SINGULARITY OF EIGENFUNCTIONS AT THE JUNCTION OF SHRINKING TUBES. PART I.

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Abstract. Consider two domains connected by a thin tube: it can be shown that, generically, the mass of a given eigenfunction of the Dirichlet Laplacian concentrates in only one of them. The restriction to the other domain, when suitably normalized, develops a singularity at the junction of the tube, as the channel section tends to zero. Our main result states that, under a nondegeneracy condition, the normalized limiting profile has a singularity of order $N-1$, where N is the space dimension. We give a precise description of the asymptotic behavior of eigenfunctions at the singular junction, which provides us with some important information about its sign near the tunnel entrance. More precisely, the solution is shown to be one-sign in a neighborhood of the singular junction. In other words, we prove that the nodal set does not enter inside the channel.

1. Introduction and statement of the main results

We are concerned with the behavior of eigenfunctions of the Dirichlet Laplacian on dumbbell domains depending on a parameter and disconnecting in some limit process. More precisely, let us consider two slightly different domains which are connected by a thin tube so that the mass of a given eigenfunction is concentrated in one of the two domains. Then the restriction of the eigenfunction to the other domain develops a singularity right at the junction of the tube, as the section of the channel shrinks to zero. The purpose of this paper is to describe the features of this singularity formation.

A strong motivation for the interest in the spectral analysis of thin branching domains comes from the theory of quantum graphs modeling waves in thin graph-like structures (narrow waveguides, quantum wires, photonic crystals, blood vessels, lungs) and having applications in nanotechnology, optics, chemistry, medicine, see e.g. [23, 12] and references therein.

The behavior of the eigenvalues and eigenfunctions of the Laplace operator in varying domains has been intensively studied in the literature starting from [7, 13, 22, 25, 26] and more recently in [4, 5, 6, 11, 14, 17], where spectral continuity is discussed under different kind of perturbations and boundary conditions (of either Dirichlet or Neumann type). The problem of rate of convergence for eigenvalues of elliptic systems was investigated in [27], while in [9] estimates of the splitting between the first two eigenvalues of elliptic operators under Dirichlet boundary conditions are provided. We also mention that some results on the behavior of eigenfunctions of the Laplace operator under singular perturbation adding a thin handle to a compact manifold have been obtained in [3]. As far as the nonlinear counterpart of the problem is concerned, the effect of the domain shape on the number of positive solutions to some nonlinear Dirichlet boundary value problems has been investigated in [15, 16], where domains constructed as connected approximations to a finite number of disjoint or touching balls have been considered, proving that the number of positive solutions which are not "large" grows with the number of the balls.

When dealing with a dumbbell domain which is going to disconnect, the spectral continuity proved e.g. in [17] implies that eigenfunctions of the approximating problem converge to the eigenfunction of some limit eigenvalue problem on a domain with two connected components, whose spectrum is therefore the union of the spectra on the two components; as a consequence, if an eigenfunction of the limit problem is supported in one of the two domains, then the corresponding

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eigenfunction of the approximating problem is going to vanish on the other domain. We are going to show that a suitable normalization of such eigenfunction develops a singularity at the junction of the tube, whose rate is related to the order of the zero that the limit eigenfunction has at the other junction (see Theorem 1.2). The description of the behavior of eigenfunctions at the junction will also provide informations about nodal sets; more precisely we will prove in Corollary 1.3 that if the limit eigenfunction has at one junction of the tube a zero of order one, then the nodal regions of the corresponding eigenfunctions on the dumbbell stay away from the other junction.

In this paper we set up a strategy to evaluate the rate to the singularity at the junction, based upon a sharp control of the transversal frequencies along the connecting tube. To this aim, we shall exploit the monotonicity method introduced by Almgren [2] in 1979 and then extended by Garofalo and Lin [21] to elliptic operators with variable coefficients in order to prove unique continuation properties. We mention that monotonicity methods were recently used in [18, 19, 20] to prove not only unique continuation but also precise asymptotics near singularities of solutions to linear and semilinear elliptic equations with singular potentials, by extracting such precious information from the behavior of the quotient associated with the Lagrangian energy.

As a paradigmatic example, let us consider the following dumbbell domain in $\mathbb{R}^N = \mathbb{R}\times\mathbb{R}^{N-1}$, $N \geqslant 3$,

$$\Omega^\varepsilon = D^- \cup \mathcal{C}_\varepsilon \cup D^+$$

where $\varepsilon \in (0,1)$,

$$\begin{aligned} D^- &= \{(x_1,x') \in \mathbb{R}\times\mathbb{R}^{N-1} : x_1 < 0\},\\ \mathcal{C}_\varepsilon &= \big\{(x_1,x') \in \mathbb{R}\times\mathbb{R}^{N-1} : 0 \leqslant x_1 \leqslant 1,\ \tfrac{x'}{\varepsilon} \in \Sigma\big\},\\ D^+ &= \{(x_1,x') \in \mathbb{R}\times\mathbb{R}^{N-1} : x_1 > 1\}, \end{aligned}$$

and $\Sigma \subset \mathbb{R}^{N-1}$ is an open bounded set with $C^{2,\alpha}$-boundary containing 0. For simplicity of notation, without loss of generality, we assume that Σ satisfies

$$\big\{x' \in \mathbb{R}^{N-1} : |x'| \leqslant \tfrac{1}{\sqrt{2}}\big\} \subset \Sigma \subset \{x' \in \mathbb{R}^{N-1} : |x'| < 1\}. \tag{1}$$

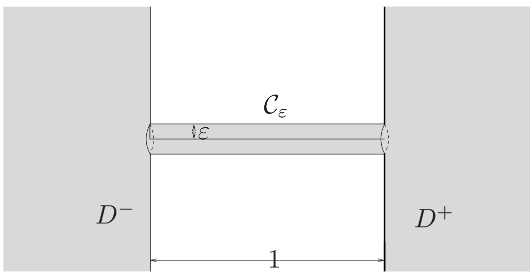


FIGURE 1. The domain Ω^ε.

We also denote, for all $t > 0$,

$$B_t^+ := D^+ \cap B(\mathbf{e}_1, t), \quad B_t^- := D^- \cap B(\mathbf{0}, t),$$

where $\mathbf{e}_1 = (1,0,\dots,0) \in \mathbb{R}^N$ and $B(P,t) := \{x \in \mathbb{R}^N : |x-P| < t\}$ denotes the ball of radius t centered at P. Let $p \in C^1(\mathbb{R}^N,\mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ satisfying

$$p \geqslant 0 \text{ a.e. in } \mathbb{R}^N,\ p \in L^{N/2}(\mathbb{R}^N),\ \nabla p(x)\cdot x \in L^{N/2}(\mathbb{R}^N),\ \frac{\partial p}{\partial x_1} \in L^{N/2}(\mathbb{R}^N), \tag{2}$$

$$\begin{cases} p \not\equiv 0 \text{ in } D^-, \quad p \not\equiv 0 \text{ in } D^+,\\ p(x) = 0 \text{ for all } x \in \{(x_1,x') \in \mathbb{R}\times\mathbb{R}^{N-1} : 1/2 \leqslant x_1 \leqslant 1,\ x' \in \Sigma\} \cup B_3^+. \end{cases} \tag{3}$$

While assumption (2) makes the problem consistent with the usual spectral theory, (3) is introduced for technical reasons; we don't believe it is necessary: its only use is in section 2, to prove some uniform estimates for approximating eigenfunctions close to the right junction uniformly with respect to the parameter ε. Possible weakening of assumption (3) is the object of a current elaboration.

By classical spectral theory, for every open set $\Omega \subset \mathbb{R}^N$ such that $p \not\equiv 0$ in Ω, the weighted eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda p\varphi, & \text{in } \Omega,\\ \varphi = 0, & \text{on } \partial\Omega, \end{cases}$$

admits a sequence of diverging eigenvalues $\{\lambda_k(\Omega)\}_{k\geqslant 1}$; in the enumeration

$$\lambda_1(\Omega) \leqslant \lambda_2(\Omega) \leqslant \cdots \leqslant \lambda_k(\Omega) \leqslant \cdots$$

we repeat each eigenvalue as many times as its multiplicity. We denote $\sigma_p(\Omega) = \{\lambda_k(\Omega) : k \geqslant 1\}$. For all $\varepsilon \in (0,1)$, we also denote

$$\lambda_k^\varepsilon = \lambda_k(\Omega^\varepsilon), \quad \sigma_p^\varepsilon = \sigma_p(\Omega^\varepsilon).$$

It is easy to verify that $\sigma_p(D^- \cup D^+) = \sigma_p(D^-) \cup \sigma_p(D^+)$. Let us assume that there exists $k_0 \geqslant 1$ such that

(4) $\lambda_{k_0}(D^+)$ is simple and the corresponding eigenfunctions have in $\mathbf{e}_1$ a zero of order 1,

(5) $\lambda_{k_0}(D^+) \notin \sigma_p(D^-)$.

In view of [24], these non degeneracy assumptions hold generically with respect to domain (and weight) variations. We can then fix an eigenfunction $\varphi_{k_0}^+ \in \mathcal{D}^{1,2}(D^+) \setminus \{0\}$ associated to $\lambda_{k_0}(D^+)$, i.e. solving

$$\begin{cases} -\Delta\varphi_{k_0}^+ = \lambda_{k_0}(D^+)p\varphi_{k_0}^+, & \text{in } D^+,\\ \varphi_{k_0}^+ = 0, & \text{on } \partial D^+, \end{cases}$$

such that

$$\frac{\partial\varphi_{k_0}^+}{\partial x_1}(\mathbf{e}_1) > 0. \tag{6}$$

Here and in the sequel, for every open set $\Omega \subseteq \mathbb{R}^N$, $\mathcal{D}^{1,2}(\Omega)$ denotes the functional space obtained as completion of $C_c^\infty(\Omega)$ with respect to the Dirichlet norm $\big(\int_\Omega |\nabla u|^2 dx\big)^{1/2}$.

We refer to [17, Example 8.2, Corollary 4.7, Remark 4.3] for the proof of the following lemma.

Lemma 1.1. *Let*

$$\begin{aligned} \bar{k} &= \operatorname{card}\big\{j \in \mathbb{N} \setminus \{0\} : \lambda_j(D^- \cup D^+) \leqslant \lambda_{k_0}(D^+)\big\}\\ &= k_0 + \operatorname{card}\big\{j \in \mathbb{N} \setminus \{0\} : \lambda_j(D^-) \leqslant \lambda_{k_0}(D^+)\big\}, \end{aligned}$$

so that $\lambda_{k_0}(D^+) = \lambda_{\bar{k}}(D^- \cup D^+)$. *Then*

$$\lambda_{\bar{k}}^\varepsilon \to \lambda_{k_0}(D^+) \quad \text{as } \varepsilon \to 0^+. \tag{7}$$

Furthermore, for every ε *sufficiently small,* $\lambda_{\bar{k}}^\varepsilon$ *is simple and there exists an eigenfunction* $\varphi_{\bar{k}}^\varepsilon$ *associated to* $\lambda_{\bar{k}}^\varepsilon$, *i.e. satisfying*

$$\begin{cases} -\Delta\varphi_{\bar{k}}^\varepsilon = \lambda_{\bar{k}}^\varepsilon p\varphi_{\bar{k}}^\varepsilon, & \text{in } \Omega^\varepsilon,\\ \varphi_{\bar{k}}^\varepsilon = 0, & \text{on } \partial\Omega^\varepsilon, \end{cases}$$

such that

$$\varphi_{\bar{k}}^\varepsilon \to \varphi_{k_0}^+ \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N) \quad \text{as } \varepsilon \to 0^+,$$

where in the above formula we mean the functions $\varphi_{\bar{k}}^\varepsilon, \varphi_{k_0}^+$ *to be trivially extended to the whole* $\mathbb{R}^N$.

We mention that uniform convergence of eigenfunctions has been established in [10, §5.2].

Henceforward, for simplicity of notation, we denote

$$u_\varepsilon = \varphi_{\bar{k}}^\varepsilon \quad \text{and} \quad u_0 = \varphi_{k_0}^+. \tag{8}$$

Hence, for small ε, u_ε solves

$$\begin{cases} -\Delta u_\varepsilon = \lambda_{\bar{k}}^\varepsilon p u_\varepsilon, & \text{in } \Omega^\varepsilon,\\ u_\varepsilon = 0, & \text{on } \partial\Omega^\varepsilon. \end{cases} \tag{9}$$

The main result of the present paper is the following theorem describing the behavior as $\varepsilon \to 0^+$ of u_ε at the junction $\mathbf{0} = (0, \dots, 0)$. For all $t > 0$, let us denote

$$\mathcal{D}_t^- := \{v \in C^\infty(D^- \setminus B_t^-) : \operatorname{supp} v \Subset D^-\}$$

and let $\mathcal{H}_t^-$ be the completion of $\mathcal{D}_t^-$ with respect to the norm $\big(\int_{D^- \setminus B_t^-} |\nabla v|^2 dx\big)^{1/2}$, i.e. $\mathcal{H}_t^-$ is the space of functions with finite energy in $D^- \setminus \overline{B_t^-}$ vanishing on ∂D^-. We also define, for all $t > 0$,

$$\Gamma_t^- = D^- \cap \partial B_t^-. \tag{10}$$

Let

$$Y_1 : \mathbb{S}_-^{N-1} \to \mathbb{R}, \quad Y_1(\theta_1, \theta_2, \dots, \theta_N) = -\frac{\theta_1}{\Upsilon_N}, \tag{11}$$

where

$$\mathbb{S}_-^{N-1} := \{\theta = (\theta_1, \theta_2, \dots, \theta_N) \in \mathbb{S}^{N-1} : \theta_1 < 0\}, \quad \Upsilon_N = \sqrt{\textstyle\int_{\mathbb{S}_-^{N-1}} \theta_1^2 d\sigma(\theta)}, \tag{12}$$

being $\mathbb{S}^{N-1}$ the unit $(N-1)$-dimensional sphere. Here and in the sequel, the notation $d\sigma$ is used to denote the volume element on $(N-1)$-dimensional surfaces. We notice that Y_1 is the first positive $L^2(\mathbb{S}_-^{N-1})$-normalized eigenfunction of $-\Delta_{\mathbb{S}^{N-1}}$ on $\mathbb{S}_-^{N-1}$ under null Dirichlet boundary conditions and satisfies $-\Delta_{\mathbb{S}^{N-1}} Y_1 = (N-1) Y_1$ on $\mathbb{S}_-^{N-1}$, where $\Delta_{\mathbb{S}^{N-1}}$ is the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{N-1}$.

Theorem 1.2. *Let us assume* (2)–(6) *hold and let u_ε as in* (8). *Then there exists $\tilde h \in (0,1)$ such that, for every sequence $\varepsilon_n \to 0^+$, there exist a subsequence $\{\varepsilon_{n_j}\}_j$, $U \in C^2(D^-) \cup \big(\bigcup_{t>0} \mathcal{H}_t^-\big)$, $U \not\equiv 0$, and $\beta < 0$ such that*

i) $$\frac{u_{\varepsilon_{n_j}}}{\sqrt{\int_{\Gamma_{\tilde h}^-} u_{\varepsilon_{n_j}}^2 d\sigma}} \to U \quad \text{as } j \to +\infty \quad \text{strongly in } \mathcal{H}_t^- \text{ for every } t > 0 \text{ and in } C^2(\overline{B_{t_2}^- \setminus B_{t_1}^-}) \text{ for all } 0 < t_1 < t_2;$$

ii) $$\lambda^{N-1} U(\lambda x) \to \beta \frac{x_1}{|x|^N} \quad \text{as } \lambda \to 0^+ \quad \text{strongly in } \mathcal{H}_t^- \text{ for every } t > 0 \text{ and in } C^2(\overline{B_{t_2}^- \setminus B_{t_1}^-}) \text{ for all } 0 < t_1 < t_2;$$

iii) $$\beta = -\frac{\int_{\mathbb{S}_-^{N-1}} U Y_1 \, d\sigma - \frac{\lambda_{k_0}(D^+)}{N} \int_{D^-} p(x) U(x) Y_1(\frac{x}{|x|}) \Big(|x| \chi_{B_1^-}(x) + \frac{\chi_{D^- \setminus B_1^-}(x)}{|x|^{N-1}}\Big) dx}{\Upsilon_N}.$$

In the forthcoming paper [1], some improvements of Theorem 1.2 will be obtained; more precisely, the dependence on the subsequence will be removed and the exact asymptotic behavior of the normalization $\sqrt{\int_{\Gamma_{\tilde h}^-} u_{\varepsilon_{n_j}}^2 d\sigma}$ will be derived.

The description of the behavior of eigenfunctions at the junction given by Theorem 1.2 provides us with some important information about the sign of u_ε near the left junction. More precisely, the nondegeneracy condition (4) on the right junction implies that the solution is one-sign in a neighborhood of the left one. In other words, the nodal set of u_ε does not enter inside the channel.

Corollary 1.3. *Let us assume* (2)–(6) *hold and let u_ε as in* (8). *Then there exists $R > 0$ such that*

for every $r \in (0, R)$ there exists $\varepsilon_r > 0$ such that $u_\varepsilon > 0$ in Γ_r^- for all $\varepsilon \in (0, \varepsilon_r)$.

The paper is organized as follows. In section 2 we prove some estimates from above and from below of eigenfunctions of the approximating problem close to the right junction uniformly with respect to the parameter ε. In section 3 we introduce a frequency function associated to the approximating problem and study its behavior at the left, in the corridor, and at the right of the domain. Sections 4 and 5 contain a blow-up analysis (at the right and at the left junction respectively) leading to some uniform bounds of the frequency function which allow describing, in section 6, the asymptotic behavior of the eigenfunctions (suitably normalized) close to the left junction of the tube, thus proving Theorem 1.2 and Corollary 1.3.

2. Estimates on u_ε on the right

This section collects some estimates of eigenfunctions u_ε close to the right junction, which will be crucial to control the frequency function at the right.

Lemma 2.1. *There exist $0 < r_0 < 3$, $\varepsilon_0 \in (0, r_0/2)$, and $C_0 > 0$ such that*

$$\frac{1}{C_0}(x_1 - 1) \leqslant u_\varepsilon(x) \leqslant C_0(x_1 - 1) \quad \textit{for all } x \in D^+ \cap \partial B^+_{r_0} \textit{ and } \varepsilon \in (0, \varepsilon_0).$$

Proof. From Lemma 1.1 and classical elliptic regularity theory,

$$u_\varepsilon \to u_0 \text{ in } C^2_{\rm loc}(\overline{D^+} \setminus \{\mathbf{e}_1\}) \text{ and } \nabla u_\varepsilon \to \nabla u_0 \text{ in } C^1_{\rm loc}(\overline{D^+} \setminus \{\mathbf{e}_1\}). \tag{13}$$

Furthermore (6) implies that there exist $C > 0$ and $r_0 \in (0, 3)$ such that

$$\frac{\partial u_0}{\partial x_1}(x) \geqslant C, \quad u_0(x) > 0, \quad \text{for all } x \in B^+_{r_0}. \tag{14}$$

Let $t_0 \in (1, 1 + r_0/4)$ such that, if $x = (x_1, x') \in \mathcal{A}_0 := \big(B^+_{r_0} \setminus B^+_{(3r_0)/4}\big) \cap \{1 < x_1 < t_0\}$, then $(1, x') \in B^+_{r_0} \setminus B^+_{r_0/2}$. By (14) and continuity of u_0, there exist $c > 0$ such that

$$u_0(x) \geqslant c \quad \text{for all } x \in \big(B^+_{r_0} \setminus B^+_{(3r_0)/4}\big) \setminus \mathcal{A}_0. \tag{15}$$

From (13), there exists $\varepsilon_0 \in (0, r_0/2)$ such that equation (9) is satisfied for $\varepsilon \in (0, \varepsilon_0)$ and

$$\left|\frac{\partial u_\varepsilon}{\partial x_1}(x) - \frac{\partial u_0}{\partial x_1}(x)\right| \leqslant \frac{C}{2} \quad \text{for all } x \in B^+_{r_0} \setminus B^+_{r_0/2} \text{ and } \varepsilon \in (0, \varepsilon_0), \tag{16}$$

$$|u_\varepsilon(x) - u_0(x)| \leqslant \frac{c}{2} \quad \text{for all } x \in \big(B^+_{r_0} \setminus B^+_{(3r_0)/4}\big) \setminus \mathcal{A}_0 \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{17}$$

Estimate (17) together with (15) implies that

$$u_\varepsilon(x) \geqslant \frac{c}{2} \quad \text{for all } x \in \big(B^+_{r_0} \setminus B^+_{(3r_0)/4}\big) \setminus \mathcal{A}_0 \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{18}$$

On the other hand, (16) together with (14) implies

$$\frac{\partial u_\varepsilon}{\partial x_1}(x) \geqslant \frac{C}{2} \quad \text{for all } x \in B^+_{r_0} \setminus B^+_{r_0/2} \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{19}$$

We notice that, if $x \in \mathcal{A}_0$ then from (19) it follows that

$$u_\varepsilon(x_1, x') = u_\varepsilon(1, x') + \int_1^{x_1} \frac{\partial u_\varepsilon}{\partial x_1}(s, x')\, ds > 0. \tag{20}$$

Combining (18) and (20) we conclude that

$$u_\varepsilon(x) > 0 \quad \text{for all } x \in B^+_{r_0} \setminus B^+_{(3r_0)/4} \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{21}$$

If $x \in D^+ \cap \partial B^+_{r_0}$ and $\varepsilon \in (0, \varepsilon_0)$, from (19) and (21) we have that

$$u_\varepsilon(x) = u_\varepsilon\Big(x - \frac{x_1 - 1}{4}\mathbf{e}_1\Big) + \int_0^1 \frac{\partial u_\varepsilon}{\partial x_1}\Big(x - \frac{(1-t)(x_1 - 1)}{4}\mathbf{e}_1\Big)\frac{x_1 - 1}{4}\, dt \geqslant \frac{C}{2}\,\frac{x_1 - 1}{4}$$

thus proving the stated lower bound. The upper bound follows combining (16), (17), and (20). □

The following iterative Brezis-Kato type argument yields a uniform L^∞-bound for $\{u_\varepsilon\}_\varepsilon$.

Lemma 2.2. *There exists $C_1 > 0$ such that*

$$|u_\varepsilon(x)| \leqslant C_1 \quad \textit{for all } x \in \Omega^\varepsilon \textit{ and } \varepsilon \in (0, \varepsilon_0).$$

Proof. Since $u_\varepsilon \to u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we have that

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \|u_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)} < \infty. \tag{22}$$

We claim that

(23) there exists a positive constant $C > 0$ independent of ε and q such that if $u_\varepsilon \in L^q(\mathbb{R}^N)$ for some $q \geqslant 2^*$ and all $\varepsilon \in (0, \varepsilon_0)$ then

$$\|u_\varepsilon\|_{L^{\frac{q2^*}{2}}(\mathbb{R}^N)} \leqslant C^{\frac{1}{q}}(q - 2)^{\frac{1}{q}}\|u_\varepsilon\|_{L^q(\mathbb{R}^N)}.$$

The claim can be proved by following the Brezis-Kato procedure [8]. For every $n \in \mathbb{N}$, we set $u_\varepsilon^n = \min\{n, |u_\varepsilon|\}$ and test (9) with $u_\varepsilon (u_\varepsilon^n)^{q-2}$ thus obtaining

$$(q-2)\int_{\Omega^\varepsilon} |\nabla u_\varepsilon^n|^2 (u_\varepsilon^n)^{q-2}\,dx + \int_{\Omega^\varepsilon} |\nabla u_\varepsilon|^2 (u_\varepsilon^n)^{q-2} dx = \lambda_\varepsilon \int_{\Omega^\varepsilon} p u_\varepsilon^2 (u_\varepsilon^n)^{q-2} dx.$$

Letting $C(q) = \min\left\{\frac{2}{q-2}, \frac{1}{2}\right\}$, we then obtain

$$C(q)\int_{\Omega^\varepsilon} |\nabla((u_\varepsilon^n)^{\frac{q}{2}-1} u_\varepsilon)|^2 dx \leqslant C(q)\int_{\Omega^\varepsilon}\left(\frac{(q-2)^2}{2}(u_\varepsilon^n)^{q-2}|\nabla u_\varepsilon^n|^2 + 2(u_\varepsilon^n)^{q-2}|\nabla u_\varepsilon|^2\right)dx$$
$$\leqslant \lambda_\varepsilon \int_{\Omega^\varepsilon} p u_\varepsilon^2 (u_\varepsilon^n)^{q-2} dx \leqslant \text{const} \int_{\Omega^\varepsilon} |u_\varepsilon|^q dx$$

for some const > 0 independent of ε and q, which, letting $n \to +\infty$, implies claim (23) by Sobolev inequality. Starting from $q = 2^*$ and iterating the estimate of claim (23), we obtain that, for all $n \in \mathbb{N}$, $n \geqslant 1$, letting $q_n = 2\left(\frac{2^*}{2}\right)^n$, there holds

$$\|u_\varepsilon\|_{L^{q_{n+1}}(\mathbb{R}^N)} \leqslant \|u_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)} C^{\sum_{k=1}^n \frac{1}{q_k}} \prod_{k=1}^n (q_k - 2)^{\frac{1}{q_k}} \leqslant \text{const}\, \|u_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}$$

for some const > 0 independent of ε and n. Letting $n \to \infty$, (22) yields the conclusion. □

We denote

$$T_1^- = \{(x_1, x') : x' \in \Sigma, x_1 \leqslant 1\}, \quad \widetilde{D} = D^+ \cup T_1^-, \tag{24}$$
$$T_\varepsilon^- = \left\{(x_1, x') : \tfrac{x'}{\varepsilon} \in \Sigma,\ x_1 \leqslant 1\right\},$$
$$T_1 = \{(x_1, x') : x_1 \in \mathbb{R},\ x' \in \Sigma\},$$

and, for $r \in \mathbb{R} \setminus (1, 2)$,

$$\widetilde{\Omega}_r = \begin{cases} \{(x_1, x') \in T_1 : x_1 < r\}, & \text{if } r \leqslant 1,\\ T_1^- \cup B_{r-1}^+, & \text{if } r \geqslant 2,\end{cases} \qquad \widetilde{\Gamma}_r = \begin{cases} \{(x_1, x') \in T_1 : x_1 = r\}, & \text{if } r \leqslant 1,\\ \Gamma_{r-1}^+, & \text{if } r \geqslant 2,\end{cases} \tag{25}$$

where, for all $t > 0$, we denote

$$\Gamma_t^+ = D^+ \cap \partial B_t^+. \tag{26}$$

Let us define

$$f : T_1 \to \mathbb{R}, \quad f(x_1, x') = e^{-\sqrt{\lambda_1(\Sigma)}(x_1 - 1)} \psi_1^\Sigma(x'), \tag{27}$$

where $\lambda_1(\Sigma)$ is the first eigenvalue of the Laplace operator on Σ under null Dirichlet boundary conditions and $\psi_1^\Sigma(x')$ is the corresponding positive $L^2(\Sigma)$-normalized eigenfunction, so that

$$\begin{cases} -\Delta_{x'}\psi_1^\Sigma(x') = \lambda_1(\Sigma)\psi_1^\Sigma(x'), & \text{in } \Sigma,\\ \psi_1^\Sigma = 0, & \text{on } \partial\Sigma,\end{cases}$$

being $\Delta_{x'} = \sum_{j=2}^N \frac{\partial^2}{\partial x_j^2}$, $x' = (x_2, \dots, x_N)$. In particular $f \in C^2(\overline{T_1})$ and satisfies

$$\begin{cases} -\Delta f = 0, & \text{in } T_1,\\ f = 0, & \text{on } \partial T_1.\end{cases}$$

Lemma 2.4 below shows how harmonic functions in D^+ can be extended (up to a finite energy perturbation) to harmonic functions in $\widetilde{D}$ with finite energy at $-\infty$. In order to prove it, the following Poincaré type inequality is needed.

Lemma 2.3. *There exists a constant $C_P = C_P(N)$ depending only on the dimension N such that for every function $v : D^+ \setminus B_1^+ \to \mathbb{R}$ satisfying*

$$v \in \bigcap_{R>1} H^1(B_R^+ \setminus B_1^+) \quad \text{and} \quad v = 0 \text{ on } \{x_1 = 1, |x'| > 1\},$$

there holds

$$\int_{B_{2R}^+ \setminus B_R^+} v^2(x)\,dx \leqslant C_P R^2 \int_{B_{2R}^+ \setminus B_R^+} |\nabla v(x)|^2\,dx \quad \text{for all } R > 1.$$

PROOF. It follows by scaling of the Poincaré inequality for functions vanishing on a portion of the boundary. □

Lemma 2.4. *For every* $\psi \in C^2(D^+) \cap C^1(\overline{D^+})$ *such that*

$$\begin{cases} -\Delta\psi = 0, & \text{in } D^+,\\ \psi = 0, & \text{on } \partial D^+, \end{cases}$$

there exists a unique function $u = \mathcal{T}(\psi)$ *such that*

$$\int_{\widetilde{\Omega}_R} \Big(|\nabla u(x)|^2 + |u(x)|^{2^*}\Big)\,dx < +\infty \text{ for all } R > 2, \tag{28}$$

$$-\Delta u = 0 \text{ in a distributional sense in } \widetilde{D}, \quad u = 0 \text{ on } \partial\widetilde{D}, \tag{29}$$

$$\int_{D^+} |\nabla(u-\psi)(x)|^2\,dx < +\infty. \tag{30}$$

Furthermore

$$\mathcal{T}(\psi) - \widetilde{\psi} \in \mathcal{D}^{1,2}(\widetilde{D}), \quad \text{where} \quad \widetilde{\psi} := \begin{cases} \psi & \text{in } D^+\\ 0 & \text{in } T_1^-. \end{cases} \tag{31}$$

PROOF. Let us define $J_\psi : \mathcal{D}^{1,2}(\widetilde{D}) \to \mathbb{R}$ as

$$J_\psi(\varphi) = \frac{1}{2}\int_{\widetilde{D}} |\nabla\varphi(x)|^2\,dx - \int_\Sigma \varphi(1,x')\Big(\frac{\partial\psi}{\partial x_1}\Big)_+(1,x')\,dx' \tag{32}$$

where $\big(\frac{\partial\psi}{\partial x_1}\big)_+(1,x') := \lim_{t\to 0^+} \frac{\psi(1+t,x')}{t}$. By standard minimization methods it is easy to prove that there exists $w \in \mathcal{D}^{1,2}(\widetilde{D})$ such that $J_\psi(w) = \min_{\mathcal{D}^{1,2}(\widetilde{D})} J_\psi$. In particular w satisfies

$$0 = dJ_\psi(w)[\varphi] = \int_{\widetilde{D}} \nabla w(x)\cdot\nabla\varphi(x)\,dx - \int_\Sigma \varphi(1,x')\Big(\frac{\partial\psi}{\partial x_1}\Big)_+(1,x')\,dx'$$

for all $\varphi \in \mathcal{D}^{1,2}(\widetilde{D})$. Hence the function $u : \widetilde{D} \to \mathbb{R}$,

$$u = \begin{cases} w + \psi, & \text{in } D^+,\\ w, & \text{in } T_1^-, \end{cases}$$

satisfies (28), (30), and, for every $\varphi \in C^\infty_c(\widetilde{D})$,

$$\begin{aligned}\int_{\widetilde{D}} \nabla u(x)\cdot\nabla\varphi(x)\,dx &= \int_{\widetilde{D}} \nabla w(x)\cdot\nabla\varphi(x)\,dx + \int_{D^+} \nabla\psi(x)\cdot\nabla\varphi(x)\,dx\\ &= \int_\Sigma \varphi(1,x')\Big(\frac{\partial\psi}{\partial x_1}\Big)_+(1,x')\,dx' - \int_\Sigma \varphi(1,x')\Big(\frac{\partial\psi}{\partial x_1}\Big)_+(1,x')\,dx' = 0\end{aligned}$$

thus implying (29). To prove uniqueness, let us assume that u_1 and u_2 both satisfy (29–30); then the difference $u = u_1 - u_2$ solves

$$-\Delta u = 0 \text{ in a distributional sense in } \widetilde{D}, \quad u = 0 \text{ on } \partial\widetilde{D}, \tag{33}$$

and satisfies

$$\int_{D^+} |\nabla u(x)|^2 dx = \int_{D^+} |\nabla(u_1-\psi)(x) - \nabla(u_2-\psi)(x)|^2 dx < +\infty. \tag{34}$$

For all $R > 2$ let η_R be a cut-off function satisfying

$$\eta_R \in C^\infty(\widetilde{D}), \quad \eta_R \equiv 1 \text{ in } \widetilde{\Omega}_R, \quad \eta_R \equiv 0 \text{ in } D^+ \setminus B^+_{2(R-1)}, \quad |\nabla\eta_R(x)| \leqslant \frac{2}{R-1} \text{ in } \widetilde{D}.$$

Multiplying (33) with $\eta_R^2 u$ and integrating by parts over $\widetilde{D}$ we obtain

$$\begin{aligned}\int_{\widetilde{D}} |\nabla u(x)|^2\eta_R^2(x)\,dx &= -2\int_{\widetilde{D}} u(x)\eta_R(x)\nabla u(x)\cdot\nabla\eta_R(x)\,dx\\ &\leqslant \frac{1}{2}\int_{\widetilde{D}} |\nabla u(x)|^2\eta_R^2(x)\,dx + 2\int_{\widetilde{D}} u^2(x)|\nabla\eta_R(x)|^2\,dx\end{aligned}$$

thus implying, in view of Lemma 2.3,

$$\begin{aligned}\frac{1}{2}\int_{\widetilde{\Omega}_R}|\nabla u(x)|^2\,dx &\leqslant \frac{1}{2}\int_{\widetilde{D}}|\nabla u(x)|^2\eta_R^2(x)\,dx\\ &\leqslant 2\int_{\widetilde{D}}u^2(x)|\nabla\eta_R(x)|^2\,dx \leqslant \frac{8}{(R-1)^2}\int_{B^+_{2(R-1)}\setminus B^+_{R-1}}u^2(x)\,dx\\ &\leqslant 8C_P\int_{B^+_{2(R-1)}\setminus B^+_{R-1}}|\nabla u(x)|^2\,dx.\end{aligned}$$

Letting $R\to+\infty$, from (34) we deduce that $\int_{\widetilde{D}}|\nabla u|^2dx=0$ and hence u must be constant on $\widetilde{D}$. Since u vanishes on $\partial\widetilde{D}$, we deduce that $u\equiv 0$ and then $u_1=u_2$ in $\widetilde{D}$ thus proving uniqueness. □

Henceforward we denote

$$\Phi_1=\mathcal{T}(x_1-1). \tag{35}$$

Since in the case $\psi(x)=x_1-1$ we have that $\big(\frac{\partial\psi}{\partial x_1}\big)_+(1,x')=1>0$, the minimum of the functional J_{x_1-1} defined in (32) is attained by a nonnegative function w. Hence we deduce that

$$\Phi_1(x_1,x')\geqslant(x_1-1)^+\quad\text{for all }(x_1,x')\in\widetilde{D}. \tag{36}$$

Hence, from the Strong Maximum Principle we deduce that

$$\Phi_1(x_1,x')>0\quad\text{for all }(x_1,x')\in\widetilde{D}. \tag{37}$$

For all $r\in\mathbb{R}$, let us denote

$$T_{1,r}=:\{(x_1,x'):x'\in\Sigma,\ x_1\leqslant r\},\quad \Gamma_r:=\{(x_1,x'):x'\in\Sigma,\ x_1=r\}, \tag{38}$$

and define $\mathcal{E}_r$ as the completion of $C^\infty_c(T_{1,r})$ with respect to the norm $\big(\int_{T_{1,r}}|\nabla v|^2dx\big)^{1/2}$ (which is actually equivalent to the norm $\big(\int_{T_{1,r}}|\nabla v|^2dx+\int_{\Gamma_r}v^2d\sigma\big)^{1/2}$), i.e. $\mathcal{E}_r$ is the space of finite energy functions in $T_{1,r}$ vanishing on $\{(x_1,x'):x_1\leqslant r\text{ and }x'\in\partial\Sigma\}$.

The following Lemma associate an Almgren type frequency function to harmonic functions in $\mathcal{E}_R$ and describe its behavior at $-\infty$.

Lemma 2.5. *Let $R\in\mathbb{R}$ and $\phi\in\mathcal{E}_R\setminus\{0\}$ satisfying*

$$\begin{cases}-\Delta\phi=0, & \text{in } T_{1,R},\\ \phi=0, & \text{on }\{(x_1,x'):x_1\leqslant R\text{ and }x'\in\partial\Sigma\},\end{cases}$$

in a weak sense, and let $N_\phi:(-\infty,R)\to\mathbb{R}$ be defined as

$$N_\phi(r):=\frac{\int_{T_{1,r}}|\nabla\phi(x)|^2dx}{\int_{\Gamma_r}\phi^2(x)\,d\sigma}.$$

Then

- i) *N_ϕ is non decreasing in $(-\infty,R)$;*
- ii) *there exists $K_0\in\mathbb{N}$, $K_0\geqslant 1$, such that*

  $$\lim_{r\to-\infty}N_\phi(r)=\sqrt{\lambda_{K_0}(\Sigma)},$$

  where $\lambda_{K_0}(\Sigma)$ is the K_0-th eigenvalue of the Laplace operator on Σ under null Dirichlet boundary conditions;
- iii) *if $N_\phi\equiv\gamma$ for some $\gamma\in\mathbb{R}$ then $\gamma=\sqrt{\lambda_{K_0}(\Sigma)}$ and $\phi(x_1,x')=e^{\sqrt{\lambda_{K_0}(\Sigma)}x_1}\psi(x')$ for some eigenfunction ψ of $-\Delta_{x'}$ in Σ associated to the eigenvalue $\lambda_{K_0}(\Sigma)$;*
- iv) *if $\phi>0$ in $T_{1,R}$, then $K_0=1$.*

PROOF. It is easy to prove that $N_\phi\in C^1(-\infty,R)$ and, for all $r\in(-\infty,R)$,

$$N'_\phi(r)=2\frac{\Big(\int_{\Gamma_r}\big|\frac{\partial\phi}{\partial x_1}\big|^2d\sigma\Big)\Big(\int_{\Gamma_r}\phi^2\,d\sigma\Big)-\Big(\int_{\Gamma_r}\phi\frac{\partial\phi}{\partial x_1}\,d\sigma\Big)^2}{\Big(\int_{\Gamma_r}\phi^2\,d\sigma\Big)^2}.$$

Hence, Schwarz's inequality implies that $N'_\phi(r) \geqslant 0$ for all $r < R$. Therefore N_ϕ is non-decreasing in $(-\infty, R)$ and statement i) is proved. By monotonicity, there exists

$$\gamma := \lim_{r\to-\infty} N_\phi(r) \in [0,+\infty). \tag{39}$$

For every $\lambda > 0$ let us define

$$\phi_\lambda(x_1,x') := \frac{\phi(x_1-\lambda,x')}{\sqrt{\int_{\Gamma_{R-\lambda}}\phi^2 d\sigma}}.$$

We have that $\phi_\lambda \in \mathcal{E}_{R+\lambda}$,

$$\int_{\Gamma_R} \phi_\lambda^2 d\sigma = 1, \tag{40}$$

and ϕ_λ weakly solves

$$\begin{cases} -\Delta\phi_\lambda = 0, & \text{in } T_{1,R+\lambda},\\ \phi_\lambda = 0, & \text{on } \{(x_1,x') : x_1 \leqslant R+\lambda \text{ and } x' \in \partial\Sigma\}. \end{cases} \tag{41}$$

Moreover, the change of variable $(x_1,x') = (y_1-\lambda,y')$ yields

$$N_\phi(r-\lambda) = \frac{\int_{T_{1,r}} |\nabla\phi_\lambda(y)|^2 dy}{\int_{\Gamma_r}\phi_\lambda^2 d\sigma} \quad \text{for all } r < R+\lambda. \tag{42}$$

In particular we have that

$$N_\phi(R-\lambda) = \int_{T_{1,R}} |\nabla\phi_\lambda(y)|^2 dy \leqslant N_\phi\Big(\frac{R}{2}\Big) \quad \text{for every } \lambda \geqslant \frac{R}{2},$$

and hence $\{\phi_\lambda\}_{\lambda\geqslant R/2}$ is bounded in $\mathcal{E}_R$. Therefore there exist a sequence $\lambda_n \to +\infty$ and some $\tilde\phi \in \mathcal{E}_R$ such that $\phi_{\lambda_n} \rightharpoonup \tilde\phi$ weakly in $\mathcal{E}_R$ and a.e. in $T_{1,R}$. From compactness of the embedding $\mathcal{E}_R \hookrightarrow L^2(\Gamma_R)$ and (40) we deduce that $\int_{\Gamma_R}\tilde\phi^2 d\sigma = 1$; in particular $\tilde\phi \not\equiv 0$. Passing to the weak limit in (41) as $\lambda_n \to +\infty$ we have that

$$\begin{cases} -\Delta\tilde\phi = 0, & \text{in } T_{1,R},\\ \tilde\phi = 0, & \text{on } \{(x_1,x') : x_1 \leqslant R \text{ and } x' \in \partial\Sigma\}. \end{cases} \tag{43}$$

By classical elliptic regularity estimates, we also have that $\phi_{\lambda_n} \to \tilde\phi$ in $C^2(T_{1,r_2}\setminus T_{1,r_1})$ for all $r_1 < r_2 < R$. Therefore, multiplying (43) by $\tilde\phi$ and integrating over $T_{1,r}$ with $r < R$, we obtain

$$\int_{\Gamma_r} \frac{\partial\phi_{\lambda_n}}{\partial x_1}\phi_{\lambda_n} d\sigma \to \int_{\Gamma_r}\frac{\partial\tilde\phi}{\partial x_1}\tilde\phi\, d\sigma = \int_{T_{1,r}} |\nabla\tilde\phi(x)|^2 dx. \tag{44}$$

On the other hand, multiplication of (41) by ϕ_{λ_n} and integration by parts over $T_{1,r}$ yield

$$\int_{T_{1,r}} |\nabla\phi_{\lambda_n}(x)|^2 dx = \int_{\Gamma_r}\frac{\partial\phi_{\lambda_n}}{\partial x_1}\phi_{\lambda_n}\, d\sigma. \tag{45}$$

From (44) and (45), we deduce that $\|\phi_{\lambda_n}\|_{\mathcal{E}_r} \to \|\tilde\phi\|_{\mathcal{E}_r}$ and then $\phi_{\lambda_n} \to \tilde\phi$ strongly in $\mathcal{E}_r$ for every $r < R$. Therefore, for every $r < R$, passing to the limit as $\lambda_n \to +\infty$ in (42) and letting γ as in (39), we obtain that

$$N_{\tilde\phi}(r) = \gamma \quad \text{for all } r < R, \tag{46}$$

where

$$N_{\tilde\phi}(r) = \frac{\int_{T_{1,r}} |\nabla\tilde\phi(y)|^2 dy}{\int_{\Gamma_r}\tilde\phi^2 d\sigma}.$$

Then

$$N'_{\tilde\phi}(r) = 2\frac{\Big(\int_{\Gamma_r}\big|\frac{\partial\tilde\phi}{\partial x_1}\big|^2 d\sigma\Big)\Big(\int_{\Gamma_r}\tilde\phi^2\, d\sigma\Big) - \Big(\int_{\Gamma_r}\tilde\phi\frac{\partial\tilde\phi}{\partial x_1}\, d\sigma\Big)^2}{\Big(\int_{\Gamma_r}\tilde\phi^2\, d\sigma\Big)^2} = 0 \quad \text{for all } r < R.$$

Since equality in the Schwarz's inequality holds only for parallel vectors, we infer that $\frac{\partial\tilde\phi}{\partial x_1}$ and $\tilde\phi$ must be parallel as vectors in $L^2(\Gamma_r)$, hence there exists some function $\eta:(-\infty,R)\to\mathbb{R}$ such that

$$\frac{\partial\tilde\phi}{\partial x_1}(x_1,x')=\eta(x_1)\tilde\phi(x_1,x')\quad\text{for all } x_1\in(-\infty,R)\text{ and } x'\in\Sigma.$$

Integration with respect to x_1 yields

$$\tilde\phi(x_1,x')=\varphi(x_1)\psi(x')\quad\text{for all } x_1\in(-\infty,R)\text{ and } x'\in\Sigma, \tag{47}$$

where $\varphi(x_1)=e^{\int_R^{x_1}\eta(s)ds}$, $\psi(x')=\tilde\phi(R,x')$. From (43) and (47), we derive

$$\varphi''(x_1)\psi(x')+\varphi(x_1)\Delta_{x'}\psi(x')=0.$$

Taking x_1 fixed, we deduce that ψ is an eigenfunction of $-\Delta_{x'}$ in Σ under homogeneous Dirichlet boundary conditions. If $\lambda_{K_0}(\Sigma)$ is the corresponding eigenvalue then $\varphi(x_1)$ solves the equation

$$\varphi''(x_1)-\lambda_{K_0}(\Sigma)\varphi(x_1)=0$$

and hence φ is of the form

$$\varphi(x_1)=c_1e^{\sqrt{\lambda_{K_0}(\Sigma)}(x_1-R)}+c_2e^{-\sqrt{\lambda_{K_0}(\Sigma)}(x_1-R)}\quad\text{for some } c_1,c_2\in\mathbb{R}.$$

Since the function $e^{-\sqrt{\lambda_{K_0}(\Sigma)}(x_1-R)}\psi(x')\notin\mathcal{E}_R$, then $c_2=0$ and $\varphi(x_1)=c_1e^{\sqrt{\lambda_{K_0}(\Sigma)}(x_1-R)}$. Since $\varphi(R)=1$, we obtain that $c_1=1$ and then

$$\tilde\phi(x_1,x')=e^{\sqrt{\lambda_{K_0}(\Sigma)}(x_1-R)}\psi(x'),\quad\text{for all } x_1\in(-\infty,R)\text{ and } x'\in\Sigma. \tag{48}$$

Substituting (48) into (46) we obtain that $\gamma=\sqrt{\lambda_{K_0}(\Sigma)}$. Hence statement ii) is proved. We notice that the above argument of classification of harmonic functions $\tilde\phi$ with constant frequency $N_{\tilde\phi}$ also proves statement iii).

In order to prove iv), let us assume that $\phi>0$ in $T_{1,R}$. Then $\phi_\lambda>0$ in $T_{1,R+\lambda}$. Hence a.e. convergence implies that $\tilde\phi\geqslant 0$ in $T_{1,R}$. From the Strong Maximum Principle we obtain that $\tilde\phi>0$ in $T_{1,R}$, which necessarily implies that $\psi>0$ in Σ. Then ψ must be the eigenfunction associated to the first eigenvalue, i.e. $\lambda_{K_0}(\Sigma)=\lambda_1(\Sigma)$. □

The previous lemma allows describing the behavior of the Almgren type frequency quotient naturally associated to the function Φ_1 introduced in (35). For all $r\in\mathbb{R}\setminus(1,2)$, let $\widetilde{\mathcal{N}}(r)=\widetilde{\mathcal{N}}_{\Phi_1}(r)$ be the frequency function associated to Φ_1, i.e.

$$\widetilde{\mathcal{N}}(r)=\widetilde{\mathcal{N}}_{\Phi_1}(r)=\frac{\Lambda_N(r)\int_{\widetilde\Omega_r}|\nabla\Phi_1(x)|^2dx}{\int_{\widetilde\Gamma_r}\Phi_1(x)\,d\sigma}, \tag{49}$$

where

$$\Lambda_N(r)=\begin{cases}\big(\frac{2}{\omega_{N-1}}\big)^{\frac{1}{N-1}}|\widetilde\Gamma_r|^{\frac{1}{N-1}}=r-1, & \text{if } r\geqslant 2,\\ \big(\frac{N-1}{\omega_{N-2}}\big)^{\frac{1}{N-1}}|\widetilde\Gamma_r|^{\frac{1}{N-1}}=1, & \text{if } r\leqslant 1,\end{cases} \tag{50}$$

$|\widetilde\Gamma_r|$ denotes the $(N-1)$-dimensional volume of $\widetilde\Gamma_r$, and ω_{N-1} is the volume of the unit sphere $\mathbb{S}^{N-1}$, i.e. $\omega_{N-1}=\int_{\mathbb{S}^{N-1}}d\sigma(\theta)$.

An immediate consequence of Lemma 2.5 and (37) is the following corollary.

Corollary 2.6. $\lim_{r\to-\infty}\widetilde{\mathcal{N}}(r)=\sqrt{\lambda_1(\Sigma)}$.

As a left counterpart of Lemma 2.4, we now construct a harmonic extension to $\widetilde D$ of the function f defined in (27) (up to a finite energy perturbation in the tube) having finite energy at the right.

Lemma 2.7. *There exists a unique function $\Phi_2:\widetilde D\to\mathbb{R}$ such that*

$$\int_{D^+}\Big(|\nabla\Phi_2(x)|^2+|\Phi_2(x)|^{2^*}\Big)\,dx<+\infty, \tag{51}$$

$$-\Delta\Phi_2=0\ \textit{in a distributional sense in}\ \widetilde D,\quad \Phi_2=0\ \textit{on}\ \partial\widetilde D, \tag{52}$$

$$\int_{T_1}|\nabla(\Phi_2-f)(x)|^2\,dx<+\infty, \tag{53}$$

where f is defined in (27). Furthermore

$$\Phi_2 \geqslant f \quad \text{in } T_1 \quad \text{and} \quad \Phi_2 \geqslant 0 \quad \text{in } \widetilde{D}. \tag{54}$$

PROOF. Let us define $J : \mathcal{D}^{1,2}(\widetilde{D}) \to \mathbb{R}$ as

$$\begin{aligned} J(\varphi) &= \frac{1}{2}\int_{\widetilde{D}} |\nabla\varphi(x)|^2\,dx + \int_{(1,+\infty)\times\partial\Sigma} \varphi\frac{\partial f}{\partial\nu}\,d\sigma \\ &= \frac{1}{2}\int_{\widetilde{D}} |\nabla\varphi(x)|^2\,dx + \int_{(1,+\infty)\times\partial\Sigma} \varphi(x_1,x')e^{-\sqrt{\lambda_1(\Sigma)}(x_1-1)}\frac{\partial\psi_1^\Sigma}{\partial\nu_{x'}}(x')\,d\sigma, \end{aligned}$$

where ν denotes the normal external unit vector to ∂T_1 and $\nu_{x'}$ the normal external unit vector to $\partial\Sigma$. It is easy to prove that $J(\varphi) \geqslant c_1\|\varphi\|^2_{\mathcal{D}^{1,2}(\widetilde{D})} - c_2$ for some constants $c_1, c_2 > 0$ and all $\varphi \in \mathcal{D}^{1,2}(\widetilde{D})$ and that J is weakly lower semi-continuous. Hence there exists $w \in \mathcal{D}^{1,2}(\widetilde{D})$ such that $J(w) = \min_{\mathcal{D}^{1,2}(\widetilde{D})} J$. Since, by the Hopf Lemma, $\frac{\partial\psi_1^\Sigma}{\partial\nu_{x'}} < 0$ on $\partial\Sigma$, we can assume that $w \geqslant 0$ (otherwise we take $|w|$ which is still a minimizer). The minimizer w satisfies

$$0 = dJ(w)[\varphi] = \int_{\widetilde{D}} \nabla w(x)\cdot\nabla\varphi(x)\,dx + \int_{(1,+\infty)\times\partial\Sigma} \varphi\frac{\partial f}{\partial\nu}\,d\sigma$$

for all $\varphi \in \mathcal{D}^{1,2}(\widetilde{D})$. Hence the function $\Phi_2 : \widetilde{D} \to \mathbb{R}$,

$$\Phi_2 = \begin{cases} w + f, & \text{in } T_1, \\ w, & \text{in } \widetilde{D}\setminus T_1, \end{cases}$$

satisfies (51), (53), (54), and, for every $\varphi \in C^\infty_{\rm c}(\widetilde{D})$,

$$\begin{aligned} \int_{\widetilde{D}} \nabla\Phi_2(x)\cdot\nabla\varphi(x)\,dx &= \int_{\widetilde{D}} \nabla w(x)\cdot\nabla\varphi(x)\,dx + \int_{T_1} \nabla f(x)\cdot\nabla\varphi(x)\,dx \\ &= -\int_{(1,+\infty)\times\partial\Sigma} \varphi\frac{\partial f}{\partial\nu}\,d\sigma + \int_{(1,+\infty)\times\partial\Sigma} \varphi\frac{\partial f}{\partial\nu}\,d\sigma = 0 \end{aligned}$$

thus implying (52). To prove uniqueness, let us assume that u_1 and u_2 both satisfy (52–53); then the difference $u = u_1 - u_2$ solves

$$-\Delta u = 0 \text{ in a distributional sense in } \widetilde{D}, \quad u = 0 \text{ on } \partial\widetilde{D}, \tag{55}$$

and satisfies

$$\int_{T_1} |\nabla u(x)|^2 dx = \int_{T_1} |\nabla(u_1 - f)(x) - \nabla(u_2 - f)(x)|^2 dx < +\infty. \tag{56}$$

For all $t < 1$ let η_t be a cut-off function satisfying

$$\eta_t \in C^\infty(\widetilde{D}), \quad \eta_1(x_1,x') = 1 \text{ if } x_1 > t, \quad \eta_t(x_1,x') = 0 \text{ if } x_1 < t-1, \quad |\nabla\eta_t(x)| \leqslant 2 \text{ in } \widetilde{D}.$$

Multiplying (55) with $\eta_t^2 u$ and integrating by parts over $\widetilde{D}$ we obtain

$$\begin{aligned} \int_{\widetilde{D}} |\nabla u(x)|^2\eta_t^2(x)\,dx &= -2\int_{\widetilde{D}} u(x)\eta_t(x)\nabla u(x)\cdot\nabla\eta_t(x)\,dx \\ &\leqslant \frac{1}{2}\int_{\widetilde{D}} |\nabla u(x)|^2\eta_t^2(x)\,dx + 2\int_{\widetilde{D}} u^2(x)|\nabla\eta_t(x)|^2\,dx \end{aligned}$$

thus implying

$$\begin{aligned} \frac{1}{2}\int_{\widetilde{D}\cap\{x_1>t\}} |\nabla u(x)|^2\,dx &\leqslant \frac{1}{2}\int_{\widetilde{D}} |\nabla u(x)|^2\eta_t^2(x)\,dx \\ &\leqslant 2\int_{\widetilde{D}} u^2(x)|\nabla\eta_t(x)|^2\,dx \leqslant 8\int_{\widetilde{D}\cap\{t-1<x_1<t\}} u^2(x)\,dx \\ &\leqslant 8\widetilde{C}_P\int_{\widetilde{D}\cap\{t-1<x_1<t\}} |\nabla u(x)|^2\,dx \end{aligned}$$

where the constant $\widetilde{C}_P > 0$ depends only on the dimension and is the best constant of the Poincaré inequality for functions on $(-1,0)\times\Sigma$ vanishing on $\partial\Sigma$. Letting $t \to -\infty$, from (56) we deduce that $\int_{\widetilde{D}} |\nabla u|^2 = 0$ and hence u must be constant on $\widetilde{D}$. Since u vanishes on $\partial\widetilde{D}$, we deduce that $u \equiv 0$ and then $u_1 = u_2$ in $\widetilde{D}$, thus proving uniqueness. □

Remark 2.8. From (54) and the Strong Maximum Principle we deduce that

$$\Phi_2(x_1, x') > 0 \quad \text{for all } (x_1, x') \in \widetilde{D}.$$

The functions Φ_1 and Φ_2 can be estimated as follows.

Lemma 2.9.

(i) *For every $\delta > 0$ there exists $c(\delta) > 0$ such that*

$$\Big|\Phi_1(x) - (x_1 - 1)^+\Big| \leqslant c(\delta)\,\frac{x_1 - 1}{|x - \mathbf{e}_1|^N} \quad \text{and} \quad \Phi_2(x) \leqslant c(\delta)\,\frac{x_1 - 1}{|x - \mathbf{e}_1|^N}$$

for all $x \in D^+ \setminus B^+_{1+\delta}$.

(ii) *There exists $C_2 > 0$ such that*

$$\Phi_1(x) \leqslant C_2 e^{\sqrt{\lambda_1(\Sigma)}\frac{x_1-1}{2}} \quad \text{for all } x \in T_1^-.$$

Proof. Let us first prove (i) for the function $w = \Phi_1(x) - (x_1 - 1)^+ = \mathcal{T}(x_1 - 1)(x) - (x_1 - 1)^+$ (the analogous estimate for Φ_2 can be proved in a similar way). We observe that w belongs to $\mathcal{D}^{1,2}(\widetilde{D})$ in view of (31) and weakly solves $-\Delta w = 0$ in $D^1 \setminus \overline{B_1^+}$ by (29); moreover $w(x) = 0$ for all $x \in \{(x_1, x') : x_1 = 1, |x - \mathbf{e}_1| > 1\}$. Therefore, its Kelvin transform

$$\widetilde{w}(x) = |x - \mathbf{e}_1|^{-(N-2)} w\bigg(\frac{x - \mathbf{e}_1}{|x - \mathbf{e}_1|^2} + \mathbf{e}_1\bigg)$$

belongs to $H^1(B_1^+)$ and weakly satisfies

$$\begin{cases} -\Delta\widetilde{w}(x) = 0, & \text{in } B_1^+, \\ \widetilde{w}(x) = 0, & \text{on } \{(x_1, x') : x_1 = 1, |x - \mathbf{e}_1| < 1\}. \end{cases}$$

By classical elliptic estimates, for any $\delta > 0$ there exists $c(\delta) > 0$ such that $\big|\frac{\partial\widetilde{w}}{\partial x_1}\big| \leqslant c(\delta)$ in $\overline{B^+_{1/(\delta+1)}}$, thus implying

$$|\widetilde{w}(x_1, x')| = \bigg|\widetilde{w}(1, x') + \int_1^{x_1} \frac{\partial\widetilde{w}}{\partial x_1}(s, x')\,ds\bigg| \leqslant \int_1^{x_1} \bigg|\frac{\partial\widetilde{w}}{\partial x_1}(s, x')\bigg|\,ds \leqslant c(\delta)(x_1 - 1)$$

for all $(x_1, x') \in \overline{B^+_{1/(\delta+1)}}$, which implies (i). To prove (ii), it is enough to observe that, in view of (1), the function $v(x_1, x') = e^{\sqrt{\lambda_1(\Sigma)}\frac{x_1-1}{2}}\psi_1^\Sigma\big(x'/2\big)$ is well-defined, harmonic and strictly positive in T_1^-, bounded from below away from 0 on $\{(x_1, x') \in T_1^- : x_1 = 1\}$, and $\int_{T_1^-}(|\nabla v|^2 + |v|^{2^*}) < +\infty$. Hence, from the Maximum Principle we deduce that $\Phi_1(x) \leqslant \mathrm{const}\, v(x)$ in T_1^-, thus implying statement (ii). □

In order to control u_ε with suitable sub/super-solutions and obtain the needed upper and lower estimates, let us introduce the following functions:

$$\Phi^\varepsilon : D^+ \cup T_\varepsilon^- \to \mathbb{R}, \quad \Phi^\varepsilon(x) = \varepsilon\Phi_1\Big(\mathbf{e}_1 + \frac{x - \mathbf{e}_1}{\varepsilon}\Big) + 2\gamma_\varepsilon\varepsilon\Phi_2\Big(\mathbf{e}_1 + \frac{x - \mathbf{e}_1}{2\varepsilon}\Big), \tag{57}$$

$$\widetilde{\Phi}^\varepsilon : D^+ \cup T_\varepsilon^- \to \mathbb{R}, \quad \widetilde{\Phi}^\varepsilon(x) = \varepsilon\Phi_1\Big(\mathbf{e}_1 + \frac{x - \mathbf{e}_1}{\varepsilon}\Big) - \sqrt{2}\widetilde{\gamma}_\varepsilon\varepsilon\Phi_2\Big(\mathbf{e}_1 + \frac{x - \mathbf{e}_1}{\sqrt{2}\varepsilon}\Big), \tag{58}$$

where

$$\gamma_\varepsilon = \bigg(2\varepsilon\exp\Big(\frac{\sqrt{\lambda_1(\Sigma)}}{4\varepsilon}\Big)\bigg)^{-1}, \quad \widetilde{\gamma}_\varepsilon = \bigg(\sqrt{2}\varepsilon\exp\Big(\frac{\sqrt{\lambda_1(\Sigma)}}{2\sqrt{2}\varepsilon}\Big)\bigg)^{-1}.$$

We notice that $\Phi^\varepsilon, \widetilde{\Phi}^\varepsilon$ are well-defined in view of (1).

Lemma 2.10. *There exists $C_3 > 0$ such that*

$$|u_\varepsilon(x)| \leqslant C_3 \Phi^\varepsilon(x) \quad \text{for all } \varepsilon \in (0,\varepsilon_0) \text{ and } x \in \mathcal{B}_\varepsilon,$$

where

$$\mathcal{B}_\varepsilon = B^+_{r_0} \cup \Big\{(x_1,x') \in \mathbb{R}^N : \tfrac{x'}{\varepsilon} \in \Sigma,\ \tfrac12 < x_1 \leqslant 1\Big\}. \tag{59}$$

Proof. Let us first observe that

$$-\Delta \Phi^\varepsilon = 0, \quad \text{in } D^+ \cup T^-_\varepsilon. \tag{60}$$

Moreover, if $x \in \Gamma^+_{r_0} = \partial B^+_{r_0} \cap D^+$ and $\varepsilon \in (0,\varepsilon_0)$, then Lemma 2.1 implies that

$$|u_\varepsilon(x)| \leqslant C_0(x_1 - 1), \tag{61}$$

while (36) and (54) ensure

$$\Phi^\varepsilon(x) \geqslant \varepsilon\, \frac{(x_1-1)^+}{\varepsilon} = (x_1-1)^+ \quad \text{in } D^+ \cup T^+_\varepsilon. \tag{62}$$

From (61–62) we deduce that

$$|u_\varepsilon(x)| \leqslant C_0 \Phi^\varepsilon(x) \quad \text{for all } x \in \Gamma^+_{r_0} \text{ and } \varepsilon \in (0,\varepsilon_0). \tag{63}$$

On the other hand, if $x = (x_1,x') \in T^-_\varepsilon$ and $x_1 = \frac12$, then from (36), (54), (27), and (1), it follows that

$$\Phi^\varepsilon(x) \geqslant 2\gamma_\varepsilon \varepsilon e^{\frac{\sqrt{\lambda_1(\Sigma)}}{4\varepsilon}} \psi_1^\Sigma\Big(\frac{x'}{2\varepsilon}\Big) \geqslant \min_{\substack{y' \in \mathbb{R}^{N-1}\\ |y'| \leqslant 1/2}} \psi_1^\Sigma(y') = C_4 > 0$$

thus implying, in view of Lemma 2.2, that

$$|u_\varepsilon(x)| \leqslant \frac{C_1}{C_4} \Phi^\varepsilon(x) \quad \text{for all } x = (x_1,x') \in T^-_\varepsilon \text{ such that } x_1 = \frac12. \tag{64}$$

From (63) and (64) we conclude that

$$|u_\varepsilon(x)| \leqslant \max\Big\{C_0, \frac{C_1}{C_4}\Big\} \Phi^\varepsilon(x) \quad \text{for all } x \in \partial \mathcal{B}_\varepsilon. \tag{65}$$

Since, from (3) and Kato's inequality, $-\Delta|u_\varepsilon| \leqslant 0$ in $\mathcal{B}_\varepsilon$, from (60), (65), and the Maximum Principle we reach the conclusion. □

Let us define

$$\widetilde{u}_\varepsilon : \widetilde{\Omega}^\varepsilon \to \mathbb{R}, \quad \widetilde{u}_\varepsilon(x) = \frac{1}{\varepsilon} u_\varepsilon\big(\mathbf{e}_1 + \varepsilon(x - \mathbf{e}_1)\big), \tag{66}$$

where

$$\widetilde{\Omega}^\varepsilon := \mathbf{e}_1 + \frac{\Omega^\varepsilon - \mathbf{e}_1}{\varepsilon} = \{x \in \mathbb{R}^N : \mathbf{e}_1 + \varepsilon(x - \mathbf{e}_1) \in \Omega^\varepsilon\}. \tag{67}$$

We observe that $\widetilde{u}_\varepsilon$ solves

$$\begin{cases} -\Delta \widetilde{u}_\varepsilon(x) = \varepsilon^2 \lambda_k^\varepsilon p\big(\mathbf{e}_1 + \varepsilon(x - \mathbf{e}_1)\big)\widetilde{u}_\varepsilon(x), & \text{in } \widetilde{\Omega}^\varepsilon, \\ \widetilde{u}_\varepsilon = 0, & \text{on } \partial\widetilde{\Omega}^\varepsilon. \end{cases} \tag{68}$$

From Lemma 2.10, the following uniform estimate on the gradient of u_ε on half-annuli with radius of order ε can de derived.

Lemma 2.11. *For every $1 < r_1 < r_2 < \frac{r_0}{\varepsilon_0}$ there exists $C_{r_1,r_2} > 0$ such that*

$$|\nabla u_\varepsilon(x)| \leqslant C_{r_1,r_2} \quad \text{for all } x \in B^+_{\varepsilon r_2} \setminus B^+_{\varepsilon r_1} \text{ and } \varepsilon \in (0,\varepsilon_0).$$

Proof. From Lemma 2.10 and (57), it follows that, letting $\widetilde{u}_\varepsilon$ as in (66–67),

$$\begin{aligned} |\widetilde{u}_\varepsilon(x)| &\leqslant \frac{C_3}{\varepsilon} \Phi^\varepsilon\big(\mathbf{e}_1 + \varepsilon(x - \mathbf{e}_1)\big) \\ &= C_3\bigg(\Phi_1(x) + 2\gamma_\varepsilon \Phi_2\Big(\frac{x + \mathbf{e}_1}{2}\Big)\bigg) \quad \text{for all } x \in B^+_{r_0/\varepsilon},\ \varepsilon \in (0,\varepsilon_0). \end{aligned} \tag{69}$$

Let us fix R_1, R_2 such that $1 < R_1 < r_1 < r_2 < R_2 < \frac{r_0}{\varepsilon_0}$. From (69) it follows that

$$|\widetilde{u}_\varepsilon(x)| \leqslant \text{const} \quad \text{for all } x \in B^+_{R_2} \setminus B^+_{R_1},\ \varepsilon \in (0, \varepsilon_0),$$

for some const > 0 independent of ε (but depending on R_1, R_2). Hence, from (68) and classical elliptic estimates, we deduce that

$$|\nabla \widetilde{u}_\varepsilon(x)| \leqslant C_{r_1, r_2} \quad \text{for all } x \in B^+_{r_2} \setminus B^+_{r_1},\ \varepsilon \in (0, \varepsilon_0),$$

thus proving the statement. □

A lower bound for u_ε can be given in terms of the function $\widetilde{\Phi}^\varepsilon$ defined in (58).

Lemma 2.12. *There exist $C_5 > 0$ and $\varepsilon_1 \in (0, \varepsilon_0)$ such that*

$$u_\varepsilon(x) \geqslant C_5 \widetilde{\Phi}^\varepsilon(x) \quad \textit{for all } \varepsilon \in (0, \varepsilon_1) \textit{ and } x \in \mathcal{B}_\varepsilon,$$

where $\mathcal{B}_\varepsilon$ is defined in (59) and $\widetilde{\Phi}^\varepsilon$ in (58).

Proof. Let us first observe that

$$-\Delta \widetilde{\Phi}^\varepsilon = 0, \quad \text{in } D^+ \cup T^-_\varepsilon. \tag{70}$$

Moreover, if $x \in \Gamma^+_{r_0}$ and $\varepsilon \in (0, \varepsilon_0)$, then Lemma 2.1 implies that

$$u_\varepsilon(x) \geqslant \frac{1}{C_0}(x_1 - 1). \tag{71}$$

Furthermore, from (54) and (58), we have that

$$\widetilde{\Phi}^\varepsilon(x) \leqslant \varepsilon \Phi_1\Big(\mathbf{e}_1 + \frac{x - \mathbf{e}_1}{\varepsilon}\Big) \quad \text{for all } x \in D^+ \cup T^-_\varepsilon. \tag{72}$$

From Lemma 2.9, there exist $C_6, C_7 > 0$ such that

$$\Phi_1(x) \leqslant (x_1 - 1)\bigg(1 + \frac{C_6}{|x - \mathbf{e}_1|^N}\bigg) \leqslant C_7(x_1 - 1) \quad \text{for all } x \in D^+ \setminus B^+_2. \tag{73}$$

Combining (72) and (73), we obtain that

$$\widetilde{\Phi}^\varepsilon(x) \leqslant C_7(x_1 - 1) \quad \text{for all } x \in D^+ \setminus B^+_{2\varepsilon},$$

which, together with (71), yields

$$\widetilde{\Phi}^\varepsilon(x) \leqslant C_0 C_7 u_\varepsilon(x) \quad \text{for all } x \in \Gamma^+_{r_0} \text{ and } 0 < \varepsilon < \varepsilon_0. \tag{74}$$

On the other hand, if $x = (x_1, x') \in T^-_\varepsilon$ and $x_1 = \frac{1}{2}$, then (58), (72), Lemma 2.9(ii), (54), (27), and Lemma 2.2 yield

$$\widetilde{\Phi}^\varepsilon(x) \leqslant C_2 \varepsilon e^{-\frac{\sqrt{\lambda_1(\Sigma)}}{4\varepsilon}} - \min_{\substack{y' \in \mathbb{R}^{N-1} \\ |y'| \leqslant 1/\sqrt{2}}} \psi_1^\Sigma(y') \leqslant -\frac{1}{2} \min_{\substack{y' \in \mathbb{R}^{N-1} \\ |y'| \leqslant 1/\sqrt{2}}} \psi_1^\Sigma(y') \leqslant \frac{u_\varepsilon(x)}{2C_1} \min_{\substack{y' \in \mathbb{R}^{N-1} \\ |y'| \leqslant 1/\sqrt{2}}} \psi_1^\Sigma(y') \tag{75}$$

provided ε is sufficiently small. Estimates (74) and (75) imply the existence of some $C_5 > 0$ and $\varepsilon_1 > 0$ such that

$$u_\varepsilon(x) \geqslant C_5 \widetilde{\Phi}^\varepsilon(x) \quad \text{for all } \varepsilon \in (0, \varepsilon_1) \text{ and } x \in \partial\mathcal{B}_\varepsilon,$$

which, together with (70) and the Maximum Principle, yields the conclusion. □

Lemma 2.13. *There exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that*

$$u_\varepsilon(x) \geqslant \frac{C_5}{2}(x_1 - 1) \quad \textit{for all } x \in B^+_{r_0} \setminus B^+_{2\varepsilon},\ \varepsilon \in (0, \varepsilon_2).$$

Proof. From (58), (36), and Lemma 2.9, it follows that, for all $x \in B^+_{r_0} \setminus B^+_{2\varepsilon}$,

$$\begin{aligned} \widetilde{\Phi}^\varepsilon(x) &= \varepsilon \Phi_1\Big(\mathbf{e}_1 + \frac{x - \mathbf{e}_1}{\varepsilon}\Big) - \sqrt{2}\widetilde{\gamma}_\varepsilon \varepsilon \Phi_2\Big(\mathbf{e}_1 + \frac{x - \mathbf{e}_1}{\sqrt{2}\varepsilon}\Big) \\ &\geqslant (x_1 - 1) - \text{const}\, \widetilde{\gamma}_\varepsilon(x_1 - 1) \geqslant \frac{1}{2}(x_1 - 1) \end{aligned} \tag{76}$$

provided ε is sufficiently small. The conclusion follows from Lemma 2.12 and (76). □

3. The frequency function

In this section we introduce an Almgren type quotient associated to problem (9) and study its monotonicity properties with the aim of uniformly controlling the transversal frequencies along the connecting tube.

For every $\varepsilon > 0$, let $\xi_\varepsilon : \mathbb{R} \setminus \big((-\varepsilon, 0) \cup (1, 1+\varepsilon)\big) \to \mathbb{R}$ be such that

$$\begin{cases} \xi_\varepsilon(r) = -r, & \text{if } r \leqslant -\varepsilon, \\ \xi_\varepsilon(r) = r, & \text{if } 0 \leqslant r \leqslant 1, \\ \xi_\varepsilon(r) = r - 1, & \text{if } r \geqslant \varepsilon + 1. \end{cases}$$

For $r \in \mathbb{R} \setminus \big((-\varepsilon, 0) \cup (1, 1+\varepsilon)\big)$, we define

$$\Omega_r^\varepsilon = \begin{cases} D^- \setminus \overline{B^-_{\xi_\varepsilon(r)}}, & \text{if } r \leqslant -\varepsilon, \\ D^- \cup \{(x_1, x') \in \mathcal{C}_\varepsilon : x_1 < r\}, & \text{if } 0 \leqslant r \leqslant 1, \\ D^- \cup \mathcal{C}_\varepsilon \cup B^+_{\xi_\varepsilon(r)}, & \text{if } r \geqslant \varepsilon + 1, \end{cases}$$

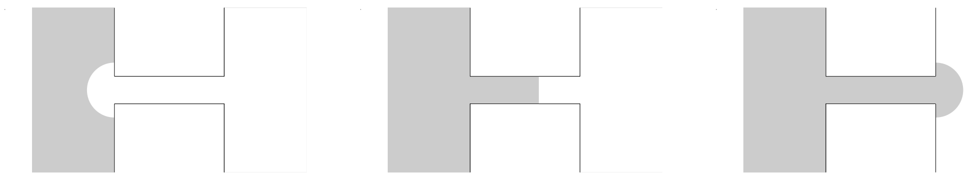
$$\Gamma_r^\varepsilon = \begin{cases} D^- \cap \partial B^-_{\xi_\varepsilon(r)}, & \text{if } r \leqslant -\varepsilon, \\ \{(x_1, x') \in \mathcal{C}_\varepsilon : x_1 = r\}, & \text{if } 0 \leqslant r \leqslant 1, \\ D^+ \cap \partial B^+_{\xi_\varepsilon(r)}, & \text{if } r \geqslant \varepsilon + 1. \end{cases}$$

We also denote

$$\Omega_r := D^- \setminus \overline{B^-_{-r}} \quad \text{for all } r < 0 \tag{77}$$

and notice that

$$\Omega_r^\varepsilon = \Omega_r \quad \text{for all } r \leqslant -\varepsilon.$$



(a) Ω_r^ε with $r \leqslant -\varepsilon$ (b) Ω_r^ε with $0 \leqslant r \leqslant 1$ (c) Ω_r^ε with $r \geqslant \varepsilon$

Figure 2. The moving domains Ω_r^ε for different values of the parameters.

A key role in the definition and in the study of the frequency associated to problem (9) is played by Lemmas 3.4 and 3.6 below, which give a Poincaré type lemma on domains Ω_{-t}, $t > 0$, for functions in $\mathcal{H}_t^-$ and, respectively, a uniform coercivity type estimate for the quadratic form associated to equation (9) in domains Ω_r^ε, $r < 1$. An important ingredient for their proof is the Kelvin transform, which is described in the following remark.

Remark 3.1. For all $R > 0$, $v \in \mathcal{H}_R^-$ if and only if its Kelvin transform $\widetilde{v}(x) = |x|^{-(N-2)} v\big(\frac{x}{|x|^2}\big)$ belongs to $H^1(B^-_{1/R})$ and has null trace on $\partial B^-_{1/R} \cap \partial D^-$; furthermore

$$\int_{B^-_{1/R}} |\nabla \widetilde{v}(x)|^2 dx + (N-2)R \int_{\Gamma^-_{1/R}} \widetilde{v}^2 d\sigma = \int_{\Omega_{-R}} |\nabla v(x)|^2 dx,$$

$$\int_{B^-_{1/R}} |\widetilde{v}(x)|^{2^*} dx = \int_{\Omega_{-R}} |v(x)|^{2^*} dx, \quad \text{and} \quad R^2 \int_{\Gamma^-_{1/R}} \widetilde{v}^2(x)\, d\sigma = \int_{\Gamma^-_R} v^2(x)\, d\sigma.$$

Functions in $\mathcal{H}_t^-$ satisfy the following Sobolev type inequality.

Lemma 3.2. *There exists a constant $C_S = C_S(N)$ depending only on the dimension N such that for all $t > 0$ and $v \in \mathcal{H}_t^-$ there holds*

$$C_S \bigg(\int_{\Omega_{-t}} |v(x)|^{2^*} dx \bigg)^{2/2^*} \leqslant \int_{\Omega_{-t}} |\nabla v(x)|^2 dx.$$

Proof. By scaling it is enough to prove the inequality for $t = 1$, which, in view of remark 3.1, is equivalent to prove that

$$C_S\bigg(\int_{B_1^-}|w(x)|^{2^*}dx\bigg)^{2/2^*}\leqslant\int_{B_1^-}|\nabla w(x)|^2dx+(N-2)\int_{\Gamma_1^-}w^2d\sigma$$

for all $w\in H^1(B_1^-)$ such that $w\equiv 0$ on $\partial B_1^-\cap\partial D^-$. Such inequality follows easily from classical Sobolev embeddings by trivially extending w in $B(\mathbf{0},1)$ and observing that

$$\int_{B(\mathbf{0},1)}|\nabla w(x)|^2dx+(N-2)\int_{\partial B(\mathbf{0},1)}w^2d\sigma$$

is an equivalent norm in $H^1(B(\mathbf{0},1))$. □

The Poincaré inequality we will state in Lemma 3.4 with its best constant is a consequence of the following lemma, which is the counterpart of Lemma 2.5 for the frequency of harmonic functions in $\mathcal{H}_t^-$.

Lemma 3.3. *Let $R>0$ and $\phi\in\mathcal{H}_R^-\setminus\{0\}$ satisfying*

$$\begin{cases}-\Delta\phi=0, & \text{in } \Omega_{-R},\\ \phi=0, & \text{on } \partial\Omega_{-R}\cap\partial D^-,\end{cases}$$

in a weak sense, and let $N_\phi^-:(R,+\infty)\to\mathbb{R}$ be defined as

$$N_\phi^-(r):=\frac{r\int_{\Omega_{-r}}|\nabla\phi(x)|^2dx}{\int_{\Gamma_r^-}\phi^2(x)\,d\sigma}.$$

Then

i) *N_ϕ^- is non-increasing in $(R,+\infty)$;*
ii) *there exists $K_0\in\mathbb{N}$, $K_0\geqslant 1$, such that*
$$\lim_{r\to\infty}N_\phi^-(r)=N-2+K_0;$$
iii) *if $N_\phi^-\equiv\gamma$ for some $\gamma\in\mathbb{R}$ then $\gamma=N-2+K_0$ and $\phi(x)=|x|^{-N+2-K_0}Y(x/|x|)$ for some eigenfunction Y of $-\Delta_{\mathbb{S}^{N-1}}$ associated to the eigenvalue $K_0(N-2+K_0)$, i.e. satisfying $-\Delta_{\mathbb{S}^{N-1}}Y=K_0(N-2+K_0)Y$ on $\mathbb{S}^{N-1}$;*
iv) *if $\phi>0$ in Ω_{-R}, then $K_0=1$.*

Proof. Let $\widetilde{\phi}\in H^1(B_{1/R}^-)$ be the Kelvin transform of ϕ, i.e. $\widetilde{\phi}(x)=|x|^{-(N-2)}\phi\big(\frac{x}{|x|^2}\big)$. Then $\widetilde{\phi}$ satisfies

$$\begin{cases}-\Delta\widetilde{\phi}=0, & \text{in } B_{1/R}^-,\\ \widetilde{\phi}=0, & \text{on } \partial B_{1/R}^-\cap\partial D^-,\end{cases}$$

and, by Remark 3.1, the frequency function N_ϕ^- can be rewritten as

$$N_\phi^-(r)=N-2+\widetilde{N}\Big(\frac{1}{r}\Big), \tag{78}$$

where

$$\widetilde{N}:\Big(0,\frac{1}{R}\Big)\to\mathbb{R},\quad \widetilde{N}(t):=\frac{t\int_{B_t^-}|\nabla\widetilde{\phi}(x)|^2dx}{\int_{\Gamma_t^-}\widetilde{\phi}^2d\sigma}.$$

Let us define

$$\widetilde{\phi}_0(x)=\widetilde{\phi}_0(x_1,x')=\begin{cases}\widetilde{\phi}(x_1,x'), & \text{if } x_1\leqslant 0,\\ -\widetilde{\phi}(-x_1,x'), & \text{if } x_1>0,\end{cases}$$

and observe that $\widetilde{\phi}_0\in H^1(B(\mathbf{0},1/R))$ satisfies $\widetilde{\phi}_0(-x_1,x')=-\widetilde{\phi}_0(x_1,x')$ and weakly solves

$$-\Delta\widetilde{\phi}_0=0,\quad \text{in } B(\mathbf{0},1/R).$$

Moreover

$$\widetilde{N}(t)=\frac{t\int_{B(\mathbf{0},t)}|\nabla\widetilde{\phi}_0(x)|^2dx}{\int_{\partial B(\mathbf{0},t)}\widetilde{\phi}_0^2d\sigma}. \tag{79}$$

From the classical Almgren monotonicity formula [2]

$$\widetilde{N}'(t) = \frac{2t\Big[\Big(\int_{\partial B(\mathbf{0},t)}\Big|\frac{\partial\widetilde{\phi}_0}{\partial\nu}\Big|^2 d\sigma\Big)\Big(\int_{\partial B(\mathbf{0},t)}\widetilde{\phi}_0^2 d\sigma\Big) - \Big(\int_{\partial B(\mathbf{0},t)}\widetilde{\phi}_0\frac{\partial\widetilde{\phi}_0}{\partial\nu}d\sigma\Big)^2\Big]}{\Big(\int_{\partial B(\mathbf{0},t)}\widetilde{\phi}_0^2 d\sigma\Big)^2}, \tag{80}$$

for all $t \in (0, 1/R)$, where $\nu = \nu(x) = \frac{x}{|x|}$, hence from Schwarz's inequality $\widetilde{N}' \geqslant 0$ and the function $t \in (0, 1/R) \mapsto \widetilde{N}(t)$ is non-decreasing, thus implying, in view of (78), that N_ϕ^- is non-increasing in $(R, +\infty)$ and proving statement i). Furthermore from [18, Theorem 1.3] there exist $K_0 \in \mathbb{N}$ and an eigenfunction Y of $-\Delta_{\mathbb{S}^{N-1}}$ associated to the eigenvalue $K_0(N-2+K_0)$, i.e. satisfying $-\Delta_{\mathbb{S}^{N-1}}Y = K_0(N-2+K_0)Y$ on $\mathbb{S}^{N-1}$, such that

$$\lim_{t\to 0^+}\widetilde{N}(t) = -\frac{N-2}{2} + \sqrt{\Big(\frac{N-2}{2}\Big)^2 + K_0(N-2+K_0)} = K_0 \tag{81}$$

and

$$\lambda^{-K_0}\widetilde{\phi}_0(\lambda\theta) \to Y(\theta) \quad \text{in } C^{1,\tau}(\mathbb{S}^{N-1}), \tag{82}$$

$$\lambda^{1-K_0}\nabla\widetilde{\phi}_0(\lambda\theta) \to K_0Y(\theta)\theta + \nabla_{\mathbb{S}^{N-1}}Y(\theta) \quad \text{in } C^{0,\tau}(\mathbb{S}^{N-1}), \tag{83}$$

as $\lambda \to 0^+$, for every $\tau \in (0,1)$. Since $\widetilde{\phi}_0$ vanishes on $B(\mathbf{0},1)\cap(\{0\}\times\mathbb{R}^{N-1})$, from (82) we infer that Y vanishes on the equator $\mathbb{S}^{N-1}\cap(\{0\}\times\mathbb{R}^{N-1})$. Therefore, Y can not be the first eigenfunction of $-\Delta_{\mathbb{S}^{N-1}}$ and hence $K_0 \geqslant 1$ necessarily. Statement ii) then follows from (78) and (81).

Let us now assume that $N_\phi^- \equiv \gamma$ for some $\gamma \in \mathbb{R}$, so that $\widetilde{N}(t) \equiv \gamma - N + 2$ in $(0, 1/R)$ and hence $\widetilde{N}'(t) = 0$ for any $t \in (0, 1/R)$. By (80) we obtain

$$\Big(\int_{\partial B(\mathbf{0},t)}\Big|\frac{\partial\widetilde{\phi}_0}{\partial\nu}\Big|^2 d\sigma\Big)\cdot\Big(\int_{\partial B(\mathbf{0},t)}\widetilde{\phi}_0^2 d\sigma\Big) - \Big(\int_{\partial B(\mathbf{0},t)}\widetilde{\phi}_0\frac{\partial\widetilde{\phi}_0}{\partial\nu}d\sigma\Big)^2 = 0 \quad \text{for all } t \in (0, 1/R),$$

i.e. $\widetilde{\phi}_0$ and $\frac{\partial\widetilde{\phi}_0}{\partial\nu}$ have the same direction as vectors in $L^2(\partial B(\mathbf{0},t))$ and hence there exists a function $\eta = \eta(t)$ such that $\frac{\partial\widetilde{\phi}_0}{\partial\nu}(t,\theta) = \eta(t)\widetilde{\phi}_0(t,\theta)$ for $t \in (0, 1/R)$ and $\theta \in \mathbb{S}^{N-1}$. After integration we obtain

$$\widetilde{\phi}_0(t,\theta) = e^{\int_{1/R}^t \eta(s)ds}\widetilde{\phi}_0\Big(\frac{1}{R},\theta\Big) = \varphi(t)\psi(\theta), \quad t \in (0, 1/R),\ \theta \in \mathbb{S}^{N-1}, \tag{84}$$

where $\varphi(t) = e^{\int_{1/R}^t \eta(s)ds}$ and $\psi(\theta) = \widetilde{\phi}_0\big(\frac{1}{R},\theta\big)$. Since $-\Delta\widetilde{\phi}_0 = 0$ in $B(\mathbf{0}, 1/R)$, (84) yields

$$\Big(-\varphi''(t) - \frac{N-1}{t}\varphi'(t)\Big)\psi(\theta) - \frac{\varphi(t)}{t^2}\Delta_{\mathbb{S}^{N-1}}\psi(\theta) = 0.$$

Taking t fixed we deduce that ψ is an eigenfunction of the operator $-\Delta_{\mathbb{S}^{N-1}}$. If $K_0(N-2+K_0)$ is the corresponding eigenvalue then $\varphi(t)$ solves the equation

$$-\varphi''(t) - \frac{N-1}{t}\varphi(t) + \frac{K_0(N-2+K_0)}{t^2}\varphi(t) = 0$$

and hence $\varphi(t)$ is of the form

$$\varphi(t) = c_1t^{K_0} + c_2t^{-(N-2)-K_0}, \quad \text{for some } c_1, c_2 \in \mathbb{R}.$$

Since the function $|x|^{-(N-2)-K_0}\psi(\frac{x}{|x|}) \notin H^1(B_{1/R})$, then $c_2 = 0$ and $\varphi(t) = c_1t^{K_0}$. Since $\varphi(\frac{1}{R}) = 1$, we obtain that $c_1 = R^{K_0}$ and then

$$\widetilde{\phi}_0(t,\theta) = R^{K_0}t^{K_0}\psi(\theta), \quad \text{for all } t \in (0, 1/R) \text{ and } \theta \in \mathbb{S}^{N-1}. \tag{85}$$

Therefore $\phi(y) = R^{K_0}|y|^{-N+2-K_0}\psi(\frac{y}{|y|})$ in Ω_{-R}. Substituting (85) into (79) and taking into account that $\widetilde{N}(t) \equiv \gamma - N + 2$, we obtain that necessarily $\gamma - N + 2 = K_0$, i.e. $\gamma = N - 2 + K_0$. Claim iii) is thereby proved.

If $\phi > 0$ in Ω_{-R}, then $\widetilde{\phi} > 0$ in $B_{1/R}^-$, and Hopf's Lemma implies that

$$\frac{\partial\widetilde{\phi}}{\partial x_1}(0, x') < 0, \quad \text{for all } x' \in \mathbb{R}^{N-1} \text{ s.t. } |x'| < \frac{1}{R}. \tag{86}$$

(86) and (83) imply that $K_0 \leqslant 1$. Hence $K_0 = 1$ and statement iv) is proved. □

We are now ready to prove the following Poincaré type inequality.

Lemma 3.4. *For all $t > 0$ and $v \in \mathcal{H}_t^-$ there holds*

$$\frac{1}{t^{N-2}} \int_{\Omega_{-t}} |\nabla v(x)|^2 dx \geqslant \frac{N-1}{t^{N-1}} \int_{\Gamma_t^-} v^2 d\sigma,$$

being $N-1$ the optimal constant.

Proof. By scaling it is enough to prove the inequality for $t = 1$, i.e. the statement of the lemma is equivalent to prove that the infimum

$$\mathcal{I} := \inf_{w \in \mathcal{H}_1^- \setminus \{0\}} \frac{\int_{\Omega_{-1}} |\nabla w(x)|^2 dx}{\int_{\Gamma_1^-} w^2 d\sigma}$$

is equal to $N-1$. By standard minimization arguments and compactness of the embedding $\mathcal{H}_1^- \hookrightarrow L^2(\Gamma_1^-)$, it is easy to prove that the infimum $\mathcal{I}$ is strictly positive and attained by some function $w_0 \in \mathcal{H}_1^- \setminus \{0\}$ satisfying

$$\begin{cases} -\Delta w_0 = 0, & \text{in } \Omega_{-1}, \\ w_0 > 0, & \text{in } \Omega_{-1}, \\ \frac{\partial w_0}{\partial \nu} = -\mathcal{I} w_0, & \text{on } \Gamma_1^-, \\ w_0 \equiv 0, & \text{on } \partial\Omega_{-1} \cap \partial D^-, \end{cases}$$

being $\nu = \frac{x}{|x|}$. Then Lemma 3.3 implies that

$$\mathcal{I} = \frac{\int_{\Omega_{-1}} |\nabla w_0(x)|^2 dx}{\int_{\Gamma_1^-} w_0^2 d\sigma} = N_{w_0}^-(1) \geqslant \lim_{r \to +\infty} N_{w_0}^-(r) \geqslant N-1.$$

On the other hand the quotient $\big(\int_{\Omega_{-1}} |\nabla w(x)|^2 dx\big)\big(\int_{\Gamma_1^-} w^2 d\sigma\big)^{-1}$ evaluated in $w(x_1, x') = \frac{x_1}{|x|^N}$ is equal to $N-1$, thus implying that $\mathcal{I} \leqslant N-1$. □

Remark 3.5. By remark 3.1, Lemma 3.4 is equivalent to

$$r \int_{B_r^-} |\nabla w(x)|^2 dx \geqslant \int_{\Gamma_r^-} w^2 d\sigma \quad \text{for all } w \in H^1(B_r^-) \text{ such that } w \equiv 0 \text{ on } \partial B_r^- \cap \partial D^-.$$

Lemma 3.6 below provides a uniform coercivity type estimate for the quadratic form associated to equation (9), whose validity is strongly related to the nondegeneracy condition (5).

Lemma 3.6.

i) *For every $f \in L^{N/2}(\mathbb{R}^N)$ and $M > 0$, there exist $\tilde{r}_{M,f} > 0$ and $\tilde{\varepsilon}_{M,f} \in (0, \varepsilon_0)$ such that for all $\varepsilon \in (0, \tilde{\varepsilon}_{M,f})$ and $r \in (\varepsilon, \tilde{r}_{M,f})$*

$$\int_{\Omega_{-r}} |\nabla u_\varepsilon(x)|^2 dx \geqslant M \int_{\Omega_{-r}} |f(x)| u_\varepsilon^2(x) dx.$$

ii) *For every $f \in L^{N/2}(\mathbb{R}^N)$ and $M > 0$, there exists $\bar{\varepsilon}_{M,f} \in (0, \varepsilon_0)$ such that for all $r \in (0,1)$ and $\varepsilon \in (0, \bar{\varepsilon}_{M,f})$*

$$\int_{\Omega_r^\varepsilon} |\nabla u_\varepsilon(x)|^2 dx \geqslant M \int_{\Omega_r^\varepsilon} |f(x)| u_\varepsilon^2(x) dx.$$

Proof. To prove i), we argue by contradiction and assume that there exist $f \in L^{N/2}(\mathbb{R}^N)$, $M > 0$, and sequences $\varepsilon_n \to 0^+$, $r_n \to 0^+$, such that $r_n > \varepsilon_n$ and, denoting $u_n = u_{\varepsilon_n}$,

$$\int_{\Omega_{-r_n}} |\nabla u_n(x)|^2 dx < M \int_{\Omega_{-r_n}} |f(x)| u_n^2(x) dx. \tag{87}$$

Let us define

$$v_n(x) = \begin{cases} u_n(x), & \text{if } x \in \Omega_{-r_n}, \\ \big(\frac{r_n}{|x|}\big)^{N-2} u_n\big(\frac{r_n^2 x}{|x|^2}\big), & \text{if } x \in B_{r_n}^-. \end{cases}$$

We notice that $v_n \in \mathcal{D}^{1,2}(D^-)$ and, by Remark 3.1,

$$\int_{B^-_{r_n}} |\nabla v_n(x)|^2 dx + \frac{N-2}{r_n} \int_{\Gamma^-_{r_n}} v_n^2 d\sigma = \int_{\Omega_{-r_n}} |\nabla u_n(x)|^2 dx,$$

thus implying

$$\int_{D^-} |\nabla v_n(x)|^2 dx \leqslant \int_{D^-} |\nabla v_n(x)|^2 dx + \frac{N-2}{r_n} \int_{\Gamma^-_{r_n}} v_n^2 d\sigma = 2\int_{\Omega_{-r_n}} |\nabla u_n(x)|^2 dx. \tag{88}$$

From (87) and (88) it follows that, if

$$w_n = \frac{v_n}{\left(\int_{\Omega_{-r_n}} |f(x)| u_n^2(x) dx\right)^{1/2}},$$

then $w_n \in \mathcal{D}^{1,2}(D^-)$ and

$$\int_{D^-} |\nabla w_n(x)|^2 dx < 2M.$$

Hence there exists a subsequence $\{w_{n_k}\}_k$ such that

$$w_{n_k} \rightharpoonup w \quad \text{weakly in } \mathcal{D}^{1,2}(D^-)$$

for some $w \in \mathcal{D}^{1,2}(D^-)$. From $\int_{D^-} |f(x)| v_n^2(x) dx \geqslant \int_{\Omega_{-r_n}} |f(x)| u_n^2(x) dx$ we deduce that

$$\int_{D^-} |f(x)| w_n^2(x) dx \geqslant 1$$

which implies that $w \not\equiv 0$. Since w_n solves

$$\begin{cases} -\Delta w_n = \lambda_{\bar{k}}^{\varepsilon_n} p w_n, & \text{in } \Omega_{-r_n}, \\ w_n = 0, & \text{on } \partial D^-, \end{cases}$$

and $r_n \to 0^+$, from (7) we conclude that w weakly solves

$$\begin{cases} -\Delta w = \lambda_{k_0}(D^+) p w, & \text{in } D^-, \\ w = 0, & \text{on } \partial D^-, \end{cases}$$

thus implying $\lambda_{k_0}(D^+) \in \sigma_p(D^-)$ and contradicting assumption (5).

Let us now prove ii). We argue by contradiction and assume that there exist $f \in L^{N/2}(\mathbb{R}^N)$, $M > 0$, and sequences $\varepsilon_n \to 0^+$, $r_n \in (0,1)$, such that denoting $u_n = u_{\varepsilon_n}$,

$$\int_{\Omega^{\varepsilon_n}_{r_n}} |\nabla u_n(x)|^2 dx < M \int_{\Omega^{\varepsilon_n}_{r_n}} |f(x)| u_n^2(x) dx. \tag{89}$$

Let us define

$$v_n(x) = \begin{cases} u_n(x), & \text{if } x \in \Omega^{\varepsilon_n}_{r_n}, \\ u_n(2r_n \mathbf{e}_1 - x), & \text{if } 2r_n \mathbf{e}_1 - x \in \Omega^{\varepsilon_n}_{r_n}, \\ 0, & \text{otherwise}. \end{cases}$$

We notice that $v_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and, by (89),

$$\int_{\mathbb{R}^N} |\nabla v_n(x)|^2 dx = 2\int_{\Omega^{\varepsilon_n}_{r_n}} |\nabla u_n(x)|^2 dx < 2M \int_{\Omega^{\varepsilon_n}_{r_n}} |f(x)| u_n^2(x) dx,$$

thus implying that, letting

$$w_n = \frac{v_n}{\left(\int_{\Omega^{\varepsilon_n}_{r_n}} |f(x)| u_n^2(x) dx\right)^{1/2}},$$

then $w_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |\nabla w_n(x)|^2 dx < 2M.$$

Hence there exist a subsequence $\{w_{n_k}\}_k$ and some $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $w_{n_k} \rightharpoonup w$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. From

$$1 = \int_{\Omega^{\varepsilon_n}_{r_n}} |f(x)| w_n^2(x)dx = \int_{D^-} |f(x)|w_n^2(x)dx + \int_{\Omega^{\varepsilon_n}_{r_n}\setminus D^-} |f(x)|w_n^2(x)dx,$$

$$\int_{\Omega^{\varepsilon_n}_{r_n}\setminus D^-} |f(x)|w_n^2(x)dx \leqslant \|w_n\|^2_{L^{2^*}(\mathbb{R}^N)}\|f\|_{L^{N/2}(\Omega^{\varepsilon_n}_{r_n}\setminus D^-)} = o(1) \quad \text{as } n \to +\infty,$$

$$\int_{D^-} |f(x)|w_{n_k}^2(x)dx = \int_{D^-} |f(x)|w^2(x)dx + o(1) \quad \text{as } k \to +\infty,$$

we deduce that

$$\int_{D^-} |f(x)|w^2(x)dx = 1$$

and hence $w \not\equiv 0$ in D^-. On the other hand, a.e. convergence of w_{n_k} to w implies that $w = 0$ on ∂D^-. Furthermore, passing to the weak limit in the equation $-\Delta w_{n_k} = \lambda_{\bar k}^{\varepsilon_{n_k}} p w_{n_k}$ satisfied by w_{n_k} in D^-, we conclude that w weakly solves

$$\begin{cases} -\Delta w = \lambda_{k_0}(D^+)pw, & \text{in } D^-,\\ w = 0, & \text{on } \partial D^-, \end{cases}$$

thus implying $\lambda_{k_0}(D^+) \in \sigma_p(D^-)$ and contradicting assumption (5). □

From Lemma 3.6 and (3), there exist $\check R \in (0,1)$ and $\check\varepsilon \in (0, \varepsilon_0)$ such that, for every $\varepsilon \in (0, \check\varepsilon)$,

$$\int_{\Omega^\varepsilon_r} \Big(|\nabla u_\varepsilon|^2 - \lambda^\varepsilon_{\bar k} p u_\varepsilon^2\Big)dx \geqslant \frac{1}{2}\int_{\Omega^\varepsilon_r} |\nabla u_\varepsilon|^2 dx \quad \text{for all } r \in (-\check R, -\varepsilon) \cup (0,1), \tag{90}$$

and

$$\int_{\Omega^\varepsilon_r} \Big(|\nabla u_\varepsilon|^2 - \lambda^\varepsilon_{\bar k} p u_\varepsilon^2\Big)dx \geqslant \frac{1}{2}\int_{\Omega^\varepsilon_{1/2}} |\nabla u_\varepsilon|^2 dx \quad \text{for all } r \in (1+\varepsilon, 4). \tag{91}$$

Estimates (90) and (91), together with equation (9) and classical unique continuation principle, imply that

$$\int_{\Gamma^\varepsilon_r} u_\varepsilon^2(x)\, d\sigma > 0 \quad \text{for all } \varepsilon \in (0,\check\varepsilon) \text{ and } r \in (-\check R, -\varepsilon) \cup (0,1) \cup (1+\varepsilon, 4).$$

Therefore, for all $\varepsilon \in (0, \check\varepsilon)$, the frequency function $\mathcal{N}_\varepsilon : (-\check R, -\varepsilon) \cup (0,1) \cup (1+\varepsilon, 4) \to \mathbb{R}$,

$$\mathcal{N}_\varepsilon(r) = \frac{\Lambda_N(r,\varepsilon)\int_{\Omega^\varepsilon_r}\Big(|\nabla u_\varepsilon(x)|^2 - \lambda^\varepsilon_{\bar k} p(x) u_\varepsilon^2(x)\Big)dx}{\int_{\Gamma^\varepsilon_r} u_\varepsilon^2(x)\, d\sigma},$$

where

$$\Lambda_N(r,\varepsilon) = \begin{cases} \big(\frac{2}{\omega_{N-1}}\big)^{\frac{1}{N-1}} |\Gamma^\varepsilon_r|^{\frac{1}{N-1}} = \xi_\varepsilon(r), & \text{if } r \in (-\infty, -\varepsilon) \cup (1+\varepsilon, +\infty),\\ \big(\frac{N-1}{\omega_{N-2}}\big)^{\frac{1}{N-1}} |\Gamma^\varepsilon_r|^{\frac{1}{N-1}} = \varepsilon, & \text{if } r \in [0,1], \end{cases}$$

and $|\Gamma^\varepsilon_r|$ denotes the $(N-1)$-dimensional volume of Γ^ε_r, is well defined.

3.1. **The frequency function at the right.**

If $\varepsilon \in (0, \check\varepsilon)$ and $r \in (1+\varepsilon, 4)$, then

$$\mathcal{N}_\varepsilon(r) = \mathcal{N}^+_\varepsilon(r-1) \tag{92}$$

where, for $t \in (\varepsilon, 3)$,

$$\mathcal{N}^+_\varepsilon(t) = \frac{D^+_\varepsilon(t)}{H^+_\varepsilon(t)}, \tag{93}$$

$$D^+_\varepsilon(t) = \frac{1}{t^{N-2}}\int_{D^-\cup\mathcal{C}_\varepsilon\cup B^+_t} \Big(|\nabla u_\varepsilon(x)|^2 - \lambda^\varepsilon_{\bar k} p u_\varepsilon^2(x)\Big)dx,$$

$$H^+_\varepsilon(t) = \frac{1}{t^{N-1}}\int_{\Gamma^+_t} u_\varepsilon^2(x)\, d\sigma,$$

with Γ^+_t as defined in (26). The behavior of $\mathcal{N}^+_\varepsilon$ for small t and ε is described by the following proposition.

Proposition 3.7. *There holds*

$$\lim_{t\to 0^+}\Big(\lim_{\varepsilon\to 0^+}\mathcal{N}_\varepsilon^+(t)\Big)=\lim_{r\to 1^+}\Big(\lim_{\varepsilon\to 0^+}\mathcal{N}_\varepsilon(r)\Big)=1.$$

PROOF. Let us first notice that the strong $\mathcal{D}^{1,2}(\mathbb{R}^N)$ convergence of u_ε to $\varphi_{k_0}^+$ ensured by Lemma 1.1 implies that, for all $t\in(0,3)$,

$$\lim_{\varepsilon\to 0^+}\mathcal{N}_\varepsilon^+(t)=\mathcal{N}^+(t) \tag{94}$$

where

$$\mathcal{N}^+(t)=\frac{t\int_{B_t^+}\Big(|\nabla\varphi_{k_0}^+(x)|^2-\lambda_{k_0}(D^+)p(x)(\varphi_{k_0}^+(x))^2\Big)dx}{\int_{\Gamma_t^+}(\varphi_{k_0}^+(x))^2\,d\sigma}.$$

Let us define

$$\varphi_0(x)=\varphi_0(x_1,x')=\begin{cases}\varphi_{k_0}^+(x_1+1,x'), & \text{if } x_1\geqslant 0,\\ -\varphi_{k_0}^+(-x_1+1,x'), & \text{if } x_1<0,\end{cases}$$

and observe that $\varphi_0\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ satisfies $\varphi_0(-x_1,x')=-\varphi_0(x_1,x')$ and weakly solves

$$-\Delta\varphi_0(x)=\lambda_{k_0}(D^+)p_0(x)\varphi_0(x),$$

where

$$p_0(x)=p_0(x_1,x')=\begin{cases}p(x_1+1,x'), & \text{if } x_1\geqslant 0,\\ p(-x_1+1,x'), & \text{if } x_1<0.\end{cases}$$

Moreover $\mathcal{N}^+$ can be rewritten as

$$\mathcal{N}^+(t)=\frac{t\int_{B(\mathbf{0},t)}\Big(|\nabla\varphi_0(x)|^2-\lambda_{k_0}(D^+)p_0(x)\varphi_0^2(x)\Big)dx}{\int_{\partial B(\mathbf{0},t)}\varphi_0^2(x)\,d\sigma}.$$

Hence, from [18, Theorem 1.3] it follows that there exist $j_0\in\mathbb{N}$ and an eigenfunction Y of $-\Delta_{\mathbb{S}^{N-1}}$ associated to the eigenvalue $j_0(N-2+j_0)$, i.e. satisfying $-\Delta_{\mathbb{S}^{N-1}}Y=j_0(N-2+j_0)Y$ on $\mathbb{S}^{N-1}$, such that

$$\lim_{t\to 0^+}\mathcal{N}^+(t)=-\frac{N-2}{2}+\sqrt{\Big(\frac{N-2}{2}\Big)^2+j_0(N-2+j_0)}=j_0 \tag{95}$$

and

$$\lambda^{-j_0}\varphi_0(\lambda\theta)\to Y(\theta)\quad\text{in } C^{1,\tau}(\mathbb{S}^{N-1}), \tag{96}$$

$$\lambda^{1-j_0}\nabla\varphi_0(\lambda\theta)\to j_0Y(\theta)\theta+\nabla_{\mathbb{S}^{N-1}}Y(\theta)\quad\text{in } C^{0,\tau}(\mathbb{S}^{N-1}), \tag{97}$$

as $\lambda\to 0^+$, for every $\tau\in(0,1)$. Since the nodal set of φ_0 is $\{0\}\times\mathbb{R}^{N-1}$, we infer that Y vanishes on the equator $\mathbb{S}^{N-1}\cap(\{0\}\times\mathbb{R}^{N-1})$. Therefore, Y can not be the first eigenfunction of $-\Delta_{\mathbb{S}^{N-1}}$ and hence $j_0\geqslant 1$ necessarily. On the other hand, (6) and (97) imply that $j_0\leqslant 1$. Hence $j_0=1$. The conclusion hence follows from (94) and (95). □

Lemma 3.8. *For all $\varepsilon\in(0,\breve\varepsilon)$ and $t\in(2\varepsilon,3)$ there holds*

$$t\int_{\Gamma_t^+}|\nabla u_\varepsilon|^2d\sigma=2\varepsilon\int_{\Gamma_{2\varepsilon}^+}\Big(|\nabla u_\varepsilon|^2-2\Big|\frac{\partial u_\varepsilon}{\partial\nu}\Big|^2\Big)+(N-2)\int_{B_t^+\setminus B_{2\varepsilon}^+}|\nabla u_\varepsilon(x)|^2dx+2t\int_{\Gamma_t^+}\Big|\frac{\partial u_\varepsilon}{\partial\nu}\Big|^2d\sigma.$$

PROOF. The stated identity follows from multiplication of equation (9) by $(x-\mathbf{e}_1)\cdot\nabla u_\varepsilon$ and integration by parts over $B_t^+\setminus B_{2\varepsilon}^+$. □

Lemma 3.9. *For all $\varepsilon\in(0,\breve\varepsilon)$, $\mathcal{N}_\varepsilon^+\in C^1(2\varepsilon,3)$ and*

$$(\mathcal{N}_\varepsilon^+)'(t)=\frac{2t\Big[\big(\int_{\Gamma_t^+}\big|\frac{\partial u_\varepsilon}{\partial\nu}\big|^2d\sigma\big)\big(\int_{\Gamma_t^+}u_\varepsilon^2d\sigma\big)-\big(\int_{\Gamma_t^+}u_\varepsilon\frac{\partial u_\varepsilon}{\partial\nu}d\sigma\big)^2\Big]}{\big(\int_{\Gamma_t^+}u_\varepsilon^2d\sigma\big)^2}+\frac{R_\varepsilon^+}{\int_{\Gamma_t^+}u_\varepsilon^2d\sigma},$$

for all $t\in(2\varepsilon,3)$, where

$$R_\varepsilon^+=\int_{\Gamma_{2\varepsilon}^+}\Big(-(N-2)u_\varepsilon\frac{\partial u_\varepsilon}{\partial\nu}+2\varepsilon|\nabla u_\varepsilon|^2-4\varepsilon\Big|\frac{\partial u_\varepsilon}{\partial\nu}\Big|^2\Big)d\sigma. \tag{98}$$

PROOF. Multiplication of equation (9) by u_ε and integration by parts over $D^- \cup \mathcal{C}_\varepsilon \cup B_t^+$ yield, for every $t > \varepsilon$,

$$\int_{D^- \cup \mathcal{C}_\varepsilon \cup B_t^+} \Big(|\nabla u_\varepsilon(x)|^2 - \lambda_{\bar k}^\varepsilon p u_\varepsilon^2(x)\Big) dx = \int_{\Gamma_t^+} \frac{\partial u_\varepsilon}{\partial \nu} u_\varepsilon \, d\sigma. \tag{99}$$

From Lemma 3.8 and (99) we deduce

$$\begin{aligned}(D_\varepsilon^+)'(t) &= \frac{d}{dt}\bigg(\frac{1}{t^{N-2}}\int_{D^- \cup \mathcal{C}_\varepsilon \cup B_{2\varepsilon}^+} \Big(|\nabla u_\varepsilon(x)|^2 - \lambda_{\bar k}^\varepsilon p u_\varepsilon^2(x)\Big) dx + \frac{1}{t^{N-2}} \int_{B_t^+ \setminus B_{2\varepsilon}^+} |\nabla u_\varepsilon(x)|^2 dx\bigg) \\ &= -\frac{N-2}{t^{N-1}} \int_{D^- \cup \mathcal{C}_\varepsilon \cup B_{2\varepsilon}^+} \Big(|\nabla u_\varepsilon(x)|^2 - \lambda_{\bar k}^\varepsilon p u_\varepsilon^2(x)\Big) dx \\ &\quad - \frac{N-2}{t^{N-1}} \int_{B_t^+ \setminus B_{2\varepsilon}^+} |\nabla u_\varepsilon(x)|^2 dx + \frac{1}{t^{N-2}} \int_{\Gamma_t^+} |\nabla u_\varepsilon|^2 d\sigma \\ &= -\frac{N-2}{t^{N-1}} \int_{\Gamma_{2\varepsilon}^+} \frac{\partial u_\varepsilon}{\partial \nu} u_\varepsilon \, d\sigma + \frac{2\varepsilon}{t^{N-1}} \int_{\Gamma_{2\varepsilon}^+} \bigg(|\nabla u_\varepsilon|^2 - 2\Big|\frac{\partial u_\varepsilon}{\partial \nu}\Big|^2\bigg) d\sigma + \frac{2}{t^{N-2}} \int_{\Gamma_t^+} \Big|\frac{\partial u_\varepsilon}{\partial \nu}\Big|^2 d\sigma \\ &= \frac{2}{t^{N-2}} \int_{\Gamma_t^+} \Big|\frac{\partial u_\varepsilon}{\partial \nu}\Big|^2 d\sigma + \frac{R_\varepsilon^+}{t^{N-1}}\end{aligned} \tag{100}$$

for all $t \in (2\varepsilon, 3)$. Furthermore

$$(H_\varepsilon^+)'(t) = \frac{2}{t^{N-1}} \int_{\Gamma_t^+} \frac{\partial u_\varepsilon}{\partial \nu} u_\varepsilon \, d\sigma \tag{101}$$

which, in view of (99), implies

$$(H_\varepsilon^+)'(t) = \frac{2}{t} D_\varepsilon^+(t) \quad \text{for all } t \in (\varepsilon, 3). \tag{102}$$

From (93) and (102) it follows that

$$(\mathcal{N}_\varepsilon^+)'(t) = \frac{(D_\varepsilon^+)'(t) H_\varepsilon^+(t) - \frac{t}{2}\big((H_\varepsilon^+)'(t)\big)^2}{(H_\varepsilon^+(t))^2}$$

which yields the conclusion in view of (100) and (101). □

Lemma 3.10. *For $\varepsilon \in (0, \check\varepsilon)$, let R_ε^+ as in (98). There exists $C_8 > 0$ such that, for all $\varepsilon \in (0, \check\varepsilon)$,*

$$|R_\varepsilon^+| \leqslant C_8 \varepsilon^N.$$

PROOF. From (98), Lemmas 2.10 and 2.11, and (57), it follows that, for all $\varepsilon \in (0, \check\varepsilon)$,

$$|R_\varepsilon^+| \leqslant \text{const} \int_{\Gamma_{2\varepsilon}^+} (\varepsilon + \Phi^\varepsilon) \, d\sigma = \text{const} \bigg(2^{N-2}\varepsilon^N \omega_{N-1} + \varepsilon^N \int_{\Gamma_2^+} \Phi_1 d\sigma + 2^N \gamma_\varepsilon \varepsilon^N \int_{\Gamma_1^+} \Phi_2 d\sigma\bigg)$$

thus implying the conclusion. □

As a consequence of the above estimates, we finally obtain the following uniform control of the frequency close to the right junction of the tube.

Lemma 3.11. *There exists $C_9 > 0$ such that, for all $\varepsilon \in (0, \min\{\varepsilon_2, \check\varepsilon\})$ and $t \in (2\varepsilon, r_0)$,*

$$(\mathcal{N}_\varepsilon^+)'(t) \geqslant -C_9 \frac{\varepsilon^N}{t^{N+1}} \tag{103}$$

PROOF. From Lemma 2.13, we deduce that, for all $t \in (2\varepsilon, r_0)$ and $\varepsilon \in (0, \min\{\varepsilon_2, \check\varepsilon\})$,

$$\int_{\Gamma_t^+} u_\varepsilon^2 d\sigma \geqslant \frac{C_5^2}{4} \int_{\Gamma_t^+} (x_1 - 1)^2 d\sigma = \frac{C_5^2}{8} t^{N+1} \int_{\mathbb{S}^{N-1}} |\theta \cdot \mathbf{e}_1|^2 d\sigma(\theta). \tag{104}$$

The conclusion follows from Lemma 3.9, Schwarz's inequality, Lemma 3.10, and (104). □

Corollary 3.12. *For all $\varepsilon \in (0, \min\{\varepsilon_2, \check\varepsilon\})$ and r_1, r_2 such that $1 + 2\varepsilon < r_1 < r_2 < 1 + r_0$ there holds*

$$\mathcal{N}_\varepsilon(r_1) \leqslant \mathcal{N}_\varepsilon(r_2) + \frac{C_9}{N}\Big(\frac{\varepsilon}{r_1 - 1}\Big)^N \leqslant \mathcal{N}_\varepsilon(r_2) + \frac{C_9}{N 2^N}.$$

PROOF. It follows from (92) and integration of (103). □

Corollary 3.13. *For every $\delta > 0$ there exist $\tilde{r}_\delta, \widetilde{R}_\delta > 0$ such that*

$$\mathcal{N}_\varepsilon(1+R\varepsilon) \leqslant 1+\delta \quad \textit{for all } R > \widetilde{R}_\delta \textit{ and } \varepsilon \in \left(0, \frac{\tilde{r}_\delta}{R}\right).$$

PROOF. Let $\delta > 0$. From Proposition 3.7 there exist $\tilde{r}_\delta \in (0, r_0)$ and $\tilde{\varepsilon}_\delta > 0$ such that

$$\mathcal{N}_\varepsilon(1+\tilde{r}_\delta) \leqslant 1+\frac{\delta}{2} \quad \text{for all } \varepsilon \in (0, \tilde{\varepsilon}_\delta). \tag{105}$$

Let $\widetilde{R}_\delta > \max\{2, \tilde{r}_\delta/\min\{\varepsilon_2, \check{\varepsilon}\}\}$ be such that $\frac{C_9}{N}\widetilde{R}_\delta^{-N} < \frac{\delta}{2}$. Then, from Corollary 3.12, for all $R > \widetilde{R}_\delta$ and $\varepsilon \in \left(0, \frac{\tilde{r}_\delta}{R}\right)$ there holds

$$\mathcal{N}_\varepsilon(1+R\varepsilon) \leqslant \mathcal{N}_\varepsilon(1+\tilde{r}_\delta) + \frac{C_9}{N}R^{-N} \leqslant \mathcal{N}_\varepsilon(1+\tilde{r}_\delta) + \frac{C_9}{N}\widetilde{R}_\delta^{-N} \leqslant \mathcal{N}_\varepsilon(1+\tilde{r}_\delta) + \frac{\delta}{2}. \tag{106}$$

The conclusion follows from (105) and (106). □

3.2. **The frequency function at the left.** If $\varepsilon \in (0, \check{\varepsilon})$ and $r \in (-\check{R}, -\varepsilon)$, then

$$\mathcal{N}_\varepsilon(r) = \mathcal{N}_\varepsilon^-(-r)$$

where, for $t \in (\varepsilon, \check{R})$,

$$\mathcal{N}_\varepsilon^-(t) = \frac{D_\varepsilon^-(t)}{H_\varepsilon^-(t)}, \tag{107}$$

$$D_\varepsilon^-(t) = \frac{1}{t^{N-2}} \int_{\Omega_{-t}} \Big(|\nabla u_\varepsilon(x)|^2 - \lambda_{\bar{k}}^\varepsilon p(x) u_\varepsilon^2(x)\Big) dx, \tag{108}$$

$$H_\varepsilon^-(t) = \frac{1}{t^{N-1}} \int_{\Gamma_t^-} u_\varepsilon^2(x)\, d\sigma, \tag{109}$$

with Γ_t^- defined in (10).

Lemma 3.14. *For $t > \varepsilon$ there holds*

$$t\int_{\Gamma_t^-} \Big(|\nabla u_\varepsilon|^2 - \lambda_{\bar{k}}^\varepsilon p u_\varepsilon^2\Big) d\sigma = 2t \int_{\Gamma_t^-} \left|\frac{\partial u_\varepsilon}{\partial \nu}\right|^2 d\sigma - (N-2)\int_{\Omega_{-t}} |\nabla u_\varepsilon(x)|^2 dx$$
$$+ \lambda_{\bar{k}}^\varepsilon \int_{\Omega_{-t}} (Np(x) + x\cdot\nabla p(x)) u_\varepsilon^2(x) dx,$$

where $\nu = \nu(x) = \frac{x}{|x|}$.

PROOF. The stated identity follows from multiplication of equation (9) by $x \cdot \nabla u_\varepsilon$ and integration by parts over Ω_{-t}. □

Lemma 3.15. *For $\varepsilon \in (0, \check{\varepsilon})$ and $t \in (\varepsilon, \check{R})$ there holds*

$$\frac{d}{dt}D_\varepsilon^-(t) = -\frac{2}{t^{N-2}}\int_{\Gamma_t^-}\left|\frac{\partial u_\varepsilon}{\partial \nu}\right|^2 d\sigma - \frac{\lambda_{\bar{k}}^\varepsilon}{t^{N-1}} \int_{\Omega_{-t}} (2p(x) + x\cdot\nabla p(x)) u_\varepsilon^2(x) dx, \tag{110}$$

$$\frac{d}{dt}H_\varepsilon^-(t) = \frac{2}{t^{N-1}}\int_{\Gamma_t^-} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu}\, d\sigma = -\frac{2}{t} D_\varepsilon^-(t), \tag{111}$$

$$\frac{d}{dt}\mathcal{N}_\varepsilon^-(t) = -2t\frac{\left(\int_{\Gamma_t^-}\left|\frac{\partial u_\varepsilon}{\partial \nu}\right|^2 d\sigma\right)\left(\int_{\Gamma_t^-} u_\varepsilon^2(x)\, d\sigma\right) - \left(\int_{\Gamma_t^-} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} d\sigma\right)^2}{\left(\int_{\Gamma_t^-} u_\varepsilon^2(x)\, d\sigma\right)^2} - \lambda_{\bar{k}}^\varepsilon \frac{\int_{\Omega_{-t}}(2p(x) + x\cdot\nabla p(x)) u_\varepsilon^2(x) dx}{\int_{\Gamma_t^-} u_\varepsilon^2(x) d\sigma}. \tag{112}$$

Proof. Since

$$\frac{d}{dt}D_\varepsilon^-(t) = -\frac{N-2}{t^{N-1}}\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2 - \lambda_{\bar k}^\varepsilon p u_\varepsilon^2(x)\Big)dx - \frac{1}{t^{N-2}}\int_{\Gamma_t^-}\Big(|\nabla u_\varepsilon|^2 - \lambda_{\bar k}^\varepsilon p u_\varepsilon^2\Big)d\sigma,$$

(110) follows from Lemma 3.14. From direct calculation, we obtain that

$$\frac{d}{dt}H_\varepsilon^-(t) = \frac{2}{t^{N-1}}\int_{\Gamma_t^-} u_\varepsilon \frac{\partial u_\varepsilon}{\partial\nu}\,d\sigma,$$

while testing equation (9) with u_ε and integration over Ω_{-t} yield

$$\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2 - \lambda_{\bar k}^\varepsilon p u_\varepsilon^2(x)\Big)dx = -\int_{\Gamma_t^-} u_\varepsilon \frac{\partial u_\varepsilon}{\partial\nu}\,d\sigma,$$

thus implying (111). Finally, (112) follows from (110), (111), and $(\mathcal{N}_\varepsilon^-)' = \frac{(D_\varepsilon^-)'H_\varepsilon^- - D_\varepsilon^-(H_\varepsilon^-)'}{(H_\varepsilon^-)^2}$. □

The following estimates strongly rely on Lemmas 3.4 and 3.6.

Lemma 3.16. *For every $\delta\in(0,1)$ there exist $\bar r_\delta\in(0,\check R)$ and $\bar\varepsilon_\delta\in(0,\check\varepsilon)$ such that, for every $\varepsilon\in(0,\bar\varepsilon_\delta)$,*

$$\frac{\frac{d}{dt}H_\varepsilon^-(t)}{H_\varepsilon^-(t)} \leqslant -\frac{2(1-\delta)(N-1)}{t} \quad \text{for all } t\in(\varepsilon,\bar r_\delta), \tag{113}$$

$$\frac{\frac{d}{dt}D_\varepsilon^-(t)}{D_\varepsilon^-(t)} \leqslant -\frac{2(1-\delta)(N-1)}{t} \quad \text{for all } t\in(\varepsilon,\bar r_\delta), \tag{114}$$

$$H_\varepsilon^-(t_1) \geqslant \left(\frac{t_2}{t_1}\right)^{2(1-\delta)(N-1)} H_\varepsilon^-(t_2) \quad \text{for all } t_1,t_2\in(\varepsilon,\bar r_\delta) \text{ such that } t_1<t_2, \tag{115}$$

$$D_\varepsilon^-(t_1) \geqslant \left(\frac{t_2}{t_1}\right)^{2(1-\delta)(N-1)} D_\varepsilon^-(t_2) \quad \text{for all } t_1,t_2\in(\varepsilon,\bar r_\delta) \text{ such that } t_1<t_2. \tag{116}$$

Proof. From Lemmas 3.15, 3.6, and 3.4, we deduce that, for every $\delta\in(0,1)$, there exist $\bar r_\delta>0$ and $\bar\varepsilon_\delta>0$ such that, for every $\varepsilon\in(0,\bar\varepsilon_\delta)$ and $t\in(\varepsilon,\bar r_\delta)$, there holds

$$\begin{aligned}\frac{d}{dt}H_\varepsilon^-(t) &= -\frac{2}{t^{N-1}}\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2 - \lambda_{\bar k}^\varepsilon p(x)u_\varepsilon^2(x)\Big)dx\\ &\leqslant -\frac{2(1-\delta)}{t^{N-1}}\int_{\Omega_{-t}}|\nabla u_\varepsilon(x)|^2dx \leqslant -\frac{2(1-\delta)(N-1)}{t}H_\varepsilon^-(t)\end{aligned}$$

which yields (113). From (111), we have that

$$\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2 - \lambda_{\bar k}^\varepsilon p(x)u_\varepsilon^2(x)\Big)dx = -\int_{\Gamma_t^-} u_\varepsilon\frac{\partial u_\varepsilon}{\partial\nu}\,d\sigma$$

which, by Schwarz's inequality, Lemmas 3.6 and 3.4, up to shrinking $\bar r_\delta>0$ and $\bar\varepsilon_\delta>0$, for every $\varepsilon\in(0,\bar\varepsilon_\delta)$ and $t\in(\varepsilon,\bar r_\delta)$ yields

$$\begin{aligned}\int_{\Gamma_t^-}\left|\frac{\partial u_\varepsilon}{\partial\nu}\right|^2 d\sigma &\geqslant \frac{\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2 - \lambda_{\bar k}^\varepsilon p(x)u_\varepsilon^2(x)\Big)dx}{\int_{\Gamma_t^-}u_\varepsilon^2\,d\sigma}\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2 - \lambda_{\bar k}^\varepsilon p(x)u_\varepsilon^2(x)\Big)dx\\ &\geqslant \frac{1-\frac{\delta}{2}}{t}\,\frac{\frac{1}{t^{N-2}}\int_{\Omega_{-t}}|\nabla u_\varepsilon(x)|^2dx}{\frac{1}{t^{N-1}}\int_{\Gamma_t^-}u_\varepsilon^2\,d\sigma}\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2 - \lambda_{\bar k}^\varepsilon p(x)u_\varepsilon^2(x)\Big)dx\\ &\geqslant \frac{(1-\frac{\delta}{2})(N-1)}{t}\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2 - \lambda_{\bar k}^\varepsilon p(x)u_\varepsilon^2(x)\Big)dx.\end{aligned} \tag{117}$$

From (110), (117), (2), and Lemma 3.6, up to shrinking $\bar r_\delta > 0$ and $\bar\varepsilon_\delta > 0$, there holds

$$\begin{aligned}
-\frac{d}{dt}D_\varepsilon^-(t) &= \frac{2}{t^{N-2}}\int_{\Gamma_t^-}\left|\frac{\partial u_\varepsilon}{\partial \nu}\right|^2 d\sigma + \frac{\lambda_{\bar k}^\varepsilon}{t^{N-1}}\int_{\Omega_{-t}}(2p(x)+x\cdot\nabla p(x))u_\varepsilon^2(x)dx,\\
&\geqslant \frac{2(1-\frac{\delta}{2})(N-1)}{t^{N-1}}\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2-\lambda_{\bar k}^\varepsilon p(x)u_\varepsilon^2(x)\Big)dx\\
&\quad-\frac{\delta(N-1)}{t^{N-1}}\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2-\lambda_{\bar k}^\varepsilon p(x)u_\varepsilon^2(x)\Big)dx\\
&\geqslant \frac{2(1-\delta)(N-1)}{t^{N-1}}\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2-\lambda_{\bar k}^\varepsilon p(x)u_\varepsilon^2(x)\Big)dx = \frac{2(1-\delta)(N-1)}{t}D_\varepsilon^-(t)
\end{aligned}$$

thus proving (114).

Estimate (115) follows by integration of (113), while (116) follows by integration of (114). □

Lemma 3.17. *For every $\delta > 0$ there exist $\check R_\delta \in (0,\check R)$, and $\check\varepsilon_\delta \in (0,\check\varepsilon)$ such that*

$$\frac{\frac{d}{dt}\mathcal N_\varepsilon^-(t)}{\mathcal N_\varepsilon^-(t)} \leqslant \delta \quad \text{and} \quad \int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon|^2-\lambda_{\bar k}^\varepsilon p u_\varepsilon^2\Big)dx \geqslant \frac12\int_{\Omega_{-t}}|\nabla u_\varepsilon|^2dx \tag{118}$$

for every $\varepsilon \in (0,\check\varepsilon_\delta)$ and $t \in (\varepsilon,\check R_\delta)$.

Proof. From Lemma 3.16, letting $\delta_0 = \frac{2N-5}{4(N-1)} \in (0,1)$, there holds

$$D_\varepsilon^-(t_1) \geqslant \left(\frac{t_2}{t_1}\right)^{N+\frac12} D_\varepsilon^-(t_2) \quad \text{for every } \varepsilon \in (0,\bar\varepsilon_{\delta_0}) \text{ and } t_1,t_2 \in (\varepsilon,\bar r_{\delta_0}) \text{ such that } t_1<t_2. \tag{119}$$

Let us fix $\delta > 0$. From (2), Lemma 3.6, (90), and (7), there exist $\breve R_\delta \in (0,\min\{\bar r_{\delta_0},\check R\})$ and $\check\varepsilon_\delta \in (0,\min\{\bar\varepsilon_{\delta_0},\check\varepsilon\})$ such that

$$\|2p+x\cdot\nabla p\|_{L^{3N}\left(B^-_{\breve R_\delta}\right)} \leqslant \left(\frac{2N}{\omega_{N-1}}\right)^{\frac{5}{3N}}\frac{C_S\delta}{8\lambda_{k_0}(D^+)}, \tag{120}$$

$$\lambda_{\bar k}^\varepsilon \leqslant 2\lambda_{k_0}(D^+) \quad \text{for all } \varepsilon \in (0,\check\varepsilon_\delta), \tag{121}$$

$$\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon|^2-\lambda_{\bar k}^\varepsilon p u_\varepsilon^2\Big)dx \geqslant \frac12\int_{\Omega_{-t}}|\nabla u_\varepsilon|^2dx,\ \text{for all } \varepsilon \in (0,\check\varepsilon_\delta),\ t\in(\varepsilon,\breve R_\delta) \tag{122}$$

$$\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon|^2-\lambda_{\bar k}^\varepsilon p u_\varepsilon^2\Big)dx \geqslant \tfrac{4\lambda_{k_0}(D^+)}{\delta}\int_{\Omega_{-t}}|2p+x\cdot\nabla p|u_\varepsilon^2dx,\ \text{for all } \varepsilon \in (0,\check\varepsilon_\delta),\ t\in(\varepsilon,\breve R_\delta). \tag{123}$$

Let $\check R_\delta = \breve R_\delta^{5/3}$. From (112), (121), and Schwarz's inequality, we have that, for all $\varepsilon \in (0,\check\varepsilon_\delta)$ and $t \in (\varepsilon,\check R_\delta)$

$$\frac{\frac{d}{dt}\mathcal N_\varepsilon^-(t)}{\mathcal N_\varepsilon^-(t)} \leqslant \mathcal I_\varepsilon(t) \tag{124}$$

where

$$\mathcal I_\varepsilon(t) := \frac{2\lambda_{k_0}(D^+)}{t}\frac{\int_{\Omega_{-t}}|2p(x)+x\cdot\nabla p(x)|u_\varepsilon^2(x)dx}{\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2-\lambda_{\bar k}^\varepsilon p(x)u_\varepsilon^2(x)\Big)dx} = \frac{2\lambda_{k_0}(D^+)}{t}\Big(I_\varepsilon(t)+II_\varepsilon(t)\Big) \tag{125}$$

with

$$I_\varepsilon(t) = \frac{\int_{\Omega_{-t}\setminus\Omega_{-t^{3/5}}}|2p(x)+x\cdot\nabla p(x)|u_\varepsilon^2(x)dx}{\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2-\lambda_{\bar k}^\varepsilon p(x)u_\varepsilon^2(x)\Big)dx},$$

$$II_\varepsilon(t) = \frac{\int_{\Omega_{-t^{3/5}}}|2p(x)+x\cdot\nabla p(x)|u_\varepsilon^2(x)dx}{\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2-\lambda_{\bar k}^\varepsilon p(x)u_\varepsilon^2(x)\Big)dx}.$$

By Hölder inequality, (122), Lemma 3.2, and (120), $I_\varepsilon(t)$ can be estimated as

$$I_\varepsilon(t) \leqslant \|2p + x\cdot\nabla p\|_{L^{3N}\left(B^-_{t^{3/5}}\right)} \left|\Omega_{-t}\setminus\Omega_{-t^{3/5}}\right|^{\frac{5}{3N}} \frac{\left(\int_{\Omega_{-t}}|u_\varepsilon(x)|^{2^*}dx\right)^{2/2^*}}{\int_{\Omega_{-t}}\left(|\nabla u_\varepsilon(x)|^2-\lambda^\varepsilon_{\bar k}p(x)u^2_\varepsilon(x)\right)dx} \tag{126}$$

$$\leqslant \frac{2}{C_S}\left(\frac{\omega_{N-1}}{2N}\right)^{\frac{5}{3N}}\|2p+x\cdot\nabla p\|_{L^{3N}\left(B^-_{\check R_\delta}\right)} t \leqslant \frac{\delta}{4\lambda_{k_0}(D^+)}\,t$$

for all $t\in(\varepsilon,\check R_\delta)$ and $\varepsilon\in(0,\check\varepsilon_\delta)$. On the other hand, from (123) and (119)

$$II_\varepsilon(t) = \frac{\int_{\Omega_{-t^{3/5}}}|2p(x)+x\cdot\nabla p(x)|u^2_\varepsilon(x)dx}{\int_{\Omega_{-t^{3/5}}}\left(|\nabla u_\varepsilon(x)|^2-\lambda^\varepsilon_{\bar k}p(x)u^2_\varepsilon(x)\right)dx}\,\frac{\int_{\Omega_{-t^{3/5}}}\left(|\nabla u_\varepsilon(x)|^2-\lambda^\varepsilon_{\bar k}p(x)u^2_\varepsilon(x)\right)dx}{\int_{\Omega_{-t}}\left(|\nabla u_\varepsilon(x)|^2-\lambda^\varepsilon_{\bar k}p(x)u^2_\varepsilon(x)\right)dx} \tag{127}$$

$$\leqslant \frac{\delta}{4\lambda_{k_0}(D^+)}\,t^{-\frac25(N-2)}\frac{D^-_\varepsilon(t^{3/5})}{D^-_\varepsilon(t)} \leqslant \frac{\delta}{4\lambda_{k_0}(D^+)}\,t^{-\frac25(N-2)}\left(\frac{t}{t^{3/5}}\right)^{N+\frac12} = \frac{\delta}{4\lambda_{k_0}(D^+)}\,t$$

for all $t\in(\varepsilon,\check R_\delta)$ and $\varepsilon\in(0,\check\varepsilon_\delta)$. (125), (126), and (127) imply that

$$\mathcal{I}_\varepsilon(t)\leqslant\delta \tag{128}$$

for all $t\in(\varepsilon,\check R_\delta)$ and $\varepsilon\in(0,\check\varepsilon_\delta)$. Estimate (118) follows from (128) and (124). □

Corollary 3.18. *For every $\delta>0$, let $\check R_\delta\in(0,1)$ and $\check\varepsilon_\delta>0$ as in Lemma 3.17. Then, for every $\varepsilon\in(0,\check\varepsilon_\delta)$ and r_1,r_2 such that $-\check R_\delta<r_1<r_2<-\varepsilon$, there holds*

$$\mathcal{N}_\varepsilon(r_1)\leqslant\mathcal{N}_\varepsilon(r_2)e^{\delta(r_2-r_1)}.$$

Proof. It follows from integration of (118). □

3.3. **The frequency function in the corridor.** If $\varepsilon\in(0,\check\varepsilon)$ and $0<r<1$, then

$$\mathcal{N}_\varepsilon(r)=\frac{\varepsilon D^c_\varepsilon(r)}{H^c_\varepsilon(r)} \tag{129}$$

where

$$D^c_\varepsilon(r)=\int_{\Omega^\varepsilon_r}\left(|\nabla u_\varepsilon(x)|^2-\lambda^\varepsilon_{\bar k}p(x)u^2_\varepsilon(x)\right)dx,\quad H^c_\varepsilon(r)=\int_{\Gamma^\varepsilon_r}u^2_\varepsilon(x)\,d\sigma.$$

Lemma 3.19. *For all $\varepsilon\in(0,\varepsilon_0)$ and $r\in(0,1)$*

$$\int_{\Gamma^\varepsilon_r}\left(|\nabla u_\varepsilon|^2-\lambda^\varepsilon_{\bar k}pu^2_\varepsilon\right)d\sigma = 2\int_{\Gamma^\varepsilon_r}\left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2d\sigma+\int_{S_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2d\sigma-\lambda^\varepsilon_{\bar k}\int_{\Omega^\varepsilon_r}\frac{\partial p}{\partial x_1}(x)u^2_\varepsilon(x)dx,$$

where $S_\varepsilon=\partial D^-\setminus\Gamma^\varepsilon_0=\left\{(0,x')\in\mathbb{R}\times\mathbb{R}^{N-1}:\frac{x'}{\varepsilon}\notin\Sigma\right\}$.

Proof. The stated identity follows from multiplication of equation (9) by $\frac{\partial u_\varepsilon}{\partial x_1}$ and integration by parts over Ω^ε_r. □

Lemma 3.20. *For all $\varepsilon\in(0,\check\varepsilon)$ and $r\in(0,1)$ there holds*

$$\frac{d}{dr}D^c_\varepsilon(r)=2\int_{\Gamma^\varepsilon_r}\left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2d\sigma+\int_{S_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2d\sigma-\lambda^\varepsilon_{\bar k}\int_{\Omega^\varepsilon_r}\frac{\partial p}{\partial x_1}(x)u^2_\varepsilon(x)dx, \tag{130}$$

$$\frac{d}{dr}H^c_\varepsilon(r)=2\int_{\Gamma^\varepsilon_r}u_\varepsilon\frac{\partial u_\varepsilon}{\partial x_1}\,d\sigma=2D^c_\varepsilon(r), \tag{131}$$

$$\frac{d}{dr}\mathcal{N}_\varepsilon(r)=\varepsilon\left[2\frac{\left(\int_{\Gamma^\varepsilon_r}\left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2d\sigma\right)\left(\int_{\Gamma^\varepsilon_r}u^2_\varepsilon\,d\sigma\right)-\left(\int_{\Gamma^\varepsilon_r}u_\varepsilon\frac{\partial u_\varepsilon}{\partial x_1}\,d\sigma\right)^2}{\left(\int_{\Gamma^\varepsilon_r}u^2_\varepsilon\,d\sigma\right)^2}+\frac{\int_{S_\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2d\sigma}{\int_{\Gamma^\varepsilon_r}u^2_\varepsilon\,d\sigma}\right] \tag{132}$$

$$-\varepsilon\lambda^\varepsilon_{\bar k}\frac{\int_{\Omega^\varepsilon_r}\frac{\partial p}{\partial x_1}(x)u^2_\varepsilon(x)dx}{\int_{\Gamma^\varepsilon_r}u^2_\varepsilon\,d\sigma}.$$

Proof. Since

$$\frac{d}{dr}D^c_\varepsilon(r) = \int_{\Gamma^\varepsilon_r} \Big(|\nabla u_\varepsilon|^2 - \lambda^\varepsilon_k p u^2_\varepsilon\Big)d\sigma,$$

(130) follows from Lemma 3.19. From direct calculation, we obtain that

$$\frac{d}{dr}H^c_\varepsilon(r) = 2\int_{\Gamma^\varepsilon_r} u_\varepsilon \frac{\partial u_\varepsilon}{\partial x_1}\,d\sigma,$$

while, testing equation (9) with u_ε and integrating over Ω^ε_r, we have that

$$\int_{\Omega^\varepsilon_r} \Big(|\nabla u_\varepsilon(x)|^2 - \lambda^\varepsilon_k p u^2_\varepsilon(x)\Big)dx = \int_{\Gamma^\varepsilon_r} u_\varepsilon \frac{\partial u_\varepsilon}{\partial x_1}\,d\sigma,$$

thus implying (131). Finally, $(\mathcal{N}_\varepsilon)' = \varepsilon\frac{(D^c_\varepsilon)'H^c_\varepsilon - D^c_\varepsilon(H^c_\varepsilon)'}{(H^c_\varepsilon)^2}$, (130), and (131) yield (132). □

Lemma 3.21. *For every $\delta > 0$ there exists $\bar\varepsilon^\delta_c \in (0, \check\varepsilon)$ such that*

$$\frac{\frac{d}{dt}\mathcal{N}_\varepsilon(r)}{\mathcal{N}_\varepsilon(r)} \geqslant -\delta \quad \textit{for all } \varepsilon \in (0, \bar\varepsilon^\delta_c) \textit{ and } r \in (0,1), \tag{133}$$

$$\mathcal{N}_\varepsilon(r_1) \leqslant \mathcal{N}_\varepsilon(r_2)e^{\delta(r_2-r_1)} \quad \textit{for all } \varepsilon \in (0, \bar\varepsilon^\delta_c) \textit{ and } 0 < r_1 < r_2 < 1. \tag{134}$$

Proof. From (132) and Schwarz's inequality we have that, for all $\varepsilon \in (0, \check\varepsilon)$ and $r \in (0,1)$,

$$\frac{d}{dr}\mathcal{N}_\varepsilon(r) \geqslant -\varepsilon\lambda^\varepsilon_k \frac{\int_{\Omega^\varepsilon_r} \frac{\partial p}{\partial x_1}(x)u^2_\varepsilon(x)dx}{\int_{\Gamma^\varepsilon_r} u^2_\varepsilon\,d\sigma}. \tag{135}$$

By part ii) of Lemma 3.6, for every $\delta > 0$ there exists $\bar\varepsilon^\delta_c \in (0, \check\varepsilon)$ such that, for every $\varepsilon \in (0, \bar\varepsilon^\delta_c)$ and $r \in (0,1)$,

$$\int_{\Omega^\varepsilon_r} \Big(|\nabla u_\varepsilon(x)|^2 - \lambda^\varepsilon_k p(x)u^2_\varepsilon(x)\Big)dx \geqslant \frac{\lambda^\varepsilon_k}{\delta}\int_{\Omega^\varepsilon_r}\left|\frac{\partial p}{\partial x_1}(x)\right|u^2_\varepsilon(x)dx. \tag{136}$$

Estimate (133) follows from (135), (136), and (129). (134) follows from integration of (133). □

4. Blow-up at the right

Throughout this section, $\widetilde{u}_\varepsilon$ will denote the scaling of u_ε introduced in (66–67). For every $R > 1$ we define as $\mathcal{H}^+_R$ the completion of

$$\mathcal{D}^+_R := \Big\{v \in C^\infty(\overline{((-\infty,1]\times\mathbb{R}^{N-1})\cup B^+_R}) : \operatorname{supp} v \Subset \mathbb{R}^N \setminus \{(1,x') \in \mathbb{R}\times\mathbb{R}^{N-1} : |x'| > R\}\Big\}$$

with respect to the norm $\big(\int_{((-\infty,1)\times\mathbb{R}^{N-1})\cup B^+_R} |\nabla v|^2dx\big)^{1/2}$ (which is actually equivalent to the norm $\big(\int_{((-\infty,1)\times\mathbb{R}^{N-1})\cup B^+_R} |\nabla v|^2dx + \int_{\Gamma^+_R} v^2d\sigma\big)^{1/2}$ by Poincaré inequality), i.e. $\mathcal{H}^+_R$ is the space of functions with finite energy in $((-\infty,1]\times\mathbb{R}^{N-1})\cup B^+_R$ vanishing on $\{(1,x') \in \mathbb{R}\times\mathbb{R}^{N-1} : |x'| \geqslant R\}$.

Lemma 4.1. *For every sequence $\varepsilon_n \to 0^+$ there exist a subsequence $\{\varepsilon_{n_k}\}_k$ and $\widetilde{u} \in \bigcup_{R>2}\mathcal{H}^+_R$ such that*

i) *$\widetilde{u}_{\varepsilon_{n_k}} \to \widetilde{u}$ strongly in $\mathcal{H}^+_R$ for every $R > 2$ and a.e.;*
ii) *$\widetilde{u} \equiv 0$ in $\mathbb{R}^N \setminus \widetilde{D}$;*
iii) *$\widetilde{u}$ weakly solves*

$$\begin{cases} -\Delta\widetilde{u}(x) = 0, & \textit{in } \widetilde{D}, \\ \widetilde{u} = 0, & \textit{on } \partial\widetilde{D}, \end{cases} \tag{137}$$

with $\widetilde{D}$ as in (24);

iv) *$\widetilde{u}(x) \geqslant \frac{C_5}{2}(x_1 - 1)$ for all $x \in D^+ \setminus B^+_2$.*

Proof. Let $R>2$. From Lemma 2.10 and (57), there exists $C_R>0$ such that

$$\int_{\Gamma_R^+}|\widetilde u_\varepsilon|^2d\sigma=\frac{1}{\varepsilon^2}\int_{\Gamma_R^+}u_\varepsilon^2(\mathbf{e}_1+\varepsilon(x-\mathbf{e}_1))d\sigma\leqslant C_3^2\int_{\Gamma_R^+}\bigg(\Phi_1(x)+2\gamma_\varepsilon\Phi_2\Big(\frac{x+\mathbf{e}_1}{2}\Big)\bigg)^2d\sigma\leqslant C_R \tag{138}$$

for all $\varepsilon\in(0,r_0/R)$. By the change of variable $x=\mathbf{e}_1+\varepsilon(y-\mathbf{e}_1)$ we have that

$$\mathcal{N}_\varepsilon(1+R\varepsilon)=\frac{R\int_{\widetilde\Omega^\varepsilon_{R+1}}\Big(|\nabla\widetilde u_\varepsilon(y)|^2-\lambda_{\bar k}^\varepsilon\varepsilon^2p(\mathbf{e}_1+\varepsilon(y-\mathbf{e}_1))\widetilde u_\varepsilon^2(y)\Big)dy}{\int_{\Gamma_R^+}\widetilde u_\varepsilon^2(y)\,d\sigma} \tag{139}$$

where

$\widetilde\Omega^\varepsilon_{R+1}:=\Big\{(y_1,y')\in\mathbb{R}\times\mathbb{R}^{N-1}:y_1<1-\frac1\varepsilon\Big\}\cup\Big\{(y_1,y')\in\mathbb{R}\times\mathbb{R}^{N-1}:1-\frac1\varepsilon\leqslant y_1\leqslant 1,\ y'\in\Sigma\Big\}\cup B_R^+.$

From Corollary 3.12

$$\mathcal{N}_\varepsilon(1+R\varepsilon)\leqslant\mathcal{N}_\varepsilon(1+r_0)+\frac{C_9}{N2^N} \tag{140}$$

for all $\varepsilon\in(0,\min\{r_0/R,\varepsilon_2\})$. From the strong $\mathcal{D}^{1,2}(\mathbb{R}^N)$ convergence of u_ε to $\varphi^+_{k_0}$ ensured by Lemma 1.1, we deduce that there exists some positive constant $C_{10}>0$ (depending on r_0 but independent of ε) such that $\mathcal{N}_\varepsilon(1+r_0)\leqslant C_{10}$ for all $\varepsilon\in(0,\varepsilon_0)$, so that (138–140) yield

$$\begin{aligned}\int_{\widetilde\Omega^\varepsilon_{R+1}}\Big(|\nabla\widetilde u_\varepsilon(y)|^2-\lambda_{\bar k}^\varepsilon\varepsilon^2p(\mathbf{e}_1+\varepsilon(y-\mathbf{e}_1))\widetilde u_\varepsilon^2(y)\Big)dy&\leqslant\Big(C_{10}+\frac{C_9}{N2^N}\Big)\frac{\int_{\Gamma_R^+}\widetilde u_\varepsilon^2(y)\,d\sigma}{R}\\&\leqslant\Big(C_{10}+\frac{C_9}{N2^N}\Big)\frac{C_R}{R}\end{aligned} \tag{141}$$

for all $\varepsilon\in\big(0,\min\{r_0/R,\varepsilon_2\}\big)$. From (141), Lemma 3.6, and assumption (3), we obtain that

$$\int_{\widetilde\Omega^\varepsilon_{R+1}}|\nabla\widetilde u_\varepsilon(y)|^2dy\leqslant 2\Big(C_{10}+\frac{C_9}{N2^N}\Big)\frac{C_R}{R} \tag{142}$$

for all $\varepsilon\in(0,\min\{r_0/R,\varepsilon_2,\bar\varepsilon_{2,2\lambda_{k_0}(D^+)p}\})$. In view of (138) and (142), we have proved that for every $R>2$ there exists $\varepsilon_R>0$ such that

$$\{\widetilde u_\varepsilon\}_{\varepsilon\in(0,\varepsilon_R)}\text{ is bounded in }\mathcal{H}_R^+. \tag{143}$$

Let $\varepsilon_n\to0^+$. From (143) and a diagonal process, we deduce that there exist a subsequence $\varepsilon_{n_k}\to0^+$ and some $\widetilde u\in\bigcup_{R>2}\mathcal{H}_R^+$ such that $\widetilde u_{\varepsilon_{n_k}}\rightharpoonup\widetilde u$ weakly in $\mathcal{H}_R^+$ for every $R>2$. In particular $\widetilde u_{\varepsilon_{n_k}}\to\widetilde u$ a.e., so that $\widetilde u\equiv0$ in $\mathbb{R}^N\setminus\widetilde D$. Passing to the weak limit in (68), we obtain that $\widetilde u$ is a weak solution to (137). By classical elliptic estimates, we also have that $\widetilde u_{\varepsilon_{n_k}}\to\widetilde u$ in $C^2(\overline{B^+_{r_2}\setminus B^+_{r_1}})$ for all $1<r_1<r_2$. Therefore, multiplying (137) by $\widetilde u$ and integrating over $T_1^-\cup B_R^+$ with T_1^- as in (24), we obtain

$$\int_{\Gamma_R^+}\frac{\partial\widetilde u_{\varepsilon_{n_k}}}{\partial\nu}\widetilde u_{\varepsilon_{n_k}}d\sigma\to\int_{\Gamma_R^+}\frac{\partial\widetilde u}{\partial\nu}\widetilde u\,d\sigma=\int_{T_1^-\cup B_R^+}|\nabla\widetilde u(x)|^2dx\quad\text{as }k\to+\infty. \tag{144}$$

On the other hand, multiplication of (68) by $\widetilde u_{\varepsilon_{n_k}}$ and integration by parts over $\widetilde\Omega^{\varepsilon_{n_k}}_{R+1}$ yield

$$\int_{\widetilde\Omega^{\varepsilon_{n_k}}_{R+1}}|\nabla\widetilde u_{\varepsilon_{n_k}}(x)|^2dx=\int_{\Gamma_R^+}\frac{\partial\widetilde u_{\varepsilon_{n_k}}}{\partial\nu}\widetilde u_{\varepsilon_{n_k}}d\sigma+\lambda_{\bar k}^{\varepsilon_{n_k}}\varepsilon_{n_k}^2\int_{\widetilde\Omega^{\varepsilon_{n_k}}_{R+1}}p(\mathbf{e}_1+\varepsilon_{n_k}(x-\mathbf{e}_1))\widetilde u^2_{\varepsilon_{n_k}}(x)\,dx. \tag{145}$$

We claim that

$$\varepsilon_{n_k}^2\int_{\widetilde\Omega^{\varepsilon_{n_k}}_{R+1}}p(\mathbf{e}_1+\varepsilon_{n_k}(x-\mathbf{e}_1))\widetilde u^2_{\varepsilon_{n_k}}(x)\,dx\to0\quad\text{as }k\to+\infty. \tag{146}$$

Indeed, from Lemma 3.6, for every $\delta>0$ there exists k_0 such that for all $k\geqslant k_0$

$$\int_{\Omega^{\varepsilon_{n_k}}_{1/2}}p(y)u^2_{\varepsilon_{n_k}}(y)dy\leqslant\delta\int_{\Omega^{\varepsilon_{n_k}}_{1/2}}|\nabla u_{\varepsilon_{n_k}}(y)|^2dy$$

and hence, from the change of variable $y=\mathbf{e}_1+\varepsilon_{n_k}(x-\mathbf{e}_1)$, assumption (3), and (142), we deduce that

$$\begin{aligned}\varepsilon_{n_k}^2\int_{\widetilde\Omega_{R+1}^{\varepsilon_{n_k}}}p(\mathbf{e}_1+\varepsilon_{n_k}(x-\mathbf{e}_1))\widetilde u_{\varepsilon_{n_k}}^2(x)\,dx&=\varepsilon_{n_k}^{-N}\int_{\Omega_{1+R\varepsilon_{n_k}}^{\varepsilon_{n_k}}}p(y)u_{\varepsilon_{n_k}}^2(y)dy\\&=\varepsilon_{n_k}^{-N}\int_{\Omega_{1/2}^{\varepsilon_{n_k}}}p(y)u_{\varepsilon_{n_k}}^2(y)dy\leqslant\delta\varepsilon_{n_k}^{-N}\int_{\Omega_{1+R\varepsilon_{n_k}}^{\varepsilon_{n_k}}}|\nabla u_{\varepsilon_{n_k}}(y)|^2dy\\&=\delta\int_{\widetilde\Omega_{R+1}^{\varepsilon_{n_k}}}|\nabla\widetilde u_{\varepsilon_{n_k}}(x)|^2dx\leqslant2\delta\bigg(C_{10}+\frac{C_9}{N2^N}\bigg)\frac{C_R}{R},\end{aligned}$$

thus proving claim (146). Combining (144), (145), and (146), we conclude that $\|\widetilde u_{\varepsilon_{n_k}}\|_{\mathcal{H}_R^+}\to\|\widetilde u\|_{\mathcal{H}_R^+}$ and then $\widetilde u_{\varepsilon_{n_k}}\to\widetilde u$ strongly in $\mathcal{H}_R^+$ for every $R>2$.

To prove iv), it is enough to observe that Lemma 2.13 implies that, for k large,

$$\widetilde u_{\varepsilon_{n_k}}(x)\geqslant\frac{C_5}{2}(x_1-1)\quad\text{for all }x\in B^+_{r_0/\varepsilon_{n_k}}\setminus B_2^+,$$

which yields iv) thanks to a.e convergence of $\widetilde u_{\varepsilon_{n_k}}$ to $\widetilde u$. □

Remark 4.2. We notice that the function $\widetilde u$ found in Lemma 4.1 satisfies

$$\int_{\widetilde D}|\nabla\widetilde u(x)|^2dx=+\infty.$$

Indeed, $\int_{\widetilde D}|\nabla\widetilde u(x)|^2dx<+\infty$ would imply, by testing (137) with $\widetilde u$, that $\widetilde u\equiv0$ in $\widetilde D$, thus contradicting statement iv) of Lemma 4.1.

Lemma 4.3. *Let $\widetilde u$ be as in Lemma 4.1 and, for $r\in\mathbb{R}\setminus(1,2)$, let $\widetilde{\mathcal N}_{\widetilde u}(r)$ be the frequency function associated to $\widetilde u$, i.e.*

$$\widetilde{\mathcal N}_{\widetilde u}(r)=\frac{\Lambda_N(r)\int_{\widetilde\Omega_r}|\nabla\widetilde u(x)|^2dx}{\int_{\widetilde\Gamma_r}\widetilde u^2(x)\,d\sigma},$$

with $\widetilde\Omega_r$ and $\widetilde\Gamma_r$ defined in (25) and $\Lambda_N(r)$ as in (50)*. Then*

i) $\lim_{r\to+\infty}\widetilde{\mathcal N}_{\widetilde u}(r)=1$*;*
ii) *there exists $\tilde c>0$ such that $\int_{D^+}|\nabla(\widetilde u-\tilde c(x_1-1))(x)|^2\,dx<+\infty$.*

Proof. We notice that $\widetilde{\mathcal N}_{\widetilde u}$ is well defined in $\mathbb{R}\setminus(1,2)$ in view of equation (137) and classical unique continuation (in particular $\widetilde u\not\equiv0$ by part iv) of Lemma 4.1)). Let us first prove that

$$\limsup_{r\to+\infty}\widetilde{\mathcal N}_{\widetilde u}(r)\leqslant1. \tag{147}$$

Indeed, letting $\varepsilon_n\to0^+$ and $\{\varepsilon_{n_k}\}_k$ as in Lemma 4.1, passing to the limit as $k\to+\infty$ in (139), and using (146), we have that

$$\lim_{k\to+\infty}\mathcal N_{\varepsilon_{n_k}}(1+R\varepsilon_{n_k})=\widetilde{\mathcal N}_{\widetilde u}(1+R)\quad\text{for every }R>0,$$

which, together with Corollary 3.13, implies for every $\delta>0$ the existence of some $\widetilde R_\delta$ such that

$$\widetilde{\mathcal N}_{\widetilde u}(1+R)\leqslant1+\delta\quad\text{ for all }R>\widetilde R_\delta,$$

thus proving claim (147).

It is easy to prove that there exists $g\in H^1_{\rm loc}(D^+)$ such that

$$\begin{cases}-\Delta g=0,&\text{in }D^+,\\g=\widetilde u,&\text{on }\partial D^+,\\\int_{D^+}|\nabla g(x)|^2dx<+\infty,\end{cases}$$

i.e. g is a finite-energy harmonic extension of $\widetilde u\big|_{\partial D^+}$ in D^+. We observe that the Kelvin transform $\widetilde g(x)=|x-\mathbf{e}_1|^{-(N-2)}g\big(\frac{x-\mathbf{e}_1}{|x-\mathbf{e}_1|^2}+\mathbf{e}_1\big)$ belongs to $H^1(B_1^+)$ and weakly satisfies

$$\begin{cases}-\Delta\widetilde g(x)=0,&\text{in }B_1^+,\\\widetilde g(x)=0,&\text{on }\{(x_1,x'):x_1=1,|x'|<1\}.\end{cases}$$

By classical elliptic estimates, there exists $c_g > 0$ such that $\left|\frac{\partial \widetilde{g}}{\partial x_1}\right| \leqslant c_g$ in $\overline{B^+_{1/2}}$, thus implying

$$|\widetilde{g}(x_1, x')| = \left|\widetilde{g}(1, x') + \int_1^{x_1} \frac{\partial \widetilde{g}}{\partial x_1}(s, x')\, ds\right| \leqslant \int_1^{x_1} \left|\frac{\partial \widetilde{g}}{\partial x_1}(s, x')\right| ds \leqslant c_g(x_1 - 1)$$

for all $(x_1, x') \in \overline{B^+_{1/2}}$. Then

$$|g(x)| \leqslant c_g \frac{x_1 - 1}{|x - \mathbf{e}_1|^N} \tag{148}$$

for all $x \in D^+ \setminus B^+_2$. Let us observe that the function $v := \widetilde{u} - g \in H^1_{\rm loc}(D^+) \setminus \{0\}$ satisfies

$$\begin{cases} -\Delta v(x) = 0, & \text{in } D^+, \\ v = 0, & \text{on } \partial D^+, \\ \int_{B^+_r} |\nabla v(x)|^2 dx < +\infty, & \text{for all } r > 0. \end{cases} \tag{149}$$

Let us define

$$\mathcal{N}_v : (0, +\infty) \to \mathbb{R}, \quad \mathcal{N}_v(t) := \frac{t \int_{B^+_t} |\nabla v(x)|^2 dx}{\int_{\Gamma^+_t} v^2(x)\, d\sigma}.$$

Direct calculations yield

$$\mathcal{N}_v'(t) = \frac{2t\Big[\Big(\int_{\Gamma^+_t} \big|\frac{\partial v}{\partial \nu}\big|^2 d\sigma\Big)\Big(\int_{\Gamma^+_t} v^2 d\sigma\Big) - \Big(\int_{\Gamma^+_t} v \frac{\partial v}{\partial \nu} d\sigma\Big)^2\Big]}{\Big(\int_{\Gamma^+_t} v^2 d\sigma\Big)^2}, \quad \text{for all } t > 0, \tag{150}$$

where $\nu = \nu(x) = \frac{x - \mathbf{e}_1}{|x - \mathbf{e}_1|}$. In particular, Schwarz's inequality implies that $\mathcal{N}_v$ is non decreasing in $(0, +\infty)$. From Remark 3.5 it follows that

$$\mathcal{N}_v(t) \geqslant \lim_{r \to 0^+} \mathcal{N}_v(r) \geqslant 1 \quad \text{for all } t > 0. \tag{151}$$

From (148) and Lemma 4.1, it follows that, if $x \in \Gamma^+_t$ and $t > 2$, then

$$\Big(1 - \frac{2c_g}{C_5 t^N}\Big)\widetilde{u}(x) \leqslant v(x) \leqslant \Big(1 + \frac{2c_g}{C_5 t^N}\Big)\widetilde{u}(x),$$

so that

$$\Big(1 - \frac{2c_g}{C_5 t^N}\Big)^2 \int_{\Gamma^+_t} \widetilde{u}^2 d\sigma \leqslant \int_{\Gamma^+_t} v^2 d\sigma \leqslant \Big(1 + \frac{2c_g}{C_5 t^N}\Big)^2 \int_{\Gamma^+_t} \widetilde{u}^2 d\sigma$$

for all $t > \max\{2, (2c_g/C_5)^{1/N}\}$. Let us fix $\delta > 0$. For every $R > 2$ there holds

$$\begin{aligned} &\int_{B^+_R} |\nabla v(x)|^2 dx - (1+\delta) \int_{\widetilde{\Omega}_{R+1}} |\nabla \widetilde{u}(x)|^2 dx \\ &\quad \leqslant \int_{B^+_R} |\nabla g(x)|^2 dx - 2\int_{B^+_R} \nabla g(x) \cdot \nabla \widetilde{u}(x) dx - \delta \int_{\widetilde{\Omega}_{R+1}} |\nabla \widetilde{u}(x)|^2 dx \\ &\quad \leqslant \Big(1 + \frac{2}{\delta}\Big) \int_{B^+_R} |\nabla g(x)|^2 dx + \frac{\delta}{2} \int_{B^+_R} |\nabla \widetilde{u}(x)|^2 dx - \delta \int_{\widetilde{\Omega}_{R+1}} |\nabla \widetilde{u}(x)|^2 dx \end{aligned}$$

and hence, for all $R > \max\{2, (2c_g/C_5)^{1/N}\}$,

$$\begin{aligned} \mathcal{N}_v(R) &\leqslant \frac{(1+\delta)}{\big(1 - \frac{2c_g}{C_5} R^{-N}\big)^2} \frac{R \int_{\widetilde{\Omega}_{R+1}} |\nabla \widetilde{u}(x)|^2 dx}{\int_{\Gamma^+_R} \widetilde{u}^2 d\sigma} \left(1 + \frac{1 + \frac{2}{\delta}}{1+\delta} \frac{\int_{B^+_R} |\nabla g(x)|^2 dx}{\int_{\widetilde{\Omega}_{R+1}} |\nabla \widetilde{u}(x)|^2 dx}\right) \\ &= \frac{(1+\delta)}{\big(1 - \frac{2c_g}{C_5} R^{-N}\big)^2} \widetilde{\mathcal{N}}_{\widetilde{u}}(R+1) \left(1 + \frac{1 + \frac{2}{\delta}}{1+\delta} \frac{\int_{B^+_R} |\nabla g(x)|^2 dx}{\int_{\widetilde{\Omega}_{R+1}} |\nabla \widetilde{u}(x)|^2 dx}\right). \end{aligned} \tag{152}$$

On the other hand, for every $R > 2$ there holds

$$\begin{aligned}\int_{B_R^+} |\nabla v(x)|^2 dx - (1-\delta)\int_{\widetilde{\Omega}_{R+1}} |\nabla \widetilde{u}(x)|^2 dx \\ = -\int_{T_1^-} |\nabla \widetilde{u}(x)|^2 dx + \int_{B_R^+} |\nabla g(x)|^2 dx - 2\int_{B_R^+} \nabla g(x)\cdot\nabla\widetilde{u}(x)dx + \delta\int_{\widetilde{\Omega}_{R+1}} |\nabla \widetilde{u}(x)|^2 dx \\ \geqslant -\int_{T_1^-} |\nabla \widetilde{u}(x)|^2 dx + \left(1-\frac{2}{\delta}\right)\int_{B_R^+} |\nabla g(x)|^2 dx\end{aligned}$$

and hence, for all $R > \max\{2, (2c_g/C_5)^{1/N}\}$,

$$\mathcal{N}_v(R) \geqslant \frac{(1-\delta)\widetilde{\mathcal{N}}_{\widetilde{u}}(R+1)}{\left(1+\frac{2c_g}{C_5}R^{-N}\right)^2}\left(1 - \frac{1}{1-\delta}\frac{\int_{T_1^-}|\nabla\widetilde{u}(x)|^2dx}{\int_{\widetilde{\Omega}_{R+1}}|\nabla\widetilde{u}(x)|^2dx} + \frac{1-\frac{2}{\delta}}{1-\delta}\frac{\int_{B_R^+}|\nabla g(x)|^2dx}{\int_{\widetilde{\Omega}_{R+1}}|\nabla\widetilde{u}(x)|^2dx}\right). \tag{153}$$

Since $\int_{B_R^+}|\nabla g(x)|^2dx = O(1)$ and $\int_{\widetilde{\Omega}_{R+1}}|\nabla\widetilde{u}(x)|^2dx \to +\infty$ as $R \to +\infty$ (see Remark 4.2), passing to lim sup and lim inf the in (152–153) we obtain that

$$\begin{aligned}(1-\delta)\limsup_{R\to\infty}\widetilde{\mathcal{N}}_{\widetilde{u}}(R) \leqslant \limsup_{R\to\infty}\mathcal{N}_v(R) \leqslant (1+\delta)\limsup_{R\to\infty}\widetilde{\mathcal{N}}_{\widetilde{u}}(R) \quad \text{for all } \delta > 0,\\ (1-\delta)\liminf_{R\to\infty}\widetilde{\mathcal{N}}_{\widetilde{u}}(R) \leqslant \liminf_{R\to\infty}\mathcal{N}_v(R) \leqslant (1+\delta)\liminf_{R\to\infty}\widetilde{\mathcal{N}}_{\widetilde{u}}(R) \quad \text{for all } \delta > 0,\end{aligned}$$

thus implying, in view of from (151),

$$\liminf_{R\to\infty}\widetilde{\mathcal{N}}_{\widetilde{u}}(R) = \liminf_{R\to\infty}\mathcal{N}_v(R) \geqslant 1, \tag{154}$$

and, in view of (147),

$$1 \geqslant \limsup_{R\to\infty}\widetilde{\mathcal{N}}_{\widetilde{u}}(R) = \limsup_{R\to\infty}\mathcal{N}_v(R). \tag{155}$$

From (154) and (155) we deduce that

$$\lim_{R\to\infty}\widetilde{\mathcal{N}}_{\widetilde{u}}(R) = \lim_{R\to\infty}\mathcal{N}_v(R) = 1, \tag{156}$$

thus proving statement i). Furthermore (156), (151), and the fact that $\mathcal{N}_v$ is non decreasing imply that

$$\mathcal{N}_v(t) \equiv 1 \text{ in } (0,+\infty). \tag{157}$$

Therefore $\mathcal{N}_v'(t) = 0$ for any $t > 0$. From (150) we obtain

$$\left(\int_{\Gamma_t^+}\left|\frac{\partial v}{\partial\nu}\right|^2 d\sigma\right)\left(\int_{\Gamma_t^+} v^2 d\sigma\right) = \left(\int_{\Gamma_t^+} v\frac{\partial v}{\partial\nu}d\sigma\right)^2 \quad \text{for all } t > 0,$$

which implies that v and $\frac{\partial v}{\partial\nu}$ are linearly dependent as vectors in $L^2(\Gamma_t^+)$, i.e. there exists a function $\eta = \eta(t)$ such that $\frac{\partial v}{\partial\nu}(\mathbf{e}_1 + t\theta) = \eta(t)v(\mathbf{e}_1 + t\theta)$ for $t > 0$. After integration we obtain

$$v(\mathbf{e}_1 + t\theta) = e^{\int_1^t \eta(s)ds}v(\mathbf{e}_1+\theta) = \varphi(t)\psi(\theta) \quad t > 0,\ \theta \in \mathbb{S}_+^{N-1},$$

where $\mathbb{S}_+^{N-1} := \{\theta = (\theta_1,\theta_2,\dots,\theta_N) \in \mathbb{S}^{N-1} : \theta_1 > 0\}$, $\varphi(t) = e^{\int_1^t \eta(s)ds}$ and $\psi(\theta) = v(\mathbf{e}_1+\theta)$. Since v satisfies (149), then

$$\left(\varphi''(t) + \frac{N-1}{t}\varphi'(t)\right)\psi(\theta) + \frac{\varphi(t)}{t^2}\Delta_{\mathbb{S}^{N-1}}\psi(\theta) = 0.$$

Taking t fixed, we deduce that ψ is an eigenfunction of the operator $-\Delta_{\mathbb{S}^{N-1}}$ on $\mathbb{S}_+^{N-1}$ under null Dirichlet boundary conditions on $\partial\mathbb{S}_+^{N-1}$, i.e. there exists $K_0 \in \mathbb{N}$, $K_0 \geqslant 1$, such that

$$\begin{cases}-\Delta_{\mathbb{S}^{N-1}}\psi = K_0(N-2+K_0)\psi, & \text{in } \mathbb{S}_+^{N-1},\\ \psi = 0, & \text{on } \partial\mathbb{S}_+^{N-1}.\end{cases} \tag{158}$$

Then $\varphi(t)$ solves the equation

$$\varphi''(t) + \frac{N-1}{t}\varphi(t) - \frac{K_0(N-2+K_0)}{t^2}\varphi(t) = 0$$

and hence φ is of the form

$$\varphi(r) = c_1 t^{K_0} + c_2 t^{-(N-2)-K_0},$$

for some $c_1, c_2 \in \mathbb{R}$. Since, by elliptic regularity theory, v is smooth in $\overline{D^+}$, c_2 must be 0 and $\varphi(t) = c_1 t^{K_0}$. Since $\varphi(1) = 1$, we obtain that $c_1 = 1$ and then

$$v(\mathbf{e}_1 + t\theta) = t^{K_0}\psi(\theta), \quad \text{for all } t > 0 \text{ and } \theta \in \mathbb{S}^{N-1}_+. \tag{159}$$

Substituting (159) into (157), we find that $1 \equiv \mathcal{N}_v(t) \equiv K_0$ and therefore $K_0 = 1$. Being $N-1$ the first eigenvalue of problem (158), ψ is simple. Hence there exists $\tilde{c} \in \mathbb{R}$ such that $\psi(\theta) = \tilde{c}\theta_1^+$ and $v(x) = \tilde{c}(x_1 - 1)^+$. Lemma 4.1 part iv) and estimate (148) imply that $\tilde{c} > 0$, thus proving ii). □

Corollary 4.4. *Let $\widetilde{u}$ be as in Lemma 4.1 and $\tilde{c}$ as in Lemma 4.3. Then*

$$\widetilde{u} = \tilde{c}\,\mathcal{T}(x_1 - 1) = \tilde{c}\,\Phi_1$$

where Φ_1 is defined in (35).

Proof. It follows from Lemmas 4.1 and 4.3, taking into account Lemma 2.4 and the fact that $\mathcal{T}(c\psi) = c\mathcal{T}(\psi)$. □

Lemma 4.5. *For every $R > 0$*

$$\lim_{\varepsilon\to 0^+} \mathcal{N}_\varepsilon(1 - R\varepsilon) = \widetilde{\mathcal{N}}(1 - R),$$

with $\widetilde{\mathcal{N}}$ as in (49).

Proof. Fix $R > 0$. Let $\varepsilon_n \to 0^+$. From Lemma 4.1 and Corollary 4.4, there exist a subsequence $\{\varepsilon_{n_k}\}_k$ and $\tilde{c} > 0$ such that $\widetilde{u}_{\varepsilon_{n_k}} \to \tilde{c}\,\Phi_1$ strongly in $\mathcal{H}_r^+$ for every $r > 2$. By the change of variable $x = \mathbf{e}_1 + \varepsilon(y - \mathbf{e}_1)$, we have that, for $\varepsilon < \frac{1}{R}$,

$$\mathcal{N}_\varepsilon(1 - R\varepsilon) = \frac{\int_{\widetilde{\Omega}^\varepsilon_{1-R}} \Big(|\nabla \widetilde{u}_\varepsilon(y)|^2 - \lambda^\varepsilon_k \varepsilon^2 p(\mathbf{e}_1 + \varepsilon(y - \mathbf{e}_1))\widetilde{u}^2_\varepsilon(y)\Big)dy}{\int_{\widetilde{\Gamma}_{1-R}} \widetilde{u}^2_\varepsilon(y)\,d\sigma} \tag{160}$$

where $\widetilde{\Gamma}_{1-R}$ is defined in (25) and

$$\widetilde{\Omega}^\varepsilon_{1-R} := \Big\{(y_1, y') \in \mathbb{R}\times\mathbb{R}^{N-1} : y_1 < 1 - \tfrac{1}{\varepsilon}\Big\} \cup \Big\{(y_1, y') \in \mathbb{R}\times\mathbb{R}^{N-1} : 1 - \tfrac{1}{\varepsilon} \leqslant y_1 \leqslant 1 - R,\ y' \in \Sigma\Big\}.$$

From strong convergence of $\widetilde{u}_{\varepsilon_{n_k}}$ to $\tilde{c}\,\Phi_1$ in $\mathcal{H}_r^+$ for every $r > 2$, passing to the limit in (160) along the subsequence $\{\varepsilon_{n_k}\}_k$ and using (146), we obtain that

$$\lim_{k\to+\infty} \mathcal{N}_{\varepsilon_{n_k}}(1 - R\varepsilon_{n_k}) = \frac{\int_{\widetilde{\Omega}_{1-R}} |\nabla(\tilde{c}\,\Phi_1)(y)|^2 dy}{\int_{\widetilde{\Gamma}_{1-R}} (\tilde{c}\,\Phi_1)^2(y)\,d\sigma} = \frac{\int_{\widetilde{\Omega}_{1-R}} |\nabla\Phi_1(y)|^2 dy}{\int_{\widetilde{\Gamma}_{1-R}} \Phi_1^2(y)\,d\sigma} = \widetilde{\mathcal{N}}(1 - R),$$

where $\widetilde{\Omega}_{1-R}$ is defined in (25). Since the limit depends neither on the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$, we conclude that the convergence actually holds as $\varepsilon \to 0^+$ thus proving the lemma. □

Lemma 4.6. *For every $R > 0$ and $\delta > 0$, there exists $\hat{\varepsilon}_{R,\delta} \in (0, \check{\varepsilon})$ such that*

$$\mathcal{N}_\varepsilon(r) < (1 + \delta)\sqrt{\lambda_1(\Sigma)} \quad \textit{for all } r \in (0, R\varepsilon] \textit{ and } \varepsilon \in (0, \hat{\varepsilon}_{R,\delta}).$$

Proof. Let $\delta > 0$ and choose $\delta' > 0$ sufficiently small such that $(1 + \delta')^2 e^{\delta'} < 1 + \delta$. From Corollary 2.6, there exists $\widehat{R}_\delta > 0$ such that

$$\widetilde{\mathcal{N}}(1 - \widehat{R}_\delta) < (1 + \delta')\sqrt{\lambda_1(\Sigma)}. \tag{161}$$

From Lemma 4.5, there exists $\varepsilon_\delta > 0$ such that

$$\mathcal{N}_\varepsilon(1 - \widehat{R}_\delta\,\varepsilon) < (1 + \delta')\widetilde{\mathcal{N}}(1 - \widehat{R}_\delta) \quad \text{for all } \varepsilon \in (0, \varepsilon_\delta). \tag{162}$$

Let $R > 0$. Letting $\bar{\varepsilon}^{\delta'}_c$ as in Lemma 3.21 and using (134), (161), and (162), for all $r \in (0, R\varepsilon)$ and $0 < \varepsilon < \min\Big\{\varepsilon_\delta, \bar{\varepsilon}^{\delta'}_c, \frac{1}{R + \widehat{R}_\delta}\Big\}$ we obtain

$$\mathcal{N}_\varepsilon(r) \leqslant \mathcal{N}_\varepsilon(1 - \widehat{R}_\delta\,\varepsilon)e^{\delta'(1 - \widehat{R}_\delta\varepsilon - r)} < (1 + \delta')^2 e^{\delta'}\sqrt{\lambda_1(\Sigma)} < (1 + \delta)\sqrt{\lambda_1(\Sigma)}.$$

The lemma is thereby proved. □

5. Blow-up at the left

Let us define

$$\widehat{u}_\varepsilon : \widehat{\Omega}^\varepsilon \to \mathbb{R}, \quad \widehat{u}_\varepsilon(x) = \frac{u_\varepsilon(\varepsilon x)}{\sqrt{\varepsilon^{1-N}\int_{\Gamma_\varepsilon^\varepsilon} u_\varepsilon^2 d\sigma}} \tag{163}$$

where

$$\widehat{\Omega}^\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Omega^\varepsilon\} = D^- \cup \{(x_1, x') \in T_1 : 0 \leqslant x_1 \leqslant 1/\varepsilon\} \cup \{(x_1, x') : x_1 > 1/\varepsilon\}.$$

We observe that $\widehat{u}_\varepsilon$ solves

$$\begin{cases} -\Delta \widehat{u}_\varepsilon(x) = \varepsilon^2 \lambda_k^\varepsilon p(\varepsilon x)\widehat{u}_\varepsilon(x), & \text{in } \widehat{\Omega}^\varepsilon, \\ \widehat{u}_\varepsilon = 0, & \text{on } \partial\widehat{\Omega}^\varepsilon. \end{cases} \tag{164}$$

We denote

$$T_1^+ = \{(x_1, x') : x' \in \Sigma, x_1 \geqslant 0\}, \quad \widehat{D} = D^- \cup T_1^+.$$

For every $R > 0$ we define

$$\widehat{\Omega}_R = D^- \cup \{(x_1, x') \in T_1^+ : x_1 < R\}, \quad \widehat{\Gamma}_R = \Gamma_R = \{(x_1, x') \in T_1^+ : x_1 = R\}, \tag{165}$$

and $\mathcal{H}_R$ as the completion of

$$\mathcal{D}_R := \Big\{ v \in C^\infty(\overline{\widehat{\Omega}_R}) : \operatorname{supp} v \Subset \widehat{D} \Big\}$$

with respect to the norm $\big(\int_{\widehat{\Omega}_R} |\nabla v|^2 dx\big)^{1/2}$ (which is equivalent to $\big(\int_{\widehat{\Omega}_R} |\nabla v|^2 dx + \int_{\widehat{\Gamma}_R} v^2 d\sigma\big)^{1/2}$), i.e. $\mathcal{H}_R$ is the space of functions with finite energy in $\widehat{\Omega}_R$ vanishing on $\{(x_1, x') \in \partial\widehat{\Omega}_R : x_1 < R\}$.

The change of variable $y' = \varepsilon x'$ yields

$$\int_{\widehat{\Gamma}_1} \widehat{u}_\varepsilon^2 d\sigma = 1. \tag{166}$$

Lemma 5.1. *For every $R > 1$, there exists $\hat{\varepsilon}_R > 0$ such that*

$$\int_{\widehat{\Gamma}_R} \widehat{u}_\varepsilon^2 d\sigma \leqslant e^{4\sqrt{\lambda_1(\Sigma)}(R-1)} \quad \textit{for all } \varepsilon \in (0, \hat{\varepsilon}_R).$$

Proof. For $R > 1$, let $\hat{\varepsilon}_R = \hat{\varepsilon}_{R,1} > 0$ as in Lemma 4.6. From Lemma 4.6, (131), and (129) it follows that

$$\frac{\frac{d}{dr} H_\varepsilon^c(r)}{H_\varepsilon^c(r)} = \frac{2}{\varepsilon}\mathcal{N}_\varepsilon(r) \leqslant \frac{4}{\varepsilon}\sqrt{\lambda_1(\Sigma)} \quad \text{for all } r \in (0, R\varepsilon] \text{ and } \varepsilon \in (0, \hat{\varepsilon}_R),$$

which after integration between ε and $R\varepsilon$ yields

$$H_\varepsilon^c(R\varepsilon) \leqslant H_\varepsilon^c(\varepsilon) e^{4\sqrt{\lambda_1(\Sigma)}(R-1)} \quad \text{for all } \varepsilon \in (0, \hat{\varepsilon}_R).$$

(163) and the change of variable $y' = \varepsilon x'$ yield

$$\int_{\widehat{\Gamma}_R} \widehat{u}_\varepsilon^2 d\sigma = \frac{H_\varepsilon^c(R\varepsilon)}{H_\varepsilon^c(\varepsilon)}$$

thus implying the conclusion. □

Lemma 5.2. *For every sequence $\varepsilon_n \to 0^+$ there exist a subsequence $\{\varepsilon_{n_k}\}_k$ and $\widehat{u} \in \bigcup_{R>1} \mathcal{H}_R$ such that*

i) *$\widehat{u}_{\varepsilon_{n_k}} \to \widehat{u}$ strongly in $\mathcal{H}_R$ for every $R > 1$ and a.e.;*
ii) *$\widehat{u} \not\equiv 0$ in $\widehat{D}$;*
iii) *$\widehat{u}$ weakly solves*

$$\begin{cases} -\Delta \widehat{u}(x) = 0, & \textit{in } \widehat{D}, \\ \widehat{u} = 0, & \textit{on } \partial\widehat{D}. \end{cases} \tag{167}$$

Proof. Let $R > 1$. By the change of variable $x = \varepsilon y$ we have that, for $\varepsilon \in (0, \min\{1/R, \breve{\varepsilon}\})$,

$$\mathcal{N}_\varepsilon(R\varepsilon) = \frac{\int_{\widehat{\Omega}_R} \Big(|\nabla \widehat{u}_\varepsilon(y)|^2 - \lambda_{\bar{k}}^\varepsilon \varepsilon^2 p(\varepsilon y)\widehat{u}_\varepsilon^2(y)\Big)\,dy}{\int_{\widehat{\Gamma}_R} \widehat{u}_\varepsilon^2(y)\,d\sigma}. \tag{168}$$

From Lemma 4.6, for every $\delta > 0$ there exists $\hat{\varepsilon}_{R,\delta} > 0$ such that

$$\mathcal{N}_\varepsilon(R\varepsilon) < (1+\delta)\sqrt{\lambda_1(\Sigma)} \quad \text{for all } \varepsilon \in (0, \hat{\varepsilon}_{R,\delta}). \tag{169}$$

Choosing $\delta = 1$, from (168), (169), and Lemma 5.1, we have that

$$\int_{\widehat{\Omega}_R} \Big(|\nabla \widehat{u}_\varepsilon(y)|^2 - \lambda_{\bar{k}}^\varepsilon \varepsilon^2 p(\varepsilon y)\widehat{u}_\varepsilon^2(y)\Big)dy \leqslant 2\sqrt{\lambda_1(\Sigma)} \int_{\widehat{\Gamma}_R} \widehat{u}_\varepsilon^2(y)\,d\sigma \leqslant 2\sqrt{\lambda_1(\Sigma)}e^{4\sqrt{\lambda_1(\Sigma)}(R-1)} \tag{170}$$

for all $\varepsilon \in (0, \hat{\varepsilon}_R)$, where $\hat{\varepsilon}_R = \hat{\varepsilon}_{R,1} > 0$ (accordingly with the notation of Lemma 5.1). From (170) and Lemma 3.6, we obtain that for all $\varepsilon \in (0, \min\{\hat{\varepsilon}_R, \bar{\varepsilon}_{2,2\lambda_{k_0}(D^+)p}\})$

$$\int_{\widehat{\Omega}_R} |\nabla \widehat{u}_\varepsilon(y)|^2 dy \leqslant 4\sqrt{\lambda_1(\Sigma)}e^{4\sqrt{\lambda_1(\Sigma)}(R-1)}. \tag{171}$$

In view of (171) and Lemma 5.1, we have that for every $R > 1$ there exists $\varepsilon_R > 0$ such that

$$\{\widehat{u}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_R)} \text{ is bounded in } \mathcal{H}_R. \tag{172}$$

Let $\varepsilon_n \to 0^+$. From (172) and a diagonal process, we deduce that there exist a subsequence $\varepsilon_{n_k} \to 0^+$ and some $\widehat{u} \in \bigcup_{R>1} \mathcal{H}_R$ such that $\widehat{u}_{\varepsilon_{n_k}} \rightharpoonup \widehat{u}$ weakly in $\mathcal{H}_R$ for every $R > 1$ and almost everywhere. From compactness of the embedding $\mathcal{H}_R \hookrightarrow L^2(\widehat{\Gamma}_1)$ and (166) we deduce that $\int_{\widehat{\Gamma}_1} \widehat{u}^2 d\sigma = 1$; in particular $\widehat{u} \not\equiv 0$.

Passing to the weak limit in (164), we obtain that $\widehat{u}$ is a weak solution to (167). By classical elliptic estimates, we also have that $\widehat{u}_{\varepsilon_{n_k}} \to \widehat{u}$ in $C^2(\{(x_1, x') \in T_1^+ : r_1 \leqslant x_1 \leqslant r_2\})$ for all $0 < r_1 < r_2$. Therefore, multiplying (167) by $\widehat{u}$ and integrating over $\widehat{\Omega}_R$, we obtain

$$\int_{\widehat{\Gamma}_R} \frac{\partial \widehat{u}_{\varepsilon_{n_k}}}{\partial x_1}\widehat{u}_{\varepsilon_{n_k}}\,d\sigma \to \int_{\widehat{\Gamma}_R} \frac{\partial \widehat{u}}{\partial x_1}\widehat{u}\,d\sigma = \int_{\widehat{\Omega}_R} |\nabla \widehat{u}(x)|^2 dx. \tag{173}$$

On the other hand, multiplication of (164) by $\widehat{u}_{\varepsilon_{n_k}}$ and integration by parts over $\widehat{\Omega}_R$ yield

$$\int_{\widehat{\Omega}_R} |\nabla \widehat{u}_{\varepsilon_{n_k}}(x)|^2 dx = \int_{\widehat{\Gamma}_R} \frac{\partial \widehat{u}_{\varepsilon_{n_k}}}{\partial x_1}\widehat{u}_{\varepsilon_{n_k}}\,d\sigma + \lambda_{\bar{k}}^{\varepsilon_{n_k}} \varepsilon_{n_k}^2 \int_{\widehat{\Omega}_R} p(\varepsilon_{n_k} x)\widehat{u}_{\varepsilon_{n_k}}^2(x)\,dx. \tag{174}$$

We claim that, for every $R > 1$,

$$\varepsilon_{n_k}^2 \int_{\widehat{\Omega}_R} p(\varepsilon_{n_k} x)\widehat{u}_{\varepsilon_{n_k}}^2(x)\,dx \to 0 \quad \text{as } n \to +\infty. \tag{175}$$

Indeed, from Lemma 3.6, for every $\delta > 0$ there exists k_0 such that for all $k \geqslant k_0$

$$\int_{\Omega_{R\varepsilon_{n_k}}^{\varepsilon_{n_k}}} p(y)u_{\varepsilon_{n_k}}^2(y)dy \leqslant \delta \int_{\Omega_{R\varepsilon_{n_k}}^{\varepsilon_{n_k}}} |\nabla u_{\varepsilon_{n_k}}(y)|^2 dy$$

and hence, from the change of variable $y = \varepsilon_{n_k} x$ and (171), we deduce that

$$\begin{aligned}
\varepsilon_{n_k}^2 \int_{\widehat{\Omega}_R} p(\varepsilon_{n_k} x)\widehat{u}_{\varepsilon_{n_k}}^2(x)\,dx &= \frac{\varepsilon_{n_k}}{\int_{\Gamma_{\varepsilon_{n_k}}^{\varepsilon_{n_k}}} u_{\varepsilon_{n_k}}^2\,d\sigma} \int_{\Omega_{R\varepsilon_{n_k}}^{\varepsilon_{n_k}}} p(y)u_{\varepsilon_{n_k}}^2(y)dy \\
&\leqslant \frac{\varepsilon_{n_k}\delta}{\int_{\Gamma_{\varepsilon_{n_k}}^{\varepsilon_{n_k}}} u_{\varepsilon_{n_k}}^2\,d\sigma} \int_{\Omega_{R\varepsilon_{n_k}}^{\varepsilon_{n_k}}} |\nabla u_{\varepsilon_{n_k}}(y)|^2 dy \\
&= \delta \int_{\widehat{\Omega}_R} |\nabla \widehat{u}_{\varepsilon_{n_k}}(x)|^2 dx \leqslant 4\delta\sqrt{\lambda_1(\Sigma)}e^{4\sqrt{\lambda_1(\Sigma)}(R-1)},
\end{aligned}$$

thus proving claim (175). Combining (173), (174), and (175), we conclude that $\|\widehat{u}_{\varepsilon_{n_k}}\|_{\mathcal{H}_R} \to \|\widehat{u}\|_{\mathcal{H}_R}$ and then $\widehat{u}_{\varepsilon_{n_k}} \to \widehat{u}$ strongly in $\mathcal{H}_R$ for every $R > 1$. □

Remark 5.3. We notice that the function $\widehat{u}$ found in Lemma 5.2 satisfies

$$\int_{\widehat{D}} |\nabla \widehat{u}(x)|^2 dx = +\infty.$$

Indeed, $\int_{\widehat{D}} |\nabla \widehat{u}(x)|^2 dx < +\infty$ would imply, by testing (167) with $\widehat{u}$, that $\widehat{u} \equiv 0$ in $\widehat{D}$, thus contradicting statement ii) of Lemma 5.2.

We also observe that, denoting as $\widehat{H}(r) = \int_{\widehat{\Gamma}_r} \widehat{u}^2 d\sigma$ for all $r > 0$, multiplication of (167) by $\widehat{u}$ and integration over $\widehat{\Omega}_r$ yield

$$\frac{d}{dr}\widehat{H}(r) = 2\int_{\widehat{\Gamma}_r} \widehat{u}\frac{\partial \widehat{u}}{\partial x_1} d\sigma = 2\int_{\widehat{\Omega}_r} |\nabla \widehat{u}(x)|^2 dx \to \int_{\widehat{D}} |\nabla \widehat{u}(x)|^2 dx = +\infty \quad \text{as } r \to +\infty,$$

thus implying that

$$\lim_{r\to+\infty} \widehat{H}(r) = \lim_{r\to+\infty} \int_{\Gamma_r} \widehat{u}^2 d\sigma = +\infty.$$

Lemma 5.4. *Let $\widehat{u}$ as in Lemma 5.2 and, for $r > 0$, let $\widehat{\mathcal{N}}_{\widehat{u}}(r)$ be the frequency function associated to $\widehat{u}$, i.e.*

$$\widehat{\mathcal{N}}_{\widehat{u}}(r) = \frac{\int_{\widehat{\Omega}_r} |\nabla \widehat{u}(x)|^2 dx}{\int_{\widehat{\Gamma}_r} \widehat{u}(x)\, d\sigma}, \quad r > 0,$$

with $\widehat{\Omega}_r$ and $\widehat{\Gamma}_r$ defined in (165). Then

i) $\lim_{r\to+\infty} \widehat{\mathcal{N}}_{\widehat{u}}(r) = \sqrt{\lambda_1(\Sigma)}$;
ii) *there exists $\hat{c} \in \mathbb{R} \setminus \{0\}$ such that $\int_{T_1} |\nabla(\widehat{u} - \hat{c}h)(x)|^2\, dx < +\infty$,*

where

$$h : T_1 \to \mathbb{R}, \quad h(x_1, x') = f(1 - x_1, x') = e^{\sqrt{\lambda_1(\Sigma)}x_1}\psi_1^\Sigma(x'), \tag{176}$$

being f defined in (27).

Proof. Letting $\varepsilon_n \to 0^+$ and $\{\varepsilon_{n_k}\}_k$ as in Lemma 5.2, passing to the limit as $k \to +\infty$ in (168), and using (175), we have that

$$\lim_{k\to+\infty} \mathcal{N}_{\varepsilon_{n_k}}(R\varepsilon_{n_k}) = \widehat{\mathcal{N}}_{\widehat{u}}(R) \quad \text{for every } R > 0,$$

which, together with (169), implies that, for every $\delta > 0$ and $R > 0$,

$$\widehat{\mathcal{N}}_{\widehat{u}}(R) \leqslant (1+\delta)\sqrt{\lambda_1(\Sigma)}.$$

Therefore

$$\widehat{\mathcal{N}}_{\widehat{u}}(R) \leqslant \sqrt{\lambda_1(\Sigma)} \quad \text{for every } R > 0. \tag{177}$$

It is easy to prove that there exists $\zeta \in H^1_{\text{loc}}(T_1) \cap L^\infty(T_1)$ such that

$$\begin{cases} -\Delta\zeta = 0, & \text{in } T_1,\\ \zeta = \widehat{u}, & \text{on } \partial T_1,\\ \int_{T_1} |\nabla\zeta(x)|^2 dx < +\infty, \end{cases}$$

i.e. ζ is a finite-energy harmonic extension of $\widehat{u}\big|_{\partial T_1}$ in T_1. Since $w(x_1, x') = e^{-\sqrt{\lambda_1(\Sigma)}\frac{x_1}{2}}\psi_1^\Sigma\big(\frac{x'}{2}\big)$ is harmonic and strictly positive in T_1, bounded from below away from 0 in $\{(x_1, x') \in T_1 : x_1 \leqslant 0\}$, and $\int_{\{(x_1,x')\in T_1 : x_1 \geqslant r\}} (|\nabla w|^2 + |w|^{2^*}) < +\infty$ for all r, from the Maximum Principle we deduce that $|\zeta| \leqslant \text{const}\, w$ in T_1, thus implying that, for some $c_\zeta > 0$,

$$|\zeta(x)| \leqslant c_\zeta e^{-\sqrt{\lambda_1(\Sigma)}\frac{x_1}{2}} \quad \text{for all } x \in T_1.$$

Let us observe that the function $\widehat{v} := \widehat{u} - \zeta \in H^1_{\text{loc}}(T_1) \setminus \{0\}$ satisfies

$$\begin{cases} -\Delta\widehat{v}(x) = 0, & \text{in } T_1,\\ \widehat{v} = 0, & \text{on } \partial T_1. \end{cases}$$

We notice that $\widehat{v} \not\equiv 0$ in view of Remark 5.3. Let

$$N_{\widehat{v}} : \mathbb{R} \to \mathbb{R}, \quad N_{\widehat{v}}(r) := \frac{\int_{T_{1,r}} |\nabla \widehat{v}(x)|^2 dx}{\int_{\Gamma_r} \widehat{v}^2(x)\, d\sigma},$$

be as in Lemma 2.5, where, for all $r \in \mathbb{R}$, $T_{1,r}$ and Γ_r are defined in (38). From Lemma 2.5 it follows that $N_{\widehat{v}}$ is non decreasing in $\mathbb{R}$ and

$$N_{\widehat{v}}(t) \geqslant \lim_{r\to-\infty} N_{\widehat{v}}(r) \geqslant \sqrt{\lambda_1(\Sigma)} \quad \text{for all } t \in \mathbb{R}. \tag{178}$$

For all $R > 0$, $\delta \in (0,1)$,

$$\begin{aligned} \int_{\Gamma_R} \widehat{v}^2 d\sigma - (1-\delta)\int_{\Gamma_R} \widehat{u}^2 d\sigma &= \int_{\Gamma_R} \zeta^2 d\sigma - 2\int_{\Gamma_R} \zeta \widehat{u} d\sigma + \delta \int_{\Gamma_R} \widehat{u}^2 d\sigma \\ &\geqslant \left(1-\frac{2}{\delta}\right)\int_{\Gamma_R} \zeta^2 d\sigma + \frac{\delta}{2}\int_{\Gamma_R} \widehat{u}^2 d\sigma \geqslant \left(1-\frac{2}{\delta}\right)\int_{\Gamma_R} \zeta^2 d\sigma \geqslant \left(1-\frac{2}{\delta}\right)\frac{\omega_{N-2}}{N-1}\, c_\zeta^2 e^{-\sqrt{\lambda_1(\Sigma)}R} \end{aligned}$$

and

$$\begin{aligned} &\int_{T_{1,R}} |\nabla \widehat{v}(x)|^2 dx - (1+\delta)\int_{\widehat{\Omega}_R} |\nabla \widehat{u}(x)|^2 dx \\ &\qquad \leqslant \int_{T_{1,R}} |\nabla \zeta(x)|^2 dx - 2\int_{T_{1,R}} \nabla\zeta(x)\cdot\nabla\widehat{u}(x) dx - \delta\int_{\widehat{\Omega}_R} |\nabla \widehat{u}(x)|^2 dx \\ &\qquad \leqslant \left(1+\frac{2}{\delta}\right)\int_{T_{1,R}} |\nabla \zeta(x)|^2 dx + \frac{\delta}{2}\int_{T_{1,R}} |\nabla \widehat{u}(x)|^2 dx - \delta\int_{\widehat{\Omega}_R} |\nabla \widehat{u}(x)|^2 dx \\ &\qquad \leqslant \left(1+\frac{2}{\delta}\right)\int_{T_{1,R}} |\nabla \zeta(x)|^2 dx, \end{aligned}$$

thus implying

$$N_{\widehat{v}}(R) \leqslant \frac{1+\delta}{1-\delta}\widehat{\mathcal{N}}_{\widehat{u}}(R) \frac{1 + \dfrac{1+\frac{2}{\delta}}{1+\delta}\dfrac{\int_{T_{1,R}} |\nabla\zeta(x)|^2 dx}{\int_{\widehat{\Omega}_R} |\nabla \widehat{u}(x)|^2 dx}}{1 + \dfrac{(1-\frac{2}{\delta})\omega_{N-2} c_\zeta^2}{(1-\delta)(N-1)\int_{\Gamma_R} \widehat{u}^2 d\sigma} e^{-\sqrt{\lambda_1(\Sigma)}R}}. \tag{179}$$

On the other hand, for all $R > 0$, $\delta \in (0,1)$,

$$\begin{aligned} \int_{\Gamma_R} \widehat{v}^2 d\sigma - (1+\delta)\int_{\Gamma_R} \widehat{u}^2 d\sigma &= \int_{\Gamma_R} \zeta^2 d\sigma - 2\int_{\Gamma_R} \zeta \widehat{u} d\sigma - \delta \int_{\Gamma_R} \widehat{u}^2 d\sigma \\ &\leqslant \left(1+\frac{2}{\delta}\right)\int_{\Gamma_R} \zeta^2 d\sigma - \frac{\delta}{2}\int_{\Gamma_R} \widehat{u}^2 d\sigma \leqslant \left(1+\frac{2}{\delta}\right)\int_{\Gamma_R} \zeta^2 d\sigma \leqslant \left(1+\frac{2}{\delta}\right)\frac{\omega_{N-2}}{N-1}\, c_\zeta^2 e^{-\sqrt{\lambda_1(\Sigma)}R} \end{aligned}$$

and

$$\begin{aligned} &\int_{T_{1,R}} |\nabla \widehat{v}(x)|^2 dx - (1-\delta)\int_{\widehat{\Omega}_R} |\nabla \widehat{u}(x)|^2 dx \\ &= -\int_{D^-\setminus T_1} |\nabla \widehat{u}(x)|^2 dx + \int_{T_{1,R}} |\nabla \zeta(x)|^2 dx - 2\int_{T_{1,R}} \nabla\zeta(x)\cdot\nabla\widehat{u}(x) dx + \delta\int_{\widehat{\Omega}_R} |\nabla \widehat{u}(x)|^2 dx \\ &\geqslant -\int_{D^-} |\nabla \widehat{u}(x)|^2 dx + \left(1-\frac{2}{\delta}\right)\int_{T_{1,R}} |\nabla \zeta(x)|^2 dx - \frac{\delta}{2}\int_{T_{1,R}} |\nabla \widehat{u}(x)|^2 dx + \delta\int_{\widehat{\Omega}_R} |\nabla \widehat{u}(x)|^2 dx \\ &\qquad \geqslant -\int_{D^-} |\nabla \widehat{u}(x)|^2 dx + \left(1-\frac{2}{\delta}\right)\int_{T_{1,R}} |\nabla \zeta(x)|^2 dx, \end{aligned}$$

thus implying

$$N_{\widehat{v}}(R) \geqslant \frac{1-\delta}{1+\delta}\widehat{\mathcal{N}}_{\widehat{u}}(R)\,\frac{1-\dfrac{\int_{D^-}|\nabla\widehat{u}(x)|^2dx}{(1-\delta)\int_{\widehat{\Omega}_R}|\nabla\widehat{u}(x)|^2dx}+\dfrac{1-\frac{2}{\delta}}{1-\delta}\dfrac{\int_{T_{1,R}}|\nabla\zeta(x)|^2dx}{\int_{\widehat{\Omega}_R}|\nabla\widehat{u}(x)|^2dx}}{1+\dfrac{(1+\frac{2}{\delta})\omega_{N-2}c_\zeta^2}{(1+\delta)(N-1)\int_{\Gamma_R}\widehat{u}^2d\sigma}e^{-\sqrt{\lambda_1(\Sigma)}R}}. \tag{180}$$

Since $\int_{T_{1,R}}|\nabla\zeta(x)|^2dx = O(1)$, $\int_{\widehat{\Omega}_R}|\nabla\widehat{u}(x)|^2dx \to +\infty$, and $\int_{\Gamma_R}\widehat{u}^2d\sigma \to +\infty$ as $R\to+\infty$ (see Remark 5.3), passing to lim sup and lim inf in (179–180) we obtain that

$$\frac{1-\delta}{1+\delta}\limsup_{R\to\infty}\widehat{\mathcal{N}}_{\widehat{u}}(R) \leqslant \limsup_{R\to\infty}N_{\widehat{v}}(R) \leqslant \frac{1+\delta}{1-\delta}\limsup_{R\to\infty}\widehat{\mathcal{N}}_{\widehat{u}}(R) \quad \text{for all } \delta>0,$$

$$\frac{1-\delta}{1+\delta}\liminf_{R\to\infty}\widehat{\mathcal{N}}_{\widehat{u}}(R) \leqslant \liminf_{R\to\infty}N_{\widehat{v}}(R) \leqslant \frac{1+\delta}{1-\delta}\liminf_{R\to\infty}\widehat{\mathcal{N}}_{\widehat{u}}(R) \quad \text{for all } \delta>0,$$

thus implying, in view of (178),

$$\liminf_{R\to\infty}\widehat{\mathcal{N}}_{\widehat{u}}(R) = \liminf_{R\to\infty}N_{\widehat{v}}(R) \geqslant \sqrt{\lambda_1(\Sigma)} \tag{181}$$

and, in view of (177),

$$\sqrt{\lambda_1(\Sigma)} \geqslant \limsup_{R\to\infty}\widehat{\mathcal{N}}_{\widehat{u}}(R) = \limsup_{R\to\infty}N_{\widehat{v}}(R). \tag{182}$$

From (181) and (182) we deduce that

$$\lim_{R\to\infty}\widehat{\mathcal{N}}_{\widehat{u}}(R) = \lim_{R\to\infty}N_{\widehat{v}}(R) = \sqrt{\lambda_1(\Sigma)}, \tag{183}$$

thus proving statement i). Furthermore (183), (178), and the fact that $N_{\widehat{v}}$ is non decreasing imply that

$$N_{\widehat{v}}(t) \equiv \sqrt{\lambda_1(\Sigma)} \quad \text{in } \mathbb{R}.$$

From Lemma 2.5 iii), it follows that there exists $\hat{c}\in\mathbb{R}\setminus\{0\}$ such that $\widehat{v}(x_1,x') = \hat{c}h(x_1,x')$ with h as in (176). Since $\int_{T_1}|\nabla(\widehat{u}-\hat{c}h)(x)|^2\,dx = \int_{T_1}|\nabla\zeta(x)|^2\,dx < +\infty$, also claim ii) is proved. □

Corollary 5.5. *Let $\widehat{u}$ be as in Lemma 5.2 and $\hat{c}$ as in Lemma 5.4. Then*

$$\widehat{u}(x_1,x') = \hat{c}\,\Phi_2(1-x_1,x')$$

where Φ_2 is as in Lemma 2.7.

Proof. It follows from Lemmas 5.2 and 5.4, taking into account Lemma 2.7. □

Let us define $\widehat{\Phi}(x_1,x') := \Phi_2(1-x_1,x')$ and, for all $r<-1$

$$\widehat{\mathcal{N}}(r) = \widehat{\mathcal{N}}_{\widehat{\Phi}}(r) = \frac{(-r)\int_{\Omega_r}|\nabla\widehat{\Phi}(x)|^2dx}{\int_{\Gamma^-_{-r}}\widehat{\Phi}(x)\,d\sigma},$$

with Ω_r as in (77) and Γ^-_{-r} as in (10), so that, according to notation of Lemma 3.3, $\widehat{\mathcal{N}}(r) = N^-_{\widehat{\Phi}}(-r)$ for all $r<-1$.

Lemma 5.6. $\lim_{r\to-\infty}\widehat{\mathcal{N}}(r) = N-1$.

Proof. The proof follows from Lemma 3.3 and Remark 2.8. □

Lemma 5.7. *For every $R>1$*

$$\lim_{\varepsilon\to0^+}\mathcal{N}_\varepsilon(-R\varepsilon) = \widehat{\mathcal{N}}(-R).$$

Proof. Fix $R>1$. Let $\varepsilon_n\to0^+$. From Lemma 5.2 and Corollary 5.5, there exist a subsequence $\{\varepsilon_{n_k}\}_k$ and $\hat{c}\neq0$ such that $\widehat{u}_{\varepsilon_{n_k}}\to\hat{c}\,\widehat{\Phi}$ strongly in $\mathcal{H}_r$ for every $r>1$. By the change of variable $x=\varepsilon y$ we have that, for $\varepsilon\in(0,\tilde{\varepsilon})$ and $R>1$,

$$\mathcal{N}_\varepsilon(-R\varepsilon) = \frac{R\int_{\Omega_{-R}}\Big(|\nabla\widehat{u}_\varepsilon(y)|^2-\lambda_{\bar{k}}^\varepsilon\varepsilon^2p(\varepsilon y)\widehat{u}_\varepsilon^2(y)\Big)dy}{\int_{\Gamma^-_R}\widehat{u}_\varepsilon^2(y)\,d\sigma} \tag{184}$$

with Ω_{-R} and Γ^-_R as in (77) and (10) respectively. From strong convergence of $\widehat{u}_{\varepsilon_{n_k}}$ to $\hat{c}\,\widehat{\Phi}$ in $\mathcal{H}_r$ for every $r > 1$, passing to the limit in (184) along the subsequence $\{\varepsilon_{n_k}\}_k$ and using (175) we obtain that

$$\lim_{k\to+\infty} \mathcal{N}_{\varepsilon_{n_k}}(-R\varepsilon_{n_k}) = \frac{R\int_{\Omega_{-R}} |\nabla(\hat{c}\,\widehat{\Phi})(y)|^2 dy}{\int_{\Gamma^-_R} (\hat{c}\,\widehat{\Phi})^2(y)\, d\sigma} = \frac{R\int_{\Omega_{-R}} |\nabla\widehat{\Phi}(y)|^2 dy}{\int_{\Gamma^-_R} \widehat{\Phi}^2(y)\, d\sigma} = \widehat{\mathcal{N}}(-R).$$

Since the limit depends neither on the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$, we conclude that the convergence actually holds as $\varepsilon \to 0^+$ thus proving the lemma. □

Lemma 5.8. *For every $\delta > 0$ there exist $K_\delta > 1$, $k_\delta \in (0,1)$, and $\rho_\delta \in \big(0, \frac{k_\delta}{K_\delta}\big)$, such that*

$$\mathcal{N}_\varepsilon(r) \leqslant N - 1 + \delta \quad \textit{and} \quad \int_{\Omega_r} \Big(|\nabla u_\varepsilon|^2 - \lambda^\varepsilon_{\bar{k}} p u_\varepsilon^2\Big) dx \geqslant \frac{1}{2}\int_{\Omega_r} |\nabla u_\varepsilon|^2 dx \tag{185}$$

for all $r \in (-k_\delta, -K_\delta\varepsilon)$ and $\varepsilon \in (0, \rho_\delta)$.

Proof. Let $\delta > 0$ and fix $\delta' \in (0,1)$ such that

$$(N - 1 + 2\delta')e^{\delta'} = N - 1 + \delta.$$

From Lemma 5.6 there exists some $K_\delta > 1$ such that $\widehat{\mathcal{N}}(-K_\delta) < N-1+\delta'$. From Lemma 5.7 there exists some $\varepsilon'_\delta > 0$ such that, for all $\varepsilon \in (0, \varepsilon'_\delta)$, $\mathcal{N}_\varepsilon(-K_\delta\varepsilon) < \widehat{\mathcal{N}}(-K_\delta) + \delta' < N - 1 + 2\delta'$. Letting $\check{R}_{\delta'}$, $\check{\varepsilon}_{\delta'}$ as in Lemma 3.17 and Corollary 3.18, we have that for all $\varepsilon \in \big(0, \min\big\{\varepsilon'_\delta, \check{\varepsilon}_{\delta'}, \check{R}_{\delta'}/K_\delta\big\}\big)$ and $r \in (-\check{R}_{\delta'}, -K_\delta\varepsilon)$

$$\mathcal{N}_\varepsilon(r) \leqslant \mathcal{N}_\varepsilon(-K_\delta\varepsilon)e^{\delta'\check{R}_{\delta'}} \leqslant (N-1+2\delta')e^{\delta'\check{R}_{\delta'}} \leqslant N-1+\delta$$

and $\int_{\Omega_r} \big(|\nabla u_\varepsilon|^2 - \lambda^\varepsilon_{\bar{k}} p u_\varepsilon^2\big)dx \geqslant \frac{1}{2}\int_{\Omega_r} |\nabla u_\varepsilon|^2 dx$. Then the lemma follows choosing $k_\delta = \check{R}_{\delta'}$ and $\rho_\delta = \min\big\{\varepsilon'_\delta, \check{\varepsilon}_{\delta'}, \check{R}_{\delta'}/K_\delta\big\}$. □

6. Asymptotics at the left junction

Throughout this section, we fix $\delta \in (0,1)$ so that $N - 1 + \delta < N$. Let us denote $\widetilde{K} = K_\delta > 1$, $\tilde{h} = k_\delta \in (0,1)$, and $\tilde{\rho} = \rho_\delta \in \big(0, \frac{\tilde{h}}{\widetilde{K}}\big)$ with $K_\delta, k_\delta, \rho_\delta$ as in Lemma 5.8, so that

$$\mathcal{N}_\varepsilon(r) \leqslant N - 1 + \delta < N \quad \text{for all } r \in (-\tilde{h}, -\widetilde{K}\varepsilon) \text{ and } \varepsilon \in (0, \tilde{\rho}). \tag{186}$$

Let us denote

$$U_\varepsilon(x) = \frac{u_\varepsilon(x)}{\sqrt{\int_{\Gamma^-_{\tilde{h}}} u_\varepsilon^2 d\sigma}} \tag{187}$$

with $\Gamma^-_{\tilde{h}}$ as in (10). Let us notice that, for $\varepsilon \in (0, \varepsilon_0)$, U_ε solves

$$\begin{cases} -\Delta U_\varepsilon = \lambda^\varepsilon_{\bar{k}} p U_\varepsilon, & \text{in } \Omega^\varepsilon, \\ U_\varepsilon = 0, & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{188}$$

and

$$\int_{\Gamma^-_{\tilde{h}}} U_\varepsilon^2 d\sigma = 1. \tag{189}$$

Proposition 6.1. *For every sequence $\varepsilon_n \to 0^+$ there exist a subsequence $\{\varepsilon_{n_k}\}_k$ and a function $U \in C^2(D^-) \cup \big(\bigcup_{t>0} \mathcal{H}^-_t\big)$ such that*

i) *$U_{\varepsilon_{n_k}} \to U$ strongly in $\mathcal{H}^-_t$ for every $t > 0$ and in $C^2(\overline{B^-_{t_2} \setminus B^-_{t_1}})$ for all $0 < t_1 < t_2$;*
ii) *$U \not\equiv 0$ in D^-;*
iii) *U solves*

$$\begin{cases} -\Delta U(x) = \lambda_{k_0}(D^+) p(x) U(x), & \textit{in } D^-, \\ U = 0, & \textit{on } \partial D^-; \end{cases} \tag{190}$$

iv) *if* $\mathcal{N}_U : (-\infty, 0) \to \mathbb{R}$ *is defined as*

$$\mathcal{N}_U(r) := \frac{(-r)\int_{\Omega_r}\Big(|\nabla U(x)|^2 - \lambda_{k_0}(D^+)p(x)U^2(x)\Big)dx}{\int_{\Gamma^-_{-r}} U^2(x)\,d\sigma}, \tag{191}$$

then

$$\mathcal{N}_U(r) \leqslant N - 1 + \delta \quad \textit{for all } r \in (-\tilde{h}, 0). \tag{192}$$

Proof. Letting $H^-_\varepsilon(t)$ as in (109), from (111), (107)–(109), and Lemma 5.8 it follows that

$$\frac{\frac{d}{dt}H^-_\varepsilon(t)}{H^-_\varepsilon(t)} = -\frac{2}{t}\mathcal{N}_\varepsilon(-t) \geqslant -\frac{2N}{t}$$

for all $t \in (\widetilde{K}\varepsilon, \tilde{h})$ and $\varepsilon \in (0, \tilde{\rho})$, which after integration yields

$$H^-_\varepsilon(t) \leqslant \tilde{h}^{2N} H^-_\varepsilon(\tilde{h}) t^{-2N} \quad \text{for all } t \in (\widetilde{K}\varepsilon, \tilde{h}). \tag{193}$$

From (187), (107)–(109), (193), and Lemma 5.8, we deduce that

$$\begin{aligned}\frac{1}{2}\int_{\Omega_{-t}} |\nabla U_\varepsilon(x)|^2 dx &\leqslant \int_{\Omega_{-t}} \Big(|\nabla U_\varepsilon(x)|^2 - \lambda^\varepsilon_{\bar{k}} p(x) U^2_\varepsilon(x)\Big)dx \\ &= \frac{\int_{\Omega_{-t}}\Big(|\nabla u_\varepsilon(x)|^2 - \lambda^\varepsilon_{\bar{k}} p(x) u^2_\varepsilon(x)\Big)dx}{\int_{\Gamma^-_{\tilde{h}}} u^2_\varepsilon d\sigma} = \frac{t^{N-2}}{\tilde{h}^{N-1}}\mathcal{N}_\varepsilon(-t)\frac{H^-_\varepsilon(t)}{H^-_\varepsilon(\tilde{h})} \leqslant N\tilde{h}^{N+1}t^{-N-2}\end{aligned}$$

for all $t \in (\widetilde{K}\varepsilon, \tilde{h})$ and $\varepsilon \in (0, \tilde{\rho})$. Hence for every $t > 0$

$$\{U_\varepsilon\}_{\varepsilon \in (0, \min\{\tilde{\rho}, t/\widetilde{K}\})} \text{ is bounded in } \mathcal{H}^-_t. \tag{194}$$

Let $\varepsilon_n \to 0^+$. From (194) and a diagonal process, there exist a subsequence $\varepsilon_{n_k} \to 0^+$ and some $U \in \bigcup_{R>0} \mathcal{H}^-_t$ such that $U_{\varepsilon_{n_k}} \rightharpoonup U$ weakly in $\mathcal{H}^-_t$ for every $t > 0$ and a.e. in D^-. From compactness of the embedding $\mathcal{H}^-_t \hookrightarrow L^2(\Gamma^-_t)$, passing to the limit in (189) we obtain that $\int_{\Gamma^-_{\tilde{h}}} U^2 d\sigma = 1$; in particular $U \not\equiv 0$ in D^-. Passing to the weak limit in (188), we obtain that U is a weak solution to (190). By classical elliptic estimates, we also have that $U_{\varepsilon_{n_k}} \to U$ in $C^2(\overline{B^-_{t_2} \setminus B^-_{t_1}})$ for all $0 < t_1 < t_2$. Therefore, multiplying (190) by U and integrating over Ω_{-t}, we obtain

$$\int_{\Gamma^-_t} \frac{\partial U_{\varepsilon_{n_k}}}{\partial\nu} U_{\varepsilon_{n_k}} d\sigma \to \int_{\Gamma^-_t} \frac{\partial U}{\partial \nu} U\, d\sigma = -\int_{\Omega_{-t}} \Big(|\nabla U(x)|^2 - \lambda_{k_0}(D^+)p(x)U^2(x)\Big)dx, \tag{195}$$

being $\nu = \nu(x) = \frac{x}{|x|}$. On the other hand, multiplication of (188) by $U_{\varepsilon_{n_k}}$ and integration by parts over Ω_{-t} yield

$$\int_{\Omega_{-t}} \Big(|\nabla U_{\varepsilon_{n_k}}(x)|^2 - \lambda^\varepsilon_{\bar{k}} p(x) U^2_\varepsilon(x)\Big)dx = -\int_{\Gamma^-_t} \frac{\partial U_{\varepsilon_{n_k}}}{\partial \nu} U_{\varepsilon_{n_k}} d\sigma. \tag{196}$$

Since weak $\mathcal{H}^-_t$-convergence of $U_{\varepsilon_{n_k}}$ to U implies that

$$\int_{\Omega_{-t}} p(x)U^2_{\varepsilon_{n_k}}(x)dx \to \int_{\Omega_{-t}} p(x)U^2(x)dx \quad \text{as } k \to +\infty, \tag{197}$$

combining (195), (196), and (197), we conclude that $\|U_{\varepsilon_{n_k}}\|_{\mathcal{H}^-_R} \to \|U\|_{\mathcal{H}^-_R}$ and then $U_{\varepsilon_{n_k}} \to U$ strongly in $\mathcal{H}^-_t$ for every $t > 0$.

Finally, we notice that strong $\mathcal{H}^-_t$-convergence of $U_{\varepsilon_{n_k}}$ to U implies that, for every $r < 0$,

$$\mathcal{N}_{\varepsilon_{n_k}}(r) = \frac{(-r)\int_{\Omega_r}\Big(|\nabla U_{\varepsilon_{n_k}}(x)|^2 - \lambda^\varepsilon_{\bar{k}} p(x) U^2_{\varepsilon_{n_k}}(x)\Big)dx}{\int_{\Gamma^-_{-r}} U^2_{\varepsilon_{n_k}}(x)\,d\sigma} \to \mathcal{N}_U(r) \quad \text{as } k \to +\infty,$$

hence, passing to the limit in (186) as $\varepsilon = \varepsilon_{n_k} \to 0$, we obtain (192) and complete the proof. □

Lemma 6.2. *Let U be as in Proposition 6.1 and let $\mathcal{N}_U : (-\infty, 0) \to \mathbb{R}$ be the frequency function associated to U defined in* (191)*. Then*

(i) $\lim_{r \to 0^-} \mathcal{N}_U(r) = N - 1$*;*

(ii) *for every sequence* $\lambda_n \to 0^+$ *there exist a subsequence* $\{\lambda_{n_k}\}_k$ *and some constant* $c \in \mathbb{R} \setminus \{0\}$ *such that*

$$\frac{U(\lambda_{n_k} x)}{\sqrt{H_U(\lambda_{n_k})}} \underset{k\to+\infty}{\longrightarrow} c\,\frac{x_1}{|x|^N}$$

strongly in $\mathcal{H}_t^-$ *for every* $t > 0$ *and in* $C^2(\overline{B_{t_2}^- \setminus B_{t_1}^-})$ *for all* $0 < t_1 < t_2$, *where*

$$H_U(\lambda) := \frac{1}{\lambda^{N-1}} \int_{\Gamma_\lambda^-} U^2(x)\, d\sigma. \tag{198}$$

PROOF. We first notice that, letting $\varepsilon_n \to 0^+$ and $\{\varepsilon_{n_k}\}_k$ as in Proposition 6.1, passing to the limit as $k \to +\infty$, from (185) and strong $\mathcal{H}_t^-$-convergence of $U_{\varepsilon_{n_k}}$ to U we obtain that

$$\int_{\Omega_r} \Big(|\nabla U|^2 - \lambda_{k_0}(D^+) p U^2\Big) dx \geqslant \frac{1}{2} \int_{\Omega_r} |\nabla U|^2 dx \tag{199}$$

for all $r \in (-\tilde h, 0)$. In particular

$$\mathcal{N}_U(r) \geqslant 0 \quad \text{for all } r \in (-\tilde h, 0). \tag{200}$$

Arguing as in the proof of Lemma 3.15, we can prove that, for all $r < 0$,

$$\frac{d}{dr}\mathcal{N}_U(r) = \nu_1(r) + \nu_2(r), \tag{201}$$

where

$$\nu_1(r) = -2r \frac{\Big(\int_{\Gamma_{-r}^-} \big|\frac{\partial U}{\partial \nu}\big|^2 d\sigma\Big)\Big(\int_{\Gamma_{-r}^-} U^2(x)\, d\sigma\Big) - \Big(\int_{\Gamma_{-r}^-} U \frac{\partial U}{\partial \nu} d\sigma\Big)^2}{\Big(\int_{\Gamma_{-r}^-} U^2(x)\, d\sigma\Big)^2} \tag{202}$$

$$\nu_2(r) = \lambda_{k_0}(D^+) \frac{\int_{\Omega_r} (2p(x) + x \cdot \nabla p(x)) U^2(x) dx}{\int_{\Gamma_{-r}^-} U^2(x) d\sigma}. \tag{203}$$

Schwarz's inequality implies that

$$\nu_1(r) \geqslant 0 \quad \text{for all } r < 0. \tag{204}$$

Furthermore

$$\frac{|\nu_2(r)|}{\mathcal{N}_U(r)} \leqslant \lambda_{k_0}(D^+) \frac{\int_{\Omega_r} |2p(x) + x \cdot \nabla p(x)| U^2(x) dx}{(-r)\int_{\Omega_r} \Big(|\nabla U(x)|^2 - \lambda_{k_0}(D^+) p(x) U^2(x)\Big) dx} \leqslant \delta, \quad \text{for all } r \in (-\tilde h, 0),$$

where the last inequality is obtained passing to the limit as $\varepsilon = \varepsilon_{n_k} \to 0^+$ in (128). Hence from (192) we obtain that

$$|\nu_2(r)| \leqslant \delta\big(N - 1 + \delta\big), \quad \text{for all } r \in (-\tilde h, 0). \tag{205}$$

From (204) and (205) it follows that $\frac{d}{dr}\mathcal{N}_U$ is the sum of a nonnegative function and of a bounded function on $(-\tilde h, 0)$. Therefore $\mathcal{N}_U(r) = \mathcal{N}_U(-\tilde h) + \int_{-\tilde h}^r (\nu_1(s) + \nu_2(s))\, ds$ admits a limit as $r \to 0^+$ which is necessarily finite in view of (192) and (200). More precisely, denoting as

$$\gamma := \lim_{r\to 0^-} \mathcal{N}_U(r), \tag{206}$$

(192) and (200) ensure that

$$\gamma \in [0, N - 1 + \delta] \subset [0, N). \tag{207}$$

For all $x \in D^-$ and $\lambda > 0$, let us consider

$$U^\lambda(x) := \frac{U(\lambda x)}{\sqrt{H_U(\lambda)}} \tag{208}$$

where $H_U(\lambda)$ is defined in (198). We notice that

$$\int_{\Gamma_1^-} U_\lambda^2 d\sigma = 1. \tag{209}$$

Furthermore, by direct calculation (see also the proof of Lemma 3.15 which is analogous), we have that

$$\frac{H_U'(\lambda)}{H_U(\lambda)} = -\frac{2}{\lambda}\mathcal{N}_U(-\lambda) \geqslant -\frac{2}{\lambda}\big(N-1+\delta\big) \quad \text{for all } \lambda \in (0,\tilde h), \tag{210}$$

which after integration yields

$$H_U(\lambda_1) \leqslant H_U(\lambda_2)\left(\frac{\lambda_2}{\lambda_1}\right)^{2(N-1+\delta)} \qquad \text{for all } 0<\lambda_1<\lambda_2<\tilde h. \tag{211}$$

From (199), (211), and (192), for every $t \in (0,1)$ and $\lambda \in (0,\tilde h/t)$, we have that

$$\begin{aligned}\int_{\Omega_{-t}} |\nabla U^\lambda(x)|^2 dx &= t^{N-2}\frac{H_U(\lambda t)}{H_U(\lambda)}\frac{\lambda t\int_{\Omega_{-\lambda t}}|\nabla U(x)|^2dx}{\int_{\Gamma^-_{\lambda t}}U^2 d\sigma} \leqslant 2t^{N-2}t^{-2(N-1+\delta)}\mathcal{N}_U(-\lambda t)\\ &\leqslant 2t^{-N-2\delta}(N-1+\delta).\end{aligned} \tag{212}$$

Hence for every $t\in(0,1)$ there exists $\lambda_t>0$ such that

$$\{U^\lambda\}_{\lambda\in(0,\lambda_t)} \text{ is bounded in } \mathcal{H}^-_t. \tag{213}$$

Let $\lambda_n \to 0^+$. From (213) and a diagonal process, we deduce that there exist a subsequence $\lambda_{n_k}\to 0^+$ and some $\widetilde U \in \bigcup_{t>0}\mathcal{H}^-_t$ such that $U^{\lambda_{n_k}} \rightharpoonup \widetilde U$ weakly in $\mathcal{H}^-_t$ for every $t>0$ and a.e. in D^-. Since U^λ solves

$$\begin{cases}-\Delta U^\lambda(x) = \lambda^2\lambda_{k_0}(D^+)p(\lambda x)U^\lambda(x), & \text{in } D^-,\\ U^\lambda = 0, & \text{on } \partial D^-,\end{cases} \tag{214}$$

passing to the weak limit in (214), we obtain that $\widetilde U$ satisfies

$$\begin{cases}-\Delta \widetilde U(x) = 0, & \text{in } D^-,\\ \widetilde U = 0, & \text{on } \partial D^-.\end{cases} \tag{215}$$

By compactness of the embedding $\mathcal{H}^-_1 \hookrightarrow L^2(\Gamma^-_1)$, passing to the limit in (209), we have that $\int_{\Gamma^-_1}\widetilde U^2 d\sigma = 1$. In particular $\widetilde U \not\equiv 0$.

From Lemma 3.6, for every $\alpha>0$ there exists $k_\alpha\in\mathbb{N}$ and $t_\alpha>0$ such that for all $k>k_\alpha$ and $t\in(\varepsilon_{n_k},t_\alpha)$

$$\int_{\Omega_{-t}}|p(x)|U^2_{\varepsilon_{n_k}}(x)dx \leqslant \alpha\int_{\Omega_{-t}}|\nabla U_{\varepsilon_{n_k}}(x)|^2dx.$$

Strong $\mathcal{H}^-_t$-convergence of $U_{\varepsilon_{n_k}}$ to U then implies that

$$\int_{\Omega_{-t}}|p(x)|U^2(x)dx \leqslant \alpha\int_{\Omega_{-t}}|\nabla U(x)|^2dx, \quad \text{for all } t\in(0,t_\alpha).$$

Hence, by the change of variable $x=\lambda y$ and (212), we obtain that, for every $s>0$,

$$\lambda^2\int_{\Omega_{-s}}|p(\lambda y)||U^\lambda(y)|^2dy \leqslant \alpha\int_{\Omega_{-s}}|\nabla U^\lambda(y)|^2dy \leqslant 2\alpha s^{-N-2\delta}(N-1+\delta), \text{ for all } \lambda<\min\left\{\frac{t_\alpha}{s},\frac{\tilde h}{s}\right\},$$

thus implying that, for every $s>0$,

$$\lambda^2\int_{\Omega_{-s}}|p(\lambda y)||U^\lambda(y)|^2dy \to 0 \quad \text{as } \lambda\to 0^+. \tag{216}$$

By classical elliptic estimates, we also have that $U^{\lambda_{n_k}} \to \widetilde U$ in $C^2(\overline{B^-_{r_2}\setminus B^-_{r_1}})$ for all $0<r_1<r_2$. Therefore, multiplying (215) by $\widetilde U$ and integrating over Ω_{-t}, we obtain

$$\int_{\Gamma^-_t}\frac{\partial U^{\lambda_{n_k}}}{\partial\nu}U^{\lambda_{n_k}}d\sigma \to \int_{\Gamma^-_t}\frac{\partial\widetilde U}{\partial\nu}\widetilde U\,d\sigma = -\int_{\Omega_{-t}}|\nabla\widetilde U(x)|^2dx, \tag{217}$$

while multiplication of (214) by $U^{\lambda_{n_k}}$ and integration by parts over Ω_{-t} yield

$$\int_{\Omega_{-t}}|\nabla U^{\lambda_{n_k}}|^2dx = -\int_{\Gamma^-_t}\frac{\partial U^{\lambda_{n_k}}}{\partial\nu}U^{\lambda_{n_k}}d\sigma + \lambda^2_{n_k}\lambda_{k_0}(D^+)\int_{\Omega_{-t}}p(\lambda_{n_k}x)|U^{\lambda_{n_k}}(x)|^2\,dx. \tag{218}$$

Combining (217), (218), and (216), we conclude that $\|U^{\lambda_{n_k}}\|_{\mathcal{H}_t^-} \to \|\widetilde{U}\|_{\mathcal{H}_t^-}$ and then $U^{\lambda_{n_k}} \to \widetilde{U}$ strongly in $\mathcal{H}_t^-$ for every $t > 0$.

From (191), strong convergence $U^{\lambda_{n_k}} \to \widetilde{U}$ in $\mathcal{H}_t^-$, and (216), we have that, for every $t > 0$,

$$(219) \qquad \mathcal{N}_U(-t\lambda_{n_k}) = \frac{t\lambda_{n_k}\int_{\Omega_{-t\lambda_{n_k}}}\Big(|\nabla U(x)|^2 - \lambda_{k_0}(D^+)p(x)U^2(x)\Big)dx}{\int_{\Gamma^-_{t\lambda_{n_k}}} U^2(x)\,d\sigma}$$
$$= \frac{t\int_{\Omega_{-t}}\Big(|\nabla U^{\lambda_{n_k}}(x)|^2 - \lambda_{n_k}^2\lambda_{k_0}(D^+)p(\lambda_{n_k}x)|U^{\lambda_{n_k}}(x)|^2\Big)dx}{\int_{\Gamma_t^-}|U^{\lambda_{n_k}}(x)|^2\,d\sigma}$$
$$\longrightarrow \frac{t\int_{\Omega_{-t}}|\nabla\widetilde{U}(x)|^2dx}{\int_{\Gamma_t^-}\widetilde{U}^2(x)\,d\sigma} \quad \text{as } k\to+\infty.$$

Combining (206) and (219) we conclude that

$$\frac{t\int_{\Omega_{-t}}|\nabla\widetilde{U}(x)|^2dx}{\int_{\Gamma_t^-}\widetilde{U}^2(x)\,d\sigma} = \gamma \quad \text{for all } t>0.$$

From Lemma 3.3 there exists $K_0 \in \mathbb{N}$, $K_0 \geqslant 1$, such that

$$(220) \qquad \gamma = N - 2 + K_0$$

and $\widetilde{U}(x) = |x|^{-N+2-K_0}Y(x/|x|)$ for some eigenfunction Y of $-\Delta_{\mathbb{S}^{N-1}}$ associated to the eigenvalue $K_0(N-2+K_0)$, i.e. satisfying $-\Delta_{\mathbb{S}^{N-1}}Y = K_0(N-2+K_0)Y$ on $\mathbb{S}^{N-1}$. From (207) and (220) we infer that necessarily $K_0 = 1$, so that

$$\gamma = N-1 \quad\text{and}\quad \widetilde{U}(x) = |x|^{-N+1}Y(x/|x|).$$

From $\widetilde{U} = 0$ on ∂D^-, we deduce that $Y \equiv 0$ on $\{\theta = (\theta_1,\theta_2,\dots,\theta_N) \in \mathbb{S}^{N-1} : \theta_1 = 0\}$, hence Y is an eigenfunction of $-\Delta_{\mathbb{S}^{N-1}}$ on $\mathbb{S}^{N-1}_- = \{\theta = (\theta_1,\theta_2,\dots,\theta_N) \in \mathbb{S}^{N-1} : \theta_1 < 0\}$ under null Dirichlet boundary conditions associated to the eigenvalue $N-1$. It is easy to verify that $N-1$ is the first eigenvalue of such eigenvalue problem and hence it is simple; furthermore an eigenfunction associated to the eigenvalue $N-1$ is $\theta = (\theta_1,\theta_2,\dots,\theta_N) \in \mathbb{S}^{N-1}_- \mapsto \theta_1$. Therefore we conclude that there exists some constant $c \in \mathbb{R}\setminus\{0\}$ such that $Y(\theta) = c\theta_1$ and then

$$\widetilde{U}(x) = c\,\frac{x_1}{|x|^N}.$$

The proof is thereby completed. □

Lemma 6.3. *Let U as in Proposition 6.1 and let $H_U : (0,+\infty) \to \mathbb{R}$ be defined in* (198)*. Then*

- (i) $H_U(\lambda) \leqslant e^{2\delta(N-1+\delta)\tilde{h}}\tilde{h}^{2(N-1)}H_U(\tilde{h})\lambda^{-2(N-1)}$ *for all* $\lambda \in (0,\tilde{h})$*;*
- (ii) *for every* $\varrho > 0$ *there exists* $\lambda_\varrho > 0$ *such that* $H_U(\lambda) \geqslant H_U(\lambda_\varrho)\lambda_\varrho^{2(N-1-\varrho)}\lambda^{-2(N-1-\varrho)}$ *for all* $\lambda \in (0,\lambda_\varrho)$*;*
- (iii) $\lim_{\lambda\to0^+}\lambda^{2(N-1)}H_U(\lambda)$ *exists and is finite.*

Proof. From Lemma 6.2 (i), (201), (204), (205), we obtain that

$$N-1-\mathcal{N}_U(-\lambda) = \int_{-\lambda}^0 \mathcal{N}_U'(s)ds \geqslant \int_{-\lambda}^0 \nu_2(s)\,ds \geqslant -\delta(N-1+\delta)\lambda \quad \text{for all } \lambda\in(0,\tilde{h})$$

where ν_2 is defined in (203), and then

$$\mathcal{N}_U(-\lambda) \leqslant N-1+\delta(N-1+\delta)\lambda \quad \text{for all } \lambda \in (0,\tilde{h}),$$

which, together with (210), yields

$$\frac{H_U'(\lambda)}{H_U(\lambda)} = -\frac{2}{\lambda}\mathcal{N}_U(-\lambda) \geqslant -\frac{2(N-1)}{\lambda} - 2\delta(N-1+\delta) \quad \text{for all } \lambda\in(0,\tilde{h}).$$

Integration of the above inequality between λ and $\tilde{h}$ proves estimate (i).

From Lemma 6.2 (i), for any $\rho > 0$ there exists $\lambda_\varrho > 0$ such that $\mathcal{N}_U(r) > N-1-\rho$ for any $r \in (-\lambda_\varrho, 0)$ and hence

$$\frac{H_U'(\lambda)}{H_U(\lambda)} = -\frac{2}{\lambda}\mathcal{N}_U(-\lambda) < -\frac{2(N-1-\rho)}{\lambda} \quad \text{for all } \lambda \in (0, \lambda_\varrho).$$

Integration over the interval $(\lambda, \lambda_\varrho)$ yields (ii).

In view of (i), to prove (iii) it is sufficient to show that the limit exists. From (210), Lemma 6.2 (i), and (201) it follows that

$$\begin{aligned}\frac{d}{d\lambda}\Big(\lambda^{2(N-1)}H_U(\lambda)\Big) &= 2\lambda^{2N-3}H_U(\lambda)\bigg(N-1+\frac{\lambda}{2}\frac{H_U'(\lambda)}{H_U(\lambda)}\bigg) = 2\lambda^{2N-3}H_U(\lambda)(N-1-\mathcal{N}_U(-\lambda)) \\ &= 2\lambda^{2N-3}H_U(\lambda)\int_{-\lambda}^0 \mathcal{N}_U'(s)ds = 2\lambda^{2N-3}H_U(\lambda)\int_{-\lambda}^0 \big(\nu_1(s)+\nu_2(s)\big)ds\end{aligned}$$

where ν_1 and ν_2 are defined in (202) and (203) respectively. By integration of the above identity we obtain that, for all $\lambda \in (0, \tilde h)$,

$$\begin{aligned}\lambda^{2(N-1)}H_U(\lambda) - \tilde h^{2(N-1)}H_U(\tilde h) = &-2\int_\lambda^{\tilde h} s^{2N-3}H_U(s)\bigg(\int_{-s}^0 \nu_1(t)\,dt\bigg)ds \\ &-2\int_\lambda^{\tilde h} s^{2N-3}H_U(s)\bigg(\int_{-s}^0 \nu_2(t)\,dt\bigg)ds.\end{aligned} \tag{221}$$

From (204) the limit

$$\lim_{\lambda\to 0^+}\int_\lambda^{\tilde h} s^{2N-3}H_U(s)\bigg(\int_{-s}^0 \nu_1(t)\,dt\bigg)ds$$

exists. On the other hand from (i) and (205) it follows that

$$s^{2N-3}H_U(s)\bigg(\int_{-s}^0 \nu_2(t)\,dt\bigg) = O(1) \quad \text{as } s \to 0^+ \tag{222}$$

thus proving in particular that $s \mapsto s^{2N-3}H_U(s)\big(\int_{-s}^0 \nu_2(t)\,dt\big) \in L^1(0, \tilde h)$. We conclude that both terms at the right hand side of (221) admit a limit as $\lambda \to 0^+$, the second one being finite in view of (222), thus completing the proof of the lemma. □

Lemma 6.4. *Let U be as in Proposition 6.1, Y_1 as in* (11)*, and let $H_U : (0,+\infty) \to \mathbb{R}$ be defined in* (198)*. Then*

(i) $\displaystyle\int_{\mathbb{S}^{N-1}_-} U(\lambda\theta)Y_1(\theta)\,d\sigma(\theta)$

$$= \lambda^{1-N}\bigg[\int_{\Gamma_1^-} U(x)Y_1(\tfrac{x}{|x|})\,dx - \tfrac{\lambda_{k_0}(D^+)}{N}\int_{D^-} p(x)U(x)Y_1(\tfrac{x}{|x|})\Big(|x|\chi_{B_1^-}(x) + \tfrac{\chi_{\Omega_{-1}}(x)}{|x|^{N-1}}\Big)dx\bigg]$$
$$+\, O(\lambda^{3-N}) \quad \textit{as } \lambda \to 0^+,$$

(ii) $\displaystyle\lim_{\lambda\to 0^+}\lambda^{2(N-1)}H_U(\lambda) > 0.$

Proof. Let us define, for all $\lambda > 0$,

$$\mu(\lambda) = \int_{\mathbb{S}^{N-1}_-} U(\lambda\theta)Y_1(\theta)\,d\sigma(\theta), \quad \varsigma(\lambda) = \lambda_{k_0}(D^+)\int_{\mathbb{S}^{N-1}_-} p(\lambda\theta)U(\lambda\theta)Y_1(\theta)\,d\sigma(\theta).$$

From (190) μ satisfies

$$-\mu''(\lambda) - \frac{N-1}{\lambda}\mu'(\lambda) + \frac{N-1}{\lambda^2}\mu(\lambda) = \varsigma(\lambda), \quad \text{in } (0,+\infty).$$

Hence there exist $c_1, c_2 \in \mathbb{R}$ such that

$$\mu(\lambda) = \lambda\bigg(c_1 + \frac{1}{N}\int_\lambda^1 \varsigma(t)dt\bigg) + \lambda^{1-N}\bigg(c_2 - \frac{1}{N}\int_\lambda^1 t^N\varsigma(t)dt\bigg) \quad \text{for all } \lambda \in (0,+\infty). \tag{223}$$

Since $p \in L^{N/2}(\mathbb{R}^N)$ and $U \in \mathcal{H}_1^-$ ensure that $\frac{p(x)U(x)Y_1(x/|x|)}{|x|^{N-1-\alpha}} \in L^1(\Omega_{-1})$ for all $\alpha \in [0, \frac{N}{2})$ and, for all $\lambda > 1$,

$$\int_\lambda^1 t^\alpha \varsigma(t)dt = -\lambda_{k_0}(D^+)\int_{B_\lambda^- \setminus B_1^-} \frac{p(x)U(x)Y_1(x/|x|)}{|x|^{N-1-\alpha}}\,dx,$$
$$\int_1^\lambda t^\alpha |\varsigma(t)|dt \leqslant \lambda_{k_0}(D^+)\int_{B_\lambda^- \setminus B_1^-} \frac{p(x)|U(x)|Y_1(x/|x|)}{|x|^{N-1-\alpha}}\,dx,$$

we deduce that $\int_\lambda^1 t^\alpha\varsigma(t)dt$ admits a finite limit and $\int_1^\lambda t^\alpha|\varsigma(t)|dt = O(1)$ as $\lambda \to +\infty$ for every $\alpha \in [0, \frac{N}{2})$. In particular $\int_\lambda^1 \varsigma(t)dt$ admits a finite limit as $\lambda \to +\infty$ and

$$\left|\int_\lambda^1 t^N\varsigma(t)dt\right| \leqslant \int_1^\lambda t^{N-1}t|\varsigma(t)|dt \leqslant \lambda^{N-1}\int_1^\lambda t|\varsigma(t)|dt = O(\lambda^{N-1}) \quad \text{as } \lambda \to +\infty.$$

Hence from (223) we deduce

$$\mu(\lambda) = \lambda\bigg(c_1 - \frac{1}{N}\int_1^{+\infty}\varsigma(t)dt + o(1)\bigg) + O(1) \quad \text{as } \lambda \to +\infty. \tag{224}$$

Since $U \in \mathcal{H}_1^-$ yields $\int_1^{+\infty} t^{N-1}|\mu(t)|^{2^*}dt < +\infty$, (224) necessarily implies that $c_1 = \frac{1}{N}\int_1^{+\infty}\varsigma(t)dt$. Then (223) becomes

$$\mu(\lambda) = \frac{\lambda}{N}\int_\lambda^{+\infty}\varsigma(t)dt + \lambda^{1-N}\bigg(c_2 - \frac{1}{N}\int_\lambda^1 t^N\varsigma(t)dt\bigg) \quad \text{for all } \lambda \in (0,+\infty). \tag{225}$$

The above formula at $\lambda = 1$ yields

$$c_2 = \mu(1) - \frac{1}{N}\int_1^{+\infty}\varsigma(t)dt = \int_{\mathbb{S}_-^{N-1}} U(\theta)Y_1(\theta)\,d\sigma(\theta) - \frac{\lambda_{k_0}(D^+)}{N}\int_{\Omega_{-1}} \frac{p(x)U(x)Y_1(\frac{x}{|x|})}{|x|^{N-1}}dx. \tag{226}$$

Since

$$|\varsigma(\lambda)| \leqslant \lambda_{k_0}(D^+)\sup_{B_1^-}|p|\sqrt{H_U(\lambda)} \quad \text{for all } \lambda \in (0,1),$$

from Lemma 6.3 (i) we deduce that

$$\varsigma(\lambda) = O(\lambda^{1-N}) \quad \text{as } \lambda \to 0^+.$$

Hence

$$\frac{\lambda}{N}\int_\lambda^{+\infty}\varsigma(t)dt = O(\lambda^{3-N}) \quad \text{as } \lambda \to 0^+, \tag{227}$$

$t^N\varsigma(t) \in L^1(0,1)$, and $t^N\varsigma(t) = O(t)$ as $t \to 0^+$, so that

$$\begin{aligned} -\frac{1}{N}\int_\lambda^1 t^N\varsigma(t)dt &= -\frac{1}{N}\int_0^1 t^N\varsigma(t)dt + \frac{1}{N}\int_0^\lambda t^N\varsigma(t)dt \\ &= -\frac{\lambda_{k_0}(D^+)}{N}\int_{B_1^-}|x|p(x)U(x)Y_1(\tfrac{x}{|x|})dx + O(\lambda^2) \text{ as } \lambda \to 0^+. \end{aligned} \tag{228}$$

Combining (225–228) we obtain statement (i).

To prove (ii), let us assume by contradiction that $\lim_{\lambda\to 0^+}\lambda^{2(N-1)}H_U(\lambda) = 0$. Since, by Schwarz's inequality, $H_U(\lambda) = \int_{\mathbb{S}_-^{N-1}} U^2(\lambda\theta)\,d\sigma(\theta) \geqslant |\mu(\lambda)|^2$, it would follow that

$$\lim_{\lambda\to 0^+}\lambda^{N-1}\mu(\lambda) = 0.$$

Hence (i) would imply that

$$\int_{\Gamma_1^-} U(x)Y_1(\tfrac{x}{|x|})\,dx - \tfrac{\lambda_{k_0}(D^+)}{N}\int_{D^-} p(x)U(x)Y_1(\tfrac{x}{|x|})\Big(|x|\chi_{B_1^-}(x) + \tfrac{\chi_{\Omega_{-1}}(x)}{|x|^{N-1}}\Big)dx = 0$$

and

$$\int_{\mathbb{S}_-^{N-1}} U(\lambda\theta)Y_1(\theta)\,d\sigma(\theta) = O(\lambda^{3-N}) \quad \text{as } \lambda \to 0^+.$$

Therefore, letting U^λ as in (208) and using Lemma 6.3 (ii) with $\varrho < 2$, we obtain that

$$\int_{\mathbb{S}^{N-1}_-} U^\lambda(\theta) Y_1(\theta)\, d\sigma(\theta) = O(\lambda^{2-\varrho}) \quad \text{as } \lambda \to 0^+. \tag{229}$$

From Lemma 6.2 (ii), for every sequence $\lambda_n \to 0^+$ there exist a subsequence $\{\lambda_{n_k}\}_k$ and some constant $c \in \mathbb{R} \setminus \{0\}$ such that

$$U^{\lambda_{n_k}} \to c\, Y_1 \text{ in } L^2(\mathbb{S}^{N-1}_-). \tag{230}$$

From (229) and (230) we infer that

$$0 = \lim_{k\to+\infty} \int_{\mathbb{S}^{N-1}_-} U^{\lambda_{n_k}}(\theta) Y_1(\theta)\, d\sigma(\theta) = c \int_{\mathbb{S}^{N-1}_-} Y_1^2(\theta)\, d\sigma(\theta) = c$$

thus reaching a contradiction and proving statement (ii). □

Proposition 6.5. *Let U be as in Proposition 6.1. Then*

$$\lambda^{N-1} U(\lambda x) \underset{\lambda\to 0^+}{\longrightarrow} \beta \frac{x_1}{|x|^N}$$

strongly in $\mathcal{H}^-_t$ for every $t > 0$ and in $C^2(\overline{B^-_{t_2} \setminus B^-_{t_1}})$ for all $0 < t_1 < t_2$, where

$$\beta = -\frac{\int_{\Gamma_1^-} U(x) Y_1(\frac{x}{|x|})\, dx - \frac{\lambda_{k_0}(D^+)}{N} \int_{D^-} p(x) U(x) Y_1(\frac{x}{|x|}) \Big(|x| \chi_{B_1^-}(x) + \frac{\chi_{\Omega_{-1}}(x)}{|x|^{N-1}} \Big) dx}{\Upsilon_N} \neq 0 \tag{231}$$

and Υ_N is defined in (12).

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ such that $\lim_{n\to+\infty} \lambda_n = 0$. Then, from part (ii) of Lemma 6.2 and part (ii) of Lemma 6.4, there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and some constant $\beta \in \mathbb{R} \setminus \{0\}$ such that

$$\lambda_{n_k}^{N-1} U(\lambda_{n_k}\theta) \underset{k\to+\infty}{\longrightarrow} \beta \frac{x_1}{|x|^N} \tag{232}$$

strongly in $\mathcal{H}^-_t$ for every $t > 0$ and in $C^2(\overline{B^-_{t_2} \setminus B^-_{t_1}})$ for all $0 < t_1 < t_2$. In particular

$$\lambda_{n_k}^{N-1} U(\lambda_{n_k}\theta) \underset{k\to+\infty}{\longrightarrow} \beta\, \theta_1 \quad \text{in } C^2(\mathbb{S}^{N-1}_-) \quad \text{as } k \to +\infty.$$

From Lemma 6.4

$$\begin{aligned}\lim_{k\to+\infty} \lambda_{n_k}^{N-1} &\int_{\mathbb{S}^{N-1}_-} U(\lambda_{n_k}\theta) Y_1(\theta)\, d\sigma(\theta) \\ &= \int_{\Gamma_1^-} U(x) Y_1(\tfrac{x}{|x|})\, dx - \tfrac{\lambda_{k_0}(D^+)}{N} \int_{D^-} p(x) U(x) Y_1(\tfrac{x}{|x|}) \Big(|x| \chi_{B_1^-}(x) + \tfrac{\chi_{\Omega_{-1}}(x)}{|x|^{N-1}} \Big) dx,\end{aligned}$$

thus implying that

$$\begin{aligned}\beta &= \frac{\int_{\Gamma_1^-} U(x) Y_1(\frac{x}{|x|})\, dx - \frac{\lambda_{k_0}(D^+)}{N} \int_{D^-} p(x) U(x) Y_1(\frac{x}{|x|}) \Big(|x| \chi_{B_1^-}(x) + \frac{\chi_{\Omega_{-1}}(x)}{|x|^{N-1}} \Big) dx}{\int_{\mathbb{S}^{N-1}_-} \theta_1 Y_1(\theta) d\sigma(\theta)} \\ &= -\frac{\int_{\Gamma_1^-} U(x) Y_1(\frac{x}{|x|})\, dx - \frac{\lambda_{k_0}(D^+)}{N} \int_{D^-} p(x) U(x) Y_1(\frac{x}{|x|}) \Big(|x| \chi_{B_1^-}(x) + \frac{\chi_{\Omega_{-1}}(x)}{|x|^{N-1}} \Big) dx}{\Upsilon_N}.\end{aligned}$$

Hence we have proved that β depends neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$, thus implying that the convergence in (232) actually holds as $\lambda \to 0^+$ and proving the proposition. □

The following lemmas investigate the sign of the β in (231), thus allowing the study of the nodal properties of u_ε close to the left junction.

Lemma 6.6. *Let U be as in Proposition 6.1 and $\beta \neq 0$ as in* (231)*. If $\beta > 0$ (respectively $\beta < 0$) then there exists $R > 0$ such that*

for every $r \in (0, R)$ there exists $\varepsilon_r > 0$ such that

$u_\varepsilon < 0$ (respectively $u_\varepsilon > 0$) in Γ_r^- for all $\varepsilon \in (0, \varepsilon_r)$.

Proof. Let us prove the lemma under the assumption $\beta > 0$ (under the assumption $\beta < 0$ the argument is exactly the same). We claim that

$$\text{there exists } R > 0 \text{ such that } U < 0 \text{ in } B_R^-. \tag{233}$$

To prove (233), let us assume by contradiction that there exist $\lambda_n \to 0^+$, $\theta_n \in \mathbb{S}_-^{N-1}$, $\bar\theta \in \overline{\mathbb{S}_-^{N-1}}$ such that $\theta_n \to \bar\theta$ and $U(\lambda_n\theta_n) \geqslant 0$. If $\bar\theta \in \mathbb{S}_-^{N-1}$ then from Proposition 6.5 we obtain that

$$0 \leqslant \lambda^{N-1}U(\lambda_n\theta_n) = \Big(\lambda^{N-1}U(\lambda_n\theta_n) - \beta(\theta_n)_1\Big) + \beta(\theta_n)_1 = \beta\bar\theta_1 + o(1) \quad \text{as } n \to +\infty$$

which yields a contradiction. On the other hand, if $\bar\theta \in \partial\mathbb{S}_-^{N-1}$, i.e. if $\bar\theta_1 = 0$, then, letting $s > 0$ sufficiently small to have $|x|^N - N|x|^{N-2}x_1^2 > c > 0$ for all $x \in A_s := \{x \in B_1^- \setminus B_{1/2}^- : x_1 > -s\}$, we have that $(\frac{t}{\lambda_n}, \theta_n') \in A_s$ for all $t \in (\lambda_n(\theta_n)_1, 0)$ and n large. Since from Proposition 6.5 $\lambda^N \frac{\partial U}{\partial x_1}(\lambda x) \to \beta\frac{|x|^N - N|x|^{N-2}x_1^2}{|x|^{2N}}$ in $C^1(\overline{A_s})$, we deduce that $\frac{\partial U}{\partial x_1}(\lambda_n x) > 0$ for all $x \in A_s$ and n large. Hence

$$U(\lambda_n\theta_n) = -\int_{\lambda_n(\theta_n)_1}^0 \frac{\partial U}{\partial x_1}(t, \lambda_n\theta_n')\,dt < 0$$

thus giving a contradiction. Claim (233) is thereby proved. It remains to prove that

$$\text{for every } r \in (0,R) \text{ there exists } \varepsilon_r > 0 \text{ such that } u_\varepsilon < 0 \text{ in } \Gamma_r^- \text{ for all } \varepsilon \in (0,\varepsilon_r). \tag{234}$$

To prove (234), let us assume by contradiction that there exist $r \in (0,R)$, $\varepsilon_n \to 0^+$, $\theta_n \in \mathbb{S}_-^{N-1}$, $\bar\theta \in \overline{\mathbb{S}_-^{N-1}}$ such that $\theta_n \to \bar\theta$ and $u_{\varepsilon_n}(r\theta_n) \geqslant 0$ (and hence $U_\varepsilon(r\theta_n) \geqslant 0$). If $\bar\theta \in \mathbb{S}_-^{N-1}$ then from Proposition 6.1 it follows that

$$0 \leqslant U_{\varepsilon_n}(r\theta_n) = \Big(U_{\varepsilon_n}(r\theta_n) - U(r\theta_n)\Big) + U(r\theta_n) = U(r\bar\theta) + o(1) \quad \text{as } n \to +\infty$$

which contradicts (233). On the other hand, if $\bar\theta \in \partial\mathbb{S}_-^{N-1}$, then by Hopf's Lemma $\frac{\partial U}{\partial x_1}(r\bar\theta) > 0$. If $t \in (r(\theta_n)_1, 0)$, Proposition 6.1 yields

$$\frac{\partial U_{\varepsilon_n}}{\partial x_1}(t, r\theta_n') = \left(\frac{\partial U_{\varepsilon_n}}{\partial x_1}(t, r\theta_n') - \frac{\partial U}{\partial x_1}(t, r\theta_n')\right) + \frac{\partial U}{\partial x_1}(t, r\theta_n') = \frac{\partial U}{\partial x_1}(r\bar\theta) + o(1)$$

as $n \to +\infty$, so that

$$\frac{\partial U_{\varepsilon_n}}{\partial x_1}(t, r\theta_n') > 0$$

provide n is sufficiently large. Therefore

$$U_{\varepsilon_n}(r\theta_n) = -\int_{r(\theta_n)_1}^0 \frac{\partial U_{\varepsilon_n}}{\partial x_1}(t, r\theta_n')\,dt < 0$$

leads to a contradiction proving claim (234). □

In fact, condition (6) forces the sign of β to be negative, as we show below.

Lemma 6.7. *Let U be as in Proposition 6.1 and $\beta \neq 0$ as in* (231)*. Then*

$$\beta < 0.$$

Proof. Let us assume by contradiction that $\beta > 0$. From Lemma 6.6, for every n (sufficiently large), there exists $\varepsilon_n \in (0, 1/n)$ such that

$$u_{\varepsilon_n} < 0 \quad \text{on } \Gamma_{1/n}^-. \tag{235}$$

Let us denote $u_{\varepsilon_n}^- := \max\{0, -u_{\varepsilon_n}\}$. From Lemma 2.13, $u_{\varepsilon_n}^- = 0$ on $\partial\Omega_{1+2\varepsilon_n}^{\varepsilon_n}$. Therefore, letting

$$v_n := \begin{cases} u_{\varepsilon_n}, & \text{in } \Omega_{-1/n}, \\ -u_{\varepsilon_n}^-, & \text{in } \Omega_{1+2\varepsilon_n}^{\varepsilon_n} \setminus \Omega_{-1/n}, \\ 0, & \text{in } \mathbb{R}^N \setminus \Omega_{1+2\varepsilon_n}^{\varepsilon_n}, \end{cases}$$

(235) ensures that $v_n \in \mathcal{D}^{1,2}(\Omega_{1+2\varepsilon_n}^{\varepsilon_n}) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$, $v_n \not\equiv 0$ in D^-.

Testing equation $-\Delta u_{\varepsilon_n} = \lambda_{\bar{k}}^{\varepsilon_n} p u_{\varepsilon_n}$ with v_n, we obtain $\int_{\Omega^{\varepsilon_n}_{1+2\varepsilon_n}} |\nabla v_n|^2 dx = \lambda_{\bar{k}}^{\varepsilon_n} \int_{\Omega^{\varepsilon_n}_{1+2\varepsilon_n}} p v_n^2 dx$, hence, defining

$$w_n := \frac{v_n}{\sqrt{\int_{\Omega^{\varepsilon_n}_{1+2\varepsilon_n}} p v_n^2 dx}},$$

we have that $w_n \in \mathcal{D}^{1,2}(\Omega^{\varepsilon_n}_{1+2\varepsilon_n}) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} p w_n^2 dx = \int_{\Omega^{\varepsilon_n}_{1+2\varepsilon_n}} p w_n^2 dx = 1, \quad \int_{\mathbb{R}^N} |\nabla w_n|^2 dx = \int_{\Omega^{\varepsilon_n}_{1+2\varepsilon_n}} |\nabla w_n|^2 dx = \lambda_{\bar{k}}^{\varepsilon_n}.$$

Hence $\{w_n\}_n$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and there exists a subsequence $\{w_{n_k}\}_k$ such that $w_{n_k} \rightharpoonup w$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $w_{n_k} \to w$ a.e. in $\mathbb{R}^N$, for some $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. Since $\operatorname{supp} w_n \subset \Omega^{\varepsilon_n}_{1+2\varepsilon_n}$, a.e. convergence implies that $\operatorname{supp} w \subset D^-$ so that $w \in \mathcal{D}^{1,2}(D^-)$. From $\int_{\mathbb{R}^N} p w_n^2 dx = 1$ we deduce that $\int_{D^-} p w^2 dx = 1$ which implies that $w \not\equiv 0$. Since w_n solves

$$\begin{cases} -\Delta w_n = \lambda_{\bar{k}}^{\varepsilon_n} p w_n, & \text{in } \Omega_{-1/n}, \\ w_n = 0, & \text{on } \partial\Omega_{-1/n} \cap \partial D^-, \end{cases}$$

weak convergence and (7) imply that w weakly solves

$$\begin{cases} -\Delta w = \lambda_{k_0}(D^+) p w, & \text{in } D^-, \\ w = 0, & \text{on } \partial D^-, \end{cases}$$

thus implying $\lambda_{k_0}(D^+) \in \sigma_p(D^-)$ and contradicting assumption (5). □

The proofs of the main results of the paper follow by combining the previous results.

Proof of Theorem 1.2. It follows by combining Propositions 6.1, 6.5 and Lemma 6.7. □

Proof of Corollary 1.3. It follows from Lemmas 6.6 and 6.7. □

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