A Constructive Algebraic Strategy for Interpolatory Subdivision Schemes Induced by Bivariate Box splines

Costanza Conti [*], Luca Gemignani [†], Lucia Romani [‡]

September 21, 2012

Abstract

This paper describes an algebraic construction of bivariate interpolatory subdivision masks induced by three-directional box spline subdivision schemes. Specifically, given a three-directional box spline, we address the problem of defining a corresponding interpolatory subdivision scheme by constructing an appropriate correction mask to convolve with the three-directional box spline mask. The proposed approach is based on the analysis of certain polynomial identities in two variables and leads to interesting new interpolatory bivariate subdivision schemes.

Key words: Interpolatory subdivision schemes, Subdivision symbols, Bezout equation, Bivariate polynomial, Matrix polynomial, Box splines.

2010 Mathematics Subject Classification: 65F05; 68W30; 65D05; 41A15

1 Introduction

Interpolatory subdivision schemes are useful practical tools to generate graphs of functions, curves and surfaces interpolating given sets of points. As any *subdivision scheme*, they consist in the repeated application of linear rules determining successive refinements of coarse initial meshes and converging to continuous (interpolatory) limits. In the "standard" bivariate situation, the refinement rules are simple average rules with the average coefficients defining the so called *subdivision mask*, a finite sequence of real numbers hereafter denoted by $\mathbf{a} = \{a_\alpha, \ \alpha \in \mathbb{Z}^2\}$. In the interpolatory case, the mask $\mathbf{a}$ satisfies $a_{2\alpha} = \delta_{\alpha,0}$, for all $\alpha \in \mathbb{Z}^2$. The subdivision mask $\mathbf{a}$ is associated with the bivariate Laurent polynomial, or *subdivision symbol*,

$$a(z_1, z_2) = \sum_{\alpha \in \mathbb{Z}^2} a_\alpha \boldsymbol{z}^\alpha, \quad \boldsymbol{z}^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}, \quad z_1, \, z_2 \in \mathbb{C} \,,$$

whose algebraic properties translate into analytical properties of the corresponding subdivision scheme and of its limit (see [2], [3], [10], [21] and references therein). For example, a convergent subdivision scheme is interpolatory if and only if its symbol satisfies

$$a(z_1, z_2) + a(-z_1, z_2) + a(z_1, -z_2) + a(-z_1, -z_2) = 4 \,,$$

or a "shifted" version of it.

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[*]Dipartimento di Energetica "Sergio Stecco", Università di Firenze, Viale Morgagni 40/44, 50134 Firenze, Italy. tel.: +39-0554796713 - fax: +39-0554224137 - email: costanza.conti@unifi.it

[†]Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy. tel.: +39-0502213291 - fax: +39-0502213224 - email: gemignan@dm.unipi.it

[‡]Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Via R. Cozzi 53, 20125 Milano, Italy. tel.: +39-0264485735 - fax: +39-0264485705 - email: lucia.romani@unimib.it

This is actually the feature that we are going to exploit in this paper to provide an algebraic procedure for constructing an interpolatory bivariate symbol from a given non–interpolatory one. In particular, in this paper we confine our attention to the case of a three-directional box spline symbol *i.e.*,

$$a(z_1, z_2) = 4 \frac{(1 + z_1)^{k_1}}{2^{k_1}} \frac{(1 + z_2)^{k_2}}{2^{k_2}} \frac{(1 + z_1 z_2)^{k_3}}{2^{k_3}}, \quad k_i > 0, \ i = 1, 2, 3,$$

since, in view of the results in [5], three-directional symbols are building blocks of any reasonable bivariate subdivision scheme. This subject was started in [19] and further developed in [20] where interpolatory subdivision masks induced by bivariate box splines were considered for the first time. There, the regularity and the interpolatory property of the underlying refinable function as well as the convergence of the corresponding subdivision scheme are established only for three-directional box splines of type $B_{k,k,k}$ belonging to the smoothness class C^{2k-2}. Later, the same problem was investigated by Jia in [15], where existence and uniqueness of interpolatory masks induced by multi-dimensional box splines of type $B_{k,k,k}$ is discussed, though no constructive strategies are given. In this paper, a family of interpolatory masks induced by a more general box spline B_{k_1,k_2,k_3} is considered.

At the core of our approach is the idea of looking for a bivariate polynomial correction $m(z_1, z_2)$ such that the modified symbol $\hat{a}(z_1, z_2) = a(z_1, z_2)m(z_1, z_2)$ fulfills the necessary condition $\hat{a}(1, 1) = 4$ and the interpolatory condition

$$\hat{a}(z_1, z_2) + \hat{a}(-z_1, z_2) + \hat{a}(z_1, -z_2) + \hat{a}(-z_1, -z_2) = 4z_1^{n_1} z_2^{n_2}, \tag{1.1}$$

for suitable even integers $n_1, \ n_2$. More specifically, our strategy may determine a *family* of bivariate polynomial corrections

$$m_\omega(z_1, z_2) = m(z_1, z_2) + \omega k(z_1, z_2), \quad \omega \in \mathbb{R},$$

with $m(z_1, z_2)$ as above and $k(z_1, z_2)$ a "kernel" polynomial such that $k(1, 1) = 0$ and $\tilde{a}(z_1, z_2) = a(z_1, z_2)k(z_1, z_2)$ satisfies

$$\tilde{a}(z_1, z_2) + \tilde{a}(-z_1, z_2) + \tilde{a}(z_1, -z_2) + \tilde{a}(-z_1, -z_2) = 0.$$

In one respect this paper is therefore a continuation of [6], [7] where the unidimensional case is treated. In fact, in [6], [7] it is shown that, under some reasonable assumptions on the degree of the correction polynomials and under some primality assumptions on the symbol of the given univariate non-interpolatory scheme, all the admissible corrections can be determined by solving a set of polynomial equations that are equivalent in a matrix setting to inverting a certain Hurwitz matrix defined by the coefficients of the given symbol. But, on the other hand, it is clear that the bivariate extension of the results in [6], [7] is not straightforward at all, since the solution of the analogue bivariate polynomial equation through the use of Hilbert Nullstellensatz generally provides high degree solutions of poor applicative value.

To circumvent this difficulty, in this paper we follow a different approach motivated by the work of Rabarison and de Villiers [12] on the characterization of interpolatory bivariate mask symbols. The proposed strategy proceeds by splitting the correction symbol $m(z_1, z_2)$ into two components which can be individually characterized as solutions of Bezout-like equations for univariate polynomials with coefficients over an integral domain. The results in [6], [7] can thus be applied for both the analysis of the equations and possibly the computation of the components. Also, the introduction of the real parameter ω and of the family of kernel polynomials $k(z_1, z_2)$ greatly extends the flexibility of the resulting algebraic construction by allowing the computation of solutions with specific requirements such as symmetry, low degree, polynomial reproduction, etc., that are important in applications. In other words specific values of the free parameter ω can be set in order the interpolatory subdivision scheme satisfies extra conditions than the one induced by the interpolatory mask. Moreover, our strategy can be easily extended to the case where a symbol factor is not of box spline type as, for example, to

$$a(z_1, z_2) = 4 \frac{(1 + z_1)^{k_1}}{2^{k_1}} \frac{(1 + z_2)^{k_2}}{2^{k_2}} b(z_1, z_2), \quad k_i > 0, \ i = 1, 2,$$

with $b(1,1) = 1$, $b(-1; z_2) \neq 0$ and $b(z_1, -1) \neq 0$ for at least one z_1, $z_2 \in \mathbb{C}$, a case which is presently under investigation.

To conclude, we mention that, as in previous works, the complete convergence and regularity analysis of the subdivision scheme associated to the constructed interpolatory mask is here conducted a posteriori due to the non linearity of the condition the correction symbol has to satisfy. Nevertheless, the condition $m(1,1) = 4$ guarantees, at least, the existence of an associated compactly supported distribution which is refinable with respect to the interpolatory mask. At last, we also point out that there are necessary and sufficient conditions for convergence and smoothness analysis of a subdivision scheme in terms of the given mask (see, [10], [14], [16], [17], [19], for example). Once we have introduced the algebraic machinery for the solution of (1.1) these techniques can be applied for the analisys of the resulting subdivision schemes as well as for the selection of specific solutions among the ones of the algebraic equation (1.1). Some examples of this approach are provided at the end of this paper where we *essentially* use the results in [10] for the convergence and smoothness analysis of some novel schemes.

The paper is organized as follows. Section 2 starts with a brief summary of the strategy proposed in [7] to determine admissible corrections for a univariate non-interpolatory symbol to become interpolatory. Then, it continues with the description of the construction of an interpolatory bivariate symbol induced by a three-directional box spline symbol. Applications to specific box spline symbols are illustrated in Section 3 where we show that the proposed algebraic construction leads to interesting new interpolatory bivariate subdivision schemes analyzed in full detail. Section 4 presents a further alternative constructive approach for the determination of the polynomial correction. Conclusions and future work are drawn in Section 5.

2 The algebraic construction

To make the paper self contained we start by briefly summarizing the univariate strategy proposed in [7] to determine admissible corrections for a non-interpolatory symbol to become interpolatory. The univariate results that we recall in the sequel will be repeatedly used in the bivariate construction.

2.1 The univariate framework

In [7] the set of minimal–degree univariate polynomials that are eligible to convert a non-interpolatory subdivision symbol $c(z) \in \mathbb{R}[z]$ into an interpolatory one is fully characterized. In particular, it is shown that these polynomials are obtained by solving a set of structured Bezout equations that are equivalent, in a matrix setting, to inverting a Hurwitz matrix defined by the coefficients of the given symbol.

Proposition 2.1. [7] *Let $c(z) = c_0 + c_1 z + \ldots c_k z^k \in \mathbb{R}[z]$ be a polynomial of degree k relatively prime with $c(-z)$. Let $\mathcal{H}^- \in \mathbb{R}^{k \times k}$ and $\mathcal{H}^+ \in \mathbb{R}^{k \times k}$ be the Hurwitz matrices defined by*

$$
\mathcal{H}^- := \begin{bmatrix}
c_1 & c_0 & 0 & \ldots & \ldots & \ldots \\
c_3 & c_2 & c_1 & c_0 & 0 & \ldots \\
c_5 & c_4 & c_3 & \ldots & & \ldots \\
\vdots & \vdots & \vdots & \vdots & & \ldots \\
\vdots & \vdots & \vdots & \vdots & & \ldots \\
c_{2(k-1)+1} & c_{2(k-1)} & c_{2(k-1)-1} & \ldots & & \ldots
\end{bmatrix},
$$

$$\mathcal{H}^+ := \begin{bmatrix} c_0 & 0 & \dots & \dots & \dots \\ c_2 & c_1 & c_0 & 0 & \dots \\ c_4 & c_3 & c_2 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ c_{2(k-1)} & c_{2(k-1)-1} & c_{2(k-1)-2} & \dots & \dots \end{bmatrix},$$

where $c_j = 0$ if $j > k$. Then $\mathcal{H}^-$ and $\mathcal{H}^+$ are invertible and, moreover, the polynomials $p_i^-(z)$ and $p_i^+(z)$, with coefficients given by the entries of the i-th column $(1 \le i \le k)$ of $(\mathcal{H}^-)^{-1}$ and $(\mathcal{H}^+)^{-1}$ are respectively the unique polynomials of degree less than k such that

$$c(z)p_i^-(z) - c(-z)p_i^-(-z) = 2\,z^{2i-1}, \qquad 1 \le i \le k, \tag{2.1}$$

and

$$c(z)p_i^+(z) + c(-z)p_i^+(-z) = 2\,z^{2i-2}, \qquad 1 \le i \le k. \tag{2.2}$$

Observe that the solution of the Bezout equations (2.1) and (2.2) is a linear problem which is customarily reformulated into a matrix setting as the solution of a Sylvester resultant linear system of order $2k$. For the very special case considered here the $k \times k$ Hurwitz matrices $\mathcal{H}^-$ and $\mathcal{H}^+$ encompass the properties of the Sylvester-based formulation by providing a simplified approach. This can also be extended to deal with polynomials having coefficients over an integral domain rather than a field.

2.2 The bivariate framework

Let us start investigating the problem of constructing a bivariate interpolatory symbol given the bivariate polynomial associated with the (approximating) three-directional subdivision symbol

$$a(z_1, z_2) = 2^{2-k_1-k_2-k_3}(1 + z_1)^{k_1}(1 + z_2)^{k_2}(1 + z_1 z_2)^{k_3}, \quad z_1, z_2 \in \mathbb{C}, \tag{2.3}$$

for some integers $k_1, k_2, k_3 \in \mathbb{N}$. Our problem consists of finding (if it exists) a correction $m(z_1, z_2)$ that is a bivariate polynomial such that the modified symbol

$$\hat{a}(z_1, z_2) = a(z_1, z_2)\,m(z_1, z_2)$$

fulfills the interpolatory condition

$$\hat{a}(z_1, z_2) + \hat{a}(-z_1, z_2) + \hat{a}(z_1, -z_2) + \hat{a}(-z_1, -z_2) = 4z_1^{n_1} z_2^{n_2}, \tag{2.4}$$

for suitable even integers n_1 and n_2. As we are interested in finding a correction of degree (in some specified sense) as low as possible, we impose the additional condition that $0 \le n_1 \le 2(k_1 - 1)$ and $0 \le n_2 \le 2(k_2 - 1)$. Such interval ranges may be translated by considering shifted solutions. The modified symbol $\hat{a}(z_1, z_2)$ can also be expressed as

$$\hat{a}(z_1, z_2) = 2^{2-k_1-k_2}(1 + z_1)^{k_1}(1 + z_2)^{k_2}\hat{b}(z_1, z_2), \tag{2.5}$$

where

$$\hat{b}(z_1, z_2) = \frac{(1 + z_1 z_2)^{k_3}}{2^{k_3}}m(z_1, z_2).$$

Let us consider the polynomial $h(z_1, z_2)$ defined by

$$h(z_1, z_2) = \hat{a}(z_1, z_2) + \hat{a}(z_1, -z_2), \quad z_1, z_2 \in \mathbb{C}.$$

From (2.4) it follows that this polynomial should satisfy

$$h(z_1, z_2) + h(-z_1, z_2) = 4z_1^{n_1} z_2^{n_2}, \quad z_1, z_2 \in \mathbb{C}. \tag{2.6}$$

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In addition, from (2.5) we obtain that

$$h(z_1, z_2) = 2^{2-k_1-k_2}(1 + z_1)^{k_1}\{(1 + z_2)^{k_2}\hat{b}(z_1, z_2) + (1 - z_2)^{k_2}\hat{b}(z_1, -z_2)\},$$

which gives

$$h(z_1, z_2) = 2^{2-k_1-k_2}(1 + z_1)^{k_1}g(z_1, z_2), \quad z_1, z_2 \in \mathbb{C}, \tag{2.7}$$

where

$$g(z_1, z_2) = (1 + z_2)^{k_2}\hat{b}(z_1, z_2) + (1 - z_2)^{k_2}\hat{b}(z_1, -z_2). \tag{2.8}$$

By combining (2.7) and (2.6) we find that

$$2^{-k_1-k_2}(1 + z_1)^{k_1}g(z_1, z_2) + 2^{-k_1-k_2}(1 - z_1)^{k_1}g(-z_1, z_2) = z_1^{n_1}z_2^{n_2}. \tag{2.9}$$

Since $c(z_1) = (1 + z_1)^{k_1}$ and $c(-z_1) = (1 - z_1)^{k_1}$ are relatively prime, we know from Subsection 2.1 that there exists a unique univariate polynomial $p_0(z_1) \in \mathbb{R}[z_1]$ of degree less than k_1 such that

$$2^{-k_1-k_2}(1 + z_1)^{k_1}p_0(z_1) + 2^{-k_1-k_2}(1 - z_1)^{k_1}p_0(-z_1) = z_1^{n_1}. \tag{2.10}$$

The general solution of (2.9) can thus be obtained from the solution $p_0(z_1)$ of (2.10) by linearity. In fact we observe that the bivariate polynomial $\hat{g}(z_1, z_2) = g(z_1, z_2) - z_2^{n_2}p_0(z_1)$ satisfies the homogeneous equation

$$2^{-k_1-k_2}(1 + z_1)^{k_1}\hat{g}(z_1, z_2) + 2^{-k_1-k_2}(1 - z_1)^{k_1}g(-z_1, z_2) = 0,$$

which says that

$$\hat{g}(z_1, z_2) = g(z_1, z_2) - z_2^{n_2}p_0(z_1) = (1 - z_1)^{k_1}t(z_1, z_2), \tag{2.11}$$

where $t(z_1, z_2)$ is any bivariate polynomial which is odd in z_1, that is,

$$t(z_1, z_2) + t(-z_1, z_2) = 0.$$

By expressing $t(z_1, z_2)$ as a univariate polynomial in z_2 whose coefficients are polynomials in z_1, it is shown that this property implies that these coefficients are also odd polynomials. The proof immediately follows by using interpolation techniques and the identity principle for univariate polynomials. By replacing (2.11) into (2.8) we find that

$$z_2^{n_2}p_0(z_1) + (1 - z_1)^{k_1}t(z_1, z_2) = (1 + z_2)^{k_2}\hat{b}(z_1, z_2) + (1 - z_2)^{k_2}\hat{b}(z_1, -z_2). \tag{2.12}$$

It is easy to see that the polynomial on the right hand side of (2.12) is even in z_2, and, therefore, since n_2 is an even integer, the same property should be fulfilled by $t(z_1, z_2)$. Summing up, $t(z_1, z_2)$ is a bivariate polynomial which is odd in z_1 and even in z_2 and, hence, it can be expressed as

$$t(z_1, z_2) = \sum_{j=1}^{\ell} p_j(z_1)z_2^{2(j-1)},$$

where $p_j(z_1)$, $1 \leq j \leq \ell$, are suitable odd polynomials and $\ell \in \mathbb{N}$, $\ell \geq k_2$.

Thus (2.12) can be rewritten as

$$z_2^{n_2}p_0(z_1) + (1 - z_1)^{k_1}\sum_{j=1}^{\ell} p_j(z_1)z_2^{2(j-1)} = (1 + z_2)^{k_2}\hat{b}(z_1, z_2) + (1 - z_2)^{k_2}\hat{b}(z_1, -z_2). \tag{2.13}$$

The relation (2.13) is a specific instance for $z_2^{n_2}p_0(z_1) = \sum_{j=1}^{\ell} f_j(z_1)z_2^{2(j-1)}$ of the more general equation of the form

$$\sum_{j=1}^{\ell}(f_j^{(0)}(z_1) + (1 - z_1)^{k_1}p_j(z_1))z_2^{2(j-1)} =$$
$$= (1 + z_2)^{k_2}(1 + z_1z_2)^{k_3}m(z_1, z_2) + (1 - z_2)^{k_2}(1 - z_1z_2)^{k_3}m(z_1, -z_2), \tag{2.14}$$

where $f_j^{(0)}(z_1) = f_j(z_1)$ are given prescribed polynomials.

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Remark 2.2. *We observe that when k_1, $k_2 \geq 1$ from (2.10) and (2.13) follows that $\hat{b}(1,1) = 1$. Moreover, if $f_j(z_1) = 0$ (which is the case to get a "kernel" solution) we have $\hat{b}(1,1) = 0$.*

When the integer ℓ and the polynomials $p_j(z_1)$, $1 \leq j \leq \ell$ are fixed, then under some additional mild assumptions we can determine a solution $m(z_1, z_2)$ which is a polynomial in the variable z_2 with coefficients that are rational functions in the variable z_1. The main question to be addressed here is if for some suitable choices of ℓ and the polynomials $p_j(z_1)$, $1 \leq j \leq \ell$, these rational functions reduce to Laurent polynomials so that $m(z_1, z_2)$ becomes a bivariate Laurent polynomial. We summarize the ultimate computational task as follows:

> Given the polynomials $f_j^{(0)}(z_1) = f_j(z_1)$, $1 \leq j \leq k_2$, we have to determine the integer $\ell \geq k_2$ and the odd polynomials $p_j(z_1)$, $1 \leq j \leq \ell$, such that (2.14) holds for a suitable bivariate Laurent polynomial $m(z_1, z_2)$. This solution is the candidate correction.

For solving (2.14) we can rely upon the natural isomorphism

$$\mathbb{R}[z_1, z_2] \ni f(z_1, z_2) \to \hat{f}(z_2) \in \mathbb{R}[z_1][z_2],$$

mapping bivariate polynomials to univariate polynomials with (univariate) polynomial coefficients. Specifically, it is worth noting that $(1 + z_1 z_2)^{k_3}$ can be viewed as a univariate polynomial in z_2 of degree k_3 with coefficients in $\mathbb{R}[z_1]$, that is,

$$(1 + z_1 z_2)^{k_3} = \sum_{i=0}^{k_3} b_i(z_1) z_2^i, \quad \text{with} \quad b_i(z_1) = \binom{k_3}{i} z_1^i.$$

There follows that the univariate procedure described in Subsection 2.1 can in principle be applied to

$$c(z_2) = (1 + z_2)^{k_2}(1 + z_1 z_2)^{k_3} = \sum_{i=0}^{k_2 + k_3} g_i(z_1) z_2^i \tag{2.15}$$

where

$$g_i(z_1) = \begin{cases} \sum_{j=0}^{i} \binom{k_2}{j} \binom{k_3}{i-j} z_1^{i-j}, & i = 0, \cdots, \min\{k_2, k_3\} \\ \sum_{j=0}^{\infty} \binom{k_2}{j} \binom{k_3}{i-j} z_1^{i-j}, & i = \min\{k_2, k_3\}, \cdots, \max\{k_2, k_3\} \\ \sum_{j=i-k_3}^{k_2} \binom{k_2}{j} \binom{k_3}{i-j} z_1^{i-j}, & i = \max\{k_2, k_3\}, \cdots, k_2 + k_3 \end{cases}$$

and the binomial coefficients in the second expression are interpreted in terms of the Gamma function. We continue by exploiting the association between the pair of univariate polynomials $c(z_2)$ and $c(-z_2)$ with the Hurwitz matrices $\mathcal{H}^+$ and $\mathcal{H}^-$ of order $n = k_2 + k_3$, having entries over the integral domain $\mathbb{F} = \mathbb{R}[z_1]$. Namely, we have

$$\mathcal{H}^+ := \begin{bmatrix} g_0(z_1) & 0 & \cdots & \cdots & \cdots \\ g_2(z_1) & g_1(z_1) & g_0(z_1) & 0 & \cdots \\ g_4(z_1) & g_3(z_1) & g_2(z_1) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ g_{2(k_2+k_3-1)}(z_1) & g_{2(k_2+k_3-1)-1}(z_1) & g_{2(k_2+k_3-1)-2}(z_1) & \cdots & \cdots \end{bmatrix},$$

and

$$\mathcal{H}^- := \begin{bmatrix} g_1(z_1) & g_0(z_1) & 0 & \cdots & \cdots & \cdots \\ g_3(z_1) & g_2(z_1) & g_1(z_1) & g_0(z_1) & 0 & \cdots \\ g_5(z_1) & g_4(z_1) & g_3(z_1) & & & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \cdots \\ g_{2(k_2+k_3-1)+1}(z_1) & g_{2(k_2+k_3-1)}(z_1) & g_{2(k_2+k_3-1)-1}(z_1) & \cdots & & \cdots \end{bmatrix}.$$

And so, virtually, we can think to attack the computation of $m(z_1, z_2)$ satisfying (2.14) by using Proposition 2.1 applied to this novel Hurwitz matrix $\mathcal{H}^+ \in \mathbb{F}^{n \times n}$. The resulting approach can be summarized as follows:

The basic idea of the correction polynomial construction

1. Set $\ell = k_2 + k_3 = n$;

2. Solve $\mathcal{H}^+ \boldsymbol{x}^{(j)} = (f_j^{(0)}(z_1) + (1 - z_1)^{k_1} p_j(z_1)) \boldsymbol{e}_j$, $1 \leq j \leq \ell$ with $f_j^{(0)}(z_1)$ and $p_j(z_1)$ specified above;

3. Define the polynomials $x_j(z_2) = \sum_{i=0}^{n-1} x_i^{(j)} z_2^i$, $1 \leq j \leq \ell$;

4. Set $m(z_1, z_2) = \sum_{j=1}^{\ell} x_j(z_2)$.

It is worth observing that, from one hand, theoretical and computational problems can be encountered at step 2 where we have to solve a linear system defined by a polynomial matrix whose solutions generally lie in the quotient field of rational functions. On the other hand, the known vectors are partially undetermined since they depend on the unknown polynomials $p_j(z_1)$, $1 \leq j \leq \ell$. Our primary goal is to exploit these two issues simultaneously in such a way to find an eligible strategy for computing odd polynomials $p_j(z_1)$ corresponding with polynomial solutions of the system. In order to investigate the properties of this system we start by proving a result about the determinant of $\mathcal{H}^+ \in \mathbb{F}^{n \times n}$. The proof follows by using a suitable generalization of the techniques employed in [7] to computations with matrices having entries over an integral domain. In particular we relate $\mathcal{H}^+ \in \mathbb{F}^{n \times n}$ and $\mathcal{H}^- \in \mathbb{F}^{n \times n}$ with the Sylvester-resultant matrix $\mathcal{S} \in \mathbb{F}^{2n \times 2n}$ generated from the coefficients of $c(z_2)$ and $c(-z_2)$, that is,

$$\mathcal{S} = \begin{bmatrix} g_0(z_1) & & & g_0(z_1) & & \\ g_1(z_1) & \ddots & & -g_1(z_1) & \ddots & \\ \vdots & & g_0(z_1) & \vdots & & g_0(z_1) \\ g_n(z_1) & & \vdots & (-1)^n g_n(z_1) & & \vdots \\ & \ddots & \vdots & & \ddots & \vdots \\ & & g_n(z_1) & & & (-1)^n g_n(z_1) \end{bmatrix}.$$

Theorem 2.3. *Let $c(z_2)$ as in (2.15) with $k_2, k_3 \in \mathbb{N}$ and let $r(z_1) = \mathrm{res}(c(z_2), c(-z_2)) = \det \mathcal{S}$ be the resultant, i.e., the determinant of the Sylvester matrix of order $2(k_2 + k_3)$, of $c(z_2)$ and $c(-z_2)$. We have that*

$$r(z_1) = \pm 2^n z_1^{k_3} (\det(\mathcal{H}^+))^2.$$

Proof. Let $P_n \in \mathbb{R}^{2n \times 2n}$, $P_n = (\delta_{i,\sigma(j)})$ be the permutation matrix associated with the "perfect shuffle" permutation given by

$$\sigma \; : \; \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}, \quad \sigma(j) = \begin{cases} (j+1)/2 + n, & \text{if } j \text{ is odd}; \\ \\ j/2, & \text{if } j \text{ is even}. \end{cases}$$

Furthermore, let $G_n \in \mathbb{R}^{2n \times 2n}$ be the matrix defined by

$$G_n := \left[\begin{array}{c|c} I_n & -D_n \\ \hline D_n & I_n \end{array} \right],$$

where $D_n := \mathrm{diag}[-1, (-1)^2, \ldots, (-1)^{n-1}, (-1)^n]$. It is straightforward to verify that

$$G_n^{-1} = \frac{1}{2} \left[\begin{array}{c|c} I_n & D_n \\ \hline -D_n & I_n \end{array} \right].$$

By premultiplying the Sylvester-resultant matrix $\mathcal{S}$ by P_n we find that

$$P_n \mathcal{S} = \left[\begin{array}{c|c} \mathcal{H}^- & \mathcal{H}^- D_n \\ \hline \mathcal{H}^+ & -\mathcal{H}^+ D_n \end{array} \right].$$

Hence, there follows that

$$P_n \mathcal{S} G_n^{-1} = \left[\begin{array}{c|c} 0_n & \mathcal{H}^- D_n \\ \hline \mathcal{H}^+ & 0_n \end{array} \right].$$

From the Binet theorem this gives

$$\pm r(z_1) \left(\frac{1}{2} \right)^n = \pm z_1^{k_3} (\det(\mathcal{H}^+))^2$$

which completes the proof. $\qquad\square$

Based on Theorem 2.3 we obtain an explicit expression for the determinant of $\mathcal{H}^+ \in \mathbb{F}^{n \times n}$. First, we make use of a classical result in elimination theory to find an explicit expression for the resultant $r(z_1) = \mathrm{res}(c(z_2), c(-z_2))$.

Theorem 2.4. *Let $c(z_2)$ as in (2.15) with $k_2, k_3 \in \mathbb{N}$. For the resultant $r(z_1) = \mathrm{res}(c(z_2), c(-z_2)) \in \mathbb{R}[z_1]$, it holds that*

$$r(z_1) = \alpha \, z_1^{k_3^2} \, (1 + z_1)^{2k_2 k_3},$$

for a suitable $\alpha \in \mathbb{R} \backslash \{0\}$.

Proof. Setting $c(z_2) = (1 + z_2)^{k_2} (1 + z_1 z_2)^{k_3} = f(z_1, z_2)$, by using well-known properties of the resultant polynomial [1], it is easy to see that

$$\{z_1 \in \mathbb{C} : r(z_1) = 0\} = \{z_1 \in \mathbb{C} : \exists z_2 \in \mathbb{C}, f(z_1, z_2) = 0 = f(z_1, -z_2)\} \cup \{z_1 \in \mathbb{C} : g_{k_2 + k_3}(z_1) = z_1^{k_3} = 0\}$$

where the roots are counted with their "multiplicity" which is the sum of the corresponding intersection multiplicities including points at infinity. From this characterization we deduce that the distinct roots of $r(z_1)$ are $z_1 = 0$ and $z_1 = -1$. In order to evaluate the corresponding multiplicities we perform a perturbative analysis. Let us start considering the root $z_1 = -1$. Observe that the polynomial system

$$\begin{cases} f(-1, z_2) = 0 \\ f(-1, -z_2) = 0, \end{cases}$$

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has two roots $z_2 = 1$ and $z_2 = -1$. For the multiplicity of $(-1, 1)$ we perform the substitution $z_1 = -1 + s$ and $z_2 = 1 + t$, obtaining the system

$$\begin{cases} f_1(s,t) = f(-1+s, 1+t) = (2+t)^{k_2}[s(1+t) - t]^{k_3} = 0 \\ f_2(s,t) = f(-1+s, -1-t) = (-1)^{k_2}t^{k_2}[2+t-s(1+t)] = 0. \end{cases}$$

A small perturbation of the coefficients of the polynomials yields the system

$$\begin{cases} \widetilde{f}_1(s,t) = (2+t)^{k_2}\{[s(1+t) - t]^{k_3} - \epsilon\} = 0 \\ \widetilde{f}_2(s,t) = (-1)^{k_2}(t^{k_2} - \epsilon)[2+t-s(1+t)] = 0, \end{cases}$$

where the common root at $s = t = 0$ is spread out into $k_2 k_3$ distinct roots and, therefore, $k_2 k_3$ is the multiplicity of $(-1, 1)$. A similar analysis applies to $(-1, -1)$, which implies that $2k_2 k_3$ is the multiplicity of $z_1 = -1$. For the root $z_1 = 0$, which annihilates the leading coefficient $g_{k_2+k_3}(z_1)$ of $c(z_2)$, see (2.15), we can proceed as follows. Let us introduce the perturbed polynomial

$$\widetilde{f}(z_1, z_2) = (1 + z_2)^{k_2}[(1 + z_1 z_2)^{k_3} - \epsilon z_2^{k_3}].$$

We find that the solutions of

$$\begin{cases} f(z_1, z_2) = 0 \\ f(z_1, -z_2) = 0, \end{cases}$$

at "infinite points" for $z_1 = 0$ are replaced by k_3 solutions of

$$\begin{cases} \widetilde{f}(z_1, z_2) = 0 \\ \widetilde{f}(z_1, -z_2) = 0, \end{cases}$$

given by $z_1^{(j)} = \epsilon^{k_3^{-1}} e^{2\pi\sqrt{-1}(j-1)/k_3}$, $1 \le j \le k_3$, each of them corresponding with k_3 distinct intersection points. In this way we conclude that the multiplicity of $z_1 = 0$ is k_3^2. $\qquad\square$

By virtue of Theorem 2.3 and Theorem 2.4, we obtain the desired expression of $\mathcal{H}^+$.

Corollary 2.5. *For the matrix $\mathcal{H}^+ \in \mathbb{F}^{n\times n}$, $\mathbb{F} = \mathbb{R}[z_1]$, generated from the coefficients of $c(z_2) = (1 + z_2)^{k_2}(1 + z_2 z_3)^{k_3}$, with $c(z_2)$ as in (2.15), it holds*

$$\det \mathcal{H}^+ = \alpha\, z_1^{k_3(k_3-1)/2}\, (1 + z_1)^{k_2 k_3},$$

for a suitable $\alpha \in \mathbb{R}\backslash\{0\}$.

It is worth noting that the scaled polynomial

$$h(z_1) = \frac{\det \mathcal{H}^+}{z_1^{k_3(k_3-1)/2}} \tag{2.16}$$

satisfies the following properties:

P.1 $h(z_1)$ and $h(-z_1)$ are relatively prime;

P.2 $h(0)h(1) \ne 0$.

These properties play a fundamental role to ensure the reliability of the four steps construction described above for computing $m(z_1, z_2)$ satisfying (2.14). The possibility of choosing the odd polynomials $p_j(z_1)$ freely is exploited in the following result.

Theorem 2.6. *Let $f(z_1) \in \mathbb{R}[z_1]$ be a given polynomial. Then for the polynomial $h(z_1)$ defined as in (2.16) there exist a polynomial $m(z_1) \in \mathbb{R}[z_1]$ and an odd polynomial $p(z_1) = z_1 \widehat{p}(z_1^2) \in \mathbb{R}[z_1]$ such that*

$$m(z_1)h(z_1) = f(z_1) + z_1(1 - z_1)^{k_1}\widehat{p}(z_1^2).$$

Proof. Since from P.2 $h(z_1)$ and $z_1(1 - z_1)^{k_1}$ are relatively prime, by using the extended Euclidean algorithm we can find polynomials $\widetilde{m}(z_1)$ and $t(z_1)$ such that

$$\widetilde{m}(z_1)h(z_1) - z_1(1 - z_1)^{k_1}t(z_1) = f(z_1). \tag{2.17}$$

Further, from Proposition 2.1 and P.1 it follows that the following equation

$$\rho(z_1)h(z_1) - \rho(-z_1)h(-z_1) = t(-z_1) - t(z_1), \tag{2.18}$$

is also solvable. This implies that $\rho(z_1)h(z_1) + t(z_1)$ is an even polynomial, that is,

$$\rho(z_1)h(z_1) + t(z_1) = \widehat{p}(z_1^2).$$

Now let us define

$$m(z_1) = \widetilde{m}(z_1) + z_1(1 - z_1)^{k_1}\rho(z_1).$$

We have

$$\begin{aligned} m(z_1)h(z_1) - f(z_1) \quad &= \widetilde{m}(z_1)h(z_1) + z_1(1 - z_1)^{k_1}\rho(z_1)h(z_1) - f(z_1) \\[2mm] &= z_1(1 - z_1)^{k_1}(t(z_1) + \rho(z_1)h(z_1)) \\[2mm] &= z_1(1 - z_1)^{k_1}\widehat{p}(z_1^2). \end{aligned}$$

$\square$

This theorem can be used to guarantee that the solutions of the systems at step 2 of the proposed construction are indeed Laurent polynomials. Recall that for an invertible square matrix A the adjoint matrix $B = Adj(A)$ satisfies $B = (\det A)A^{-1}$. If A is a polynomial matrix then B is a polynomial matrix and, therefore, if we require that

$$f_j^{(0)}(z_1) + (1 - z_1)^{k_1}p_j(z_1) = f_j^{(0)}(z_1) + z_1(1 - z_1)^{k_1}\widehat{p}_j(z_1^2) = m_j(z_1)h(z_1), \quad 1 \le j \le \ell, \tag{2.19}$$

then the solution $\boldsymbol{x}^{(j)}$ can be expressed as

$$\boldsymbol{x}^{(j)} = \frac{m_j(z_1)}{z_1^{k_3(k_3-1)/2}}Adj(\mathcal{H}^+)\boldsymbol{e}_j, \quad 1 \le j \le \ell,$$

which is a vector whose entries are Laurent polynomials. By incorporating this modification into our strategy we arrive at the following.

The revised idea of the correction polynomial construction

1. Set $\ell = k_2 + k_3 = n$ and $h(z_1)$ as defined in (2.16);

2. Compute the polynomials $p_j(z_1)$ and $m_j(z_1)$, $1 \le j \le \ell$, satisfying (2.19);

3. Solve $\mathcal{H}^+\boldsymbol{x}^{(j)} = (f_j^{(0)}(z_1) + (1 - z_1)^{k_1}p_j(z_1))\boldsymbol{e}_j = h(z_1)m_j(z_1)\boldsymbol{e}_j$, $1 \le j \le \ell$;

4. Define the polynomials $x_j(z_2) = \sum_{i=0}^{n-1} x_i^{(j)} z_2^i$, $1 \le j \le \ell$;

5. Set $m(z_1, z_2) = \sum_{j=1}^{\ell} x_j(z_2)$.

In light of Remark 2.2 we easily see that the described construction defines a correction $m(z_1, z_2)$ satisfying the necessary condition $m(1,1) = 1$ as formalized in the following

Proposition 2.7. *If $\sum_{j=1}^{\ell} f_j^{(0)}(1) = 2^{k_2+k_3}$ and $k_1, k_2 > 1$ then the polynomial $m(z_1, z_2)$ is such that $m(1,1) = 1$.*

Remark 2.8. *It is important to remark that the degree of polynomial reproduction of the interpolatory subdivision scheme associated with a three-directional box spline obtained by the algebraic strategy just described, is at least the degree of polynomial generation of the box spline subdivision scheme it is built upon. This result simply follows from the algebraic construction of the interpolatory scheme and from the Leibniz rule for the derivatives of a product of symbols (see, for example, [4]).*

Remark 2.9. *From a computational point of view the solution of the linear systems at step 3 raises several interesting issues. The approach considered above is based on the multiplication of the inverse matrix by the known vector. On the contrary, the typical numerical strategy employs recursive factorization techniques. An alternative recursive method exploiting the factorization properties of Hurwitz matrices is described in Section 4. Furthermore, the case where $f_j^{(0)}(z_1) = 0$, $1 \le j \le \ell$, providing the so-called "kernel" solutions deserves special attention. Although both solution methods –recursive and inverse-based– can be used, a complete description of the kernel of the associated Hurwitz matrices requires more sophisticated tools like the Smith normal form of polynomial matrices [13]. Applications of this form are presented in the next section.*

In the next section we discuss the properties of some interpolatory schemes generated by means of this procedure.

3 Application examples

This section is devoted to the application of the described algebraic procedure to four different three-directional box spline symbols. In particular, we are going to derive the polynomial corrections needed to construct the interpolatory symbols corresponding to the B_{111}, B_{221}, B_{222} (and B_{331}) box splines. For the corresponding interpolatory subdivision schemes a convergence and smoothness analysis is also conducted together with a polynomial reproduction investigation. For all related theoretical results and definitions we refer the reader to [4] (multivariate extension of the results contained in the papers [8], [9]).

The convergence and smoothness analysis of the considered novel schemes follows here from some results in [10] exploiting functional properties of the associated symbols. Alternative approaches based on a linear algebra technology exploiting the properties of the eigen-decomposition of certain matrices directly constructed from the associated masks have been proposed in [19]. Although these techniques are useful to enlighten several properties of the considered schemes, our preference here is motivated by the intent to focus on the polynomial framework.

3.1 Interpolatory subdivision schemes associated with the Courant hat function

We start with the simplest three-directional box spline symbol (the so-called Courant hat function)

$$a(z_1, z_2) = (1 + z_1)(1 + z_2)\frac{1 + z_1 z_2}{2},$$

which is already interpolatory, though with an associated basic limit function which is only continuous. Therefore, the described construction is used to obtain an interpolatory scheme with increased regularity. In particular, we look for a polynomial correction $m(z_1, z_2)$ such that the modified symbol

$a(z_1, z_2)m(z_1, z_2)$ fulfills the interpolatory condition (2.4) with $n_1 = n_2 = 2$. It is straightforward to see that for $c(z) = (1+z)^2$, the general solution (2.11) can be expressed as

$$g(z_1, z_2) = z_2^2 p_0(z_1) + (1 - z_1)t(z_1, z_2) = 2z_1 z_2^2 + (1 - z_1)t(z_1, z_2),$$

for a suitable bivariate polynomial $t(z_1, z_2)$ which is odd with respect to z_1. Since $\ell = 1 + 1 = 2$, we are now going to determine a bivariate polynomial $m(z_1, z_2)$ of the form

$$m(z_1, z_2) = \sum_{j=0}^{1} m_j(z_1)z_2^j,$$

satisfying (2.14) for

$$f_1^{(0)}(z_1) = 0, \quad f_2^{(0)}(z_1) = 2z_1.$$

The coefficients of the polynomial $c(z_2) = (1 + z_2)\dfrac{1 + z_1 z_2}{2}$ are used to generate the Hurwitz matrix

$$\mathcal{H}^+ = \begin{bmatrix} 1/2 & 0 \\ z_1/2 & 1/2 + 1/2\, z_1 \end{bmatrix},$$

whose determinant is $h(z_1) = 4^{-1}(1 + z_1)$ in accordance with Corollary 2.5. It is easily observed that

$$\mathcal{H}^+ \begin{bmatrix} 0 \\ 2z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ -z_1(1 - z_1) \end{bmatrix},$$

which gives $m(z_1, z_2) = z_1 z_2$.

Some other interesting solutions can be obtained by linearly combining $m(z_1, z_2)$ with bivariate polynomials satisfying (2.14) with $f_j^{(0)}(z_1) = 0$, $1 \le j \le \ell$. A simple modification is obtained by considering an admissible known vector of the form

$$\boldsymbol{f} = [z_1(1 - z_1)\{(1 + z_1)(1 - z_1)p_0(z_1)\}, z_1(1 - z_1)\{(1 + z_1)(1 - z_1)p_1(z_1)\}]^T,$$

where $p_0(z_1)$ and $p_1(z_1)$ are suitable even polynomials. Then it follows that the solution of the linear system $\mathcal{H}^+ \boldsymbol{x} = \boldsymbol{f}$ satisfies

$$\begin{aligned}
\boldsymbol{x} &= ((1 + z_1)(\mathcal{H}^+)^{-1}) \left[z_1(1 - z_1)^2 p_0(z_1), z_1(1 - z_1)^2 p_1(z_1) \right]^T \\[2mm]
&= 4Adj(\mathcal{H}^+) \left[z_1(1 - z_1)^2 p_0(z_1), z_1(1 - z_1)^2 p_1(z_1) \right]^T \\[2mm]
&= 2z_1(1 - z_1)^2 \left[(1 + z_1)p_0(z_1), p_1(z_1) - z_1 p_0(z_1) \right].
\end{aligned}$$

This gives a parameterized family of symbols defined by

$$\hat{a}_\omega(z_1, z_2) = a(z_1, z_2)\{z_1 z_2 + \omega\, 2z_1(1 - z_1)^2[(1 + z_1)p_0(z_1) + z_2(p_1(z_1) - z_1 p_0(z_1))]\}.$$

More general modifications can be obtained by computing the Smith normal form [13] of a Hurwitz matrix. For instance, let us consider the Hurwitz matrix $\mathcal{H}^+ \in \mathbb{F}^{6\times 6}$ generated by the coefficients of the polynomial $c(z_2) = (1 + z_2)\dfrac{1 + z_1 z_2}{2}$, namely

$$\mathcal{H}^+ = 2^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ z_1 & 1 + z_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & z_1 & 1 + z_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & z_1 & 1 + z_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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Then by computing the Smith normal form of $\mathcal{H}^+$ we get that

$$U\mathcal{H}^+V = \mathcal{D},$$

where U and V are unimodular matrices given by

$$U = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ -2z_1 & 2 & 0 & 0 & 0 & 0 \\ 2z_1^2 & -2z_1 & 2 & 0 & 0 & 0 \\ -2z_1^3 & 2z_1^2 & -2z_1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1-z_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & z_1+z_1^2 & -1-z_1 \\ 0 & 0 & 0 & 1 & -z_1^2 & z_1 \end{bmatrix},$$

and $\mathcal{D}$ is a diagonal matrix defined by

$$\mathcal{D} = \mathrm{diag}[1, 1, 1, 1+z_1, 0, 0].$$

In this way we obtain that the general solution of

$$\mathcal{H}^+\boldsymbol{x} = \boldsymbol{f} = [0, (1-z_1)^2(1+z_1)z_1p_0(z_1), 0, (1-z_1)^2(1+z_1)z_1p_1(z_1), 0, 0]^T,$$

for suitable even polynomials $p_0(z_1)$ and $p_1(z_1)$ can be determined as

$$\boldsymbol{x} = V\{\mathrm{diag}[1, 1, 1, (1+z_1)^{-1}, 0, 0]\, U\, \boldsymbol{f} + q_0(z_1)\boldsymbol{e}_5 + q_1(z_1)\boldsymbol{e}_6\},$$

where $q_0(z_1)$ and $q_1(z_1)$ are any polynomials. This means that $m_1(z_1, z_2) = x_1 + x_2z_2 + x_3z_2^2 + x_4z_2^3 + x_5z_2^4 + x_6z_2^5$ satisfies (2.14) with $f_j^{(0)}(z_1) = 0$, $1 \le j \le 6$. If $p_0(z_1) = 1$ and $p_1(z_1) = z_1^2$, for a certain choice of $q_0(z_1)$ and $q_1(z_1)$, the symbol

$$\hat{a}_\omega(z_1, z_2) = a(z_1, z_2)\{z_1z_2 + \omega m_1(z_1, z_2)\}$$

defines the Butterfly scheme on the three-directional grid [11] which is C^1 for $\omega \in (0, 0.08)$. In general, the convergence of the subdivision scheme with symbol $\hat{a}_\omega(z_1, z_2)$ depends on the specific properties of ω and $m_1(z_1, z_2)$. Nevertheless, in view of Remark 2.8, whenever the scheme is C^0, it certainly reproduces at least linear polynomials.

3.2 Interpolatory subdivision schemes associated with the B_{221} box spline

In this second example we consider the non-interpolatory box spline symbol defined by

$$a(z_1, z_2) = 2^{-2}(1+z_1)^2(1+z_2)^2\frac{1+z_1z_2}{2},$$

and we look for a polynomial correction $m(z_1, z_2)$ such that the modified symbol fulfills the interpolatory condition (2.4) with $n_1 = n_2 = 2$. Since for $c(z) = (1+z)^2$ we have

$$\mathcal{H}^+ = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix},$$

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we find that
$$(\mathcal{H}^+)^{-1} = \begin{bmatrix} 1 & 0 \\ -1/2 & 1/2 \end{bmatrix},$$

and, therefore, for $n_1 = 2$ the general solution (2.11) can be expressed as
$$g(z_1, z_2) = z_2^2 p_0(z_1) + (1 - z_1)^2 t(z_1, z_2) = 4z_1 z_2^2 + (1 - z_1)^2 t(z_1, z_2),$$

for a suitable bivariate polynomial $t(z_1, z_2)$ which is odd with respect to z_1. Since $\ell = 2 + 1 = 3$, we are now going to determine a bivariate polynomial $m(z_1, z_2)$ of the form

$$m(z_1, z_2) = \sum_{j=0}^{2} m_j(z_1) z_2^j,$$

satisfying (2.14) for
$$f_1^{(0)}(z_1) = 0, \quad f_2^{(0)}(z_1) = 4z_1, \quad f_3^{(0)}(z_1) = 0.$$

The coefficients of the polynomial $c(z_2) = (1 + z_2)^2 \dfrac{1 + z_1 z_2}{2}$ are used to generate the Hurwitz matrix

$$\mathcal{H}^+ = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/2 + z_1 & 1 + z_1/2 & 1/2 \\ 0 & z_1/2 & 1/2 + z_1 \end{bmatrix},$$

whose determinant is $h(z_1) = 4^{-1}(1 + z_1)^2$ in accordance with Corollary 2.5 (i.e., $k_3(k_3 - 1) = 0$ and $k_2 k_3 = 2$). For the solution of (2.19) we find that

$$4z_1 = 4(1 + z_1)^2 \frac{z_1(2 - z_1)}{4} - z_1(1 - z_1)^2(-2 - z_1),$$

which gives
$$\widetilde{m}(z_1) = 4z_1(2 - z_1), \quad t(z_1) = -2 - z_1.$$

Then we have that $\rho(z_1) = 2$ fulfills (2.18) and, therefore, we get
$$m(z_1) = 4z_1(2 - z_1) + 2z_1(1 - z_1)^2 = 2z_1(5 - 4z_1 + z_1^2).$$

Hence, we obtain
$$\boldsymbol{x}^{(2)} = m(z_1) Adj(\mathcal{H}^+) \boldsymbol{e}_2,$$

which gives
$$\boldsymbol{x}^{(2)} = m(z_1) \frac{1}{4} \begin{bmatrix} 2(1 + z_1)^2 & 0 & 0 \\ -(1 + 2z_1)^2 & 1 + 2z_1 & -1 \\ z_1(1 + 2z_1) & -z_1 & 2 + z_1 \end{bmatrix} \boldsymbol{e}_2.$$

In this way we arrive at
$$m(z_1, z_2) = \frac{z_1(5 - 4z_1 + z_1^2)}{4}\{(1 + 2z_1)z_2 - z_1 z_2^2\}.$$

It can be easily verified that
$$\hat{a}(z_1, z_2) = a(z_1, z_2) m(z_1, z_2),$$

is such that
$$\hat{a}(z_1, z_2) + \hat{a}(-z_1, z_2) + \hat{a}(z_1, -z_2) + \hat{a}(-z_1, -z_2) = 8z_1^2 z_2^2.$$

As described in the previous example the properties of the computed solution can be improved by performing a linear combination with some polynomials in the "kernel" of (2.14). In this case a simpler approach comes by reducing the coefficients of $m(z_1, z_2)$ modulo $(1 - z_1)^2$. From

$$(5 - 4z_1 + z_1^2)(1 + 2z_1) = (-3 + 2z_1)(1 - z_1)^2 + (8 - 2z_1),$$

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and
$$-z_1(5 - 4z_1 + z_1^2) = (2 - z_1)(1 - z_1)^2 - 2,$$

we deduce that
$$m(z_1, z_2) = z_1 z_2(4 - z_1 - z_2) + m_1(z_1, z_2),$$

where
$$m_1(z_1, z_2) = z_1 z_2(1 - z_1)^2\{(-3 + 2z_1) + (2 - z_1)z_2\}.$$

It is straightforward to verify that $m_1(z_1, z_2)$ satisfies (2.14) with $f_j^{(0)}(z_1) = 0$, $1 \leq j \leq 3$, and, hence, we conclude that

$$m_2(z_1, z_2) = \frac{1}{2}z_1 z_2(4 - z_1 - z_2), \tag{3.1}$$

also defines a suitable correction so that $\hat{a}(z_1, z_2) = a(z_1, z_2)m_2(z_1, z_2)$ fulfills

$$\hat{a}(z_1, z_2) + \hat{a}(-z_1, z_2) + \hat{a}(z_1, -z_2) + \hat{a}(-z_1, -z_2) = 4z_1^2 z_2^2.$$

Similarly, by employing the shifted solution $p_0(z_1) = z_1(2z_1 - 1)$ it is found that the polynomial

$$m_3(z_1, z_2) = \frac{1}{2}z_1 z_2(4z_1 z_2 - z_1 - z_2),$$

modifies the symbol in such a way to satisfy the interpolatory condition (2.4) with $n_1 = n_2 = 4$. As noticed in [6] (though in the univariate setting), any affine combination of the corrections $m_2(z_1, z_2)$ and $m_3(z_1, z_2)$ suitably shifted, also generates an interpolatory scheme. In particular the symbol

$$\begin{aligned}
\hat{a}(z_1, z_2) &= \tfrac{1}{4}a(z_1, z_2)\left(m_2(z_1, z_2) + z_1^{-2}z_2^{-2}m_3(z_1, z_2)\right) \\
&= \tfrac{1}{4}a(z_1, z_2)\left(4 - z_1^{-1} - z_2^{-1} + 4z_1 z_2 - z_1^2 z_2 - z_1 z_2^2\right),
\end{aligned} \tag{3.2}$$

is such that

$$\hat{a}(z_1, z_2) + \hat{a}(-z_1, z_2) + \hat{a}(z_1, -z_2) + \hat{a}(-z_1, -z_2) = 4z_1^2 z_2^2 \quad \text{and} \quad \hat{a}(z_1, z_2) = \hat{a}(z_2, z_1).$$

The mask of the new interpolatory subdivision scheme is

$$\hat{\mathbf{a}} = \frac{1}{32}\begin{pmatrix}
0 & 0 & 0 & -1 & -2 & -1 & 0 \\
0 & 0 & -1 & 0 & 2 & 0 & -1 \\
0 & -1 & 4 & 17 & 16 & 2 & -2 \\
-1 & 0 & 17 & 32 & 17 & 0 & -1 \\
-2 & 2 & 16 & 17 & 4 & -1 & 0 \\
-1 & 0 & 2 & 0 & -1 & 0 & 0 \\
0 & -1 & -2 & -1 & 0 & 0 & 0
\end{pmatrix}, \tag{3.3}$$

and the corresponding basic limit function is displayed in Figure 1.

Remark 3.1. *It is worthwhile to observe that $m_2(z_1, z_2)$ in (3.1) is the only solution with symmetric coefficients and degrees (in each variable) less than or equal to 2, a fact that can be proved via the Smith form. Our computational experience with many other symbols of the form B_{k_1, k_1, k_3} indicates that a similar property holds more generally. For instance, in the case $B_{3,3,1}$ after reduction our procedure generates the correction $m(z_1, z_2)$ given by*

$$m(z_1, z_2) = \frac{1}{8}z_1 z_2\left(2 - 9z_2 + 3z_2^2 + z_1(-9 + 34z_2 - 9z_2^2) + z_1^2(3 - 9z_2 + 2z_2^2)\right),$$

which is the only solution with symmetric coefficients and degrees (in each variable) less than or equal to $\ell - 1 = 3 + 1 - 1 = 3$. This property extends the uniqueness results in [19] for symbols of the form $B_{k,k,k}$. A theoretical proof of the property as well as the analysis of the associated interpolatory schemes is an ongoing work.
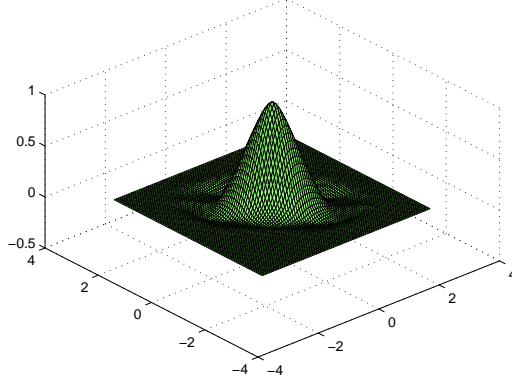
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Figure 1: Graph of the basic limit function of the subdivision scheme with mask in (3.3).

3.2.1 Analysis of the symmetric scheme

The interpolatory subdivision scheme with symbol (3.2) is symmetric, C^1 and reproduces polynomials up to degree 2. For its smoothness analysis we make use of [10, Theorem 4.26] exploiting the existence of the three-directional factors $(1 + z_1)^2(1 + z_2)^2(1 + z_1 z_2)$ in the subdivision symbol.

Proposition 3.2. *The interpolatory subdivision scheme with symbol*

$$\hat{a}(z_1, z_2) = \frac{1}{32}(1 + z_1)^2(1 + z_2)^2(1 + z_1 z_2)\left(4 - z_1^{-1} - z_2^{-1} + 4z_1 z_2 - z_1^2 z_2 - z_1 z_2^2\right) ,$$

is C^1 and reproduces polynomials in Π_2.

Proof. Following [10, Section 4], to prove that the scheme with symbol $\hat{a}(z_1, z_2)$ is C^1 we show that any two of the schemes with symbols

$$2\frac{\hat{a}(z_1, z_2)}{(1 + z_1)}, \quad 2\frac{\hat{a}(z_1, z_2)}{(1 + z_2)}, \quad 2\frac{\hat{a}(z_1, z_2)}{(1 + z_1 z_2)},$$

are C^0. In particular we consider the first and the second symbols above, that is

$$\frac{1}{16}(1 + z_1)(1 + z_2)^2(1 + z_1 z_2)\left(4 - z_1^{-1} - z_2^{-1} + 4z_1 z_2 - z_1^2 z_2 - z_1 z_2^2\right) ,$$

and

$$\frac{1}{16}(1 + z_1)^2(1 + z_2)(1 + z_1 z_2)\left(4 - z_1^{-1} - z_2^{-1} + 4z_1 z_2 - z_1^2 z_2 - z_1 z_2^2\right) ,$$

and check that their difference schemes are contractive. To this purpose, for

$$b_1(z_1, z_2) = \frac{1}{16}(1 + z_1)(1 + z_2)(1 + z_1 z_2)\left(4 - z_1^{-1} - z_2^{-1} + 4z_1 z_2 - z_1^2 z_2 - z_1 z_2^2\right),$$

and

$$b_2(z_1, z_2) = \tfrac{1}{16}(1 + z_1)^2(1 + z_2)\left(4 - z_1^{-1} - z_2^{-1} + 4z_1 z_2 - z_1^2 z_2 - z_1 z_2^2\right),$$

$$b_3(z_1, z_2) = \tfrac{1}{16}(1 + z_1)(1 + z_2)^2\left(4 - z_1^{-1} - z_2^{-1} + 4z_1 z_2 - z_1^2 z_2 - z_1 z_2^2\right),$$

we verify that $b_i(z_1, z_2)b_i(z_1^2, z_2^2)$, $i = 1, 2, 3$ are indeed contractive.
The polynomial reproduction properties of the scheme follow from the polynomial generation properties of the box spline subdivision scheme $a(z_1, z_2) = \frac{1}{8}(1 + z_1)^2(1 + z_2)^2(1 + z_1 z_2)$. $\qquad\square$

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The properties of the new interpolatory subdivision scheme have been analyzed by relying on the results in [10]. Obviously, the same conclusions can be derived regardless the approach taken in the analysis. As a matter of fact, we have also generated the 49×49 matrix H determined by the mask elements (that is the 7×7 block matrix with blocks $H_{k,\ell}$, $k, \ell = 1, \ldots, 7$ defined as $H_{k,\ell}(i,j) = \hat{a}(2j - i, 2\ell - k)$, $i, j = 1, \ldots, 7$). It is easily found that this matrix has 1 as a simple eigenvalue with $(1, \cdots, 1)^T$ as its eigenvector. Hence, from [19, Theorem 1.2] we can conclude the existence of a function refinable with respect to the given mask, which is a fundamental solution for cardinal interpolation.

3.3   Interpolatory subdivision schemes associated with the B_{222} box spline

As third application example we consider the approximating box spline symbol defined by

$$a(z_1, z_2) = 2^{-2}(1 + z_1)^2 (1 + z_2)^2 \left(\frac{1 + z_1 z_2}{2} \right)^2 .$$

Again, we look for a polynomial correction $m(z_1, z_2)$ such that the modified symbol fulfills the interpolatory condition (2.4) with $n_1 = n_2 = 4$. It is found that the general solution (2.11) can be expressed as

$$g(z_1, z_2) = z_2^4 p_0(z_1) + (1 - z_1)^2 t(z_1, z_2),$$

where

$$p_0(z_1) = z_1(-1 + 2z_1 + 31z_1^2),$$

and $t(z_1, z_2)$ is a suitable bivariate polynomial which is odd with respect to z_1. Then, by using the same approach described in the previous example, after some simplifications, the correction $m(z_1, z_2)$ given by

$$m(z_1, z_2) = -\frac{z_1 z_2}{4} \left(1 + z_1 + z_2 - 10z_1 z_2 + z_1^2 z_2 + z_1 z_2^2 + z_1^2 z_2^2 \right) , \tag{3.4}$$

is computed. The mask of the new interpolatory subdivision scheme $\hat{a}(z_1, z_2) = m(z_1, z_2) a(z_1, z_2)$ is

$$\hat{\mathbf{a}} = \frac{1}{64} \begin{pmatrix} 0 & 0 & 0 & -1 & -3 & -3 & -1 \\ 0 & 0 & -3 & 0 & 6 & 0 & -3 \\ 0 & -3 & 6 & 33 & 33 & 6 & -3 \\ -1 & 0 & 33 & 64 & 33 & 0 & -1 \\ -3 & 6 & 33 & 33 & 6 & -3 & 0 \\ -3 & 0 & 6 & 0 & -3 & 0 & 0 \\ -1 & -3 & -3 & -1 & 0 & 0 & 0 \end{pmatrix} . \tag{3.5}$$

Note that the interpolatory mask in (3.5) is the one obtained for B_{222} in [19].

3.3.1   Analysis of the new interpolatory subdivision scheme

Also in this case, the algebraic construction leads to a new interesting interpolatory bivariate subdivision scheme: the interpolatory scheme with correction (3.4) is symmetric, C^1 and reproducing Π_3.

Proposition 3.3. *The interpolatory subdivision scheme with symbol*

$$\hat{a}(z_1, z_2) = \frac{1}{64} z_1 z_2 (1 + z_1)^2 (1 + z_2)^2 (1 + z_1 z_2)^2 \left(10z_1 z_2 - 1 - z_1 - z_2 - z_1^2 z_2 - z_1 z_2^2 - z_1^2 z_2^2 \right) ,$$

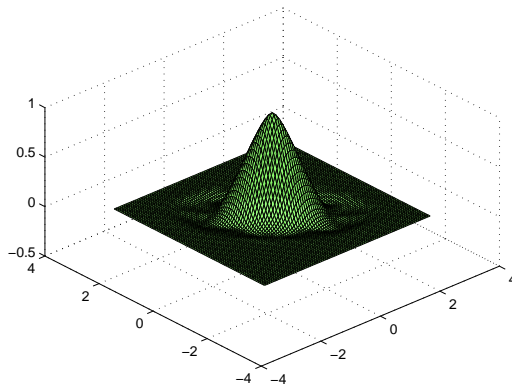*is C^1 and reproduces polynomials in Π_3.*

Figure 2: Graph of the basic limit function of the subdivision scheme with mask in (3.5).

Proof. Following [10, Section 4], to prove that the scheme with symbol $\hat{a}(z_1, z_2)$ is C^1 we show that any two of the schemes with symbols

$$2\frac{\hat{a}(z_1, z_2)}{(1 + z_1)}, \quad 2\frac{\hat{a}(z_1, z_2)}{(1 + z_2)}, \quad 2\frac{\hat{a}(z_1, z_2)}{(1 + z_1 z_2)},$$

are C^0. In particular we consider the first and the second symbols above that is

$$\frac{1}{32} z_1 z_2 (1 + z_1)(1 + z_2)^2 (1 + z_1 z_2)^2 \left(10 z_1 z_2 - 1 - z_1 - z_2 - z_1^2 z_2 - z_1 z_2^2 - z_1^2 z_2^2\right),$$

and

$$\frac{1}{32} z_1 z_2 (1 + z_1)^2 (1 + z_2)(1 + z_1 z_2)^2 \left(10 z_1 z_2 - 1 - z_1 - z_2 - z_1^2 z_2 - z_1 z_2^2 - z_1^2 z_2^2\right).$$

For these symbols we check whether their difference symbols

$$b_1(z_1, z_2) = \tfrac{1}{32} z_1 z_2 (1 + z_1)(1 + z_2)(1 + z_1 z_2)^2 \left(10 z_1 z_2 - 1 - z_1 - z_2 - z_1^2 z_2 - z_1 z_2^2 - z_1^2 z_2^2\right),$$

$$b_2(z_1, z_2) = \tfrac{1}{32} z_1 z_2 (1 + z_1)(1 + z_2)^2 (1 + z_1 z_2) \left(10 z_1 z_2 - 1 - z_1 - z_2 - z_1^2 z_2 - z_1 z_2^2 - z_1^2 z_2^2\right), \quad (3.6)$$

$$b_3(z_1, z_2) = \tfrac{1}{32} z_1 z_2 (1 + z_1)^2 (1 + z_2)(1 + z_1 z_2) \left(10 z_1 z_2 - 1 - z_1 - z_2 - z_1^2 z_2 - z_1 z_2^2 - z_1^2 z_2^2\right),$$

are contractive. Indeed it holds that the norm of $b_i(z_1, z_2) b_i(z_1^2, z_2^2)$, for all $i = 1, 2, 3$ is less than 1. The polynomial reproduction properties of the scheme follow from the polynomial generation properties of the box spline subdivision scheme $a(z_1, z_2) = \frac{1}{64}(1 + z_1)^2(1 + z_2)^2(1 + z_1 z_2)^2$. $\square$

Remark 3.4. *We remark that the scheme with mask (3.5), due to the extra smoothing factor $(1 + z_1 z_2)$, is somehow smoother than the one with mask (3.3), even if this does not emerge form the conducted regularity analysis.*

4   A recursive approach for solving Hurwitz systems over integral domains

The constructions presented in the previous section basically reduce the computation of a polynomial correction $m(z_1, z_2)$ to solving some linear systems whose coefficient matrix is a Hurwitz matrix with entries over the integral domain $\mathbb{F} = \mathbb{R}[z_1]$. The computation of the determinant of a polynomial matrix as well as the solution of a linear system can be carried out by using the Hermite normal form of the matrix [18] or some other triangularization procedure working on the integral domain $\mathbb{R}[z]$. In

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the case of Hurwitz and Sylvester matrices it is common to reinterpret matrix factorization methods in the context of a polynomial framework. In this section we provide a polynomial method for solving (2.14) as a supplementary material. The computational interest of the procedure is that it generally leads to a different solution and would make possible to exploit special properties of the symbol.

For the sake of illustration, let us start by continuing the discussion with the example presented in Subsection 3.2. Recall that our task is to compute univariate polynomials $m_j(z_1)$, $j = 0, 1, 2$, such that

$$
\begin{bmatrix}
1/2 & 0 & 0 \\
1/2 + z_1 & 1 + z_1/2 & 1/2 \\
0 & z_1/2 & 1/2 + z_1
\end{bmatrix}
\begin{bmatrix}
m_0(z_1) \\
m_1(z_1) \\
m_2(z_1)
\end{bmatrix}
=
\begin{bmatrix}
(1 - z_1)^2 z_1 p_1(z_1^2) \\
4z_1 + (1 - z_1)^2 z_1 p_2(z_1^2) \\
(1 - z_1)^2 z_1 p_3(z_1^2)
\end{bmatrix},
$$

for suitable polynomials $p_j(z_1)$, $j = 1, 2, 3$. The first equation can be satisfied by setting $m_0(z_1) = 0$ and $p_1(z_1) = 0$. This yields the reduced system

$$
\begin{bmatrix}
1 + z_1/2 & 1/2 \\
z_1/2 & 1/2 + z_1
\end{bmatrix}
\begin{bmatrix}
m_1(z_1) \\
m_2(z_1)
\end{bmatrix}
=
\begin{bmatrix}
4z_1 + (1 - z_1)^2 z_1 p_2(z_1^2) \\
(1 - z_1)^2 z_1 p_3(z_1^2)
\end{bmatrix}.
$$

The coefficient matrix on the left hand side can be reduced to a triangular form by multiplication on the right by the triangular matrix

$$
G =
\begin{bmatrix}
1 & -1/2 \\
0 & 1 + z_1/2
\end{bmatrix}.
$$

We have that

$$
\begin{bmatrix}
1 + z_1/2 & 1/2 \\
z_1/2 & 1/2 + z_1
\end{bmatrix}
G =
\begin{bmatrix}
1 + z_1/2 & 0 \\
z_1/2 & \dfrac{(1 + z_1)^2}{2}
\end{bmatrix}.
$$

Since

$$
\begin{bmatrix}
1 & -1/2 \\
0 & 1 + z_1/2
\end{bmatrix}
\begin{bmatrix}
1 + z_1/2 & 1/2 \\
0 & 1
\end{bmatrix}
= (1 + z_1/2) I_2,
$$

then we can rewrite the reduced system in the following form

$$
\begin{bmatrix}
1 + z_1/2 & 0 \\
z_1/2 & \dfrac{(1+z_1)^2}{2}
\end{bmatrix}
\begin{bmatrix}
m_1^{(1)}(z_1) \\
m_2^{(1)}(z_1)
\end{bmatrix}
= (1 + z_1/2)
\begin{bmatrix}
4z_1 + (1 - z_1)^2 z_1 p_2(z_1^2) \\
(1 - z_1)^2 z_1 p_3(z_1^2)
\end{bmatrix},
$$

where

$$
\begin{bmatrix}
m_1^{(1)}(z_1) \\
m_2^{(1)}(z_1)
\end{bmatrix}
=
\begin{bmatrix}
1 + z_1/2 & 1/2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
m_1(z_1) \\
m_2(z_1)
\end{bmatrix}.
\tag{4.1}
$$

We look for a solution of the reduced system where the polynomials $m_j^{(1)}(z_1)$, $j = 1, 2$, are such that $m_j^{(1)}(z_1) = (1 + z_1/2)\hat{m}_j^{(1)}(z_1)$, $j = 1, 2$. In this way from the first equation we obtain that

$$
(1 + z_1/2)\hat{m}_1^{(1)}(z_1) = 4z_1 + (1 - z_1)^2 z_1 p_2(z_1^2).
$$

This equation has the form (2.19) and it can be solved by setting

$$
\hat{m}_1^{(1)}(z_1) = \frac{8}{9} z_1(4 - z_1), \quad p_2(z_1) = -\frac{4}{9}.
$$

Hence, by replacing $\hat{m}_1^{(1)}(z_1)$ into the second equation it is found that

$$
\frac{(1 + z_1)^2}{2}\hat{m}_2^{(1)}(z_1) = \frac{4}{9} z_1^2(z_1 - 4) + (1 - z_1)^2 z_1 p_3(z_1^2).
$$

This is again a polynomial equation of the form (2.19) and a solution is given by

$$
\hat{m}_2^{(1)}(z_1) = z_1\left(-1 + \frac{4}{9} z_1 - \frac{1}{9} z_1^2\right), \quad p_3(z_1) = \frac{1}{2} + \frac{1}{18} z_1.
$$

Now we are in position to determine the polynomial coefficients $m_1(z_1)$ and $m_2(z_1)$. By using (4.1) we deduce that

$$(1 + z_1/2) \begin{bmatrix} \hat{m}_1^{(1)}(z_1) \\ \hat{m}_2^{(1)}(z_1) \end{bmatrix} = \begin{bmatrix} 1 + z_1/2 & 1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1(z_1) \\ m_2(z_1) \end{bmatrix}.$$

Hence, we find that

$$(1 + z_1/2) \begin{bmatrix} \hat{m}_1^{(1)}(z_1) \\ \hat{m}_2^{(1)}(z_1) \end{bmatrix} = \begin{bmatrix} 1 + z_1/2 & 1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1 + z_1/2 \end{bmatrix} \begin{bmatrix} \hat{m}_1^{(1)}(z_1) \\ \hat{m}_2^{(1)}(z_1) \end{bmatrix},$$

which gives

$$\begin{bmatrix} 1 & -1/2 \\ 0 & 1 + z_1/2 \end{bmatrix} \begin{bmatrix} \hat{m}_1^{(1)}(z_1) \\ \hat{m}_2^{(1)}(z_1) \end{bmatrix} = \begin{bmatrix} m_1(z_1) \\ m_2(z_1) \end{bmatrix},$$

and, therefore,

$$\begin{bmatrix} m_1(z_1) \\ m_2(z_1) \end{bmatrix} = z_1 \begin{bmatrix} \dfrac{73}{18} - \dfrac{10z_1}{9} + \dfrac{z_1^2}{18} \\ -1 - \dfrac{z_1}{18} + \dfrac{z_1^2}{9} - \dfrac{z_1^3}{18} \end{bmatrix}.$$

This solution can be simplified by operating modulo $(1 - z_1)^2$. The remainder of $m_1(z_1)/z_1$ and $m_2(z_1)/z_1$ by $(1 - z_1)^2$ is $4 - z_1$ and -1, respectively. Therefore, in this way we arrive at the same solution $m_2(z_1, z_2)$ determined in Subsection 3.2.

The approach described above for solving (2.14) in the case of the B_{221} box spline symbol can be suitably generalized and reinterpreted in a polynomial framework.

Let $\mathcal{H}_0^+ \in \mathbb{F}^{n \times n}$, $\mathbb{F} = \mathbb{R}[z_1]$, $n = \ell = k_2 + k_3$, be the Hurwitz matrix generated from the coefficients of $c^{(0)}(z_2) = c(z_2) = \sum_{i=0}^{k_2+k_3} g_i(z_1)z_2^i$, namely

$$\mathcal{H}_0^+ := \begin{bmatrix} g_0(z_1) & 0 & \cdots & \cdots & \cdots \\ g_2(z_1) & g_1(z_1) & g_0(z_1) & 0 & \cdots \\ g_4(z_1) & g_3(z_1) & g_2(z_1) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ g_{2(n-1)}(z_1) & g_{2(n-1)-1}(z_1) & g_{2(n-1)-2}(z_1) & \cdots & \cdots \end{bmatrix}.$$

We have to solve a linear system

$$\mathcal{H}_0^+ \begin{bmatrix} m_0^{(0)}(z_1) \\ \vdots \\ m_{n-1}^{(0)}(z_1) \end{bmatrix} = \begin{bmatrix} f_1^{(0)}(z_1) + (1 - z_1)^2 z_1 p_1(z_1^2) \\ \vdots \\ f_n^{(0)}(z_1) + (1 - z_1)^2 z_1 p_n(z_1^2) \end{bmatrix},$$

where $f_j^{(0)}(z_1)$ are given and, moreover, $p_j(z_1)$ have to be suitably chosen in such a way to admit a polynomial solution. The first step is trying to solve the first equation having the form of (2.19), that is

$$g_0(z_1)m_0(z_1) = f_1^{(0)}(z_1) + (1 - z_1)^2 z_1 p_1(z_1^2). \tag{4.2}$$

If we succeed to do that, then we can reduce the size of the system by obtaining

$$\mathcal{H}_0^+(2 \colon n; 2 \colon n) \begin{bmatrix} m_1^{(0)}(z_1) \\ \vdots \\ m_{n-1}^{(0)}(z_1) \end{bmatrix} = \begin{bmatrix} f_2^{(1)}(z_1) + (1 - z_1)^2 z_1 p_1(z_1^2) \\ \vdots \\ f_n^{(1)}(z_1) + (1 - z_1)^2 z_1 p_n(z_1^2) \end{bmatrix}.$$

Let $G^+ \in \mathbb{F}^{2 \times 2}$ and $G^- \in \mathbb{F}^{2 \times 2}$ be the triangular matrices defined by

$$G^+ = \begin{bmatrix} 1 & -g_0(z_1) \\ 0 & g_1(z_1) \end{bmatrix}, \quad G^- = \begin{bmatrix} g_1(z_1) & g_0(z_1) \\ 0 & 1 \end{bmatrix}.$$

Observe that

$$G^+ G^- = G^- G^+ = g_1(z_1) I_2. \tag{4.3}$$

According to the parity of $n-1$ we introduce the cumulative elimination matrix $\mathcal{G}^+ \in \mathbb{F}^{(n-1) \times (n-1)}$ defined as follows: if $n-1$ is even then

$$\mathcal{G}^+ = G^+ \oplus G^+ \oplus \ldots \oplus G^+,$$

otherwise, if $n-1$ is odd

$$\mathcal{G}^+ = G^+ \oplus G^+ \oplus \ldots \oplus G^+ \oplus 1.$$

Similarly, we define $\mathcal{G}^- \in \mathbb{F}^{(n-1) \times (n-1)}$ by setting

$$\mathcal{G}^- = G^- \oplus G^- \oplus \ldots \oplus G^-,$$

or

$$\mathcal{G}^- = G^- \oplus G^- \oplus \ldots \oplus G^- \oplus g_1(z_1).$$

From (4.3) there follows that

$$\mathcal{G}^+ \mathcal{G}^- = \mathcal{G}^- \mathcal{G}^+ = g_1(z_1) I_{n-1}.$$

By using this relation we can rewrite the reduced system into the equivalent form

$$\mathcal{H}_0^+(2\colon n; 2\colon n) \mathcal{G}^+ \mathcal{G}^- \begin{bmatrix} m_1^{(0)}(z_1) \\ \vdots \\ m_{n-1}^{(0)}(z_1) \end{bmatrix} = g_1(z_1) \begin{bmatrix} f_2^{(1)}(z_1) + (1 - z_1)^2 z_1 p_1(z_1^2) \\ \vdots \\ f_n^{(1)}(z_1) + (1 - z_1)^2 z_1 p_n(z_1^2) \end{bmatrix}.$$

The matrix $\mathcal{H}_1^+ = \mathcal{H}_0^+(2\colon n; 2\colon n) \mathcal{G}^+$ is the Hurwitz matrix associated with a transformed polynomial $c^{(1)}(z_2)$ generated according to the following rules: given the even–odd decomposition of $c^{(0)}(z_2)$, i.e.,

$$c^{(0)}(z_2) = c_e^{(0)}(z_2^2) + z_2 c_o^{(0)}(z_2^2),$$

then we obtain

$$c^{(1)}(z_2) = c_o^{(0)}(z_2^2) + z_2 \frac{c_o^{(0)}(0) c_e^{(0)}(z_2^2) - c_e^{(0)}(0) c_o^{(0)}(z_2^2)}{z_2^2},$$

which gives the transformation rules

$$c_e^{(1)}(z_2) = c_o^{(0)}(z_2), \quad c_o^{(1)}(z_2) = \frac{c_o^{(0)}(0) c_e^{(0)}(z_2) - c_e^{(0)}(0) c_o^{(0)}(z_2)}{z_2}. \tag{4.4}$$

It is worth noting that the degree of $c^{(1)}(z_2)$ is at least 1 minus the degree of $c^{(0)}(z_2)$. The solution of the reduced system of order $n-1$ is denoted by

$$\begin{bmatrix} m_1^{(1)}(z_1) \\ \vdots \\ m_{n-1}^{(1)}(z_1) \end{bmatrix} := \mathcal{G}^- \begin{bmatrix} m_1^{(0)}(z_1) \\ \vdots \\ m_{n-1}^{(0)}(z_1) \end{bmatrix}.$$

Suppose that we are looking for polynomials $m_j^{(1)}(z_1)$ satisfying $m_j^{(1)}(z_1) = g_1(z_1) \hat{m}_j^{(1)}(z_1)$, $j = 1, \ldots, n-1$ and then define $m_1(z_1, z_2) = \sum_{j=1}^{n-1} \hat{m}_j^{(1)}(z_1) z_2^{j-1}$. The coefficient of this correction polynomial solves

$$\mathcal{H}_1^+ \begin{bmatrix} \hat{m}_1^{(1)}(z_1) \\ \vdots \\ \hat{m}_{n-1}^{(1)}(z_1) \end{bmatrix} = \begin{bmatrix} f_2^{(1)}(z_1) + (1 - z_1)^2 z_1 p_1(z_1^2) \\ \vdots \\ f_n^{(1)}(z_1) + (1 - z_1)^2 z_1 p_n(z_1^2) \end{bmatrix},$$

which is equivalent in a polynomial setting to the equation of reduced degree

$$\sum_{j=1}^{n-1}(f_{j+1}^{(1)}(z_1) + (1-z_1)^{k_1}p_j(z_1))z_2^{2(j-1)} = c^{(1)}(z_2)m_1(z_1,z_2) + c^{(1)}(-z_2)m_1(z_1,-z_2).$$

The transformation from (2.14) to this equation of reduced degree can be summarized as follows:

The polynomial reduction process

1. Compute $m_0(z_1)$ by solving (4.2);

2. Compute the polynomial $f_j^{(1)}(z_1) = f_j^{(0)}(z_1) - m_0(z_1)g_{2(j-1)}(z_1)$, $2 \le j \le n$;

3. Compute the coefficients of $c^{(1)}(z_2) = \displaystyle\sum_{i=0}^{k_2+k_3-1} g_i^{(1)}(z_1)z_2^i$ from (4.4).

This process can in principle be continued until we arrive at a polynomial equation in one single variable (i.e., of degree zero). After that, we have to reconstruct the coefficients of the initial correction $m_0(z_1, z_2)$ starting from the coefficients of the polynomials $m_j(z_1, z_2)$ generated step-by-step in the degree-reduction procedure. Let us suppose that $m_1(z_1, z_2)$ is known and we have to proceed with the reconstruction of $m_0(z_1, z_2)$. From

$$\mathcal{G}^-\begin{bmatrix} m_1^{(0)}(z_1) \\ \vdots \\ m_{n-1}^{(0)}(z_1) \end{bmatrix} = \begin{bmatrix} m_1^{(1)}(z_1) \\ \vdots \\ m_{n-1}^{(1)}(z_1) \end{bmatrix} = g_1(z_1)\begin{bmatrix} \hat{m}_1^{(1)}(z_1) \\ \vdots \\ \hat{m}_{n-1}^{(1)}(z_1) \end{bmatrix} = \mathcal{G}^-\mathcal{G}^+\begin{bmatrix} \hat{m}_1^{(1)}(z_1) \\ \vdots \\ \hat{m}_{n-1}^{(1)}(z_1) \end{bmatrix},$$

we deduce that

$$\begin{bmatrix} m_1^{(0)}(z_1) \\ \vdots \\ m_{n-1}^{(0)}(z_1) \end{bmatrix} = \mathcal{G}^+\begin{bmatrix} \hat{m}_1^{(1)}(z_1) \\ \vdots \\ \hat{m}_{n-1}^{(1)}(z_1) \end{bmatrix},$$

which makes possible to compute the coefficients $m_j^{(0)}$ of $m(z_1, z_2) = m_0(z_1, z_2)$ starting from the coefficients $\hat{m}_j^{(1)}$ of $m_1(z_1, z_2)$ by performing polynomial multiplications only.

5 Conclusions and future work

In this paper we devise a general algebraic strategy for the construction of interpolatory bivariate subdivision schemes induced by the three-directional box spline symbols. The proposed approach is able to generate large families of interpolatory schemes depending on the algebraic properties of the Hurwitz matrices associated with the symbols. Future work includes the systematic analysis of these schemes as well as the characterization of the properties of the Hurwitz matrices in terms of the properties of the related box spline symbols. Also, it is our intention to extend these results to the case where a symbol factor is not of box spline type.

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