Construction and characterization of non-uniform local interpolating polynomial splines

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Abstract

This paper presents a general framework for the construction of piecewise-polynomial local interpolants with given smoothness and approximation order, defined on non-uniform knot partitions. We design such splines through a suitable combination of polynomial interpolants with either polynomial or rational, compactly supported blending functions. In particular, when the blending functions are rational, our approach provides spline interpolants having low, and sometimes minimum degree.

Thanks to its generality, the proposed framework also allows us to recover uniform local interpolating splines previously proposed in the literature, to generalize them to the non-uniform case, and to complete families of arbitrary support width. Furthermore it provides new local interpolating polynomial splines with prescribed smoothness and polynomial reproduction properties.

Keywords: Local interpolation; Polynomial splines; Non-uniform knot-partition; Blending; Rational functions.

1. Introduction

In the design of spline bases for local interpolation, there are typically several degrees of freedom, which can be set to identify a specific interpolant. So, many different criteria exist for making such choice, such as requiring minimal support, maximizing accuracy or smoothness, leaving free shape parameters and certainly possible others.

In this paper we consider the problem of defining the basis starting from the desired approximation and continuity orders for a given support width. A similar idea was pursued in the context of uniform interpolation in [2] and [4]. The first paper presents a family of piecewise polynomial functions that solve an interpolation problem of equally spaced data, given degree, smoothness, approximation order and support width. The second work proposes a general framework for the design of uniform, local interpolating splines as a combination of a suitable approximation operator, with known continuity and approximation order, and a general interpolation operator. Both proposals address the construction of local interpolating splines with uniform knots, while to the best of our knowledge no similar method is available in the non-uniform setting. Although in principle the idea in [4] could be extended to the non-uniform case, its generalization would require to know blending functions, for arbitrarily spaced knots, with given compact support, approximation order and continuity.

Thus we address the construction of non-uniform, local interpolating polynomial splines with desired locality, polynomial reproduction and smoothness following an alternative approach. For an arbitrarily chosen support width $w \geq 3$, we exploit a suitable combination of degree-m polynomial interpolants with blending functions of support width $n + 1$ and assigned continuity, such that $m + n + 1 = w$. In this way, the spline interpolant naturally inherits the degree of polynomial reproduction from the polynomial interpolants and the smoothness from the blending functions.

The design of a local interpolant via blending techniques is certainly not a new idea in computational mathematics. In fact, starting from the pioneer paper by Catmull and Rom [3], where a local spline basis is obtained through a

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combination of B-spline blending functions and interpolating polynomials, in the last three decades several authors exploited the idea of blending local interpolants (see, e.g., [5, 6, 7, 9, 12, 13, 14, 16]). However, so far anybody has never tackled the problem of completing a general framework for the construction and the detailed characterization of piecewise-polynomial local interpolants defined on non-uniform knot partitions.

One of the key aspects of the presented method is that the exploited blending functions can be of either polynomial or rational kind. In both cases they are determined through an original constructive procedure, that guarantees the desired smoothness and support width. In particular, the use of rational blending functions allows for obtaining spline interpolants of considerably low and sometimes even minimum degree against high polynomial reproduction and smoothness of the resulting splines.

The remainder of the paper is organized as follows. In Section 2 we provide a constructive method that, given a set of distinct and ordered knots, allows us to determine a sequence of well-defined non-uniform blending functions, i.e., non-negative, compactly supported and normalized functions, of assigned continuity and support width. In Section 3 we define general interpolants that belong to the proposed framework, characterize their properties of polynomial reproduction, smoothness and locality and discuss which is the lowest achievable degree of the related spline basis. At the end of the section we also illustrate how these splines can be efficiently evaluated. Then, in Section 4 we show that our general framework includes known uniform local interpolating splines and generalizes them both to the non-uniform setting and to complete families with arbitrary support width. Moreover it also allows for designing new fundamental functions of minimum degree with prescribed smoothness and polynomial reproduction. Finally, in Section 5 we conclude by illustrating some application examples of local interpolating spline curves obtained by the proposed framework.

2. The design of blending functions

In this section we present a general procedure for designing compactly supported blending functions, defined on a knot partition $\mathbf{t} = \{t_j\}$ of distinct knots, in ascending order. The i-th blending function $B_{i,n+1}$, $n + 1 \geq 2$, belongs to $C^k(\mathbb{R})$, $k \geq 0$, and satisfies the following properties

$$(i) \; B_{i,n+1}(x) > 0, \;\; x \in (t_i, t_{i+n+1}), \qquad (ii) \; B_{i,n+1}(x) = 0, \;\; x \notin (t_i, t_{i+n+1}), \qquad (iii) \sum_i B_{i,n+1}(x) = 1, \;\; \forall x \in \mathbb{R}.$$

The most popular blending functions are the well-known B-splines, that are piecewise polynomials of degree n, support width $n + 1$ and smoothness C^{n-1}. One way of defining a non-uniform degree-n polynomial B-spline consists in applying $n - 1$ times an integral recurrence relation [15], that starts from piecewise constant blending functions (of support width 1 and smoothness C^{-1}) and proceeds through subsequent integrations and subtractions, obtaining at each new step blending functions with degree, support and continuity increased by one. Analogously, here we design several classes of blending functions as differences of *transition functions*, having a similar behavior to that of the integral functions in the definition of B-splines. In particular we give the following definition.

Definition 1. *Given an increasing sequence of knots* $\mathbf{t} = \{t_j\}_{j=1,...,N}$, *the* transition functions f_i, $i = 1, \ldots, N - n$, *relative to the knot partition* $\mathbf{t}$, *are the piecewise functions defined on* $\mathbb{R}$ *such that*

$$A) \; f_i(x) = \begin{cases} 0 & x < t_i, \\ 1 & x \geq t_{i+n}, \end{cases} \qquad B) \; f_i \in C^k(\mathbb{R}), \qquad C) \; 0 \leq f_i \leq 1, \qquad D) \; f_i \geq f_{i+1}.$$

In the sequel we refer to the i-th transition function f_i with the term n-piece transition function to emphasize its piecewise formulation on the interval $[t_i, t_{i+n}]$ where its expression is not trivially 0 or 1.

The i-th blending function, for $i = 1, \ldots, N - n - 1$, can thus be obtained as

$$B_{i,n+1}(x) = f_i(x) - f_{i+1}(x), \quad \forall x \in \mathbb{R}. \tag{1}$$

By construction, the functions $B_{i,n+1}$ have compact support of width $n + 1$ given by $[t_i, t_{i+n+1}]$, are linearly independent, at least C^k continuous and non-negative. Moreover on each interval $[t_\ell, t_{\ell+1})$ they satisfy the partition of unity property, in fact

$$\sum_{i=\ell-n}^{\ell} B_{i,n+1}(x) = \sum_{i=\ell-n}^{\ell} (f_i(x) - f_{i+1}(x)) = f_{\ell-n}(x) - f_{\ell+1}(x) = 1. \tag{2}$$

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Remark 1. *It can be easily verified that polynomial B-splines can be generated following the proposed approach. In this case the transition functions are degree-n piecewise polynomials uniquely determined by imposing the maximum smoothness at the knots, namely by the conditions*

$$\begin{cases} f_i^{(j)}(t_i) = 0, & j = 0, \ldots, n-1, \\ f_i^{(j)}(t_\ell^-) = f_i^{(j)}(t_\ell^+), & j = 0, \ldots, n-1, \quad \ell = i+1, \ldots, i+n-1, \\ f_i^{(j)}(t_{i+n}) = \delta_{0,j}, & j = 0, \ldots, n-1. \end{cases} \tag{3}$$

To design general blending functions we thus proceed by determining the related transition functions as piecewise functions with maximum smoothness at the knots whereas other degrees of freedom that may occur in the definition of the pieces and are not sufficient to increase the global smoothness may be exploited to satisfy additional requirements.

This procedure allows us to explicitly derive the expression of the pieces of each transition function, however note that conditions C) and D) must be verified for any arbitrary knot-partition to ensure well-definedness of the blending functions $B_{i,n+1}$.

Remark 2. *The proposed blending functions determine an approximation operator (hereinafter denoted by A) that has minimum approximation order 1 (as the constant function is certainly reproduced), but possibly higher, as in the case of B-splines.*

We observe that the proposed method allows for designing blending functions that on each interval may be of different kind, e.g., polynomial, rational, trigonometric, hyperbolic and so forth. It also allows for arbitrarily playing with different orders and numbers of degrees of freedom, e.g., it is possible to define transition functions with different smoothness at the knots, multiple knots and relaxed continuity constraints, as well as general support width independent of the smoothness order.

When designing polynomial blending functions, in the remainder of the paper the j-th polynomial piece of the transition function f_i is represented in the Bernstein basis using the notation

$$q(x|a_0^{i,j}, \ldots, a_{d_{i,j}}^{i,j}) := \sum_{k=0}^{d_{i,j}} a_k^{i,j} \binom{d_{i,j}}{k} R_{i+j-1}^{d_{i,j}-k}(x) L_{i+j-1}^k(x), \qquad x \in [t_{i+j-1}, t_{i+j}),$$

with $L_r(x) = \dfrac{x - t_r}{t_{r+1} - t_r}$ and $R_r(x) = \dfrac{t_{r+1} - x}{t_{r+1} - t_r}$.

Thus in the polynomial case, the i-th transition function has the expression

$$f_i(x) = \begin{cases} 0, & x < t_i, \\ q(x|a_0^{i,j}, \ldots, a_{d_{i,j}}^{i,j}), & x \in [t_{i+j-1}, t_{i+j}), \quad j = 1, \ldots, n, \\ 1, & x \geq t_{i+n}. \end{cases}$$

Differently, when designing rational blending functions, the pieces of the transition functions can be either polynomial or rational, and can also have different degrees in the numerators and denominators. Thus in each piece we use the notation $d_{i,j}/e_{i,j}$, where $d_{i,j}$ and $e_{i,j}$ are the degrees of the numerators and denominators respectively. We observe that to guarantee that the resulting blending functions are symmetric, when so is the knot partition, the transition functions f_i must be antisymmetric with respect to the midpoint of the interval $[t_i, t_{i+n}]$, namely $f_i(x) = 1 - f_i(t_i + t_{i+n} - x)$. This relation means that it is possible to design the transition function on the half of such interval.

To illustrate in more detail our constructive approach, in the following we design three different blending functions with support width 3: a quintic C^3-continuous function, a cubic one which is C^2 at the outermost knots and C^1 at the internal knots of its support, and finally a rational C^2 function of degree 4/1. These will be used in the following sections to generate local spline interpolants belonging to the proposed framework.

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2.1. A C^3 quintic blending function

Following the proposed approach we want to define a two-piece transition function of degree 5, such that

$$f_i(x) = \begin{cases} 0, & x < t_i, \\ q(x|a_0^{i,j}, \ldots, a_5^{i,j}), & x \in [t_{i+j-1}, t_{i+j}), \quad j = 1, 2, \\ 1, & x \geq t_{i+2}. \end{cases}$$

By imposing the third-order smoothness conditions (3) at the knots t_i, t_{i+1} and t_{i+2}, we get a square linear system of equations of order 12 that has unique solution for distinct knots. In particular, it provides the following coefficients of the transition function

$$a_k^{i,1} = 0 \quad k = 0, \ldots, 3, \quad a_4^{i,1} = \frac{(t_{i+1} - t_i)^2}{(t_{i+2} - t_i)^2}, \quad a_5^{i,1} = \frac{(t_{i+1} - t_i)^2(3t_{i+2} - 2t_{i+1} - t_i)}{(t_{i+2} - t_i)^3};$$

$$a_0^{i,2} = a_5^{i,1}, \quad a_1^{i,2} = \frac{(t_{i+1} - t_i)(2t_{i+2} - t_{i+1} - t_i)}{(t_{i+2} - t_i)^2}, \quad a_k^{i,2} = 1 \quad k = 2, \ldots, 5.$$

Figure 1 shows f_i for $x \in [t_i, t_{i+2}]$ in case of uniform knots.
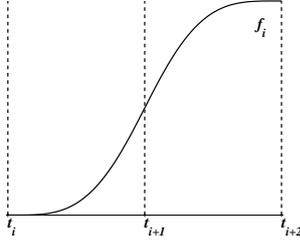
Figure 1: A C^3, quintic, two-piece transition function over uniform knots.
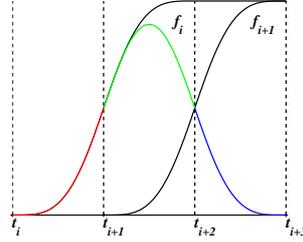
Figure 2: C^3 quintic transition functions and related blending function.
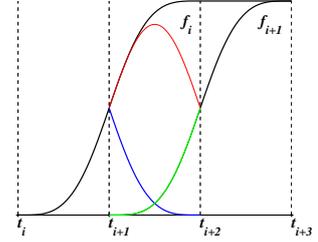
Figure 3: C^3 quintic transition functions and blending functions in a knot interval.

In order to define a blending function through difference of subsequent transition functions f_i, as in equation (1), we now show that conditions C) and D) hold. Property C) follows from the fact that, for any sequence of distinct and ordered knots,

$$0 \leq a_k^{i,1}, a_k^{i,2} \leq 1 \quad k = 0, \ldots, 5, \tag{4}$$

and from the convex hull property of the Bernstein representation. To check condition D), we need to confirm the inequality $f_i - f_{i+1} \geq 0$, which is equivalent to $a_k^{i,1} \geq 0$, $a_k^{i,2} - a_k^{i+1,1} \geq 0$, $1 - a_k^{i+1,2} \geq 0$, $k = 0, \ldots, 5$. The first and last inequalities always hold as a consequence of (4), while the second can be trivially proven given the expressions of $a_k^{i,2}$ and $a_k^{i+1,1}$.

By applying formula (1) to the transition functions we can now work out the related blending functions, that are C^3-continuous by construction and have compact support of width 3. Figure 2 shows the i-th blending function, supported on $[t_i, t_{i+3}]$, given explicitly as

$$B_{i,3}(x) = \begin{cases} f_i(x), & x \in [t_i, t_{i+1}), \\ f_i(x) - f_{i+1}(x), & x \in [t_{i+1}, t_{i+2}), \\ 1 - f_{i+1}(x), & x \in [t_{i+2}, t_{i+3}). \end{cases}$$

On the single interval $[t_{i+1}, t_{i+2})$ the set of non-vanishing blending functions (see Figure 3) is easily computed through the relations

$$B_{i-1,3} = 1 - f_i, \quad B_{i,3} = f_i - f_{i+1}, \quad B_{i+1,3} = f_{i+1}.$$

2.2. Cubic blending function with mixed C^2C^1 continuity

We are now interested in determining cubic blending functions of support width 3, with different continuity at the knots. To this aim, we consider transition functions of the form

$$f_i(x) = \begin{cases} 0, & x < t_i, \\ q(x|a_0^{i,j},\ldots,a_3^{i,j}), & x \in [t_{i+j}, t_{i+1+j}), \; j = 0,1 \\ 1, & x \geq t_{i+2}, \end{cases} \quad \text{such that} \quad \begin{cases} f_i^{(j)}(t_i) = 0, & j = 0,1,2, \\ f_i^{(j)}(t_{i+1}^-) = f_i^{(j)}(t_{i+1}^+), & j = 0,1, \\ f_i^{(j)}(t_{i+2}) = \delta_{0,j}, & j = 0,1,2. \end{cases}$$

The 8×8 linear system above has the unique solution

$$a_k^{i,1} = 0 \quad k = 0,1,2, \quad a_3^{i,1} = \frac{t_{i+1} - t_i}{t_{i+2} - t_i}; \qquad a_0^{i,2} = a_3^{i,1}, \qquad a_k^{i,2} = 1 \quad k = 1,2,3.$$

As suggested in the previous section, a similar argument can be used to observe that conditions C) and D) are trivially verified by the set of coefficients above. The related blending functions, obtained through (1), are piecewise cubic polynomials of support width 3. At the outermost knots of the support these functions are obtained as difference of C^2 functions and thus are C^2 continuous. At any interior knot the blending functions are defined as difference between a C^2 and a C^1 continuous function and thus they inherit the lower continuity, i.e., C^1.

2.3. Rational quartic/linear, C^2 blending functions

Let us consider the rational transition function

$$f_i(x) = \begin{cases} 0, & x < t_i, \\ \dfrac{q(x|a_0^{i,1},\ldots,a_4^{i,1})}{R_{i+1}(x)}, & x \in [t_i, t_{i+1}), \\ \dfrac{q(x|a_0^{i,2},\ldots,a_4^{i,2})}{L_i(x)}, & x \in [t_{i+1}, t_{i+2}), \\ 1, & x \geq t_{i+2}, \end{cases}$$

defined so that there is no singularity inside each interval of definition. We observe that a C^2 join at the knots leads to 9 conditions against 10 degrees of freedom. To ensure antisymmetry in the uniform case, we can exploit this additional degree of freedom to impose the condition $f_i(t_{i+1}^-) = f_i(t_{i+1}^+) = \dfrac{t_{i+1} - t_{i-1}}{t_{i+2} - t_{i-1}}$. The linear system obtained in this way has the unique solution

$$a_k^{i,1} = 0 \quad k = 0,1,2, \quad a_3^{i,1} = \frac{1}{16}\frac{(t_{i+1} - t_i)(7t_{i+2} - 3t_{i+1} - 4t_i)}{(t_{i+2} - t_{i+1})(t_{i+2} - t_{i-1})}, \quad a_4^{i,1} = \frac{t_{i+1} - t_i}{t_{i+2} - t_i};$$

$$a_0^{i,2} = a_4^{i,1}, \quad a_1^{i,2} = \frac{1}{16}\frac{13t_{i+2} + 3t_{i+1} - 16t_i}{t_{i+2} - t_i}, \quad a_2^{i,2} = \frac{1}{2}\frac{t_{i+2} + t_{i+1} - 2t_i}{t_{i+1} - t_i}, \quad a_3^{i,2} = \frac{1}{4}\frac{3t_{i+2} + t_{i+1} - 4t_i}{t_{i+1} - t_i}, \quad a_4^{i,2} = \frac{t_{i+2} - t_i}{t_{i+1} - t_i},$$

which satisfies conditions C) and D). The blending functions computed by the difference formula (1) are thus well-defined, supported on three intervals and trivially C^2 continuous.

3. A general framework for the construction and characterization of non-uniform local interpolating splines

Given a set of interpolation points $\{(x_j, p_j) \in \mathbb{R}^2\}$ with an increasing sequence of nodes $\mathbf{x} = \{x_j\}$, we denote by $p_{i,m}$ a degree-m polynomial interpolating the $m + 1$ consecutive data (x_j, p_j), $j = i, \ldots, i + m$. Let $\mathbf{t} = \{t_j\}$ be an increasing sequence of knots, not necessarily coinciding with $\mathbf{x}$. More precisely we confine ourselves to considering the following two knot configurations, termed respectively *even* and *odd*

- $\mathbf{t} \equiv \mathbf{x}$ (even configuration);

- $\mathbf{t} \neq \mathbf{x}$ and $x_i < t_i < x_{i+1}$, $\forall i$ (odd configuration).

Moreover, let $B_{i,n+1}$ be a C^k, $k \geq -1$, blending function defined on the knot partition **t**, with compact support $[t_i, t_{i+n+1}]$. We consider the class of spline functions defined by

$$F(x) = \sum_i p_{i-\sigma,m}(x) B_{i,n+1}(x), \quad \text{with } \sigma \in \mathbb{Z}. \tag{5}$$

Due to the compact support of the blending functions $B_{i,n+1}$, the above equation provides a general framework for the definition of local interpolating or approximating spline functions related to any odd or even knot configuration. In the following we are interested in characterizing which functions represented by (5) are interpolating and in this case what are the remarkable properties of such interpolant, i.e., exact smoothness, polynomial reproduction degree and locality.

To this aim, an interpolation condition can be easily obtained from the compactness of the support of the blending functions. In particular, if we observe that inside each knot interval there are exactly $n + 1$ non-vanishing blending functions $B_{i,n+1}$ and only n of them are non-zero at each knot t_j, the next result follows straightforwardly.

Proposition 1. *The spline function F defined in* (5) *interpolates the value p_j at x_j if and only if the polynomials $p_{i-\sigma,m}$, multiplying the non-vanishing blending functions $B_{i,n+1}$ at the node x_j, are all interpolating the value p_j.*

Corollary 1. *A necessary condition for defining an interpolant F passing through the whole set of data (x_j, p_j) is*

(a) $m + 1 \geq n$, in case of even knot configuration;

(b) $m + 1 \geq n + 1$, in case of odd knot configuration.

Proof. The result trivially follows by observing that for each value p_j the number of blending functions that do not vanish at x_j must be smaller than or equal to the number of polynomials interpolating (x_j, p_j). □

In other words, Corollary 1 asserts that the nodes x_j not interpolated by $p_{i-\sigma,m}$ cannot lie in the support of $B_{i,n+1}$, that is (t_i, t_{i+n+1}).

Remark 3. *From Proposition 1 and Corollary 1 follows that F defined in* (5) *interpolates the values p_j at x_j if and only if the integer $\sigma \in \{-1, \ldots, m - n\}$ and $\sigma \neq -1$ if $n = 0$.*

The following corollary points out the least polynomial reproduction degree and smoothness of F, that are given by construction.

Corollary 2. *Under the hypotheses of Proposition 1, F reproduces polynomials of degree at least m and, if $B_{i,n+1}$ are C^k-continuous, is at least C^k-continuous.*

Having determined under which condition the expression (5) defines an interpolant to the considered data, it is useful to investigate the exact smoothness and polynomial reproduction degree of such interpolating function. Obviously, any considered F obtained by combining C^k blending functions and suitable polynomial interpolants is at least C^k continuous. However, in case of even knot configuration, it can be proven that the interpolant F gains one more order of smoothness.

Proposition 2 (Exact smoothness). *Let F be a spline interpolant defined by* (5)*, with knots in even configuration. Moreover assume that the $n + 2$ polynomials associated with the non-vanishing blending functions on the interval $[t_{\ell-1}, t_{\ell+1})$ are all interpolating the data (t_ℓ, p_ℓ). Then F is C^{k+1}-continuous.*

Proof. To prove the thesis, it is sufficient to show that F is C^{k+1} at the knot t_ℓ. In the interval $[t_{\ell-1}, t_{\ell+1})$ we have a finite number of non-vanishing blending functions and in particular

$$F(x) = \sum_{i=\ell-n-1}^{\ell} p_{i-\sigma,m}(x) B_{i,n+1}(x). \tag{6}$$

Moreover, by the partition of unity property (2) of the blending functions, we can write $B_{\ell,n+1} = 1 - \sum_{i=\ell-n-1}^{\ell-1} B_{i,n+1}$, which, substituted in (6), gives

$$F(x) = p_{\ell-\sigma,m}(x) + \sum_{i=\ell-n-1}^{\ell-1} B_{i,n+1}(x)(p_{i-\sigma,m}(x) - p_{\ell-\sigma,m}(x)).$$

To conclude the proof we show that the order-$(k+1)$ right and left derivatives of F are equal at t_ℓ. By using Leibniz's formula, we have

$$
\begin{aligned}
D_+^{(k+1)}F(x) &= p_{\ell-\sigma,m}^{(k+1)}(x) + \sum_{i=\ell-n-1}^{\ell-1}\sum_{j=0}^{k+1}\binom{k+1}{j}D_+^{(j)}B_{i,n+1}(x)(p_{i-\sigma,m}^{(k+1-j)}(x) - p_{\ell-\sigma,m}^{(k+1-j)}(x)) \\
&= p_{\ell-\sigma,m}^{(k+1)}(x) + \sum_{i=\ell-n-1}^{\ell-1}\sum_{j=0}^{k}\binom{k+1}{j}B_{i,n+1}^{(j)}(x)(p_{i-\sigma,m}^{(k+1-j)}(x) - p_{\ell-\sigma,m}^{(k+1-j)}(x)) \\
&\quad + \sum_{i=\ell-n-1}^{\ell-1}D_+^{(k+1)}B_{i,n+1}(x)(p_{i-\sigma,m}(x) - p_{\ell-\sigma,m}(x)),
\end{aligned}
$$

and $D_-^{(k+1)}F(x)$ can be computed analogously. Now, evaluating the two derivative expressions at $x = t_\ell$ and subtracting the first from the second, we get

$$D_-^{(k+1)}F(x)|_{x=t_\ell} - D_+^{(k+1)}F(x)|_{x=t_\ell} = \sum_{i=\ell-n-1}^{\ell-1}(p_{i-\sigma,m}(t_\ell) - p_{\ell-\sigma,m}(t_\ell))\left(D_-^{(k+1)}B_{i,n+1}(x)|_{x=t_\ell} - D_+^{(k+1)}B_{i,n+1}(x)|_{x=t_\ell}\right) = 0,$$

since by hypothesis all the polynomials $p_{i-\sigma,m}$ for $i = \ell - n - 1, \ldots, \ell$, interpolate p_ℓ at the node t_ℓ. □

Remark 4. *Differentiating one more time, we observe that $D_-^{(k+2)}F(x)|_{x=t_\ell} - D_+^{(k+2)}F(x)|_{x=t_\ell} \neq 0$, that is F is not C^{k+2}.*

A proof similar to that proposed above can be used to show that in the odd knot configuration the smoothness of the function F remains exactly C^k. As a consequence of the above proposition, the following result also holds.

Corollary 3. *Let F be a spline interpolant defined by (5), on an even knot configuration with C^k blending functions of support width $n + 1$. If at least one of the $n + 2$ polynomials associated with the non-vanishing blending functions on the interval $[t_{\ell-1}, t_{\ell+1}]$ does not interpolate (t_ℓ, p_ℓ), then F is C^{k+1} at t_ℓ if and only if the corresponding blending functions are C^{k+1} in t_ℓ.*

Proof. Since F is interpolating, if $p_{j-\sigma,m}$ is a polynomial associated to a non vanishing blending function in $[t_{\ell-1}, t_{\ell+1}]$ and does not interpolate (t_ℓ, p_ℓ), then it must necessarily be either $j = \ell - n - 1$ or $j = \ell$ (since only $B_{\ell-n-1,n+1}$ and $B_{\ell,n+1}$ vanish at t_ℓ).
By the last equation in the proof of Proposition 2 we have that if $p_{\ell-n-1-\sigma,m}$ is the polynomial not interpolating (t_ℓ, p_ℓ), then the term $(p_{\ell-n-1-\sigma,m}(t_\ell) - p_{\ell-\sigma,m}(t_\ell))\left(D_-^{(k+1)}B_{\ell-n-1,n+1}(x)|_{x=t_\ell} - D_+^{(k+1)}B_{\ell-n-1,n+1}(x)|_{x=t_\ell}\right)$ can be zero if and only if $D_-^{(k+1)}B_{\ell-n-1,n+1}(x)|_{x=t_\ell} = D_+^{(k+1)}B_{\ell-n-1,n+1}(x)|_{x=t_\ell}$, that is F is C^{k+1} in t_ℓ and it must be $B_{\ell-n-1,n+1}^{(k+1)}(t_\ell) = 0$.
If the polynomial not interpolating (t_ℓ, p_ℓ) is $p_{\ell-\sigma,m}$, then we can write F differently than above, e.g., $F(x) = p_{j-\sigma,m}(x) + \sum_{i=\ell-n-1, i\neq j}^{\ell} B_{i,n+1}(x)(p_{i-\sigma,m}(x) - p_{j-\sigma,m}(x))$, with $p_{j-\sigma,m}$ interpolating (t_ℓ, p_ℓ), and proceed analogously. □

The following proposition characterizes the exact polynomial reproduction property of the interpolating spline F for any knot configuration.

Proposition 3 (Exact polynomial reproduction degree). *The exact degree of polynomial reproduction of a local interpolating spline F is $\max(m, l)$, where m is the degree of the blended interpolating polynomials and l is the polynomial reproduction degree of the blending functions.*

Proof. If the blending functions reproduce polynomials of degree $l \leq m$, then, by Corollary 2, the degree of polynomial reproduction of the local interpolating spline F is trivially m.
If the blending functions reproduce polynomials of degree $l > m$, then we can always write

$$F = I(f - Af) + Af,$$

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with A the approximating operator of the data f defined by the blending functions of our scheme, and I a suitable interpolating operator of the error $f - Af$. From this formula we can easily see that F inherits the polynomial reproduction degree of the approximating operator A: in fact if the data f belong to a degree-l polynomial, then $Af = f$ and therefore $F = f$. Thus the thesis follows straightforwardly. $\qquad\square$

To fully characterize the spline function F another remarkable property is concerned with the locality of the interpolant. As it is well-known, such property is related to the support width of the related cardinal basis functions, often referred to as fundamental functions when the knot partition is non-uniform and defined as follows.

Definition 2. *We call* fundamental functions *associated with a class of local interpolating spline functions and a given set of nodes* $\mathbf{x} = \{x_j\}$, *the functions* ψ_i, *which belong to the considered class, satisfying* $\psi_i(x_j) = \delta_{i,j}$.

The proposition below allows us to determine the exact support width of the fundamental functions of the interpolating spline F given in (5), for a specific choice of blending functions and polynomials.

Proposition 4 (Support width). *Let* $p_{j-\sigma,m}$ *be the degree-m polynomials interpolating $m + 1$ values at consecutive nodes and $B_{j,n+1}$ the blending functions with support width $n + 1$. If F is interpolatory, then the support width of the associated fundamental functions ψ_i is $m + n + 1$.*

Proof. To prove the statement, observe that the number of consecutive degree-m polynomials interpolating $(x_i, 1)$ is $m + 1$. Then the support width of ψ_i is given by the union of the supports of the blending functions associated with the $m + 1$ polynomials $p_{j-\sigma,m}$ interpolating the value $(x_i, 1)$, that is $m + n + 1$. $\qquad\square$

To further characterize the shape of the fundamental functions we introduce the following definition.

Definition 3. *The fundamental functions are called* centered *if they assume the value 1*

   (i) *at the central knot of their support* $[t_{i-\frac{m+n+1}{2}}, t_{i+\frac{m+n+1}{2}}]$, *in case of even support width;*

   (ii) *at the midpoint of the central knot interval of their support* $[t_{i-\lceil\frac{m+n+1}{2}\rceil}, t_{i+\lfloor\frac{m+n+1}{2}\rfloor}]$, *in case of odd support width.*

Centered fundamental functions can be obtained when the central point of the support of each $B_{i,n+1}$ and the central point of the interval containing the nodes x_j associated with the values p_j interpolated by $p_{i-\sigma,m}$ coincide.

Remark 5. *It is easy to deduce from Definition 3 that a centered fundamental function has even (respectively odd) support width in case of even (respectively odd) knot configuration.*
Moreover, a centered fundamental function is symmetric with respect to the central knot if the blending functions used for its construction are also symmetric.

An interesting feature of the proposed framework is that it does not impose limitations on the type of blending functions used, though in this work we have confined ourselves to consider only polynomial or rational blending functions. In particular, the possibility of using rational blending functions is a key property of the proposed approach, as it allows for considerably reducing the degree of the interpolating polynomial spline. In fact, inside a single knot interval we can write the interpolating spline as

$$F(x) = \sum_{i=\ell-n}^{\ell} p_{i-\sigma,m}(x)B_{i,n+1}(x), \qquad x \in [t_\ell, t_{\ell+1}). \tag{7}$$

From partition of unity (2), $B_{\ell-n,n+1} = 1 - \sum_{i=\ell-n+1}^{\ell} B_{i,n+1}$, and substituting in (7) we get

$$F(x) = p_{\ell-n-\sigma,m}(x)\left(1 - \sum_{i=\ell-n+1}^{\ell} B_{i,n+1}(x)\right) + \sum_{i=\ell-n+1}^{\ell} p_{i-\sigma,m}(x)B_{i,n+1}(x) \tag{8}$$

$$= p_{\ell-n-\sigma,m}(x) + \sum_{i=\ell-n+1}^{\ell} B_{i,n+1}(x)\left(p_{i-\sigma,m}(x) - p_{\ell-n-\sigma,m}(x)\right) = p_{\ell-n-\sigma,m}(x) + \sum_{i=\ell-n+1}^{\ell} B_{i,n+1}(x) \sum_{j=i}^{\ell-n+1} \left(p_{j-\sigma,m}(x) - p_{j-1-\sigma,m}(x)\right)$$

$$= p_{\ell-n-\sigma,m}(x) + \sum_{i=\ell-n+1}^{\ell} \left(p_{i-\sigma,m}(x) - p_{i-1-\sigma,m}(x)\right) \sum_{j=i}^{\ell} B_{j,n+1}(x) = p_{\ell-n-\sigma,m}(x) + \sum_{i=\ell-n+1}^{\ell} \left(p_{i-\sigma,m}(x) - p_{i-1-\sigma,m}(x)\right) f_i(x),$$

where in the last equality we have applied relation (1) to the blending functions $B_{j,n+1}$.

Since two degree-m interpolating polynomials with consecutive indices pass through a common set of m points, we can represent their difference as

$$p_{r,m}(x) - p_{r-1,m}(x) = c_r \prod_{j=r}^{r+m-1} (x - x_j), \qquad (9)$$

with some constant c_r, and then

$$F(x) = p_{\ell-n-\sigma,m}(x) + \sum_{i=\ell-n+1}^{\ell} c_{i-\sigma} f_i(x) \prod_{j=i-\sigma}^{i-\sigma+m-1} (x - x_j), \quad x \in [t_\ell, t_{\ell+1}). \qquad (10)$$

Looking at formulation (10) it is evident how we can obtain the polynomial spline interpolant F having the minimum degree provided by the framework. In fact, a rational transition function f_i that on the interval $[t_\ell, t_{\ell+1})$ has in the denominator one or more of the factors $(x - x_j)$, $j = i - \sigma, ..., i - \sigma + m - 1$, with $j \neq \ell, \ell + 1$ or $j \neq \ell + 1$ in order to avoid singularities in the denominator, leads to a polynomial interpolating spline F whose final degree is lower than the sum of the degrees of $p_{i-\sigma,m}$ and $B_{i,n+1}$. Therefore for all $i = \ell - n + 1, ..., \ell$ we can conveniently design the piece of f_i restricted to $[t_\ell, t_{\ell+1})$ such that its denominator simplifies as much as possible with the factors $(x - x_j)$, $j = i - \sigma, ..., i - \sigma + m - 1$, and does not vanish in such interval.

The results collected in this section establish a general framework for the construction of local interpolating polynomial splines. Based on the discussion above, to characterize the salient properties of the resulting interpolants we use the acronym $D^g C^{\bar{k}} P^{\bar{m}} S^w$, where $g, \bar{k}, \bar{m}, w$ are integers identifying respectively degree (g), exact smoothness ($\bar{k}$), exact degree of polynomial reproduction ($\bar{m}$) and support width of the fundamental function (w). In our framework we can consider any choice of m and n such that $m + n + 1 = w$, with m, n subject to the interpolation constraint $m + 1 \geq n$ when w is even, and $m + 1 \geq n + 1$ when w is odd. Once m and n have been fixed, the support width is identified and the exact smoothness and the exact degree of polynomial reproduction are given by Propositions 2 and 3, respectively. The degree g of the interpolating spline depends on the degree and type of blending functions. In particular, when using polynomial blending functions, g is equal to the sum of the degrees of the interpolating polynomials and the blending functions. Oppositely, in case of rational blending functions the degree is obviously greater than or equal to $\bar{m}$, but, as emphasized above, it can be conveniently minimized by suitably designing the transition functions.
As we will see in the following section, when using rational transition functions we can obtain

$$g = \begin{cases} \max(m, \bar{d} + 2) & \text{for even knot configuration,} \\ \max(m, \bar{d} + 1) & \text{for odd knot configuration,} \end{cases} \qquad (11)$$

with $\bar{d} = \max_{j=1,...,n} d_{i,j}$, for an arbitrary i. Thus, comparing the odd and even support width cases, we observe that, when the knot configuration is even, the spline interpolant can achieve an additional order of continuity with respect to that of the blending functions (see Proposition 2), but the spline degree can be reduced at most of $m - 2$; oppositely, for an odd knot configuration, the spline interpolant has exactly the same continuity as the blending functions, but its degree can be reduced by one comparing to the even case.

Tables 1, 2 and 3 show the centered, local interpolating splines of support width respectively 4, 6, 8 and lowest possible degree that our framework can provide for any fixed m (degree of the interpolating polynomial) and k (smoothness of the blending function). For the listed interpolants the degrees of the numerator and denominator of the pieces of the transition functions are shown in brackets. Note that if all the pieces are of the same kind, we specify only one piece; if they are different, we indicate the degrees of the first $\lceil n/2 \rceil$ pieces, being the transition function defined antisymmetrically. For instance, to design the interpolating spline $D^5 C^3 P^2 S^6$ in Table 2 one needs to exploit a rational transition function which is C^3 continuous at the outermost knots and C^2 at the internal knots, made of three pieces of degree $4/1, 3, 4/1$ respectively, so as to generate on each interval a spline interpolant of degree 5.
Looking at the tables it is evident that, moving across columns, we can always increase the smoothness of the interpolating spline by increasing its degree, while moving across rows we can increase the polynomial reproduction degree of the interpolating spline only up to $w - 1$ since the polynomial reproduction degree is always strictly lower than the

$m, n+1$ \ k	-1	0	1
1, 3			$D^4C^2P^1$ (3)
2, 2		$D^3C^1P^2$ (1)	$D^5C^2P^2$ (3)
3, 1	$D^3C^0P^3$ (0)	-	-

Table 1: Centered, local interpolating splines of support width 4 and lowest degree, given m and k, i.e., the degree of the interpolating polynomials and the smoothness of the blending functions, respectively.

$m, n+1$ \ k	-1	0	1	2
2, 4				$D^5C^3P^2$ (4/1, 3)
3, 3			$D^4C^2P^3$ (2/1)	$D^6C^3P^3$ (4/1)
4, 2		$D^4C^1P^4$ (1/2)	$D^5C^2P^4$ (3/2)	$D^7C^3P^4$ (5/2)
5, 1	$D^5C^0P^5$ (0)	-	-	-

Table 2: Centered, local interpolating splines of support width 6 and lowest degree, given m and k, i.e., the degree of the interpolating polynomials and the smoothness of the blending functions, respectively.

$m, n+1$ \ k	-1	0	1	2	3
3, 5					$D^6C^4P^3$ (5/2, 4/1)
4, 4				$D^5C^3P^4$ (3/2)	$D^7C^4P^4$ (5/2)
5, 3			$D^5C^2P^5$ (2/3)	$D^6C^3P^5$ (4/3)	$D^7C^4P^5$ (5/3)
6, 2		$D^6C^1P^6$ (1/4)	$D^6C^2P^6$ (3/4)	$D^7C^3P^6$ (5/4)	$D^8C^4P^6$ (7/4)
7, 1	$D^7C^0P^7$ (0)	-	-	-	-

Table 3: Centered, local interpolating splines of support width 8 and lowest degree, given m and k, i.e., the degree of the interpolating polynomials and the smoothness of the blending functions, respectively.

support width (i.e., $m < w$). Moreover, when $m = w - 1$ (maximal reproduction degree), it can be easily noticed that the smoothness is necessarily C^0 if w is even (while only C^{-1} if w is odd).

We observe that, when $g = m$ the spline degree is trivially minimal, where, also in the following, with *minimum degree* we mean that there does not exist any local interpolating spline with same smoothness, polynomial reproduction and support width, but lower degree.

We also remark that the interpolating spline $D^4C^2P^1S^4$ in Table 1 is obtained by the transition functions computed in Section 2.2, while the interpolating spline $D^6C^3P^3S^6$ in Table 2 by those in Section 2.3.

To conclude this section, we present an efficient evaluation algorithm for the spline interpolant that fits into the proposed framework. In particular, we observe that, from a computational point of view, expression (10) for the local interpolating spline F on the interval $[t_\ell, t_{\ell+1})$, coincides with its most convenient form. In fact, if we define the interpolating polynomials $p_{i-\sigma,m}$ in Newton form and, in particular, we represent the interpolating polynomials with consecutive indices appearing in (8) in the same Newton basis, their difference can be written as in (9) with

$$c_r = [p_r, \ldots, p_{r+m-1}, p_{r+m}] - [p_r, \ldots, p_{r+m-1}, p_{r-1}]. \tag{12}$$

The evaluation procedure of F thus consists in determining the divided differences of order m for all points (x_j, p_j) in order to be used for representing the interpolating polynomials $p_{i-\sigma,m}$ and their differences.

If the rational transition functions are designed to minimize the degree of F, as previously discussed, we can simplify their denominators with the factors $(x - x_j)$, $j = i - \sigma, \ldots, i - \sigma + m - 1$, and $j \neq \ell, \ell + 1$ in the even case, while $j \neq \ell + 1$ in the odd case, thus obtaining

$$F(x) = p_{\ell-n-\sigma,m}(x) + \sum_{i=\ell-n+1}^{\ell} c_{i-\sigma} \, K_i(x) \, q(x|a_0^{i,\ell-i+1}, \dots, a_{d_{i,\ell-i+1}}^{i,\ell-i+1}), \quad x \in [t_\ell, t_{\ell+1}), \tag{13}$$

where $K_i(x)$ can be of the following types:

$$K_i(x) = \begin{cases} \begin{cases} (x - x_\ell) \\ (x - x_\ell)(x - x_{\ell+1}) & \text{for even knot configuration,} \\ (x - x_{\ell+1}) \end{cases} \\ (x - x_{\ell+1}) & \text{for odd knot configuration.} \end{cases} \tag{14}$$

From the expression above it can also be easily observed that the lowest possible degree for F is as in (11).

4. Revisiting known classes of local interpolating splines and designing new ones

In this section we show that the presented interpolation framework includes known local interpolating splines and allows for generalizing them to a complete family with arbitrary support width. At the same time it also provides a constructive method for designing new local interpolating splines of low, and sometimes minimum, degree given polynomial reproduction and smoothness.

In the following we limit ourselves to consider local interpolating splines of the form (7) where the integer σ is always set to the central value of its range of variability $[-1, m-n]$, denoted by $\sigma_c := \lfloor \frac{m-n-1}{2} \rfloor$. Both for even and odd knot partitions, this choice guarantees to obtain centered fundamental functions, which, as previously discussed, is a necessary symmetry condition in case of symmetric knot configurations.

Using the acronym $D^g C^k P^m S^w$ introduced in Section 3, where k and m now stand for the exact smoothness and the exact polynomial reproduction degree, in Table 4 we classify families of centered, local interpolating splines of fixed support. In particular, the first column of Table 4 contains three families of known local interpolating splines of support S^{2k}, with $k \geq 2$, that can be recovered by our framework: the IMOMS splines [2], the Catmull-Rom splines [3] and the Z-splines [1, 17]. In particular, IMOMS splines, originally proposed only in the uniform case for image processing purposes, are here generalized to the non-uniform setting by degree-$(2k-1)$ interpolating polynomials and piecewise constant blending functions of support width 1; Catmull-Rom splines are obtained by blending interpolating polynomials of degree k with degree-$(k-1)$ B-splines; finally, Z-splines are obtained through polynomial interpolants of degree $2k-2$ and rational blending functions of support width 2, generated by the transition functions

$$f_i(x) = \begin{cases} 0, & x < t_i, \\ \dfrac{q(x|a_0^{i,1}, \dots, a_{2k-3}^{i,1})}{\displaystyle\prod_{j=1}^{k-2} L_{i-j}(x) \prod_{j=1}^{k-2} R_{i+j}(x)}, & t_i \leq x < t_{i+1}, \\ 1, & x \geq t_{i+1}, \end{cases}$$

with q a degree-$(2k-3)$ polynomial on the interval $[t_i, t_{i+1})$, satisfying the conditions for C^{k-2} smoothness with $k \geq 2$. Among these three families of splines, that share the same support and degree, IMOMS have the highest polynomial reproduction degree, even if their smoothness is only C^0. Then, after IMOMS, Z-splines have the highest polynomial reproduction degree (which is thus the sub-optimal for such support), at the same smoothness of Catmull-Rom splines. Moreover, as regards Z-splines, we observe that the related rational blending functions provide splines of minimum degree, which trivially derives from the uniqueness property of the Z-spline basis [1].

In the remainder of this section, we show how the proposed framework can be exploited to construct new classes of non-uniform local interpolating splines. As discussed in the previous section, rational blending functions allow for obtaining spline interpolants of lower degree, with respect to the polynomial case. Thus, it is interesting to investigate whether there exists a family of interpolants with the same smoothness, polynomial reproduction and support width of Catmull-Rom splines, but lower degree (and desirably the lowest possible). Indeed such a family can be constructed through our framework and is identified by $D^{k+1} C^{k-1} P^k S^{2k}$ for any $k \geq 2$, as indicated in the last line of the first

	$S^{2k}, k \geq 2$	$S^{2k-1}, k \geq 2$	$S^w, w \geq 3$
Catmull-Rom family	$D^{2k-1}C^{k-1}P^k$		
Z-splines family	$D^{2k-1}C^{k-1}P^{2k-2}$	$D^{2k-2}C^{k-2}P^{2k-3}$	$D^{w-1}C^{\lfloor \frac{w}{2} \rfloor -1}P^{w-2}$
I-MOMS family	$D^{2k-1}C^0 P^{2k-1}$	$D^{2k-2}C^{-1}P^{2k-2}$	$D^{w-1}C^{-1}P^{w-1}$
Jenkins family	$D^{k+1}C^{k-1}P^k$	$D^k C^{k-2}P^{k-1}$	$D^{\lfloor \frac{w}{2} \rfloor +1}C^{\lfloor \frac{w}{2} \rfloor -1}P^{\lfloor \frac{w}{2} \rfloor}$

Table 4: Classification and characterization of families of centered, local interpolating splines of fixed support.

column of Table 4. The blending functions at the basis of this new class are rational and have support width k. The associated transition functions are $(k-1)$-piece rational polynomials of degree $(k-1)/(k-2)$, joining C^{k-2} continuously. We remark that this choice of blending functions allows to obtain splines of minimum degree: in fact an easy computation shows that a spline with the same properties, but lower degree, does not even provide enough degrees of freedom to impose the interpolation constraints and the desired smoothness. In the table we refer to this new class as the Jenkins family because the degree-4 member of this class, the local interpolating spline $D^4 C^2 P^3 S^6$, represents a generalization to an arbitrary knot partition of the Jenkins's uniform interpolant [11], whose construction is illustrated in the following.

$D^4 C^2 P^3 S^6$ Jenkins local interpolating spline - According to the acronym used, we seek a quartic, C^2 local interpolating spline of support width 6, which reproduces polynomials up to degree 3, defined over an even knot partition $t_i = x_i \; \forall i$. Since these interpolating splines must reproduce cubic polynomials, we use interpolating polynomials of degree $m = 3$ and since the support width must be $w = 6$, we deduce that the support of the blending functions has to be $n + 1 = 3$ (see Figure 4). Over each knot interval $[t_i, t_{i+1})$, the interpolating spline F is thus of the form

$$
\begin{aligned}
F(x) &= p_{i-2,3}(x)B_{i-2,3}(x) + p_{i-1,3}(x)B_{i-1,3}(x) + p_{i,3}(x)B_{i,3}(x) \\
&= p_{i-2,3}(x) + f_{i-1}(x)\left(p_{i-1,3}(x) - p_{i-2,3}(x)\right) + f_i(x)\left(p_{i,3}(x) - p_{i-1,3}(x)\right).
\end{aligned}
$$
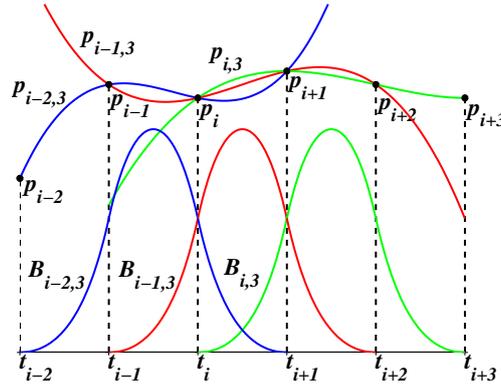
Figure 4: Illustration of the given points (x_i, p_i) and their interpolating polynomials $p_{i,3}$ with the associated compactly supported blending functions $B_{i,3}$.

Since $p_{i-2,3}$ and $p_{i-1,3}$ both interpolate (x_j, p_j) for $j = i-1, i, i+1$, their difference is of the form

$$
p_{i-1,3}(x) - p_{i-2,3}(x) = c_{i-1}(x - t_{i-1})(x - t_i)(x - t_{i+1}). \tag{15}
$$

Analogously, since $p_{i-1,3}$ and $p_{i,3}$ both interpolate (x_j, p_j) for $j = i, i+1, i+2$, their difference has the expression

$$
p_{i,3}(x) - p_{i-1,3}(x) = c_i(x - t_i)(x - t_{i+1})(x - t_{i+2}). \tag{16}
$$

In equations (15) and (16) c_{i-1} and c_i are given by (12). Then, we define two-piece, C^1 rational transition functions of degree 2/1 where the linear denominator is the only non-vanishing term among those in (15) and (16), so that the

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final interpolant F is a piecewise quartic. More precisely,

$$f_{i-1}(x) = \begin{cases} 0, & x < t_{i-1}, \\ \dfrac{q(x|a_0^{i-1,1}, a_1^{i-1,1}, a_2^{i-1,1})}{R_i(x)}, & x \in [t_{i-1}, t_i), \\ \dfrac{q(x|a_0^{i-1,2}, a_1^{i-1,2}, a_2^{i-1,2})}{L_{i-1}(x)}, & x \in [t_i, t_{i+1}), \\ 1, & x \geq t_{i+1}, \end{cases} \quad \text{and} \quad f_i(x) = \begin{cases} 0, & x < t_i, \\ \dfrac{q(x|a_0^{i,1}, a_1^{i,1}, a_2^{i,1})}{R_{i+1}(x)}, & x \in [t_i, t_{i+1}), \\ \dfrac{q(x|a_0^{i,2}, a_1^{i,2}, a_2^{i,2})}{L_i(x)}, & x \in [t_{i+1}, t_{i+2}), \\ 1, & x \geq t_{i+2}. \end{cases}$$

We can now compute the coefficients for f_i by solving the 6×6 linear system obtained by imposing C^1 smoothness at the knots, which, for distinct knots, has the unique solution

$$a_0^{i,1} = a_1^{i,1} = 0, \quad a_2^{i,1} = \frac{1}{3} \frac{t_{i+2} + t_{i+1} - 2t_i}{t_{i+2} - t_i}; \quad a_0^{i,2} = a_2^{i,1}, \quad a_1^{i,2} = \frac{1}{2} \frac{t_{i+2} + t_{i+1} - 2t_i}{t_{i+1} - t_i}, \quad a_2^{i,2} = \frac{t_{i+2} - t_i}{t_{i+1} - t_i}.$$

Shifting the knots indices by -1, the explicit expression of f_{i-1} can be obtained straightforwardly. Thus, by equation (1) we can get the expressions of all the non-vanishing blending functions on $[t_i, t_{i+1})$

$$B_{i-2,3}(x) = 1 - f_{i-1}(x) = \frac{2t_{i+1} - t_i - t_{i-1}}{3(t_{i+1} - t_{i-1})} \frac{R_i(x)^2}{L_{i-1}(x)},$$
$$B_{i-1,3}(x) = f_{i-1}(x) - f_i(x) = 1 - B_{i-2,3}(x) - B_{i,3}(x),$$
$$B_{i,3}(x) = f_i(x) = \frac{t_{i+2} + t_{i+1} - 2t_i}{3(t_{i+2} - t_i)} \frac{L_i(x)^2}{R_{i+1}(x)},$$

and the explicit polynomial form of F turns out to be

$$\begin{aligned} F(x) &= p_{i-2,3}(x) + q(x|a_0^{i-1,2}, a_1^{i-1,2}, a_2^{i-1,2})(t_i - t_{i-1})\left([p_{i-1}, p_i, p_{i+1}, p_{i+2}] - [p_{i-1}, p_i, p_{i+1}, p_{i-2}]\right)(x - t_i)(x - t_{i+1}) \\ &\quad - q(x|a_0^{i,1}, a_1^{i,1}, a_2^{i,1})(t_{i+2} - t_{i+1})\left([p_i, p_{i+1}, p_{i+2}, p_{i+3}] - [p_i, p_{i+1}, p_{i+2}, p_{i-1}]\right)(x - t_i)(x - t_{i+1}), \qquad x \in [t_i, t_{i+1}) \end{aligned}$$

with

$$p_{i-2,3}(x) = [p_{i-2}] + [p_{i-2}, p_{i-1}](x - t_{i-2}) + [p_{i-2}, p_{i-1}, p_i](x - t_{i-2})(x - t_{i-1}) + [p_{i-2}, p_{i-1}, p_i, p_{i+1}](x - t_{i-2})(x - t_{i-1})(x - t_i).$$

Many other examples of known local interpolating splines (and the new proposals of their non-uniform versions) can be recovered by our framework, such as the German's interpolant $D^4 C^1 P^4 S^6$ [8] and Han's interpolant $D^4 C^2 P^1 S^4$ [10]. Note that the latter also appears in Table 1 and has minimum degree.

To the best of our knowledge, the literature also suffers from a lack of proposals of families of non-uniform local interpolating splines of odd support. Thanks to our very general framework, we can easily generalize the Z-splines, IMOMS and Jenkins families such that their support width is S^{2k-1} for any $k \geq 2$, and characterize them as summarized in the second column of Table 4. For the sake of completeness, we show in the third column of Table 4 the complete characterization of such families for an arbitrary support width $w \geq 3$.

We conclude this section by showing the details of the construction of the degree-5 member in the Jenkins family, namely the local interpolating spline $D^5 C^3 P^4 S^8$, which to the authors' knowledge was never discussed before and presents interesting properties for modeling purposes.

$D^5 C^3 P^4 S^8$ Jenkins local interpolating spline - In order to reproduce polynomials up to degree 4, the construction combines quartic polynomials with at least C^2-continuous blending functions of support width $n + 1 = 4$. Inside each knot interval $[t_i, t_{i+1})$ the interpolant is represented as

$$\begin{aligned} F(x) &= p_{i-3,4}(x)B_{i-3,3}(x) + p_{i-2,4}(x)B_{i-2,3}(x) + p_{i-1,4}(x)B_{i-1,3}(x) + p_{i,4}(x)B_{i,3}(x) \\ &= p_{i-3,4}(x) + f_{i-2}(x)\left(p_{i-2,4}(x) - p_{i-3,4}(x)\right) + f_{i-1}(x)\left(p_{i-1,4}(x) - p_{i-2,4}(x)\right) + f_i(x)\left(p_{i,4}(x) - p_{i-1,4}(x)\right) \quad x \in [t_i, t_{i+1}). \end{aligned} \tag{17}$$

Since $p_{i-j,4}$ and $p_{i-j-1,4}$, for $j = 0, 1, 2$, interpolate the common sequence of data (x_{i-j+h}, p_{i-j+h}), $h = 0, \ldots, 3$, their difference can be represented by

$$p_{i-j,4}(x) - p_{i-j-1,4}(x) = c_{i-j}(x - t_{i-j})(x - t_{i-j+1})(x - t_{i-j+2})(x - t_{i-j+3}), \tag{18}$$

with c_{i-j} given by (12). Now, to ensure that the sought spline interpolant has minimum degree and C^3 smoothness we need to design a set of transition functions, each of which is supported on three intervals and at least C^2. Thus we can conveniently exploit a rational cubic transition function supported on three intervals, that provides 12 degrees of freedom. On each interval, the denominator is computed as the product of the two factors in (18) that do not vanish at t_i and t_{i+1}, so as to get a quintic interpolant. Based on this reasoning, the function f_i has the form

$$
f_i(x) = \begin{cases}
0 & x < t_i, \\[2mm]
\dfrac{q(x|a_0^{i,1},\ldots,a_3^{i,1})}{R_{i+1}(x)R_{i+2}(x)} & x \in [t_i, t_{i+1}), \\[3mm]
\dfrac{q(x|a_0^{i,2},\ldots,a_3^{i,2})}{L_i(x)R_{i+2}(x)} & x \in [t_{i+1}, t_{i+2}), \\[3mm]
\dfrac{q(x|a_0^{i,3},\ldots,a_3^{i,3})}{L_i(x)L_{i+1}(x)} & x \in [t_{i+2}, t_{i+3}), \\[3mm]
1 & x \geq t_{i+3}.
\end{cases}
$$

Moreover, if we now require that the pieces join up to the second derivative, we get a 12×12 linear system with unique solution

$$
a_k^{i,1} = 0 \quad k = 0,1,2, \quad a_3^{i,1} = a_0^{i,2} = \frac{1}{8}\frac{\beta_i}{(t_{i+3}-t_{i+2})^2(t_{i+2}-t_i)}\left(1 + 9\frac{(t_{i+2}-t_i)(t_{i+3}-t_{i+1})}{\alpha_i}\right),
$$

$$
a_1^{i,2} = \frac{1}{6}\frac{\beta_i}{(t_{i+3}-t_{i+2})^2(t_{i+1}-t_i)}\left(1 + 9\frac{(t_{i+2}-t_i)(t_{i+3}-t_{i+1})}{\alpha_i}\right), \quad a_2^{i,2} = \frac{(t_{i+2}-t_i)\beta_i}{(t_{i+1}-t_i)(t_{i+3}-t_{i+2})\alpha_i},
$$

$$
a_3^{i,2} = a_0^{i,3} = \frac{1}{4}\frac{\gamma_i}{(t_{i+3}-t_{i+1})(t_{i+1}-t_i)} + \frac{3}{4}\frac{(t_{i+2}-t_i)\beta_i}{(t_{i+1}-t_i)(t_{i+3}-t_{i+1})\alpha_i},
$$

$$
a_1^{i,3} = \frac{1}{3}\frac{\gamma_i}{(t_{i+1}-t_i)(t_{i+2}-t_{i+1})}, \quad a_2^{i,3} = \frac{1}{3}\frac{\delta_i}{(t_{i+1}-t_i)(t_{i+2}-t_{i+1})}, \quad a_3^{i,3} = \frac{(t_{i+3}-t_i)(t_{i+3}-t_{i+1})}{(t_{i+1}-t_i)(t_{i+2}-t_{i+1})}
$$

where

$$
\alpha_i = 4(t_it_{i+2} + t_{i+1}t_{i+3}) + 5(t_{i+1}t_{i+2} + t_it_{i+3}) - 9(t_{i+2}t_{i+3} + t_it_{i+1}),
$$
$$
\beta_i = (t_{i+1} + t_{i+3} - 2t_i)(t_{i+1}t_{i+3} + t_{i+2}^2) + (t_{i+1}^2 + t_{i+3}^2 - t_{i+1}t_{i+3})(3t_i - 4t_{i+2}) + 2t_it_{i+2}(t_{i+1} + t_{i+3}) - 3t_it_{i+1}t_{i+3},
$$
$$
\gamma_i = (t_{i+2} - t_i)(t_{i+3} + 2t_{i+2} - 3t_{i+1}) + (t_{i+3} - t_{i+2})(t_{i+2} - t_{i+1}),
$$
$$
\delta_i = (2t_{i+3} + t_{i+2} - 3t_{i+1})(t_{i+3} - t_i) - (t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2}).
$$

5. Application examples

This section provides examples of 2D parametric curves generated through some of the discussed non-uniform spline interpolants with knots given by the centripetal parametrization. To select such examples, we have chosen curves which represent a good trade-off between low degree, high smoothness and approximation order. In Figures 5 and 6 top-left we show the local interpolating splines $D^3C^1P^2S^4$, $D^4C^2P^3S^6$, $D^5C^2P^4S^6$, $D^5C^3P^4S^8$ and $D^6C^3P^5S^8$ passing through a given set of highly non uniformly distributed points. In the other five pictures of Figures 5 and 6 we illustrate the curvature combs of these curves, using the same color for the interpolated data as ready-reference to the first snapshot. As it is well-known, the parametrization highly influences the overall quality of the interpolant. In particular, if non-uniform and suitably chosen, it may help avoiding or minimizing the occurrence of interpolation artifacts, that are clearly visible in the uniform setting. So it is easy to understand why all the examples provide a good-looking interpolant, despite the different smoothness and approximation orders, that can be freely fixed based on the requirement of a specific application.

6. Conclusions

This work presents a general framework for the construction and characterization of local interpolating splines on non-uniform knot partitions with fixed continuity and polynomial reproduction degree. To this aim, we have defined

the sought spline functions as a combination of degree-m polynomial interpolants with arbitrary blending functions of compact support width $n+1$ and smoothness C^k, generating local fundamental bases of compact support $w = m+n+1$. We have also provided a general and simple method for designing either polynomial or rational blending functions of any required support width and smoothness to be used in such construction. One of the most remarkable features of the proposed approach is the possibility of exploiting rational blending functions, which provides a significant advantage with respect to using polynomial ones: in fact, in the former setting, the spline has considerably lower and sometimes minimum degree.
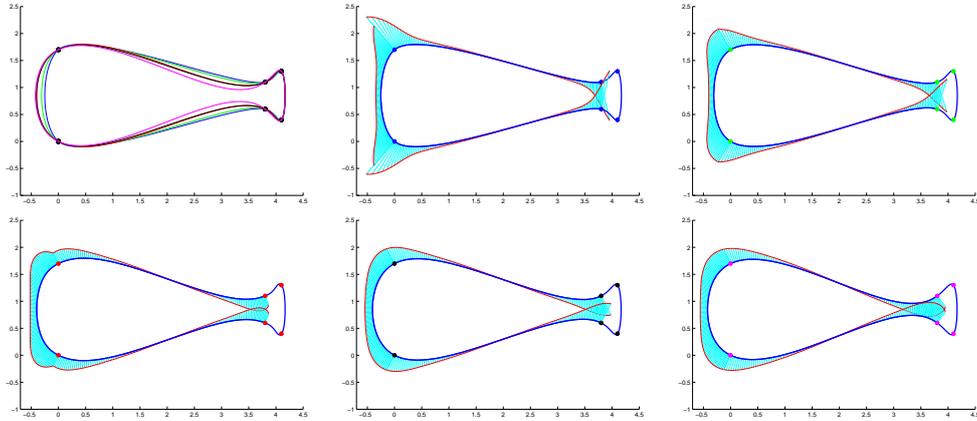
Figure 5: The local interpolating splines $D^3C^1P^2S^4$, $D^4C^2P^3S^6$, $D^5C^2P^4S^6$, $D^5C^3P^4S^8$ and $D^6C^3P^5S^8$ passing through the same set of highly non uniformly distributed points (in blue, green, red, black and magenta respectively), and the corresponding curvature combs.
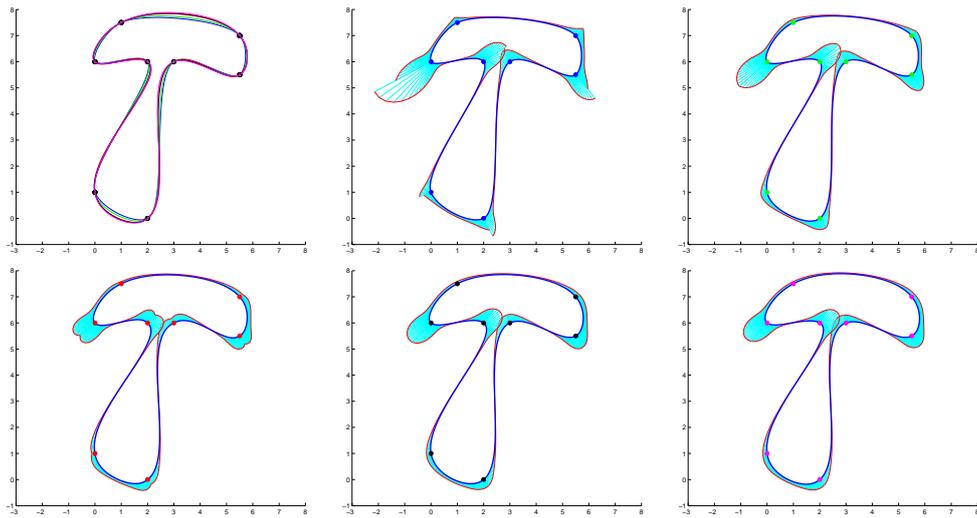
Figure 6: The local interpolating splines $D^3C^1P^2S^4$, $D^4C^2P^3S^6$, $D^5C^2P^4S^6$, $D^5C^3P^4S^8$ and $D^6C^3P^5S^8$ passing through the same set of highly non uniformly distributed points (in blue, green, red, black and magenta respectively), and the corresponding curvature combs.

References

[1] J.T. Becerra Sagredo. Z-splines: moment conserving cardinal spline interpolation of compact support for arbitrarily spaced data. Research report no. 2003-10, Eidgenössische Technische Hochschule, Zurich, Switzerland (2003).

[2] T. Blu, P. Thévenaz, M. Unser. Complete parameterization of piecewise-polynomial interpolation kernels. IEEE Trans. on Image Processing 11(12) (2003) 1297-1309.

[3] E. Catmull, R. Rom. A class of local interpolants. In: Barnhill, R., E., Riesenfeld, R., F. (Eds.). Computer Aided Geometric Design. Academic Press (1974) 317-326.

[4] C.K. Chui, H. Diamond. A general framework for local interpolation. Numer. Math. 58 (1991) 569-581.

[5] M.S. Floater, K. Hormann. Barycentric rational interpolation with no poles and high rates of approximation. Numer. Math. 107(2)(2007) 315-331.

[6] R. Franke. Scattered data interpolation: Test of some methods. Math. Comp. 38(157) (1982) 181-200.

[7] R. Franke, G. Nielson. Smooth interpolation of large sets of scattered data. International Journal for Numerical Methods in Engineering, 15(11) (1980) 1691-1704.

[8] I. German. Shoer kernel fifth-order interpolation. IEEE Trans. Signal Processing 45 (1997) 1355-1359.

[9] W.J. Gordon, J.A. Wixom, Shepard's method of "metric interpolation" to bivariate and multivariate interpolation. Math. Comp. 32(141) (1978) 253-264.

[10] X. Han. A class of general quartic spline curves with shape parameters. Comput. Aided Geom. Design 3(28) (2011) 151-163.

[11] W.A. Jenkins. Osculatory interpolation: new derivation and formulae. Record of the American Institute of Actuaries 15 (1926), Pages 87.

[12] P. Lancaster, K. Salkauskas. Surfaces generated by moving least squares methods. Math. Comp. 37(155) (1981) 141-158.

[13] D. Levin. The approximation power of moving least-squares. Math. Comp. 67(224) (1998) 1517-1531.

[14] Y. Ohtake, A. Belyaev, M. Alexa, G. Turk, H.P. Seidel. Multi-level partition of unity implicits. ACM Trans. Graph. 3(22) (2003) 463-470.

[15] E. Quak. About B-splines. Twenty answers to one question: What is the cubic B-spline for the knots $-2, -1, 0, 1, 2$?. Technical report, SINTEF Applied Mathematics, Oslo Norway (2004).

[16] D. Shepard. A two-dimensional interpolation function for irregularly-spaced data. In Proceedings of the 23rd ACM national conference, pp.517-524 (1968).

[17] T. Ueno, S. Truscott, M. Okada. New spline basis functions for sampling approximations. Numer. Algor. 45 (2007) 283-293.