Dual univariate m-ary subdivision schemes of de Rham-type

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Abstract

In this paper, we present an algebraic perspective of the de Rham transform of a binary subdivision scheme and propose an elegant strategy for constructing dual m-ary approximating subdivision schemes of de Rham-type, starting from two primal schemes of arity m and 2, respectively. On the one hand, this new strategy allows us to show that several existing dual corner-cutting subdivision schemes fit into a unified framework. On the other hand, the proposed strategy provides a straightforward algorithm for constructing new dual subdivision schemes having higher smoothness and higher polynomial reproduction capabilities with respect to the two given primal schemes.

Keywords: Linear subdivision; Arity m; Primal and dual parametrization; Smoothness; Polynomial reproduction

1. Introduction

Univariate *subdivision schemes* are iterative methods for representing smooth curves via the specification of a coarse polygon $\mathbf{f}^{(0)} := \{f_i^{(0)},\ i \in \mathbb{Z}\}$ and a set of refinement rules mapping the sequence of points $\mathbf{f}^{(k)}$, $k \geq 0$, into the denser sequence of points $\mathbf{f}^{(k+1)}$. If the k-th refinement step consists of m refinement rules, the subdivision scheme is said to be of *arity* m and, if the refinement rules are linear combinations of the coarser points, the subdivision scheme is called *linear*. Let a_i, $i \in \mathbb{Z}$, be the coefficients appearing in the linear combination. Then, for each $k \geq 0$ the m refinement rules read as

$$f_{mi+\ell}^{(k+1)} := \sum_{j\in\mathbb{Z}} a_{m(i-j)+\ell}\, f_j^{(k)}, \quad \ell = 0, \cdots, m-1.$$

The set of coefficients $\{a_i,\ i \in \mathbb{Z}\}$ is called *subdivision mask* and is denoted by $\mathbf{a}$. The associated subdivision scheme is denoted by $S_{\mathbf{a}}$ and can be equivalently seen as the repeated application of the subdivision matrix $S = \{s(i, j) = a_{i-mj} : i, j \in \mathbb{Z}\}$. To establish a notion of convergence, we associate to the sequence of refined data $\{\mathbf{f}^{(k)},\ k \geq 0\}$ a sequence of *parameter values* $\mathbf{t}^{(k)} = \{t_i^{(k)},\ i \in \mathbb{Z}\}$ with $t_{i+1}^{(k)} - t_i^{(k)} = m^{-k}$, and we define the piecewise linear function $F^{(k)}$ that interpolates the data $\mathbf{f}^{(k)}$ at the parameters $\mathbf{t}^{(k)}$. If, for every initial data $\mathbf{f}^{(0)}$, the sequence $\{F^{(k)},\ k \geq 0\}$ is convergent to a continuous function $g_{\mathbf{f}^{(0)}}$, then the subdivision scheme is said to be C^0-*convergent*. Moreover, the scheme is said to be C^r-*convergent* if $g_{\mathbf{f}^{(0)}}$ is a $C^r(\mathbb{R})$ function. In this paper, we only consider subdivision schemes that are convergent and *non-singular*, so that $g_{\mathbf{f}^{(0)}} = 0$ if and only if $\mathbf{f}^{(0)} = 0$. Assuming that the *support* of the subdivision mask $\mathbf{a}$ is $[0, N] \cap \mathbb{Z}$ (*i.e.* $a_i = 0$ for $i < 0$ and $i > N$ as well as $a_0, a_N \neq 0$), then the support of the limit function is given by $\left[0, \frac{N}{m-1}\right]$ (see for example [8, Section 1]).

Support size and smoothness are considered mutually conflicting properties of a subdivision scheme because a high degree of smoothness generally requires a large support, thus leading to a more global influence of each initial data value on the limit function. Raising the arity of the subdivision scheme provides a way to overcome this dilemma

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to some extents. For example, the ternary and quaternary 4-point schemes discussed in [18, 21] and [23], respectively, have smaller support and higher smoothness than the classical binary 4-point scheme in [15, 17], and all three schemes reproduce cubic polynomials by construction. The latter means that, whenever starting from data on a cubic polynomial, their limits are exactly that cubic polynomial. So, subdivision schemes of arity $m > 2$, although much less known in the literature, are potentially more useful because smoother but with smaller support than their binary counterparts.

Another way of increasing the smoothness of a subdivision scheme in exchange of a slight increase of its support width is to use *dual* subdivision schemes instead of *primal* ones. From a geometric point of view, primal schemes are those that retain or modify the old vertices and create $m-1$ new vertices at each old edge of the control polygon. Dual schemes, instead, create m new points at the old edges and discard the old vertices. The importance of dual schemes in practical applications is due to the fact that they can be smoother than the primal schemes having the same degree of polynomial reproduction, in exchange of a slight increase of the support width. This is the case, for instance, of the dual C^2 four-point subdivision scheme presented in [14]. In fact, the support width of its subdivision mask is increased only by one with respect to that of the interpolatory (primal) C^1 four-point scheme in [15, 17], and they both reproduce cubic polynomials.

The first univariate, linear subdivision scheme appeared in the literature is the arity-2 (binary) scheme having mask

$$\mathbf{a} = [w,\ 1-w,\ 1-w,\ w] \quad \text{with} \quad w \in \left(0, \frac{1}{2}\right). \tag{1}$$

The particular choice of $w = \frac{1}{3}$ corresponds to the de Rham scheme [11] and that of $w = \frac{1}{4}$ to the Chaikin's scheme [1]. Since the refinement rules associated to (1) are

$$f_{2i}^{(k+1)} = (1-w)f_i^{(k)} + wf_{i+1}^{(k)}, \qquad f_{2i+1}^{(k+1)} = wf_i^{(k)} + (1-w)f_{i+1}^{(k)}, \qquad k \geq 0,\ i \in \mathbb{Z},$$

the $(k+1)$-level new vertices $f_{2i}^{(k+1)}$ and $f_{2i+1}^{(k+1)}$ are constructed at points w and $1-w$ of the way along each edge of the k-level control polygon and so each line segment connecting $f_i^{(k)}$ and $f_{i+1}^{(k)}$ is partitioned with the ratio $w : (1-2w) : w$. The mask in (1) thus provides a family of 2-point corner-cutting subdivision schemes.

In the recent paper [12], Dubuc observed that a single step of the binary subdivision scheme having mask (1) with $w = \frac{1}{4}$ can be seen as the subsequent application of two steps of the linear binary B-spline scheme with mask $[\frac{1}{2},\ 1,\ \frac{1}{2}]$, followed by the selection of the obtained points with odd index only (see Figure 1, first row). Prompted by this observation, Dubuc also defined the *de Rham transform* of a binary subdivision scheme with matrix $S = \{s(i, j) : i, j \in \mathbb{Z}\}$ as the subdivision scheme with matrix $\tilde{S} = \{s_2(2i+1, j) : i, j \in \mathbb{Z}\}$, where $s_2(i, j)$ are the entries of S^2. He observed that, for any binary subdivision scheme, we can define its de Rham transform which generalizes the de Rham and Chaikin corner cutting. In particular, in [12] he applies the de Rham transform to three families of interpolatory subdivision schemes and shows that, although the interpolatory property is lost, the de Rham transform can provide subdivision schemes that are smoother than the original ones. In [13], this idea was originally applied to binary interpolatory Hermite subdivision schemes in order to define new smoother non-interpolatory Hermite schemes.

Exploiting the notion of *subdivision symbol*, in this paper we provide an algebraic perspective of the de Rham transform. More precisely, denoted by $a(z) = \sum_{i\in\mathbb{Z}} a_i z^i,\ z \in \mathbb{C}\backslash\{0\}$ the symbol of the binary subdivision scheme $S_\mathbf{a}$ with mask $\mathbf{a} = \{a_i,\ i \in \mathbb{Z}\}$, we can show that the symbol of the de Rham transform of $S_\mathbf{a}$ is the product of two special Laurent polynomial factors. In fact, for a given sequence of vertices $\mathbf{f}^{(k)} = \{f_i^{(k)},\ i \in \mathbb{Z}\}$, the double application of the subdivision operator $S_\mathbf{a}$ provides the points $\tilde{f}_i = \sum_{j\in\mathbb{Z}} a_{i-2j} \left(\sum_{\ell\in\mathbb{Z}} a_{j-2\ell} f_\ell^{(k)}\right)$, $i \in \mathbb{Z}$. Thus, defining the sequence of points at level $k+1$ as the subset of the odd-indexed points $\tilde{f}_{2i+1}$, we get

$$\mathbf{f}^{(k+1)} = \left\{ f_i^{(k+1)} = \sum_{\ell\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} a_{2i+1-4\ell-2j}\, a_j \right) f_\ell^{(k)}, \quad i \in \mathbb{Z} \right\}.$$

Hence, denoting by $S_\mathbf{c}$ the de Rham transform of $S_\mathbf{a}$, i.e. the subdivision scheme mapping $\mathbf{f}^{(k)}$ into $\mathbf{f}^{(k+1)}$, it turns out that the subdivision mask of $S_\mathbf{c}$ is given by $\mathbf{c} = \left\{c_i = \sum_{j\in\mathbb{Z}} a_{2i+1-2j}\, a_j,\ i \in \mathbb{Z}\right\}$ and its symbol is

$$c(z) = \sum_{i\in\mathbb{Z}} c_i z^i = \sum_{r\in\mathbb{Z}} a_{2r+1} z^r \sum_{j\in\mathbb{Z}} a_j z^j = a_{odd}(z) a(z).$$

This algebraic interpretation of the de Rham transform allows us to extend the results in [12] in two different directions: firstly, we replace the double step of a primal binary scheme by the subsequent application of two different binary schemes; secondly, we allow one of the two primal schemes to be of arity different from 2. The so obtained extended de Rham-type strategy provides a corner cutting scheme of arbitrary arity. For example, the ternary scheme in [24, pag.53] (known as the "neither" scheme), is a corner cutting scheme which can be interpreted as a de Rham-type scheme obtained by applying first a linear ternary B-spline scheme, then a linear binary B-spline scheme, both followed by a selection of the odd entries only (see Figure 1, second row).

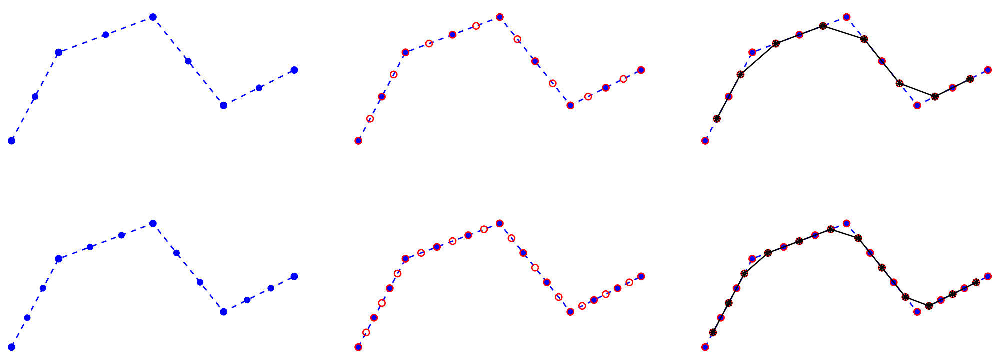

Figure 1: First row: interpretation of Chaikin's scheme as the subsequent application of two steps of the linear binary B-spline scheme, followed by the selection of the obtained points with odd index only. Second row: interpretation of the ternary "neither" scheme as a de Rham-type scheme obtained by applying first a linear ternary B-spline scheme, then a linear binary B-spline scheme, both followed by a selection of the odd entries only.

In this paper we will show that any de Rham-type m-ary subdivision scheme is *dual* by construction and each subdivision step of such a scheme can be obtained by applying to the k-level vertices first a primal m-ary subdivision operator $S_{\mathbf{b}}$, then a primal binary subdivision operator $S_{\mathbf{a}}$, both followed by a decimation step consisting in the selection of the odd elements only of the refined sequence. The symbol of the de Rham-type subdivision scheme is thus given by $a_{odd}(z)b(z)$. This clearly provides a generalization of the de Rham transform in [12] which offers the following advantages. Firstly, it establishes an algebraic interpretation of the de Rham transform, which turns out to be useful to derive the refinement rules of de Rham-type schemes at a very low computational cost. Secondly, it allows us to describe many existing dual corner-cutting schemes in a unified way: for instance, Chaikin's scheme in [1] and more generally all binary schemes associated to degree-$(2k+2)$ B-splines, the dual 4-point schemes in [14] and [4], the dual ν-point ($\nu \geq 3$) schemes in [20], the ternary "neither" scheme in [24, pag.53] and the quaternary 4-point approximating schemes in [23]. Finally, it provides a long needed tool for constructing new approximating subdivision schemes of arbitrary arity with required smoothness and reproduction capabilities.

The paper is organized as follows. Section 2 is devoted to a detailed description of the de Rham-type strategy for the construction of dual approximating m-ary subdivision schemes starting from two primal subdivision schemes of arity m and 2, respectively. For the sake of clarity the sketch of an algorithm is also given. While convergence and smoothness of de Rham-type subdivision schemes is discussed in Section 3 together with their polynomial reproduction properties, in Section 4 the derived theoretical results are exploited to define novel dual m-ary subdivision schemes of arity 2, 3 and 4, respectively, as well as to revisit some known approximating schemes via the de Rham approach. Conclusions are drawn in Section 5.

2. An algorithm for the construction of dual de Rham-type m-ary approximating schemes

To describe our construction of a dual de Rham-type m-ary approximating scheme, we first need to introduce the notion of *parametrization* of a subdivision scheme. According to [16], the parametrization of a symmetric subdivision scheme may be either *primal* or *dual*, and this provides a useful criterion to classify subdivision schemes. From a

geometric point of view, primal schemes are those that retain or modify the old vertices and create $m-1$ new vertices at each old edge of the control polygon. Dual schemes, instead, create m new points at the old edges and discard the old vertices. For example, the binary cubic B-spline scheme is primal, while Chaikin's scheme for generating quadratic B-splines is dual. The reason why mostly no other parametrizations are considered is due to the fact that dual and primal parametrizations are the only ones that guarantee reproduction of linear polynomials for a symmetric, convergent subdivision scheme generating linear polynomials (see [8, Corollary 5.7]). Reproduction of linear polynomials means that, if the starting sequence $\mathbf{f}^{(0)}$ is sampled from a linear polynomial, its limit is exactly that polynomial, while the generation of polynomials means that the limit can be any linear polynomial.
Following [8], the sets of parameters $\mathbf{t}^{(k)} := \{t_i^{(k)},\ i \in \mathbb{Z}\}$ providing the so-called *parametrization* of a convergent, m-ary subdivision scheme are defined as

$$t_i^{(k)} := m^{-k}\left(i + \frac{\tau}{m-1}\right), \quad i \in \mathbb{Z}, \quad k \geq 0\,, \tag{2}$$

where $\tau \in \mathbb{R}$ is a *shift parameter* given by

$$\tau = \frac{(D^1 a(z))|_{z=1}}{m}.$$

There follows that, in case of a *primal* parametrization, $\tau = 0$ and thus

$$t_i^{(k)} := m^{-k}\, i, \quad i \in \mathbb{Z}, \quad k \geq 0\,.$$

As a consequence,

$$t_{mi+\ell}^{(k+1)} = \frac{m-\ell}{m}\, t_i^{(k)} + \frac{\ell}{m}\, t_{i+1}^{(k)}, \quad \ell = 0, \cdots, m-1\,, \quad i \in \mathbb{Z}, \quad k \geq 0\,.$$

Accordingly, each subdivision step replaces the old vertices $f_i^{(k)}$ attached to $t_i^{(k)}$ by the new vertices $f_{mi}^{(k+1)}$ attached to $t_{mi}^{(k+1)}$, and inserts the new vertices $f_{mi+\ell}^{(k+1)}$, $\ell = 1, \cdots, m-1$ uniformly between the old vertices, *i.e.*, with relative distance $\frac{1}{m}$ of the distance between the neighbors $f_i^{(k)}$ and $f_{i+1}^{(k)}$.
On the contrary, for a *dual* parametrization, $\tau = -\frac{1}{2}$ and thus

$$t_i^{(k)} := m^{-k}\left(i - \frac{1}{2(m-1)}\right), \quad i \in \mathbb{Z}, \quad k \geq 0\,. \tag{3}$$

It follows that the relationship between parameters of consecutive levels is therefore

$$t_{mi+\ell}^{(k+1)} = \frac{2m-1-2\ell}{2m}\, t_i^{(k)} + \frac{1+2\ell}{2m}\, t_{i+1}^{(k)}, \quad \ell = 0, \cdots, m-1\,, \quad i \in \mathbb{Z}, \quad k \geq 0.$$

This means that, a dual subdivision scheme at each subdivision step replaces the old vertices $f_i^{(k)}$ attached to $t_i^{(k)}$ by the two new vertices $f_{mi-1}^{(k+1)}$ and $f_{mi}^{(k+1)}$ attached to $t_{mi-1}^{(k+1)}$ and $t_{mi}^{(k+1)}$, respectively (that is one to the left, the other to the right of $f_i^{(k)}$, and both at $\frac{1}{2m}$ the distance to the neighboring vertices $f_{i-1}^{(k)}$ and $f_{i+1}^{(k)}$). It also inserts $m-2$ new vertices uniformly between the old vertices $f_i^{(k)}$ and $f_{i+1}^{(k)}$, *i.e.*, with relative distance $\frac{1}{m}$ of the distance between the neighbors $f_i^{(k)}$ and $f_{i+1}^{(k)}$.
Corner-cutting schemes are therefore associated to a dual parametrization.

Remark 1. *We remark that, any scheme with shift parameter $\tau = n$ is also primal since multiplication of the symbol by z^{-n} yields a scheme with $\tau = 0$. Similarly, any scheme with shift parameter $\tau = -\frac{1}{2} + n$ is dual (see [8, Corollary 5.1]).*

In the following we will show that a dual corner-cutting subdivision scheme $S_{\mathbf{c}}$ of arity m can be obtained by generating the new sequence of points $\mathbf{f}^{(k+1)}$ from the previous one $\mathbf{f}^{(k)}$, in the following way:

1. apply a primal m-ary subdivision scheme $S_{\mathbf{b}}$ to the k-level data $\mathbf{f}^{(k)}$;

2. apply a primal binary subdivision scheme $S_{\mathbf{a}}$ to the data obtained in step 1.;

3. select from the result of step 2. only the data with odd indices, and let the obtained subset be the $(k+1)$-level data $\mathbf{f}^{(k+1)}$.

In fact, given a sequence of vertices $\mathbf{f}^{(k)}$ associated to the parameters $\mathbf{t}^{(k)}$, since we first apply the primal subdivision scheme $S_{\mathbf{b}}$ of arity m, the elements of $\tilde{\mathbf{f}}^{(k+1)} := S_{\mathbf{b}}\mathbf{f}^{(k)}$ are attached to the parameters

$$\tilde{t}^{(k+1)}_{mi+\ell} = \frac{m-\ell}{m}\, t^{(k)}_i + \frac{\ell}{m}\, t^{(k)}_{i+1}, \quad \ell = 0, \cdots, m-1, \;\; i \in \mathbb{Z}, \;\; k \geq 0.$$

Next, since we apply the primal binary subdivision scheme $S_{\mathbf{a}}$ to the resulting sequence $\tilde{\mathbf{f}}^{(k+1)}$, the elements of $\tilde{\mathbf{f}}^{(k+2)} := S_{\mathbf{a}}\, \tilde{\mathbf{f}}^{(k+1)} = S_{\mathbf{a}}\, S_{\mathbf{b}}\mathbf{f}^{(k)}$ are attached to the parameters

$$\begin{aligned} &\tilde{t}^{(k+2)}_{2(mi+\ell)} = \tilde{t}^{(k+1)}_{mi+\ell} = \tfrac{m-\ell}{m}\, t^{(k)}_i + \tfrac{\ell}{m}\, t^{(k)}_{i+1}\,, \\ &\tilde{t}^{(k+2)}_{2(mi+\ell)+1} = \tfrac{1}{2}\, \tilde{t}^{(k+1)}_{mi+\ell} + \tfrac{1}{2}\, \tilde{t}^{(k+1)}_{mi+\ell+1} = \tfrac{m-\ell}{2m}\, t^{(k)}_i + \tfrac{\ell}{2m}\, t^{(k)}_{i+1} + \tfrac{m-\ell-1}{2m}\, t^{(k)}_i + \tfrac{\ell+1}{2m}\, t^{(k)}_{i+1} = \tfrac{2m-1-2\ell}{2m}\, t^{(k)}_i + \tfrac{2\ell+1}{2m}\, t^{(k)}_{i+1}\,, \end{aligned} \qquad i \in \mathbb{Z}, \;\; k \geq 0.$$

The sequence $\tilde{\mathbf{f}}^{(k+2)}$ is then decimated by taking the odd entries only, so providing $\mathbf{f}^{(k+1)}$, which is therefore associated to the dual parameter values

$$t^{(k+1)}_{mi+\ell} = \frac{2m-1-2\ell}{2m}\, t^{(k)}_i + \frac{2\ell+1}{2m}\, t^{(k)}_{i+1}, \quad \ell = 0, \cdots, m-1, \quad i \in \mathbb{Z}, \quad k \geq 0\,,$$

meaning that the corresponding subdivision scheme is dual in the sense of (3). In terms of values we have

$$\tilde{f}^{(k+1)}_i = \sum_{j \in \mathbb{Z}} b_{i-mj}\, f^{(k)}_j\,, \qquad \tilde{f}^{(k+2)}_i = \sum_{j \in \mathbb{Z}} a_{i-2j}\, \tilde{f}^{(k+1)}_j\,, \qquad f^{(k+1)}_i = \tilde{f}^{(k+2)}_{2i+1}\,, \qquad i \in \mathbb{Z}.$$

Therefore, there is an underlined m-ary subdivision scheme such that

$$f^{(k+1)}_i = \sum_{r \in \mathbb{Z}} c_{i-mr}\, f^{(k)}_r,$$

where

$$c_i = \sum_{j \in \mathbb{Z}} a_{2i+1-2j}\, b_j, \quad i \in \mathbb{Z}. \tag{4}$$

In fact,

$$f^{(k+1)}_i = \tilde{f}^{(k+2)}_{2i+1} = \sum_{j \in \mathbb{Z}} a_{2i+1-2j} \sum_{r \in \mathbb{Z}} b_{j-mr}\, f^{(k)}_r = \sum_{r \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} a_{2i+1-2j}\, b_{j-mr} \right) f^{(k)}_r = \sum_{r \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} a_{2i+1-2mr-2j}\, b_j \right) f^{(k)}_r = \sum_{r \in \mathbb{Z}} c_{i-mr}\, f^{(k)}_r.$$

In order to construct from the subdivision mask $\mathbf{c} = \{c_i,\ i \in \mathbb{Z}\}$ the Laurent polynomial $c(z) = \sum_{i \in \mathbb{Z}} c_i\, z^i$, $z \in \mathbb{C}\backslash\{0\}$, we recall that any symbol $a(z)$ can be decomposed as

$$a(z) = a_{even}(z^2) + z\, a_{odd}(z^2)$$

with

$$a_{odd}(z) = \sum_{i \in \mathbb{Z}} a_{2i+1}\, z^i \qquad \text{and} \qquad a_{even}(z) = \sum_{i \in \mathbb{Z}} a_{2i}\, z^i.$$

Hence, from (4) we write

$$\sum_{i \in \mathbb{Z}} c_i\, z^i = \sum_{i \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} a_{2i+1-2j}\, b_j \right) z^i = \sum_{i \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} a_{2(i-j)+1}\, z^{i-j} \right) b_j z^j = \sum_{r \in \mathbb{Z}} a_{2r+1}\, z^r \sum_{j \in \mathbb{Z}} b_j\, z^j,$$

that is

$$c(z) = a_{odd}(z)\, b(z)\,. \tag{5}$$

Therefore, the de Rham-type strategy, previously described in terms of generated sequences of points, offers the following simple algebraic perspective in terms of subdivision symbols.

Algorithm for the construction of a dual de Rham-type m-ary approximating scheme

Input: $a(z)$ symbol of a primal, binary scheme; $b(z)$ symbol of a primal, m-ary scheme;

1. Decompose $a(z)$ as $a(z) = a_{even}(z^2) + z a_{odd}(z^2)$;
2. Extract the odd sub-symbol $a_{odd}(z)$;
3. Construct the new symbol $c(z) = a_{odd}(z) b(z)$;

Output: $c(z)$, symbol of a dual, m-ary scheme.

Remark 2. *Note that, even if $a_{odd}(z)$ is not the odd rule of a primal subdivision scheme $S_{\mathbf{a}}$ but simply an odd-degree Laurent polynomial, the above algorithm still produces a dual m-ary subdivision scheme.*

As a side effect, the proposed de Rham-type strategy allows to put existing isolated constructions of dual approximating schemes (e.g. [1, 4, 14, 20, 23]) in a general framework. In Table 1 we show the odd sub-symbol of the binary subdivision schemes $S_{\mathbf{a}}$ and the symbol of the m-ary subdivision scheme $S_{\mathbf{b}}$ that should be given as input to generate these existing corner-cutting schemes via the above algorithm.

Existing corner-cutting schemes	m	$a_{odd}(z)$	$b(z)$
Chaikin's scheme in [1]	2	$z^{-1}\frac{1+z}{2}$	$z^{-1}\frac{(1+z)^2}{2}$
degree-$(2k+2)$ B-splines, $k \geq 0$	2	$z^{-1}\frac{1+z}{2}$	$z^{-(k+1)}\frac{(1+z)^{2k+2}}{2^{2k+1}}$
dual 4-point schemes in [4]	2	$z^{-1}\frac{1+z}{2}$	$z^{-3}\left(\frac{1+z}{2}\right)^2 \frac{-16wz^4+(64w-3)z^3+(14-96w)z^2+(64w-3)z-16w}{4}, \quad w \in \mathbb{R}$
dual 4-point scheme in [14]	2	$z^{-1}\frac{1+z}{2}$	$z^{-3}\left(\frac{1+z}{2}\right)^4 \frac{-5+18z-5z^2}{4}$
dual ν-point ($\nu \geq 3$) schemes in [20]	2	$z^{-1}\frac{1+z}{2}$	$z^{-(\nu-1)}\left(\frac{1+z}{2}\right)^{2\nu-4} \frac{-(2\nu-3)+2(2\nu+1)z-(2\nu-3)z^2}{4}$
"neither" scheme in [24, pag.53]	3	$z^{-1}\frac{1+z}{2}$	$z^{-2}\frac{(z^2+z+1)^2}{3}$
4-point approximating schemes in [23]	4	$\frac{1}{4}z^{-2}(z+1)(z^2+1)$	$z^{-6}\frac{(z+1)^4(z^2+1)^3}{32}\Big(2(1-w)z^2+(4w-3)z+2(1-w)\Big), \; w \in \mathbb{R}$

Table 1: Classification of existing corner-cutting schemes as de Rham-type subdivision schemes.

In Section 4, the well-known reproduction and smoothness properties of all the subdivision schemes included in Table 1 will be revisited in relationship to the corresponding properties of the subdivision schemes $S_{\mathbf{a}}$ and $S_{\mathbf{b}}$ that originate them. We will see that dual de Rham-type subdivision schemes not only can possess higher regularity with respect to the primal schemes used in their construction, but they can also reproduce polynomials of higher degree.

3. Properties of de Rham-type subdivision schemes

3.1. Preliminary notions

The notion of *polynomial reproduction* (see also [10]) is directly related to the approximation order of a subdivision scheme. Indeed, if a scheme reproduces the space of polynomials of degree d (hereinafter denoted by Π_d), then it has approximation order $d+1$ (see [22]).

Definition 1. *A subdivision scheme $S_{\mathbf{a}}$* reproduces *polynomials of degree d if it is convergent and if $g_{\mathbf{f}^{(0)}} = p$ for initial data $f_i^{(0)} = p\left(t_i^{(0)}\right)$, $i \in \mathbb{Z}$ and for any polynomial $p \in \Pi_d$. Instead, a subdivision scheme $S_{\mathbf{a}}$* generates *polynomials of degree d if, for the initial data $f_i^{(0)} = p\left(t_i^{(0)}\right)$, $i \in \mathbb{Z}$, $p \in \Pi_d$, it is convergent and $g_{\mathbf{f}^{(0)}} = \tilde{p}$ with $\tilde{p} \in \Pi_d$, $p - \tilde{p} \in \Pi_{d-1}$.*

Polynomial reproduction can be checked via algebraic conditions on the subdivision symbol $a(z)$ as the following proposition shows. For the sake of conciseness, we first introduce the following definition of frequent use in the remainder of the paper.

Definition 2. *We denote by* $\zeta_m^j = \exp(\frac{2\pi \mathrm{i}}{m} j)$*,* $j = 1, \ldots, m-1$ *the* m*-th roots of unity.*

Proposition 1. *[8, Theorem 4.3] A convergent subdivision scheme* $S_{\mathbf{a}}$ *reproduces polynomials of degree* d *with respect to the parametrization in* (2) *if and only if*

$$a(1) = m, \quad (D^n a(z))|_{z=1} = m \prod_{l=0}^{n-1} (\tau - l) \quad \textit{for} \quad n = 1, \ldots, d,$$

$$(D^n a(z))|_{z=\zeta_m^j} = 0 \quad \forall j = 1, \ldots, m-1 \quad \textit{for} \quad n = 0, \ldots, d,$$

with ζ_m^j *as in Definition 2.*

We continue by presenting an important theoretical result concerning the properties of subdivision sub-symbols. It connects two types of sub-symbols, associated with a subdivision symbol $a(z)$, *i.e.*,

$$a_\ell(z) = \sum_{i \in \mathbb{Z}} a_{mi+\ell}\, z^i, \quad \ell = 0, \cdots, m-1, \qquad \text{see, for example, [2, 3, 5, 7]} \tag{6}$$

and

$$\tilde{a}_\ell(z) = \sum_{i \in \mathbb{Z}} a_{mi+\ell}\, z^{mi+\ell}, \quad \ell = 0, \cdots, m-1, \qquad \text{see, for example, [8, 16].}$$

These sub-symbols, both used in the literature, are obviously related by the equations $\tilde{a}_\ell(z) = z^\ell\, a_\ell(z^m)$, $\ell = 0, \cdots, m-1$, $z \in \mathbb{C} \backslash \{0\}$.

Lemma 1. *Assume* ζ_m^j*,* $j = 1, \cdots, m-1$ *as in Definition 2. The* n*-th derivative of a subdivision symbol* $a(z)$ *satisfies*

$$(D^n a(z))|_{z=\zeta_m^j} = 0, \qquad \forall j = 1, \ldots, m-1,$$

if and only if the n*-th derivative of all its sub-symbols* $a_\ell(z)$ *in (6) assume the same value at* $z = 1$*, namely*

$$(D^n a_\ell(z))|_{z=1} = \frac{(-1)^n}{m} \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} p_{m,i}(\ell)\, p_{m,n-i}(-1)\, (D^{n-i} a(z))|_{z=1}, \quad \forall \ell = 0, \ldots, m-1\,, \tag{7}$$

where

$$p_{m,0}(x) := 1, \ \forall x \in \mathbb{R} \qquad \textit{and} \qquad p_{m,i}(x) := \prod_{l=0}^{i-1} \left(\frac{x}{m} + l \right), \ \forall i \geq 1. \tag{8}$$

Proof. From $\tilde{a}_\ell(z) = z^\ell a_\ell(z^m)$ we can easily write $a_\ell(z) = z^{-\frac{\ell}{m}}\, \tilde{a}_\ell(z^{\frac{1}{m}})$ so that, using Leibniz rule we have

$$\begin{aligned} D^n a_\ell(z) \ &= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} D^i z^{-\frac{\ell}{m}}\, D^{n-i} \tilde{a}_\ell(z^{\frac{1}{m}}) = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} (-1)^i\, p_{m,i}(\ell)\, z^{-\frac{\ell}{m}-i} \left(D^{n-i} \tilde{a}_\ell(y) \right) \Big|_{y=z^{\frac{1}{m}}} D^{n-i} z^{\frac{1}{m}} \\ &= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} (-1)^i\, p_{m,i}(\ell)\, z^{-\frac{\ell}{m}-i} \left(D^{n-i} \tilde{a}_\ell(y) \right) \Big|_{y=z^{\frac{1}{m}}} (-1)^{n-i} \cdot p_{m,n-i}(-1)\, z^{\frac{1}{m}-n+i}\,, \end{aligned}$$

and therefore, evaluating at $z = 1$,

$$(D^n a_\ell(z))|_{z=1} = (-1)^n \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} p_{m,i}(\ell)\, p_{m,n-i}(-1)\, (D^{n-i} \tilde{a}_\ell(y))|_{y=1}\,.$$

Now, taking into account that in view of [8, Lemma 2.1] the condition $(D^{n-i} a(z))|_{z=\zeta_m^j} = 0\ \forall j = 1, \ldots, m-1$, is equivalent to

$$(D^{n-i} \tilde{a}_\ell(z))|_{z=1} = \frac{1}{m} (D^{n-i} a(z))|_{z=1}, \quad \forall \ell = 0, \cdots, m-1\,,$$

the proof is completed. $\square$

Remark 3. *We remark that, accordingly to the sub-symbol notation in* (6)*, for a binary subdivision scheme* $a_{even}(z) \equiv a_0(z)$ *and* $a_{odd}(z) \equiv a_1(z)$*. Though, since we believe that the use of* $a_{even}(z)$ *and* $a_{odd}(z)$ *is simpler and more consistent with the existing literature of binary subdivision schemes, in this paper we have opted for this notation. We also observe that in case* $a(z)$ *contains a shift factor* z^{-n}*, the shift factor in* $a_{odd}(z)$ *becomes* $z^{-\frac{n}{2}}$ *for n even, and* $z^{-\frac{n+1}{2}}$ *for n odd.*

The auxiliary result in Lemma 1 will be used in the next subsection to show that the subdivision operator $S_{\mathbf{c}}$ (associated to the symbol in (5)) reproduce linear polynomials with respect to the dual parametrization.

3.2. Analysis of de Rham-type subdivision schemes

We start by investigating the capability of de Rham-type subdivision schemes of reproducing polynomials.

Proposition 2. *Let* $a(z) = a_{even}(z^2) + za_{odd}(z^2)$ *be the symbol of a primal binary subdivision scheme reproducing linear polynomials, and let* $b(z)$ *be the symbol of a convergent, primal m-ary subdivision scheme also reproducing linear polynomials. The correct parametrization for the m-ary subdivision scheme with symbol* $c(z) = a_{odd}(z)\, b(z)$ *is the dual one.*

Proof. We start by observing that $S_{\mathbf{c}}$ generates linear polynomials, namely $(D^k c(z))|_{z=\zeta_m^\ell} = 0$ for $\ell = 1, \cdots, m-1$, $k = 0, 1$, since $S_{\mathbf{b}}$ generates linear polynomials. Denoted by τ_b the shift parameter characterizing the parametrization of $S_{\mathbf{b}}$, from Proposition 1 we know also that $b(1) = m$ and $(D^1 b(z))|_{z=1} = m\, \tau_b$. Moreover, τ_c is the correct parameter for $c(z)$ if and only if $\tau_c = \frac{1}{m}\,(D^1 c(z))|_{z=1}$. Since $c(z)$ has the expression in (5), we have $D^1 c(z) = (D^1 a_{odd}(z))\, b(z) + a_{odd}(z)\,(D^1 b(z))$, which evaluated at $z = 1$ yields

$$(D^1 c(z))|_{z=1} = (D^1 a_{odd}(z))|_{z=1}\, b(1) + a_{odd}(1)\,(D^1 b(z))|_{z=1},$$

that is $\tau_c = (D^1 a_{odd}(z))|_{z=1} + a_{odd}(1)\, \tau_b$. Now, since $a(z)$ reproduces constants, $a_{odd}(1) = 1$, while due to Lemma 1, we obtain from (7)-(8) with $n = 1$, $m = 2$ and $\ell = 1$ that $(D^1 a_{odd}(z))|_{z=1} = \frac{\tau_a}{2} - \frac{1}{2}$. Therefore, $\tau_c = \tau_b + \frac{\tau_a}{2} - \frac{1}{2}$. At this point, taking into account that $S_{\mathbf{a}}$ and $S_{\mathbf{b}}$ are primal subdivision schemes reproducing linear functions, we have $\tau_a = \tau_b = 0$, from which $\tau_c = -\frac{1}{2}$ follows and thus the claim is shown. □

In view of Proposition 1, we can also show that, when the dual parametrization is assumed, a de Rham-type subdivision scheme reproduces polynomials whose degree is at least the minimum between the polynomial reproduction degrees of the two primal schemes it originates from.

Proposition 3. *Let* $S_{\mathbf{a}}$ *be a primal binary subdivision scheme reproducing polynomials up to degree* $d_a \geq 0$ *with respect to the parameter* $\tau_a = 0$*. Let* $S_{\mathbf{b}}$ *be a primal m-ary subdivision scheme reproducing polynomials up to degree* $d_b \geq 0$ *with respect to the parameter* $\tau_b = 0$*. Then the de Rham subdivision scheme* $S_{\mathbf{c}}$*, with symbol* $c(z)$ *as in* (5)*, reproduces polynomials up to degree* $d_c = \min\{d_a, d_b\}$ *with respect to the parameter* $\tau_c = -\frac{1}{2}$*.*

Proof. We start by observing that

$$(D^j c(z))|_{z=\zeta_m^\ell} = 0, \quad j = 0, \cdots, d_b,\ \ell = 1, \cdots, m-1,$$

since $(D^j b(z))|_{z=\zeta_m^\ell} = 0$, $j = 0, \cdots, d_b$, $\ell = 1, \cdots, m-1$. Then, we observe that being $a_{odd}(1) = 1$ and $b(1) = m$, we get $c(1) = a_{odd}(1)b(1) = m$. In addition, using Leibniz rule, we can write the j-th derivative of $c(z) = a_{odd}(z)b(z)$ evaluated at $z = 1$ as

$$(D^j c(z))|_{z=1} = \sum_{i=0}^{j} \binom{j}{i} (D^i a_{odd}(z))|_{z=1}\, (D^{j-i} b(z))|_{z=1}\,.$$

Recalling that, by assumption, $b(1) = m$ and $(D^k b(z))|_{z=1} = 0$, $k = 1, \cdots, d_b$ (since $\tau_b = 0$), only the term $i = j$ is left in the sum so that

$$(D^j c(z))|_{z=1} = (D^j a_{odd}(z))|_{z=1}\, m\,.$$

Now, in view of Lemma 1, from (7)-(8) with $m = 2$ we have

$$(D^j a_{odd}(z))|_{z=1} = \frac{(-1)^j}{2} \sum_{i=0}^{j} \binom{j}{i} p_{2,i}(1) p_{2,j-i}(-1)\, (D^{j-i} a(z))|_{z=1}\,,$$

so that, due to the reproduction properties of $S_{\mathbf{a}}$, *i.e.* $a(1) = 2$, $(D^k a(z))|_{z=1} = 0$, $k = 1, \cdots, d_a$, by setting $\tau_c = -\frac{1}{2}$, we get

$$(D^j c(z))|_{z=1} = m(-1)^j p_{2,j}(1) = m \prod_{l=0}^{j-1} (\tau_c - l), \quad j = 1, \cdots, d_a,$$

which shows the claim. □

In the next section we will show that, when dealing with special case studies, the assumptions in Proposition 3 can be relaxed to show that the polynomial reproduction degree of $S_{\mathbf{c}}$ can be even higher.

Now, we continue by investigating the convergence and smoothness properties of de Rham-type subdivision schemes in relationship to the analogous properties of the primal schemes they are originated from.

Proposition 4. *Let $S_{\mathbf{a}}$, $S_{\mathbf{b}}$ be a primal binary and a primal m-ary subdivision scheme, respectively. Let $a(z)$, $b(z)$ be the associated subdivision symbols such that for a fixed $r \geq 1$*

$$b(z) = \frac{(1 + z + \cdots + z^{m-1})^r}{m^{r-1}}\, b^{[r]}(z) \quad \textit{with} \quad b^{[r]}(1) = 1$$

and $a(z)$, satisfying $a(1) = 2$, $a(-1) = 0$, is decomposed as

$$a(z) = a_{even}(z^2) + z a_{odd}(z^2)\,.$$

If one of the following conditions is satisfied,

(i) $\|\mathbf{a}_{odd}\|_1 \cdot \|S_{\mathbf{b}^{[r]}}\|_\infty < 1$, *with* $\|\mathbf{a}_{odd}\|_1 = \sum_{i\in\mathbb{Z}} |a_{2i+1}|$ *and* $\|S_{\mathbf{b}^{[r]}}\|_\infty = \max\{\sum_{i\in\mathbb{Z}} |b^{[r]}_{2i}|, \sum_{i\in\mathbb{Z}} |b^{[r]}_{2i+1}|\}$,

(ii) $\|\mathbf{v}\|_1 \cdot \|S^2_{\mathbf{b}^{[r]}}\|_\infty < 1$, *with $v(z) = a_{odd}(z) a_{odd}(z^m)$ and $\mathbf{v}$ the associated mask,*

then the de Rham-type subdivision scheme with symbol $c(z) = a_{odd}(z)\, b(z)$ is C^{r-1}.

Proof. The symbol of the de Rham-type subdivision scheme can be written as

$$c(z) = \frac{(1 + z + \cdots + z^{m-1})^r}{m^{r-1}}\, c^{[r]}(z), \quad c^{[r]}(z) = a_{odd}(z) b^{[r]}(z)\,.$$

Therefore, the associated subdivision operator $S_{\mathbf{c}}$ satisfies the necessary conditions for convergence $c(1) = m$, $c(\zeta_m^\ell) = 0$, $\ell = 1, \cdots, m-1$. Moreover, since $c_i^{[r]} = \sum_{j\in\mathbb{Z}} a_{2j+1}\, b^{[r]}_{i-j}$, in case ($i$) we have that for all $\ell = 1, \cdots, m-1$

$$\textstyle\sum_{i\in\mathbb{Z}} |c^{[r]}_{mi+\ell}| = \sum_{i\in\mathbb{Z}} \left|\sum_{j\in\mathbb{Z}} a_{2j+1}\, b^{[r]}_{mi+\ell-j}\right| \leq \|\mathbf{a}_{odd}\|_1 \cdot \|S_{\mathbf{b}^{[r]}}\|_\infty\,,$$

from which the claim follows. Differently, in case (ii), after setting

$$w(z) := b(z)b(z^m) = \frac{\left(\sum_{i=0}^{m-1} z^i\right)^r \left(\sum_{j=0}^{m-1} z^{mj}\right)^r}{m^{2(r-1)}}\, b^{[r]}(z)\, b^{[r]}(z^m),$$

we consider the subdivision scheme $S_{\mathbf{q}}$ of arity m^2 having symbol

$$q(z) = v(z)w(z) = \frac{\left(\sum_{i=0}^{m-1} z^i\right)^r \left(\sum_{j=0}^{m-1} z^{mj}\right)^r}{m^{2(r-1)}}\, v(z)\, w^{[r]}(z)$$

with $w^{[r]}(z) = b^{[r]}(z)\, b^{[r]}(z^m)$, $w^{[r]}(1) = 1$. Thus, if $\|\mathbf{v}\|_1 \cdot \|S^2_{\mathbf{b}^{[r]}}\|_\infty = \|\mathbf{v}\|_1 \cdot \|S_{\mathbf{w}^{[r]}}\|_\infty < 1$, in view of the same arguments used to show (i), we can conclude that the subdivision scheme $S_{\mathbf{q}}$ of arity m^2 is C^{r-1}, that is the subdivision scheme $S_{\mathbf{c}}$ of arity m is C^{r-1} too. □

We conclude the discussion concerning the convergence and smoothness properties of de Rham-type subdivision schemes with a useful proposition dealing with the interesting case when $a_{odd}(z)$ is a smoothing factor for the m-ary subdivision scheme $S_{\mathbf{b}}$. As observed in Remark 2, being $a(z)$ primal, the highest power of $a_{odd}(z)$ turns out to be odd. Since the proof of this result is very simple and well-known in the theory of subdivision, it is omitted.

Proposition 5. *Let $a(z)$, $b(z)$ be the symbols of the schemes $S_{\mathbf{a}}$, $S_{\mathbf{b}}$ satisfying the hypotheses of Proposition 4 that are in common to the $(i)-(ii)$ cases. Moreover, let $a(z)$ be such that, for a fixed $s \geq 1$,*

$$a_{odd}(z) = z^{-\frac{s(m-1)+1}{2}} \frac{(1+z+\cdots+z^{m-1})^s}{m^s},$$

where $s(m-1)$ is odd due to the fact that $S_{\mathbf{a}}$ is primal.
If there exists $L \geq 1$ such that $\|S^L_{\mathbf{b}^{[r]}}\|_\infty < 1$, then the de Rham-type subdivision scheme with symbol $c(z) = a_{odd}(z)\, b(z)$ is C^{r+s-1}.

4. Applications and examples

Aim of this section is to exploit the strategy proposed in Section 2 to construct new dual approximating subdivision schemes of arity m, and to show that the generation and reproduction properties of any de Rham-type subdivision scheme $S_{\mathbf{c}}$ are related with those of the two primal schemes $S_{\mathbf{a}}$, $S_{\mathbf{b}}$ used for its construction. In some cases, the special choice of these primal schemes even allows to increase the degree of polynomial reproduction of the corresponding dual de Rham-type scheme. Of course, the reproduction properties of the de Rham-type scheme $S_{\mathbf{c}}$ could be also analyzed by checking directly the algebraic conditions in [8] on the associated symbol $c(z)$. However, the understanding of the link between the properties of $c(z)$ and the symbols of the two primal schemes used to construct it, is certainly more interesting.
In particular, in the next two propositions we illustrate how to choose the primal schemes in order to construct a de Rham-type subdivision scheme reproducing polynomials up to degree 3. While the first proposition provides the mutual relationship that, for any choice of m, the derivatives of the symbols of the two primal schemes should satisfy, the second proposition identifies the algebraic conditions that, for an even m, the symbol of the m-ary scheme $S_{\mathbf{b}}$ has to satisfy when $a_{odd}(z)$ is its smoothing factor.

Proposition 6. *Let $S_{\mathbf{b}}$ be a primal m-ary subdivision scheme reproducing linear polynomials with respect to the parameter $\tau_b = 0$ and generating cubic polynomials. Assume also that*

$$a_{odd}(1) = 1, \ (D^1 a_{odd}(z))|_{z=1} = -\frac{1}{2}.$$

Then the de Rham subdivision scheme $S_{\mathbf{c}}$ with symbol $c(z) = a_{odd}(z)b(z)$ reproduces linear polynomials with respect to the parameter $\tau_c = -\frac{1}{2}$. Moreover, if the second and third derivatives of $b(z)$ and $a_{odd}(z)$ are such that

$$(D^2 b(z))|_{z=1} = m\left(\frac{3}{4} - (D^2 a_{odd}(z))|_{z=1}\right),$$

$$(D^3 b(z))|_{z=1} = -m\left(\frac{3}{4} + \frac{3}{2}(D^2 a_{odd}(z))|_{z=1} + (D^3 a_{odd}(z))|_{z=1}\right),$$

the de Rham subdivision scheme $S_{\mathbf{c}}$ reproduces cubic polynomials with respect to the parameter $\tau_c = -\frac{1}{2}$.

Proposition 7. *Let $S_{\mathbf{b}}$ be a primal m-ary (with m even) subdivision scheme reproducing linear polynomials with respect to the parameter $\tau_b = 0$, and denote by (i) and (ii) the following conditions on the second and third derivative of the symbol $b(z)$:*

(i) $(D^2 b(z))|_{z=1} = \frac{m}{12}(1-m^2)$,

(ii) $(D^2 b(z))|_{z=\zeta_m^j} = 0$, $j = 1, \cdots, m-1$ *and* $(D^3 b(z))|_{z=1} = -\frac{m}{4}(1-m^2)$.

Assuming that

$$a_{odd}(z) = z^{-\frac{m}{2}} \frac{1+z+\cdots+z^{m-1}}{m} = m^{-1} z^{-\frac{m}{2}} \frac{1-z^m}{1-z},$$

if condition (i) is satisfied then the de Rham subdivision scheme $S_{\mathbf{c}}$ with symbol $c(z) = a_{odd}(z)b(z)$ reproduces quadratic polynomials with respect to the parameter $\tau_c = -\frac{1}{2}$, while if conditions (i) and (ii) are both satisfied, then it reproduces also cubic polynomials.

The proofs of Propositions 6 and 7 are technical (essentially based on the use of Leibniz rule) but trivial, and therefore omitted. In the next subsections these two results will be exploited to analyze the reproduction properties of the derived de Rham-type subdivision schemes.

4.1. Dual binary de Rham-type subdivision schemes

We start by reading dual binary B-spline subdivision schemes (even-degree B-splines) as de Rham-type subdivision schemes originated from a linear B-spline and a primal (odd-degree) B-spline. In fact, for $k \geq 0$, taking

$$a(z) = z^{-1}\frac{(1+z)^2}{2} \quad \text{and} \quad b(z) = z^{-(k+1)}\frac{(1+z)^{2k+2}}{2^{2k+1}},$$

the dual de Rham-type subdivision scheme has symbol

$$c(z) = a_{odd}(z)\, b(z) = z^{-(k+2)}\frac{(1+z)^{2k+3}}{2^{2k+2}}, \tag{9}$$

which is the symbol of the C^{2k+1} degree-$(2k+2)$ B-spline. From Proposition 4-(i), since $\|\mathbf{a}_{odd}\|_1 \cdot \|S_{\mathbf{b}^{[2k+1]}}\|_\infty = \frac{1}{2}$, we know that $S_{\mathbf{c}}$ is at least C^{2k}. On the other hand, since $a_{odd}(z) = z^{-1}\,\frac{1+z}{2}$ is a smoothing factor, by Proposition 5 we have that $S_{\mathbf{c}}$ is actually C^{2k+1}. In addition, from Proposition 3 we know that $S_{\mathbf{c}}$ reproduces Π_1 with respect to the shift parameter $\tau_c = -\frac{1}{2}$.

Other interesting examples of de Rham-type subdivision schemes are the dual ν-point ($\nu \geq 3$) subdivision schemes in [20]. They can be obtained combining the linear B-spline symbol with a parameter dependent symbol, given respectively by

$$a(z) = z^{-1}\frac{(1+z)^2}{2},$$

and

$$b(z) = z^{-(\nu-1)}\left(\frac{1+z}{2}\right)^{2\nu-4} \frac{-(2\nu-3)+2(2\nu+1)z-(2\nu-3)z^2}{4},$$

with $S_{\mathbf{a}}$ and $S_{\mathbf{b}}$ both reproducing Π_1 with respect to the shift parameter $\tau = 0$. Since, for $v(z) = a_{odd}(z)a_{odd}(z^2)$, it results $\|\mathbf{v}\|_1 \cdot \|S^2_{\mathbf{b}^{[\nu-2]}}\|_\infty < 1$, the dual scheme $S_{\mathbf{c}}$ with symbol

$$c(z) = a_{odd}(z)\, b(z) = z^{-\nu}\left(\frac{1+z}{2}\right)^{2\nu-3} \frac{-(2\nu-3)+2(2\nu+1)z-(2\nu-3)z^2}{4}, \tag{10}$$

for any $\nu \geq 3$ is $C^{\nu-3}$ and reproduces Π_1 (at least) in view of Proposition 4-(ii) and Proposition 3, respectively. Yet, once again $a_{odd}(z) = z^{-1}\,\frac{z+1}{2}$ is a smoothing factor, so that due to Proposition 5 $S_{\mathbf{c}}$ is indeed $C^{\nu-2}$ for all $\nu \geq 3$. We conclude by observing that, since $(D^2 b(z))|_{z=1} = -\frac{1}{2}$, then in view of Proposition 7 (with $m = 2$) $S_{\mathbf{c}}$ reproduces Π_2. Moreover, since when $\nu \geq 4$ the subdivision scheme $S_{\mathbf{b}}$ generates polynomials of degree $2\nu - 5 \geq 3$ and $(D^3 b(z))|_{z=1} = \frac{3}{2}$, applying Proposition 7 (with $m = 2$) we are able to show that the de Rham subdivision scheme $S_{\mathbf{c}}$ reproduces Π_3 with respect to the shift parameter $\tau_c = -\frac{1}{2}$ whenever $\nu \geq 4$.
In Figure 2 the basic limit functions of the dual ν-point de Rham-type subdivision schemes corresponding to (10) with $\nu = 3, 4$ are illustrated.

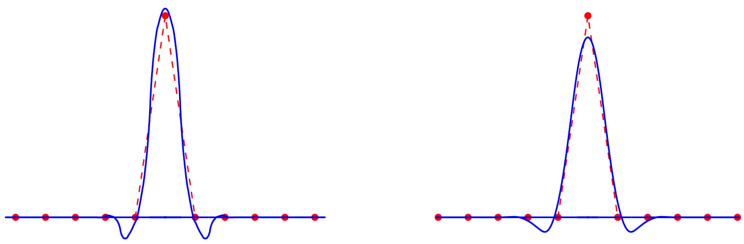

Figure 2: Basic limit function of the dual binary ν-point de Rham-type subdivision scheme having symbol (10) with $\nu = 3$ (left) and $\nu = 4$ (right).

As a last interesting example, we show how to exploit the de Rham approach to define a new family of dual 4-point subdivision schemes depending on two free parameters. More precisely, starting from the primal schemes $S_{\mathbf{a}}$ and $S_{\mathbf{b}}$ having symbols

$$a(z) = z^{-1}\,\frac{(1+z)^2}{2}$$

and

$$b(z) = z^{-3}\left(\frac{1+z}{2}\right)^2 \frac{1}{4}\Big(-16wz^4 - (2v-3)z^3 + 2(2v+1+16w)z^2 - (2v-3)z - 16w\Big), \quad v, w \in \mathbb{R},$$

respectively, being $a_{odd}(z) = z^{-1}\,\frac{1+z}{2}$, we obtain

$$c(z) = a_{odd}(z)\,b(z) = z^{-4}\left(\frac{1+z}{2}\right)^3 \frac{1}{4}\Big(-16wz^4 - (2v-3)z^3 + 2(2v+1+16w)z^2 - (2v-3)z - 16w\Big), \quad v, w \in \mathbb{R}. \tag{11}$$

Since $S_{\mathbf{a}}$ and $S_{\mathbf{b}}$ both reproduce Π_1 with respect to the shift parameter $\tau = 0$, in view of Proposition 3, $S_{\mathbf{c}}$ also reproduces Π_1 with respect to the shift parameter $\tau = -\frac{1}{2}$, independently of the value of v and w. As to higher reproduction capabilities, we fix one of the two free parameters in such a way that the resulting one-parameter family presents interesting features. First, for $v = 3-32w$, condition (i) of Proposition 7 ($m = 2$) is satisfied, and therefore $S_{\mathbf{c}}$ reproduces Π_2, even if $S_{\mathbf{b}}$ reproduces only Π_1. With this choice of the parameter v the de Rham subdivision scheme coincides with the one-parameter family of dual 4-point schemes proposed in [4]. If we further assume $w = \frac{5}{64}$, then condition (ii) of Proposition 7 ($m = 2$) is satisfied too, and $S_{\mathbf{c}}$ reproduces Π_3. This family member coincides exactly with the dual 4-point scheme proposed in [14], whose basic limit function is displayed in Figure 2(right). Concerning the regularity, when $v = 3-32w$ $S_{\mathbf{c}}$ is C^2 for all $w \in \left(\frac{1}{192}(35-\sqrt{433}),\ \frac{1}{128}(-13+3\sqrt{97})\right)$. In fact, with w in this range, introducing the symbol $v(z) = a_{odd}(z)a_{odd}(z^2)$, the inequality $\|\mathbf{v}\|_1 \cdot \|S^2_{\mathbf{b}^{[2]}}\|_\infty < 1$ is verified and thus, taking into account that $a_{odd}(z)$ is a smoothing factor, we can use Proposition 4-(ii) and Proposition 5. We now fix $v = \frac{1}{2}(3-32w)$ such that $S_{\mathbf{b}}$ reduces to the interpolatory 4-point scheme with free parameter proposed in [15], which is C^1 for $w \in (0, 0.19273)$ [19]. As a consequence, by means of Proposition 5, the de Rham scheme $S_{\mathbf{c}}$ is C^2 for w in the same range. The only dual 4-point scheme of this one-parameter family being capable of reproducing quadratic polynomials is obtained by $w = \frac{3}{32}$. In fact, this is the only value such that condition (i) of Proposition 7 is satisfied. Note that this value of w is inside the range $(0, 0.19273)$, and so the resulting scheme is C^2. In Figure 3(left) we illustrate the basic limit function of such a scheme.

As a last case study we set $v = \frac{1}{2}$. Whenever $w \in \left(-\frac{3}{16}, \frac{1}{16}\right)$, by means of Propositions 4-(i) and 5 we can prove that the resulting de Rham-type subdivision scheme is a one-parameter family of C^3 dual 4-point subdivision schemes. In Figure 3(right) the basic limit function corresponding to the choice $v = \frac{1}{2}$ and $w = -\frac{1}{8}$ is illustrated. As already shown in general, this one-parameter family reproduces Π_1. The only way of reproducing polynomials of degree higher than 1 is setting $w = \frac{5}{64}$. As previously pointed out, this choice provides the dual 4-point scheme in [14] which is Π_3-reproducing, but only C^2.

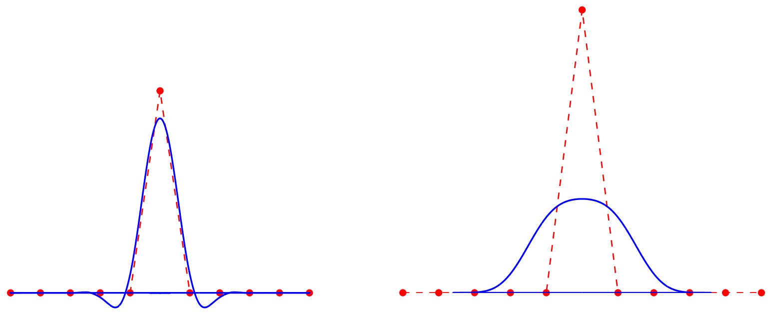

Figure 3: Basic limit function of the dual binary 4-point de Rham-type subdivision scheme having symbol (11) with $w = \frac{3}{32}$, $v = 0$ (left) and $v = \frac{1}{2}$, $w = -\frac{1}{8}$ (right).

4.2. Dual ternary de Rham-type subdivision schemes

In the ternary setting we start by applying the de Rham strategy to the primal schemes associated to binary linear B-splines and ternary B-splines of degree $k \geq 0$, having symbols

$$a(z) = z^{-1}\frac{(z+1)^2}{2} \quad \text{and} \quad b(z) = z^{-(k+1)}\frac{(z^2+z+1)^{k+1}}{3^k},$$

respectively. The associated subdivision schemes $S_{\mathbf{a}}$ and $S_{\mathbf{b}}$ reproduce Π_1 only, with respect to the parameter $\tau = 0$. The dual de Rham subdivision scheme resulting from them has symbol

$$c(z) = a_{odd}(z)\, b(z) = z^{-(k+2)}\frac{1}{2\, 3^k}(z+1)(z^2+z+1)^{k+1}\,, \tag{12}$$

and, due to Proposition 4-(i), since $\|\mathbf{a}_{odd}\|_1 \cdot \|S_{\mathbf{b}^{[k]}}\|_\infty < 1$ we already know that $S_{\mathbf{c}}$ is C^{k-1}. But, though $a_{odd}(z)$ is not a smoothing factor for the ternary scheme $S_{\mathbf{b}}$, it plays somehow the same role, and $S_{\mathbf{c}}$ turns out to be actually C^k (in fact, the subdivision scheme associated to the symbol $\frac{c(z)\, z^{k+2}\, 3^k}{(z^2+z+1)^{k+1}} = \frac{1}{2}(z+1)$ is certainly contractive). Moreover, from Proposition 3 we are able to show that $S_{\mathbf{c}}$ is Π_1-reproducing with respect to the shift parameter $\tau_c = -\frac{1}{2}$.
In Figure 4 we plot the basic limit function of the dual ternary de Rham-type subdivision schemes having symbol (12) with $k = 1, 2, 3$.

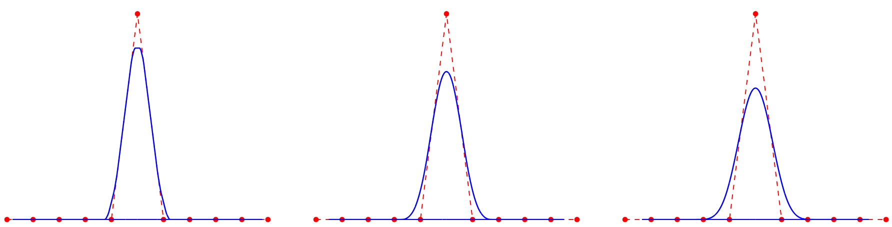

Figure 4: Basic limit function of the dual ternary de Rham-type subdivision schemes having symbol (12) with $k = 1$ (left), $k = 2$ (center), $k = 3$ (right).

As a second example, we apply the de Rham strategy to the binary primal subdivision scheme $S_{\mathbf{a}}$ having symbol

$$a(z) = z^{-3}(z^2+z+1)\left(\alpha z^4 + \frac{1}{2}z^3 - 2\alpha z^2 + \frac{1}{2}z + \alpha\right), \quad \alpha \in \mathbb{R},$$

and the ternary primal subdivision scheme $S_{\mathbf{b}}$ associated to cubic B-splines, whose symbol is given by

$$b(z) = z^{-4}\frac{(z^2+z+1)^4}{27}\,.$$

It is not difficult to see that $S_{\mathbf{a}}$ is C^0 for $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and that

$$a_{odd}(z) = z^{-2}\left(\alpha + \left(\frac{1}{2} - \alpha\right)z + \left(\frac{1}{2} - \alpha\right)z^2 + \alpha z^3\right) = z^{-2}\,(z+1)\left(\alpha + \left(\frac{1}{2} - 2\alpha\right)z + \alpha z^2\right)$$

satisfies $a_{odd}(1) = 1$, $(D^1 a_{odd}(z))|_{z=1} = -\frac{1}{2}$ for all $\alpha \in \mathbb{R}$. On the other hand, as it is well-known, $S_{\mathbf{b}}$ is C^2 and Π_1-reproducing. The resulting de Rham-type dual subdivision scheme is a new ternary dual scheme with symbol

$$c(z) = a_{odd}(z)b(z) = z^{-6}\frac{(z^2+z+1)^4}{27}\,(z+1)\left(\alpha + \left(\frac{1}{2} - 2\alpha\right)z + \alpha z^2\right). \tag{13}$$

In view of Proposition 4-(i), since $\|\mathbf{a}_{odd}\|_1 \cdot \|S_{\mathbf{b}^{[3]}}\|_\infty < 1$, we know that $S_{\mathbf{c}}$ is C^2 for all $\alpha \in \mathbb{R}$, and due to Proposition 3 we are able to show that $S_{\mathbf{c}}$ is Π_1-reproducing, at least. Yet, suitably setting the parameter α, we can achieve higher smoothness and higher degrees of polynomial reproduction. For example, for any $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, the de Rham subdivision scheme $S_{\mathbf{c}}$ is still Π_1-reproducing but C^3-continuous because $\frac{c(z)\,z^6\,27}{(z^2+z+1)^4} = (z+1)(\alpha + \left(\frac{1}{2} - 2\alpha\right)z + \alpha z^2)$ is contractive. Differently, for $\alpha = -\frac{35}{48}$, the de Rham scheme has symbol

$$c(z) = z^{-6}\frac{1}{1296}(z+1)(-35z^2 + 94z - 35)(z^2+z+1)^4,$$

which is C^2-continuous and Π_3-reproducing. This can be shown by using Proposition 6 with $m = 3$.
Figure 5 shows the basic limit functions of the dual ternary de Rham-type subdivision schemes having symbol (13) with $\alpha = \frac{1}{32}$ and $\alpha = -\frac{35}{48}$, respectively.

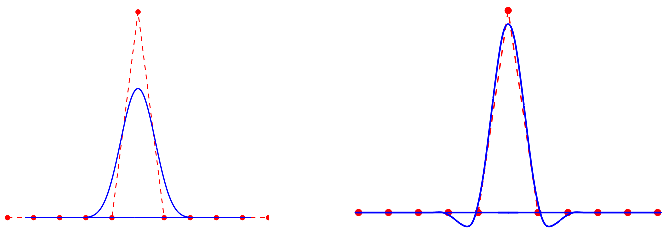

Figure 5: Basic limit function of the dual ternary de Rham-type subdivision scheme having symbol (13) with $\alpha = \frac{1}{32}$ (left) and $\alpha = -\frac{35}{48}$ (right).

4.3. Dual quaternary de Rham-type subdivision schemes

This last subsection is devoted to the construction of quaternary dual de Rham-type subdivision schemes. In this case, we observe that if the final goal is the construction of a subdivision scheme $S_{\mathbf{c}}$ which is smoother than the two primal schemes $S_{\mathbf{a}}$, $S_{\mathbf{b}}$ it is built upon, we can select $a(z)$ so that $a_{odd}(z)$ is a smoothing factor for the quaternary scheme $S_{\mathbf{b}}$. This is possible by taking

$$a(z) = z^{-3}\Big(\frac{1}{4}z^6 + \alpha z^5 + \frac{1}{4}z^4 + \beta z^3 + \frac{1}{4}z^2 + \gamma z + \frac{1}{4}\Big),$$

with $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha + \beta + \gamma = 1$. With the above observation in mind, we select $\gamma = \alpha$ and $\beta = 1 - 2\alpha$, so obtaining

$$a(z) = z^{-3}\frac{1}{4}(z+1)^2\Big(z^4 + 2(2\alpha-1)z^3 + 4(1-2\alpha)z^2 + 2(2\alpha-1)z + 1\Big), \quad \alpha \in \mathbb{R},$$

that is

$$a_{odd}(z) = z^{-2}\frac{1}{4}(1 + z + z^2 + z^3) = \frac{1}{4}z^{-2}(z+1)(z^2+1).$$

Next we choose $b(z)$ as

$$b(z) = \frac{1}{32}z^{-6}(z+1)^4(z^2+1)^3\Big(2(1-w)z^2 + (4w-3)z + 2(1-w)\Big), \quad \text{with} \quad w \in \mathbb{R}.$$

The resulting dual quaternary de Rham-type subdivision scheme has symbol

$$c(z) = \frac{1}{128}z^{-8}(z+1)^5(z^2+1)^4\Big(2(1-w)z^2 + (4w-3)z + 2(1-w)\Big), \quad w \in \mathbb{R}, \tag{14}$$

and thus coincides with the family of approximating schemes recently proposed in [23]. Applying Proposition 4-(i), since $\|\mathbf{a}_{odd}\|_1 \cdot \|S_{\mathbf{b}^{[2]}}\|_\infty < 1$ for all $w \in (-\frac{3}{2}, \frac{5}{2})$ and $\|\mathbf{a}_{odd}\|_1 \cdot \|S_{\mathbf{b}^{[3]}}\|_\infty < 1$ for all $w \in (0, \frac{3}{2})$, we can show that $S_{\mathbf{c}}$ is C^1 for all $w \in (-\frac{3}{2}, \frac{5}{2})$ and C^2 for all $w \in (0, \frac{3}{2})$. Due to the fact that $a_{odd}(z)$ is a smoothing factor for the quaternary scheme $S_{\mathbf{b}}$, the de Rham scheme $S_{\mathbf{c}}$ turns out to be actually C^2 for $w \in (-\frac{3}{2}, \frac{5}{2})$ and C^3 for $w \in (0, \frac{3}{2})$, in view of Proposition 5. Moreover, since $a_{odd}(1) = 1$, $(D^1 a_{odd}(z))|_{z=1} = -\frac{1}{2}$ and $S_{\mathbf{b}}$ is Π_1-reproducing for all $w \in \mathbb{R}$, due to Proposition 6 $S_{\mathbf{c}}$ is at least Π_1-reproducing. Furthermore, for $w = \frac{37}{16}$ $S_{\mathbf{b}}$ generates quadratic polynomials and $(D^2 b(z))|_{z=1} = -5$, $(D^3 b(z))|_{z=1} = 15$. Thus, using Proposition 7 with $m = 4$ we have that $S_{\mathbf{c}}$ is even Π_3-reproducing.
In Figure 6 we display the basic limit function of the de Rham scheme with symbol (14) corresponding to the choice $w = \frac{1}{2}$ and $w = \frac{37}{16}$.

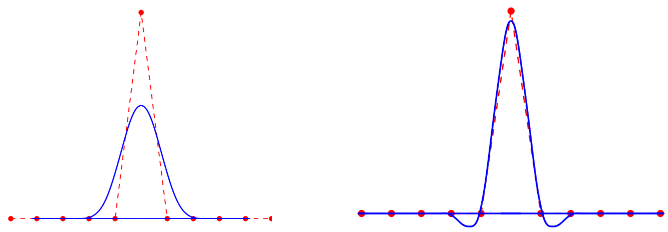

Figure 6: Basic limit function of the dual quaternary de Rham-type subdivision scheme having symbol (14) with $w = \frac{1}{2}$ (left) and $w = \frac{37}{16}$ (right).

5. Conclusions

In this paper we generalize the de Rham transform of a binary subdivision scheme presented in [12], to construct dual approximating subdivision schemes of arity m via the step-by-step application of a primal m-ary subdivision operator followed by a primal binary one, and successively by a decimation step aimed at discarding all even-indexed points from the obtained sequence. Dual de Rham-type subdivision schemes generated by this construction are usually smoother than the two primal schemes they are built upon, in exchange of a slight increase of the support width. This is particularly true when we start from interpolatory subdivision schemes, as shown in the numerical examples. Furthermore, de Rham-type subdivision schemes can also achieve a degree of polynomial reproduction that is higher than that of the two primal schemes they are built upon, by simply requiring that the symbols associated to the given primal schemes satisfy certain algebraic properties.
We conclude by mention that the proposed de Rham-type strategy could also be easily applied in the non-stationary setting to construct dual approximating non-stationary subdivision schemes, for example starting from known primal interpolatory non-stationary subdivision schemes [6, 9].

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