conditions hold true only with respect to a compact positively invariant subset of the domain [7, 92]. We also remark that sometimes a map f on X is named topologically transitive if there exists a dense orbit for f in X [78]. In the case of compact metric spaces without isolated points, the two definitions turn out to be equivalent, but in general they are independent. The precise

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[10]We recall that, in the special case of intervals, also the density of periodic points is superfluous according to [158] and therefore Devaney chaos coincides with transitivity in the one-dimensional framework.

relationship between such notions can be found in [143, Proposition (1.1)]: in any metric space without isolated points the existence of a dense orbit implies the topological transitivity; vice versa, in separable and second category metric spaces, the topological transitivity implies the existence of a dense orbit. According to [8, Lemma 3], another case in which the two definitions of transitivity coincide is when f is onto and this happens, for instance, in any dynamical system. Such fact will find an application in Theorem 2.3.3.

About the sensitivity, we notice that it can be equivalently defined using neighborhoods instead of sequences: given a metric space (X, d_X), the continuous map $f : X \to X$ is said to display sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in X$ and for every open set $X \supseteq O_x \ni x$, there exist $y \in O_x$ and a positive integer m with $d_X(f^m(x), f^m(y)) \geq \delta$ [143]. We also recall that it is possible to give a pointwise version of the definition of sensitivity, using neighborhoods or sequences. As regards this latter characterization, any $x \in X$ admitting a sequence $(x_i)_{i \in \mathbb{N}}$ of points of X with $x_i \to x$ for $i \to \infty$ and a sequence $(m_i)_{i \in \mathbb{N}}$ of positive integers with $d_X(f^{m_i}(x_i), f^{m_i}(x)) \geq \delta$, $\forall i \in \mathbb{N}$ and for some $\delta > 0$, is sometimes called δ-*unstable* (or simply *unstable*) [8]. For an equivalent definition of unstable point based on neighborhoods, see [20], where it is also observed that, in general, the instability of f at every point of X does not imply f to be sensitive on X. Indeed there could be no positive δ such that all points of X are δ-unstable. However, the previous inference is true if the map is e.g. transitive [20], such as in Theorem 2.3.3, where we use the concept of sensitivity in both its equivalent versions and also the definition of unstable point based on sequences.

The positivity of the topological entropy and the sensitivity on initial conditions are someway related, since both are signals of a certain instability of the system. The topological entropy is however a "locally detectable" feature, in the sense that, according to (2.2.2), it is sufficient to find a (positively) invariant subset of the domain where it is positive in order to infer its positivity on the whole domain. Therefore, in general, we cannot expect the system to be sensitive at each point if the entropy is positive. Adding a global property, such as transitivity, then this implication holds true. Indeed in [20] it is argued that any transitive map with positive topological entropy displays sensitivity with respect to initial data. On the other hand, if we are content with the presence of sensitivity only on an invariant subset of the domain, then some authors have obtained corresponding results by considering, instead of the positivity of the entropy, the stronger property of chaos in the sense of coin-tossing (cf. Section 2.1), or at least the semi-conjugacy to the Bernoulli shift σ for the map f defining the dynamical system or one of its iterates. For example, the already cited [69, Lemma 4] (Chaos Lemma), under hypotheses

for a map f similar to the ones for ψ in Theorem 2.1.3, establishes the existence of a compact f-invariant set Q_*, on which f is sensitive and such that each forward itinerary on m symbols is realized by the itinerary generated by some point of Q_* (here m is the crossing number introduced in Subsection 1.1.1). A similar result is mentioned, without proof, by Aulbach and Kieninger in [7] and it asserts that if a continuous self-map f of a compact metric space X is *chaotic in the sense of Block and Coppel*, that is, if there exist an iterate f^k (with $k \geq 1$) of f and a compact subset $Y \subseteq X$ positively f^k-invariant, such that $f^k \upharpoonright_Y$ is semi-conjugate to the one-sided Bernoulli shift on two symbols [11], then $f \upharpoonright_Z$ is Block-Coppel chaotic, transitive and sensitive on Z, where Z is a suitable compact positively f-invariant subset of X. As suggested in [7], such conclusions rely on [8, Theorem 3]. Since the proof of this latter result is only sketched in [8], we furnish the details in Theorem 2.3.3 below for the sake of completeness. Notice that the original requirement on the existence of a dense orbit in the statement of [8, Theorem 3] has been here replaced with the one of transitivity from Definition 2.3.2, since such notions coincide in every dynamical system (cf. [8, Lemma 3]). Moreover, we have replaced the pointwise instability in [8, Theorem 3] with the (generally) stronger property of sensitivity: indeed, as mentioned before, such concepts turn out to be equivalent in any transitive system [20] [12]. Finally, we stress that the set X_0 in the statement of [8, Theorem 3] is claimed to be only positively invariant. However, after a look at its proof, since by construction X_0 is the ω-limit set of a certain point of X, it turns out to be invariant by the compactness of X (cf. [160, Theorem 5.5]).

Theorem 2.3.3 (Auslander-Yorke)**.** *Let (X, f, d_X) and (Y, g, d_Y) be dynamical systems and let $\pi : X \to Y$ be a continuous and surjective map such that $\pi \circ f = g \circ \pi$. Assume that g is sensitive and transitive on Y. Then there exists a closed f-invariant subset $X_0 \subseteq X$ such that $\pi(X_0) = Y$ and such that f is sensitive and transitive on X_0.*

Proof. First of all we use Zorn Lemma to show the existence of a closed positively f-invariant subset $X_0 \subseteq X$ such that $\pi(X_0) = Y$ and X_0 is minimal with respect to these properties. Indeed, let us consider a chain $\mathcal{C} = \{X_i\}_{i \in \mathcal{F}}$, whose elements are closed positively f-invariant subsets of X and $\pi(X_i) = Y$, $\forall i \in \mathcal{F}$.

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[11]We observe that, although the definition of Block-Coppel chaos is presented in [7] with respect to the shift on two symbols, in all the related results stated below, one could deal with the more general case in which $m \geq 2$ symbols are involved. Indeed, the situation is exactly the same such as for Definitions 2.1.1 and 2.1.2, where the difference is only a formal matter.

[12]Actually, the sensitivity on initial conditions is equivalent to the instability of a point with dense orbit [8], as it will emerge from the proof of Theorem 2.3.3.

Then we have to check that $X_\infty := \bigcap_{i \in \mathcal{F}} X_i$ is still closed, positively f-invariant and $\pi(X_\infty) = Y$. Notice that X_∞ is nonempty by the strong Cantor property, valid for compact spaces (in fact, equivalent to compactness in any topological space)[13]. Since X_∞ is the intersection of closed sets, it is closed, too. The positive f-invariance of X_∞ follows easily by observing that

$$f(X_\infty) \subseteq \bigcap_{i \in \mathcal{F}} f(X_i) \subseteq \bigcap_{i \in \mathcal{F}} X_i = X_\infty.$$

In order to verify that $\pi(X_\infty) = Y$, we have to check two inclusions. Since $\pi(X_i) = Y$, $\forall i \in \mathcal{F}$, then $\pi(X_\infty) \subseteq \bigcap_{i \in \mathcal{F}} \pi(X_i) = \bigcap_{i \in \mathcal{F}} Y = Y$. For the reverse inclusion, let us fix $y \in Y$ and consider the compact sets $C_i := \pi^{-1}(y) \cap X_i$, $i \in \mathcal{F}$. Notice that they are nonempty as $\pi(X_i) = Y$, $\forall i \in \mathcal{F}$. Moreover the family $\{C_i\}_{i \in \mathcal{F}}$ has the finite intersection property: indeed, recalling that $\mathcal{C}$ is a chain, given $C_{i_1}, \ldots, C_{i_k}$, with $i_1, \ldots, i_k \in \mathcal{F}$, it holds that

$$\bigcap_{j=1}^{k} C_{i_j} = C_{i^*}, \quad \text{for some} \quad i^* = i_1, \ldots, i_k$$

and thus such intersection is nonempty. By the strong Cantor property, there exists

$$x \in \bigcap_{i \in \mathcal{F}} C_i = \bigcap_{i \in \mathcal{F}} \left(\pi^{-1}(y) \cap X_i \right) = \pi^{-1}(y) \cap \left(\bigcap_{i \in \mathcal{F}} X_i \right) = \pi^{-1}(y) \cap X_\infty,$$

that is, there exists $x \in X_\infty$ such that $\pi(x) = y$. Hence, $Y \subseteq \pi(X_\infty)$. The existence of the minimal set X_0 is then ensured by Zorn Lemma.

Notice that, since $\pi(X_0) = Y$ and $f(X_0) \subseteq X_0$, the relation $\pi \restriction_{X_0} \circ f \restriction_{X_0} = g \circ \pi \restriction_{X_0}$ holds. Indeed, for any $x_0 \in X_0 \subseteq X$, we have $\pi(f \restriction_{X_0} (x_0)) = \pi(f(x_0)) = g(\pi(x_0)) = g(\pi \restriction_{X_0} (x_0))$. On the other hand, since $f(x_0) \in X_0$, then $\pi(f \restriction_{X_0} (x_0)) = \pi \restriction_{X_0} (f \restriction_{X_0} (x_0))$, from which $\pi \restriction_{X_0} (f \restriction_{X_0} (x_0)) = g(\pi \restriction_{X_0} (x_0))$. The validity of $\pi \restriction_{X_0} \circ f \restriction_{X_0} = g \circ \pi \restriction_{X_0}$ is thus checked.

By the transitivity of the onto map g on Y, there exists $y^* \in Y$ with dense g-orbit, that is, $\overline{\gamma(y^*)} = Y$, for $\gamma(y^*) := \{g^n(y^*) : n \in \mathbb{N}\}$. Let $x_0^* \in X_0$ be such that $\pi(x_0^*) = y^*$. We will show that

$$\pi(\omega(x_0^*)) = \omega(y^*) = Y,$$

where, for a dynamical system (Z, l) with $z \in Z$, we have

$$\omega(z) := \{x \in Z : \exists n_j \nearrow \infty \text{ with } l^{n_j}(z) \to x\}.$$

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[13]We recall that a topological space X has the strong Cantor property if, for any family $\{C_\alpha\}_{\alpha \in \mathcal{I}}$ of closed subsets of X with the finite intersection property (i.e. such that, for every finite subset $\mathcal{J} \subseteq \mathcal{I}$, it holds that $\bigcap_{\alpha \in \mathcal{J}} C_\alpha \neq \emptyset$), we have $\bigcap_{\alpha \in \mathcal{I}} C_\alpha \neq \emptyset$.

According to [160, Theorem 5.5], it holds that $\omega(z)$, called ω-*limit set of* z, is closed nonempty and invariant by the compactness of Z. Let us start with the verification of $\omega(y^*) = Y$. By the surjectivity of g on Y, there exists $\tilde{y} \in Y$ such that $g(\tilde{y}) = y^*$. Since the g-orbit of y^* is dense in Y and recalling that $\overline{\gamma(y^*)} = \gamma(y^*) \cup \omega(y^*)$, there are two possibilities for $\tilde{y}$, that is, $\tilde{y} \in \gamma(y^*)$ or $\tilde{y} \in \omega(y^*)$. In the former case, we find $\tilde{y} = g^k(y^*)$, for some $k \geq 0$. Since $g(\tilde{y}) = y^*$, this means that $g^{k+1}(y^*) = y^*$ and thus y^* is a periodic point, from which $Y = \overline{\gamma(y^*)} = \gamma(y^*) = \omega(y^*)$ is a finite set. On the other hand, if $\tilde{y} \in \omega(y^*)$, it follows that $g(\tilde{y}) = y^* \in \omega(y^*)$, by the invariance of $\omega(y^*)$. Recalling that it is also closed, then $Y = \overline{\gamma(y^*)} \subseteq \omega(y^*)$. Therefore we find $\omega(y^*) = Y$ also in the latter case.

As regards the equality $\pi(\omega(x_0^*)) = \omega(y^*)$, at first we observe that, since $\pi \restriction_{X_0} \circ f \restriction_{X_0} = g \circ \pi \restriction_{X_0}$, then it follows that $\pi \restriction_{X_0} \circ (f \restriction_{X_0})^m = g^m \circ \pi \restriction_{X_0}$, $\forall m \geq 2$. Let us start with the inclusion $\pi(\omega(x_0^*)) \subseteq \omega(y^*)$. If $x^* \in \omega(x_0^*)$, then there exists a sequence $n_j \nearrow \infty$ such that $f^{n_j}(x_0^*) \to x^*$. By the continuity of the map π, it holds that $\pi(f^{n_j}(x_0^*)) \to \pi(x^*)$ for $j \to \infty$, but $\pi(f^{n_j}(x_0^*)) = g^{n_j}(\pi(x_0^*)) = g^{n_j}(y^*)$, i.e. $\lim_{j\to\infty} g^{n_j}(y^*) = \pi(x^*)$ and thus $\pi(x^*) \in \omega(y^*)$. For the reverse inclusion, we have to check that any $\bar{y} \in \omega(y^*)$ can be written as $\bar{y} = \pi(\bar{x})$, with $\bar{x} \in \omega(x_0^*)$. Indeed, if $\bar{y} \in \omega(y^*)$, then there exists a sequence $m_k \nearrow \infty$ such that $g^{m_k}(y^*) \to \bar{y}$. Therefore $\pi(f^{m_k}(x_0^*)) = g^{m_k}(\pi(x_0^*)) = g^{m_k}(y^*) \to \bar{y}$ as $k \to \infty$. Since $(f^{m_k}(x_0^*))_{k\in\mathbb{N}}$ is a sequence contained in the compact set X_0, there exist a subsequence $m_{k_j} \nearrow \infty$ of $(m_k)_{k\in\mathbb{N}}$ and $\bar{x} \in X_0$ such that $f^{m_{k_j}}(x_0^*) \to \bar{x}$ for $j \to \infty$. This means that $\bar{x} \in \omega(x_0^*)$. Notice that also for the subsequence $(m_{k_j})_{j\in\mathbb{N}}$ it holds that $\pi(f^{m_{k_j}}(x_0^*)) = g^{m_{k_j}}(\pi(x_0^*)) = g^{m_{k_j}}(y^*) \to \bar{y}$ as $j \to \infty$. Hence, passing to the limit, we find $\pi(\bar{x}) = \lim_{j\to\infty} g^{m_{k_j}}(\pi(x_0^*)) = \bar{y}$, from which one obtains $\bar{y} = \pi(\bar{x})$, with $\bar{x} \in \omega(x_0^*)$. The equality $\pi(\omega(x_0^*)) = \omega(y^*)$ is thus established.

Then, since $\pi(\omega(x_0^*)) = Y$ and recalling that $\omega(x_0^*)$ is a closed f-invariant subset of X_0, by the minimality of X_0 it follows that $\omega(x_0^*) = X_0$. By the invariance of $\omega(x_0^*)$, then $f(\omega(x_0^*)) = \omega(x_0^*)$ and so X_0 is invariant, too.

Moreover, from $\overline{\gamma(x_0^*)} = \gamma(x_0^*) \cup \omega(x_0^*)$, we find that the f-orbit of x_0^* is dense in X_0. In fact, by the minimality of X_0, the f-orbit of any point of X_0 is dense in X_0 [14].

Let us check now that f is sensitive on X_0. By the sensitivity of g on Y, there exists $\varepsilon > 0$ such that for every $y \in Y$ with dense g-orbit (actually, for any point of Y), there exist a sequence $(y_j)_{j\in\mathbb{N}}$ of points of Y and a sequence $(n_j)_{j\in\mathbb{N}}$ of positive integers such that $y_j \to y$, but $d_Y(g^{n_j}(y), g^{n_j}(y_j)) > \varepsilon$, $\forall j \in \mathbb{N}$. Let

---

[14]A subset M of the dynamical system (Z, l) is called *minimal* if it is closed nonempty positively l-invariant and contains no proper closed nonempty positively l-invariant subsets. An equivalent definition is that M is minimal if each of its points has dense l-orbit in M.

$(x_j)_{j\in\mathbb{N}}$ be a sequence in X_0 such that $\pi(x_j) = y_j$, $\forall j \in \mathbb{N}$, and let $(x_{j_k})_{k\in\mathbb{N}}$ be a converging subsequence of $(x_j)_{j\in\mathbb{N}}$, with $x_{j_k} \to \overline{x_0}$, for some $\overline{x_0} \in X_0$. By the continuity of π it follows that $\pi(\overline{x_0}) = y$. We want to show that $\overline{x_0}$ is unstable, i.e that there exists $\bar{\delta} > 0$ such that $d_X(f^{n_{j_k}}(\overline{x_0}), f^{n_{j_k}}(x_{j_k})) > \bar{\delta}$, $\forall k \in \mathbb{N}$, where $(n_{j_k})_{k\in\mathbb{N}}$ is the subsequence of $(n_j)_{j\in\mathbb{N}}$ corresponding to $(y_{j_k})_{k\in\mathbb{N}}$. Indeed, if by contradiction there exists a subsequence of (n_{j_k}), that by notational convenience we still denote by (n_{j_k}), such that $d_X(f^{n_{j_k}}(\overline{x_0}), f^{n_{j_k}}(x_{j_k})) \to 0$ for $k \to \infty$, by the continuity of π, we would find $0 < \varepsilon < d_Y(g^{n_{j_k}}(y), g^{n_{j_k}}(y_{j_k})) = d_Y(g^{n_{j_k}}(\pi(\overline{x_0})), g^{n_{j_k}}(\pi(x_{j_k}))) = d_Y(\pi(f^{n_{j_k}}(\overline{x_0})), \pi(f^{n_{j_k}}(x_{j_k}))) \to 0$. Therefore $d_X(f^{n_{j_k}}(\overline{x_0}), f^{n_{j_k}}(x_{j_k})) > \bar{\delta}$, $\forall k \in \mathbb{N}$, for some $\bar{\delta} > 0$, and $\overline{x_0} \in X_0$ is an unstable point. Moreover, since X_0 is minimal, the f-orbit of $\overline{x_0}$ is dense in X_0. Let us prove that these two facts about $\overline{x_0}$ are sufficient to conclude that there exists a $\tilde{\delta} > 0$ such that any point of X_0 is $\tilde{\delta}$-unstable.

By the continuity of f, since $\overline{x_0}$ is $\bar{\delta}$-unstable, then also $f(\overline{x_0})$ is and hence the whole f-orbit $\gamma(\overline{x_0})$ is $\bar{\delta}$-unstable. Calling $\mathcal{U}_{\bar{\delta}} \subseteq X_0$ the set of $\bar{\delta}$-unstable points of X_0, we show that $\overline{\mathcal{U}_{\bar{\delta}}} \subseteq \mathcal{U}_{\bar{\delta}/2}$. If this is true, since $X_0 = \overline{\mathcal{U}_{\bar{\delta}}}$, the thesis is achieved for $\tilde{\delta} := \bar{\delta}/2$. So let $x \in \overline{\mathcal{U}_{\bar{\delta}}}$. Then there exists a sequence $(x_i)_{i\in\mathbb{N}}$ in $\mathcal{U}_{\bar{\delta}}$ with $x_i \to x$. For any such x_i and for every open set $X_0 \supseteq O_i \ni x_i$, there exist $\widetilde{x_i} \in O_i$ and a positive integer m_i with $d_X(f^{m_i}(x_i), f^{m_i}(\widetilde{x_i})) \geq \bar{\delta}$. In particular we can choose $O_i := B(x_i, 1/i) \cap X_0$, so that $d_X(x_i, \widetilde{x_i}) < 1/i \to 0$, as $i \to \infty$. Now there are two alternatives: if $d_X(f^{m_i}(x_i), f^{m_i}(x)) \geq \bar{\delta}/2$, then x is $\bar{\delta}/2$-unstable and we are done. Otherwise it holds that $d_X(f^{m_i}(x), f^{m_i}(\widetilde{x_i})) \geq \bar{\delta}/2$. Indeed, if this would fail, then $d_X(f^{m_i}(x_i), f^{m_i}(\widetilde{x_i})) \leq d_X(f^{m_i}(x_i), f^{m_i}(x)) + d_X(f^{m_i}(x), f^{m_i}(\widetilde{x_i})) < \bar{\delta}$, a contradiction. Since $\widetilde{x_i} \to x$, it follows also in this latter case that $x \in \mathcal{U}_{\bar{\delta}/2}$. The proof is complete. $\qquad\square$

By the similarity between our point of view and the frameworks in [7, 69], it is not difficult to show that both the above quoted results on chaotic invariant sets obtained therein still hold in the setting of Definition 2.1.2. Therefore the sensitivity is for us ensured at least on some subset of the domain. More precisely, as discussed in Section 2.1, since our notion of chaos in Definition 2.1.2 is stricter than the one considered in [69, 71], we can prove the existence of the chaotic invariant set Q_* mentioned above, exactly as in [69, Lemma 4]. On the other hand, with reference to [7] and [8], we notice that any map chaotic according to Definition 2.1.1 is also chaotic in the sense of Block-Coppel, thanks to Theorem 2.1.4, conclusion (ii). Thus, following the suggestion in [7], we have used Theorem 2.3.3 to get the existence of the set Λ in Theorem 2.1.4, conclusion (v). More precisely, we point out that in Theorem 2.1.4, conclusion (ii), a property stronger than the notion of chaos in the sense of Block-Coppel is obtained. Indeed, the semi-conjugacy with the Bernoulli shift is established for the map ψ itself and not for one of its iterates. The same remark applies to

Theorem 2.1.4, conclusion (v), where again a sharper feature than the Block-Coppel chaos is deduced.

Pursuing further this discussion on the Block-Coppel chaos, we notice that every system (X, f) Block-Coppel chaotic has positive topological entropy. Indeed, by the postulated semi-conjugacy between an iterate f^k (with $k \geq 1$) of the map f, restricted to a suitable positively invariant subset of the domain, and the one-sided Bernoulli shift on two symbols, by (2.2.3) and (2.2.5) it follows that $kh_{\mathrm{top}}(f) = h_{\mathrm{top}}(f^k) \geq \log(2)$, from which $h_{\mathrm{top}}(f) \geq \log(2)/k > 0$. Thus, by the previously quoted [20, Theorem 2.3], any such system is also Li-Yorke chaotic. However, we observe that the Block-Coppel chaos is strictly weaker than chaos in the sense of coin-tossing and, a fortiori, also than the concept in Definition 2.1.1: indeed, according to [7, Remark 3.2], there exist systems (X, f) such that f^2 restricted to some f-invariant subset of X is semi-conjugate to the one-sided Bernoulli shift on two symbols, while such property does not hold for f.

Returning back to the analysis on the relationship between the topological entropy and the sensitivity on initial conditions, we recall that, except for the special case of continuous self-mappings of compact intervals, in general the sensitivity does not imply a positive entropy [77]. Actually, even more can be said. Indeed, if instead of the sensitivity alone, we take into account the definition of Devaney chaos in its completeness, then it is possible to prove that the Devaney chaoticity and the positivity of topological entropy are independent, as none of the two implies the other [9, 85]. In particular this means that, in generic metric spaces, Devaney chaos does not imply chaos in the sense of Definition 2.1.2 (because, otherwise, the topological entropy would be positive in any Devaney chaotic system). Vice versa it is not clear if our notion of chaos implies the one by Devaney. Indeed, on the one hand, in [7] it is presented a dynamical system Block-Coppel chaotic, but not chaotic according to Devaney, since it has no periodic points. On the other hand, we have already noticed that our notion of chaos is strictly stronger than the one in the Block-Coppel sense.

When confining ourselves to the one-dimensional case, most of the definitions of chaos are known to be equivalent, while it is in the higher dimensional setting that the relationship among them becomes more involved. Indeed, in [92] it was proved that for a continuous self-mapping f of a compact interval to have positive topological entropy is equivalent to be chaotic in the sense of Devaney on some closed (positively) invariant subset of the domain. The positivity of the topological entropy is also equivalent to the fact that some iterate of f is turbulent (or even strictly turbulent, cf. Section 2.2 for the corresponding definitions) or to the chaoticity in the Block-Coppel sense [7].

Such equivalence, however, does not extend to chaos in the sense of Li-Yorke: indeed, as already pointed out, there exist interval maps Li-Yorke chaotic, but with zero topological entropy [146].

When moving to more general frameworks, the previous discussion should suggest that several links among the various notions of chaos get lost. Nonetheless something can still be said. In addition to the facts already expounded (e.g., chaos according to Definition 2.1.1 $\Rightarrow$ chaos in the sense of coin-tossing $\Rightarrow$ Block-Coppel chaos $\Rightarrow h_{\text{top}} > 0 \Rightarrow$ Li-Yorke chaos), we mention that Devaney chaos implies Li-Yorke chaos in any compact metric space: actually, in order to prove this implication, the hypothesis on the density of periodic points in Definition 2.3.2 could be replaced with the weaker condition that at least one periodic point does exist [65]. On the other hand this weaker requirement is necessary, because transitivity and sensitivity alone are not sufficient to imply Li-Yorke chaos and vice versa, as shown in [20]. We recall that a dynamical system that is both sensitive and transitive is sometimes named *Auslander-Yorke chaotic* [8, 20]. Therefore we can rephrase the previous sentence by saying that the concepts of Li-Yorke chaos and Auslander-Yorke chaos are independent.

Of course, the above treatment is just meant to give an idea of the intricate network of connections among some of the most well-known definitions of chaos. A useful and almost complete survey on the existing relationships can be found in [111].

2.4 Linked Twist Maps

In this last section we present a different geometrical context where it is possible to apply our method of "stretching along the paths" from Section 1.1 and the corresponding results on chaotic dynamics from Section 2.1. More precisely we will be concerned with the study of the so-called *Linked Twist Maps* (for short, LTMs). In such framework, instead of considering a single map that expands the arcs along a domain homeomorphic to a rectangle, one deals with a geometric configuration characterized by the alternation of two planar homeomorphisms (or diffeomorphisms) which twist two circular annuli (or two families of them [132]) intersecting in two disjoint generalized rectangles $\mathcal{A}$ and $\mathcal{B}$. Each annulus is turned onto itself by a homeomorphism which leaves the boundaries of the annulus invariant. Both the maps act in their domain so that a twist effect is produced. This happens, for instance, when the angular speed is monotone with respect to the radius. Considering the composition of the two movements in the common regions, we obtain a resulting function which is what we call a "linked twist map" (see [163, 164] for a detailed description of the geometry of the domain of a LTM). Such kind of

maps furnish a geometrical setting for the existence of Smale horseshoes: in fact, under certain conditions, it is possible to prove the presence of a Smale horseshoe inside $\mathcal{A}$ and $\mathcal{B}$ [40]. Usual assumptions on the twist mappings require, among others, their smoothness, monotonicity of the angular speed with respect to the radial coordinate and preservation of the Lebesgue measure. On the other hand, since our approach is purely topological, we just need a twist condition on the boundary (cf. Example 2.4.2).

In the past decades a growing interest has concerned LTMs. In the 80s they were studied from a theoretical point of view by Devaney [40], Burton and Easton [28] and Przytycki [131, 132] (just to cite a few contributions in this direction), proving some mathematical properties like ergodicity, hyperbolicity and conjugacy to the Bernoulli shift. However, as observed in [40], such maps naturally appear in various different applicative contexts, like for instance in mathematical models for particle motions in a magnetic field, as well as in differential geometry, in the study of diffeomorphisms of surfaces. Special configurations related to LTMs can also be found in the restricted three-body problem [112, pp.90–94]. In more recent years, thanks to the work of Ottino, Sturman and Wiggins, significant applications of LTMs have been performed in the area of fluid mixing [153, 154, 163, 164].

Our purpose is to adapt the general results from Sections 1.1–2.1 to a geometrical framework which is connected to and generalizes (in a direction explained in Example 2.4.3) the case of the LTMs. The corresponding Theorems 2.4.1 and 2.4.4 below will be applied to some nonlinear ODEs with periodic coefficients in Chapter 4.

Theorem 2.4.1. *Let X be a metric space and assume that $\varphi : X \supseteq D_\varphi \to X$ and $\psi : X \supseteq D_\psi \to X$ are continuous maps defined on the sets D_φ and D_ψ, respectively. Let also $\widetilde{\mathcal{A}} := (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} := (\mathcal{B}, \mathcal{B}^-)$ be oriented rectangles of X. Suppose that the following conditions are satisfied:*

(H_φ) *There are $m \geq 2$ pairwise disjoint compact sets $\mathcal{H}_0 , \dots , \mathcal{H}_{m-1} \subseteq \mathcal{A} \cap D_\varphi$ such that $(\mathcal{H}_i, \varphi) : \widetilde{\mathcal{A}} \Longleftrightarrow \widetilde{\mathcal{B}}$, for $i = 0, \dots, m-1$;*

(H_ψ) $\mathcal{B} \subseteq D_\psi$ *and* $\psi : \widetilde{\mathcal{B}} \Longleftrightarrow \widetilde{\mathcal{A}}$.

Then the map $\phi := \psi \circ \varphi$ induces chaotic dynamics on m symbols in the set $\mathcal{H} \cap \varphi^{-1}(\mathcal{B})$, where $\mathcal{H} := \bigcup_{i=0}^{m-1} \mathcal{H}_i$, and thus satisfies properties (i)-(v) from Theorem 2.1.4 (or the stronger properties (i)-(v) from Theorem 2.1.6, if ϕ is also injective on $\mathcal{H} \cap \varphi^{-1}(\mathcal{B})$ [15]).

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[15]This is for instance the case in Chapter 4, since there we deal with the Poincaré map, that is a homeomorphism.

Proof. We show that

$$(\mathcal{H}_i \cap \varphi^{-1}(\mathcal{B}), \phi) : \widetilde{\mathcal{A}} \Leftrightarrow \widetilde{\mathcal{A}}, \quad \forall\, i = 0, \ldots, m-1, \qquad (2.4.1)$$

from which the thesis about the chaotic dynamics is an immediate consequence of Theorem 2.1.3.

To check condition (2.4.1), let us consider a path $\gamma : [0,1] \to \mathcal{A}$ such that $\gamma(0) \in \mathcal{A}_\ell^-$ and $\gamma(1) \in \mathcal{A}_r^-$ (or with $\gamma(0) \in \mathcal{A}_r^-$ and $\gamma(1) \in \mathcal{A}_\ell^-$) and let us fix $i \in \{0, \ldots, m-1\}$. By (H_φ), there exists a compact interval $[t', t''] \subseteq [0,1]$ such that $\gamma(t) \in \mathcal{H}_i$ and $\varphi(\gamma(t)) \in \mathcal{B}$, for every $t \in [t', t'']$, with $\varphi(\gamma(t'))$ and $\varphi(\gamma(t''))$ belonging to different components of $\mathcal{B}^-$. Define now

$$\omega : [t', t''] \to \mathcal{B}, \quad \omega(t) := \varphi(\gamma(t)).$$

By (H_ψ) there is a compact interval $[s', s''] \subseteq [t', t'']$ such that $\psi(\omega(t)) \in \mathcal{A}$, for every $t \in [s', s'']$, with $\psi(\omega(s'))$ and $\psi(\omega(s''))$ belonging to different components of $\mathcal{A}^-$.

Rewriting all in terms of γ, we have thus proved that

$$\gamma(t) \in \mathcal{H}_i \cap \varphi^{-1}(\mathcal{B}) \quad \text{and} \quad \phi(\gamma(t)) \in \mathcal{A}, \quad \forall\, t \in [s', s''],$$

with $\phi(\gamma(s'))$ and $\phi(\gamma(s''))$ belonging to different components of $\mathcal{A}^-$. The continuity of the composite mapping $\phi = \psi \circ \varphi$ on $\mathcal{H}_i \cap \varphi^{-1}(\mathcal{B})$ follows from the continuity of φ on $D_\varphi \supseteq \mathcal{H}_i$ and from the continuity of ψ on $D_\psi \supseteq \mathcal{B}$. By the arbitrariness of the path γ and of $i \in \{0, \ldots, m-1\}$, the verification of (2.4.1) is complete. $\qquad\square$

We remark that in condition (H_φ) we could consider the case in which there exists a set $\mathcal{D} \subseteq \mathcal{A} \cap D_\varphi$ with $\mathcal{H} = \bigcup_{i=0}^{m-1} \mathcal{H}_i \subseteq \mathcal{D}$, in analogy with Theorem 2.1.3. From the proof of Theorem 2.4.1 it is also clear that the continuity of φ on D_φ and of ψ on D_ψ could be weakened requiring that φ is continuous only on $\mathcal{H}$ and ψ on $\mathcal{B}$. For simplicity's sake and in view of the applications in Section 4.2, we have preferred to present the easier framework.

Moreover, as regards condition (H_ψ), we could assume that the stretching condition therein holds with respect to a compact set $\mathcal{K} \subseteq \mathcal{B} \subseteq D_\psi$, that is, $(\mathcal{K}, \psi) : \widetilde{\mathcal{B}} \Leftrightarrow \widetilde{\mathcal{A}}$. Then, the chaotic dynamics would be localized in the set

$$\mathcal{H}^* := \bigcup_{i=0}^{m-1} \left(\mathcal{H}_i \cap \varphi^{-1}(\mathcal{K}) \right).$$

Such case is analyzed in the more general Theorem 2.4.4.

We present now two simple examples for the application of Theorem 2.4.1, which are meant to show how this result is well-fit for studying LTMs. Example 2.4.2 is fairly classical as it concerns two overlapping annuli subject to twist rotations, while Example 2.4.3 describes the composition of a map which twists an annulus with a longitudinal motion along a strip. A similar geometric setting was already considered by Kennedy and Yorke in [70] in the framework of the theory of fluid mixing, studying planar functions obtained as composition of a squeezing map and a stirring rotation.

We stress that such examples are only of "pedagogical" nature and, in fact, the chaotic-like dynamics that we obtain could be also proved using different approaches already developed in various papers (like, for instance, [108, 148, 172, 176]).

Example 2.4.2. A classical kind of LTM is represented by the composition of two planar maps φ and ψ which act as twist rotations around two given points. For instance, a possible choice is that of considering two continuous functions expressed by means of complex variables as

$$\varphi(z) := -r + (z + r) \, e^{\imath(c_1 + d_1 |z+r|)} \tag{2.4.2}$$

and

$$\psi(z) := r + (z - r) \, e^{\imath(c_2 + d_2 |z-r|)}, \tag{2.4.3}$$

where $\imath$ is the imaginary unit, while $r > 0$, c_j $(j = 1, 2)$ and $d_j \neq 0$ are real coefficients. Such maps twist around the centers $(-r, 0)$ and $(r, 0)$, respectively. We denote by p_1 and p_2 the inner and the outer radii for the annulus around $(-r, 0)$ and by q_1 and q_2 the inner and the outer radii for the annulus around $(r, 0)$.

For a suitable choice of r, of the radii $p_1 < p_2$ and $q_1 < q_2$, as well as of the parameters determining φ and ψ, it is possible to apply Theorem 2.4.1 in order to obtain chaotic dynamics (see Figures 2.2–2.4).
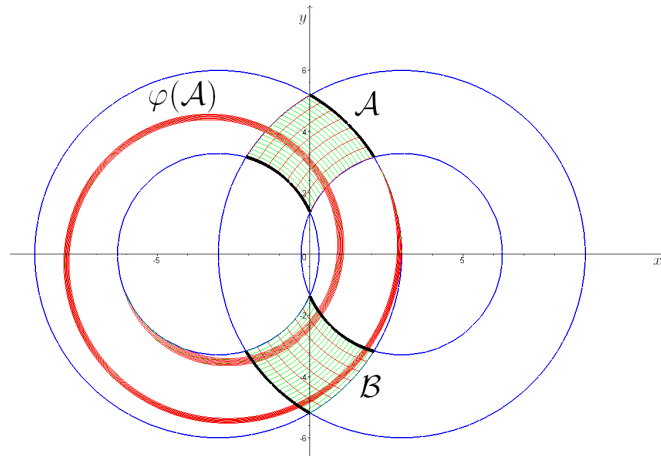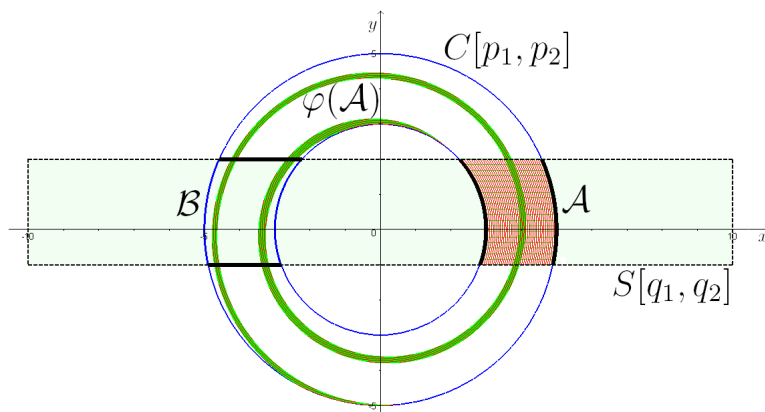
Figure 2.2: A pictorial comment to condition (H_φ) of Theorem 2.4.1, with reference to Example 2.4.2. In order to determine the two annuli, we have set $r = 3$, $p_1 = q_1 = 3.3$ and $p_2 = q_2 = 6$ (this choice is just to simplify the explanation, since no special symmetry is needed). For the map φ in (2.4.2) we have taken $c_1 = -1.5$ and $d_1 = 3.3$. We choose as generalized rectangle $\mathcal{A}$ in Theorem 2.4.1 the upper intersection of the two annuli and select as components of $\mathcal{A}^-$ the intersections of $\mathcal{A}$ with the inner and outer boundaries of the annulus at the left-hand side. The set $\mathcal{B}$ is defined as the the lower intersection of the two annuli and the two components of $\mathcal{B}^-$ are the intersections of $\mathcal{B}$ with the inner and outer boundaries of the annulus at the right-hand side. The sets $\mathcal{A}^-$ and $\mathcal{B}^-$ are drawn with thicker lines. The narrow strip spiralling inside the left annulus is the image of $\mathcal{A}$ under φ. Clearly, any path in $\mathcal{A}$ joining the two components of $\mathcal{A}^-$ is transformed by φ onto a path crossing $\mathcal{B}$ twice in the correct manner (that is, from a side of $\mathcal{B}^-$ to the other side).
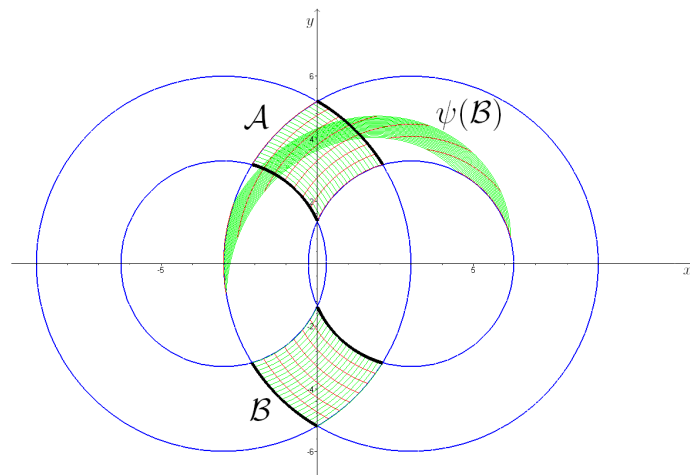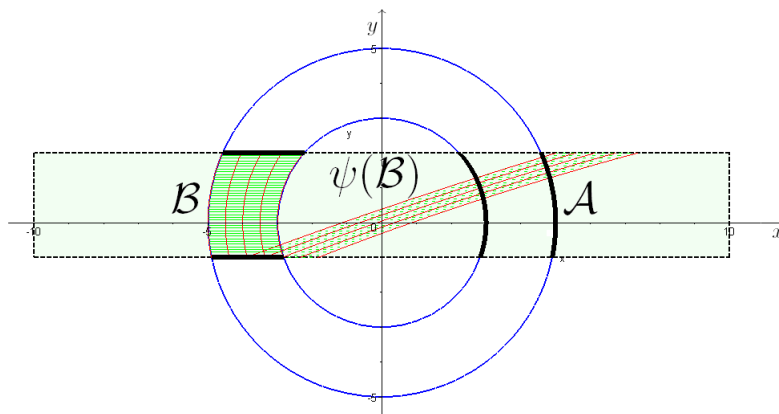

Figure 2.3: A pictorial comment to condition (H_ψ) of Theorem 2.4.1, with reference to Example 2.4.2. For r, p_1, p_2, q_1, q_2 as in Figure 2.2, we have also fixed the parameters of ψ in (2.4.3) by taking $c_2 = 0$ and $d_2 = 0.9$. It is evident that the image of $\mathcal{B}$ under ψ crosses $\mathcal{A}$ once in the correct way.
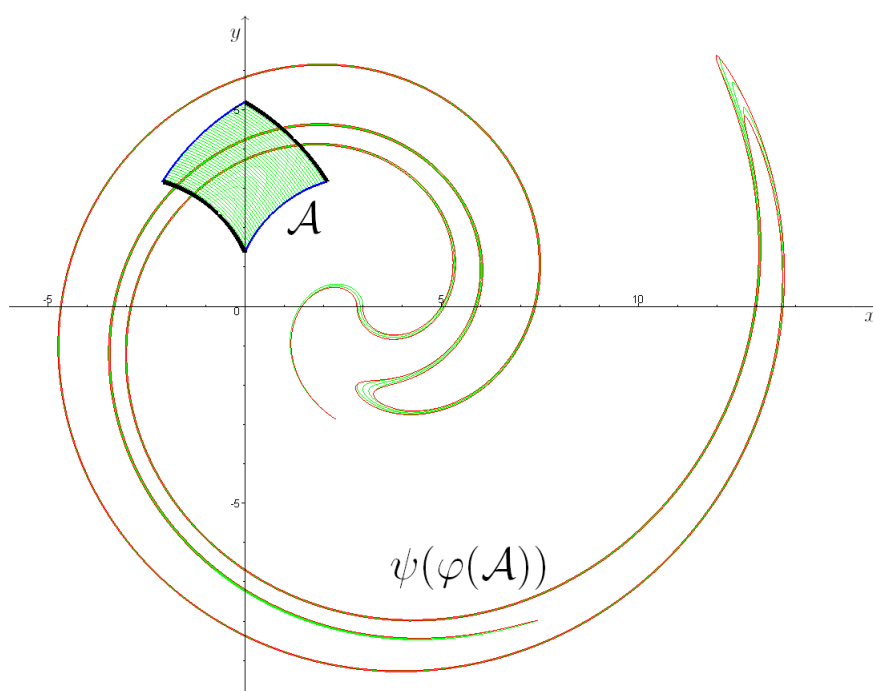
Figure 2.4: For $r, p_1, p_2, q_1, q_2, \varphi$ as in Figure 2.2 and ψ as in Figure 2.3, we show the image of $\mathcal{A}$ under the composite map $\psi \circ \varphi$. It is clear that $\psi(\varphi(\mathcal{A}))$ crosses $\mathcal{A}$ twice in the correct way.

Example 2.4.3. A nonstandard LTM is represented by the composition of two planar maps φ and ψ, where φ is a twist rotation around a given point and ψ produces a longitudinal motion along a strip with different velocities on the upper and lower components of the boundary of the strip.

First we introduce, for any pair of real numbers $a < b$, the real valued function

$$Pr_{[a,b]}(t) := \frac{1}{b-a} \, \min\{b-a, \max\{0, t-a\}\},$$

and then we set

$$f(t) := c_1 + d_1 Pr_{[p_1,p_2]}(t), \quad g(t) := c_2 + d_2 Pr_{[q_1,q_2]}(t),$$

where c_j $(j = 1, 2)$ and $d_j \neq 0$ are real coefficients, while

$$0 < p_1 < p_2 \quad \text{and} \quad -p_1 < q_1 < q_2 < p_1$$

are real parameters which determine the inner and the outer radii of the circular annulus

$$C[p_1, p_2] := \{(x, y) : p_1^2 \leq x^2 + y^2 \leq p_2^2\}$$

and the position of the strip

$$S[q_1, q_2] := \{(x, y) : q_1 \leq y \leq q_2\}.$$

Finally we define two continuous maps expressed by means of complex variables as

$$\varphi(z) := z \, e^{\imath f(|z|)} \tag{2.4.4}$$

(which twists around the origin) and, for $\Im(z)$ the imaginary part of z,

$$\psi(z) := z + g(\Im(z)) \tag{2.4.5}$$

(which shifts the points along the horizontal lines).

For a suitable choice of the parameters determining φ and ψ, it is possible to apply Theorem 2.4.1 in order to obtain chaotic dynamics (see Figures 2.5–2.7).

Figure 2.5: A pictorial comment to condition (H_φ) of Theorem 2.4.1, with reference to Example 2.4.3. We have set $p_1 = 3$, $p_2 = 5$, $q_1 = -1$ and $q_2 = 2$ in order to determine the circular annulus $C[p_1, p_2]$ centered at the origin and the strip $S[q_1, q_2]$ which goes across it. For the map φ in (2.4.4) we have taken $c_1 = 0.4\pi$ and $d_1 = 3\pi$. We choose as set $\mathcal{A}$ in Theorem 2.4.1 the right-hand side intersection of the annulus with the strip and select as components of $\mathcal{A}^-$ the intersections of $\mathcal{A}$ with the inner and outer boundaries of the annulus. The set $\mathcal{B}$ is defined as the left-hand side intersection of the annulus with the strip and the two components of $\mathcal{B}^-$ are the intersections of $\mathcal{B}$ with the lower and upper sides of the strip. The sets $\mathcal{A}^-$ and $\mathcal{B}^-$ are drawn with thicker lines. The narrow band spiralling inside the annulus is the image of $\mathcal{A}$ under φ. Clearly, any path in $\mathcal{A}$ joining the two components of $\mathcal{A}^-$ is transformed by φ onto a path crossing $\mathcal{B}$ twice in the correct manner.

Figure 2.6: A pictorial comment to condition (H_ψ) of Theorem 2.4.1, with reference to Example 2.4.3. The sets $C[p_1, p_2]$ and $S[q_1, q_2]$ and the oriented rectangles $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ are as in Figure 2.5. For the map ψ in (2.4.5), we have chosen $c_2 = 1$ and $d_2 = 8.6$. The parallelogram-shaped narrow figure inside the strip $S[q_1, q_2]$ is the image of $\mathcal{B}$ under ψ. Any path in $\mathcal{B}$ joining the two components of $\mathcal{B}^-$ is transformed by ψ onto a path crossing $\mathcal{A}$ once in the right way.
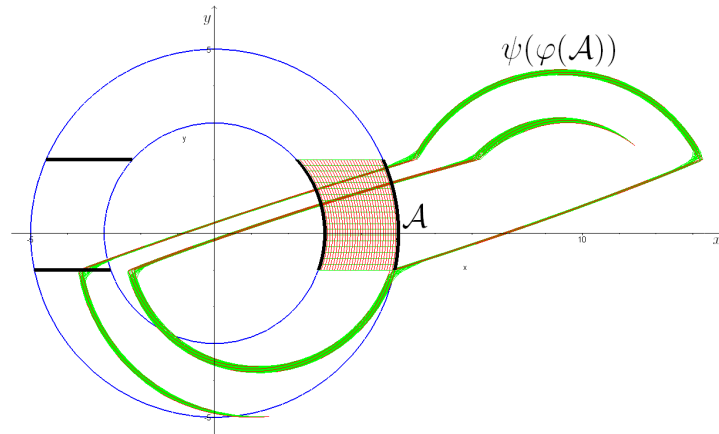
Figure 2.7: For p_1, p_2, q_1, q_2 and φ as in Figure 2.5 and ψ as in Figure 2.6, we show the image of $\mathcal{A}$ under the composite map $\psi \circ \varphi$. It is evident that $\psi(\varphi(\mathcal{A}))$ crosses $\mathcal{A}$ twice in the correct manner.
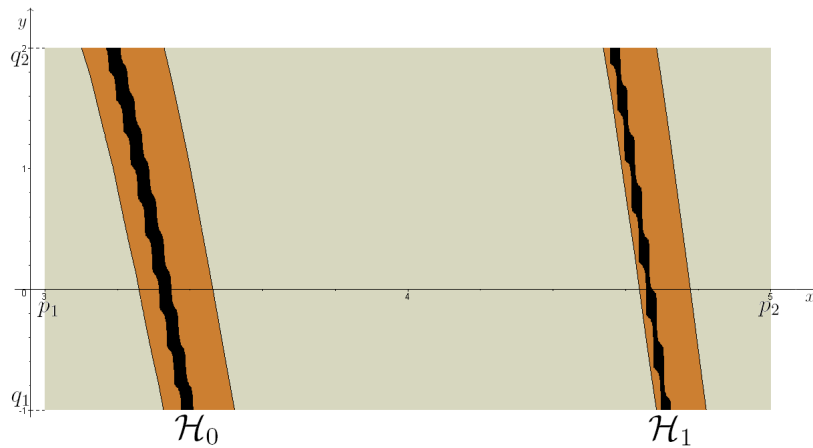
Figure 2.8: In the simple case of Example 2.4.3, for φ and ψ with the coefficients chosen for drawing Figure 2.7, we can determine the sets $\mathcal{H}_0$ and $\mathcal{H}_1$ from Theorem 2.4.1. Indeed, inside the set $\mathcal{A}$, transformed in the rectangle $[p_1, p_2] \times [q_1, q_2]$ by a suitable change of coordinates, we have found two subsets (drawn with a darker color) whose image under φ is contained in $\mathcal{B}$. Inside these subsets, we have localized two smaller darker domains such that their images under $\psi \circ \varphi$ are contained again in $\mathcal{A}$. Clearly, any path contained in the rectangle $[p_1, p_2] \times [q_1, q_2]$ and joining the left and the right sides contains two subpaths (inside the smaller darker subsets) which are stretched by the composite mapping onto paths connecting again the left and the right sides of the rectangle. Such smaller subsets can thus be taken as $\mathcal{H}_0$ and $\mathcal{H}_1$. Up to a homeomorphism between $\mathcal{A}$ and $[p_1, p_2] \times [q_1, q_2]$, this is precisely what happens in $(\mathcal{A}, \mathcal{A}^-)$ for $\psi \circ \varphi$.

With a straightforward modification in the proof of Theorem 2.4.1, one could check that (H_φ) and (H_ψ) imply the existence of chaotic dynamics also for $\varphi \circ \psi$. Actually, both Theorem 2.4.1 and such variant of it can be obtained as corollaries of the following more general result, whose proof is only sketched, by the similarity with the verification of Theorem 2.4.1.

Theorem 2.4.4. *Let X be a metric space and assume that $\varphi : X \supseteq D_\varphi \to X$ and $\psi : X \supseteq D_\psi \to X$ are continuous maps defined on the sets D_φ and D_ψ, respectively. Let also $\widetilde{\mathcal{A}} := (\mathcal{A}, \mathcal{A}^-)$ and $\widetilde{\mathcal{B}} := (\mathcal{B}, \mathcal{B}^-)$ be oriented rectangles of X. Suppose that the following conditions are satisfied:*

(H_φ) *There are $m \geq 1$ pairwise disjoint compact sets $\mathcal{H}_0, \ldots, \mathcal{H}_{m-1} \subseteq \mathcal{A} \cap D_\varphi$ such that $(\mathcal{H}_i, \varphi) : \widetilde{\mathcal{A}} \mathrel{\Longleftrightarrow} \widetilde{\mathcal{B}}$, for $i = 0, \ldots, m-1$;*

(H_ψ) *There exist $l \geq 1$ pairwise disjoint compact sets $\mathcal{K}_0, \ldots, \mathcal{K}_{l-1} \subseteq \mathcal{B} \cap D_\psi$ such that $(\mathcal{K}_i, \psi) : \widetilde{\mathcal{B}} \mathrel{\Longleftrightarrow} \widetilde{\mathcal{A}}$, for $i = 0, \ldots, l-1$.*

If at least one between m and l is greater or equal than 2, then the composite map $\phi := \psi \circ \varphi$ induces chaotic dynamics on $m \times l$ symbols in the set

$$\mathcal{H}^* := \bigcup_{\substack{i=0,\ldots,m-1 \\ j=0,\ldots,l-1}} \mathcal{H}'_{i,j}, \quad \text{with } \mathcal{H}'_{i,j} := \mathcal{H}_i \cap \varphi^{-1}(\mathcal{K}_j),$$

and thus satisfies properties (i)-(v) from Theorem 2.1.4.

Proof. In order to get the thesis, it suffices to show that

$$(\mathcal{H}'_{i,j}, \phi) : \widetilde{\mathcal{A}} \mathrel{\Longleftrightarrow} \widetilde{\mathcal{A}}, \quad \forall\, i = 0, \ldots, m-1 \text{ and } j = 0, \ldots, l-1.$$

Such condition can be checked by steps analogous to the ones in Theorem 2.4.1. The details are omitted since they are straightforward. $\qquad\square$

As we shall see, this latter result will find an application in Section 4.1 when dealing with a periodic version of the Volterra predator-prey model.

Part II

Applications

Chapter 3

Applications to difference equations

In order to present possible applications of the "stretching along the paths" method introduced in Part I, we start with some discrete-time economic models, for which we show the presence of a chaotic behaviour in a rigorous manner, without the need of computer simulations. In fact, when studying problems arising in different areas of research, there is plenty of numerical evidence of complex dynamics [2, 37, 50, 54, 62, 80, 91, 95, 157, 168, 173], but, if we except the simpler one-dimensional case, few rigorous proofs of the existence of chaos in one or the other of its main characterizations can be found in the literature. In more details, in Section 3.1 we deal with some models of the "Overlapping Generations" class (henceforth OLG). This name comes from the fact that one considers an economy with a population divided into "young" and "old". In order to make the comprehension of our techniques easier, we start with the analysis of some one-dimensional cases in Subsection 3.1.1 and we move to planar systems in Subsection 3.1.2.

On the other hand, in Section 3.2 we consider a bidimensional model of a "Duopoly Game" (from now on DG) where the players have heterogeneous expectations. By "duopoly", economists denote a market form characterized by the presence of two firms. The term "game" refers to the fact that the players - in our case the firms - make their decisions, or "moves", reacting to each other actual or expected moves, following a more or less sophisticated strategy. In the present framework, we deal with a dynamic game where moves are repeated in time, at discrete, uniform intervals. Finally, the expression "heterogeneous expectations" simply means that, in the absence of precise information about its competitor's future decisions, each firm is assumed to employ a different strategy.

In the next sections we will mainly focus on the mathematical aspects of the

treatment, limiting the explanation of the economic interpretation to what is barely necessary to the understanding of our main argument.

3.1 OLG Models

3.1.1 Unidimensional examples

Just to fix ideas, we start by presenting a one-dimensional application of the Stretching Along the Paths method to a basic pure exchange overlapping generations (OLG) model. In this simple case, we do not expect to establish any new result but the exercise is intended to introduce the reader to our techniques in a clear and intuitive manner.

The model deals with an economy with a constant population living two periods of unit time length and, at each moment in time, being divided into two equally numerous classes of persons, labeled respectively "young" and "old". Because individuals in each class are assumed to be otherwise identical, we shall describe the situation in terms of "the young (old) representative agent". There is no production but constant, nonnegative endowments of the unique, perishable consumption good are distributed at the beginning of each period to young and old.

For each generation, the young representative agent maximizes a concave utility function over a two-period life, subject to a budget constraint. We assume that the market for the consumption good is always in equilibrium (demand = supply) and that agents' expectations are fulfilled (perfect foresight).

Let the notation be as follows:

$c_t \geq 0$: young agent's consumption at time t

$g_t \geq 0$: old agent's consumption at time t

$w_0 \geq 0$: young agent's endowment

$w_1 \geq 0$: old agent's endowment

$\rho_t > 0$: interest factor at time t

= exchange rate between present and future consumption

$U(c_t, g_{t+1}) = u_1(c_t) + u_2(g_{t+1})$: utility functions (the same for all agents).

A mathematical formulation of such problem can be written as

$$\max_{c_t, g_{t+1}} \{u_1(c_t) + u_2(g_{t+1})\}$$
$$\text{s.t. } g_{t+1} \leq w_1 + \rho_t(w_0 - c_t), \tag{P1}$$
$$c_t, \ g_{t+1} \geq 0, \ \forall t \geq 0$$

and the market-clearing condition is

$$c_t + g_t = w_0 + w_1, \ \forall t \geq 0. \tag{3.1.1}$$

From the first order conditions of (P1) and from (3.1.1), we deduce that the optimal choice for the young agent's current saving (dis-saving), $(w_0 - c_t)$, equal to the old agent's current dis-saving (saving), $(g_t - w_1)$, must satisfy the equation

$$H(c_{t+1}, c_t) = \mathcal{U}(c_{t+1}) + \mathcal{V}(c_t) = 0, \tag{3.1.2}$$

where $\mathcal{U}(c) = u_2'(c)(w_1 - g)$ and $\mathcal{V}(c) = u_1'(c)(c - w_0)$.

Whether or not from (3.1.2) we can derive a difference equation moving forward in time depends on whether the function $\mathcal{U}$ is invertible. To visualize the situation, let us adopt a quasilinear-quadratic pair of utility functions following [17], namely

$$u_1(c) = ac - (b/2)c^2, \quad u_2(g) = g.$$

If, for simplicity's sake, we put $a = b = \mu$ and, rescaling endowments, $w_0 = 0$, we obtain the difference equation

$$c_{t+1} = F(c_t) = \mu c_t(1 - c_t), \tag{3.1.3}$$

i.e. the much-studied "logistic map". Starting from an arbitrary initial condition c_0, equation (3.1.3) determines sequences of young agents' consumption forward in time. Through the equilibrium condition (3.1.1), it will determine old agents' consumption as well. According to [52], this is the "classical case", where the young agents dis-save and borrow and the old ones save and lend[1]. Alternatively, using the pair of functions

$$u_1(c) = -\mu e^{-c}, \quad u_2(g) = g,$$

we get the difference equation

$$c_{t+1} = F(c_t) = \mu c_t e^{-c_t}. \tag{3.1.4}$$

The logistic map has already been analyzed in Section 2.2 and thus we deal with it only briefly here. As we have seen, if we consider the second iterate of the function F in (3.1.3), by a simple geometric argument it can be established that the conditions of Theorem 2.1.3 are verified for sufficiently large values of the parameter $\mu \leq 4$. For instance, in Figure 2.1 we visualized F^2 for $\mu = 3.88$: with reference to that picture, using the terminology from Theorem 2.1.3, we showed that F^2 induces chaotic dynamics on two symbols and consequently possesses all the chaotic properties listed in Theorem 2.1.4.

We recall that the proof that $F : [0, 1] \to \mathbb{R}$ satisfies the hypotheses of Theorem 2.1.3 and generates chaotic dynamics on two symbols is trivial for $\mu > 4$. Unfortunately, in this case almost all points of $[0, 1]$ limit to $-\infty$ through the iterates of F.

---

[1]Notice that, whereas w_0 is unimportant here, w_1 must be chosen in such a way that, whenever $c_t \geq 0$, so is g_t, i.e. $w_1 \geq c_{\max} = F_\mu(1/2)$.
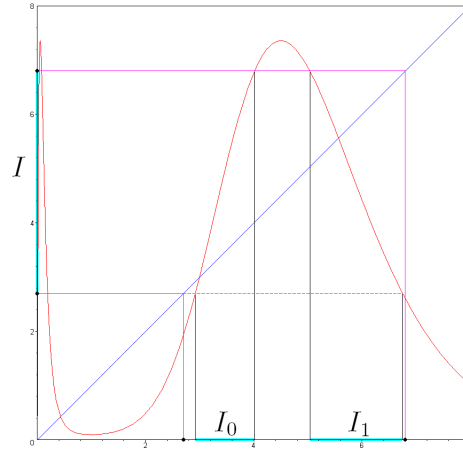
Figure 3.1: Partial representation of the graph of the second iterate of the map F in (3.1.4), with $\mu = 20$.

An entirely analogous line of reasoning can be followed for the map F from (3.1.4), for which we draw the graph of the second iterate in Figure 3.1 with $\mu = 20$. We enter the framework of Theorem 2.1.3 with the positions

$$\widetilde{\mathcal{R}} = \widetilde{I} = (I, I^-), \quad \mathcal{K}_0 = I_0 \quad \text{and} \quad \mathcal{K}_1 = I_1 \,,$$

where I, I_0 and I_1 are the intervals highlighted on the coordinate axes in Figure 3.1, so that we can promptly verify that

$$(I_i, F^2) : \widetilde{I} \Longleftrightarrow \widetilde{I}, \quad i = 0, 1.$$

Hence, the map F^2 in (3.1.4) induces chaotic dynamics on two symbols relatively to I_0 and I_1 and, consequently, has the chaotic features described in Theorem 2.1.4.

3.1.2   A planar model

We continue our treatment with a bidimensional OLG model with production, such as that discussed in [104] and [135].

Economically, there are two basic differences between this framework and the one discussed in the previous subsection. The first one is that there are no endowments. Instead, in each period t, a certain amount of the single homogeneous output x (call it "harvest corn") is produced by means of inputs of current labor l and capital ("seed corn") k is saved and invested in the previous period. The stock of capital depreciates entirely during one period of time.

The second difference concerns the fact that only young people work, earning a unit wage w, and save, while only old people consume. The supply of labor at each time t is determined as a solution of a problem of intertemporal constrained maximization of the young agent's utility function, whose arguments are labor (at time t) and consumption (at time $t + 1$) [2].

To simplify the analysis, we assume a linear, fixed coefficient technology, i.e., formally, we adopt the production function

$$x_t = \min\{a\,l_t, b\,k_{t-1}\}, \tag{3.1.5}$$

where the constants a and b denote the output/labor and the output/capital coefficients, respectively, and

$$k_t = x_t - c_t. \tag{3.1.6}$$

In order to ensure that the economy is "viable", we shall suppose that $b > 1$, i.e. that the fixed requirement of "seed corn" per unit of "harvest corn" is less than one. Moreover, to economize in the use of parameters and without loss of generality, we shall choose the unit of measure of labor so that $a = 1$.

Under these hypotheses, the young agent's problem can be formalized as follows:

$$
\begin{aligned}
\max_{c_{t+1}, l_t} & \{u(c_{t+1}) - v(l_t)\} \\
\text{s.t. } & c_{t+1} \le \rho_{t+1}(w_t\,l_t - c_t), \\
& c_t,\, l_t,\, k_t \ge 0,\ \forall t \ge 0,
\end{aligned}
\tag{P2}
$$

where u and $-v$ are the two separable components of the utility function with

$$u'(c) > 0,\ u''(c) \le 0 \quad \text{and} \quad v'(l) > 0,\ v''(l) > 0.$$

From the first order conditions of problem (P2), we get

$$\mathcal{U}(c_{t+1}) = \mathcal{V}(l_t), \tag{3.1.7}$$

where $\mathcal{U}(c) = u'(c)\,c$ and $\mathcal{V}(l) = v'(l)\,l$. Moreover, from (3.1.5)–(3.1.6), we have

$$l_{t+1} = b(l_t - c_t). \tag{3.1.8}$$

Considering that, under the postulated assumptions, $\mathcal{V}(l)$ is always invertible, whether conditions (3.1.7)–(3.1.8) can actually define forward discrete-time dynamics of the state variables (c, l) or not depends entirely on the properties

---

[2]To give the problem a beginning, an exception must be made for the first generation, which is assumed to be born old at $t = 1$ and to be endowed with a given amount of "harvest corn", entirely consumed in that period.

of $\mathcal{U}(c)$ and therefore on those of $u(c)$.

If $\mathcal{U}(c)$ is invertible, then simple calculations lead to the difference equation

$$(c_{t+1}, l_{t+1}) = F(c_t, l_t) = [\mathcal{U}^{-1}(\mathcal{V}(l_t)), b(l_t - c_t)]. \tag{3.1.9}$$

Although for this case there is a rigorous proof of the existence of periodic solution in [135], the most interesting dynamics are found when $\mathcal{U}(c)$ is not invertible. We thus concentrate on the latter situation and, in particular, we assume the "constant absolute risk aversion (CARA)" utility function

$$u(c) = -\mu\, e^{-c},$$

where μ is a positive parameter, whence

$$\mathcal{U}(c) = u'(c)c = \mu\, c\, e^{-c}.$$

We also adopt the labor (dis)utility function

$$v(l) = \frac{1}{\beta}\, l^{\beta},$$

with $\beta > 1$. Now $\mathcal{U}(c)$ has a "one-hump" form and consequently it does not admit a global inverse $\mathcal{U}^{-1}(c)$. Hence, we cannot define forward dynamics as we did with (3.1.9). Since $\mathcal{V}(l)$ is invertible, however, we could write the system of equations

$$(c_t, l_t) = F(c_{t+1}, l_{t+1}) = \begin{cases} g_0(c_{t+1}) - \frac{1}{b} l_{t+1} \\ g_0(c_{t+1}), \end{cases} \tag{3.1.10}$$

where

$$g_0(c) := \mathcal{V}^{-1}[\mathcal{U}(c)]. \tag{3.1.11}$$

Given the initial conditions for l and c, system (3.1.10) determines the dynamics of the variables (c, l) "in the past".

In what follows, we apply the stretching along the paths method to prove that, for certain parameter configurations, the dynamical system (3.1.10) (actually, a generalization of it) exhibits chaos. We will then suggest how to use Inverse Limit Theory [106] to show that chaotic dynamics backward in time imply chaotic dynamics forward in time.

In order to make reference to the treatment of Section 1.1 easier and to deal with a more general case, we rewrite system (3.1.10), defining the continuous planar map $F : (\mathbb{R}^+)^2 \to \mathbb{R}^2$ of the first quadrant as

$$F(x, y) = (F_1(x, y), F_2(x, y)) := \left(g(x) - \frac{y}{b}, g(x) \right), \tag{3.1.12}$$

where $b > 1$ is a constant and $g(x)$ is a continuous real-valued function assuming nonnegative values for $x \in [0, +\infty)$ and vanishing in 0. Moreover, we suppose that g has a positive maximum value M which is achieved at a unique $\bar{x} \in (0, +\infty)$. Notice that the required properties for g are satisfied by any unimodal map with $g(0) = 0$ and therefore they do not necessarily restrict the choice to the map g_0 in (3.1.11).

In Theorem 3.1.1 below, we prove that the stretching along the paths property for the map F in (3.1.12) is satisfied when taking as generalized rectangle a member of the family of trapezoidal regions described analytically by

$$\mathcal{R} = \mathcal{R}(K, M) := \left\{ (x, y) \in \mathbb{R}^2 : 0 \le x \le K, \ x \le y \le M \right\}, \qquad (3.1.13)$$

with $M = g(\bar{x})$ the maximum of the map g as discussed above and K a suitable parameter. The stretching property will be checked under the technical hypothesis that there exists $K > \bar{x}$ such that

$$bg(K) \le K < M\left(1 - \frac{1}{b}\right). \qquad (3.1.14)$$

Notice that, in order for (3.1.14) to hold for some value of $K > \bar{x}$, we must have

$$M = g(\bar{x}) > \frac{b\bar{x}}{b - 1} \ .$$
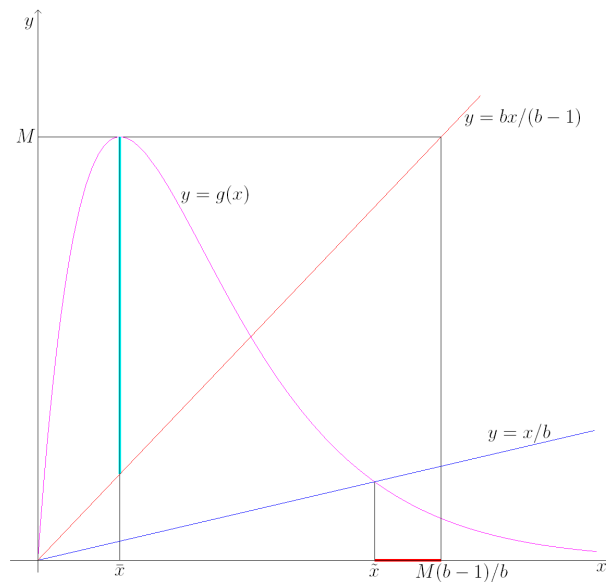
Figure 3.2: A visual representation of conditions (3.1.14).

Figure 3.2 provides a graphical illustration of conditions (3.1.14). Let $\tilde{x}$ be the maximal solution of the equation $x = bg(x)$. If the maximum M of the map g lies on the thicker vertical half-line, that is, if $M = g(\bar{x}) > b\bar{x}/(b-1)$, and so $\tilde{x} > \bar{x} > 0$, then any point in the segment $[\tilde{x}, M(b-1)/b)$ on the x-axis can be taken as the right edge K for the trapezoid $\mathcal{R}$.

In the special case of system (3.1.10) with the choice (3.1.11), the function g has the form

$$g(x) = g_0(x) := (\mu x \exp(-x))^{1/\beta}, \tag{3.1.15}$$

with $\mu > 0$ and $\beta > 1$. Thus $\bar{x} = 1$ and $M = \left(\frac{\mu}{e}\right)^{1/\beta}$. It follows that, for any given value of $b > 1$, the condition $M = g(\bar{x}) > b\bar{x}/(b-1)$ is satisfied for sufficiently large values of μ, i.e. when the unimodal curve of g is sufficiently steep. Also (3.1.14) holds true for every μ large enough. For example, if we fix $b = 2$, $\beta = 1.3$, the conditions of Theorem 3.1.1 are fulfilled for $\mu = 80$, $b = 2$, $\beta = 1.3$ (to which there corresponds $M = g(1) \approx 13.48474370$) and $K = 6$. These are indeed the parameters used in Figures 3.3–3.5.

The generalized rectangle in (3.1.13) can be oriented by setting

$$\mathcal{R}_\ell^- := \{0\} \times [0, M] \quad \text{and} \quad \mathcal{R}_r^- := \{K\} \times [K, M], \tag{3.1.16}$$

and, according to our notation, their union will be denoted by $\mathcal{R}^-$.

Our main result for system (3.1.10) (and, more generally, for the map in (3.1.12)) is the following:

Theorem 3.1.1. *For the map F defined in* (3.1.12) *and for any generalized rectangle $\mathcal{R} = \mathcal{R}(K, M)$ belonging to the family described in* (3.1.13), *with*

$$M = g(\bar{x}), \quad bg(K) \leq K < M\left(1 - \frac{1}{b}\right), \quad K > \bar{x}, \tag{3.1.17}$$

and oriented as in (3.1.16), *there are two disjoint compact subsets $\mathcal{K}_0 = \mathcal{K}_0(\mathcal{R})$ and $\mathcal{K}_1 = \mathcal{K}_1(\mathcal{R})$ of $\mathcal{R}$ such that*

$$(\mathcal{K}_i, F) : \widetilde{\mathcal{R}} \Leftrightarrow \widetilde{\mathcal{R}}, \ \text{for } i = 0, 1. \tag{3.1.18}$$

Hence, the map F induces chaotic dynamics on two symbols on $\mathcal{R}$ relatively to $\mathcal{K}_0$ and $\mathcal{K}_1$ and has all the properties listed in Theorem 2.1.4.

Proof. We are going to show that the conditions on the parameters K and M in (3.1.17) are sufficient to guarantee that the image under the map F of any path $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to \mathcal{R}$, joining in $\mathcal{R}$ the sides $\mathcal{R}_\ell^-$ and $\mathcal{R}_r^-$ defined in (3.1.16), satisfies the following:

$(C1)$ $F_1(\gamma(0)) \leq 0$;

$(C2)$ $F_1(\gamma(1)) \leq 0$;

$(C3)$ $\exists\,\bar{t} \in (0,1) : F_1(\gamma(\bar{t})) > K$;

$(C4)$ $F_2(\gamma(t)) \subseteq [F_1(\gamma(t)), M]$, $\forall t \in [0,1]$.

The first three requirements together imply an expansion along the x-axis which is accompanied by a folding. Indeed, the image $F \circ \gamma$ of any path γ joining $\mathcal{R}_\ell^-$ and $\mathcal{R}_r^-$ crosses a first time the trapezoid $\mathcal{R}$, for $t \in (0, \bar{t})$, and then crosses $\mathcal{R}$ back again, for $t \in (\bar{t}, 1)$. Condition $(C4)$ implies a contraction along the y-axis.

In order to simplify the exposition, we will replace $(C3)$ with the stronger condition

$(C3')$ $F_1(\bar{x}, y) > K$, $\forall\, y \in [\bar{x}, M]$,

i.e. we ask that the inequality in $(C3)$ holds for any $\bar{t} \in (0,1)$ such that $\gamma_1(\bar{t}) = \bar{x}$. Notice that, since $K > \bar{x}$, the vertical segment $\{\bar{x}\} \times [\bar{x}, M]$ is contained in $\mathcal{R}$.

Let us now define

$$\mathcal{R}_0 := \left\{ (x,y) \in \mathbb{R}^2 : 0 \leq x \leq \bar{x},\ x \leq y \leq M \right\},$$
$$\mathcal{R}_1 := \left\{ (x,y) \in \mathbb{R}^2 : \bar{x} \leq x \leq K,\ x \leq y \leq M \right\}$$

and

$$\mathcal{K}_0 := \mathcal{R}_0 \cap F(\mathcal{R}),\qquad \mathcal{K}_1 := \mathcal{R}_1 \cap F(\mathcal{R}).$$

Then, we claim that conditions $(C1), (C2), (C3')$ and $(C4)$ imply the stretching property in (3.1.18).

First of all consider that, by condition $(C3')$, the sets $\mathcal{K}_0$ and $\mathcal{K}_1$ are disjoint because $\{\bar{x}\} \times [\bar{x}, M]$ is mapped by F outside $\mathcal{R}$. Furthermore $(C1), (C2)$ and $(C3')$ ensure that, for every path $\gamma : [0,1] \to \mathcal{R}$ with $\gamma(0) \in \mathcal{R}_\ell^-$ and $\gamma(1) \in \mathcal{R}_r^-$ (or $\gamma(0) \in \mathcal{R}_r^-$ and $\gamma(1) \in \mathcal{R}_\ell^-$), there exist two disjoint subintervals $[t_0', t_0''], [t_1', t_1''] \subseteq [0,1]$ such that

$$\gamma(t) \in \mathcal{K}_0,\ \forall\, t \in [t_0', t_0''],\quad \gamma(t) \in \mathcal{K}_1,\ \forall\, t \in [t_1', t_1''],$$

with $F(\gamma(t_0'))$ and $F(\gamma(t_0''))$, as well as $F(\gamma(t_1'))$ and $F(\gamma(t_1''))$, belonging to different components of $\mathcal{R}^-$. Finally, from $(C4)$ it follows that $F(\gamma(t)) \in \mathcal{R}$, $\forall\, t \in [t_0', t_0''] \cup [t_1', t_1'']$. Recalling Definition 1.1.1, we can conclude that, under the postulated conditions, (3.1.18) is verified and thus the last part of the statement is a straightforward consequence of Theorem 2.1.3. Observe that F is continuous on $\mathcal{K}_0 \cup \mathcal{K}_1$ because it is continuous on $(\mathbb{R}^+)^2$.

Next, we want to prove that (3.1.17) implies conditions $(C1)$–$(C3')$ and $(C4)$.

For this purpose, let us consider any path $\gamma = (\gamma_1, \gamma_2) : [0,1] \rightarrow \mathcal{R}$ such that $\gamma(0) \in \mathcal{R}_\ell^-$ and $\gamma(1) \in \mathcal{R}_r^-$ (the case in which $\gamma(0) \in \mathcal{R}_r^-$ and $\gamma(1) \in \mathcal{R}_\ell^-$ can be treated analogously).

Since, in the present situation,

$$F_1(\gamma(0)) = -\frac{\gamma_2(0)}{b} \leq 0,$$

$(C1)$ is automatically satisfied.

For the validity of $(C2)$, we need to establish that

$$F_1(\gamma(1)) = g(K) - \frac{\gamma_2(1)}{b} \leq 0,$$

but, since $\gamma(1) \in \mathcal{R}_r^-$, it follows that $\dfrac{\gamma_2(1)}{b} = \dfrac{K}{b}$ and therefore $(C2)$ holds if

$$F_1(\gamma(1)) = g(K) - \frac{K}{b} \leq 0,$$

that is, if

$$bg(K) \leq K. \tag{3.1.19}$$

To prove $(C3')$, we must show that

$$g(\bar{x}) - \frac{\gamma_2(\bar{t})}{b} > K,$$

$\forall \bar{t} \in (0,1)$ with $\gamma_1(\bar{t}) = \bar{x}$. Recalling that $M = g(\bar{x})$ is the maximum value for the map g and noticing that

$$M - \frac{\gamma_2(\bar{t})}{b} > M\left(1 - \frac{1}{b}\right),$$

$(C3')$ is fulfilled if

$$M\left(1 - \frac{1}{b}\right) > K. \tag{3.1.20}$$

Finally, for $(C4)$ we have to consider two inequalities. First, since both b and $\gamma_2(t)$, $t \in [0,1]$, are nonnegative, the inequality $F_2(\gamma(t)) \geq F_1(\gamma(t))$, $\forall t \in [0,1]$, or equivalently

$$g(\gamma_1(t)) \geq g(\gamma_1(t)) - \frac{\gamma_2(t)}{b}, \ \forall t \in [0,1],$$

is satisfied. Second, the inequality $F_2(\gamma(t)) \leq M$, $\forall t \in [0,1]$, or equivalently

$$g(\gamma_1(t)) \leq M, \ \forall t \in [0,1],$$

is always fulfilled by the choice of M.

Summing up, in view of (3.1.19) and (3.1.20), the assumptions sufficient to imply $(C1)$, $(C2)$, $(C3')$ and $(C4)$ are

$$\begin{cases} g(K) - \frac{K}{b} \leq 0 \\ M\left(1 - \frac{1}{b}\right) > K, \end{cases}$$

which are fulfilled for any positive K such that

$$bg(K) \leq K < M\left(1 - \frac{1}{b}\right),$$

as postulated in (3.1.17). The proof is complete. $\qquad\qquad\qquad\square$

Figures 3.3–3.5 provide a visual representation of the stretching along the paths property for system (3.1.10).
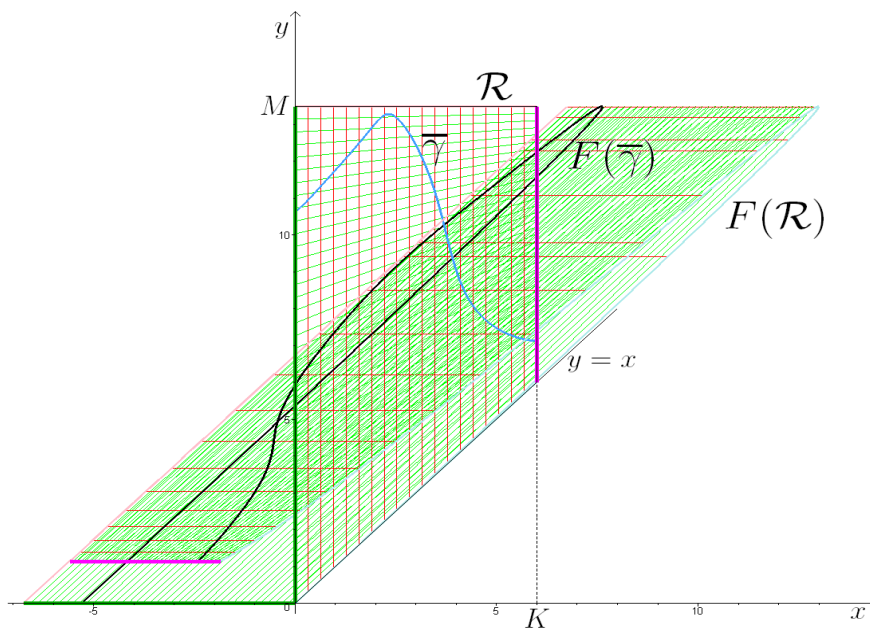
Figure 3.3: The trapezoid $\mathcal{R}$, oriented in the usual left-right manner, and its image under the map F in (3.1.12), with g as in (3.1.15). An arbitrary path γ joining in $\mathcal{R}$ the two components of the boundary set $\mathcal{R}^-$ is transformed by F so that its image intersects $\mathcal{R}$ twice.
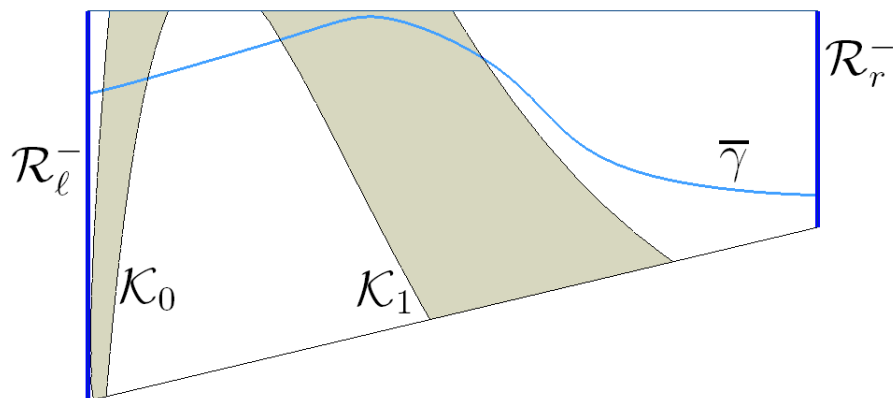
Figure 3.4: A zoom (with a magnification in the x-direction) of the generalized rectangle $\mathcal{R}$ from Figure 3.3, showing the intersections between the path γ and the two compact sets $\mathcal{K}_0$ and $\mathcal{K}_1$, which are drawn here with a darker color.
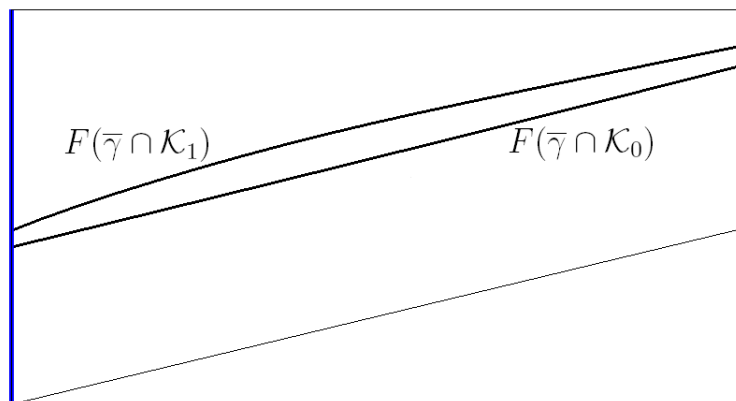
Figure 3.5: An illustration of the components $F(\overline{\gamma} \cap \mathcal{K}_0)$ and $F(\overline{\gamma} \cap \mathcal{K}_1)$ of the image of the path γ under the map F.

As we have already observed while introducing Theorem 3.1.1, for the particular choice of $g = g_0$ adopted in (3.1.15), condition (3.1.17) is always satisfied when μ is sufficiently large.

The inequalities in (3.1.17) define geometrical conditions depending on the choice of the generalized rectangle $\mathcal{R}$ (i.e. the trapezoid in (3.1.13)).

The question naturally arises how a different $\mathcal{R}$ would affect the corresponding conditions on the system parameters. A systematic investigation of this question would lead us afar. Hence, we limit ourselves to the analysis of an interesting alternative choice for $\mathcal{R}$, specifically related to the crucial function $g = g_0$ in (3.1.15).

The new $\mathcal{R}$, depicted in Figure 3.6, is a subset of the region below the graph of the map $x \mapsto bg(x)$, whose definition is

$$\mathcal{R} = \mathcal{R}(\nu, d) := \left\{ (x, y) \in \mathbb{R}^2 : 0 \le y \le bg(x),\ x \le y \le \nu x + d \right\}, \quad (3.1.21)$$

with ν and d suitably fixed parameters. Such domain can be oriented by taking as $\mathcal{R}_\ell^-$ and $\mathcal{R}_r^-$ the left and the right components of the intersection between the graph of $y = bg(x)$ and $\mathcal{R}$, respectively. An illustration of this choice for $\mathcal{R}$ with the parameters $\mu = 14.5$, $b = 2$, $\beta = 1/0.95$, $\nu = 0.6$ and $d \approx 3.662313254$ is furnished by Figure 3.6. Moreover, Figure 3.7 shows that the image under F of an arbitrary path γ joining $\mathcal{R}_\ell^-$ and $\mathcal{R}_r^-$ in $\mathcal{R}$ intersects twice $\mathcal{R}$ and suggests a strong similarity between this case and the one discussed in Theorem 3.1.1. In order to avoid tedious repetitions, however, we will not provide here a proof that the stretching property and all its implications indeed obtain for $\mathcal{R}$ in (3.1.21).
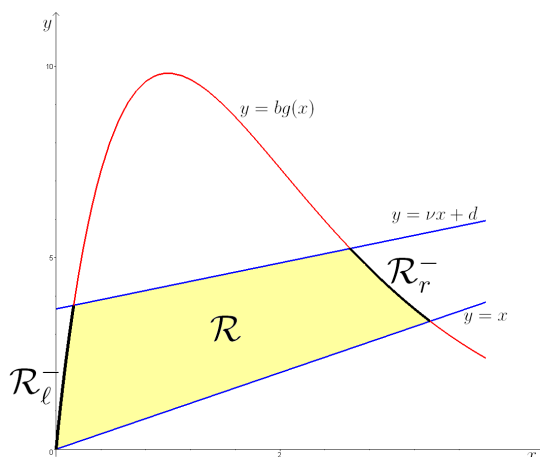
Figure 3.6: A different choice of the oriented rectangle $\widetilde{\mathcal{R}}$ for system (3.1.10) with the two components of $\mathcal{R}^-$ indicated with thicker lines.
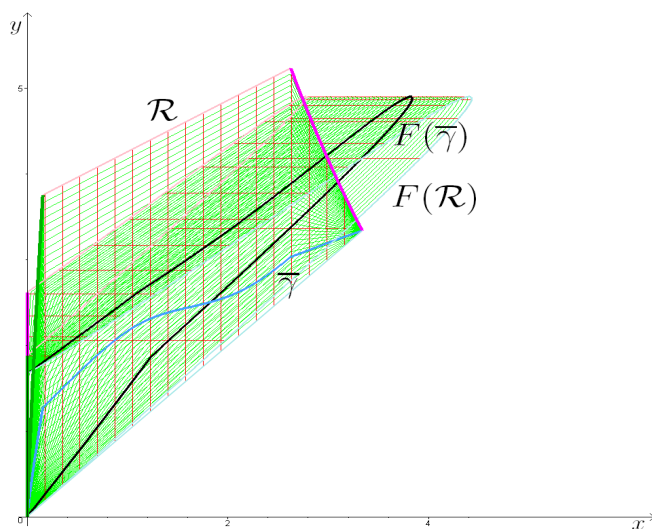
Figure 3.7: The generalized rectangle $\mathcal{R}$ represented in Figure 3.6 together with an arbitrary path γ joining the two components of the boundary set $\mathcal{R}^-$ and their images under F.

To conclude, we have thus shown that, for a suitable choice of the generalized rectangle $\mathcal{R}$, the dynamics of the map F controlling our OLG model are therein chaotic in the sense of Theorems 2.1.3 and 2.1.4. The economic interpretation of this result, however, is complicated by the fact that the iterations of F move *backward in time*, while economic agents care about the future not the past. Moreover, in the case under investigation, F is non-invertible and therefore the study of the *forward in time* dynamics implicitly defined by F is not trivial. Inverse Limit Theory provides the ideal mathematical tool to solve such problem, as it allows, under certain hypotheses, to transfer the chaotic features of the map F to a forward-moving map σ [3]. The asymptotic behaviour of the iterations σ^n of σ, in the limit for $n \to \infty$, can be taken as a representation of the long-run dynamics of the system. The details of this process are described in [105]. We also mention that Inverse Limit Theory allows to deal with another question related to the chaotic dynamics of the model above. Indeed, so far we have only discussed the *existence* of chaotic sets, not their *attractiveness*. Here, we would like to add the following brief observation. We have proved the existence of backward moving chaos for values of parameters of the map F for which the numerical simulations "explode" and the chaotic set Λ from Theorem 2.1.4 is therefore not observable on the computer screen. This fact suggests

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[3]We have chosen to name this map as the classical Bernoulli shift (cf. Section 2.2), as it acts as a (one-sided) shift on a suitable space of sequences. The application-oriented reader can find a brief introduction to the subject and basic bibliography in [106].

that Λ is a repeller for the map F. Hence, the corresponding inverse limit set is a natural candidate for an attractor, in one or the other of the various existing definitions, of the forward moving map σ. The cautious formulation of such statement is motivated by the fact that, in general, F may have many (even infinitely many) repellers, while only one, or some of the corresponding subsets of the inverse limit space are attractors for σ. This question is discussed at great length in [106], where general results are found for the case in which the bonding maps are unimodal functions of the interval.

3.2   Duopoly Games

We now apply our stretching along the paths method to detect chaotic dynamics for a planar economic model belonging to the class of "duopoly games", drawn from Agiza and Elsadany's paper [2].

The economy consists of two firms producing an identical commodity at a constant unit cost, not necessarily equal for the two firms. The commodity is sold in a single market at a price which depends on total output through a given "inverse demand function", known to both firms. The goal of each firm is the maximization of profits, i.e. the difference between revenue and cost. The problem of any firm is to decide at each time t how much to produce at time $(t+1)$ on the basis of the limited information available and, in particular, the expectations about its competitor's future decision.

Different hypotheses about demand and cost functions and firms' expectations lead to different formulations of the problem. For the model discussed here the following notation and assumptions are postulated:

1. **Notation**

   x_t : output of Firm 1 at time t ;
   y_t : output of Firm 2 at time t ;
   p : unit price of the single commodity .

2. **Inverse demand function**

   $p = a - b(x + y)$, where a and b are positive constants .

3. **Technology**

   The unit cost of production for firm i is equal to c_i, $i = 1, 2$, where c_1, c_2 are positive constants, not necessarily equal.

4. Expectations

In the presence of incomplete information concerning its competitor's future decisions (and therefore about future prices), each firm is assumed to use a different strategy, namely:

Firm 1: bounded rationality. At each time t, Firm 1 changes its output by an amount proportional to the marginal profit (= derivative of profit calculated at the known values of output and price at time t).

Firm 2: adaptive expectations. At each time t, Firm 2 changes its output by an amount proportional to the difference between the previous output y_t and the "naïve expectations value" (calculated by maximizing profits on the assumption that both firms will keep output unchanged).

The result of these assumptions is a system of two difference equations in the variables x and y, as follows:

$$\begin{cases} x_{t+1} = x_t(1 + \alpha a - \alpha b y_t - \alpha c_1 - 2\alpha b x_t) \\ y_{t+1} = (1-\nu)y_t + \frac{\nu}{2b}(a - c_2 - bx_t), \end{cases} \quad (DG)$$

where α is a positive parameter denoting the speed of Firm 1's adjustment to changes in profit, $\nu \in [0,1]$ is Firm 2's rate of adaptation and a, b, c_1, c_2 are positive constants.

In [2], Agiza and Elsadany discuss the equilibrium solutions of system (DG) and their stability and provide numerical evidence of the existence of chaotic dynamics. Here we integrate those authors' analysis, rigorously proving that, for certain parameter configurations, system (DG) exhibits chaotic behaviour in the precise sense discussed in Section 2.1. Notice, however, that we only prove *existence* of invariant, chaotic sets, not their *attractiveness*.

In order to apply our method to the analysis of system (DG), it is expedient to represent it in the form of a continuous planar map $F = (F_1, F_2) : (\mathbb{R}^+)^2 \to \mathbb{R}^2$, with components

$$\begin{aligned} F_1(x,y) &:= x(1 + \alpha a - \alpha b y - \alpha c_1 - 2\alpha b x), \\ F_2(x,y) &:= (1-\nu)y + \frac{\nu}{2b}(a - c_2 - bx). \end{aligned} \quad (3.2.1)$$

In Theorem 3.2.1 below we show that the stretching property for the map F is satisfied when taking a generalized rectangle $\mathcal{R} = \mathcal{R}(P,Q)$ from the family of rectangular trapezia of the first quadrant described analytically by

$$\mathcal{R} := \left\{ (x,y) \in \mathbb{R}^2 : P \le y \le Q,\ 0 \le x \le \frac{a - c_1 + 1/\alpha}{2b} - \frac{y}{2} \right\}, \quad (3.2.2)$$

with a, b, c_1 and α as in (DG) and P, Q satisfying the restrictions

$$0 \le P := \frac{a + c_1 - 2c_2 - 1/\alpha}{3b} < Q := \frac{a - c_2}{2b} < \frac{a - c_1 + 1/\alpha}{b}. \quad (3.2.3)$$

Notice that, for this choice, the set $\mathcal{R}$ is nonempty.

A visual illustration of the above conditions is furnished by Figure 3.8, where unexplained notation is as in the proof of Theorem 3.2.1. In analogy to [2], in Figures 3.8–3.10 we have fixed the parameters as $a = 10$, $b = 0.5$, $c_1 = 3$, $c_2 = 5$, $\alpha = 26/27$ and $\nu = 0.5$.
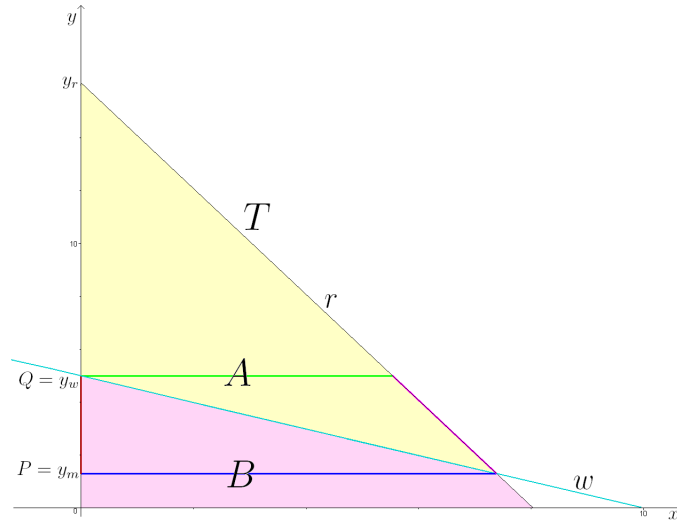

Figure 3.8: Geometric representation of some conditions in Theorem 3.2.1. The triangular region T is delimited by the coordinate axes and by the straight line r. The "watershed" line w separates T into the two regions A (yellow, in the colored version) and B (pink). The values of P and Q are defined by the intersections of w with r and the y-axis, respectively.

A trapezoid $\mathcal{R}$ as in (3.2.2) can be oriented by setting

$$\mathcal{R}_\ell^- := \{0\} \times [P, Q], \ \mathcal{R}_r^- := \left\{ \left(\frac{a - c_1 + 1/\alpha}{2b} - \frac{y}{2}, y \right) : y \in [P, Q] \right\}. \quad (3.2.4)$$

The choice of $\mathcal{R}^- = \mathcal{R}_\ell^- \cup \mathcal{R}_r^-$ is depicted in Figure 3.9.

Our result on system (DG) can be stated as follows:

Theorem 3.2.1. *If the parameters of the map F defined in* (3.2.1) *satisfy the conditions*

$$a - 2c_1 + c_2 > \frac{26}{3\alpha}, \qquad a + c_1 - 2c_2 - \frac{1}{\alpha} \geq 0, \quad (3.2.5)$$

then, for any generalized rectangle $\mathcal{R} = \mathcal{R}(P, Q)$ belonging to the family described in (3.2.2)*, with*

$$P := \frac{a + c_1 - 2c_2 - 1/\alpha}{3b} \quad and \quad Q := \frac{a - c_2}{2b}, \quad (3.2.6)$$
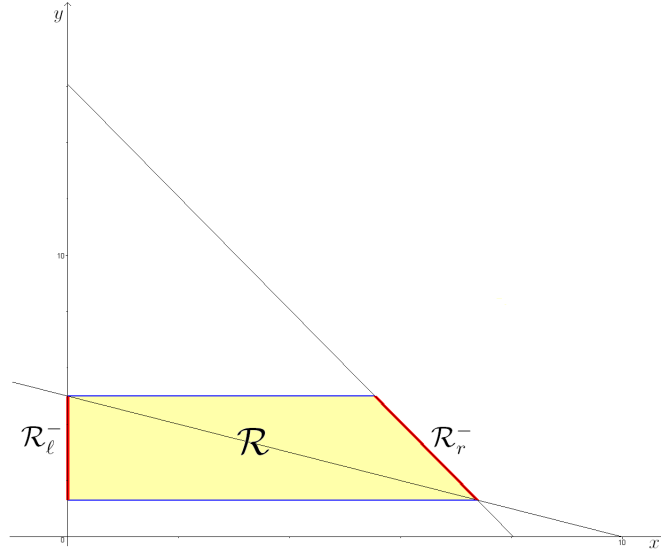
Figure 3.9: A suitable generalized rectangle $\mathcal{R}$ for system (DG), according to conditions (3.2.2) and (3.2.3). It is oriented in the usual left-right manner, by taking as $[\cdot]^-$-set the two "vertical" segments $\mathcal{R}_\ell^-$ and $\mathcal{R}_r^-$ defined in (3.2.4).

and oriented as in (3.2.4)*, there exist two disjoint compact subsets* $\mathcal{K}_0 = \mathcal{K}_0(\mathcal{R})$ *and* $\mathcal{K}_1 = \mathcal{K}_1(\mathcal{R})$ *of* $\mathcal{R}$ *such that*

$$(\mathcal{K}_i, F) : \widetilde{\mathcal{R}} \Lleftarrow\!\!\dashrightarrow \widetilde{\mathcal{R}}, \ for \ i = 0, 1. \tag{3.2.7}$$

Hence, the map F *induces chaotic dynamics on two symbols in* $\widetilde{\mathcal{R}}$ *relatively to* $\mathcal{K}_0$ *and* $\mathcal{K}_1$ *and has all the properties listed in Theorem 2.1.4.*

Proof. We are going to show that, when choosing P and Q as in (3.2.6), conditions (3.2.5) are sufficient to guarantee that the image under the map F of any path $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to \mathcal{R}$, joining in $\mathcal{R}$ the sides $\mathcal{R}_\ell^-$ and $\mathcal{R}_r^-$ defined in (3.2.4), satisfies the following:

$(C1)$ $F_1(\gamma(0)) \leq 0$;

$(C2)$ $F_1(\gamma(1)) \leq 0$;

$(C3)$ $\exists\, t^* \in (0, 1) : F_1(\gamma(t^*)) > x_M := \max\{x : (x, y) \in \mathcal{R}_r^-\}$;

$(C4)$ $F_2(\gamma(t)) \subseteq [P, Q], \forall t \in [0, 1]$.

Broadly speaking, conditions $(C1)$–$(C3)$ describe an expansion with folding along the x-coordinate. In fact, the image under F of any path γ joining $\mathcal{R}_\ell^-$

and $\mathcal{R}_r^-$ in $\mathcal{R}$ crosses a first time the trapezoid $\mathcal{R}$, for $t \in (0, t^*)$, and then crosses $\mathcal{R}$ back again, for $t \in (t^*, 1)$. Condition $(C4)$ implies a contraction along the y-coordinate.

In order to simplify the exposition, we replace $(C3)$ with the stronger condition

$$(C3') \quad F_1\left(\frac{a - c_1 + 1/\alpha}{4b} - \frac{y}{4}, y\right) > x_M, \ \forall y \in [P, Q],$$

i.e. we require that the inequality in $(C3)$ holds for any $t^* \in (0, 1)$ such that $\gamma(t^*) = (\frac{a - c_1 + 1/\alpha}{4b} - \frac{y}{4}, y)$, for some $y \in [P, Q]$.

Recalling that $\mathcal{R}_r^- := \left\{\left(\frac{a - c_1 + 1/\alpha}{2b} - \frac{y}{2}, y\right) : y \in [P, Q]\right\}$, taking P, Q as in (3.2.3) and considering that $0 < \frac{a - c_1 + 1/\alpha}{4b} - \frac{y}{4} < \frac{a - c_1 + 1/\alpha}{2b} - \frac{y}{2}, \ \forall y \in [P, Q]$, we can conclude that the segment

$$S := \left\{\left(\frac{a - c_1 + 1/\alpha}{4b} - \frac{y}{4}, y\right) : y \in [P, Q]\right\} \tag{3.2.8}$$

is contained in $\mathcal{R}$. A graphical illustration of this fact is provided by Figure 3.10.

Defining

$$\mathcal{R}_0 := \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \frac{a - c_1 + 1/\alpha}{4b} - \frac{y}{4}, \ P \leq y \leq Q\right\},$$

$$\mathcal{R}_1 := \left\{(x, y) \in \mathbb{R}^2 : \frac{a - c_1 + 1/\alpha}{4b} - \frac{y}{4} \leq x \leq \frac{a - c_1 + 1/\alpha}{2b} - \frac{y}{2}, P \leq y \leq Q\right\}$$

and

$$\mathcal{K}_0 := \mathcal{R}_0 \cap F(\mathcal{R}), \qquad \mathcal{K}_1 := \mathcal{R}_1 \cap F(\mathcal{R}),$$

we claim that $(C1), (C2), (C3')$ and $(C4)$ together imply (3.2.7).

Firstly, notice that $\mathcal{K}_0$ and $\mathcal{K}_1$ are disjoint because, by condition $(C3')$, the segment S in (3.2.8) is mapped by F outside $\mathcal{R}$. Furthermore, $(C1), (C2)$ and $(C3')$ imply that, for any path $\gamma : [0, 1] \to \mathcal{R}$ with $\gamma(0) \in \mathcal{R}_\ell^-$ and $\gamma(1) \in \mathcal{R}_r^-$ (or $\gamma(0) \in \mathcal{R}_r^-$ and $\gamma(1) \in \mathcal{R}_\ell^-$), there exist two disjoint subintervals $[t_0', t_0''], [t_1', t_1''] \subseteq [0, 1]$ such that

$$\gamma(t) \in \mathcal{K}_0, \ \forall t \in [t_0', t_0''], \quad \gamma(t) \in \mathcal{K}_1, \ \forall t \in [t_1', t_1''],$$

with $F(\gamma(t_0'))$ and $F(\gamma(t_0''))$, as well as $F(\gamma(t_1'))$ and $F(\gamma(t_1''))$, belonging to different components of $\mathcal{R}^-$. Finally from $(C4)$ it follows that $F(\gamma(t)) \in \mathcal{R}$, $\forall t \in [t_0', t_0''] \cup [t_1', t_1'']$. Recalling Definition 1.1.1, the stretching condition (3.2.7) is achieved and our claim is thus checked. The last part of the statement is now
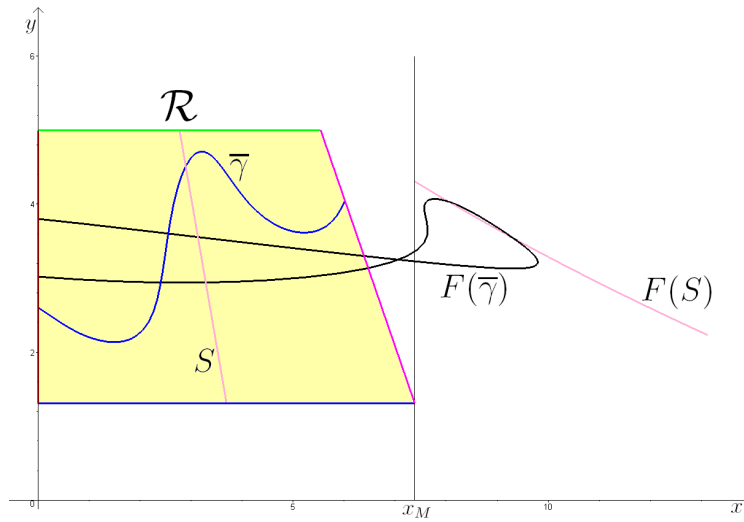
Figure 3.10:   With reference to the trapezoid $\mathcal{R}$ in Figure 3.9, reproduced here at a different scale, we show that the image under F of an arbitrary path γ joining in $\mathcal{R}$ the two components of the boundary set $\mathcal{R}^-$ intersects twice $\mathcal{R}$. This is due to the fact that the vertical sides $\mathcal{R}_\ell^-$ and $\mathcal{R}_r^-$ are mapped by F into the y-axis and the segment S, joining the middle points of the bases of the trapezoid, is mapped by F beyond the vertical line $y = x_M$, in conformity with condition $(C3')$.

an immediate consequence of Theorem 2.1.3. Observe that F is continuous on $\mathcal{K}_0 \cup \mathcal{K}_1$, as it is continuous on $(\mathbb{R}^+)^2$.

The next step is to verify that a choice of the parameters as in (3.2.5) implies conditions $(C1), (C2), (C3')$ and $(C4)$. In so doing, we will prove that the inequalities in $(C1)$ and $(C2)$ are indeed equalities.

First of all, we have to check that the set $\mathcal{R}(P, Q)$ defined in (3.2.2) and (3.2.3) is contained in the region $\{(x, y) \in (\mathbb{R}^+)^2 : F(x, y) \in (\mathbb{R}^+)^2\}$ [4].

For this purpose, let us first consider F_1. Setting

$$M := 1 + \alpha a - \alpha b y - \alpha c_1 \quad \text{and} \quad N := 2\alpha b,$$

for any given value of y it is possible to rewrite F_1 as the continuous, one-dimensional map

$$\phi : \mathbb{R}_+ \to \mathbb{R}, \quad \phi(x) := x(M - Nx),$$

which can be looked at as a parabola intersecting the x-axis in 0 and M/N and lying in the first quadrant in between. Notice that, under the postulated

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[4]Such condition guarantees that the invariant chaotic set discussed in Theorem 2.1.4 lies entirely in the first quadrant and therefore makes economic sense for the application in question.

assumptions on the model parameters, N is positive, while $M = M(y) > 0$ if $y < (a - c_1 + 1/\alpha)/b$. This inequality and the requirement that y is nonnegative define the restriction

$$0 \le y < y_r := \frac{a - c_1 + 1/\alpha}{b} \,.$$

If such condition is satisfied, M/N is positive and we can restrict our map ϕ to the interval $[0, M/N]$, a geometric configuration analogous to that of the logistic map discussed in Section 2.2. In terms of the planar coordinates (x, y), restricting ϕ to $[0, M/N]$ means that we consider only the values of x such that

$$0 \le x \le \frac{M}{N} = \frac{1 + \alpha a - \alpha b y - \alpha c_1}{2 \alpha b} = \frac{a - c_1 + 1/\alpha}{2b} - \frac{y}{2} \,.$$

The conditions on the variables x and y determine a triangular region T in the first quadrant, lying between the coordinate axes and the straight line $x = \frac{a - c_1 + 1/\alpha}{2b} - \frac{y}{2}$, or equivalently

$$y = \frac{a - c_1 + 1/\alpha}{b} - 2x \,,$$

a straight line that we will denote by r (cf. Figure 3.8). In view of the above discussion, the point $(0, y_r)$ does not belong to T. Observe that the trapezoid $\mathcal{R}$ in (3.2.2) lies is this region for any $0 \le P < Q < y_r$ [5] and thus $F_1(x, y)$ is nonnegative on $\mathcal{R}$. Furthermore, since $\phi(0) = \phi(M/N) = 0$, for any path $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to \mathbb{R}^2$ with $\gamma_1(0) = 0$ and $\gamma(1)$ belonging to the straight line r it follows that $F_1(\gamma(0)) = 0 = F_1(\gamma(1))$. Since $\mathcal{R}_\ell^-$ is contained in the y-axis and $\mathcal{R}_r^-$ in the line r, this implies that $(C1)$ and $(C2)$ are fulfilled for every choice of P and Q in $[0, y_r)$ (with $P < Q$).

In order to understand when $F_2(x, y) \ge 0$, let us consider the stronger condition $F_2(x, y) = (1 - \nu)y + \frac{\nu}{2b}(a - c_2 - bx) \ge y$, which is satisfied if the point (x, y) lies below the straight line denoted by w in Figure 3.8 and defined by the equation

$$y = \frac{a - c_2}{2b} - \frac{x}{2} \,. \tag{3.2.9}$$

Recalling that we are confined to the triangular region T, in order to have an intersection between w and the two segments

$$\{0\} \times [0, y_r) \quad \text{and} \quad \left\{ \left(\frac{a - c_1 + 1/\alpha}{2b} - \frac{y}{2}, y \right) : y \in [0, y_r) \right\}, \tag{3.2.10}$$

---

[5]The case $P = Q$ has to be excluded, because otherwise the generalized rectangle $\mathcal{R}$ would reduce to a segment and it could not be homeomorphic to the unit square of $\mathbb{R}^2$. For the same reason we will impose strict inequalities for the y-coordinates also in (3.2.11). Similar considerations led us to require $M/N > 0$, instead of $M/N \ge 0$.

we have to impose the following conditions on the y-coordinate

$$y_r = \frac{a - c_1 + 1/\alpha}{b} > y_w := \frac{a - c_2}{2b} > y_m \geq 0 \,, \tag{3.2.11}$$

where

$$(x_m, y_m) := \left(\frac{a - 2c_1 + c_2 + 2/\alpha}{3b} \,, \frac{a + c_1 - 2c_2 - 1/\alpha}{3b} \right) \tag{3.2.12}$$

is the point of intersection of r and w and $(0, y_w)$ is the point of intersection between w and the y-axis. Conditions (3.2.11) yield

$$a - 2c_1 + c_2 + \frac{2}{\alpha} > 0$$

and

$$a + c_1 - 2c_2 - \frac{1}{\alpha} \geq 0. \tag{3.2.13}$$

As we will see below, the intersection between w and the segments in (3.2.10) is fundamental for the validity of condition $(C4)$. Calling

$$\mathcal{R}_d^+ := \left[0, \frac{a - c_1 + 1/\alpha}{2b} - \frac{P}{2} \right] \times \{P\}$$

and

$$\mathcal{R}_u^+ := \left[0, \frac{a - c_1 + 1/\alpha}{2b} - \frac{Q}{2} \right] \times \{Q\}$$

the lower and upper sides of the trapezoid $\mathcal{R}$, respectively, we notice that, for P, Q in $[0, y_r)$, the arcs $\mathcal{R}_d^+$ and $\mathcal{R}_u^+$ are horizontal segments contained in T. Moreover, denoting by $\partial \mathcal{R}$ the boundary of $\mathcal{R}$ (that in our case coincides with the contour $\vartheta \mathcal{R}$), it holds that

$$\partial \mathcal{R} = \mathcal{R}_d^+ \cup \mathcal{R}_r^- \cup \mathcal{R}_u^+ \cup \mathcal{R}_\ell^-.$$

Choosing $\mathcal{R}_d^+$ and $\mathcal{R}_u^+$ as particular paths $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to \mathcal{R}$, joining in $\mathcal{R}$ the sides $\mathcal{R}_\ell^-$ and $\mathcal{R}_r^-$, in order to have condition $(C4)$ fulfilled, we need

$$F_2(x, P) \geq P, \quad \forall x \in \left[0, \frac{a - c_1 + 1/\alpha}{2b} - \frac{P}{2} \right],$$

as well as

$$F_2(x, Q) \leq Q, \quad \forall x \in \left[0, \frac{a - c_1 + 1/\alpha}{2b} - \frac{Q}{2} \right].$$

But it is possible only if $\mathcal{R}_d^+$ lies below the "watershed" line w in (3.2.9) and for $\mathcal{R}_u^+$ above it: more precisely, in view also of (3.2.11),

$$y_r > Q \geq y_w > y_m \geq P \geq 0 \,. \tag{3.2.14}$$

We will prove that such conditions are sufficient for the validity of $(C4)$.
Before that, however, we return for a while to the problem of the nonnegativity of F_2 in $\mathcal{R}$. For this purpose, let us denote by A the subset of T above the line w and by B its complement in T (see Figure 3.8). Since $F_2(x, y) \geq y \geq 0$ in B, here there are no further conditions to impose in order to have F_2 nonnegative. On the other hand, as regards the region A, recalling the notation from (3.2.12), we require that

$$
\begin{aligned}
\min_{(x,y)\in A} F_2(x, y) &= (1 - \nu)y_m + \tfrac{\nu}{2b}(a - c_2 - bx_m) = \\[2ex]
&= (1 - \nu)\left(\tfrac{a+c_1-2c_2-1/\alpha}{3b}\right) + \nu\left(\tfrac{a+c_1-2c_2-1/\alpha}{3b}\right) = \\[2ex]
&= \tfrac{a+c_1-2c_2-1/\alpha}{3b} \geq 0 \,,
\end{aligned} \tag{3.2.15}
$$

which follows from (3.2.13) and therefore it does not add any new restriction on the parameters.
Having settled the question of the sign of F_2 on $\mathcal{R}$, we return to $(C4)$ and notice that it is equivalent to

$$\max_{(x,y)\in\mathcal{R}} F_2(x, y) \leq Q \quad \text{and} \quad \min_{(x,y)\in\mathcal{R}} F_2(x, y) \geq P \,.$$

Since $F_2(x, y) \leq y$ on A and $F_2(x, y) \geq y$ on B, it suffices to ask that

$$\max_{(x,y)\in B\cap\mathcal{R}} F_2(x, y) \leq Q \quad \text{and} \quad \min_{(x,y)\in A\cap\mathcal{R}} F_2(x, y) \geq P \,.$$

From the previous discussion, it follows that $\mathcal{R}_u^+$ must lie in A and $\mathcal{R}_d^+$ in B, respectively, and thus both $A \cap \mathcal{R}$ and $B \cap \mathcal{R}$ are nonempty. Actually, instead of the above inequalities, we will investigate the stricter conditions

$$\max_{(x,y)\in B} F_2(x, y) \leq Q \quad \text{and} \quad \min_{(x,y)\in A} F_2(x, y) \geq P \,,$$

which, as we shall see, do not introduce any new restriction on the model parameters.
As concerns $\max_{(x,y)\in B} F_2(x, y) \leq Q$, recalling the expression of F_2 from (3.2.1), we have

$$\max_{(x,y)\in B} F_2(x, y) = (1-\nu)y_w + \frac{\nu}{2b}(a - c_2) = (1-\nu)\frac{a - c_2}{2b} + \frac{\nu}{2b}(a - c_2) = \frac{a - c_2}{2b} \,,$$

from which it follows that

$$y_w \left(= \frac{a - c_2}{2b} \right) \leq Q,$$

as in (3.2.14). As concerns $\min_{(x,y) \in A} F_2(x, y) \geq P$, from (3.2.15) we obtain

$$\min_{(x,y) \in A} F_2(x, y) = y_m = \frac{a + c_1 - 2c_2 - 1/\alpha}{3b} \geq P,$$

again as in (3.2.14).

Combining all the restrictions on the construction of $\mathcal{R}$ found so far, we can write

$$0 \leq x \leq \frac{a - c_1 + 1/\alpha}{2b} - \frac{y}{2}, \quad \forall y \in [P, Q],$$

where

$$0 \leq P \leq y_m < y_w \leq Q < y_r \,.$$

Such conditions are satisfied under the following restrictions on the model parameters:

$$a - 2c_1 + c_2 + \frac{2}{\alpha} > 0 \quad \text{and} \quad a + c_1 - 2c_2 - \frac{1}{\alpha} \geq 0. \qquad (3.2.16)$$

At this point, we focus on the remaining problem, that is, the validity of $(C3')$, and we start by checking that the choice

$$P = y_m := \frac{a + c_1 - 2c_2 - 1/\alpha}{3b} \quad \text{and} \quad Q = y_w := \frac{a - c_2}{2b}$$

is in some sense optimal among those allowed by (3.2.14). To such aim, we will leave for the moment $P, Q \in [0, y_r)$ unspecified, as in (3.2.14). Recalling that

$$\mathcal{R}_r^- := \left\{ \left(\frac{a - c_1 + 1/\alpha}{2b} - \frac{y}{2}, y \right) : y \in [P, Q] \right\},$$

we can write

$$x_M := \max\{x : (x, y) \in \mathcal{R}_r^-\} = \frac{a - c_1 + 1/\alpha}{2b} - \frac{P}{2} \,.$$

Clearly, the smaller x_M, the more likely is $(C3')$ to hold. Since $0 \leq P \leq y_m$, the optimal choice for P is $P = y_m$, as claimed. As concerns Q, let us consider it as a parameter $\bar{y}$ varying in $[y_w, y_r)$. An easy calculation shows that the

maximum of the function F_1 on the horizontal segment $[0, \frac{a-c_1+1/\alpha}{2b} - \frac{\bar{y}}{2}] \times \{\bar{y}\}$ is attained in correspondence of its middle point $\bar{x}$, that is for

$$\bar{x} := \frac{a - c_1 + 1/\alpha}{4b} - \frac{\bar{y}}{4} \; .$$

Evaluating the function F_1 in $(\bar{x}, \bar{y})$, we find

$$F_1(\bar{x}, \bar{y}) = \frac{\alpha \, (a - b\bar{y} - c_1 + 1/\alpha)^2}{8b} \; .$$

To fulfill condition $(C3')$, we must have $F_1(\bar{x}, \bar{y}) > x_M$, for any $\bar{y} \in [P, Q]$. Hence, Q should be chosen as small as possible, i.e. $Q = y_w$.
At this point, having fixed P and Q, from the above considerations we gather that $(C3')$ is satisfied if, for any $\bar{y} \in [P, Q]$,

$$\max\left\{F_1(x, \bar{y}) : x \in \left[0, \frac{a - c_1 + 1/\alpha}{2b} - \frac{\bar{y}}{2}\right]\right\} \quad = F_1(\bar{x}, \bar{y}) > x_M \quad =$$

$$= \tfrac{a-c_1+1/\alpha}{2b} - \tfrac{y_m}{2} \quad = x_m \; .$$

This requirement yields

$$\frac{\alpha \, (a - by_w - c_1 + 1/\alpha)^2}{8b} = \frac{\alpha \, (a - 2c_1 + c_2 + 2/\alpha)^2}{32 \, b} > \frac{a - 2c_1 + c_2 + 2/\alpha}{3b}$$

and, in view of (3.2.16),

$$a - 2c_1 + c_2 > \frac{26}{3\alpha} \; , \tag{3.2.17}$$

which corresponds to the first condition in (3.2.5). Notice that (3.2.17) implies the first inequality in (3.2.16). Thus we conclude that, from (3.2.17) and the second condition in (3.2.16) (which are precisely (3.2.5)), the validity of $(C1), (C2), (C3')$ and $(C4)$ follows.
The proof is complete. $\qquad\square$

Chapter 4

Examples from the ODEs

In the present chapter we apply the results from Part I to some nonlinear ODE models with periodic coefficients. In particular we deal with planar systems of the first order or with second order scalar equations: since any second order equation can be written as a system of two equations of the first order, in this introductory discussion we will generally refer to planar systems.

The systems we consider are studied through a combination of a careful but elementary phase-plane analysis with the results on chaotic dynamics for LTMs from Section 2.4. We recall that similar strategy has already been employed in [122, 170] with respect to second order equations. More precisely, we prove in a rigorous way (i.e. without the need of computer assistance, differently from several works on related topics [12, 14, 53, 55, 98, 109, 129, 165, 169, 173]) the presence of infinitely many periodic solutions for our systems, as well as of a chaotic behaviour in the sense of Definition 2.1.2 for the associated Poincaré map Ψ. In fact, a classical approach (see [79]) to show the existence of periodic solutions (harmonics or subharmonics) of non-autonomous differential systems like

$$\dot{\zeta} = f(t, \zeta), \tag{4.0.1}$$

where $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous vector field which is T-periodic in the time-variable, that is, $f(t + T, z) = f(t, z)$, $\forall (t, z) \in \mathbb{R} \times \mathbb{R}^N$, under the assumption of uniqueness of the solutions for the Cauchy problems, is based on the search of the fixed points for the Poincaré map $\Psi = \Psi_T$ or for its iterates, where

$$\Psi : z \mapsto \zeta(t_0 + T; t_0, z)$$

and $\zeta(\cdot\,; t_0, z)$ is the solution of (4.0.1) satisfying the initial condition $\zeta(t_0) = z \in \mathbb{R}^N$. As a consequence of the fundamental theory of ODEs, it turns out that Ψ is a homeomorphism of its domain (which is an open subset of $\mathbb{R}^N$) onto its image. Applying our Stretching Along the Paths method to Ψ, we are

led back to work with discrete dynamical systems. The difference with the examples from Chapter 3 is that, since the Poincaré map is a homeomorphism, it is possible to prove a semi-conjugacy to the two-sided Bernoulli shift (cf. Remark 2.1.5), while we recall that the controlling functions of the models in Chapter 3 are not injective.

We notice that the kind of chaos in Definition 2.1.2, when considered in relation to the case of the Poincaré map, looks similar to other ones detected in the literature on complex dynamics for ODEs with periodic coefficients (see, for instance, [33, 150]).

In more details, in Section 4.1 we furnish an application of the results on LTMs from Section 2.4 to a modified version of Volterra predator-prey model, in which a periodic harvesting is included [126]. Indeed, when the seasons with fishing alternate with the ones without harvesting in a periodic fashion, it is possible to prove the presence of chaotic features for the system, provided that we have the freedom to tune the switching times between the two regimes. Analogous conclusions could be drawn for those time-periodic planar Kolmogorov systems [76]

$$x' = X(t, x, y), \quad y' = Y(t, x, y),$$

which possess dynamical features similar to the ones of Volterra model.

On the other hand, in Section 4.2 we employ the tools from Section 2.4 to show a complex behaviour for a nonlinear second order scalar equation of the form

$$x'' + f(t, x) = 0,$$

with $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a map T-periodic in the t-variable and satisfying the Carathéodory assumptions. In particular, we will deal with a simplified version of the Lazer-McKenna suspension bridges model [120, 121], belonging to the class of the periodically perturbed Duffing equations

$$x'' + g(x) = p(t), \tag{4.0.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and $p(\cdot) : \mathbb{R} \to \mathbb{R}$ is a locally integrable function such that $p(t + T) = p(t)$, for almost every $t \in \mathbb{R}$. However, we stress that our approach is in principle applicable also to more general equations, including examples in which a term depending on x' is present in (4.0.2).

As a final remark, we recall that the detection of chaotic dynamics for ODEs is the "natural" field of applicability for the stretching along the paths method. Indeed its development has been motivated by the analysis in [116] of the Poincaré map associated to the nonlinear scalar Hill's type second order equation

$$x'' + a(t)g(x) = 0, \tag{4.0.3}$$

in the case that $a(t)$ is a sign-changing continuous weight and $g : \mathbb{R} \to \mathbb{R}$ is a function with superlinear growth at infinity. Looking in the phase-plane at the solutions $(x(t), x'(t))$ of (4.0.3), a stretching property along the paths was detected. Namely, two topological planar rectangles were found such that every path joining two opposite sides of any one of the two rectangles contained a subpath which was expanded by the flow across the same rectangle or the other one. This was the key lemma in [116] in order to prove the presence of a complicated oscillatory behaviour for the solutions of (4.0.3).

4.1 Predator-Prey Model

4.1.1 The effects of a periodic harvesting

The classical Volterra predator-prey model concerns the first order planar differential system

$$\begin{cases} x' = x(a - by) \\ y' = y(-c + dx), \end{cases} \tag{E_0}$$

where a, b, c, $d > 0$ are constant coefficients. The study of system (E_0) is confined to the open first quadrant $(\mathbb{R}_0^+)^2$ of the plane, since $x(t) > 0$ and $y(t) > 0$ represent the size (number of individuals or density) of the prey and the predator populations, respectively. Such model was proposed by Vito Volterra in 1926 answering D'Ancona's question about the percentage of selachians and food fish caught in the northern Adriatic Sea during a range of years covering the period of the World War I (see [26, 102] for a more detailed historical account).

System (E_0) is conservative and its phase-portrait is that of a global center at the point

$$P_0 := \left(\frac{c}{d}, \frac{a}{b} \right),$$

surrounded by periodic orbits (run in the counterclockwise sense), which are the level lines of the first integral

$$\mathcal{E}_0(x, y) := dx - c \log x + by - a \log y,$$

that we call "energy" in analogy to mechanical systems. The choice of the sign in the definition of the first integral implies that $\mathcal{E}_0(x, y)$ achieves a strict absolute minimum at the point P_0.

According to Volterra's analysis of (E_0), the average of a periodic solution $(x(t), y(t))$, evaluated over a time-interval corresponding to its natural period, coincides with the coordinates of the point P_0.

In order to include the effects of fishing in the model, one can suppose that, during the harvesting time, both the prey and the predator populations are reduced at a rate proportional to the size of the population itself. This assumption leads to the new system

$$\begin{cases} x' = x(a_\mu - by) \\ y' = y(-c_\mu + dx), \end{cases} \qquad (E_\mu)$$

where

$$a_\mu := a - \mu \quad \text{and} \quad c_\mu := c + \mu$$

are the modified growth coefficients which take into account the fishing rates $-\mu x(t)$ and $-\mu y(t)$, respectively. The parameter μ is assumed to be positive but small enough ($\mu < a$) in order to prevent the extinction of the populations. System (E_μ) has the same form like (E_0) and hence its phase-portrait is that of a global center at

$$P_\mu := \left(\frac{c + \mu}{d}, \frac{a - \mu}{b} \right).$$

The periodic orbits surrounding P_μ are the level lines of the first integral

$$\mathcal{E}_\mu(x, y) := dx - c_\mu \log x + by - a_\mu \log y.$$

The coordinates of P_μ coincide with the average values of the prey and the predator populations under the effect of fishing (see Figure 4.1). A comparison between the coordinates of P_0 and P_μ motivates the conclusion (*Volterra's principle*) that a moderate harvesting has a favorable effect for the prey population [26].

If we are interested in incorporating the consequences of a cyclic environment in the Volterra original model (E_0), we can assume a seasonal effect on the coefficients, which leads us to consider a system of the form

$$\begin{cases} x' = x(a(t) - b(t)y) \\ y' = y(-c(t) + d(t)x), \end{cases} \qquad (E)$$

where $a(\cdot), b(\cdot), c(\cdot), d(\cdot) : \mathbb{R} \to \mathbb{R}$ are periodic functions with a common period $T > 0$. In such a framework, it is natural to look for harmonic (i.e. T-periodic) or m-th order subharmonic (i.e. mT-periodic, for some integer $m \geq 2$, with mT the minimal period in the set $\{jT : j = 1, 2, \dots\}$) solutions with range in the open first quadrant (*positive solutions*).

The past forty years have witnessed a growing attention towards such kind of models and several results have been obtained about the existence, multiplicity and stability of periodic solutions for Lotka-Volterra type predator-prey
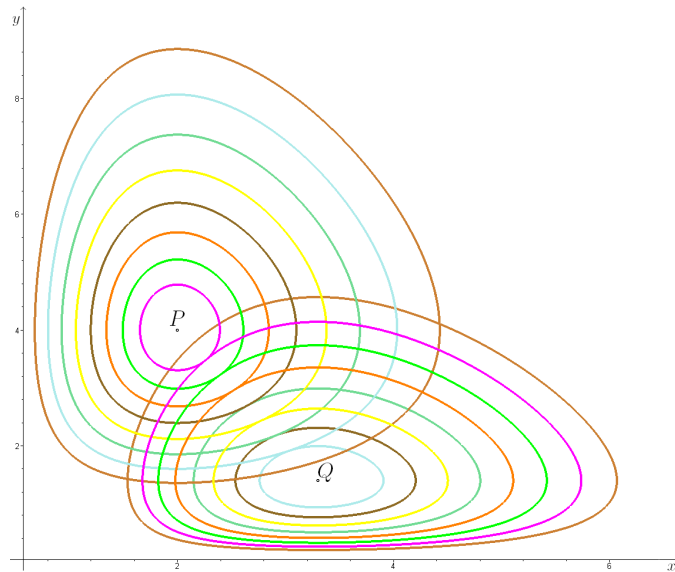
Figure 4.1: In this picture we show some periodic orbits of the Volterra system (E_0) with center at $P = P_0$, as well as of the perturbed system (E_μ) with center at $Q = P_\mu$ (for a certain $\mu \in \,]0, a[$).

systems with periodic coefficients [6, 29, 31, 38, 43, 44, 60, 61, 97, 156]. Further references are available in [126], where the reader can also find historical details about the fortune of this model as well as some related results from [43, 44].

Let us come back for a moment to the original Volterra system with constant coefficients and suppose that the interaction between the two populations is governed by system (E_0) for a certain period of the season (corresponding to a time-interval of length r_0) and by system (E_μ) for the rest of the season (corresponding to a time-interval of length r_μ). Assume also that such alternation between (E_0) and (E_μ) occurs in a periodic fashion, so that

$$T := r_0 + r_\mu$$

is the period of the season. In other terms, first we consider system (E_0) for $t \in [0, r_0[$. Next we switch to system (E_μ) at time r_0 and assume that (E_μ) rules the dynamics for $t \in [r_0, T[$. Finally, we suppose that we switch back to system (E_0) at time $t = T$ and repeat the cycle with T-periodicity.

Such two-state alternating behaviour can be equivalently described in terms of system (E), by assuming

$$a(t) = \hat{a}_\mu(t) := \begin{cases} a & \text{for } 0 \le t < r_0 \, , \\ a - \mu & \text{for } r_0 \le t < T \, , \end{cases}$$

$$c(t) = \hat{c}_\mu(t) := \begin{cases} c & \text{for } 0 \le t < r_0\,, \\ c + \mu & \text{for } r_0 \le t < T\,, \end{cases}$$

as well as

$$b(t) \equiv b, \;\; d(t) \equiv d,$$

with a, b, c, d positive constants and μ a parameter with $0 < \mu < a$. Hence we can consider the system

$$\begin{cases} x' = x(\hat{a}_\mu(t) - by) \\ y' = y(-\hat{c}_\mu(t) + dx), \end{cases} \tag{E^*}$$

where the piecewise constant functions $\hat{a}_\mu$ and $\hat{c}_\mu$ are supposed to be extended to the whole real line by T-periodicity.

It is our aim now to prove that (E^*) generates chaotic dynamics in the sense of Definition 2.1.2. To this end, as explained at the beginning of Chapter 4, we apply the results on LTMs from Section 2.4 to the Poincaré map

$$\Psi : (\mathbb{R}_0^+)^2 \to (\mathbb{R}_0^+)^2, \quad \Psi(z) := \zeta(T, z),$$

where $\zeta(\cdot, z) = (x(\cdot, z), y(\cdot, z))$ is the solution of system (E^*) starting from $z = (x_0, y_0) \in (\mathbb{R}_0^+)^2$ at the time $t = 0$. As a consequence, we will have ensured all the chaotic features listed in Theorem 2.1.6 (like, for instance, a semi-conjugacy to the two-sided Bernoulli shift and thus positive topological entropy, sensitivity with respect to initial conditions, topological transitivity, a compact invariant set containing a dense subset of periodic points).

With the aid of Figure 4.2 we try to explain how to enter the setting of Theorem 2.4.4 for the switching system (E^*).

As a first step, we take two closed overlapping annuli consisting of level lines of the first integrals associated to system (E_0) and (E_μ), respectively. In particular, the inner and outer boundaries of each annulus are closed trajectories surrounding the equilibrium point ($P = P_0$ for system (E_0) and $Q = P_\mu$ for system (E_μ)). Such annuli, that we call from now on $\mathcal{A}_P$ and $\mathcal{A}_Q$, intersect in two compact disjoint sets $\mathcal{R}_1$ and $\mathcal{R}_2$, which are generalized rectangles. The way in which we label the two regions (as $\mathcal{R}_1/\mathcal{R}_2$) is completely arbitrary. However, the choice of an order will effect some details in the argument that we describe below. Whenever we enter a framework like that visualized in Figure 4.2, we say that the annuli $\mathcal{A}_P$ and $\mathcal{A}_Q$ are *linked together*. Technical conditions on the energy level lines defining $\mathcal{A}_P$ and $\mathcal{A}_Q$, sufficient to guarantee the linking between them, are presented in Subsection 4.1.2.

As a second step, we give an "orientation" to $\mathcal{R}_i$ (for $i = 1, 2$) by selecting the boundary sets $\mathcal{R}_i^- = \mathcal{R}_{i,\ell}^- \cup \mathcal{R}_{i,r}^-$. In the specific example of Figure 4.2, we take as $\mathcal{R}_1^-$ the intersection of $\mathcal{R}_1$ with the inner and outer boundaries of $\mathcal{A}_P$ and
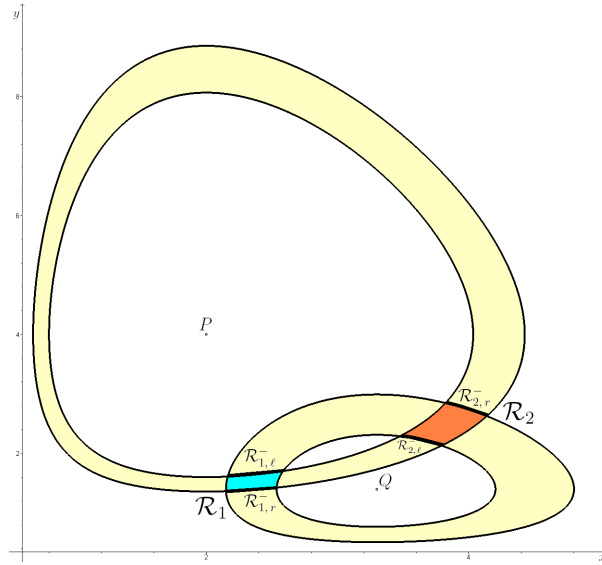
Figure 4.2: The two annular regions $\mathcal{A}_P$ and $\mathcal{A}_Q$ (centered at P and Q, respectively) are linked together. We have drawn with a darker color the two rectangular sets $\mathcal{R}_1$ and $\mathcal{R}_2$ where they meet. The boundary sets $\mathcal{R}_1^- = \mathcal{R}_{1,\ell}^- \cup \mathcal{R}_{1,r}^-$ and $\mathcal{R}_2^- = \mathcal{R}_{2,\ell}^- \cup \mathcal{R}_{2,r}^-$ have been indicated with a thicker line.

as $\mathcal{R}_2^-$ the intersection of $\mathcal{R}_2$ with the inner and outer boundaries of $\mathcal{A}_Q$. The way in which we name (as left/right) the two components of $\mathcal{R}_i^-$ is inessential for the rest of the discussion. Just to fix ideas, let us say that we choose as $\mathcal{R}_{1,\ell}^-$ the component of $\mathcal{R}_1^-$ which is closer to P and as $\mathcal{R}_{2,\ell}^-$ the component of $\mathcal{R}_2^-$ which is closer to Q (of course, the "right" components $\mathcal{R}_{1,r}^-$ and $\mathcal{R}_{2,r}^-$ are the remaining ones).

As a third step, we observe that the Poincaré map associated to (E^*) can be decomposed as

$$\Psi = \Psi_\mu \circ \Psi_0\,,$$

where Ψ_0 is the Poincaré map of system (E_0) on the time-interval $[0, r_0]$ and Ψ_μ is the Poincaré map for (E_μ) on the time-interval $[0, r_\mu] = [0, T - r_0]$. Consider a path $\gamma : [0, 1] \to \mathcal{R}_1$ with $\gamma(0) \in \mathcal{R}_{1,\ell}^-$ and $\gamma(1) \in \mathcal{R}_{1,r}^-$. As we will see in Subsection 4.1.2, the points of $\mathcal{R}_{1,\ell}^-$ move faster than those belonging to $\mathcal{R}_{1,r}^-$ under the action of system (E_0). Hence, for a choice of the first switching time r_0 large enough, it is possible to make the path

$$[0, 1] \ni s \mapsto \Psi_0(\gamma(s))$$

turn in a spiral-like fashion inside the annulus $\mathcal{A}_P$ and cross at least twice the rectangular region $\mathcal{R}_2$ from $\mathcal{R}_{2,\ell}^-$ to $\mathcal{R}_{2,r}^-$. Thus, we can select two subintervals of $[0, 1]$ such that $\Psi_0 \circ \gamma$ restricted to each of them is a path contained in

$\mathcal{R}_2$ and connecting the two components of $\mathcal{R}_2^-$. We observe that the points of $\mathcal{R}_{2,\ell}^-$ move faster than those belonging to $\mathcal{R}_{2,r}^-$ under the action of system (E_μ). Therefore, we can repeat the same argument as above and conclude that, for a suitable choice of $r_\mu = T - r_0$ sufficiently large, we can transform, via Ψ_μ, any path in $\mathcal{R}_2$ joining the two components of $\mathcal{R}_2^-$ onto a path which crosses at least once $\mathcal{R}_1$ from $\mathcal{R}_{1,\ell}^-$ to $\mathcal{R}_{1,r}^-$.

As a final step, we complete the proof about the existence of chaotic-like dynamics by applying Theorem 2.4.1. Actually, in order to obtain a more complex behaviour, in place of Theorem 2.4.1 it is possible to employ the more general Theorem 2.4.4.

In fact, our main result can be stated as follows.

Theorem 4.1.1. *For any choice of positive constants a, b, c, d, μ with $\mu < a$ and for every pair $(\mathcal{A}_P, \mathcal{A}_Q)$ of linked together annuli, the following conclusion holds:*

For every integer $m \geq 2$, there exist two positive constants α and β such that, for each

$$r_0 > \alpha \quad and \quad r_\mu > \beta,$$

the Poincaré map associated to system (E^*) induces chaotic dynamics on m symbols in $\mathcal{R}_1$ and $\mathcal{R}_2$.

As remarked in [120], if we consider Definition 2.1.2 and its consequences in the context of concrete examples of ODEs (for instance when ψ turns out to be the Poincaré map), condition (2.1.2) may sometimes be interpreted in terms of the oscillatory behaviour of the solutions. Such situation occurred in [117, 121] and takes place also for system (E^*). Indeed, as it will be clear from the proof of Theorem 4.1.1, it is possible to draw more precise conclusions in the statement of our main result. Namely, the following additional properties can be obtained:

For every decomposition of the integer $m \geq 2$ as

$$m = m_1 m_2, \quad with \ m_1, m_2 \in \mathbb{N},$$

there exist integers $\kappa_1, \kappa_2 \geq 1$ (with $\kappa_1 = \kappa_1(r_0, m_1)$ and $\kappa_2 = \kappa_2(r_\mu, m_2)$) such that, for each two-sided sequence of symbols

$$\mathbf{s} = (s_i)_{i \in \mathbb{Z}} = (p_i, q_i)_{i \in \mathbb{Z}} \in \{0, \ldots, m_1 - 1\}^{\mathbb{Z}} \times \{0, \ldots, m_2 - 1\}^{\mathbb{Z}},$$

there exists a solution

$$\zeta_{\mathbf{s}}(\cdot) = \big(x_{\mathbf{s}}(\cdot), y_{\mathbf{s}}(\cdot)\big)$$

of (E^*) with $\zeta_{\mathbf{s}}(0) \in \mathcal{R}_1$ such that $\zeta_{\mathbf{s}}(t)$ crosses $\mathcal{R}_2$ exactly $\kappa_1 + p_i + 1$ times for $t \in]iT, r_0 + iT[$ and crosses $\mathcal{R}_1$ exactly $\kappa_2 + q_i + 1$ times for $t \in]r_0 + iT, (i+1)T[$.

Moreover, if $(s_i)_{i \in \mathbb{Z}} = (p_i, q_i)_{i \in \mathbb{Z}}$ is a periodic sequence, that is, $s_{i+k} = s_i$, for some $k \geq 1$, then $\zeta_{\mathbf{s}}(t + kT) = \zeta_{\mathbf{s}}(t)$, $\forall\, t \in \mathbb{R}$.
In particular, taking $m = m_1 \geq 2$ and $m_2 = 1$, we obtain the dynamics on the set of m symbols $\{0, \ldots, m-1\} \equiv \{0, \ldots, m-1\} \times \{0\}$ as in the statement of Theorem 4.1.1.

The solutions $(x(t), y(t))$ are meant in the Carathéodory sense, i.e. $(x(t), y(t))$ is absolutely continuous and satisfies system (E^*) for almost every $t \in \mathbb{R}$. Of course, such solutions are of class C^1 if the coefficients are continuous.

The constants α and β in Theorem 4.1.1, representing the lower bounds for r_0 and r_μ, can be estimated in terms of m_1 and m_2 and other geometric parameters, such as the fundamental periods of the orbits bounding the linked annuli (see (4.1.1) and (4.1.4)).

We end this introductory discussion with a few observations about our main result.

First of all we notice that, according to Theorem 4.1.1, there is an abundance of chaotic regimes for system (E^*), provided that the time-interval lengths r_0 and r_μ (and, consequently, the period T) are sufficiently large. More precisely, we are able to prove the existence of chaotic invariant sets inside each intersection of two annular regions linked together. One could conjecture the presence of Smale horseshoes contained in such intersections, like in the classical case of linked twist maps with circular domains [40], even if we recall that our purely topological approach just requires to check a twist hypothesis on the boundary, without the need of verifying any hyperbolicity condition. This does not prevent the possibility of a further deeper analysis using more complex computations.

We also stress that our result is stable with respect to small perturbations of the coefficients. In fact, as it will emerge from the proof, whenever $r_0 > \alpha$ and $r_\mu > \beta$ are chosen so as to achieve the conclusion of Theorem 4.1.1, it follows that there exists a constant $\varepsilon > 0$ such that Theorem 4.1.1 applies to equation (E) too, provided that

$$\int_0^T |a(t) - \hat{a}_\mu(t)| \, dt < \varepsilon, \quad \int_0^T |c(t) - \hat{c}_\mu(t)| \, dt < \varepsilon,$$

$$\int_0^T |b(t) - b| \, dt < \varepsilon, \quad \int_0^T |d(t) - d| \, dt < \varepsilon.$$

Here the T-periodic coefficients may be in $L^1([0, T])$ or even continuous or smooth functions, possibly of class C^∞.

A final remark concerns the fact that, in our model, we have assumed that the harvesting period starts and ends for both the species at the same moment.

With this respect, one could face a more general situation in which some phase-shift between the two harvesting intervals occurs. Such cases have been already explored in some biological models, mostly from a numerical point of view: see [113] for an example on competing species and [136] for a predator-prey system. If we assume a phase-shift in the periodic coefficients, that is, if we consider

$$a(t) := \hat{a}_\mu(t - \theta_1) \quad \text{and} \quad c(t) := \hat{c}_\mu(t - \theta_2),$$

for some $0 < \theta_1, \theta_2 < T$, and we also suppose that the length r_0 of the time-intervals without harvesting may differ for the two species (say $r_0 = r_a \in]0, T[$ in the definition of $\hat{a}_\mu$ and $r_0 = r_c \in]0, T[$ in the definition of $\hat{c}_\mu$), then the geometry of our problem turns out to be a combination of linked twist maps on two, three or four annuli, which are mutually linked together. In this manner, we increase the possibility of chaotic configurations, provided that the system is subject to the different regimes for sufficiently long time. For a pictorial comment, see Figure 4.3, where all the possible links among four annuli are realized.
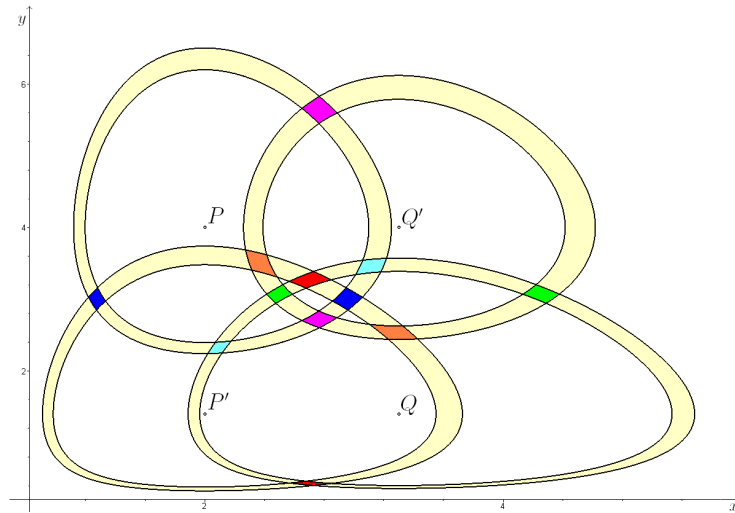
Figure 4.3: We have depicted four linked annular regions bounded by energy level lines corresponding to Volterra systems with centers at $P = P_0$, $P' = (c/d, a_\mu/b)$, $Q = P_\mu$ and $Q' = (c_\mu/d, a/b)$, by putting in evidence the regions of mutual intersection, where it is possible to locate the chaotic invariant sets.

4.1.2 Technical details and proofs

Let us consider system (E_0) and let

$$\ell > \chi_0 := \mathcal{E}_0(P_0) = \min\{\mathcal{E}_0(x, y) : x > 0, y > 0\}.$$

The level line
$$\Gamma_0(\ell) := \left\{ (x, y) \in (\mathbb{R}_0^+)^2 : \mathcal{E}_0(x, y) = \ell \right\}$$

is a closed orbit (surrounding P_0) which is run counterclockwise, completing one turn in a fundamental period that we denote by $\tau_0(\ell)$. According to classical results on the period of the Lotka-Volterra system [140, 159], the map

$$\tau_0 :]\chi_0, +\infty[\to \mathbb{R}$$

is strictly increasing with $\tau_0(+\infty) = +\infty$ and satisfies

$$\lim_{\ell \to \chi_0^+} \tau_0(\ell) = T_0 := \frac{2\pi}{\sqrt{ac}}.$$

Similarly, considering system (E_μ) with $0 < \mu < a$, we denote by $\tau_\mu(h)$ the minimal period associated to the orbit

$$\Gamma_\mu(h) := \left\{ (x, y) \in (\mathbb{R}_0^+)^2 : \mathcal{E}_\mu(x, y) = h \right\},$$

for
$$h > \chi_\mu := \mathcal{E}_\mu(P_\mu) = \min\{\mathcal{E}_\mu(x, y) : x > 0, y > 0\}.$$

Also in this case the map $h \mapsto \tau_\mu(h)$ is strictly increasing with $\tau_\mu(+\infty) = +\infty$ and

$$\lim_{h \to \chi_\mu^+} \tau_\mu(h) = T_\mu := \frac{2\pi}{\sqrt{a_\mu c_\mu}}.$$

Before giving the details for the proof of our main result, we describe conditions on the energy level lines of two annuli $\mathcal{A}_P$ and $\mathcal{A}_Q$, centered at $P = P_0 = \left(\frac{c}{d}, \frac{a}{b} \right)$ and $Q = P_\mu = \left(\frac{c+\mu}{d}, \frac{a-\mu}{b} \right)$, respectively, sufficient to ensure that they are linked together. With this respect, we have to consider the intersections among the closed orbits around the two equilibria and the straight line r passing through the points P and Q, whose equation is $by + dx - a - c = 0$. We introduce an orientation on such line by defining an order "$\preceq$" among its points. More precisely, we set $A \preceq B$ (resp. $A \prec B$) if and only if $x_A \leq x_B$ (resp. $x_A < x_B$), where $A = (x_A, y_A)$, $B = (x_B, y_B)$. In this manner, the order on r is that inherited from the oriented x-axis, by projecting the points of r onto the abscissa. Assume now we have two closed orbits $\Gamma_0(\ell_1)$ and $\Gamma_0(\ell_2)$ for system (E_0), with $\chi_0 < \ell_1 < \ell_2$. Let us call the intersection points among

r and such level lines $P_{1,-}, P_{1,+}$, with reference to ℓ_1, and $P_{2,-}, P_{2,+}$, with reference to ℓ_2, where

$$P_{2,-} \prec P_{1,-} \prec P \prec P_{1,+} \prec P_{2,+}.$$

Analogously, when we consider two orbits $\Gamma_\mu(h_1)$ and $\Gamma_\mu(h_2)$ for system (E_μ), with $\chi_\mu < h_1 < h_2$, we name the intersection points among r and these level lines $Q_{1,-}, Q_{1,+}$, with reference to h_1, and $Q_{2,-}, Q_{2,+}$, with reference to h_2, where

$$Q_{2,-} \prec Q_{1,-} \prec Q \prec Q_{1,+} \prec Q_{2,+}.$$

Then the two annuli $\mathcal{A}_P$ and $\mathcal{A}_Q$ turn out to be linked together if

$$P_{2,-} \prec P_{1,-} \preceq Q_{2,-} \prec Q_{1,-} \preceq P_{1,+} \prec P_{2,+} \preceq Q_{1,+} \prec Q_{2,+}.$$

Proof of Theorem 4.1.1. Consistently with the notation introduced above, we denote by $\Gamma_0(\ell)$, with $\ell \in [\ell_1, \ell_2]$, for some $\chi_0 < \ell_1 < \ell_2$, the level lines filling $\mathcal{A}_P$, so that

$$\mathcal{A}_P = \bigcup_{\ell_1 \leq \ell \leq \ell_2} \Gamma_0(\ell).$$

Analogously, we indicate the level lines filling $\mathcal{A}_Q$ by $\Gamma_\mu(h)$, with $h \in [h_1, h_2]$, for some $\chi_\mu < h_1 < h_2$, so that we can write

$$\mathcal{A}_Q = \bigcup_{h_1 \leq h \leq h_2} \Gamma_\mu(h).$$

By construction, such annular regions turn out to be invariant for the dynamical systems generated by (E_0) and (E_μ), respectively. We consider now the two regions in which each annulus is cut by the line r (passing through P and Q) and we call such sets $\mathcal{A}_P^t, \mathcal{A}_P^b, \mathcal{A}_Q^t$ and $\mathcal{A}_Q^b$, in order to have $\mathcal{A}_P = \mathcal{A}_P^t \cup \mathcal{A}_P^b$ and $\mathcal{A}_Q = \mathcal{A}_Q^t \cup \mathcal{A}_Q^b$, where the sets with superscript t are the "upper" ones and the sets with superscript b are the "lower" ones, with respect to the line r. We name $\mathcal{R}^b$ the rectangular region in which $\mathcal{A}_P^b$ and $\mathcal{A}_Q^b$ meet and analogously we denote by $\mathcal{R}^t$ the rectangular region belonging to the intersection between $\mathcal{A}_P^t$ and $\mathcal{A}_Q^t$ (see Figure 4.4).
Let

$$m_1 \geq 2 \quad \text{and} \quad m_2 \geq 1$$

be two fixed integers. The case $m_1 = 1$ and $m_2 \geq 2$ can be treated in a similar manner and therefore is omitted.
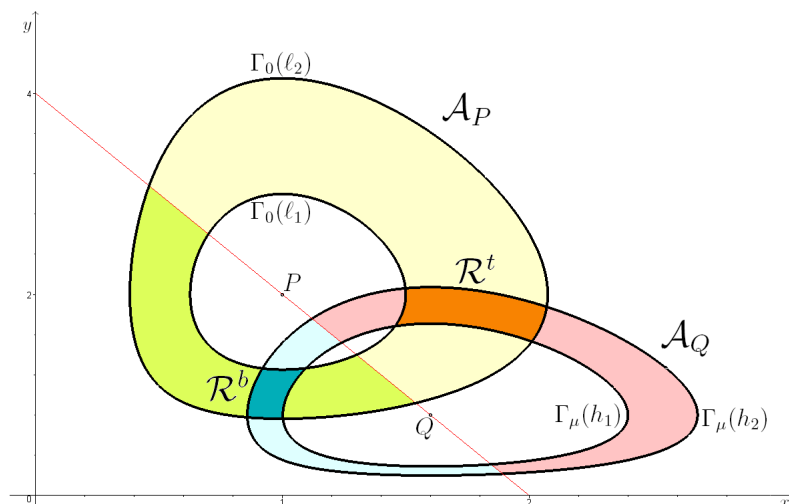As a first step, we are interested in the solutions of system (E_0) starting from

Figure 4.4: For the two linked annuli in the picture (one around P and the other around Q), we have drawn in different colors the upper and lower parts (with respect to the dashed line r), as well as the intersection regions $\mathcal{R}^t$ and $\mathcal{R}^b$ between them. As a guideline for the proof, we recall that $\mathcal{A}_P$ is the annulus around P, having as inner and outer boundaries the energy level lines $\Gamma_0(\ell_1)$ and $\Gamma_0(\ell_2)$. Similarly, $\mathcal{A}_Q$ is the annulus around Q, having as inner and outer boundaries the energy level lines $\Gamma_\mu(h_1)$ and $\Gamma_\mu(h_2)$.

$\mathcal{A}_P^b$ and crossing $\mathcal{A}_P^t$ at least m_1 times. After having performed the rototranslation of the plane $\mathbb{R}^2$ that brings the origin to the point P and makes the x-axis coincide with the line r, whose equations are

$$\begin{cases} \tilde{x} = (x - \frac{c}{d}) \cos \vartheta + (y - \frac{a}{b}) \sin \vartheta \\ \tilde{y} = (\frac{c}{d} - x) \sin \vartheta + (y - \frac{a}{b}) \cos \vartheta, \end{cases}$$

where $\vartheta := \arctan(\frac{d}{b})$, it is possible to use the Prüfer transformation and introduce generalized polar coordinates, so that we can express the solution $\zeta(\cdot, z) = (x(\cdot, z), y(\cdot, z))$ of system (E_0) with initial point in $z = (x_0, y_0) \in \mathcal{A}_P^b$ through the radial coordinate $\rho(t, z)$ and the angular coordinate $\theta(t, z)$. Thus we can assume that $\theta(0, z) \in [-\pi, 0]$. For any $t \in [0, r_0]$ and $z \in \mathcal{A}_P^b$, let us introduce also the *rotation number*, that is the quantity

$$\text{rot}_0(t, z) := \frac{\theta(t, z) - \theta(0, z)}{2\pi},$$

that indicates the normalized angular displacement along the orbit of system (E_0) starting at z, during the time-interval $[0, t]$. The continuous dependence of the solutions from the initial data implies that the function $(t, z) \mapsto \theta(t, z)$ and consequently the map $(t, z) \mapsto \text{rot}_0(t, z)$ are continuous. From the definition of

rotation number and the star-shapedness of the level lines of $\mathcal{E}_0$ with respect to the point P, we have that for every $z \in \Gamma_0(\ell)$ the following properties hold:

$$\forall\, j \in \mathbb{Z} : \;\; \mathrm{rot}_0(t, z) = j \qquad\qquad \Longleftrightarrow\; t = j\,\tau_0(\ell)\,,$$
$$\forall\, j \in \mathbb{Z} : \;\; j < \mathrm{rot}_0(t, z) < j + 1 \;\; \Longleftrightarrow\; j\,\tau_0(\ell) < t < (j + 1)\,\tau_0(\ell)$$

(if the annuli were not star-shaped, the inference " $\Longleftarrow$ " would still be true). Although we have implicitly assumed that $\ell_1 \leq \ell \leq \ell_2$, such properties hold for every $\ell > \chi_0$.

Observe that, thanks to the fact that the time-map τ_0 is strictly increasing, we know that $\tau_0(\ell_1) < \tau_0(\ell_2)$. We shall use this condition to show that a twist property for the rotation number holds for sufficiently large time-intervals. Indeed we claim that, if we choose a switching time $r_0 \geq \alpha$, where

$$\alpha := \frac{(m_1 + 3 + \frac{1}{2})\,\tau_0(\ell_1)\,\tau_0(\ell_2)}{\tau_0(\ell_2) - \tau_0(\ell_1)}\,, \qquad\qquad (4.1.1)$$

then, for any path $\gamma : [0, 1] \to \mathcal{A}_P$, with $\gamma(0) \in \Gamma_0(\ell_1)$ and $\gamma(1) \in \Gamma_0(\ell_2)$, the interval inclusion

$$[\theta(r_0, \gamma(1)), \theta(r_0, \gamma(0))] \supseteq [2\pi n^*, 2\pi(n^* + m_1) - \pi] \qquad (4.1.2)$$

is fulfilled for some $n^* = n^*(r_0) \in \mathbb{N}$. To check our claim, at first we notice that, for a path $\gamma(s)$ as above, it holds that $\mathrm{rot}_0(t, \gamma(0)) \geq \lfloor t/\tau_0(\ell_1) \rfloor$ and $\mathrm{rot}_0(t, \gamma(1)) \leq \lceil t/\tau_0(\ell_2) \rceil$, for every $t > 0$, and so

$$\mathrm{rot}_0(t, \gamma(0)) - \mathrm{rot}_0(t, \gamma(1)) > t\,\frac{\tau_0(\ell_2) - \tau_0(\ell_1)}{\tau_0(\ell_1)\,\tau_0(\ell_2)} - 2, \quad \forall\, t > 0.$$

Hence, for $t \geq \alpha$, with α defined as in (4.1.1), we obtain

$$\mathrm{rot}_0(t, \gamma(0)) > m_1 + 1 + \tfrac{1}{2} + \mathrm{rot}_0(t, \gamma(1))\,,$$

which, in turns, implies

$$\theta(t, \gamma(0)) - \theta(t, \gamma(1)) > 2\pi(m_1 + 1), \quad \forall\, t \geq \alpha.$$

Therefore, recalling the bound $2\pi(\lceil t/\tau_0(\ell_2) \rceil - \frac{3}{2}) < \theta(t, \gamma(1)) \leq 2\pi\lceil t/\tau_0(\ell_2) \rceil$, the interval inclusion (4.1.2) is achieved for

$$n^* = n^*(r_0) := \left\lceil \frac{r_0}{\tau_0(\ell_2)} \right\rceil.$$

This proves our claim.

By the continuity of the composite mapping $[0,1] \ni s \mapsto \text{rot}_0(r_0, \gamma(s))$, it follows that

$$\{\theta(r_0, \gamma(s)), \, s \in [0,1]\} \supseteq [2\pi n^*, 2\pi(n^* + m_1 - 1) + \pi].$$

As a consequence, by the Bolzano Theorem, there exist m_1 pairwise disjoint maximal intervals $[t_i', t_i''] \subseteq [0,1]$, for $i = 0, \ldots, m_1 - 1$, such that

$$\theta(r_0, \gamma(s)) \in [2\pi n^* + 2\pi i, 2\pi n^* + \pi + 2\pi i], \, \forall s \in [t_i', t_i''], \, i = 0, \ldots, m_1 - 1,$$

with $\theta(r_0, \gamma(t_i')) = 2\pi n^* + 2\pi i$ and $\theta(r_0, \gamma(t_i'')) = 2\pi n^* + \pi + 2\pi i$. Setting

$$\mathcal{R}_1 := \mathcal{R}^b \quad \text{and} \quad \mathcal{R}_2 := \mathcal{R}^t,$$

we orientate such rectangular regions by choosing

$$\mathcal{R}_{1,\ell}^- := \mathcal{R}_1 \cap \Gamma_0(\ell_1) \quad \text{and} \quad \mathcal{R}_{1,r}^- := \mathcal{R}_1 \cap \Gamma_0(\ell_2),$$

as well as

$$\mathcal{R}_{2,\ell}^- := \mathcal{R}_2 \cap \Gamma_\mu(h_1) \quad \text{and} \quad \mathcal{R}_{2,r}^- := \mathcal{R}_2 \cap \Gamma_\mu(h_2).$$

See the caption of Figure 4.4 as a reminder for the corresponding sets. Introducing at last the m_1 nonempty and pairwise disjoint compact sets

$$\mathcal{H}_i := \left\{ z \in \mathcal{A}_P^b : \theta(r_0, z) \in [2\pi n^* + 2\pi i, 2\pi n^* + \pi + 2\pi i] \right\},$$

for $i = 0, \ldots, m_1 - 1$, we are ready to prove that

$$(\mathcal{H}_i, \Psi_0) : \widetilde{\mathcal{R}}_1 \Leftrightarrow \widetilde{\mathcal{R}}_2, \quad \forall i = 0, \ldots, m_1 - 1, \tag{4.1.3}$$

where we recall that Ψ_0 is the Poincaré map associated to system (E_0). Indeed, let us take a path $\gamma : [0,1] \to \mathcal{R}_1$, with $\gamma(0) \in \mathcal{R}_{1,\ell}^-$ and $\gamma(1) \in \mathcal{R}_{1,r}^-$. For $r_0 \geq \alpha$ and fixing $i \in \{0, \ldots, m_1 - 1\}$, there is a subinterval $[t_i', t_i''] \subseteq [0,1]$, such that $\gamma(t) \in \mathcal{H}_i$ and $\Psi_0(\gamma(t)) \in \mathcal{A}_P^t$, $\forall t \in [t_i', t_i'']$. Noting that $\Gamma_\mu(\Psi_0(\gamma(t_i'))) \leq h_1$ and $\Gamma_\mu(\Psi_0(\gamma(t_i''))) \geq h_2$, then there is a subinterval $[t_i^*, t_i^{**}] \subseteq [t_i', t_i'']$, such that $\Psi_0(\gamma(t)) \in \mathcal{R}_2$, $\forall t \in [t_i^*, t_i^{**}]$, with $\Psi_0(\gamma(t_i^*)) \in \mathcal{R}_{2,\ell}^-$ and $\Psi_0(\gamma(t_i^{**})) \in \mathcal{R}_{2,r}^-$. Hence, condition (4.1.3) is verified.

Let us turn to system (E_μ). This time we focus our attention on the solutions of such system starting from $\mathcal{A}_Q^t$ and crossing $\mathcal{A}_Q^b$ at least m_2 times. Similarly as before, we assume to have performed a rototranslation of the plane that makes the x-axis coincide with the line r and that brings the origin to the point Q, so that we can express the solution $\zeta(\cdot, w)$ of system (E_μ) with starting point in $w \in \mathcal{A}_Q^t$ through polar coordinates $(\tilde{\rho}, \tilde{\theta})$. In particular, it holds that

$\tilde{\theta}(0, w) \in [0, \pi]$. For any $t \in [0, r_\mu] = [0, T - r_0]$ and $w \in \mathcal{A}_Q^t$, the rotation number is now defined as

$$\mathrm{rot}_\mu(t, w) := \frac{\tilde{\theta}(t, w) - \tilde{\theta}(0, w)}{2\pi}.$$

Since the time-map τ_μ is strictly increasing, it follows that $\tau_\mu(h_1) < \tau_\mu(h_2)$. We claim that, choosing a switching time $r_\mu \geq \beta$, with

$$\beta := \frac{(m_2 + 3 + \frac{1}{2}) \tau_\mu(h_1) \tau_\mu(h_2)}{\tau_\mu(h_2) - \tau_\mu(h_1)}, \tag{4.1.4}$$

then, for any path $\omega : [0, 1] \to \mathcal{A}_Q$, with $\omega(0) \in \Gamma_\mu(h_1)$ and $\omega(1) \in \Gamma_\mu(h_2)$, the interval inclusion

$$\left[\tilde{\theta}(r_\mu, \omega(1)), \tilde{\theta}(r_\mu, \omega(0)) \right] \supseteq [\pi(2n^{**} + 1), 2\pi(n^{**} + m_2)] \tag{4.1.5}$$

is satisfied for some $n^{**} = n^{**}(r_\mu) \in \mathbb{N}$. The claim can be proved with arguments analogous to the ones employed above and thus its verification is omitted. The nonnegative integer n^{**} has to be chosen as

$$n^{**} := \left\lceil \frac{r_\mu}{\tau_\mu(h_2)} \right\rceil.$$

By (4.1.5) and the continuity of the composite map $[0, 1] \ni s \mapsto \mathrm{rot}_\mu(r_\mu, \omega(s))$, it follows that

$$\left\{ \tilde{\theta}(r_\mu, \omega(s)), \ s \in [0, 1] \right\} \supseteq [2\pi n^{**} + \pi, 2\pi(n^{**} + m_2)].$$

As a consequence, Bolzano Theorem ensures the existence of m_2 pairwise disjoint maximal intervals $[s_i', s_i''] \subseteq [0, 1]$, for $i = 0, \ldots, m_2 - 1$, such that

$$\tilde{\theta}(r_\mu, \omega(s)) \in [2\pi n^{**} + \pi + 2\pi i, 2\pi n^{**} + 2\pi + 2\pi i], \ \forall s \in [s_i', s_i''], \ i = 0, \ldots, m_2 - 1,$$

with $\tilde{\theta}(r_\mu, \omega(s_i')) = 2\pi n^{**} + \pi + 2\pi i$ and $\tilde{\theta}(r_\mu, \omega(s_i'')) = 2\pi n^{**} + 2\pi + 2\pi i$. For $\widetilde{\mathcal{R}}_1$ and $\widetilde{\mathcal{R}}_2$ as above and introducing the m_2 nonempty, compact and pairwise disjoint sets

$$\mathcal{K}_i := \left\{ w \in \mathcal{A}_Q^t : \tilde{\theta}(r_\mu, w) \in [2\pi n^{**} + \pi + 2\pi i, 2\pi n^{**} + 2\pi + 2\pi i] \right\},$$

with $i = 0, \ldots, m_2 - 1$, we are in position to check that

$$(\mathcal{K}_i, \Psi_\mu) : \widetilde{\mathcal{R}}_2 \Leftcurlyeqarrow \widetilde{\mathcal{R}}_1, \forall \, i = 0, \ldots, m_2 - 1, \tag{4.1.6}$$

where Ψ_μ is the Poincaré map associated to system (E_μ). Indeed, taking a path $\omega : [0,1] \to \mathcal{R}_2$, with $\omega(0) \in \mathcal{R}_{2,\ell}^-$ and $\omega(1) \in \mathcal{R}_{2,r}^-$, for $r_\mu \geq \beta$ and for any $i \in \{0, \dots, m_2 - 1\}$ fixed, there exists a subinterval $[s_i', s_i''] \subseteq [0,1]$, such that $\omega(t) \in \mathcal{K}_i$ and $\Psi_\mu(\omega(t)) \in \mathcal{A}_Q^b$, $\forall t \in [s_i', s_i'']$. Since $\Gamma_0(\Psi_\mu(\omega(s_i'))) \leq \ell_1$ and $\Gamma_0(\Psi_\mu(\omega(s_i''))) \geq \ell_2$, there exists a subinterval $[s_i^*, s_i^{**}] \subseteq [s_i', s_i'']$ such that $\Psi_\mu(\omega(t)) \in \mathcal{R}_1$, $\forall t \in [s_i^*, s_i^{**}]$, with $\Psi_\mu(\omega(s_i^*)) \in \mathcal{R}_{1,\ell}^-$ and $\Psi_\mu(\omega(s_i^{**})) \in \mathcal{R}_{1,r}^-$. Hence, condition (4.1.6) is proved.

The stretching properties in (4.1.3) and (4.1.6) allow to apply Theorem 2.4.4 and the thesis follows immediately. $\qquad\square$

We observe that in the proof we have chosen as $\mathcal{R}_1$ the "lower" set $\mathcal{R}^b$ and as $\mathcal{R}_2$ the "upper" set $\mathcal{R}^t$. However, since the orbits of both systems (E_0) and (E_μ) are closed, the same argument works (by slightly modifying some constants, if needed) for $\mathcal{R}_1 = \mathcal{R}^t$ and $\mathcal{R}_2 = \mathcal{R}^b$.

4.2 Suspension Bridges Model

In this section we illustrate a possible application of the results from Section 2.4 to a periodically perturbed Duffing equation of the form

$$x'' + kx^+ = p_{q,s}(t), \tag{4.2.1}$$

where k, q, s, r_q, r_s are positive real numbers and $p_{q,s} : \mathbb{R} \to \mathbb{R}$ is a periodic function of period

$$T = T_{q,s} := r_q + r_s,$$

defined on $[0, T[$ by

$$p_{q,s}(t) := \begin{cases} q & \text{for } 0 \leq t < r_q, \\ -s & \text{for } r_q \leq t < r_q + r_s, \end{cases}$$

and then extended to the real line by T-periodicity.

Equation (4.2.1) can be regarded as a simplified version of the Lazer-McKenna suspension bridges model [88, 89] and it is also meaningful from the point of view of the study of ODEs with "jumping nonlinearities" and the periodic Dancer-Fučik spectrum. See [103] for an interesting survey concerning recent developments in such area.

The results that we are going to quote have been obtained in [121] and are recalled in [120], where only the main arguments employed in the proofs are presented, while the technical details are omitted. This is the scheme that we follow in the treatment below.

The main achievement in [121] concerns the existence of infinitely many periodic solutions as well as the presence of complex dynamics for equation (4.2.1).

Here we provide a simplified version of the original [121, Theorem 1.2], where precise information about the oscillatory behaviour of the solutions was given, too.

Theorem 4.2.1. *For any choice of $k, q, s > 0$ and for every positive integer $m \geq 2$, it is possible to find two intervals $]a_q^m, b_q^m[$ and $]a_s^m, b_s^m[$ such that, for every forcing term $p_{q,s}$ with $r_q \in]a_q^m, b_q^m[$ and $r_s \in]a_s^m, b_s^m[$, the Poincaré map associated to (4.2.1) induces chaotic dynamics on m symbols in the plane.*

We stress that, according to [121, Theorem 1.3], the above result is stable with respect to small perturbations in the sense that, once we have proved the existence of chaotic-like dynamics in a certain range of parameters for equation (4.2.1), then the same conclusions still hold for

$$x'' + cx' + kx^+ = p(t),$$

provided that $|c|$ and $\int_0^T |p(t) - p_{q,s}(t)| \, dt$ (with $T = r_q + r_s$) are sufficiently small. See [121] for more details.

We describe now the geometry associated to the phase-portrait of (4.2.1), in order to understand how to apply Theorem 2.4.1 from Section 2.4. Similar arguments could be also employed for equation

$$x'' + g(x) = p(t),$$

with g a monotone nondecreasing function bounded from below and such that $g'(+\infty) = k > 0$.

With reference to (4.2.1), first of all, we observe that the Poincaré map Ψ associated to the equivalent planar system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -kx^+ + p_{q,s}(t) \end{cases} \tag{4.2.2}$$

can be decomposed as

$$\Psi = \Psi_s \circ \Psi_q \,,$$

where Ψ_q and Ψ_s are the Poincaré maps

$$\Psi_q : z_0 \mapsto \zeta_q(r_q, z_0), \quad \Psi_s : z_0 \mapsto \zeta_s(r_s, z_0)$$

along the time intervals $[0, r_q]$ and $[0, r_s]$ associated to the dynamical systems defined by

$$(E_q) \qquad\qquad\qquad \begin{cases} \dot{x} = y \\ \dot{y} = -kx^+ + q \end{cases}$$

and

$$(E_s) \qquad\qquad \begin{cases} \dot{x} = y \\ \dot{y} = -kx^+ - s. \end{cases}$$

Here $\zeta_q(\cdot, z_0)$ denotes the solution $(x(\cdot), y(\cdot))$ of (E_q) with $(x(0), y(0)) = z_0$. Clearly, $\zeta_s(\cdot, z_0)$ has the same meaning with respect to system (E_s).
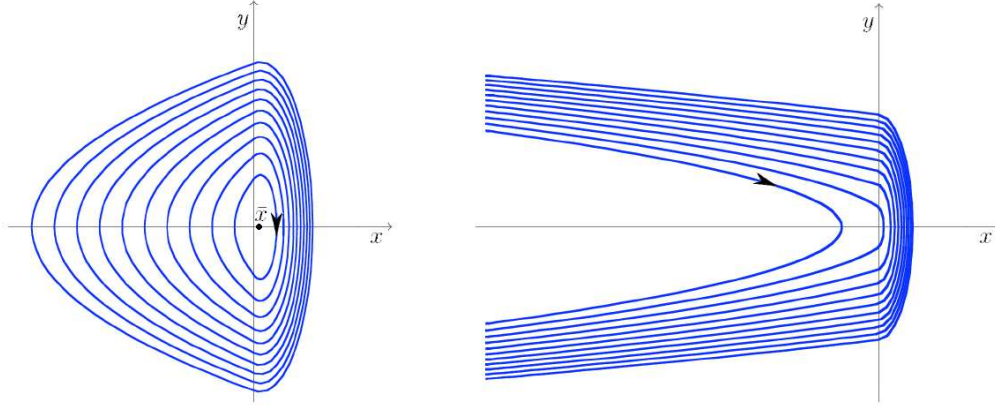
Figure 4.5: Trajectories of system (E_q) (left) and system (E_s) (right). In this picture we have chosen $k = 10$, $q = 4$ and $s = 0.5$.

The orbits of system (E_q) correspond to the level lines of the energy

$$\mathcal{E}_q(x, y) := \frac{1}{2}y^2 + \frac{k}{2}(x^+)^2 - qx.$$

Except for the equilibrium point $\bar{x} := (\frac{q}{k}, 0)$, they are closed curves which are run in the clockwise sense. Moreover, their fundamental period $\tau_q(\cdot)$ is a monotonically nondecreasing function with respect to the energy (in particular, strictly increasing on $[0, +\infty)$) and it limits to $+\infty$ as the energy tends to $+\infty$. On the other hand, the orbits of system (E_s) correspond to the level lines of the energy

$$\mathcal{E}_s(x, y) := \frac{1}{2}y^2 + \frac{k}{2}(x^+)^2 + sx$$

and they are run with $y' < 0$.

Since our trajectories will switch from system (E_q) to system (E_s) and vice versa, in order to study the effect of the forcing term $p_{q,s}(t)$, we are led to overlap the energy level lines of the two systems. Intersecting two level lines of (E_q) with two level lines of (E_s), we can determine two rectangular regions, like in Figure 4.6.

To get periodic and chaotic solutions for system (4.2.2) and thus for equation (4.2.1), we employ the following elementary procedure. We construct a set $\mathcal{R}_1$
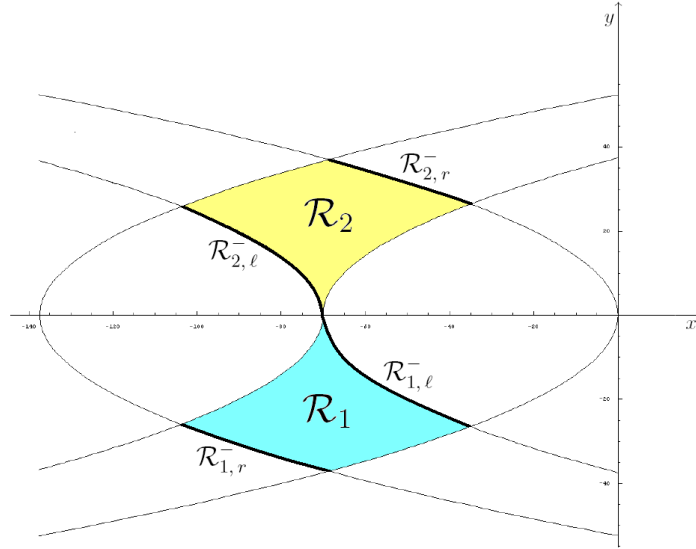
Figure 4.6: Level lines of system (E_q) overlapped with the level lines of system (E_s) in the left half plane $x \le 0$. The sets $\mathcal{R}_1$ and $\mathcal{R}_2$ are those bounded by the four different level lines and contained in the lower half plane $y \le 0$ and in the upper half plane $y \ge 0$, respectively. Orienting them by choosing as $\mathcal{R}_1^- = \mathcal{R}_{1,\ell}^- \cup \mathcal{R}_{1,r}^-$ and $\mathcal{R}_2^- = \mathcal{R}_{2,\ell}^- \cup \mathcal{R}_{2,r}^-$ the boundary sets drawn with a thicker line, we enter the setting of Theorem 2.4.1 and thus the presence of chaotic dynamics follows.

in the third quadrant bounded by two level lines of (E_q) and two level lines of (E_s). Since the period $\tau_q(\cdot)$ of the closed orbits of system (E_q) is a strictly monotone increasing function on $[0, +\infty)$, we define the two components $\mathcal{R}_{1,\ell}^-$ and $\mathcal{R}_{1,r}^-$ of $\mathcal{R}_1^-$ as the intersections of $\mathcal{R}_1$ with the level lines of $\mathcal{E}_q$, so that points on the inner boundary move faster than points on the outer boundary. Just to fix ideas, we choose as $\mathcal{R}_{1,\ell}^-$ the inner boundary and as $\mathcal{R}_{1,r}^-$ the outer boundary. Hence, if we take a path γ contained in the third quadrant and joining $\mathcal{R}_{1,\ell}^-$ to $\mathcal{R}_{1,r}^-$, we find that, after a sufficiently long time r_q, its image under Ψ_q turns out to be a spiral-like line crossing the second quadrant at least twice. In particular, we can select two subpaths ω_0 and ω_1 of γ, whose images under Ψ_q cross the second quadrant. Now we define a rectangular subset $\mathcal{R}_2$ in the second quadrant, which is just the reflection of $\mathcal{R}_1$ with respect to the x-axis, having as components of $\mathcal{R}_2^-$ the intersections of $\mathcal{R}_2$ with two level lines of $\mathcal{E}_s$. Selecting as $\mathcal{R}_{2,\ell}^-$ the intersection with the inner level line and as $\mathcal{R}_{2,r}^-$ the intersection with the outer one, it follows that $\Psi_q(\omega_0)$ and $\Psi_q(\omega_1)$ admit subpaths contained in $\mathcal{R}_2$ and joining $\mathcal{R}_{2,\ell}^-$ to $\mathcal{R}_{2,r}^-$. Switching to system (E_s), it is possible to prove that the image of such subpaths under Ψ_s crosses the set $\mathcal{R}_1$ from $\mathcal{R}_{1,\ell}^-$ to $\mathcal{R}_{1,r}^-$. Then we can conclude by applying Theorem 2.4.1,

provided we are free to choose the switching times t_q and t_s.

The technical details for this argument can be found in [121].

Similar conclusions can be drawn for the second order ODE

$$x'' + q(t)g(x) = 0,$$

with g a nonlinear term and $q(\cdot)$ a periodic step function (or a suitable small perturbation of a step function).

The examples of ODEs we have presented suggest the possibility of proving in an elementary but rigorous manner the presence of chaotic dynamics for a broad class of second order nonlinear ODEs with periodic coefficients, which are piecewise constant or at least not far from the piecewise constant case, as long as we have the freedom to tune some switching time parameters within a certain range depending on the coefficients governing the equations. With this respect, our approach shows some resemblances with different techniques based on more sophisticated tools, like for instance on modifications of the methods by Shilnikov or Melnikov, in which one looks for a transverse homoclinic point [16, 58, 115]. Indeed, also in such cases, it is possible to prove the presence of complex dynamics for differential equations, provided that the period of the time-dependent coefficients tends to infinity. See [4, 5, 19, 39], where alternative techniques are employed. In [63] the differentiability hypothesis on the Melnikov function is replaced with sign-changing conditions and the proofs make use of the Poincaré-Miranda Theorem.

We conclude by observing that the same arguments employed above seem to work also for more general time-dependent coefficients. However, a rigorous proof in such cases would require a more delicate analysis or possibly the aid of computer assistance (as in [12, 53, 109, 129, 165, 169, 173]) and this is beyond the aims of the present thesis.

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