Università degli Studi di Milano Bicocca
Dottorato di Ricerca in Matematica Pura e Applicata
Université Paris-Est — École Doctorale MSTIC
Laboratoire d'Analyse et de Mathématiques Appliquées – UMR CNRS 8050

ASYMPTOTIC PROPERTIES OF THE DYNAMICS NEAR STATIONARY SOLUTIONS FOR SOME NONLINEAR SCHRÖDINGER EQUATIONS

(Proprietés asymptotiques de la dynamique dans un voisinage des solutions stationnaires de certaines équations de Schrödinger)

Supervisors
Dott. Riccardo Adami *Dott. Diego Noja* *Prof. Galina Perelman*
Referees
Prof. Alexander Komech *Prof. Joachim Krieger*
Candidate
Cecilia Ortoleva

Defence Committee:

Prof. Hajer Bahouri	Université Paris-Est internal member
Prof. Dario Bambusi	External member
Prof. Alexander Komech	Referee
Prof. Joachim Krieger	Referee
Dott. Diego Noja	Supervisor
Prof. Galina Perelman	Supervisor

Thesis submitted for the degree of "Doctor Philosophiæ"
February 18, 2013 Academic Year 2012/2013

Abstract

The present thesis is devoted to the investigation of certain aspects of the large time behavior of the solutions of two nonlinear Schrödinger equations in dimension three in some suitable perturbative regimes.
The first model consist in a Schrödinger equation with a concentrated nonlinearity obtained considering a point (or contact) interaction with strength α, which consists of a singular perturbation of the Laplacian described by a selfadjoint operator H_α, and letting the strength α depend on the wave function: $i\frac{du}{dt} = H_\alpha u$, $\alpha = \alpha(u)$. It is well-known that the elements of the domain of a point interaction in three dimensions can be written as the sum of a regular function and a function that exhibits a singularity proportional to $|x - x_0|^{-1}$, where x_0 is the location of the point interaction. If q is the so-called charge of the domain element u, i.e. the coefficient of its singular part, then, in order to introduce a nonlinearity, we let the strength α depend on u according to the law $\alpha = -\nu|q|^\sigma$, with $\nu > 0$. This characterizes the model as a focusing NLS with concentrated nonlinearity of power type. In particular, we study orbital and asymptotic stability of standing waves for such a model. We prove the existence of standing waves of the form $u(t) = e^{i\omega t}\Phi_\omega$, which are orbitally stable in the range $\sigma \in (0,1)$, and orbitally unstable for $\sigma \geq 1$. Moreover, we show that for $\sigma \in (0, \frac{1}{\sqrt{2}}) \cup \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right)$ every standing wave is asymptotically stable, in the following sense. Choosing an initial data close to the stationary state in the energy norm, and belonging to a natural weighted L^p space which allows dispersive estimates, the following resolution holds: $u(t) = e^{i\omega_\infty t + il(t)}\Phi_{\omega_\infty} + U_t * \psi_\infty + r_\infty$, where U_t is the free Schrödinger propagator, $\omega_\infty > 0$ and ψ_∞, $r_\infty \in L^2(\mathbb{R}^3)$ with $\|r_\infty\|_{L^2} = O(t^{-p})$ as $t \to +\infty$, $p = \frac{5}{4}$, $\frac{1}{4}$ depending on $\sigma \in (0, 1/\sqrt{2})$, $\sigma \in (1/\sqrt{2}, 1)$, respectively, and finally $l(t)$ is a logarithmic increasing function that appears when $\sigma \in (\frac{1}{\sqrt{2}}, \sigma^*)$, for a certain $\sigma^* \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right]$. Notice that in the present model the admitted nonlinearities for which asymptotic stability of solitons is proved, are subcritical in the sense that it does not give rise to blow up, regardless of the chosen initial data.
The second model is the energy critical focusing nonlinear Schrödinger equation $i\frac{du}{dt} = -\Delta u - |u|^4 u$. In this case we prove, for any ν and α_0 sufficiently small, the existence of radial finite energy solutions of the form $u(t,x) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + e^{i\Delta t}\zeta^* + o_{\dot{H}^1}(1)$ as $t \to +\infty$, where $\alpha(t) = \alpha_0 \ln t$, $\lambda(t) = t^\nu$, $W(x) = (1 + \frac{1}{3}|x|^2)^{-1/2}$ is the ground state and ζ^* is arbitrarily small in $\dot{H}^1$.

Keywords
Nonlinear Schrödinger equation, soliton, asymptotic stability, energy critical, focusing, radial solution.

Résumé

Cette thèse est consacrée à l'étude de certaine aspects du comportement en temps longs des solutions de deux équations de Schrödinger non-linéaires en dimension trois dans des régimes perturbatives convenables.
Le premier modèle consiste en une équation de Schrödinger avec une non-linéarité concentrée obtenue en considérant une interaction ponctuelle de force α, c'est-à-dire une perturbation singulière du Laplacien décrite par un opérateur autoadjoint H_α, où la force α dépend de la fonction d'onde : $i\frac{du}{dt} = H_\alpha u$, $\alpha = \alpha(u)$. Il est bien connu que les éléments du domaine d'une interaction ponctuelle en trois dimensions peuvent être décrits comme la somme d'une fonction régulière et d'une fonction ayant une singularité proportionnelle à $|x - x_0|^{-1}$, où x_0 est l'emplacement du point d'interaction. Si q est la charge d'un élément du domaine u, c'est-à-dire le coefficient de sa partie singulière, alors pour introduire une non-linéarité, on fait dépendre la force α de u selon la loi $\alpha = -\nu|q|^\sigma$, avec $\nu > 0$. Ce modèle est défini comme une équation de Schrödinger non-linéaire focalisant de type puissance avec une non-linéarité concentrée en x_0. Notre étude porte sur la stabilité orbitale et asymptotique des ondes stationnaires de ce modèle. Nous prouvons l'existence d'ondes stationnaires de la forme $u(t) = e^{i\omega t}\Phi_\omega$, qui soient orbitalement stables pour $\sigma \in (0, 1)$ et orbitalement instables quand $\sigma \geq 1$. De plus nous montrons que si $\sigma \in (0, \frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, 1)$, alors chaque onde stationnaire est asymptotiquement stable, à savoir que pour des données initiales proches d'un état stationnaire dans la norme d'énergie et appartenant à un espace L^p pondéré où les estimations dispersives sont valides, l'affirmation suivante est vérifiée : il existe $\omega_\infty > 0$ et $\psi_\infty \in L^2(\mathbb{R}^3)$ tel que $\psi_\infty = O_{L^2}(t^{-p})$ quand $t \to +\infty$, tel que $u(t) = e^{i\omega_\infty t + il(t)}\Phi_{\omega_\infty} + U_t * \psi_\infty + r_\infty$, où U_t est le propagateur de Schrödinger libre, $p = \frac{5}{4}$, $\frac{1}{4}$ respectivement en fonction de $\sigma \in (0, 1/\sqrt{2})$, $\sigma \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right)$, et $l(t)$ est une fonction à croissance logarithmique qui apparaît quand $\sigma \in (\frac{1}{\sqrt{2}}, \sigma^*)$, où $\sigma^* \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right]$. Notons que dans ce modèle les non-linéarités pour lesquelles on a la stabilité asymptotique sont sous-critiques dans le sens où quelle que soit la donnée initiale il n'y a pas de solutions explosives.
Quant au deuxième modèle, il s'agit de l'équation de Schrödinger non-linéaire focalisant à énergie critique : $i\frac{du}{dt} = -\Delta u - |u|^4 u$. Pour ce cas, nous prouvons, pour tout ν et α_0 suffisamment petits, l'existence de solutions radiales à énergie finie de la forme $u(t, x) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + e^{i\Delta t}\zeta^* + o_{\dot{H}^1}(1)$ tout $t \to +\infty$, où $\alpha(t) = \alpha_0 \ln t$, $\lambda(t) = t^\nu$, $W(x) = (1 + \frac{1}{3}|x|^2)^{-1/2}$ est l'état stationnaire et ζ^* est arbitrairement petit en $\dot{H}^1$.

Mots clés
Équation de Schrödinger, soliton, stabilité asimptotique, energie critique, focalisant, solution radiale.

Contents

Introduction

The purpose of this thesis is to understand certain aspects of the large time behavior of the solutions to some nonlinear Schrödinger (NLS) equations of the form

$$i\frac{du}{dt} = -\triangle u - f(x, |u|^2)u, \qquad x \in \mathbb{R}^3. \tag{1}$$

Let us note that it is also possible to consider abstract equations with other self-adjoint operators in place of the Laplacian. Anyway, the local and global well-posedness of the associated Cauchy problem have been largely investigated for a wide family of nonlinearities in dimension three as well as in the generic Euclidean space $\mathbb{R}^n$, $n \geq 1$ (for example see [24], [25], [26], [12] and references therein). In particular, under suitable hypotheses on f, this equation has a unique solution once an initial datum is fixed.

Broadly speaking, the evolution turns out to be a competition between the linear part of the equation (which tends to disperse the solution) and the nonlinear part (which can either focus or defocus the solution depending on the sign of the nonlinear function f). Therefore, one might expect the dynamics to be a combination of three phenomena. The first one is a linearly dominated behavior which occurs when the effects of the linear part dominate those of the nonlinear one. In such a case, the solution is global and at large times converges to a solution to the linear Schrödinger equation that is known to disperse to zero. One can also have a nonlinear dominated behavior when the nonlinear effects are stronger than the linear ones. In this situation, if equation (1) is *focusing* (as will be in this thesis), then the solution can develop singularities at finite times. Finally, the linear and nonlinear effects may be in balance. In the focusing case one of the most classical manifestations of this regime is the existence of soliton type solutions.

To be more precise in the definition of soliton let us notice that the inhomogeneity given by the x dependence of f in (1) destroys the translation invariance but the dynamics still enjoys the phase shift invariance. As a consequence, it is well known that under suitable assumptions equation (1) admits a branch of non-trivial solutions of the form

$$u(t, x) = e^{i\omega t}\Phi_\omega(x),$$

with ω in some interval and Φ_ω satisfying

$$-\triangle\Phi_\omega + \omega\Phi_\omega - f(x, |\Phi_\omega|^2)\Phi_\omega = 0.$$

Existence and uniqueness as well as the properties of the solutions of this equation, which are called *solitary waves* or *solitons*, have been largely inspected, see for example [7] and [12].

Solitons appear in a wide class of nonlinear dispersive partial differential equations such as the wave equation, the Korteweg-de Vries equation or the Klein-Gordon equation. One could believe that when the nonlinear effects are not strong enough to produce finite time blow up, solutions with generic initial data should eventually resolve into a superposition of a radiation component (which

behaves like a solution to the linear Schrödinger equation) plus a finite number of modulated nonlinear bound states. This statement is known as **soliton resolution conjecture**.
As far as NLS type equations are concerned, the only case where this conjecture is proved rigorously is the cubic NLS in dimension one, that can be integrated by means of the inverse scattering method. In the non-integrable case the conjecture is in general widely open. However, there are certain important perturbative regimes that are accessible to the analysis.
Two examples of such perturbative regimes are considered in this thesis both of them being related to small initial perturbations of a single solitary wave. More precisely, in part I we study the orbital and asymptotic stability of solitary waves of some three-dimensional NLS with concentrated nonlinearities opportunely defined, and in part II we exhibit some "exotic" regimes in the vicinity of the ground state of the NLS in the energy critical regime.

Orbital and asymptotic stability for standing waves of a NLS equation with concentrated nonlinearity in dimension three

The first part of this work is devoted to the analysis of orbital and asymptotic stability of the solitary waves of a Schrödinger equation with concentrated nonlinearity in dimension three. Such a model was proposed and constructed by Adami, Dell'Antonio, Figari, and Teta in [1] and [2]. For the analogous one-dimensional model constructed by Adami and Teta in [5] these stability properties are studied by Buslaev, Komech, Kopylova, and Stuart in [8] and [33].
By Schrödinger operator with concentrated nonlinearity is meant a dynamical generator whose nonlinear part is localized at one point. More precisely, the considered model is defined through the nonlinear operator H_α defined on a suitable subspace of $L^2(\mathbb{R}^n)$, $n = 1, 2, 3$, where α is a fixed functional acting on the element domain precisely defined below. The action of the operator H_α when restricted to regular functions vanishing in 0 is that of the Laplacian. On the other hand, when α is a constant one gets a family of operators known as pointwise interaction (the topic is treated in the book of Albeverio et alii [6]). In [37], Noja and Posilicano give a general definition of concentrated nonlinearities in the case $n = 3$ that is considered here. In this particular case, the subspace of $L^2(\mathbb{R}^3)$ which turns out to be the operator domain of H_α is

$$D(H_\alpha) = \Big\{ u \in L^2(\mathbb{R}^3) : \ u(x) = \phi(x) + q\frac{1}{4\pi|x|} \text{ with } \phi \in H^2_{loc}(\mathbb{R}^3),\ \Delta\phi \in L^2(\mathbb{R}^3),$$

$$q \in \mathbb{C}, \lim_{x\to 0}\left(u(x) - q\frac{1}{4\pi|x|}\right) = \alpha(u)q \Big\},$$

while the action of the operator is described by

$$H_\alpha u = -\triangle\phi.$$

The complex number q is sometimes called *charge*. In particular we consider the case

$$\alpha(u) = -\nu|q|^{2\sigma}, \qquad \nu > 0,\ \sigma > 0. \tag{2}$$

For this nonlinearity local and global well-posedness of the dynamics and blow up properties of the equation

$$i\frac{du}{dt} = H_\alpha u, \qquad u \in D(H_\alpha), \tag{3}$$

have been studied by Adami, Dell'Antonio, Figari, and Teta in [1] and [2].

The solitary waves (or *standing waves*) of Equation (3)exist if and only if $\omega > 0$ and their analytic expression is known (see Section 1.2.2).
Notice that equation (3) is phase shift invariant (but not translationally invariant since this symmetry is broken by the pointwise interaction): this prevents the solitons from being stable in the sense of Lyapunov.
Hence, the natural notion to be used in this context is that of *orbital stability*, which roughly speaking, is Lyapunov stability up to symmetries. More precisely, one can define the orbit of a soliton Φ_ω as $\mathcal{O}(\Phi_\omega) = \{e^{i\theta}\Phi_\omega(x),\ \theta \in [0, 2\pi)\}$. Thus, by definition, the state Φ_ω is *orbitally stable in the future* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|u(0) - \Phi_\omega\|_V < \delta \quad \Rightarrow \quad d(u(t), \mathcal{O}(\Phi_\omega)) < \epsilon \quad \forall t > 0,$$

where

$$d(u, \mathcal{O}(\Phi_\omega)) = \inf_{v \in \mathcal{O}(\Phi_\omega)} \|u - v\|_V,$$

and $\|\cdot\|_V$ is the norm in the energy space. A stationary state is said to be *orbitally unstable* if it is not orbitally stable. This type of investigations can be done following two different approaches: the first one is based on variational and compactness argument (see the paper of Cazenave and Lions [13] for details), while the second one is based on the idea of constructing a sort of Lyapunov function (see the paper of Weinstein [53], [52] and those of Grillakis, Shatah and Strauss [28], [29]). In our setting one can observe that the hypotheses of the results of Weinstein [53] and of Grillakis, Shatah, and Strauss [28] are satisfied then we prove the following theorem.

Theorem 0.1. ***(Orbital stability)*** *Consider equation* (3) *with concentrated power nonlinearity* (2)*, then for all* $\omega > 0$

(a) the standing wave Φ_ω *is orbitally stable when* $0 < \sigma < 1$*,*

(b) the standing wave Φ_ω *is orbitally unstable when* $\sigma > 1$*.*

Finally, in the case $\sigma = 1$ instability by blow up is proved exploiting the additional pseudoconformal transformation. Roughly speaking, for each solitary wave Φ_ω in any neighbourhood of initial data there is a (non global) solution of equation (3) whose charge diverges as the time goes to infinity. Hence the standing wave is orbitally unstable.
A more challenging and subtle task is the study of asymptotic stability. One says that a soliton is *asymptotically stable* if it has a neighbourhood of initial data such that the corresponding solutions converges in some suitable weighted Lebesgue space to some soliton which is in general different from the initial one. Hence, one expects that the solution to the NLS equation (3) can be decomposed as

$$u(t, x) = e^{i\Theta(t)} \left(\Phi_{\omega(t)}(x) + \chi(t, x)\right),$$

where the real functions $\omega(t) \sim \dot{\Theta}(t)$ behave as a precise constant as the time goes to infinity, while the function $\chi(t)$ disperses. This implies that, for large times, the solution $u(t)$ is approximated by a soliton which might not be the initial one. Under some restriction on the nonlinearity, asymptotic stability of solitary waves of equation (1) in some fixed dimension were proved by Soffer and Weinstein [42], [43], and Buslaev and Perelman [9], [10]. In the cited papers the techniques nowadays classical in dealing with this type of problems are also developed . These results have been extended to higher dimension; in this direction some meaningful works are [14, 48, 51, 30, 22, 23, 15].
The first step in the asymptotic stability analysis is the study of the spectrum of the operator L which comes out linearizing the NLS equation (3) around the solitary wave Φ_ω. Exploiting the explicit expression of the resolvent of the linearization L the spectrum $\sigma(L)$ satifies:

if $\sigma = 1$, then L has just the eigenvalue 0 with algebraic multiplicity 4,

if $\sigma \in (1, +\infty)$, then L has two simple real eigenvalues $\pm\mu = \pm 2\sigma\sqrt{\sigma^2 - 1}\omega$ and the eigenvalue 0 with algebraic multiplicity 2.

In the case $\sigma = 1/\sqrt{2}$ the endpoints of the essential spectrum $\pm i\omega$ are resonances for the linearized problem.
The second fundamental ingredient for the study of asymptotic stability consists in the so-called modulation equations that describe the evolution of the parameter $\omega(t)$, of the phase $\Theta(t)$, and, in case of presence of the purely imaginary eigenvalues, of the coefficients of the corresponding eigenfunctions. Such equations are obtained constructing a solution $u(t)$ of the NLS equation (3) close to the stationary wave $\Phi_{\omega(t)}$ for all $t > 0$ and such that the reminder $u(t) - \Phi_{\omega(t)}$ is symplectically orthogonal to the generalized kernel of the linearized operator $L(t)$ at every positive time.
In order to obtain information about the asymptotic behavior of the solution of the NLS, we are interested in determine the behavior of the solutions of the modulation equations as $t \to +\infty$. To this purpose, one studies the behavior of the propagator of the operator L. In particular, some dispersive estimates for the propagator of L are proved. As it often happens in establishing such estimates, the structure of the resolvent of the linearized operator (in this case it is explicitly known) imposes to chose the initial data in some suitable weighted $L^1(\mathbb{R}^3)$. Let us denote this weight by w.
In the thesis only the spectral cases $\sigma \in (0, 1/\sqrt{2})$ and $\sigma \in (\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}})$ are studied. The first case correspond to the absence of non-vanishing eigenvalues while in the second case purely imaginary eigenvalues $\pm i\xi$ with the condition $2\xi > \omega$ appear. We do not consider the case $\sigma = \frac{1}{\sqrt{2}}$ where there is a resonance at the endpoint of the continuous spectrum. In the first case the steps described above lead to the following result.

Theorem 0.2. ***(Asymptotic stability in case the point spectrum only consists in the eigenvalue** 0)* *Assume that $u(t) \in C(\mathbb{R}^+, V)$ is a solution to* (3) *with concentrated power nonlinearity* (2) *where $\sigma \in (0, 1/\sqrt{2})$. Moreover, suppose that $u(0) = u_0 \in V \cap L^1_w(\mathbb{R}^3)$. Denoting*

$$d = \|u_0 - e^{i\theta_0}\Phi_{\omega_0}\|_{V \cap L^1_w},$$

for some $\omega_0 > 0$ and $\theta_0 \in \mathbb{R}$, then, provided d is sufficiently small, the solution $u(t)$ can be asymptotically decomposed as

$$u(t) = e^{i\omega_\infty t}\Phi_{\omega_\infty} + U_t * \psi_\infty + r_\infty(t),$$

where $\omega_\infty > 0$ and ψ_∞, $r_\infty \in L^2(\mathbb{R}^3)$ with

$$\|r_\infty(t)\|_{L^2} = O(t^{-5/4}) \quad as \ t \to +\infty,$$

in $L^2(\mathbb{R}^3)$.

Finally, in the second spectral case ($\sigma \in (\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}})$), the presence of the two purely imaginary eigenvalues slows down the speed of decay of the reminder r_∞. This slower decay can be observed by the behavior of the parameters whose evolution is described by the modulation equations. Hence, in order to deal with the modulation equations, it is necessary to consider also the quadratic and the cubic terms of the nonlinearity and, later, to exploit a change of variables to have a normal form of the modulation equations to proceed with the dispersive estimates and the asymptotic behavior analysis. This makes more complicate the study of the properties of ψ_∞ and r_∞.
In order to formulate the last result we denote by Ψ_1, Ψ_2 the eigenfunctions corresponding to the purely complex eigenvalues, and by z_0 the associated coefficient in the initial datum.

Theorem 0.3. ***(Asymptotic stability in the case of purely imaginary eigenvalues)*** *Assume that $u(t) \in C(\mathbb{R}^+, V)$ is a solution to (3) with concentrated power nonlinearity (2) where $\sigma \in (\frac{1}{\sqrt{2}}, \sigma^*)$, for a certain $\sigma^* \in (\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}]$. Moreover, suppose that the initial datum*

$$u(0) = u_0 = e^{i\omega_0 + \gamma_0}\Phi_{\omega_0} + e^{i\omega_0+\gamma_0}[(z_0 + \overline{z_0})\Psi_1 + i(z_0 - \overline{z_0})\Psi_2] + f_0 \in V \cap L^1_w(\mathbb{R}^3),$$

with $\omega_0 > 0$, γ_0, $z_0 \in \mathbb{R}$, and $f_0 \in L^2(\mathbb{R}^3) \cap L^1_w(\mathbb{R}^3)$ is close to a stationary wave, i.e.

$$|z_0| \leq \epsilon^{1/2} \qquad \text{and} \qquad \|f_0\|_{L^1_w} \leq c\epsilon^{3/2},$$

where c, $\epsilon > 0$.
Then, provided ϵ is sufficiently small, the solution $u(t)$ can be asymptotically decomposed as

$$u(t) = e^{i\omega_\infty t + ib_1 \log(1+\epsilon k_\infty t)}\Phi_{\omega_\infty} + U_t * \psi_\infty + r_\infty(t),$$

where ω_∞, $\epsilon k_\infty > 0$, $b_1 \in \mathbb{R}$, and ψ_∞, $r_\infty \in L^2(\mathbb{R}^3)$ such that

$$\|r_\infty(t)\|_{L^2} = O(t^{-1/4}) \quad \text{as } t \to +\infty,$$

in $L^2(\mathbb{R}^3)$.

Nondispersive vanishing and blow up at infinity for the energy critical nonlinear Schrödinger equation in $\mathbb{R}^3$

In the second part of the thesis we study the equation (also called *energy critical* NLS equation)

$$\begin{cases} i\frac{du}{dt} = -\triangle u - |u|^4 u & x \in \mathbb{R}^3 \\ u(0) = u_0 \in \dot{H}^1(\mathbb{R}^3) \end{cases} . \tag{4}$$

This Cauchy problem is known to be locally well-posed: for any initial datum $u_0 \in \dot{H}^1(\mathbb{R}^3)$ there exists a unique solution u defined on a maximal interval of definition $I = (T_-, T_+)$ such that $u \in C(I, \dot{H}^1(\mathbb{R}^3)) \cap L^{10}(\mathcal{I} \times \mathbb{R}^3)$ for any compact interval $\mathcal{I} \subset I$. If $T_+ < +\infty$ (or $T_- > -\infty$), then $\|u\|_{L^{10}((0,T_+)\times\mathbb{R}^3)} = +\infty$ (respectively $\|u\|_{L^{10}((T_-,0)\times\mathbb{R}^3)} = +\infty$), and one says that the solution blows up in finite time.
During their lifespan the solutions to (4) satisfy the conservation of energy

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t,x)|^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |u(t,x)|^6 dx = E(u(0)).$$

Both the energy and the equation are invariant under the scaling

$$u(t,x) \longmapsto \lambda^{-1/2} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right), \qquad \forall \lambda > 0.$$

The existence of this invariance is the reason of the name "energy critical NLS".
If the initial data are sufficiently small, the solution is global and scatters as $t \to \infty$. For large data, the exitence of finite time blow up solution can be proved by mean of the viral identity

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^3} |x|^2 |u(t,x)|^2 dx = 8E(u) - \frac{16}{3}\int_{\mathbb{R}^3} |u(t,x)|^6 dx,$$

which shows that localized initial data with negative energy must break down in finite time.

Moreover, equation (4) admits a stationary state in $\dot{H}^1(\mathbb{R}^3)$, namely a solution of

$$-\triangle W - |W|^4 W = 0.$$

A particular solution to the above equation is the so-called Talenti-Aubin solution

$$W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-1/2},$$

which belongs to $\dot{H}^1(\mathbb{R}^3)$ but not to $L^2(\mathbb{R}^3)$.
In [31], Kenig and Merle show that the energy of the ground state W is critical in the following sense: for any $u(t)$ a radial solution to (4) such that $E(u(0)) < E(W)$ one has

if $\|u(0)\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, then the solution is global and scatters as $t \to \infty$;

if $\|u(0)\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}$ and $u(0) \in L^2(\mathbb{R}^3)$, then the solution blows up in finite time.

The behavior of radial solutions with critical energy was classified by Duyckaerts and Merle in [19]. In this case, in addition to finite time blow up and scattering to zero (and W itself), one has solutions that as $t \to \infty$ converge in $\dot{H}^1(\mathbb{R}^3)$ to a rescaled ground state. In the case where $E(u(0)) > E(W)$ the dynamics is expected to be richer and to include the solution that as $t \to \infty$ behave as a modulated ground state $e^{i\alpha(t)}\lambda^{\frac{1}{2}}(t)W(\lambda(t)x)$ with fairly general $\alpha(t)$ and $\lambda(t)$.
For a closely related model of the critical wave equation, the solutions of this type with $\lambda(t) \to \infty$ as $t \to \infty$ (blow up at infinity) and $\lambda(t) \to 0$, $t\lambda(t) \to \infty$ as $t \to \infty$ (non-dispersive vanishing) were recently constructed by Donninger and Krieger (see [17]). The goal of the second part of this thesis is to prove an analogous result for the NLS equation (4). More precisely we show the following theorem.

Theorem 0.4. *There exists $\beta_0 > 0$ such that for any ν, $\alpha_0 \in \mathbb{R}$ with $|\nu| + |\alpha_0| \le \beta_0$ and any $\delta > 0$ there exist $T > 0$ and a radial solution $u \in C([T, +\infty), \dot{H}^1 \cap \dot{H}^2)$ to (4) of the form:*

$$u(t,x) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + \zeta(t,x), \tag{5}$$

where $\lambda(t) = t^\nu$, $\alpha(t) = \alpha_0 \ln t$, and $\zeta(t)$ verifies:

$$\begin{aligned} &\|\zeta(t)\|_{\dot{H}^1 \cap \dot{H}^2} \le \delta, \\ &\|\zeta(t)\|_{L^\infty} \le C t^{-\frac{1+\nu}{2}}, \\ &\| < \lambda(t)x >^{-1} \zeta(t)\|_{L^\infty} \le C t^{-1-\frac{3}{2}\nu}, \end{aligned} \tag{6}$$

for all $t \ge T$. The constants C here and below are independent of ν, α_0 and δ.
Furthermore, there exists $\zeta^* \in \dot{H}^s$, $\forall s > \frac{1}{2} - \nu$, such that, as $t \to +\infty$, $\zeta(t) - e^{it\Delta}\zeta^* \to 0$ in $\dot{H}^1 \cap \dot{H}^2$.

As mentioned above, a similar result is obtained for the energy critical wave equation by Donninger and Krieger in [17]. Their construction was inspired by the previous work of Krieger, Schlag, and Tataru [35] where the case of finite time blow up was considered. Both these papers and the references therein have been a source of inspiration for parts of the techniques to construct solutions to equation (4) as in the previous theorem.
The first step in proving this kind of results is to construct an approximate solution to the NLS equation (4) with an error that decay sufficiently fast in time. In order to do that it is useful to split the space $\mathbb{R}^3$ in three regions related to three different space scales: the inner region with the

scale $t^\nu|x| \lesssim 1$, the self-similar region where $|x| = O(t^{1/2})$, and, finally, the remote region where $|x| = O(t)$. In the inner region the solution will be constructed as a perturbation of the profile $e^{i\alpha_0\nu\ln t}t^{\nu/2}W(t^\nu x)$. While, the self-similar and remote regions are the regions where the solution is small and described essentially by the linear equation $i\frac{du}{dt} = -\triangle u$.
The second step consists in considering the linearization of (4) around W and prove the boundedness of the propagator of the linearized operator along its essential spectrum in the $H^1(\mathbb{R}^3)$. To achieve this result we use the distorted Fourier transform and some of its properties. In such arguments, some of the techniques are from Buslaev and Perelman [9], and Krieger and Schlag [34].
Finally, in the third and last step the results of the previous steps are exploited in order to prove, by a fixed point argument, the existence of an exact solution on the NLS equation (4) that satisfies the properties claimed in the theorem.

The results presented here form the core of three papers:

- R. Adami, D. Noja, and C. O., *Orbital and asymptotic stability for standing waves of a NLS equation with concentrated nonlinearity in dimension three*, to appear in Journal of Mathematical Physics, avaible at arxiv.org/pdf/1207.5677.
- R. Adami, D. Noja, and C. O., *Orbital and asymptotic stability for standing waves of a NLS equation with concentrated nonlinearity in dimension three. II*, in preparation.
- C. O., G. Perelman, *Nondispersive vanishing and blow up at infinity for the energy critical nonlinear Schrödinger equation in* $\mathbb{R}^3$, to appear in St. Petersburg Mathematical Journal.

Introduction

Le but de cette thèse est de comprendre certains aspects du comportement en temps longs des solutions des équations de Schrödinger non-linéaires (NLS) de la forme

$$i\frac{du}{dt} = -\triangle u - f(x, |u|^2)u, \qquad x \in \mathbb{R}^3. \tag{7}$$

Notons qu'on peut aussi considérer des equations abstraites avec des opérateurs autoadjoints autre que le Laplacien. L'existence locale et globale pour le problème de Cauchy associé a été amplement examinée pour une grand famille de nonlinéarités (pour exemple voir [24], [25], [26], [12] et leur références). En particulier, sous des hypothèses convenables sur f, cette équation a une solution unique une fois que la donnée initiale est fixée.
De manière générale, l'évolution se révèle être une compétition entre la partie linéaire de l'équation (qui tend à disperser la solution) et la partie non-linéaire (qui peut être soit focalisante, soit défocalisante en fonction du signe de la fonction f). On pourrait ainsi penser que la dynamique se caractérise par la combinaison de trois phénomènes. Le premiér est un comportement linéairement dominé qui apparait quand les effets de la partie linéaire dominent ceux de la non-linéarité. Dans ce cas, la solution est globale et en temps longs elle converge vers une solution de l'équation de Schrödinger linéaire qui nous le savons, se disperse vers zero. Si les effets non-linéaires sont plus forts que les effets linéaires, on peut avoir un comportement complètement non-linéaire. En ce cas, si l'équation (7) est *focalisante* (cas étudié dans cette thèse), alors la solution peut développer des singularités en temps fini. Enfin les effets linéaires et non-linéaires peuvent être en équilibre. Dans le cas focalisant une des manifestations les plus classiques de ce régime est l'existence de solutions solitoniques.
Pour définir d'une façon plus précise la notion de soliton on observe que la non-homogénéité, qui vient de la dépendance de f en x dans l'équation (7), détruit l'invariance par rapport aux translations mais la dynamique est toujours invariante par rapport à la variation de phase. Par conséquent, il est bien connu que, sous des hypothèses convenables, l'équation (7) admet une famille de solutions de la forme

$$u(t, x) = e^{i\omega t}\Phi_\omega(x),$$

avec ω appartenant à un intervalle et Φ_ω satisfaisant

$$-\triangle\Phi_\omega + \omega\Phi_\omega - f(x, |\Phi_\omega|^2)\Phi_\omega = 0.$$

L'existence, l'unicité et les propriétés des solutions de cette équation, qui sont appelées *ondes solitaires* ou *solitons*, ont été largement inspectées (voir par example [7] et [12]).
Les solitons apparaissent dans une large classe d'équations aux dérivées partielles non-linéaires dispersives comme l'équation des ondes, l'équation de Korteweg-de Vries ou l'équation de Klein-Gordon. On peut penser que si les effets non-linéaires ne sont pas assez forts pour produire des solutions explosives en temps fini, les solutions avec des données initiales générales devront finalement se réduire à une superposition d'un composant de radiation (qui se comporte comme

une solution de l'équation de Schrödinger linéaire) plus un nombre fini des étas liés non-linéaires modulés. Cette affirmation est connue sous le nom de **conjecture de résolution en solitons** . Pour des équations du type Schrödinger non-linéaire le seul cas où cette conjecture est rigoureusement démontrée est celui de l'équation de Schrödinger non-linéaire cubique en dimension un, qui peut être intégrée par la méthode du scattering inverse. Dans les cas non-intégrables, la conjecture est généralement largement ouverte. Il existe cependant certains régimes perturbatifs importants accessibles à l'analyse.
Dans cette thèse nous considérons deux exemples de ces régimes perturbatifs, tous les deux correspondant à de petites perturbations initiales d'une seule onde solitaire.Nous étudirons tout d'abord la stabilité orbitale et asymptotique des ondes stationnaires pour certaines équations de Schrödinger non-linéaires avec des non-linéarités concentrées (définies opportunément) en dimension trois et dans une seconde partie nous exposerons des régimes "exotiques" dans le voisinage de l'état fondamental de l'équation de Schrödinger non-linéaire à énergie critique.

Stabilité orbitale et asymptotique des ondes stationnaires pour des l'équations de Schrödinger avec des non-linéarités concentrése en dimension trois

La première partie de cette thèse est dédiée à l'analyse de la stabilité orbitale et asymptotique des ondes solitaires de l'équation de Schrödinger avec des non-linéarités concentrées en dimension trois. Ce modèle a été introduit par Adami, Dell'Antonio, Figari et Teta ([1] et [2]). Les propriétés de stabilité du modèle analogue en dimension un, construit par Adami et Teta ([5]), ont été étudiées par Buslaev, Komech, Kopylova et Stuart ([8] et [33]).
Un opérateur de Schrödinger à non-linéarité concentrée est un générateur de dynamique dont la partie non-linéaire est localisée en un point. Plus précisément, le modèle considéré est défini à l'aide de l'opérateur non-linéaire H_α défini sur un sous-espace approprié de $L^2(\mathbb{R}^n)$, $n = 1$, 2, 3, où α est une fonctionnel fixée agissant sur un élément du domaine défini précisément ci-dessous.. Dans le cas $n = 3$ (cas étudié dans cette thèse) une définition générale des non-linéarités concentrées a été donnée par Noja et Posilicano [37]. Dans ce cas, le domaine de l'opérateur H_α est le sous-espace suivant de $L^2(\mathbb{R}^3)$:

$$D(H_\alpha) = \Big\{ u \in L^2(\mathbb{R}^3) : \; u(x) = \phi(x) + q\frac{1}{4\pi|x|} \text{ avec } \phi \in H^2_{loc}(\mathbb{R}^3),\, \Delta\phi \in L^2(\mathbb{R}^3),$$

$$q \in \mathbb{C}, \lim_{x\to 0}\left(u(x) - q\frac{1}{4\pi|x|}\right) = \alpha(u)q \Big\},$$

l'action de l'opérateur étant décrite par

$$H_\alpha u = -\triangle\phi.$$

Le nombre complexe q est parfois appelé *charge*. Dans cette thèse on considère le cas

$$\alpha(u) = -\nu|q|^{2\sigma}, \qquad \nu > 0,\ \sigma > 0. \tag{8}$$

Pour cette non-linéarité l'existence locale et globale de la dynamique ainsi que les propriétés des solutions explosives pour l'équation

$$i\frac{du}{dt} = H_\alpha u, \qquad u \in D(H_\alpha), \tag{9}$$

ont été étudiée par Adami, Dell'Antonio, Figari, and Teta en [1] et [2].
Les ondes solitaires (ou *ondes stationnaires*) de l'Équation (9) existent si et seulement si $\omega > 0$ et leur expression analytique est alors connue (voir Section 1.2.2).
Notons que l'Équation (9) est invariante par changements de phase ce qui l'empêche la stabilité des solitons au sens de Lyapounov.
Un état Φ_ω est dit *orbitalement stable dans le futur* si pour tout $\epsilon > 0$ il existe $\delta > 0$ tel que

$$\|u(0) - \Phi_\omega\|_V < \delta \quad \Rightarrow \quad d(u(t), \mathcal{O}(\Phi_\omega)) < \epsilon \quad \forall t > 0,$$

où $\|\cdot\|_V$ est la norme dans l'espace d'énergie, $\mathcal{O}(\Phi_\omega) = \{e^{i\theta}\Phi_\omega(x),\ \theta \in [0, 2\pi)\}$ est l'orbite de Φ_ω et $d(u, \mathcal{O}(\Phi_\omega)) = \inf_{v\in\mathcal{O}(\Phi_\omega)} \|u - v\|_V,$. Un état stationnaire est dit *orbitalement instable* s'il n'est pas orbitalement stable. L'étude de la stabilité orbitale peut être mené selon deux approches différentses : le première est basé sur des arguments variationnels dans l'esprit des travaux pionnières de Cazenave et Lions [13], et la seconde repense sur le construction d'une fonction de Lyapounov (voir Weinstein [53], [52] et Grillakis, Shatah et Strauss [28], [29]). Les résultats obtenus par ces derniers s'appliquent bien au modèle ici considéré et nous permettent de démontrer le théorème suivant.

Theorem 0.5. ***(Stabilité orbitale)*** *Considérons l'équation* (9) *avec non-linéarité puissance concentrée* (8)*, alors pour tout* $\omega > 0$

(a) l'onde stationnaire Φ_ω *est orbitalement stable si* $0 < \sigma < 1$*,*

(b) l'onde stationnaire Φ_ω *est orbitalement instable si* $\sigma > 1$*.*

Finalement pour le cas $\sigma = 1$ l'instabilité par explosions se démontre en exploitant la transformation pseudo-conforme.
Une tache plus difficile et délicate est l'analyse de la stabilité asymptotique. On dit qu'un soliton est *asymptotiquement stable* s'il existe un voisinage de données initiales tels que les solutions correspondantes convergent dans un espace de Lebesgue convenablement pondéré vers un soliton qui généralement est différent du soliton initial dont les paramètres sont proches des paramètres initiales. Plus précisément la solution de l'équation de Schrödinger non-linéaire (9) se décompose comme

$$u(t,x) = e^{i\Theta(t)}\left(\Phi_{\omega(t)}(x) + \chi(t,x)\right),$$

où les fonctions réeles $\omega(t) \sim \dot{\Theta}(t)$ converge vers une constante précise quand $t \to +\infty$, tandis que la fonction $\chi(t)$ se disperse. Sous quelques restrictions sur la non-linéarité la stabilité asymptotique de l'équation (7) en certain dimension fixée est démontrée par Soffer et Weinstein en [42], [43], et par Buslaev et Perelman en [9], [10]. Les techniques développées dans ces articles aujourd'hui sont considérées comme classiques pour ces type de problèmes. Ces résultats ont aussi été prouvés en dimensions supérieurs ([14, 48, 51, 30, 22, 23, 15]).
Une première étape dans l'analyse de la stabilité asymptotique consiste en l'étude du spectre de l'opérateur L provenant de la linéarisation de l'équation de Schrödinger non-linéaire (9) autour d'une onde stationnaire Φ_ω. Exploitant l'expression explicite de la résolvante de l'opérateur L on peut montrer que :

si $\sigma = 1$, alors L a seulement une valeur propre 0 avec multiplicité algébrique 4 ;

si $\sigma \in (1, +\infty)$, alors L a deux valeurs propres réels $\pm\mu = \pm 2\sigma\sqrt{\sigma^2 - 1}\omega$ et la valeur propre 0 avec multiplicité algébrique 2.

Dans le cas $\sigma = 1/\sqrt{2}$ les extrémités du spectre essentiel $\pm i\omega$ sont des résonances pour le probleme linéarisé.
Le deuxième ingrédient fondamental pour l'étude de stabilité asymptotique en l'établissent des équations de modulation qui décrivent l'évolution de paramètre $\omega(t)$, de la phase $\Theta(t)$ et, dans le cas où les valeurs propres purement imaginaire sont présentes, des coefficientsdes fonctions propres correspondantes. Ces équations sont obtenues à partir de la décomposition $u(t) = e^{i\Theta(t)}\Phi_{\omega(t)}+\chi(t)$ avec $\chi(t)$ symplectiquement orthogonal au noyau généralisé de l'opérateur linéarisé L.
L'étude du comportement asymptotique des solutions des équations de modulation et donc du comportement asymptotique de l'équation de Schrödinger non-linéaire repose sur les propriétés dispersives du propagateur de L restreint au spectre essentiel. Lors de l'établissent des estimations dispersives, le structure de le résolvant de l'opérateur linéarisé (explicite dans notre cas) impose sauvent de choisir les données initiales dans un espace $L^1(\mathbb{R}^3)$ pondéré convenablement. Dénotons ce poids par w.
Dans cette thèse on étudie seulement les cas spectraux $\sigma \in (0, 1/\sqrt{2})$ et $\sigma \in (\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}})$, c'est-à-dire les cas non-résonants où on a la stabilité orbitale et en présence de valeurs propres purement imaginaires $\pm i\xi$ la condition $2\xi > \omega$ est satisfaite.. Dans le premier cas la stratégie exposée ci-dessus permet d'établir le résultat suivant.

Theorem 0.6. ***(Stabilité asymptotique quand le spectre ponctuel se compose seulement de la valeur propre** 0)* *Soit* $\sigma \in (0, 1/\sqrt{2})$. *Soit* $u \in C(\mathbb{R}^+, V)$ *une solution de* (9) *avec* $u(0) = u_0 \in V \cap L^1_w(\mathbb{R}^3)$ *et* $\omega_0 > 0$, $\theta_0 \in \mathbb{R}$. *On note* $d = \|u_0 - e^{i\theta_0}\Phi_{\omega_0}\|_{V\cap L^1_w}$. *Alors, si* d *est suffisamment petit, la solution* $u(t)$ *se décompose en somme*

$$u(t) = e^{i\omega_\infty t}\Phi_{\omega_\infty} + U_t * \psi_\infty + r_\infty(t),$$

où $\omega_\infty > 0$ *et* $\psi_\infty \in L^2(\mathbb{R}^3)$ *et le reste* $r_\infty(t)$ *vérifie*

$$\|r_\infty(t)\|_{L^2} = O(t^{-5/4}) \quad \textit{quand} \ t \to +\infty.$$

Dans le deuxième cas spectral ($\sigma \in (\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}})$), la présence de deux valeurs propres purement imaginaires ralentit la vitesse de décroissance du reste r_∞. Cette décroissance plus lente peut être observée à travers le comportement des paramétrés dont l'évolution est décrite par les équations des modulation. Pour étudier les équations de modulation il est, dans ce cas, nécessaire de tenir compte des termes quadratiques et cubiques de la non-linéarité et exploiter un changement de variables, afin de réduire les équations à une forme normale, ce qui permet ensuite, à l'aide des estimations dispersives, de procéder à l'analyse du comportement asymptotique. Cela complexifié l'étude des propriétés de ψ_∞ et r_∞.
Afin de formuler le dernier résultat notons Ψ_1, Ψ_2 les fonctions propres correspondantes aux valeurs propres purement complexes et z_0 le coefficient dans la donnée initiale.

Theorem 0.7. ***(Stabilité asymptotique en présence de valeurs propres purement imaginaires)*** *Soit* $\sigma \in (\frac{1}{\sqrt{2}}, \sigma^*)$, *où* $\sigma^* \in (\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}]$. *Soit* $u(t) \in C(\mathbb{R}^+, V)$ *une solution de* (9) *avec* $u(0) = u_0 \in V \cap L^1_w$ *de la forme*

$$u(0) = u_0 = e^{i\omega_0+\gamma_0}\Phi_{\omega_0} + e^{i\omega_0+\gamma_0}[(z_0+\overline{z_0})\Psi_1 + i(z_0-\overline{z_0})\Psi_2] + f_0 \in V \cap L^1_w(\mathbb{R}^3),$$

où $\omega_0 > 0$, γ_0, $z_0 \in \mathbb{R}$, *et* $f_0 \in L^2(\mathbb{R}^3) \cap L^1_w(\mathbb{R}^3)$ *sont tels que*

$$|z_0| \le \epsilon^{1/2} \qquad \textit{and} \qquad \|f_0\|_{L^1_w} \le c\epsilon^{3/2},$$

avec c, $\epsilon > 0$.

Alors, si ϵ est suffisamment petit, la solution $u(t)$ se en somme

$$u(t) = e^{i\omega_\infty t + ib_1 \log(1+\epsilon k_\infty t)}\Phi_{\omega_\infty} + U_t * \psi_\infty + r_\infty(t), \quad \textit{as } t \to +\infty,$$

où ω_∞, $\epsilon k_\infty > 0$, $b_1 \in \mathbb{R}$ et $\phi_\infty \in L^2(\mathbb{R}^3)$ et $r_\infty(t)$ vérifie

$$\|r_\infty(t)\|_{L^2} = O(t^{-1/4}) \quad \textit{quand } t \to +\infty,$$

en $L^2(\mathbb{R}^3)$.

Relaxation non-dispersive et explosion à l'infini pour l'équation de Schrödinger non-linéaire à énergie critique en dimension trois

Dans la deuxiéme partie de cette thése nous étudierons l'équation (appelée équation de Schrödinger non-linéaire à énergie critique)

$$\text{(10)} \qquad \begin{array}{l} i\frac{du}{dt} = -\triangle u - |u|^4 u \quad x \in \mathbb{R}^3 \\ u(0) = u_0 \in \dot{H}^1(\mathbb{R}^3) \end{array}.$$

Ce probléme de Cauchy est bien posé localement en temps : pour tout donnée initiale $u_0 \in \dot{H}^1(\mathbb{R}^3)$ il existe une unique solution u définie sur un intervalle maximal de définition $I = (T_-, T_+)$ tel que $u \in C(I, \dot{H}^1(\mathbb{R}^3)) \cap L^{10}(\mathcal{I} \times \mathbb{R}^3)$ pour tout intervalle compact $\mathcal{I} \subset I$. Si $T_+ < +\infty$ (ou $T_- > -\infty$), alors $\|u\|_{L^{10}((0,T_+)\times\mathbb{R}^3)} = +\infty$ (respectivement $\|u\|_{L^{10}((T_-,0)\times\mathbb{R}^3)} = +\infty$) et on dit que la solution explose en temps fini.
Pendant sa durée de vie la solution de (10) conserve l'énergie :

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t,x)|^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |u(t,x)|^6 dx = E(u(0)).$$

L'énergie et l'équation sont toutes les deux invariantes par changement d'échelle

$$u(t,x) \longmapsto \lambda^{-1/2} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right), \qquad \forall \lambda > 0.$$

Si les données initiales sont suffisamment petites, la solution est globale et se disperse quand $t \to \infty$. Pour des données grandes on peut démontrer l'existence de solutions explosives en temps fini à l'aide de l'identité de viriel

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^3} |x|^2 |u(t,x)|^2 dx = 8E(u) - \frac{16}{3}\int_{\mathbb{R}^3} |u(t,x)|^4 dx,$$

qui montre que le solutions avec des données initiales localisés avec énergie négative ne peuvent pas vivre qu'un temps fini.
De plus l'équation (10) admet un état stationnaire en $\dot{H}^1(\mathbb{R}^3)$, c'est-à-dire une solution de

$$-\triangle W - |W|^4 W = 0.$$

Une solution particulière de l'équation ci-dessus est la solution de Talenti-Aubin

$$W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-1/2},$$

qui appartient à $\dot{H}^1(\mathbb{R}^3)$ mais non à $L^2(\mathbb{R}^3)$.
En [31], Kenig et Merle montrent que l'énergie de l'état stationnaire W est critique au sens suivant : pour chaque $u(t)$ solution radiale de (10) tel que $E(u(0)) < E(W)$ on a

si $\|u(0)\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, alors la solution est globale et se disperse pour $t \to \infty$;

si $\|u(0)\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}$ et $u(0) \in L^2(\mathbb{R}^3)$, alors la solution explose en temps fini.

Le comportement des solutions radiales à énergie critique a été classifié par Duyckaerts et Merle [19]. Dans ce cas, en plus de l'explosion en temps fini et dispersion à zéro (et à W même), on a des solutions qui quand $t \to \infty$ convergent dans $\dot{H}^1(\mathbb{R}^3)$ vers un état stationnaire re-écaillé. Quand $E(u(0)) > E(W)$, on s'attende à ce que la dynamique soit plus riche et inclue des solutions qui quand $t \to \infty$ se comportent comme un état stationnaire modulé $e^{i\alpha(t)}\lambda^{\frac{1}{2}}(t)W(\lambda(t)x)$ avec $\alpha(t)$ et $\lambda(t)$ assez généraux.
Pour le modèle trés proche de l'équation des ondes critique les solutions de ce type ont été récemment construites par Donninger et Krieger (voir [17]) avec $\lambda(t) \to \infty$ quand $t \to \infty$ (explosion à l'infini) et $\lambda(t) \to 0$, $t\lambda(t) \to \infty$ pour $t \to \infty$ (relaxation). Le but de la deuxiéme partie de cette thése est de démontrer un résultat similaire pour l'équation de Schrödinger non-linéaire (10). Plus précisément on prouvera le théorème suivant.

Theorem 0.8. *Il existe $\beta_0 > 0$ tel que pour tout ν, $\alpha_0 \in \mathbb{R}$ avec $|\nu| + |\alpha_0| \leq \beta_0$ et tout $\delta > 0$, il existe $T > 0$ et une solution radiale $u \in C([T, +\infty), \dot{H}^1 \cap \dot{H}^2)$ de (10) de la forme :*

$$u(t,x) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + \zeta(t,x), \tag{11}$$

où $\lambda(t) = t^\nu$, $\alpha(t) = \alpha_0 \ln t$, et $\zeta(t)$ vérifie:

$$\begin{aligned} &\|\zeta(t)\|_{\dot{H}^1 \cap \dot{H}^2} \leq \delta, \\ &\|\zeta(t)\|_{L^\infty} \leq Ct^{-\frac{1+\nu}{2}}, \\ &\| < \lambda(t)x >^{-1} \zeta(t)\|_{L^\infty} \leq Ct^{-1-\frac{3}{2}\nu}, \end{aligned} \tag{12}$$

pour tout $t \geq T$. Les constantes C ici et dessous sont indépendantes de ν, α_0 et δ.
De plus il existe $\zeta^* \in \dot{H}^s$, $\forall s > \frac{1}{2} - \nu$, tel que, quand $t \to +\infty$, $\zeta(t) - e^{it\Delta}\zeta^* \to 0$ dans $\dot{H}^1 \cap \dot{H}^2$.

Comme mentionné ci-dessus, un résultat similaire pour l'équation des ondes à énergie critique a été obtenu par Donninger et Krieger [17]. Cette construction a été inspirée par l'article précédent de Krieger, Schlag et Tataru [35], où le cas d'explosions en temps fini a été traité. Ces deux articles, ont été une source d'inspiration pour partie des techniques employées dans la démonstration du théorème précédent.

Les résultés présentés ici vont à former trois publications :

- R. Adami, D. Noja, C. O., *Orbital and asymptotic stability for standing waves of a NLS equation with concentrated nonlinearity in dimension three*, accepté par Journal of Mathematical Physics, disponible sur arxiv.org/pdf/1207.5677.
- R. Adami, D. Noja, C. O., *Orbital and asymptotic stability for standing waves of a NLS equation with concentrated nonlinearity in dimension three. II*, en préparation.
- C. O., G. Perelman, *Nondispersive vanishing and blow up at infinity for the energy critical nonlinear Schrödinger equation in* $\mathbb{R}^3$, accepté par St. Petersburg Mathematical Journal.

Part I

Orbital and asymptotic stability for standing waves of a NLS equation with concentrated nonlinearity in dimension three

Chapter 1

Absence of nonvanishing eigenvalues

1.1 Introduction

In this chapter we begin a systematic analysis of the stability of solitary waves for a nonlinear Schrödinger equation with a nonlinearity concentrated in space dimension three. In particular, we show that the standing waves of the model are asymptotically stable in the sense that at large times, the evolution decomposes as the sum of a standing wave (possibly with different parameters from those of the reference initial soliton), a free linear wave, and a small remainder with a spatial decay stronger than the linear dispersive one.
An analogous study concerning the NLS equation with a concentrated nonlinearity in dimension one was given in [8] and [33]. These papers have been a source of inspiration for the present work, in particular for what concerns the general scheme of analysis and for some proofs. However, the one and the three-dimensional models are different, in particular the latter is strongly singular and its energy space is not contained in $H^1(\mathbb{R}^3)$. This fact prevents us from following step by step the techniques and the results of the cited papers; in particular, no formal manipulations with delta distributions are possible, and the full definition of a delta interaction as a point perturbation of the Laplacian is needed in the analysis. We shall comment on that along the paper.
We start by giving a presentation of the model. According to [1], we construct a Schrödinger equation with concentrated nonlinearities in dimension three by starting from the standard three-dimensional *linear* Schrödinger operator with a so-called point or delta interaction ([6]). Point interactions are widely used in Quantum Mechanics as models of contact or zero-range interactions and they are intended to describe strongly concentrated potentials at a point. In order to rigorously define a delta interaction located at the origin of $\mathbb{R}^3$ we first consider the Laplacian restricted to the set $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ and obtain a symmetric non selfadjoint operator with deficiency indices $(1,1)$. Second, by the classical Von Neumann-Krejn theory there exists a one-parameter family of selfadjoint extensions, which we denote by H_α. The operator H_α is defined on the domain

$$D(H_\alpha) = \{u \in L^2(\mathbb{R}^3) : \ u(x) = \phi(x) + qG_0(x) \, \text{with} \, \phi \in L^2_{loc}(\mathbb{R}^3) \ , \nabla\phi \in L^2(\mathbb{R}^3) \ , \Delta\phi \in L^2(\mathbb{R}^3),$$

$$q \in \mathbb{C}, \lim_{x\to 0}(u(x) - qG_0(x)) = \alpha q\}, \tag{1.1}$$

where G_0 is the Green's function of the Laplacian in three dimensions, i.e.

$$G_0(x) = \frac{1}{4\pi|x|}, \tag{1.2}$$

and the action is given by $H_\alpha u(x) = -\Delta\phi(x)$, $x \in \mathbb{R}^3$. To summarize, any element of the domain decomposes in a regular part ϕ and a singular (Coulombian) part; the coefficient q of the singular part is conventionally called *charge*, and the boundary condition imposes a relation between the charge and the value of the regular part at the origin depending on the so-called *strength* α of the point interaction, which is the parameter that fixes the selfadjoint extension.
An alternative equivalent and perhaps more direct construction, which better justifies the interpretation and the physical meaning of H_α, can be given by defining H_α as a suitable scaling limit (in norm resolvent sense) of a family of Schrödinger operators of the form $-\Delta + V_\epsilon$, where V_ϵ is a short range potential that approximates a delta distribution as $\epsilon \to 0$. Performing such limit requires a rescaling procedure in order to yield a non-trivial result, and the parameter α appearing in the above definition characterizes the particular selfadjoint extension and is related to zero energy resonances of the approximating operators. For details and further information see [6].
Whatever the definition given to the operator H_α is, we recall that, for $\alpha \geq 0$ (repulsive delta interaction), H_α is positive and its spectrum is purely absolutely continuous and coincides with $[0, +\infty)$, while for $\alpha < 0$ (attractive delta interaction) an isolated simple negative eigenvalue $\lambda = -(4\pi\alpha)^2$ appears, corresponding to a bound state. A second property relevant to the physical interpretation of the model and related to the value of α is that the scattering length of a delta interaction of strength α is given by $-(4\pi\alpha)^{-1}$. The closed and lower bounded quadratic form associated to H_α is

$$\mathbf{H}_\alpha(u) = \int_{\mathbb{R}^3} |\nabla\phi(x)|^2 dx + \alpha|q|^2, \tag{1.3}$$

defined on the domain of *finite energy states*

$$V = \{u \in L^2(\mathbb{R}^3) : \ u(x) = \phi(x) + q_u G_0(x), \text{ with } \phi \in L^2_{loc}(\mathbb{R}^3), \ \nabla\phi \in L^2(\mathbb{R}^3), \ q \in \mathbb{C}\}, \tag{1.4}$$

which is a Hilbert space endowed with the norm

$$\|u\|_V^2 = \|\nabla\phi\|_{L^2} + |q|^2. \tag{1.5}$$

Note that for a generic element u of the form domain the charge q and its regular part ϕ are independent of each other. determined by u; for example, the relation between the element u and its charge is given by Note also that the energy domain is strictly larger than $H^1(\mathbb{R}^3)$. So, the linear problem cannot be considered as a small perturbation of the standard free problem in the sense of the quadratic forms (at variance with the one-dimensional case). An equivalent representation of the energy space is obtained, fixed $\lambda > 0$, by

$$V = \left\{ u = \phi_\lambda + qG_\lambda, \text{ with } \phi_\lambda \in H^1(\mathbb{R}^3), \ q \in \mathbb{C} \ , \ G_\lambda(x) = \frac{e^{-\lambda|x|}}{4\pi|x|} \right\}, \tag{1.6}$$

and one can define an equivalent energy norm by

$$\|u\|_V^2 = \|\nabla\phi_\lambda\|_{L^2}^2 + |q|^2, \quad \forall u \in V.$$

Notice that $G_\lambda \in L^2(\mathbb{R}^3)$ and $\phi_\lambda \in H^1(\mathbb{R}^3)$, while in the representation (1.4) the regular part was just in the *homogeneous* Sobolev space $D^1(\mathbb{R}^3)$ only.
Following [1], the nonlinear model can be defined by allowing the strength α to depend on u as $\alpha(u) = -\nu|q|^{2\sigma}$, with $\nu > 0, \sigma > 0$, so that

$$D(H_\alpha) = \{u \in L^2(\mathbb{R}^3) : \ u(x) = \phi(x) + qG_0(x) \text{ with } \phi \in H^2_{loc}(\mathbb{R}^3), \ \Delta\phi \in L^2(\mathbb{R}^3),$$

$$q \in \mathbb{C}, \lim_{x \to 0}(u(x) - qG_0(x)) = -\nu|q|^{2\sigma} q\},$$

and $H_\alpha u = -\Delta\phi$. In the following sections, we often omit the notation $H_{\alpha(u)}$ in favour of H_α if no risk of confusion exists between the linear and the nonlinear operator. We stress that the nonlinearity we are considering is *focusing*. It can be interpreted as modeling the action of a defect in a medium which exerts a nonlinear response to the propagation. We remark that a more general definition of concentrated nonlinearities (with applications to the case of the wave equation) is given in [37].

We consider the evolution generated by the nonlinear operator $H_{\alpha(u)}$, i.e.

$$i\frac{du}{dt} = H_\alpha u. \tag{1.7}$$

In the present literature, there is some physical and numerical analysis of Schrödinger dynamics in presence of nonlinear defects, mainly focused on the milder one-dimensional case ([36],[45],[18]). The more technical construction of the three-dimensional problem has hindered extended modeling study, numerical work as well as rigorous analysis. Moreover, a certain amount of literature is devoted to NLS equation with nonhomogeneous (i.e. x-dependent and decaying) nonlinearities, yet with a relatively low decay at infinity (see [20, 23] and references therein).

Local (for any $\sigma > 0$) and global (for $\sigma < 1$) well-posedness of the Cauchy problem associated to the nonlinear Schrödinger equation (1.7) in the space V have been established in [1] and [2]. In particular, (1.7) admits two conserved quantities called *mass* and *energy*, defined as

$$M(u(t)) = \|u(t)\|^2_{L^2}, \qquad E(u(t)) = \tfrac{1}{2}\|\nabla\phi(t)\|^2_{L^2} - \tfrac{\nu}{2\sigma+2}|q(t)|^{2\sigma+2}.$$

In Section 1.2 we prove that equation (1.7) admits standing waves, i.e. solutions of the form $u(x,t) = e^{i\omega t}\Phi_\omega(x)$, where the profile or amplitude Φ_ω up to a phase factor $e^{i\theta}$ is given by

$$\Phi_\omega(x) = \left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{\frac{1}{2\sigma}} \frac{e^{-\sqrt{\omega}|x|}}{4\pi|x|} . \tag{1.8}$$

The set of standing waves is called the *solitary manifold* $\mathcal{M}$, and the main concern of this chapter consists in the study of the large-time evolution of initial data in the vicinity of $\mathcal{M}$. A first result concerns stability and instability of standing waves. Stability has to be intended as *orbital* stability, i.e. Lyapunov stability up to symmetries of the equation, in this case up to gauge ($U(1)$) invariance. The orbit of Φ_ω is then $\mathscr{O}(\Phi_\omega) = \{e^{i\theta}\Phi_\omega(x),\ \theta \in \mathbb{R}\}$. Thus, by definition, the state Φ_ω is orbitally stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(\psi(0), \mathscr{O}(\Phi_\omega)) < \delta \quad \Rightarrow \quad d(\psi(t), \mathscr{O}(\Phi_\omega)) < \epsilon \qquad \forall t > 0,$$

where $d(\psi, \mathscr{O}(\Phi_\omega)) = \inf_{u \in \mathscr{O}(\Phi_\omega)} \|\psi - u\|_V$. A stationary state is said to be unstable if it is not stable. Then, we have the following result, proved in Section 1.3:

Theorem (Orbital Stability) Let us consider (1.7). Then, for every $\omega > 0$,
(a) if $0 < \sigma < 1$, then the state Φ_ω is orbitally stable
(b) if $\sigma \geq 1$, then Φ_ω is orbitally unstable.

The result directly follows from Weinstein [53] and Grillakis-Shatah-Strauss [28] theory for the case $\sigma \neq 1$, while for the case $\sigma = 1$ the pseudoconformal invariance of the equation gives the instability by blow-up.

The core of the chapter is devoted to the study of the asymptotic stability of the family of stationary states. Asymptotic stability means, loosely speaking, that the solution $u(t)$ corresponding to an initial datum $u(0)$ close to the family of orbits, approaches some element of the family of orbits as $t \to \infty$. The analysis makes use of the representation

$$u(t,x) = e^{i\Theta(t)}\left(\Phi_{\omega(t)}(x) + \chi(t,x)\right), \tag{1.9}$$

where $\Theta(t) = \int_0^t \omega(s)ds + \gamma(t)$, and $\gamma(t)$ is a suitable phase. Namely, the solution is represented at every time as a modulated solitary wave, with time dependent parameters, up to a fluctuating remainder χ which has to be controlled. Asymptotic stability of the family of standing waves means that the modulating parameters $\omega(t)$ and $\gamma(t)$ have a limit as $t \to \infty$, and the fluctuation χ is in some sense a small and decaying dispersive correction; the radiation damping through dispersion is responsible for the "dissipative" asymptotic behavior of the solution u around the family of relative equilibria $\mathscr{O}(\Phi_\omega)$. Notice that, however, in general the solution does not converge to the solitary wave to which it was close initially.
The subject of asymptotic stability of solitary waves was pioneered by Soffer and Weinstein ([42], [43]), and Buslaev and Perelman ([9], [10]), who developed the main strategies and techniques, nowadays classical; a more recent presentation is contained in [11]. Many relevant later contributions refining and enlarging the hypotheses in the original papers, as well as concerning the kind of initial admitted data and nonlinearities, are contained in [14, 48, 51, 30, 22, 23, 15]. According to this consolidated analysis, one must preliminarily indagate the spectrum of the linearization of equation (1.7) around the solitary solution. Writing $u = e^{i\omega t}(\Phi_\omega + R)$ and identifying R with the vector of its real and imaginary part, we obtain that it satisfies the canonical system

$$J\frac{dR}{dt} = \begin{bmatrix} H_{\alpha_1} + \omega & 0 \\ 0 & H_{\alpha_2} + \omega \end{bmatrix} R \equiv DR$$

where H_{α_j} are (linear) delta interaction hamiltonian operators with fixed strength α_j that depend on the stationary state Φ_ω (through its charge) and on the parameters of the model ν, σ (see eq. (1.17)). So the dynamics of the linearization of the NLS equation around the standing wave Φ_ω is controlled by the nonselfadjoint (Hamiltonian) matrix operator $L = JD$. The explicit characterization of the spectrum of the linearization L is possible due to the detailed knowledge of the properties of operators H_{α_j}. Such feature is infrequent and allows to avoid further spectral assumptions. The complete result is given in Section 1.4, Theorem 1.10. Here it is sufficient to recall that in this chapter we study asymptotic stability of standing waves in the range $\sigma \in (0, 1/\sqrt{2})$ only, which corresponds to L having no eigenvalues different from zero and no resonances at the threshold of the essential spectrum. The following chapter will treat the case $\sigma \in (1/\sqrt{2}, 1)$, where two simple eigenvalues $\pm i2\sigma\sqrt{1-\sigma^2}\omega$ appear.
Let us notice that the representation (1.9) amounts in fact to a change of coordinates from the original global u to the new set $\{\omega, \gamma, \chi\}$, with a finite dimensional component given by $\{\omega, \gamma\}$, that describes the solitary manifold and an infinite dimensional one described by χ. However, the representation is not unique, because any choice of ω, γ gives a corresponding choice of χ such that u given by (1.9) is a solution of (1.7); so one has to restrict in some way the behavior of the new parameters $\{\omega, \gamma, \chi\}$ of the solution. To this end, we exploit the fact that the solitary manifold can be naturally endowed with a symplectic structure (see Section 1.2.1) and it turns out that its tangent space T_{Φ_ω} coincides with the generalized kernel of the linearization L. The generalized kernel is in turn non trivial, so the propagator e^{-tL} has a component growing in time. A parametrization of the running approximate solitary wave in the neighborhood of the solitary manifold suitable for asymptotic analysis is hence obtained through a symplectic splitting in a component along the solitary manifold and a component transversal (symplectically orthogonal) to it. Requiring that the infinite dimensional component χ is purely transversal, i.e. projecting to zero on the directions of the generalized kernel of the linearization, provides the set of the

so called modulation (coupled) equations for the parameters $\omega(t)$ and $\gamma(t)$, as well as a partial differential equation for χ (see [21] for an enlightening description of the symplectic projection method). The goal is to establish the asymptotic behavior of the solutions to the modulation equations with a simultaneous control of the decay of the nonlinear part χ, through the so-called majorant's method (see [9, 10, 11]).
The main result of this chapter is the following, and it is proven in Section 1.7.
Theorem (**Asymptotic stability**) Assume $\sigma \in (0, 1/\sqrt{2})$. Let $u \in C(\mathbb{R}^+, V)$ be a solution of equation (1.7) with $u(0) = u_0 \in V \cap L^1_w$ and denote $d = \|u_0 - e^{i\theta_0}\Phi_{\omega_0}\|_{V\cap L^1_w}$,
for some $\omega_0 > 0$ and $\theta_0 \in \mathbb{R}$. If d is sufficiently small, then the solution $u(t)$ decomposes asimptotically as follows

$$u(t) = e^{i\omega_\infty}\Phi_{\omega_\infty} + U_t * u_\infty + r_\infty, \quad t \to +\infty,$$

where $\omega_\infty > 0$ and u_∞, $r_\infty \in L^2(\mathbb{R}^3)$ with $\|r_\infty\|_{L^2} = O(t^{-5/4})$ as $t \to +\infty$.

In the previous statement, L^1_w is defined in Section 1.4.2 and is a weighted space of integrable functions. The weight guarantees the validity of the dispersive estimates needed in order to control the decay of the transversal evolution, and it seems at present unavoidable in view of the singularity of finite energy states. Moreover, it imposes a certain localization on the the admitted initial data, which seems to be a technical requirement.
Concerning the treatment of the modulation equations, one of the main additional difficulties with respect to standard models, and in particular with the case of concentrated nonlinearities in one dimension treated in [8] and [33], is that the equations controlling the evolution of the transversal part χ have domains that change with time. This fact forced us to make use of the variational formulation (i.e. in terms of quadratic forms) instead of the traditional strong formulation (i.e. in terms of operators). The same problem propagates to the proof of the asymptotics given in the above theorem. A last remark concerns the seemingly anomalous value of the nonlinearities where asymptotic stability is proven; this because in the typical situations, when standard NLS with or without potential is treated, it is difficult to have information about subcritical nonlinearities (but see the notably exception in [32]), and in particular pure power. On the other hand, the present model corresponds to an inhomogeneous (space dependent and strongly singular) nonlinearity; this seems to indicate that the analysis of specific models can give results not accessible to general theory, at least at present.

1.2 Preliminaries

1.2.1 Hamiltonian structure

We consider $L^2(\mathbb{R}^3, \mathbb{C})$ as a real Hilbert space endowed with the scalar product

$$(u, v)_{L^2} = \mathrm{Re}\int_{\mathbb{R}^3} u\overline{v}\,dx = \int_{\mathbb{R}^3} (\mathrm{Re}\,v\,\mathrm{Re}\,u + \mathrm{Im}\,v\,\mathrm{Im}\,u)dx. \tag{1.10}$$

It is sometimes convenient to shift from the complex valued representation of u to the vector real valued one through the identification $u = \mathrm{Re}\,u + i\,\mathrm{Im}\,u \mapsto (\mathrm{Re}\,u, \mathrm{Im}\,u) = (u_1, u_2)$. As a consequence, $H^s(\mathbb{R}^3, \mathbb{C}) \cong H^s(\mathbb{R}^3, \mathbb{R}^2)$, while multiplication by i is equivalent to multiplication by the matrix $-J$, where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{1.11}$$

The space $L^2(\mathbb{R}^3)$ is also a symplectic manifold when endowed with the symplectic form

$$\Omega(u,v) = \operatorname{Im}\int_{\mathbb{R}^3} u\overline{v}\,dx = \int_{\mathbb{R}^3} (\operatorname{Re} v \operatorname{Im} u - \operatorname{Im} v \operatorname{Re} u)dx = \int_{\mathbb{R}^3} (u_2 v_1 - u_1 v_2)dx. \tag{1.12}$$

Along the chapter we often shift between real and complex representation when no ambiguity occurs.
In our model the Hamiltonian functional coincides with the total energy and it is given by

$$E(u(t)) = \frac{1}{2}\|\nabla\phi\|^2_{L^2} - \frac{\nu}{2\sigma+2}|q|^{2\sigma+2}, \quad u = \phi + qG_0 \in V. \tag{1.13}$$

Correspondingly, the NLS equation (1.7) takes the hamiltonian form

$$\frac{du}{dt}(t) = J\ E^{'}(u(t))\ . \tag{1.14}$$

1.2.2 Standing waves

Standing waves are solutions of the form $u(x,t) = e^{i\omega t}\Phi_\omega(x) \in V$. It immediately follows that if a standing wave exists, then the amplitude Φ_ω satisfies the following nonlinear equation in weak form

$$H_\alpha \Phi_\omega + \omega\Phi_\omega = 0. \tag{1.15}$$

Proposition 1.1. *Standing waves for equation* (1.7) *exist if and only if* $\nu > 0$. *In such a case the set of solitary waves is given by the two-dimensional manifold*

$$\mathcal{M}\ =\ \left\{\ e^{i\Theta}\ \Phi_\omega\ ,\ \omega > 0\ ,\ \Theta \in [0,2\pi)\right\}, \tag{1.16}$$

where the function Φ_ω *reads*

$$\Phi_\omega(x) = \left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{\frac{1}{2\sigma}} \frac{e^{-\sqrt{\omega}|x|}}{4\pi|x|}$$

and the parameters ω *and* Θ *play the role of local coordinates.*

Proof. Recall that the function G_0 defined in (1.2) satisfies the equation $-\triangle G_0 = \delta$
where δ is the Dirac's delta distribution centred at $x = 0$. Hence, for $x \neq 0$ equation (1.15) is equivalent to $-\triangle\Phi_\omega(x) + \omega\Phi_\omega(x) = 0$. Consider the corresponding equation in spherical coordinates, namely

$$-\frac{\partial^2 u}{\partial^2 r} - \frac{2}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial^2\phi} - \frac{\cos\phi}{r^2\sin\phi}\frac{\partial u}{\partial\phi} - \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial^2\theta} + \omega u = 0,$$

and exploit the spherical harmonics expansion of the solution $u(r,\theta,\phi) = \sum_{l=0}^{+\infty}\sum_{j=-l}^{l} u_{l,j}(r) Y_{l,j}(\theta,\phi)$, where $Y_{l,j}$ denotes the set of spherical harmonics which is an orthonormal basis of $L^2([0,\pi]\times[0,2\pi], \sin\theta d\theta d\phi)$. Since

$$\frac{\partial^2 Y_{l,j}}{\partial\phi^2} + \frac{\cos\phi}{\sin\phi}\frac{\partial Y_{l,j}}{\partial\phi} + \frac{1}{\sin^2\phi}\frac{\partial^2 Y_{l,j}}{\partial\theta^2} = -\lambda Y_{l,j}, \qquad \text{for some } \lambda \in \mathbb{C},$$

one has that λ belongs to the set $\{\lambda_l := l(l+1),\ l \in \mathbb{N}\}$, and so the functions $u_{l,j}$ solves $-u''_{l,j}(r) - \frac{2}{r}u'_{l,j}(r) + \left(\omega - \frac{\lambda_l}{r^2}\right)u_{l,j}(r) = 0$. Then, from formula 8.491.6 in [27],

$$u_{j,l}(r) = \frac{1}{\sqrt{r}} Z_{\sqrt{\frac{1}{4}+\lambda}}(\sqrt{\omega}r),$$

where Z_ν is a generic Bessel's function. By the asymptotic expansions 8.443 and 8.451.1 in [27] one immediately has that if $\lambda \neq 0$, then $u_{j,l}$ cannot belong to $L^2(\mathbb{R}^+, r^2 dr)$. Hence, we fix $\lambda = 0$ and denote $\Phi_\omega(x) = \frac{u(r)}{r}$, $r = |x|$. Thus u has to be a $L^2(\mathbb{R}^+, r^2 dr)$ solution of $u''(r) - \omega u(r) = 0$, and

$$\Phi_\omega(x) = \frac{q e^{-\sqrt{\omega}|x|}}{4\pi |x|},$$

for some $q \in \mathbb{C}$ and $\omega > 0$.
Writing (1.15) in weak form and separating regular and singular part of the test function, one obtains

$$-\nu |q|^{2\sigma} q \overline{q_v} + \frac{\sqrt{\omega}}{4\pi} q \overline{q_v} = 0,$$

for all $q_v \in \mathbb{C}$ which coincides with the boundary condition for H_α. Supposing $\nu \neq 0$ one obtains $|q|^{2\sigma} = \frac{\sqrt{\omega}}{4\pi\nu}$. This requires $\nu > 0$, so

$$\Phi_\omega(x) = \left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{\frac{1}{2\sigma}} \frac{e^{-\sqrt{\omega}|x|}}{4\pi|x|}$$

which, up to a phase factor, gives the stated result. In the case $\nu = 0$, from boundary condition we get $q = 0$ or $\omega = 0$. If $q = 0$, then the function u vanishes. If $\omega = 0$, then one has $u(x) = \frac{1}{4\pi|x|}$, which is the resonance function of the delta interaction with vanishing strength, but it is not an element of the operator domain, and it does not solve the stationary equation (1.15). So for $\nu = 0$ standing waves do not exist. □

In the following we denote $q_\omega = \left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{\frac{1}{2\sigma}}$.

Remark 1.2. From the proof above, it turns out that a finite energy standing wave is in fact an element of $D(H_\alpha)$.

1.2.3 Linearization of $H_{\alpha(u)}$ around Φ_ω

The linearization of equation (1.7) around a stationary solution is not completely obvious, due to the fact that the nonlinearity is embodied in the domain of the operator $H_{\alpha(u)}$ *and not* in the action of the operator itself. Nevertheless, we can consider the Hamiltonian associated to equation (1.7) given by formula (1.14) and notice that the nonlinearity no longer appears in the domain V but directly in the Hamiltonian functional. So we derive the linear operator which approximates $H_{\alpha(u)}$ from the quadratic form which approximates $E(\Phi_\omega)$ and obtain the following result.

Proposition 1.3. *The Hessian $E''(\Phi_\omega)$ of the functional E can be represented as $E''(\Phi_\omega)(h,k) = \langle H_{\alpha,Lin}\ h, k\rangle$, where $H_{\alpha,Lin}$ is the operator given by*

$$H_{\alpha,Lin} = \begin{bmatrix} H_{\alpha_1} & 0 \\ 0 & H_{\alpha_2} \end{bmatrix},$$

where H_{α_1} and H_{α_2} are the selfadjoint operators on $L^2(\mathbb{R}^3)$ defined in the introduction (see (1.1)), and

$$\alpha_1 = -\nu(2\sigma+1)|q_\omega|^{2\sigma} = -\frac{2\sigma+1}{4\pi}\sqrt{\omega}, \qquad \alpha_2 = -\nu|q_\omega|^{2\sigma} = -\frac{\sqrt{\omega}}{4\pi}. \tag{1.17}$$

$H_{\alpha,lin}$ is selfadjoint with respect to the real scalar product in $L^2(\mathbb{R}^3, \mathbb{C})$.

Proof. The first Gâteaux derivative of $E(u)$ reads

$$E'(u)[h] = \frac{d}{d\epsilon}\{E(u+\epsilon h)\}_{\epsilon=0} = \mathrm{Re}\int_{\mathbb{R}^3} \nabla\phi_u(x)\cdot\overline{\nabla\phi_h(x)}dx - \nu|q_u|^{2\sigma}\,\mathrm{Re}(q_u\overline{q_h}) \quad \forall u,h\in V, \tag{1.18}$$

while the second Gâteaux derivative at Φ_ω reads

$$\frac{\partial^2}{\partial\epsilon\partial\lambda}\{E(\Phi_\omega+\epsilon h+\lambda k)\}_{\epsilon=0,\lambda=0} = \mathrm{Re}\int_{\mathbb{R}^3}\nabla\phi_h(x)\cdot\overline{\nabla\phi_k(x)}dx - \frac{\partial^2}{\partial\epsilon\partial\lambda}\{\frac{\nu}{2\sigma+2}|q_u|^{2\sigma+2}\}_{\epsilon=0,\lambda=0}\ .$$

The last term gives, after some calculation, the contribution (here $h=(h_1,h_2),\ k=(k_1,k_2)$)

$$\frac{\partial^2}{\partial\epsilon\partial\lambda}\{-\frac{\nu}{2\sigma+2}|q_u|^{2\sigma+2}\}_{\epsilon=0,\lambda=0} = -\nu|q_\omega|^{2\sigma}[(2\sigma+1)q_{h_1}q_{k_1}+q_{h_2}q_{k_2}]\ .$$

So $E''(\Phi_\omega)$ is given by the direct sum of two quadratic forms: one is acting on the real part of the functions h and k, and the other on the imaginary part. The term related to the real part is a lower bounded quadratic form whose corresponding selfadjoint operator is H_{α_1}, while the quadratic form related to the imaginary part corresponds to the operator H_{α_2} (α_1 and α_2 have been defined in (1.17)). Then, the operator $H_{\alpha,Lin}$ that represents the entire quadratic form $E''(\Phi_\omega)$ is self-adjoint and the proof is complete. □

Now, to get the linearized equation set $u(t)=e^{i\omega t}(\Phi_\omega+R(t))$ and obtain

$$\frac{d}{dt}R = J(E'(\Phi_\omega)+\omega\Phi_\omega)+J(E''(\Phi_\omega)+\omega)R + \text{higher order terms} \simeq J(H_{\alpha,Lin}+\omega)R\ .$$

Summing up, the linearized equation (1.7) becomes

$$\frac{dR}{dt} = JDR, \tag{1.19}$$

where $D=\begin{bmatrix} L_1 & 0\\ 0 & L_2\end{bmatrix}$, with

$$L_j = H_{\alpha_j}+\omega, \tag{1.20}$$

$j=1,\,2$. Notice that the operator

$$JD := L = \begin{bmatrix} 0 & L_2\\ -L_1 & 0\end{bmatrix}, \tag{1.21}$$

is not selfadjoint nor skew adjoint. Nevertheless, a standard application of Hille-Yosida theorem and a simple analysis of the resolvent of L which takes into account the factorized structure $L=JD$ with D s.a. shows that it generates a semigroup of linear operators with (at most) exponential growth in time. A more precise analysis of the resolvent of the operator L will be given in Theorem 1.10 and in the appendix 1.11 we will prove that the semigroup has in fact a linear growth (see Theorem 1.32) in the case here interesting, i.e. $\sigma\in(0,1/\sqrt{2})$.

1.3 Orbital stability

In order to prove the orbital stability of the stationary solutions to equation (1.7), we apply Grillakis-Shatah-Strauss theory, and in particular Theorem 2 in [28]. As a first step, we recall the following known fact proved in [6].

Proposition 1.4. *If $\alpha(u) = \alpha$ where $\alpha < 0$ is a constant, then*

$$\sigma(H_\alpha) \equiv \{-(4\pi\alpha)^2\} \cup [0, +\infty). \tag{1.22}$$

Thanks to the last proposition one can prove the following lemma which implies the spectral properties needed to verify Assumption 3 in [28].

Lemma 1.5. *The spectrum of the operator D is*

$$\sigma(D) = \{-4\sigma(\sigma+1)\omega, 0\} \cup [\omega, +\infty),$$

and $\ker(D) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}\right\}$.

Proof. Since D is the direct sum of the operators L_1 and L_2 acting on $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, its spectrum is given by the union of $\sigma(L_1)$ and $\sigma(L_2)$. From (1.22) follows

$$\sigma(H_{\alpha_1}) = \{-(2\sigma+1)^2\omega\} \cup [0, +\infty), \quad \sigma(H_{\alpha_2}) = \{-\omega\} \cup [0, +\infty).$$

Then

$$\sigma(L_1) = \sigma(H_{\alpha_1}) + \omega = \{-4\sigma(\sigma+1)\omega\} \cup [\omega, +\infty), \quad \sigma(L_2) = \sigma(H_{\alpha_2}) + \omega = \{0\} \cup [\omega, +\infty).$$

Hence, $\ker(L_1) = \{0\}$ and $\ker(L_2) = \operatorname{span}\{\Phi_\omega\}$, which concludes the proof. □

We can now prove the following

Theorem 1.6. (Orbital stability) *For each $\omega > 0$, if $0 < \sigma < 1$, then Φ_ω is orbitally stable. If $\sigma > 1$, then Φ_ω is orbitally unstable.*

Proof. Well-posedness and existence of a branch of standing waves, i.e. Assumptions 1 and 2 in [28], are proved in [1] and [2] and in the previous section, while Assumption 3 is true thanks to Lemma 1.5. Hence, from Theorem 3 in [28] we have orbital stability if $\frac{d}{d\omega}\|\Phi_\omega(x)\|^2_{L^2} > 0$ and orbital instability if $\frac{d}{d\omega}\|\Phi_\omega(x)\|^2_{L^2}) < 0$. In order to inspect the sign of $\frac{d}{d\omega}\|\Phi_\omega(x)\|^2_{L^2}$, we compute

$$\|\Phi_\omega\|^2_{L^2} = \left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{\frac{1}{\sigma}} \frac{1}{8\pi\sqrt{\omega}},$$

hence $\frac{d}{d\omega}\|\Phi_\omega(x)\|^2_{L^2} = \frac{1}{8\pi(4\pi\nu)^{1/\sigma}} \frac{1-\sigma}{2\sigma} \omega^{\frac{1-3\sigma}{2\sigma}}$, which concludes the proof. □

1.3.1 The case $\sigma = 1$

Since Theorem 3 in [28] does not give information about orbital stability of the stationary state $e^{i\omega t}\Phi_\omega$ when $\frac{d}{d\omega}\|\Phi_\omega(x)\|^2_{L^2} = 0$, we need to inspect the case $\sigma = 1$ apart. In such case, equation (1.7) exhibits one additional symmetry (see [2]).

Remark 1.7. Equation (1.7) is invariant under the pseudoconformal transformation

$$u^T_{pc}(t,x) = \frac{e^{-i\frac{|x|^2}{4(T-t)}}}{(T-t)^{3/2}} u\left(\frac{1}{T-t}, \frac{|x|}{T-t}\right).$$

In [1] it is proved that equation (1.7) may have some non global solutions which blow up, in the following sense: the solution $u(t)$ of equation (1.7) blows up (in the future) at time $T < +\infty$ if

$$\limsup_{t \to T^-} \|\nabla \phi\|_{L^2} = +\infty,$$

where ϕ is the regular part of the function u according to decomposition (1.1). Due to the conservation of the energy this condition is equivalent to $\limsup_{t\to T^-} |q_u(t)| = +\infty$.
Thanks to the pseudoconformal invariance we prove that in any neighbourhood (in energy norm) of each standing wave there are initial data of a blow up solution.

Theorem 1.8. *Fix $\sigma = 1$ and $\omega > 0$. For any $\delta > 0$ there exists a blow up solution $u(t) \in V$ such that $\|u(0) - \Phi_\omega\|_V < \delta$.*

Proof. Applying the pseudoconformal transformation to the solitary wave $e^{i\widetilde{\omega}t}\Phi_{\widetilde{\omega}}$ one gets that for any $T > 0$, the function

$$u_{\widetilde{\omega},T}(t,x) = e^{i\frac{\widetilde{\omega}}{T-t}} \frac{\widetilde{\omega}^{1/4}}{\sqrt{4\pi\nu}} \frac{e^{-\frac{\sqrt{\widetilde{\omega}}|x|}{T-t}}}{4\pi\sqrt{T-t}|x|} e^{-i\frac{|x|^2}{4(T-t)}}$$

is a solution to equation (1.7). Thus, for any $T > 0$, the initial datum $u_T(x) = e^{i\frac{\widetilde{\omega}}{T}} \frac{\widetilde{\omega}^{1/4}}{\sqrt{4\pi\nu}} \frac{e^{-\frac{\sqrt{\widetilde{\omega}}|x|}{T}}}{4\pi\sqrt{T-t}|x|} e^{-i\frac{|x|^2}{4T}}$ gives rise to a solution that blows up at time T. Now, let $\widetilde{\omega}$ depend on T as $\widetilde{\omega} = \omega T^2$, so that $u_T(x) = e^{-i\frac{|x|^2}{4T}}\Phi_\omega(x)$.
We prove the theorem by showing that $\|(e^{-i\frac{|\cdot|^2}{4T}} - 1)\Phi_\omega\|_V \to 0$ as $T \to +\infty$. Indeed, noting that the function $(e^{-i\frac{|\cdot|^2}{4T}} - 1)\Phi_\omega$ belongs to $H^1(\mathbb{R}^3)$,

$$\|(e^{-i\frac{|\cdot|^2}{4T}} - 1)\Phi_\omega\|_V = \|\nabla((e^{-i\frac{|\cdot|^2}{4T}} - 1)\Phi_\omega)\|_{L^2} \le \frac{1}{2T}\||\cdot|\Phi_\omega\|_{L^2} + \frac{1}{4T}\||\cdot|^2\nabla\Phi_\omega\|_{L^2} \to 0, \quad T \to +\infty.$$

□

1.4 Spectral and dispersive properties of linearization L

Here we study the long time behaviour of equation (1.19), that is the linearization of (1.7) around the stationary solution $e^{i\omega t}\Phi_\omega$.
The generalized kernel of the operator L (see (1.21)) is defined as $N_g(L) = \bigcup_{k\in\mathbb{N}} \ker(L^k)$.

In what follows let us denote

$$\begin{aligned}
\varphi_\omega(x) &= \frac{d\Phi_\omega}{d\omega}(x) = \frac{1}{4\sigma\omega}\left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{\frac{1}{2\sigma}} \frac{e^{-\sqrt{\omega}|x|}}{4\pi|x|} - \frac{1}{2\sqrt{\omega}}\left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{\frac{1}{2\sigma}} \frac{e^{-\sqrt{\omega}|x|}}{4\pi},\\
g_\omega(x) &= \frac{\omega^{\frac14}}{\sqrt{4\pi\nu}}|x|\frac{e^{-\sqrt{\omega}|x|}}{4\pi},\\
h_\omega(x) &= \frac{\omega^{\frac14}}{\sqrt{4\pi\nu}}\left(-\frac{1}{4\omega^{\frac32}}\frac{e^{-\sqrt{\omega}|x|}}{4\pi|x|} + \frac{1}{2\omega}\frac{e^{-\sqrt{\omega}|x|}}{4\pi} + \frac{1}{2\sqrt{\omega}}|x|\frac{e^{-\sqrt{\omega}|x|}}{4\pi} + \frac13|x|^2\frac{e^{-\sqrt{\omega}|x|}}{4\pi}\right).
\end{aligned}$$

In Appendix 1.9 we prove the following theorem.

Theorem 1.9. *If the nonlinearity power σ is different from 1, then* $N_g(L) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}, \begin{pmatrix} \varphi_\omega \\ 0 \end{pmatrix}\right\}$. *Moreover, if $\sigma = 1$, then* $N_g(L) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}, \begin{pmatrix} \varphi_\omega \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g_\omega \end{pmatrix}, \begin{pmatrix} h_\omega \\ 0 \end{pmatrix}\right\}$.

In the following section we provide an explicit description of the spectrum of the non-selfadjoint operator L and the dispersive estimates for the action of the propagator e^{-Lt} upon the absolutely continuous subspace.

1.4.1 The resolvent and the spectrum of the linearized operator

The purpose of this section is to prove an explicit formula for the resolvent of the linearized operator. For later convenience we denote

$$G_{\omega\pm i\lambda}(x) = \frac{e^{i\sqrt{-\omega\mp i\lambda}|x|}}{4\pi|x|} \qquad \omega > 0, \lambda \in \mathbb{C}, \tag{1.23}$$

with the prescription $\operatorname{Im}\sqrt{-\omega \pm i\lambda} > 0$.
Furthermore, we make use of the notation $\langle g, h\rangle := \int_{\mathbb{R}^3} g(x)h(x)\,dx$.
We prove the following

Theorem 1.10. *The resolvent $R(\lambda) = (L - \lambda I)^{-1}$ of the operator L defined in* (1.21) *is given by*

$$R(\lambda) = \begin{bmatrix} -\lambda\mathcal{G}_{\lambda^2}* & -\Gamma_{\lambda^2}* \\ \Gamma_{\lambda^2}* & -\lambda\mathcal{G}_{\lambda^2}* \end{bmatrix} + \frac{4\pi}{W(\lambda^2)} i \begin{bmatrix} \Lambda_1 & i\Sigma_2 \\ -i\Sigma_1 & \Lambda_2 \end{bmatrix}, \tag{1.24}$$

where

$$W(\lambda^2) = 32\pi^2\alpha_1\alpha_2 - 4i\pi(\alpha_1 + \alpha_2)\left(\sqrt{-\omega + i\lambda} + \sqrt{-\omega - i\lambda}\right) - 2\sqrt{-\omega + i\lambda}\sqrt{-\omega - i\lambda},$$

and formula (1.24) *holds for all* $\lambda \in \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \ W(\lambda^2) = 0, \ \ or \ \ \operatorname{Re}(\lambda) = 0 \ and \ |\operatorname{Im}(\lambda)| \geq \omega\}$. *Furthermore, the symbol $*$ in* (1.24) *denotes the convolution and*

$$\mathcal{G}_{\lambda^2}(x) = \frac{1}{2i\lambda}\left(G_{\omega-i\lambda}(x) - G_{\omega+i\lambda}(x)\right), \quad \Gamma_{\lambda^2}(x) = \frac{1}{2}\left(G_{\omega-i\lambda}(x) + G_{\omega+i\lambda}(x)\right). \tag{1.25}$$

Finally, the entries of the second matrix are finite rank operators whose action on $f \in L^2(\mathbb{R}^3)$ reads

$$\begin{aligned}
\Lambda_1 f &= [i\lambda(4\pi\alpha_2 - i\sqrt{-\omega + i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle - (4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega+i\lambda} + \\
&\quad + [i\lambda(4\pi\alpha_2 - i\sqrt{-\omega - i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle + (4\pi\alpha_1 - i\sqrt{-\omega - i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega-i\lambda}, \\
\Lambda_2 f &= [i\lambda(4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle - (4\pi\alpha_2 - i\sqrt{-\omega + i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega+i\lambda} + \\
&\quad + [i\lambda(4\pi\alpha_1 - i\sqrt{-\omega - i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle + (4\pi\alpha_2 - i\sqrt{-\omega - i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega-i\lambda}, \\
\Sigma_1 f &= -[i\lambda(4\pi\alpha_2 - i\sqrt{-\omega + i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle - (4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega+i\lambda} + \\
&\quad + [i\lambda(4\pi\alpha_2 - i\sqrt{-\omega - i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle + (4\pi\alpha_1 - i\sqrt{-\omega - i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega-i\lambda}, \\
\Sigma_2 f &= -[i\lambda(4\pi\alpha_1 - i\sqrt{-\omega + i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle - (4\pi\alpha_2 - i\sqrt{-\omega + i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega+i\lambda} + \\
&\quad + [i\lambda(4\pi\alpha_1 - i\sqrt{-\omega - i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle + (4\pi\alpha_2 - i\sqrt{-\omega - i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega-i\lambda}.
\end{aligned} \tag{1.26}$$

The spectrum of the operator L can be decomposed into an essential and a discrete part,

$$\sigma(L) = \sigma_{ess}(L) \cup \sigma_d(L), \tag{1.27}$$

where the essential spectrum is

$$\sigma_{ess}(L) = \mathcal{C}_+ \cup \mathcal{C}_- = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) = 0 \text{ and } \operatorname{Im}(\lambda) \geq \omega\} \cup \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) = 0 \text{ and } \operatorname{Im}(\lambda) \leq -\omega\},$$

and the discrete spectrum depends on the paremeter σ as follows:

(a) if $\sigma \in (0, 1/\sqrt{2}]$, then the only eigenvalue of L is 0 with algebraic multiplicity 2.

(b) if $\sigma \in (1/\sqrt{2}, 1)$, then L has two simple eigenvalues $\pm i 2\sigma\sqrt{1-\sigma^2}\omega$ and the eigenvalue 0 with algebraic multiplicity 2.

(c) if $\sigma = 1$, then the only eigenvalue of L is 0 with algebraic multiplicity 4.

(d) if $\sigma \in (1, +\infty)$, then L has two simple eigenvalues $\pm 2\sigma\sqrt{\sigma^2-1}\omega$ and the eigenvalue 0 with algebraic multiplicity 2.

Before giving the proof, we need two preliminary lemmas.

Lemma 1.11. *For any $\mu \in \mathbb{C}$, $\omega > 0$, the Green's function $\mathcal{G}_\mu$ of the operator $\mathcal{H}_\mu$, defined by*

$$D(\mathcal{H}_\mu) = H^4(\mathbb{R}^3), \quad \mathcal{H}_\mu = \mu + (-\triangle + \omega)^2,$$

reads

$$\mathcal{G}_\mu(x) = \frac{1}{2i\sqrt{\mu}} \left(G_{\omega - i\sqrt{\mu}}(x) - G_{\omega + i\sqrt{\mu}}(x) \right). \tag{1.28}$$

Proof. By definition of Green's function, $\mathcal{G}_\mu$ solves the equation $[\mu + (-\triangle + \omega)^2]\mathcal{G}_\mu(x) = \delta(x)$. Taking the Fourier transform, one gets

$$\widehat{\mathcal{G}_\mu}(k) = \frac{1}{(2\pi)^{3/2}(\mu + (k^2+\omega)^2)} = \frac{1}{2i\sqrt{\mu}} \left(\widehat{G_{\omega - i\sqrt{\mu}}}(k) - \widehat{G_{\omega + i\sqrt{\mu}}}(k) \right),$$

where the function $G_{\omega \pm i\sqrt{\mu}}$ was defined in (1.23). The proof is complete. □

Remark 1.12. The function $\mathcal{G}_\mu$ is an element of $H^s(\mathbb{R}^3)$ for any $s < 7/2$.

Let us denote

$$\mathcal{H}^{21}_\mu = \mu + L_2 L_1,$$

where L_2 and L_1 were defined in (1.20). Applying elementary rules on composition of operators, one can easily see that the domain of the operator $\mathcal{H}^{21}_\mu$, which coincides with the domain of $L_2 L_1$, is given by

$$\begin{aligned} D(L_2L_1) = \Big\{ & u \in L^2(\mathbb{R}^3) : \ u = \xi + pG_{\omega+i\sqrt{\mu}} + qG_{\omega-i\sqrt{\mu}}, \text{ with } \xi \in H^4(\mathbb{R}^3), \ p, q \in \mathbb{C}, \\ & \xi(0) + ip\frac{\sqrt{-\omega - i\sqrt{\mu}}}{4\pi} + iq\frac{\sqrt{-\omega + i\sqrt{\mu}}}{4\pi} = \alpha_1(p+q), \\ & (-\triangle + \omega)\xi(0) + \sqrt{\mu}p\frac{\sqrt{-\omega - i\sqrt{\mu}}}{4\pi} - \sqrt{\mu}q\frac{\sqrt{-\omega + i\sqrt{\mu}}}{4\pi} = \alpha_2 i\sqrt{\mu}(q-p) \Big\}. \end{aligned} \tag{1.29}$$

In the following lemma the inverse operator of $\mathcal{H}^{21}_\mu$ is constructed.

Lemma 1.13. *For each $\mu \in \mathbb{C}$, the inverse of the operator $\mathcal{H}^{21}_\mu$ is given by*

$$(\mathcal{H}^{21}_\mu)^{-1} : L^2(\mathbb{R}^3) \to D(\mathcal{H}^{21}_\mu) \qquad f \mapsto \mathcal{G}_\mu * f \ + p(f) G_{\omega+i\sqrt{\mu}} + q(f) G_{\omega-i\sqrt{\mu}}, \tag{1.30}$$

where the functionals p,$q : L^2(\mathbb{R}^3) \to \mathbb{C}$ act as

$$\begin{aligned} p(f) &= \tfrac{4\pi}{i\sqrt{\mu}W(\mu)}[i\sqrt{\mu}(4\pi\alpha_2 - i\sqrt{-\omega+i\sqrt{\mu}})\langle \mathcal{G}_\mu, f\rangle - (4\pi\alpha_1 - i\sqrt{-\omega+i\sqrt{\mu}})\langle \Gamma_\mu, f\rangle] \\ q(f) &= \tfrac{4\pi}{i\sqrt{\mu}W(\mu)}[i\sqrt{\mu}(4\pi\alpha_2 - i\sqrt{-\omega-i\sqrt{\mu}})\langle \mathcal{G}_\mu, f\rangle + (4\pi\alpha_1 - i\sqrt{-\omega-i\sqrt{\mu}})\langle \Gamma_\mu, f\rangle], \end{aligned} \tag{1.31}$$

with $\mathcal{G}_\mu$ and Γ_μ are given by (1.25)*, and*

$$W(\mu) = 2(4\pi)^2\alpha_1\alpha_2 - 4i\pi(\alpha_1+\alpha_2)\left(\sqrt{-\omega+i\sqrt{\mu}} + \sqrt{-\omega-i\sqrt{\mu}}\right) - 2\sqrt{-\omega+i\sqrt{\mu}}\sqrt{-\omega-i\sqrt{\mu}}.$$

Proof. First we show that the definition of the functionals p and q ensures

$$\mathcal{G}_\mu * f + p(f)G_{\omega+i\sqrt{\mu}} + q(f)G_{\omega-i\sqrt{\mu}} \ \in \ D(\mathcal{H}^{21}_\mu) = D(L_2L_1)$$

for all $f \in L^2(\mathbb{R}^3)$. Indeed, $p(f)$ and $q(f)$ solve the algebraic system given by the bounday condition in the definition of the domain (1.29), namely

$$\begin{cases} \langle \mathcal{G}_\mu, f\rangle + ip\frac{\sqrt{-\omega-i\sqrt{\mu}}}{4\pi} + iq\frac{\sqrt{-\omega+i\sqrt{\mu}}}{4\pi} = \alpha_1(p+q) \\ \langle \Gamma_\mu, f\rangle + \sqrt{\mu}p\frac{\sqrt{-\omega-i\sqrt{\mu}}}{4\pi} - \sqrt{\mu}q\frac{\sqrt{-\omega+i\sqrt{\mu}}}{4\pi} = \alpha_2 i\sqrt{\mu}(q-p). \end{cases}$$

Now, denote by $\widehat{H}_0$ the operator that acts as the Laplacian on the subspace of the Schwartz functions in $\mathbb{R}^3$ that vanish in a neighbourhood of the origin. It is well-known (see [6]), that both selfadjoint operators H_{α_1} and H_{α_2} defined in Proposition 1.3 are restrictions of $\widehat{H}_0^*$ (i.e. the adjoint of $\widehat{H}_0$ as an operator in $L^2(\mathbb{R}^3)$), whose action on $G_{\omega\pm i\sqrt{\mu}}$ yields

$$[\mu + (\widehat{H}_0^* + \omega)^2]G_{\omega\pm i\sqrt{\mu}} \ = \ 0. \tag{1.32}$$

Recalling that $\mathcal{G}_\mu \in H^4(\mathbb{R}^3)$, it follows, for any $f \in L^2(\mathbb{R}^3)$,

$$\mathcal{H}^{21}_\mu(\mathcal{G}_\mu * f + p(f)G_{\omega+i\sqrt{\mu}} + q(f)G_{\omega-i\sqrt{\mu}}) \ = \ (\mu + (\widehat{H}_0^* + \omega)^2)(\mathcal{G}_\mu * f + p(f)G_{\omega+i\sqrt{\mu}} + q(f)G_{\omega-i\sqrt{\mu}}) =$$

$$= (\mu + (-\triangle+\omega)^2)(\mu + (-\triangle+\omega)^2)^{-1}f = f.$$

To conclude the proof one has to show

$$\mathcal{G}_\mu * (\mathcal{H}^{21}_\mu f) + p(\mathcal{H}^{21}_\mu f)G_{\omega+i\sqrt{\mu}} + q(\mathcal{H}^{21}_\mu f)G_{\omega-i\sqrt{\mu}} = f$$

for any $f \in D(\mathcal{H}_{21})$. To this purpose let us set $f = \xi + aG_{\omega+i\sqrt{\mu}} + bG_{\omega-i\sqrt{\mu}}$ for some $\xi \in H^4(\mathbb{R}^3)$ and a, $b \in \mathbb{C}$ such that the boundary condition in (1.29) are satisfied, then, by (1.32)

$$\mathcal{H}^{21}_\mu f \ = \ [\mu + (-\Delta+\omega)^2]\xi$$

and, by system (1.31)

$$p(f) = a, \qquad q(f) = b.$$

The proof is complete. □

Remark 1.14. *The inverse of the operator $\mathcal{H}^{12}_\mu = \mu + L_1 L_2$ is obtained exchanging α_1 and α_2 in the expression of $(\mathcal{H}^{21}_\mu)^{-1}$.*

Now we can turn to the proof of Theorem 1.10.

Proof. We preliminarily observe that

$$\Gamma_\mu(x) = (-\triangle + \omega)\mathcal{G}_\mu(x) = \frac{e^{i\sqrt{-\omega+i\sqrt{\mu}}|x|} + e^{i\sqrt{-\omega-i\sqrt{\mu}}|x|}}{8\pi|x|} = \frac{1}{2}\left(G_{\omega-i\sqrt{\mu}}(x) + G_{\omega+i\sqrt{\mu}}(x)\right).$$

As proven in Appendix 1.10, the following identity holds:

$$R(\lambda) = (L-\lambda I)^{-1} = \begin{bmatrix} -\lambda(\lambda^2 + L_2L_1)^{-1} & -L_2(\lambda^2 + L_1L_2)^{-1} \\ L_1(\lambda^2 + L_2L_1)^{-1} & -\lambda(\lambda^2 + L_1L_2)^{-1} \end{bmatrix} = \begin{bmatrix} -\lambda(\mathcal{H}^{21}_{\lambda^2})^{-1} & -L_2(\mathcal{H}^{12}_{\lambda^2})^{-1} \\ L_1(\mathcal{H}^{21}_{\lambda^2})^{-1} & -\lambda(\mathcal{H}^{12}_{\lambda^2})^{-1} \end{bmatrix},$$

with λ in the resolvent set of L, to be specified.
In order to find the explicit expressions for Λ_1 and Λ_2 given in (1.26), one sets $\lambda = \sqrt{\mu}$ and then applies Lemma 1.13, Remark 1.14, and uses the definition of p and q given in (1.31). Besides, the operators Σ_1 and Σ_2 can be obtained applying L_1 and L_2 to $(\mathcal{H}^{21}_{\lambda^2})^{-1}$ and $(\mathcal{H}^{12}_{\lambda^2})^{-1}$, respectively, and using some trivial algebra.
The statement about the essential spectrum of L is a consequence of Weyl's theorem (Theorem XIII.4 in [40]). On the other hand, the eigenvalues of L are given by the poles of the resolvent (1.24), or equivalently by the complex roots of the function $W(\lambda)$; these can be computed through a lengthy but elementary calculation, here omitted. □

Remark 1.15. *As a by-product, the previous analysis of the complex roots of $W(\lambda)$ reveals the presence of a resonance at the endpoints of essential spectrum for the case $\sigma = \frac{1}{\sqrt{2}}$.*

1.4.2 Dispersive estimates for the linearized problem in the case $\sigma \in (0, 1/\sqrt{2})$

In this section we focus on the case $\sigma \in (0, 1/\sqrt{2})$ and study the behaviour for large t of the propagator e^{-Lt} restricted to the subspace associated to the essential spectrum of the operator L. In order to achieve an effective estimate, the following weighted L^p spaces are needed

$$L^1_w(\mathbb{R}^3) = \left\{ f : \mathbb{R}^3 \to \mathbb{C} : \int_{\mathbb{R}^3} w(x)|f(x)|dx < +\infty \right\},$$

and

$$L^\infty_{w^{-1}}(\mathbb{R}^3) = \left\{ f : \mathbb{R}^3 \to \mathbb{C} : \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} (w(x))^{-1}|f(x)| < +\infty \right\},$$

where $w(x) = 1 + \frac{1}{|x|}$. The use of such spaces is due to the singularity of the elements of (1.1). A similar choice was made in [16] for the sake of deriving dispersive estimates in the case of N delta interactions in $\mathbb{R}^3$.

Theorem 1.16. *There exists a constant $C > 0$ such that*

$$\left| \frac{1}{2\pi i} \int_{\mathbb{R}^3} \int_{\mathcal{C}_+\cup\mathcal{C}_-} (R(\lambda+0) - R(\lambda-0))(x) e^{-\lambda t} f(y)\, d\lambda dy \right| \le C\left(1 + \frac{1}{|x|}\right) t^{-\frac{3}{2}} \int_{\mathbb{R}^3} \left(1 + \frac{1}{|y|}\right) |f(y)| dy$$

for any $f \in L^1_w(\mathbb{R}^3)$, where

$$\mathcal{C}_+ = \{\lambda \in \mathbb{C} : \ \mathrm{Re}(\lambda) = 0 \text{ and } \ \mathrm{Im}(\lambda) \ge \omega\}, \quad \mathcal{C}_- = \{\lambda \in \mathbb{C} : \ \mathrm{Re}(\lambda) = 0 \text{ and } \ \mathrm{Im}(\lambda) \le -\omega\}.$$

Proof. One can compute the propagator e^{-Lt} as the inverse Laplace transform of the resolvent of L. In particular, by Theorem 1.10 and applying the residue theorem, it follows that for $t > 0$

$$e^{-Lt} = \frac{1}{2\pi i}\int_{i\mathbb{R}+0} R(\lambda)e^{-\lambda t}d\lambda = \frac{1}{2\pi i}\int_{|\lambda|=r} R(\lambda)e^{-\lambda t}d\lambda + \frac{1}{2\pi i}\int_{\mathcal{C}_+\cup\mathcal{C}_-}(R(\lambda+0)-R(\lambda-0))e^{-\lambda t}d\lambda,$$

with $r \in (0,\omega)$ and $R(\lambda \pm 0) = \lim_{\epsilon\to 0^+} R(\lambda \pm \epsilon)$.
We show the computations only for the component $R_{1,1}(\lambda)$ of the resolvent whose analytic expression is given in (1.24) and (1.26), since the other components can be handled in the same way.
Recalling the definition of α_1 and α_2 given in equation (1.17), $R_{1,1}(\lambda)$ can be written as an integral kernel, namely

$$R_{1,1}(\lambda;x,y) = i\frac{e^{i\sqrt{-\omega+i\lambda}|x-y|} - e^{i\sqrt{-\omega-i\lambda}|x-y|}}{8\pi|x-y|}+ \tag{1.33}$$

$$+i\frac{-\sigma\sqrt{\omega}e^{i\sqrt{-\omega-i\lambda}|y|}e^{i\sqrt{-\omega+i\lambda}|x|} + [(\sigma+1)\sqrt{\omega}+i\sqrt{-\omega+i\lambda}]e^{i\sqrt{-\omega-i\lambda}(|x|+|y|)}}{8\pi|x||y|[(2\sigma+1)\omega+i(\sigma+1)\sqrt{\omega}\left(\sqrt{-\omega-i\lambda}+\sqrt{-\omega+i\lambda}\right)-\sqrt{-\omega-i\lambda}\sqrt{-\omega+i\lambda}]}+$$

$$-i\frac{[(\sigma+1)\sqrt{\omega}+i\sqrt{-\omega+i\lambda}]e^{i\sqrt{-\omega+i\lambda}(|x|+|y|)} - \sigma\sqrt{\omega}e^{i\sqrt{-\omega+i\lambda}|y|}e^{i\sqrt{-\omega-i\lambda}|x|}}{8\pi|x||y|[(2\sigma+1)\omega+i(\sigma+1)\sqrt{\omega}\left(\sqrt{-\omega-i\lambda}+\sqrt{-\omega+i\lambda}\right)-\sqrt{-\omega-i\lambda}\sqrt{-\omega+i\lambda}]}.$$

Since from equation (1.33) it is clear that the computation of the integral on $\mathcal{C}_+$ and on $\mathcal{C}_-$ are analogous, we treat the cut $\mathcal{C}_+$ only. On $\mathcal{C}_+$, $\sqrt{-\omega+i\lambda}$ is continuous while, by the prescription $\mathrm{Im}(\sqrt{-\omega\pm i\lambda}) > 0$, considering ϵ as a real parameter, one has

$$\lim_{\epsilon\to 0^+}\sqrt{-\omega-i(\lambda+\epsilon)} = -\lim_{\epsilon\to 0^+}\sqrt{-\omega-i(\lambda-\epsilon)} = -\sqrt{-\omega-i\lambda}.$$

Performing the change of variable $k=\sqrt{-\omega-i\lambda}$, one can write

$$\int_{\mathcal{C}_+}(R_{1,1}(\lambda+0)-R_{1,1}(\lambda-0))e^{-\lambda t}d\lambda = ie^{-i\omega t}\int_{-\infty}^{+\infty}F(k)2ke^{-itk^2}dk,$$

where F is the function $R(\lambda+0)-R(\lambda-0)$ expressed in the variable k.
The function $R_{1,1}$ defined in (1.33) is the sum of a convolution summand $R_{*,1,1}$ and a multiplication summand $R_{m,1,1}$, where

$$R_{*,1,1}(\lambda;x,y) = i\frac{e^{i\sqrt{-\omega+i\lambda}|x-y|} - e^{i\sqrt{-\omega-i\lambda}|x-y|}}{8\pi|x-y|}$$

and

$$R_{m,1,1}(\lambda;x,y) = i\frac{-\sigma\sqrt{\omega}e^{i\sqrt{-\omega-i\lambda}|y|}e^{i\sqrt{-\omega+i\lambda}|x|} + [(\sigma+1)\sqrt{\omega}+i\sqrt{-\omega+i\lambda}]e^{i\sqrt{-\omega-i\lambda}(|x|+|y|)}}{8\pi|x||y|[(2\sigma+1)\omega+i(\sigma+1)\sqrt{\omega}\left(\sqrt{-\omega-i\lambda}+\sqrt{-\omega+i\lambda}\right)-\sqrt{-\omega-i\lambda}\sqrt{-\omega+i\lambda}]}+ \tag{1.34}$$

$$-i\frac{[(\sigma+1)\sqrt{\omega}+i\sqrt{-\omega+i\lambda}]e^{i\sqrt{-\omega+i\lambda}(|x|+|y|)} - \sigma\sqrt{\omega}e^{i\sqrt{-\omega+i\lambda}|y|}e^{i\sqrt{-\omega-i\lambda}|x|}}{8\pi|x||y|[(2\sigma+1)\omega+i(\sigma+1)\sqrt{\omega}\left(\sqrt{-\omega-i\lambda}+\sqrt{-\omega+i\lambda}\right)-\sqrt{-\omega-i\lambda}\sqrt{-\omega+i\lambda}]}.$$

Then we can define

$$F_*(k) = R_{*,1,1}(\lambda+0)-R_{*,1,1}(\lambda-0), \quad \text{and} \quad F_m(k) = R_{m,1,1}(\lambda+0)-R_{m,1,1}(\lambda-0).$$

One can easily compute F_* and gets $F_*(k) = -\frac{\sin(|x-y|k)}{4\pi|x-y|}$. Thus, by formula 3.851 in [27],

$$\int_{-\infty}^{+\infty} F_*(k) 2ke^{-itk^2} dk = \sin(|x-y|k)dk - i\frac{1+i}{16\sqrt{\pi}} t^{-\frac{3}{2}} e^{i\frac{|x-y|^2}{4t}},$$

for any $t > 0$. Hence

$$\left|\frac{1}{2\pi i}\int_{\mathbb{R}^3}\int_{-\infty}^{+\infty} F_*(k;y)dkf(y)dy\right| \leq \frac{1}{8\sqrt{2\pi}} t^{-\frac{3}{2}} \int_{\mathbb{R}^3} |f(y)|dy. \tag{1.35}$$

Let us estimate $\int_{-\infty}^{+\infty} F_m(k)2ke^{-itk^2}\, dk$. One can notice that $F_m(k)$ is the sum of terms of the form $\frac{i}{8\pi|x||y|} g(k)e^{\pm iks}$, where $g(k)$ is a rational function of k and $\sqrt{-2\omega - k^2}$ possibly multiplied by $e^{i\sqrt{-2\omega-k^2}s}$, and s can be 0, $|x|$, $|y|$ or $|x| + |y|$. Let us consider the term

$$g(k)e^{-ik(|x|+|y|)} =$$

$$= \frac{(\sigma+1)\sqrt{\omega} + i\sqrt{-2\omega - k^2}}{(2\sigma+1)\omega + i(\sigma+1)\sqrt{\omega}(\sqrt{-2\omega-k^2} - k) + k\sqrt{-2\omega-k^2}} e^{-ik(|x|+|y|)},$$

which results from the second term in (1.34) referred to $R_{m,1,1}(\lambda + 0)$.
Notice that $g \in C^1(\mathbb{R}, \mathbb{C})$ and $|g(k)| \sim \frac{1}{ik}$ as $k \to +\infty$, hence $g \in L^2(\mathbb{R})$. Moreover,

$$\frac{dg}{dk}(k) = \frac{-ik}{\left[(2\sigma+1)\omega + i(\sigma+1)\sqrt{\omega}(\sqrt{-2\omega-k^2} - k) + k\sqrt{-2\omega-k^2}\right]\sqrt{-2\omega-k^2}} +$$

$$- \frac{(\sigma+1)\sqrt{\omega} + i\sqrt{-2\omega-k^2}}{[(2\sigma+1)\omega + i(\sigma+1)\sqrt{\omega}(\sqrt{-2\omega-k^2} - k) + k\sqrt{-2\omega-k^2}]^2} \cdot$$

$$\cdot \left(-\frac{i(\sigma+1)\sqrt{\omega}k}{\sqrt{-2\omega-k^2}} - i(\sigma+1)\sqrt{\omega} + \sqrt{-2\omega-k^2} - \frac{k^2}{\sqrt{-2\omega-k^2}}\right),$$

which belongs to $L^2(\mathbb{R})$ too, so g is an element of $H^1(\mathbb{R})$, and as consequence $\check{g} \in L^1(\mathbb{R})$, where $\check{g}$ is the inverse Fourier transform of g. Furthermore, one can compute the inverse Fourier transform of $2ke^{-itk^2}$ as

$$U_t(s) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} 2ke^{-itk^2}e^{-iks}dk = \frac{1}{(4\pi it)^{\frac{3}{2}}} e^{-\frac{s^2}{4it}}.$$

From the last identity it follows

$$\left|\frac{1}{2\pi i}\int_{\mathbb{R}^3}\int_{-\infty}^{+\infty} \frac{i}{8\pi|x||y|} g(k)e^{-ik(|x|+|y|)} 2ke^{-itk^2}dkf(y)dy\right| =$$

$$= \left|\int_{\mathbb{R}^3}\int_{-\infty}^{+\infty} \frac{1}{8\pi|x||y|}\check{g}(u)U_t(u - |x| - |y|)duf(y)dy\right| \leq C\frac{1}{|x|} t^{-\frac{3}{2}} \int_{\mathbb{R}^3} \frac{|f(y)|}{|y|}dy, \tag{1.36}$$

where the last inequality follows from Hölder inequality and $C > 0$. The other terms in $F_m(k)$ are handled in an analogous way so we do not give details.

Summing up, let $f \in L^1_w(\mathbb{R}^3)$. Then

$$\left|\frac{1}{2\pi i}\int_{\mathbb{R}^3}\int_{\mathcal{C}_+\cup\mathcal{C}_-} (R(\lambda+0) - R(\lambda-0))e^{-\lambda t}d\lambda f(y)dy\right| \leq$$

$$\leq \frac{1}{2\pi}\left(\int_{\mathbb{R}^3}\left|\int_{\mathcal{C}_+}(R(\lambda+0)-R(\lambda-0))e^{-\lambda t}d\lambda f(y)\right|dy+\right.$$

$$\left.+\int_{\mathbb{R}^3}\left|\int_{\mathcal{C}_-}(R(\lambda+0)-R(\lambda-0))e^{-\lambda t}d\lambda f(y)\right|dy\right)=\frac{1}{2\pi}(I+II).$$

Let us estimate the integral I. Thanks to the estimates (1.35) and (1.36) one has

$$I=\int_{\mathbb{R}^3}\left|\int_{-\infty}^{+\infty}F(k)2ke^{-itk^2}dkf(y)\right|dy\leq$$

$$\leq\int_{\mathbb{R}^3}\left|\int_{-\infty}^{+\infty}F_*(k)2ke^{-itk^2}dkf(y)\right|dy+\int_{\mathbb{R}^3}f(y)\left|\int_{-\infty}^{+\infty}F_m(k)2ke^{-itk^2}dk\right|dy\leq$$

$$\leq Ct^{-3/2}\left(\int_{\mathbb{R}^3}|f(y)|dy+\frac{1}{|x|}\int_{\mathbb{R}^3}\frac{|f(y)|}{|y|}dy\right)\leq C\left(1+\frac{1}{|x|}\right)t^{-3/2}\int_{\mathbb{R}^3}|f(y)|\left(1+\frac{1}{|y|}\right)dy.$$

The integral II can be estimated in the same way, which completes the proof. □

Remark 1.17. *Evaluating the propagator* e^{-Lt} *at* $t=0$ *one gets*

$$1=\frac{1}{2\pi i}\int_{|\lambda|=r}R(\lambda)d\lambda+\frac{1}{2\pi i}\int_{\mathcal{C}_+\cup\mathcal{C}_-}(R(\lambda+0)-R(\lambda-0))d\lambda=P_0+P_c.$$

From Lemma 1.18 it will follow that the operators P_0 *and* P_c *are symplectic projectors onto the subspaces associated to generalized kernel and to the continuous spectrum respectively. Finally, let us note that explicitly integrating the resolvent around its poles it turns out that the dynamics along the generalized kernel grows linearly in time. This fact is proved in Appendix 1.11.*

1.5 Modulation equations

In this section we restrict to the case $\sigma\in(0,1/\sqrt{2})$, summarize the main technical steps and give some preliminary results towards the proof of asymptotic stability of standing waves. In particular, we write the so-called *modulation equations* that rule the evolution of a perturbed standing wave when splitted in a solitary component and a fluctuating one. We recall once more that the scalar product we adopt is the real scalar product on the Hilbert space $L^2(\mathbb{R}^3,\mathbb{C})$ defined in (1.10). In order to make the reading easier, let us give a brief outline of the strategy to be employed. We follow the roadmap of the classical papers [42],[43],[9],[10], [11], also adopted for the model with concentrated nonlinearity in dimension one in [8] and [33]. More specifically, we decompose the dynamics in the neighbourhood of the solitary manifold in a "longitudinal" and a "transversal" component with respect to the generalized kernel $N_g(L)$, given in Theorem 1.9, of the linearized operator L. In order to perform the required analysis, we exploit the symplectic structure introduced in Section 1.2.1. Let us begin by noticing that the solitary manifold $\mathcal{M}$ defined in (1.16) is a symplectic submanifold of $(L^2(\mathbb{R}^3,\mathbb{C}),\Omega)$, invariant under the flow of (1.7). Its tangent space at the standing wave Φ_ω is two-dimensional and is generated by the vectors $\frac{d}{d\theta}\{e^{i\theta}\Phi_\omega\}_{\theta=0}$ and $\frac{d}{d\omega}\{\Phi_\omega\}_{\omega=0}$, in real representation given by

$$\frac{d}{d\theta}\{e^{i\theta}\Phi_\omega\}_{\theta=0}\ \mapsto\ e_1=\begin{pmatrix}0\\ \Phi_\omega\end{pmatrix}\quad\text{and}\quad\frac{d}{d\omega}\{\Phi_\omega\}\ \mapsto\ e_2=\begin{pmatrix}\varphi_\omega\\ 0\end{pmatrix},$$

where $\varphi_\omega=\frac{d}{d\omega}\Phi_\omega$ was defined in Section 1.4. However, when no confusion arises, we use the shorthand expressions Φ_ω and φ_ω with the meaning of the corresponding *real* representative

vectors (second component vanishing). As already remarked the couple of vectors $\{e_1, e_2\}$ is a basis for $N_g(L)$. It is immediately seen that $\Omega(e_1, e_2) = \frac{1}{2}\frac{d}{d\omega}\|\Phi_\omega\|^2 \neq 0$, thanks to the condition $\sigma \in (0, 1/\sqrt{2})$ guaranteeing orbital stability. So the symplectic form is nondegenerate on the solitary manifold $\mathcal{M}$, which is a symplectic submanifold. By its very definition, $\mathcal{M}$ is invariant for the flow of (1.7).
The following lemma establishes the relation between the spectral projection P_0 introduced in Remark 1.17 and the symplectic projection onto the solitary manifold.

Lemma 1.18. *Let* $\Delta = \frac{1}{2}\frac{d}{d\omega}\|\Phi_\omega\|^2_{L^2}$, *then for any* $f \in L^2(\mathbb{R}^3)$

$$P_0 f = \frac{1}{\Delta}\Omega\left(f, \varphi_\omega\right) J\Phi_\omega - \frac{1}{\Delta}\Omega\left(f, J\Phi_\omega\right)\varphi_\omega \ , \tag{1.37}$$

where $\Omega(\cdot,\cdot)$ *was defined in* (1.12).

Proof. The explicit expression of the spectral projection $P_0 = \frac{1}{2\pi i}\int_{|\lambda|=r} R(\lambda)d\lambda$ can be recovered by Appendix 1.11, and the equivalence with the r.h.s. follows by straightforward calculations. □

Notice that the given representation of P_0 is well defined thanks to the fact that $\Delta > 0$, again as a consequence of the choice $\sigma \in (0, 1/\sqrt{2})$. Moreover, P_0 is a symplectically orthogonal projection, in the sense that given a couple $\{\zeta, f\}$ with $\zeta \in \text{Im } P_0$ and $f \in \text{Ker } P_0$, one has $\Omega(\zeta, f) = 0$. In particular, it is useful to note that due to the definition of symplectic form Ω, a state f with vanishing component along the continuous spectrum of L is orthogonal to the vectors Je_1 and Je_2, or in complex notation, to Φ_ω and $i\frac{d}{d\omega}\Phi_\omega \ = i\varphi_\omega$.
After these preliminaries, as anticipated in formula (1.9), we write the solution to (1.7) as

$$u(t,x) = e^{i\Theta(t)}\left(\Phi_{\omega(t)}(x) + \chi(t,x)\right), \quad \Theta(t) = \int_0^t \omega(s)ds + \gamma(t), \tag{1.38}$$

with the final goal of proving that the solution decomposes in the sum of a solitary component and a dispersive one.

The local splitting of the invariant symplectic manifold $(L^2(\mathbb{R}^3, \mathbb{C}), \Omega)$ in two symplectically orthogonal manifolds, the finite dimensional solitary manifold $\mathcal{M}$ and the infinite dimensional range of the spectral projection on the continuous spectrum, suggests to symplectically project the flow according to this decomposition (see also Remark 1.17), in order to obtain the so called modulation equations. The projection along $\mathcal{M}$ ("longitudinal") gives rise to two ordinary differential equations for the frequency ω and the phase γ of the solitary wave, depending parametrically on the fluctuating component χ; while the projection on the continuous spectrum ("transversal") gives a partial differential equation for the remainder χ (with coefficients depending on γ and ω). The solution to the equation for the χ component will be shown to decay in time in suitable norms. As a consequence, one has the asymptotic behavior of the solutions for the parameters ω and γ of the solitary wave, to be shown in Section 6, and finally asymptotic stability, which will be the subject of Section 7.
To deduce the modulation equations it proves convenient to make use of the variational formulation of equation (1.7)

$$\left(i\frac{du}{dt}(t), v\right)_{L^2} = E'[u(t)](v) \quad \forall v \in V. \tag{1.39}$$

To begin with, we replace in the previous equation the Ansatz (1.38).

By equation (1.15) and Proposition 1.3, equation (1.39) can be rephrased as
(1.40)
$$\left(i\frac{d\chi}{dt}(t),v\right)_{L^2}=Q_{\alpha,Lin}(\chi(t),v)+\dot\gamma(t)(\Phi_{\omega(t)}+\chi(t),v)_{L^2}+\dot\omega(t)\left(-i\frac{d\Phi_{\omega(t)}}{d\omega},v\right)_{L^2}+N(q_\chi(t),q_v)$$
for any $v\in V$.
Here $Q_{\alpha,Lin}$ is the quadratic form of the operator D defined in (1.19) and acting as

$$Q_{\alpha,Lin}(\chi,v)=(\nabla\phi_\chi,\nabla\phi_v)_{L^2}-\frac{\sqrt{\omega}}{4\pi}\operatorname{Re}(q_\chi\overline{q_v})-\sigma\frac{\sqrt{\omega}}{2\pi}\operatorname{Re}q_\chi\operatorname{Re}q_v+\omega(\chi,v)_{L^2}\ ,$$

and the nonlinear remainder $N(q_\chi,q_v)$ is given by

$$N(q_\chi,q_v)=-\nu|q_\chi+q_\omega|^{2\sigma}\operatorname{Re}((q_\chi+q_\omega)\overline{q_v})+\nu(2\sigma+1)|q_\omega|^{2\sigma}\operatorname{Re}q_\chi\operatorname{Re}q_v+\nu|q_\omega|^{2\sigma}\operatorname{Im}q_\chi\operatorname{Im}q_v+\nu|q_\omega|^{2\sigma}\operatorname{Re}(q_\omega\overline{q_v}),$$

where, according to Section 1.2.2, $q_\omega=\left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{\frac{1}{2\sigma}}$.

Remark 1.19. *The remainder $N(q_\chi,q_v)$ depends nonlinearly on χ (and ω) and it is real linear in v; so, by Riesz representation theorem and with a slight abuse of notation, there exist a vector $N(q_\chi)$ such that $N(q_\chi,q_v)=\operatorname{Re}N(q_\chi)\overline{q_v}$. The dependence just on the charges of χ and v is a peculiarity of this model. Moreover, by its very definition, the remainder is the difference between the action of the complete vector field and its linear part at the solitary wave, and so it is quadratic in q_χ near $\chi=0$.*

Corresponding expressions can be given with obvious modification in purely real form, which we omit for the sake of brevity. Since ω, γ and χ are all unknown the Ansatz (1.38) makes the problem underdetermined, and a supplementary condition is needed to give a unique representation of the solution; a way to close the system for ω, γ and χ is to require that the χ component is decoupled from the discrete spectrum, i.e. $P_0\chi=0$, or equivalently to project equation (1.40) onto the symplectically orthogonal complement of the generalized kernel of L. The corresponding modulation equations take different forms according to the way one writes the projection and we give two of them for future reference. In the following we denote by Q_L the bilinear form associated to the linear nonselfadjoint operator L.

Theorem 1.20. (Modulation equations I) *Let χ be a solution to equation (1.40) such that $P_0\chi(t)=0$ for all $t\geq 0$, and let the functions ω and γ belong to $C^1(\mathbb{R})$; then ω and γ solve the equations*

(1.41)
$$\dot\omega=\frac{\operatorname{Re}\ (JN(q_\chi)\overline{q_{P_0^*(\Phi_\omega+\chi)}})}{\left(\varphi_\omega-\frac{dP_0}{d\omega}\chi,\Phi_\omega+\chi\right)_{L^2}},$$

and

(1.42)
$$\dot\gamma=\frac{\operatorname{Re}\ (JN(q_\chi)\overline{q_{J(\varphi_\omega-\frac{dP_0}{d\omega}\chi)}})}{\left(\varphi_\omega-\frac{dP_0}{d\omega}\chi,\Phi_\omega+\chi\right)_{L^2}}.$$

Proof. We adapt the reasoning in [11]. Equation (1.40) is equivalent to

(1.43)
$$\left(\frac{d}{dt}(\Phi_\omega+\chi),v\right)_{L^2}=Q_L(\chi,v))+\dot\gamma\,(J(\Phi_\omega+\chi),v)_{L^2}+\operatorname{Re}(JN(q_\chi)\overline{q_v})\quad\forall v\in V\ .$$

Set $v = P_0^*(\Phi_\omega+\chi)$ where P_0^* is the adjoint in $L^2(\mathbb{R}^3)$ of the operator P_0; notice that differentiating in time $P_0\chi = 0$, one has

$$P_0\frac{d}{dt}(\Phi_\omega + \chi) = \dot\omega\left(\varphi - \frac{dP_0}{d\omega}\chi\right) ,$$

where expressions such as $\frac{dP_0}{d\omega}\chi$ are computed from the representation given in (1.37).
Moreover, one immediately has the identities

$$Q_L(\chi, P_0^*(\Phi_\omega + \chi)) = Q_L(P_0\chi, (\Phi_\omega + \chi)) = 0$$

and, using $P_0J = JP_0^*$,

$$(J(\Phi_\omega + \chi), P_0^*(\Phi_\omega + \chi))_{L^2} = \big(J(\Phi_\omega + \chi), (P_0^*)^2(\Phi_\omega + \chi)\big)_{L^2} = (JP_0^*(\Phi_\omega + \chi), P_0^*(\Phi_\omega + \chi))_{L^2} = 0\,.$$

So one remains with

$$\left(P_0\frac{d}{dt}(\Phi_\omega + \chi), \Phi_\omega + \chi\right)_{L^2} = \mathrm{Re}(JN(q_\chi)\overline{q_{P_0^*(\Phi_\omega+\chi)}})$$

from which the equation for $\dot\omega$ follows.
Now let us consider the test function $JP_0\frac{d}{dt}(\Phi_\omega + \chi)$, and notice the following facts, in which use is made of $JP_0 = P_0^*J$.

$$\left(\frac{d}{dt}(\Phi_\omega + \chi), JP_0(\Phi_\omega + \chi)\right)_{L^2} = \left(\frac{d}{dt}(\Phi_\omega + \chi), JP_0^2(\Phi_\omega + \chi)\right)_{L^2} = \left(P_0\frac{d}{dt}(\Phi_\omega + \chi), JP_0(\Phi_\omega + \chi)\right)_{L^2} = 0\,;$$

$$Q_L(\chi, JP_0\frac{d}{dt}(\Phi_\omega + \chi)) = 0\ .$$

It follows from the weak equation (1.43)

$$\dot\gamma\left(\Phi_\omega + \chi, P_0\frac{d}{dt}(\Phi_\omega + \chi)\right)_{L^2} = \mathrm{Re}(JN(q_\chi)\overline{q_{JP_0\frac{d}{dt}(\Phi_\omega+\chi)}}$$

and hence, after substituting the expression of $P_0\frac{d}{dt}(\Phi_\omega + \chi)$ determined above and cancellation of $\dot\omega$ the equation for $\dot\gamma$ follows. This ends the proof. □

Two properties of the modulation equations which will be useful in the subsequent analysis are the following.

Corollary 1.21. *Under the hypotheses of Theorem 1.20, and if it is known that $\|\chi\|_{L^1_w}$ is sufficiently small, the right hand sides of* (1.41) *and* (1.42) *are smooth and there exists a continuous function $\mathcal{R} = \mathcal{R}(\omega, \|\chi\|_{L^1_w})$ such that, for any $t \geq 0$,*

$$|\dot\omega(t)| \leq \mathcal{R}|q_\chi(t)|^2 \quad \textit{and} \quad |\dot\gamma(t)| \leq \mathcal{R}|q_\chi(t)|^2.$$

The proof of the previous result is a consequence of two facts. In the first place $(\varphi_\omega, \Phi_\omega)_{L^2} = \frac{1}{2}\frac{d}{dt}\|\Phi_\omega\|^2 > 0$ by condition $\sigma \in (0, 1/\sqrt{2})$ which gives orbital stability; secondarily, the nonlinear part in (1.40) actually depends only on the charges q_χ and q_v; provided that $|q_\chi| \leq c$, there exists a positive constant $C > 0$ such that the denominators in (1.41) and (1.42) are strictly away from zero and

$$|N(q_\chi)| \leq C|q_\chi|^2, \quad \forall \chi \in V.$$

The second property concerns the compatibility of the orthogonality condition of the fluctuating part χ with arbitrary choices of initial data. The following lemma assures in fact that the orthogonality condition $P_0\chi = 0$ can be satisfied at the initial time in the neighbourhood of the solitary manifold without loss of generality.

Lemma 1.22. *Let $u \in C(\mathbb{R}^+, V)$ be a solution to equation* (1.7) *with $u(0) = u_0 \in V \cap L^1_w$ and assume*

$$d = \|u_0 - e^{i\theta_0}\Phi_{\omega_0}\|_{V\cap L^1_w} \ll 1,$$

for some $\omega_0 > 0$ and $\theta_0 \in \mathbb{R}$.
Then, there exists a stationary wave $e^{i\widetilde{\theta}_0}\Phi_{\widetilde{\omega}_0}$, and $\chi_0(x)$ with $P_0(\widetilde{\omega}_0)\chi_0 = 0$ such that $u_0(x) = e^{i\widetilde{\theta}_0}\left(\Phi_{\widetilde{\omega}_0}(x) + \chi_0(x)\right)$, and $\|\chi_0\|_{V\cap L^1_w} = O(d) \quad as\ d \to 0$.

The result is commonly stated as a preliminary step in the analysis of modulation equations (see for example [30],[32] and [8]). The proof is an application of the implicit function theorem making use again of the condition $\frac{d}{dt}\|\Phi_\omega\|^2 \neq 0$; we omit details and refer to the cited references. As a consequence of the previous lemma, in all proofs in the rest of the chapter we can assume $P_0\chi_0 = 0$ where $\chi_0 = \chi(0)$.
An equivalent form of the modulation equations for the soliton parameters ω and γ can be obtained exploiting the characterization of the condition $P_0\chi = 0$ through the (Hilbert) orthogonality $(\chi, \Phi_\omega)_{L^2} = 0 = (\chi, i\varphi_\omega)_{L^2}$. In some respects they are more transparent and we give them making use of the complex writing.

Theorem 1.23. (Modulation equations II) *Let χ be a solution to equation* (1.40) *such that $P_0\chi(t) = 0$ for all $t \geq 0$, and let the functions ω and γ belong to $\mathrm{C}^1(\mathbb{R})$; then ω and γ satisfy the equations*

$$\dot{\omega} = \frac{((\chi, \varphi_\omega)_{L^2} + (\varphi_\omega, \Phi_\omega)_{L^2})N(\chi, i\Phi_\omega) - (\chi, i\Phi_\omega)_{L^2}N(\chi, \varphi_\omega)}{(\varphi_\omega, \Phi_\omega)^2_{L^2} - (\chi, \varphi_\omega)^2_{L^2}} \tag{1.44}$$

$$\dot{\gamma} = \frac{((\chi, \varphi_\omega)_{L^2} - (\varphi_\omega, \Phi_\omega)_{L^2})N(\chi, \varphi_\omega) + (\chi, i\frac{d}{d\omega}\varphi_\omega)_{L^2}N(\chi, i\Phi_\omega)}{(\varphi_\omega, \Phi_\omega)^2_{L^2} - (\chi, \varphi_\omega)^2_{L^2}} \tag{1.45}$$

Proof. Differentiating in time the orthogonality conditions $(\chi, \Phi_\omega)_{L^2} = 0 = (\chi, i\varphi_\omega)_{L^2}$, it easily follows that

$$(i\dot{\chi}, i\Phi_\omega)_{L^2} = -\dot{\omega}(\chi, \varphi_\omega)_{L^2}, \qquad (i\dot{\chi}, \varphi_\omega)_{L^2} = \dot{\omega}\left(\chi, i\frac{d}{d\omega}\varphi_\omega\right)_{L^2}.$$

So testing the weak equation for χ with $i\Phi_\omega$ and φ and taking into account properties of operators L_1 and L_2 and orthogonality conditions again, one obtains the system

$$\begin{aligned} \dot{\omega}((\chi, \varphi_\omega)_{L^2} - (\Phi_\omega, \varphi_\omega)_{L^2}) + \dot{\gamma}(\chi, i\Phi_\omega)_{L^2} &= -N(\chi, i\Phi_\omega) \\ \dot{\omega}\left(\chi, \frac{d}{d\omega}\varphi_\omega\right)_{L^2} - \dot{\gamma}((\Phi_\omega, \varphi_\omega)_{L^2} + (\chi, \varphi_\omega)_{L^2}) &= N(\chi, \varphi_\omega). \end{aligned}$$

The thesis follows solving for $\dot{\omega}$ and $\dot{\gamma}$.

□

Notice that to this second form of modulation equations apply similar remarks to the ones made for the first form. In particular, if a priori estimates on smallness of χ are known, the modulation equations are well defined thanks to the condition $\frac{d}{d\omega}\|\Phi_\omega\|^2 > 0$, and the analogous of Lemma 1.22 holds true.

1.6 Time decay of weak solutions

The goal of this section is to provide the time decay of the transversal component χ of the solution u (see (31)) to equation (1.7); the result we achieve shows that χ is in fact not only a fluctuation, but also a decaying dispersive remainder and it paves the way to the proof of asymptotic stability of standing waves, that is given in the next section. To this end we follow the idea developed in [9],[10],[11] for the standard NLS and applied in [8] to the case of 1-d concentrated nonlinearities. For any $T > 0$, define preliminarily the so-called *majorant*

$$M(T) = \sup_{0 \le t \le T} \left[(1+t)^{3/2} \|\chi(t)\|_{L^\infty_{w^{-1}}} + (1+t)^3 (|\dot\gamma(t)| + |\dot\omega(t)|) \right] . \tag{1.46}$$

We aim at proving that the majorant is uniformly bounded in T by a constant $\overline{M} = O(d)$, where d is the size of the dispersive component χ. The proof of such bound is the content of the following theorem.

Theorem 1.24. *Let $u \in C(\mathbb{R}^+, V)$ be a solution to equation (1.7) with $u(0) = u_0 \in V \cap L^1_w$ and define $d := \|u_0 - e^{i\theta_0}\Phi_{\omega_0}\|_{V \cap L^1_w}$, for some $\omega_0 > 0$ and $\theta_0 \in \mathbb{R}$. Then, if d is sufficiently small, there are $\omega, \gamma \in C^1(\mathbb{R}^+)$ which satisfy (1.41)-(1.42), and such that the solution u can be written as in (1.38).*
Moreover, there is a positive constant $\overline{M} > 0$, depending only on the initial data, such that, for any $T > 0$, one has $M(T) \le \overline{M}$ and $\overline{M} = O(d)$ as $d \to 0$. In particular

$$\|\chi(t)\|_{L^\infty_{w^{-1}}} \le \overline{M}\ (1+t)^{-3/2} \qquad \forall t > 0, \tag{1.47}$$

$$|\dot\gamma(t)| + |\dot\omega(t)| \le \overline{M}\ (1+t)^{-3} \qquad \forall t > 0. \tag{1.48}$$

The previous theorem is implied by the following proposition that is proven in Section 1.6.3 by using the results given in Sections 1.6.1 and 1.6.2, and the dispersive properties of the linearization operator L given in Section 1.4.2.

Proposition 1.25. *Under the hypotheses of the previous theorem, assume that there exist some $t_1 > 0$ and $\rho > 0$ such that $M(t_1) \le \rho$. Then there are two positive numbers d_1 and ρ_1, independent of t_1, such that if $d = \|\chi_0\|_{V \cap L^1_w} < d_1$ and $\rho < \rho_1$, then $M(t_1) \le \frac{\rho}{2}$.*

Indeed, if Proposition 1.25 were true, then Theorem 1.24 would follow from the next argument: let $\mathcal{I} \subset [0, +\infty)$ be defined as

$$\mathcal{I} = \{t_1 \ge 0 : \ \omega, \gamma \in C^1([0, t_1]),\ M(t_1) \le \rho\}.$$

$\mathcal{I}$ is obviously relatively closed in $[0, +\infty)$ with the topology induced by considering it as a subspace of $\mathbb{R}$ with the standard Euclidean topology. On the other hand, the thesis of Proposition 1.25 and the estimates of Corollary 1.21 imply that $\mathcal{I}$ is also relatively open. Hence, the uniform estimate of Theorem 1.24 follows from the fact that $\sup \mathcal{I} = +\infty$.

1.6.1 Frozen linearized problem

Note that the equation (1.40) is non autonomous. In order to make its study simpler, it is useful to exploit a further reparametrization of the solution $\chi(t)$. We fix a time $t_1 > 0$ and denote $\omega_1 = \omega(t_1)$ and $\gamma_1 = \gamma(t_1)$. Now define (in vector notation; we recall that J corresponds to $-i$)

$$e^{-J\Theta(t)}\chi(t,x) = e^{-J\tilde\Theta(t)}\eta(t,x) \ , \quad \text{where} \quad \tilde\Theta(t) = \omega_1 t + \gamma_1 \ . \tag{1.49}$$

The function η satisfies the equation

$$\left(e^{J(\Theta-\tilde{\Theta})}\frac{d\eta}{dt}, v\right)_{L^2} = Q_L(e^{J(\Theta-\tilde{\Theta})}\eta, v) + (\omega_1-\omega)(J\eta, v)_{L^2} + \dot{\gamma}(J\Phi_\omega, v)_{L^2}$$
$$-\dot{\omega}\left(\frac{d\Phi_\omega}{d\omega}, v\right)_{L^2} + JN(e^{J(\Theta-\tilde{\Theta})}q_\eta)\overline{q_v} \quad \forall v \in V$$

We need a further manipulation which allows to rewrite the previous equation in a form which makes the role of reparametrization clear. To this end we need the following identities, which can be obtained from straightforward computations

- $Je^{J(\Theta-\tilde{\Theta})} = e^{J(\Theta-\tilde{\Theta})}J$;
- $Q_L(e^{J(\Theta-\tilde{\Theta})}u, v) - e^{J(\Theta-\tilde{\Theta})}Q_L(u,v) = \dfrac{(\sigma+1)\sqrt{\omega}}{2\pi}\sin(\Theta-\tilde{\Theta})\sigma_3 q_u\overline{q_v}$, for any u, $v \in V$, where

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Making use of the previous identities, one rewrites the equation for η as

$$\left(\frac{d\eta}{dt}, v\right)_{L^2} = (\omega_1-\omega)(J\eta, v)_{L^2} + Q_L(\eta, v) + \left(e^{-J(\Theta-\tilde{\Theta})}\left(\dot{\gamma}J\Phi_\omega - \dot{\omega}\frac{d\Phi_\omega}{d\omega}\right), v\right)_{L^2} + \tag{1.50}$$
$$+e^{-J(\Theta-\tilde{\Theta})}\frac{(\sigma+1)\sqrt{\omega}}{2\pi}\sin(\Theta-\tilde{\Theta})\sigma_3 q_\eta\overline{q_v} + e^{-J(\Theta-\tilde{\Theta})}JN(e^{J(\Theta-\tilde{\Theta})}q_\eta)\overline{q_v}, \quad \forall v \in V\ .$$

Let us define the linearization frozen at time t_1 as $L_I = L(\omega_1)$, and observe that for all u, $v \in V$

$$Q_L(u,v) - Q_{L_I}(u,v) = \frac{\sqrt{\omega}-\sqrt{\omega_1}}{4\pi}\mathbb{T}q_u\overline{q_v} - (\omega_1-\omega)(Ju, v)_{L^2},$$

where $\mathbb{T} = \begin{bmatrix} 0 & -1 \\ 2\sigma+1 & 0 \end{bmatrix}$. Hence, equation (1.50) becomes

$$\left(\frac{d\eta}{dt}, v\right)_{L^2} = Q_{L_I}(\eta, v) + N_I(t, \omega, q_\eta, q_v) \quad \forall v \in V \tag{1.51}$$

where, for all $v \in V$, the time dependent nonlinear remainder (including now "dragging" terms due to reparametrization) is given by

$$N_I(t,\omega,q_\eta,q_v) = \left(e^{-J(\Theta-\tilde{\Theta})}\left(\dot{\gamma}J\Phi_\omega - \dot{\omega}\frac{d\Phi_\omega}{d\omega}\right), v\right)_{L^2} + \frac{\sqrt{\omega}-\sqrt{\omega_1}}{4\pi}\mathbb{T}q_\eta\overline{q_v} \tag{1.52}$$
$$+\, e^{-J(\Theta-\tilde{\Theta})}\frac{(\sigma+1)\sqrt{\omega}}{2\pi}\sin(\Theta-\tilde{\Theta})\sigma_3 q_\eta\overline{q_v} + e^{-J(\Theta-\tilde{\Theta})}JN(e^{J(\Theta-\tilde{\Theta})}q_\eta)\overline{q_v}\ . \tag{1.53}$$

The gain in changing from original (1.40) for the dispersive component to equation (1.51) is that the latter is still non autonomous, but now the generator of the evolution is (in weak form) a sum of a fixed linear vector field (the frozen linearization L_I) and a nonlinear time dependent perturbation (see also [9]). This allows to use the known dispersive properties of linearization operator L described in 1.4.2.

1.6.2 Duhamel's representation

In this subsection we write the equation (1.51) in Duhamel's representation to better exploit the dispersive properties of the propagator $e^{L_I t}$. This is not a completely trivial task since our frozen equation is a variational equation and cannot be written in strong form. In order to reach our purpose, we consider (1.51) separating in the test function v the regular and singular part accordingly to (1.6). So we begin by setting $v = \phi_v^\lambda \in H^1(\mathbb{R}^3)$. We get

$$\left(\frac{d\eta}{dt}(t), \phi_v^\lambda\right)_{L^2} = (L_I\eta(t) + f_I(t), \phi_v^\lambda)_{L^2},$$

where $f_I(t) = e^{-J(\Theta(t)-\widetilde{\Theta}(t))}\left(\dot{\gamma}(t)J\Phi_{\omega(t)} - \dot{\omega}(t)\frac{d\Phi_{\omega(t)}}{d\omega}\right)$. Hence, by Duhamel's principle one gets

$$(\eta, \phi_v^\lambda)_{L^2} = \left(e^{L_I t}\eta_0 + \int_0^t e^{L_I(t-s)} f_I(s)ds, \phi_v^\lambda\right)_{L^2}.$$

If one considers the same equation with $v = q_v G_\lambda$ where $q_v \in \mathbb{C}$, one has

$$\left(\frac{d\eta}{dt}(t), q_v G_\lambda\right)_{L^2} = (L_I\eta(t) + f_I(t) + g_I(t), q_v G_\lambda)_{L^2},$$

where

$$g_I(t) = e^{-J(\Theta(t)-\widetilde{\Theta}(t))}\Big(4\sqrt{\lambda}(\sigma+1)\sqrt{\omega(t)}\sin(\Theta(t)-\widetilde{\Theta}(t))\sigma_3 q_\eta(t)G_\lambda +$$
$$+8\pi\sqrt{\lambda}JN(e^{J(\Theta(t)-\widetilde{\Theta}(t))}q_\eta(t))G_\lambda\Big) + 2\sqrt{\lambda}(\sqrt{\omega(t)} - \sqrt{\omega_1})\mathbb{T}q_\eta(t)G_\lambda,$$

where q_η is the charge of the function η. And hence,

$$(\eta, q_v G_\lambda)_{L^2} = \left(e^{L_I t}\eta_0 + \int_0^t e^{L_I(t-s)}(f_I(s) + g_I(s))ds, q_v G_\lambda\right)_{L^2}.$$

Summing up, for any $v \in V$, the solution to equation (1.51) can be written as

$$(\eta, v)_{L^2} = \left(e^{L_I t}\eta_0 + \int_0^t e^{L_I(t-s)} f_I(s)ds, v\right)_{L^2} + \left(\int_0^t e^{L_I(t-s)} g_I(s)ds, q_v G_\lambda\right)_{L^2}.$$

In what follows we will use the following estimate on the function g_I.

Lemma 1.26. *Under the hypotheses of Proposition 1.25, there exists a constant $C > 0$ such that*

$$\|g_I(t)\|_{V\cap L^1_w} \leq C(|q_\eta|^2 + \rho|q_\eta|),$$

for any $t \leq t_1$.

Proof. First of all let us notice that it is possible to chose t_1 in such a way that $\omega(t) \geq c > 0$ for any $0 \leq t \leq t_1$, then

$$|\sqrt{\omega(t)} - \sqrt{\omega_1}| \leq C|\omega(t) - \omega_1| \leq C\int_t^{t_1}|\dot{\omega}(s)|ds \leq C\sup_{0\leq t\leq t_1}\left[(1+t)^3|\dot{\omega}(t)|\right]\int_t^{t_1}(1+s)^{-3}ds \leq C\rho,$$

and

$$|\Theta(t) - \widetilde{\Theta}(t)| \leq \int_0^t\int_s^{t_1}|\dot{\omega}(\tau)|d\tau ds + \int_t^{t_1}|\dot{\gamma}(s)|ds \leq C\rho\int_0^t\int_s^{t^1}(1+\tau)^{-3}d\tau ds + C\rho\int_t^{t_1}(1+s)^{-3}ds \leq C\rho.$$

The result follows since

$$\|g_I(t)\|_{V\cap L^1_w} \leq C(|\Theta(t) - \widetilde{\Theta}(t)||q_\eta(t)| + |\sqrt{\omega(t)} - \sqrt{\omega_1}||q_\eta(t)| + |q_\eta(t)|^2).$$

□

We end the section with a technical result that allows to transfer dispersive estimates on the frozen fluctuating component $P_c(L_I)\eta = P_c(\omega_1)\eta$ into estimates on η. This is needed because η appears in the integral Duhamel's equation where estimates have to be done, but the dispersive behavior is at our disposal for $P_c(\omega_1)\eta$. This is stated in the following lemma (see for analogous construction, for example, [22] and [8]).

Lemma 1.27. *Let the hypotheses of Proposition 1.25 hold true and suppose that the quantity*

$$\sup_{0\le t\le t_1} (|\omega(t)-\omega_1| + |\Theta(t)-\widetilde{\Theta}(t)|) = \delta$$

is sufficiently small; then, for any $t \in [0,t_1]$ there is a bounded linear operator $\Pi(t) : P_c(\omega_1)(V \cap L^\infty_{w^{-1}}) \to V \cap L^\infty_{w^{-1}}$, and a positive constant $C = C(\delta,\omega_1) > 0$ such that $\eta(t) = \Pi(t)h(t)$, and

$$C(\delta,\omega_1)^{-1}\|h\|_{V\cap L^\infty_{w^{-1}}} \le \|\eta\|_{V\cap L^\infty_{w^{-1}}} \le C(\delta,\omega_1)\|h\|_{V\cap L^\infty_{w^{-1}}}.$$

Proof. We give only a sketch of the standard proof, referring for details to the literature cited above. Set $\eta(t) = P_0(\omega_1)\eta + P_c(\omega_1)\eta = ik_1(t)\Phi_{\omega_1} + k_2(t)\frac{d}{d\omega_1}\Phi_{\omega_1} + h(t)$. The condition $P_0\chi = 0$ makes time dependent functions k_1 and k_2 to satisfy a linear system with a source term depending on h; the coefficient matrix has an inverse uniformly bounded in t and t_1 thanks to the conditions $(\Phi_\omega, \frac{d}{d\omega_1}\Phi_{\omega_1})_{L^2} > \text{const} > 0$ and $(\Phi_{\omega_1}, \frac{d}{d\omega}\Phi_\omega)_{L^2} > \text{const} > 0$ valid for $|\omega-\omega_1|$ small enough. This gives a representation of k_1 and k_2 in terms of h and as a consequence the required bound on the finite dimensional component. Now define $\Pi(t)h(t) = \eta(t) - ik_1\Phi_{\omega_1} - k_2\frac{d}{d\omega_1}\Phi_{\omega_1}$ and the complete bound follows. □

1.6.3 Proof of Proposition 1.25

`Estimate of` $|\dot\gamma| + |\dot\omega|$.

Lemma 1.28. *If $\eta \in V \cap L^\infty_{w^{-1}}$, then the charge q_η of the function η satisfies $|q_\eta| \le 4\pi\|\eta\|_{L^\infty_{w^{-1}}}$.*

Proof. Since $\eta \in L^\infty_{w^{-1}}(\mathbb{R}^3)$ then $\|\eta\|_{L^\infty_{w^{-1}}} = \sup_{x\in\mathbb{R}^3}\left|\frac{|x|}{1+|x|}\phi_\eta(x) + \frac{q_\eta}{4\pi(1+|x|)}\right| \ge \frac{1}{4\pi}|q_\eta|$.

□

From the last lemma and Corollary 1.21 one gets

$$|\dot\gamma(t)| + |\dot\omega(t)| \le c|q_\eta(t)|^2 \le c_1\|\eta(t)\|^2_{L^\infty_{w^{-1}}} \le c_1(1+t)^{-3}M(t)^2, \quad \forall t \in [0,t_1],$$

with c_1 independent of t_1. Hence, one can choose $\rho_1^2 < \frac{1}{4c_1}$ and get $(1+t)^3(|\dot\gamma(t)| + |\dot\omega(t)|) \le c_1\rho^2 \le \frac{\rho}{4}, \quad \forall t \in [0,t_1]$.

`Estimate of` $\|\eta\|_{L^\infty_{w^{-1}}}$.

As explained in the previous section, for any $t \in [0,t_1]$ we have $\eta(t) = P_0(\omega_1)\eta(t) + P_c(\omega_1)\eta(t)$ (for the definitions of P_0 and P_c see Remark 1.17) and thanks to Lemma 1.27 we have $\eta(t) = \Pi h(t)$ where $\Pi(t) : P_c(\omega_1)(V \cap L^\infty_{w^{-1}}) \to V \cap L^\infty_{w^{-1}}$ is bounded.

In order to estimate $\|\eta\|_{L^\infty_{w^{-1}}}$ we make use of the equation for h. For all $v \in V$, h is a solution to

$$\left(\frac{dh}{dt}, v\right)_{L^2} = Q_{L_I}(h,v) + (P_c(\omega_1)f_I, v)_{L^2} + (P_c(\omega_1)g_I, g_vG_\lambda)_{L^2},$$

where f_I and g_I were defined at the beginning of Section 1.4.2, hence, for any $v \in V$, h satisfies

$$(h,v)_{L^2} = \left(e^{L_It}h_0 + \int_0^t e^{L_I(t-s)}P_c(\omega_1)f_I(s)ds, v\right)_{L^2} + \left(\int_0^t e^{L_I(t-s)}P_c(\omega_1)g_I(s)ds, q_vG_\lambda\right)_{L^2}.$$

In addition let us assume that $v \in V \cap L^1_w$, hence by Hölder inequality

$$(h,v)_{L^2} \leq \left(\|e^{L_I t} h_0\|_{V\cap L^\infty_{w^{-1}}} + \left\| \int_0^t e^{L_I(t-s)} P_c(\omega_1) f_I(s) ds \right\|_{V\cap L^\infty_{w^{-1}}} \right) \|v\|_{L^1_w} +$$

$$+ \left\| \int_0^t e^{L_I(t-s)} P_c(\omega_1) g_I(s) ds \right\|_{V\cap L^\infty_{w^{-1}}} \|q_v G_\lambda\|_{L^1_w}.$$

Now we can apply the dispersive estimate proved in Theorem 1.16 and get

$$\|e^{L_I t} h_0\|_{V\cap L^\infty_{w^{-1}}} \leq c(1+t)^{-3/2} \|h_0\|_{V\cap L^1_w} \leq c(1+t)^{-3/2} d,$$

where d was defined in the statement of the present proposition. Furthermore, again by Theorem 1.16,

$$\left\| \int_0^t e^{L_I(t-s)} P_c(\omega_1) f_I(s) ds \right\|_{V\cap L^\infty_{w^{-1}}} \leq c \int_0^t (1+t-s)^{-3/2} \|f_I(s)\|_{V\cap L^1_w} ds \leq$$

$$\leq c \int_0^t (1+t-s)^{-3/2} (|\dot\gamma(s)| + |\dot\omega(s)|) ds \leq c \int_0^t (1+t-s)^{-3/2} \|\eta(s)\|^2_{L^\infty_{w^{-1}}} ds.$$

Analogously, using Lemma 1.25 and Theorem 1.16,

$$\left\| \int_0^t e^{L_I(t-s)} P_c(\omega_1) g_I(s) ds \right\|_{V\cap L^\infty_{w^{-1}}} \leq c \int_0^t (1+t-s)^{-3/2} \|g_I(s)\|_{V\cap L^1_w} ds \leq$$

$$\leq c \int_0^t (1+t-s)^{-3/2} (\|\eta(s)\|^2_{L^\infty_{w^{-1}}} + \rho \|\eta(s)\|_{L^\infty_{w^{-1}}}) ds.$$

Let us define

$$m(t) = \sup_{s\in[0,t]} (1+s)^{3/2} \|\eta(s)\|_{L^\infty_{w^{-1}}}.$$

Now, using the above inequalities, Lemma 1.25, and exploiting the duality paring defined by the inner product in L^2, it holds

$$(1+t)^{3/2} \|\eta(t)\|_{L^\infty_{w^{-1}}} = (1+t)^{3/2} \sup_{0\neq v\in L^1_w} \frac{(\eta(t),v)_{L^2}}{\|v\|_{L^1_w}} \leq$$

$$\leq c \left(\|e^{L_I t} h_0\|_{V\cap L^\infty_{w^{-1}}} + \left\| \int_0^t e^{L_I(t-s)} P_c(\omega_1) f_I(s) ds \right\|_{V\cap L^\infty_{w^{-1}}} + \left\| \int_0^t e^{L_I(t-s)} P_c(\omega_1) g_I(s) ds \right\|_{V\cap L^\infty_{w^{-1}}} \right) \leq$$

$$\leq c \int_0^t (1+t-s)^{-3/2} (\|\eta(s)\|^2_{L^\infty_{w^{-1}}} + \rho \|\eta(s)\|_{L^\infty_{w^{-1}}}) ds,$$

$$\leq c \left(d + m^2(t) \int_0^t (1+t)^{3/2} (1+s)^{-3} (1+t-s)^{-3/2} ds + \rho m(t) \int_0^t (1+t)^{3/2} (1+s)^{-3/2} (1+t-s)^{-3/2} ds \right).$$

Observe that the constant c and both integrals appearing in the last inequality are bounded independently of t, and this implies that for any $t \in [0, t_1]$ we have

$$m(t) \leq c(d + m^2(t_1) + \rho m(t_1)) \leq c(d + \rho_1^2) \leq c_2 d,$$

provided d and ρ are small enough. Since the constant c_2 does not depend on t_1, we can choose $d < \frac{\rho}{4c_2}$ and finally get

$$m(t_1) \leq \frac{\rho}{4},$$

concluding the proof of Proposition 1.25.

1.7 Asymptotic stability

Now we are in the position to prove the asymptotic stability result as stated in the next theorem. Before formulating the result, let us denote by U_t the integral kernel which defines the propagator of the free Laplacian in $\mathbb{R}^3$, namely $U_t(x) = (4\pi i t)^{-3/2} e^{i\frac{|x|^2}{4t}}$.

Theorem 1.29. *Assume $\sigma \in (0, 1/\sqrt{2})$. Let $u \in C(\mathbb{R}^+, V)$ be a solution to equation (1.7) with $u(0) = u_0 \in V \cap L^1_w$ and denote $d = \|u_0 - e^{i\theta_0}\Phi_{\omega_0}\|_{V\cap L^1_w}$, for some $\omega_0 > 0$ and $\theta_0 \in \mathbb{R}$. Then, if d is sufficiently small, the solution u can be decomposed as follows*

$$u = e^{i\omega_\infty t}\Phi_{\omega_\infty} + U_t * \psi_\infty + r_\infty, \tag{1.54}$$

where $\omega_\infty > 0$ and ψ_∞, $r_\infty \in L^2(\mathbb{R}^3)$, with $\|r_\infty\|_{L^2} = O(t^{-5/4})$ as $t \to +\infty$.

Proof. Along the proof we assume that $P_0(u_0 - e^{i\theta_0}\Phi_{\omega_0}) = 0$, and we recall from Lemma 1.22 that there is no loss of generality in this choice. First of all let us notice that Theorem 1.24 implies $\omega(t) \to \omega_\infty$, and $\Theta(t) - \omega_\infty t \to 0$, as $t \to +\infty$. Next, let us define the modulated soliton as

$$s(t,x) = e^{i\Theta(t)}\Phi_{\omega(t)}(x),$$

and the function

$$z(t,x) = u(t,x) - s(t,x). \tag{1.55}$$

By equation (1.7) and (1.15) one has that, for any $v \in V$, $z(t)$ is also a solution to

$$\left(i\frac{dz}{dt}, v\right)_{L^2} = \mathrm{Re}\int_{\mathbb{R}^3} \nabla\phi_z \cdot \overline{\nabla\phi_v}dx - \nu\,\mathrm{Re}((|q_u|^{2\sigma}q_u - |q_s|^{2\sigma}q_s)\overline{q_v}) + \left(\dot\gamma s - i\dot\omega\frac{ds}{d\omega}, v\right)_{L^2}.$$

As one can verify by direct differentiation, the solution of the last equation can be expressed as

$$z(t,x) = U_t * z_0(x) + i\int_0^t U_{t-\tau}(x)q_z(\tau)d\tau - i\int_0^t U_{t-\tau} * f(s(\tau))d\tau, \tag{1.56}$$

where we denoted $f(s) = \dot\gamma s - i\dot\omega\frac{ds}{d\omega}$ and, according to (3.77), $q_z(t) = q_u(t) - q_s(t)$. Let us consider the last integral in formula (1.56)

$$\int_0^t U_{t-\tau} * f(s(\tau))d\tau = U_t * \int_0^\infty U_{-\tau} * f(s(\tau))d\tau - \int_t^\infty U_{t-\tau} * f(s(\tau))d\tau,$$

and note that the regularity of $s(t,x)$ implies $\psi_1(x) = \int_0^\infty U_{-\tau} * f(s(\tau))d\tau \in L^2(\mathbb{R}^3)$, and $r_1(t,x) = -\int_t^\infty U_{t-\tau} * f(s(\tau))d\tau \in L^2(\mathbb{R}^3)$. Moreover, from Theorem 1.24 and the unitarity of the evolution group of the free Laplacian we have $\|r_1(t)\|_{L^2} = O(t^{-2})$, $t \to +\infty$.
To conclude the proof it is left to prove a similar asymptotic decomposition for the first integral in the formula (1.56). As before, one can write

$$\int_0^t U_{t-\tau}(x)q_z(\tau)d\tau = U_t * \int_0^\infty U_{-\tau}(x)q_z(\tau)d\tau - \int_t^\infty U_{t-\tau}(x)q_z(\tau)d\tau.$$

First of all one needs to show that $\psi_0(x) = \int_0^\infty U_{-\tau}(x)q_z(\tau)d\tau$ belongs to $L^2(\mathbb{R}^3)$. To this aim, let us observe that $\psi_0(x) = \frac{1}{(4\pi i)^{3/2}}h\left(\frac{r^2}{4}\right)$, with $h(y) = \int_0^\infty e^{-iy/\tau}\tau^{-3/2}q_z(\tau)\,d\tau$, hence

$$\|\psi_0\|^2_{L^2} = \frac{1}{(4\pi)^2}\int_0^\infty \left|h\left(\frac{r^2}{4}\right)\right|^2 r^2dr = \frac{1}{(2\pi)^2}\int_0^\infty |h(y)|^2\sqrt{y}dy.$$

From the first and the last terms one gets $\psi_0 \in L^2(\mathbb{R}^3)$ if and only if $h \in L^2(\mathbb{R}^+, \sqrt{y}dy)$. On the other hand, one can perform the change of variable $u = \frac{1}{\tau}$ in the integral function h and get

$$h(y) = \int_0^\infty e^{-iyu} \frac{1}{\sqrt{u}} q_z\left(\frac{1}{u}\right) du = \int_0^\infty e^{-iyu} \frac{1}{u} q_z\left(\frac{1}{u}\right) \sqrt{u} du,$$

where we set $y = \frac{|x|^2}{4}$. Then $\widehat{h}(u) = \frac{1}{u} q_z\left(\frac{1}{u}\right)$. Moreover, by Theorem 1.24, $\left|\frac{1}{u} q_z\left(\frac{1}{u}\right)\right|^2 \sqrt{u} \leq \frac{u^{3/2}}{(1+u)^3}$ then $\widehat{h} \in L^2(\mathbb{R}^+, \sqrt{u}du)$ and hence, by Plancherel's identity $h \in L^2(\mathbb{R}^+, \sqrt{y}dy)$.
Finally, let us denote $r_0 = \int_t^\infty U_{t-\tau}(x) q_z(\tau) d\tau$. As before, we have $r_0(x) = g\left(\frac{r^2}{4}\right)$, with $g(y) = \int_0^\infty e^{-iy/(t-\tau)}(t-\tau)^{-3/2} q_z(\tau)\, d\tau$. Moreover, we can set $y = \frac{|x|^2}{4}$ exploit the change of variables $u = -\frac{1}{t-\tau}$ in order to get

$$g(y) = \int_0^\infty e^{-iyu} \frac{i}{u} q_z\left(t + \frac{1}{u}\right) \sqrt{u} du.$$

Again, Theorem 1.24 implies that $\widehat{g}(u) = \frac{i}{u} q_z\left(t + \frac{1}{u}\right) \in L^2(\mathbb{R}^+, \sqrt{u}du)$, for any $t \geq 0$. In particular,

$$\|g\|^2_{L^2(\mathbb{R}^+, \sqrt{u}du)} \leq \tilde{c} \int_0^\infty \frac{u^{3/2}}{((1+t)u+1)^3} du \leq c(1+t)^{-5/2},$$

for any $t \geq 0$, with $\tilde{c}$, $c > 0$ independent of time. Summing up, Plancherel's identity allows us to conclude $\|r_0\|_{L^2} = O(t^{-5/4})$ as $t \to +\infty$.
Hence the theorem follows with $\psi_\infty = z_0 + \psi_0 + \psi_1$, and $r_\infty = r_0 + r_1$. □

1.8 Appendices

1.9 The generalized kernel of the operator L

The aim of this appendix is to provide the proof or Theorem 1.9.

Proof. It is easy to see that $c\Phi_\omega$, with $c \in \mathbb{C}$, is the unique family of distributional solutions to the equation

$$-\triangle u + \omega u = 0.$$

Furthermore, Φ_ω belongs to $D(H_{\alpha_2})$ but not to $D(H_{\alpha_1})$ since the boundary condition is not satisfied. Hence

$$\ker(L) = \text{span}\left\{\begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}\right\}.$$

Let us now consider the operator

$$L^2 = \begin{bmatrix} -L_2 L_1 & 0 \\ 0 & -L_1 L_2 \end{bmatrix}.$$

Since the operator L_1 is invertible, the following holds

$$u \in \ker(L_1 L_2) \Leftrightarrow u \in \ker(L_2), \quad \text{then} \quad \ker(L_1 L_2) = \text{span}\{\Phi_\omega\},$$

$$u \in \ker(L_2 L_1) \Leftrightarrow \exists u \in D(H_{\alpha_1}) \quad \text{such that} \quad L_1 u = \Phi_\omega.$$

Solving the former equation one gets that $\ker(L_1 L_2) = \text{span}\ \{\varphi_\omega\}$. From this follows

$$\ker(L^2) = \text{span}\left\{\begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}, \begin{pmatrix} \varphi_\omega \\ 0 \end{pmatrix}\right\}.$$

The operator L^3 has the following form

$$L^3 = \begin{bmatrix} 0 & -L_2L_1L_2 \\ L_1L_2L_1 & 0 \end{bmatrix}.$$

As before

$$u \in \ker(L_1L_2L_1) \Leftrightarrow L_1u \in \ker(L_1L_2) = \text{span }\{\Phi_\omega\} \Leftrightarrow \ker(L_1L_2L_1) = \text{span }\{\varphi_\omega\},$$

$$u \in \ker(L_2L_1L_2) \Leftrightarrow u \in \ker(L_2) = \text{span }\{\Phi_\omega\} \quad \text{or} \quad L_2u \in \ker(L_2L_1) = \text{span }\{\varphi_\omega\}.$$

Let us notice that the equation

$$-\triangle u + \omega u = \varphi_\omega$$

has a unique family of distributional solutions given by

$$u(x) = \left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{\frac{1}{2\sigma}} \left[\left(\frac{c_2}{2\sqrt{\omega}} + \frac{1}{16\sigma^2\omega^2}\right)\frac{e^{-\sqrt{\omega}|x|}}{4\pi|x|} + \frac{c_1}{2\sqrt{\omega}}\frac{e^{\sqrt{\omega}|x|}}{4\pi|x|} + \right.$$

$$\left. -\frac{1}{8\omega}|x|\frac{e^{-\sqrt{\omega}|x|}}{4\pi} + \left(\frac{1}{8\sigma\omega^{\frac{3}{2}}} - \frac{1}{8\omega^{\frac{3}{2}}}\right)\frac{e^{-\sqrt{\omega}|x|}}{4\pi}\right].$$

Notice that one must impose that u belongs to $D(H_{\alpha_2})$ which means that $u \in L^2(\mathbb{R}^3)$ and satisfies the boundary condition. This is equivalent to ask the following algebraic conditions to be verified

$$\begin{cases} c_1 = 0 \\ c_2 = \frac{\sigma-1}{8\sigma\omega^{\frac{3}{2}}}. \end{cases}$$

Therefore, if $\sigma \neq 1$, then $\ker(L_2L_1L_2) = \text{span }\{\Phi_\omega\}$. Hence

$$\ker(L^3) = \ker(L^2),$$

which concludes the first part of the theorem.
In the case $\sigma = 1$ we get $\ker(L_2L_1L_2) = \{\Phi_\omega, g_\omega\}$, then

$$\ker(L^3) = \text{span}\left\{\begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}, \begin{pmatrix} \varphi_\omega \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g_\omega \end{pmatrix}\right\}.$$

With analogous computations one can prove that

$$\ker(L^4) = \ker(L^5) = \text{span}\left\{\begin{pmatrix} 0 \\ \Phi_\omega \end{pmatrix}, \begin{pmatrix} \varphi_\omega \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g_\omega \end{pmatrix}, \begin{pmatrix} h_\omega \\ 0 \end{pmatrix}\right\},$$

which concludes the proof. □

1.10 Proof of the resolvent formula

In this appendix we prove that the operator $(L - \lambda I)^{-1}$ is given by

$$R(\lambda) = \begin{bmatrix} -\lambda(\lambda^2 + L_2L_1)^{-1} & -L_2(\lambda^2 + L_1L_2)^{-1} \\ L_1(\lambda^2 + L_2L_1)^{-1} & -\lambda(\lambda^2 + L_1L_2)^{-1} \end{bmatrix}$$

for the resolvent of the linear operator L. More precisely, we prove the following proposition.

Proposition 1.30. *If* $\lambda \in \mathbb{C}\backslash\sigma(L)$, *then* $R(\lambda)(L-\lambda I)u = u, \quad \forall u \in D(L)$, *and* $(L-\lambda I)R(\lambda)f = f$ *for any* $f \in (L^2(\mathbb{R}^3))^2$.

Before proving the former proposition, let us prove the following lemma.

Lemma 1.31. *For any* $\lambda \in \mathbb{C} \setminus \sigma(L)$ *the following identities hold*

1. $(\lambda^2 + L_2 L_1)^{-1} L_1^{-1} = L_1^{-1}(\lambda^2 + L_1 L_2)^{-1}$,
2. $(\lambda^2 + L_1 L_2)^{-1} = (\lambda^2 L_1^{-1} + L_2)^{-1} L_1^{-1}$,
3. $(\lambda^2 + L_1 \widetilde{L_2})^{-1} \widetilde{L_2}^{-1} = \widetilde{L_2}^{-1}(\lambda^2 + \widetilde{L_2} L_1)^{-1}$,

where $\widetilde{L_2}$ *is the restriction of the operator* L_2 *to the projection of its domain onto the subspace of* $L^2(\mathbb{R}^3)$ *associated to the continuous spectrum of* L_2.

Proof. First of all, let us notice that all the inverse operators are well defined since λ is not allowed to be a spectral point of L, L_1 is invertible and L_2 is restricted to a subspace on which it is invertible too.
In order to prove 1, we prove the following claim

$$(\lambda^2 + L_2 L_1)^{-1} L_1^{-1} = (\lambda^2 L_1 + L_1 L_2 L_1)^{-1} = L_1^{-1}(\lambda^2 + L_1 L_2)^{-1}.$$

To this purpose, let us take any $\xi \in L^2(\mathbb{R}^3)$, then one has

$$(\lambda^2 + L_2 L_1)^{-1} L_1^{-1}\xi \in D(L_2 L_1) \quad \text{and} \quad L_1^{-1}\xi \in D(L_1).$$

Hence, the following chain of identities holds

$$(\lambda^2 L_1 + L_1 L_2 L_1)(\lambda^2 + L_2 L_1)^{-1} L_1^{-1}\xi = L_1(\lambda^2 + L_2 L_1)(\lambda^2 + L_2 L_1)^{-1} L_1^{-1}\xi = L_1 L_1^{-1}\xi = \xi.$$

On the other hand, let us take $\eta \in D(L_1 L_2 L_1)$, and observe that, in particular, $\eta \in D(L_2 L_1)$. This justifies the following identities

$$(\lambda^2 + L_2 L_1)^{-1} L_1^{-1}(\lambda^2 L_1 + L_1 L_2 L_1)\eta =$$

$$= (\lambda^2 + L_2 L_1)^{-1} L_1^{-1} L_1(\lambda^2 + L_2 L_1)\eta = (\lambda^2 + L_2 L_1)^{-1}(\lambda^2 + L_2 L_1)\eta = \eta,$$

which concludes the proof of the first identity of the claim. The second one is proved in the same way.
The proof of 3. can be done in the same way exganging L_1 with $\widetilde{L_2}$ and L_2 with L_1.
It is left to prove 2.. To do that, let ξ be in $L^2(\mathbb{R}^3)$, then $(\lambda^2 L_1^{-1} + L_2)^{-1} L_1^{-1}\xi \in D((\lambda^2 L_1^{-1} + L_2))$ and $L_1^{-1}\xi \in D(L_1)$. Hence, we have

$$(\lambda^2 + L_1 L_2)(\lambda^2 L_1^{-1} + L_2)^{-1} L_1^{-1}\xi = L_1(\lambda^2 L_1^{-1} + L_2)(\lambda^2 L_1^{-1} + L_2)^{-1} L_1^{-1}\xi = \xi.$$

On the other hand, for any $\eta \in D(L_1 L_2)$ one has $\eta \in D(L_2) \subset L^2(\mathbb{R}^3) = D(L_1^{-1})$, which justifies

$$(\lambda^2 L_1^{-1} + L_2)^{-1} L_1^{-1}(\lambda^2 + L_1 L_2)\eta = (\lambda^2 L_1^{-1} + L_2)^{-1} L_1^{-1} L_1(\lambda^2 L_1^{-1} + L_2)\eta = \eta.$$

□

We can now prove the proposition.

Proof. `I step: proof of the first identity.`
Let us recall that for $u \in D(L)$ holds

$$R(\lambda)(L-\lambda I)u =$$
$$= \begin{bmatrix} -\lambda(\lambda^2+L_2L_1)^{-1} & -L_2(\lambda^2+L_1L_2)^{-1} \\ L_1(\lambda^2+L_2L_1)^{-1} & -\lambda(\lambda^2+L_1L_2)^{-1} \end{bmatrix} \begin{bmatrix} -\lambda & L_2 \\ -L_1 & -\lambda \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where

$$w_1 = \lambda^2(\lambda^2+L_2L_1)^{-1}u_1 + L_2(\lambda^2+L_1L_2)^{-1}L_1u_1 - \lambda(\lambda^2+L_2L_1)^{-1}L_2u_2 + \lambda L_2(\lambda^2+L_1L_2)^{-1}u_2,$$

and

$$w_2 = \lambda^2(\lambda^2+L_1L_2)^{-1}u_2 + L_1(\lambda^2+L_2L_1)^{-1}L_2u_2 + \lambda(\lambda^2+L_1L_2)^{-1}L_1u_1 - \lambda L_1(\lambda^2+L_2L_1)^{-1}u_1.$$

We will concentrate on the first component w_1, because the second one can be treated in the same way.
The spectrum of the selfadjoint operator L_2 is ([6])

$$\sigma(L_2) = \{0\} \cup [\omega, +\infty),$$

where 0 is a simple eigenvalue and $\ker(L_2) = \operatorname{span}\{\Phi_\omega\}$. Hence, any $u_2 \in D(L_2)$ can be decomposed as

$$u_2 = a\Phi_\omega + g_2,$$

where $a \in \mathbb{C}$ and g_2 belongs to the projection of $D(L_2)$ onto the continuous spectrum of L_2.
Moreover, since $L_2\Phi_\omega = 0$, one gets $\Phi_\omega \in D(L_1L_2)$ and

$$\Phi_\omega = \frac{1}{\lambda^2}(\lambda^2+L_1L_2)\Phi_\omega = (\lambda^2+L_1L_2)\left(\frac{1}{\lambda^2}\Phi_\omega\right),$$

which is equivalent to $(\lambda^2+L_1L_2)^{-1}\Phi_\omega \in \ker(L_2)$.
As a consequence, since L_1 and $\widetilde{L_2}$ are invertible on their domains, one has

$$w_1 = \lambda^2(\lambda^2+L_2L_1)^{-1}L_1^{-1}L_1u_1 + L_2(\lambda^2+L_1L_2)^{-1}L_1u_1 +$$
$$-\lambda(\lambda^2+\widetilde{L_2}L_1)^{-1}\widetilde{L_2}g_2 + \lambda\widetilde{L_2}(\lambda^2+L_1\widetilde{L_2})^{-1}\widetilde{L_2}^{-1}\widetilde{L_2}g_2,$$

hence, by lemma 1.31 it follows

$$w_1 = (\lambda^2L_1^{-1}+L_2)(\lambda^2+L_1L_2)^{-1}L_1u_1 - \lambda(\lambda^2+\widetilde{L_2}L_1)^{-1}\widetilde{L_2}g_2 + \lambda\widetilde{L_2}\widetilde{L_2}^{-1}(\lambda^2+\widetilde{L_2}L_1)^{-1}\widetilde{L_2}g_2 =$$
$$= (\lambda^2L_1^{-1}+L_2)(\lambda^2L_1^{-1}+L_2)^{-1}L_1^{-1}L_1u_1 = u_1.$$

Summing up, we proved

$$R(\lambda)(L-\lambda I)u = u \quad \forall u \in D(L).$$

`II step: proof of the second identity.`
First of all let us recall that for $f \in (L^2(\mathbb{R}^3))^2$ one has

$$(\lambda^2+L_2L_1)^{-1}f_1 \in D(L_2L_1) \quad \text{and} \quad (\lambda^2+L_1L_2)^{-1}f_2 \in D(L_1L_2).$$

Hence, the following identities hold

$$(L-\lambda I)R(\lambda)f = \begin{bmatrix} -\lambda & L_2 \\ -L_1 & -\lambda \end{bmatrix} \begin{bmatrix} -\lambda(\lambda^2+L_2L_1)^{-1} & -L_2(\lambda^2+L_1L_2)^{-1} \\ L_1(\lambda^2+L_2L_1)^{-1} & -\lambda(\lambda^2+L_1L_2)^{-1} \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} =$$
$$= \begin{pmatrix} (\lambda^2+L_2L_1)(\lambda^2+L_2L_1)^{-1}f_1 \\ (\lambda^2+L_1L_2)(\lambda^2+L_1L_2)^{-1}f_2 \end{pmatrix} = f,$$

which concludes the proof. □

1.11 The dynamics generated by L along the generalized kernel

In this appendix we estimate the behaviour of the propagator of L around the eigenvalue 0. This is achieved in the following theorem in which it is proved that the dynamics has a linear growth in time along the generalized kernel.

Theorem 1.32. *For any $r \in (0, \omega)$ the following identity holds*

$$\frac{1}{2\pi i}\int_{|\lambda|=r} R(\lambda; x, y)e^{-\lambda t}d\lambda =$$

$$= \begin{bmatrix} \frac{\sqrt{\omega}}{1-\sigma}\frac{e^{-\sqrt{\omega}(|x|+|y|)}}{2\pi|x||y|}(2\sigma\sqrt{\omega}|x|-1) & 0 \\ i\frac{2\omega^{\frac{3}{2}}\sigma}{1-\sigma}\frac{e^{-\sqrt{\omega}(|x|+|y|)}}{\pi|x||y|}t & \frac{\sqrt{\omega}}{1-\sigma}\frac{e^{-\sqrt{\omega}(|x|+|y|)}}{2\pi|x||y|}(2\sigma\sqrt{\omega}|y|-1) \end{bmatrix},$$

for any $x, y \in \mathbb{R}^3$.

Proof. Since the convolution term of the resolvent $R(\lambda)$ is continuous in zero it suffices to compute the integral of the multiplication term. First of all, let us note that the function

$$f(\lambda) = \frac{4\pi i}{W(\lambda^2)}\Lambda_1(\lambda)e^{-\lambda t} = i\frac{e^{-\lambda t}}{W(\lambda^2)}\cdot$$

$$\cdot\left[\frac{(4\pi\alpha_2 - i\sqrt{-\omega+i\lambda})e^{i\sqrt{-\omega-i\lambda}|x|} + (4\pi\alpha_2 - i\sqrt{-\omega-i\lambda})e^{i\sqrt{-\omega+i\lambda}|x|}}{8\pi|x||y|}\left(e^{i\sqrt{-\omega+i\lambda}|y|} - e^{i\sqrt{-\omega-i\lambda}|y|}\right) + \right.$$

$$\left. + \frac{-(4\pi\alpha_1 - i\sqrt{-\omega+i\lambda})e^{i\sqrt{-\omega-i\lambda}|x|} + (4\pi\alpha_1 - i\sqrt{-\omega-i\lambda})e^{i\sqrt{-\omega+i\lambda}|x|}}{8\pi|x||y|}\left(e^{i\sqrt{-\omega+i\lambda}|y|} + e^{i\sqrt{-\omega-i\lambda}|y|}\right)\right] =$$

$$= \frac{i}{8\pi|x||y|}\left[2(4\pi\alpha_2 + \sqrt{\omega})e^{-\sqrt{\omega}|x|}\left(\frac{i|y|}{\sqrt{\omega}}e^{-\sqrt{\omega}|y|}\lambda + o(\lambda^2)\right) + 8\pi\alpha_1 e^{-\sqrt{\omega}|y|}\left(\frac{i|x|}{\sqrt{\omega}}e^{-\sqrt{\omega}|x|}\lambda + o(\lambda^2)\right) + \right.$$

$$\left. + 2ie^{-\sqrt{\omega}|y|}\left(\left(\frac{1}{\sqrt{\omega}} + |x|\right)e^{-\sqrt{\omega}|x|}\lambda + o(\lambda^2)\right)\right]\left(\frac{1-\sigma}{2\omega}\lambda^2 + o(\lambda^4)\right)^{-1} \sim$$

$$\sim -\frac{\sqrt{\omega}}{1-\sigma}\frac{e^{-\sqrt{\omega}(|x|+|y|)}}{2\pi|x||y|}[(4\pi\alpha_2 + \sqrt{\omega})|y| + (4\pi\alpha_1 + \sqrt{\omega})|x| + 1]\frac{1}{\lambda}.$$

as $\lambda \to 0$. Hence the function $f(\lambda)$ has a pole of order one in zero. Then, by the Cauchy theorem one gets

$$\frac{1}{2\pi i}\int_{|\lambda|=r}\frac{4\pi i}{W(\lambda^2)}\Lambda_1(\lambda)e^{-\lambda t}d\lambda = -\frac{\sqrt{\omega}}{1-\sigma}\frac{e^{-\sqrt{\omega}(|x|+|y|)}}{2\pi|x||y|}[(4\pi\alpha_2 + \sqrt{\omega})|y| + (4\pi\alpha_1 + \sqrt{\omega})|x| + 1] =$$

$$= -\frac{\sqrt{\omega}}{1-\sigma}\frac{e^{-\sqrt{\omega}(|x|+|y|)}}{2\pi|x||y|}[-2\sigma\sqrt{\omega}|x| + 1].$$

Switching α_1 to α_2 and vice versa, it follows

$$\frac{1}{2\pi i}\int_{|\lambda|=r}\frac{4\pi i}{W(\lambda^2)}\Lambda_2(\lambda)e^{-\lambda t}d\lambda = -\frac{\sqrt{\omega}}{1-\sigma}\frac{e^{-\sqrt{\omega}(|x|+|y|)}}{2\pi|x||y|}[(4\pi\alpha_1 + \sqrt{\omega})|y| + (4\pi\alpha_2 + \sqrt{\omega})|x| + 1] =$$

$$= -\frac{\sqrt{\omega}}{1-\sigma}\frac{e^{-\sqrt{\omega}(|x|+|y|)}}{2\pi|x||y|}[-2\sigma\sqrt{\omega}|y| + 1].$$

On the other hand, the function

$$\frac{4\pi i}{W(\lambda^2)}\Sigma_1(\lambda)e^{-\lambda t}$$

is the sum of a continuous function and a function with a pole of second order in zero, namely

$$g(\lambda)e^{-\lambda t} =$$

$$= i\frac{(4\pi\alpha_1 - i\sqrt{-\omega+i\lambda})e^{i\sqrt{-\omega-i\lambda}|x|} + (8\pi\alpha_1 - i\sqrt{-\omega-i\lambda})e^{i\sqrt{-\omega+i\lambda}|x|}}{W(\lambda^2)4\pi|x||y|}(e^{i\sqrt{-\omega+i\lambda}|y|}+e^{i\sqrt{-\omega-i\lambda}|y|})e^{-\lambda t}.$$

Note that $g(\lambda) = \sum_{k=2}^{+\infty} a_k\lambda_k$ with

$$a_{-2} = i\frac{\omega}{1-\sigma}\frac{4\pi\alpha_1+\sqrt{\omega}}{\pi|x||y|}e^{-\sqrt{\omega}(|x|+|y|)}, \quad a_{-1} = 0,$$

then, by residue theorem,

$$\frac{1}{2\pi i}\int_{|\lambda|=r}\frac{4\pi i}{W(\lambda^2)}\Sigma_1(\lambda)e^{-\lambda t}d\lambda = -i\frac{\omega}{1-\sigma}\frac{4\pi\alpha_1+\sqrt{\omega}}{\pi|x||y|}e^{-\sqrt{\omega}(|x|+|y|)}t = i\frac{2\sigma\omega^{\frac{3}{2}}}{(1-\sigma)\pi|x||y|}e^{-\sqrt{\omega}(|x|+|y|)}t.$$

In the same way

$$\frac{1}{2\pi i}\int_{|\lambda|=r}\frac{4\pi i}{W(\lambda^2)}\Sigma_2(\lambda)e^{-\lambda t}d\lambda = 0,$$

which concludes the proof. □

Chapter 2

Presence of purely imaginary eigenvalues

2.1 Introduction

In the previous chapter we have studied the asymptotic stability of standing waves for a nonlinear Schrödinger equation with a nonlinearity concentrated at the origin in the case in which the discrete spectrum of the linearized operator is made just by the eigenvalue 0 with algebraic multiplicity 2. We recall that this component of the discrete spectrum exists in any case due to the $U(1)$ invariance of the dynamics, related through Noether Theorem to mass (or L^2-norm) conservation. Here we go on with the analysis of the asymptotic stability in the case in which a couple of two purely imaginary simple eigenvalues $\pm i\xi$ is present in the spectrum of the linearized operator with the further condition that $\pm 2i\xi$ belongs to the continuous spectrum. This case corresponds to the nonlinearities where $\sigma \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right)$. The asymptotic stability result is achieved following the outline of [11] and [33]. In particular, in [33] the same problem for the analogous one-dimensional model is studied.

Nevertheless, the three-dimensional case presents some differences. The first one is that, as explained in Chapter 1, the concentrated nonlinearity imposes to develop the analysis at the form level. This means that the estimates on the evolution of the initial data are more delicate.

The second main difference is the faster decay of the propagator of the free Laplacian. This allows to develop the the analysis using just the structural weight $w = 1 + \frac{1}{|x|}$ which arises from the dispersive estimate (once again see the previous chapter) instead of introducing new weighted spaces as done in the one-dimensional case.

Finally, the eigenfunctions associated to the purely imaginary eigenvalues do not have any oscillating term as in the one-dimensional case but they exponentially decrease as $|x| \to +\infty$. This fact will be very useful in order to get the decay in time of the radiation term.

On the other hand, comparing with the case in Chapter 1, and in parallel with the already known one-dimensional case, the presence of the two purely imaginary eigenvalues slows down the speed of decay of the remainder. This slower decay can be observed from the behavior of the parameters whose evolution is described by the modulation equations; these include an extra equation describing the evolution of the coefficients of the eigenfunctions associated to the purely imaginary eigenvalues. Hence, in order to deal with the modulation equations, it is necessary to consider also the quadratic and the cubic terms of the nonlinearity and, later, exploit a change of variables to have a normal form of the modulation equation to go on with the estimates. This makes more complicate the analysis of the integrability and of the decay of the terms in the asymptotic decomposition. Denoting by $N_2(q,q)$ the quadratic terms coming from the Taylor expansion of the nonlinearity, and by $\Psi(\omega_0) = \begin{pmatrix} \Psi_1(\omega_0) \\ \Psi_2(\omega_0) \end{pmatrix}$ the eigenfunction of the linearized operator associated to $i\xi_0$. Usually, when investigating asymptotic stability in presence of purely imaginary eigenvalues, one assumes that the following non-degeneracy condition holds:

$$JN_2(q_{\Psi(\omega_0)}, q_{\Psi(\omega_0)})\overline{q_{\Psi_+(2i\xi_0)}} \neq 0, \tag{2.1}$$

where $\Psi_+(2i\xi_0)$ is the generalized eigenfunction associated to $+2i\xi_0$. The previous condition can be considered as a nonlinear version of the Fermi Golden Rule (see for example [42], [43], [41], [44], [50], [51], [49], [10], and [11]). It is necessary to guarantee a time decay of the normal modes related to the discrete spectrum of the linearization; the decay is due to coupling with the continuous spectrum given by FGR, and consequent dispersion. Thanks to the explicit character of our model, we are able to directly verify that the decay of the discrete modes holds for σ in the range $\left(\frac{1}{\sqrt{2}}, \sigma^*\right)$, for a certain $\sigma^* \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right]$ (see Section 2.3.4). The numerical evidence is that this is true on the whole interval $\left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right)$. Eventually, we proved the following result.

Theorem (Asymptotic stability in the case of purely imaginary eigenvalues) Assume that $u(t) \in C(\mathbb{R}^+, V)$ is a solution to (3) with concentrated power nonlinearity (2) where $\sigma \in (\frac{1}{\sqrt{2}}, \sigma^*)$, for a certain $\sigma^* \in (\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}]$. Moreover, suppose that the initial datum

$$u(0) = u_0 = e^{i\omega_0+\gamma_0}\Phi_{\omega_0} + e^{i\omega_0+\gamma_0}[(z_0+\overline{z_0})\Psi_1 + i(z_0-\overline{z_0})\Psi_2] + f_0 \in V \cap L^1_w(\mathbb{R}^3),$$

with $\omega_0 > 0$, γ_0, $z_0 \in \mathbb{R}$, and $f_0 \in L^2(\mathbb{R}^3) \cap L^1_w(\mathbb{R}^3)$ is close to a stationary wave, i.e.

$$|z_0| \leq \epsilon^{1/2} \qquad \text{and} \qquad \|f_0\|_{L^1_w} \leq c\epsilon^{3/2},$$

where c, $\epsilon > 0$.
Then, provided ϵ is sufficiently small, the solution $u(t)$ can be asymptotically decomposed as

$$u(t) = e^{i\omega_\infty t + ib_1 \log(1+\epsilon k_\infty t)}\Phi_{\omega_\infty} + U_t * \psi_\infty + r_\infty, \quad \text{as } t \to +\infty,$$

where ω_∞, $\epsilon k_\infty > 0$, $b_1 \in \mathbb{R}$, and ψ_∞, $r_\infty \in L^2(\mathbb{R}^3)$ such that

$$\|r_\infty\|_{L^2} = O(t^{-1/4}) \quad \text{as } t \to +\infty,$$

in $L^2(\mathbb{R}^3)$.

Notice that the range of the admitted nonlinearities σ implies that $\pm 2i\xi$ is in the essential spectrum of the linearized operator.
A last comment of general nature is in order. As in the one dimensional case studied by Buslaev, Komech, Kopylova, and Stuart in [8] and Komech, Kopylova, and Stuart in [33], and the three dimensional model analyzed in the previous chapter, the analysis of a specific model allows to obtain asymptotic stability of standing waves without a priori assumptions. In particular the nonlinearity is fixed, of power type and subcritical, no smallness of initial data is required (in the sense that we give results for every standing wave of the model and initial data near the family of standing waves). Moreover, while Komech, Kopylova, and Suart find a link between the Fermi Golden Rule and the decay of normal modes, here such decay is directly verified. This fact seems to indicate that some of these assumptions or hypotheses are in fact unnecessary when enough information about the model is known.

For the sake of completeness, in this chapter we will repeat proofs requiring some modifications because of the facts mentioned above; on the contrary, where the arguments hold unchanged, just a reference will be given.
Recall that we are considering the following nonlinear evolution problem associated to the operator H_α, i.e.

$$i\frac{du}{dt} = H_\alpha u, \tag{2.2}$$

with an initial datum $u(0) = u_0$. The action of the operator H_α is defined in Section 1.1 and the existence of the solitary waves manifold

$$\mathcal{M} = \left\{ \Phi_\omega(x) = \left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{\frac{1}{2\sigma}} \frac{e^{-\sqrt{\omega}|x|}}{4\pi|x|} \in D : \ \omega > 0 \right\}.$$

is proved in Section 1.2.2. Furthermore, in Section 1.4 we describe some spectral properties of the linearized operator

$$L = \begin{bmatrix} 0 & L_2 \\ -L_1 & 0 \end{bmatrix},$$

where $L_j = H_{\alpha_j} + \omega$ for $j = 1, 2$, where $\alpha_1 = -(2\sigma+1)\frac{\sqrt{\omega}}{4\pi}$ and $\alpha_2 = -\frac{\sqrt{\omega}}{4\pi}$.
Let us stress that in the case $\sigma \in (1/\sqrt{2}, 1)$ the discrete spectrum of L consist in the eigenvalue 0 with algebraic multiplicity 2 and two purely imaginary eigenvalues $\pm i\xi$ with

$$\xi = 2\sigma\sqrt{1-\sigma^2}\omega. \tag{2.3}$$

As it is proved in Appendix 2.6.1, the eigenfunction Ψ associated to the eigenvalue $i\xi$ can be chosen such that its first component is real and its second component is purely imaginary. Hence, one gets that the eigenfunction associated to $-i\xi$ is

$$\Psi^* = \begin{pmatrix} \Psi_1 \\ -\Psi_2 \end{pmatrix}.$$

As a consequence, the domain of the operator L can be decomposed in three symplectic subspaces, more precisely

$$D(L) = X^0 \oplus X^1 \oplus X^c,$$

where X^0, X^1, and X^c are the generalized kernel, the eigensubspace corresponding to the eigenfunctions Ψ and Ψ^*, and the continuous spectral subspace respectively.
The projection operators from $L^2(\mathbb{R}^3)$ onto X^0, X^1 and X^c are

$$\begin{aligned} P^0 f &= -\frac{2}{\Delta}\Omega\left(f, \frac{d\Phi_\omega}{d\omega}\right) J\Phi_\omega + \frac{2}{\Delta}\Omega\left(f, J\Phi_\omega\right)\frac{d\Phi_\omega}{d\omega}, \quad && \Delta = \tfrac{d}{d\omega}\|\Phi_\omega\|_{L^2}, \\ P^1 f &= \frac{\Omega(f,\Psi)}{\kappa}\Psi + \frac{\Omega(f,\Psi^*)}{\kappa}\Psi^*, && \kappa = \Omega(\Psi,\Psi^*), \\ P^c f &= f - P^0 f - P^1 f, \end{aligned}$$

respectively. Moreover, we denote with $\Pi^\pm$ the projections onto the branches $\mathcal{C}_\pm$ of the continuous spectrum separately.
Finally note that the dispersive estimate in Theorem 1.16 still holds true since there are no embedded eigenvalues nor threshold resonances and the eigenvalue 0 has the same algebraic multiplicity as in the case $\left(0, \frac{1}{\sqrt{2}}\right)$.

2.2 Modulation equations

Since the operators we are dealing with are all different in domain while the forms associated to them have all the same domain, namely

$$V = \left\{ u = \phi_\lambda + qG_\lambda, \text{ with } \phi_\lambda \in D^1(\mathbb{R}^3),\, q \in \mathbb{C} \right\},$$

it makes sense to do the following computations at the form level as done in Section 1.5. In order to do that let us recall that the variational formulation of equation (2.2) is

$$\left(i\frac{du}{dt}(t), v \right)_{L^2} = Q_\alpha(u(t), v) \quad \forall v \in V. \tag{2.4}$$

Note that the last equation makes sense because V is independent on the positive parameter λ and it is a Hilbert space with the norm

$$\|u\|_V^2 = \|\nabla \phi_\lambda\|_{L^2}^2 + |q|^2, \quad \forall u \in V.$$

In order to inspect the asymptotic stability of equation (2.2) it is useful to solve it with the ansatz

$$u(t,x) = e^{i\Theta(t)} \left(\Phi_{\omega(t)}(x) + \chi(t,x) \right), \tag{2.5}$$

where

$$\chi(t,x) = z(t)\Psi(t,x) + \overline{z(t)}\Psi^*(t,x) + f(t,x) = \psi(t,x) + f(t,x), \tag{2.6}$$

with $\psi \in X^1$, $f \in X^c$, and

$$\Theta(t) = \int_0^t \omega(s)ds + \gamma(t),$$

with $\omega(t)$, $\gamma(t)$ to be chosen in a suitable way.
Hence, we are constructing a solution of equation (2.2) close at each time to a solitary wave. Let us notice that the solitary wave does not need to be the same at every time, which means that the parameters $\omega(t)$ and $\Theta(t)$ are free to vary in time.
As in the case in which $\sigma \in \left(0, \frac{1}{\sqrt{2}}\right)$ (see Section 1.5) the function χ solves

$$\left(i\frac{d\chi}{dt}(t), v \right)_{L^2} = Q_{\alpha,Lin}(\chi(t), v) + \dot\gamma(t)(\Phi_{\omega(t)} + \chi(t), v)_{L^2} + \tag{2.7}$$

$$+\dot\omega(t) \left(-i\frac{d\Phi_{\omega(t)}}{d\omega}, v \right)_{L^2} + N(q_\chi(t), q_v),$$

for all $v \in V$, where $N(q_\chi(t), q_v)$ is the nonlinear part of the variational formulation of equation (2.2) defined together with $Q_{\alpha,Lin}(\chi(t), v)$ in Section 1.5.
Since $\omega(t)$, $\gamma(t)$, and $\chi(x,t)$ are unknown and the propagator grows in time along the directions of the generalized kernel of the operator L, the idea is to get a determined system requiring the function $\chi(t)$ to be orthogonal to the generalized kernel of L at any time $t \geq 0$. Hence, one obtains that ω, γ, z, and f must solve the following system of equations.

Theorem 2.1. (Modulation equations) *If $\chi(t)$ is a solution of equation* (2.7) *such that $P_0\chi(t) = 0$ for all $t \geq 0$ and $\omega(t)$ and $\gamma(t)$ are continuously differentiable in time, then ω and γ are solutions of*

$$\dot{\omega} = \frac{\mathrm{Re}\ (JN(q_\chi)\overline{q_{P_0^*(\Phi_\omega+\chi)}})}{\left(\varphi_\omega - \frac{dP_0}{d\omega}\chi, \Phi_\omega + \chi\right)_{L^2}}, \tag{2.8}$$

$$\dot{\gamma} = \frac{\mathrm{Re}\ (JN(q_\chi)\overline{q_{J(\varphi_\omega - \frac{dP_0}{d\omega}\chi)}})}{\left(\varphi_\omega - \frac{dP_0}{d\omega}\chi, \Phi_\omega + \chi\right)_{L^2}}, \tag{2.9}$$

and z and f satisfy

$$(\Psi, J\Psi)_{L^2}(\dot{z} - i\xi z) = \mathrm{Re}(JN(q_\chi)\overline{q_{J\Psi}}) + \dot{\omega}\left[\left(f, J\frac{d\Psi}{d\omega}\right)_{L^2} - \left(\frac{d\psi}{d\omega}, J\Psi\right)_{L^2}\right] + \dot{\gamma}(\chi, \Psi)_{L^2}, \tag{2.10}$$

$$\left(\frac{df}{dt}, v\right)_{L^2} = Q_L(f, v) + \left(-\dot{\omega}\left(zP^c\frac{d\Psi}{d\omega} + \overline{z}P^c\frac{d\Psi^*}{d\omega}\right) + \dot{\gamma}P^cJ\chi, v\right)_{L^2} + \tag{2.11}$$
$$+(8\pi\sqrt{\lambda}P^cJN(q_\chi)G_\lambda, q_vG_\lambda)_{L^2},$$

for all $v \in V$.

Proof. Equations (2.8) and (2.9) can be proved with the same argument exploited in the case $\sigma \in \left(0, \frac{1}{\sqrt{2}}\right)$.
Equation (2.10) can be obtained taking $v = J\Psi$ as test function and noting that

- $\frac{d\chi}{dt} = \dot{z}\Psi + \dot{\overline{z}}\Psi^* + \dot{\omega}\left(z\frac{d\Psi}{d\omega} + \overline{z}\frac{d\Psi^*}{d\omega}\right) + \frac{df}{dt}$,
- $(\Psi^*, J\Psi)_{L^2} = 0$,
- $\left(\frac{d\Phi_\omega}{d\omega}, J\Psi\right)_{L^2} = 0$,
- $\left(\frac{df}{dt}, J\Psi\right)_{L^2} = -\dot{\omega}\left(f, J\frac{d\Psi}{dt}\right)_{L^2}$, and
- $\dot{\omega}\left(\frac{d\Psi^*}{d\omega}, J\Psi\right)_{L^2} = -\left(\Psi^*, J\frac{d\Psi}{dt}\right)_{L^2}$.

Finally, equation (2.11) follows taking the projection onto the continuous spectrum P^c of both side of equation (2.7) and recalling that $f \in X^c$. □

2.2.1 Frozen spectral decomposition

The goal of this subsection is to get an autonomous linearized equation for the component f, as done in Section 1.6.1.
Let us fix some $T > 0$, then for any $t \in [0, T]$ one can decompose $f(t) \in X^c = X^c(t)$ as

$$f = g + h \qquad \text{with} \quad g \in X_T^d = X_T^0 \oplus X_T^1,\ h \in X_T^c,$$

where the subscript T means that the time is fixed at $t = T$.

Denote $P_T^d = P_T^0 + P_T^1$ and $\omega_T = \omega(T)$. Moreover, let us define

$$L_T = L(\omega_T),$$

then

$$Q_L(u,v) - Q_{L_T}(u,v) = \frac{\sqrt{\omega}-\sqrt{\omega_T}}{4\pi}\,\mathrm{Re}(\mathbb{T}q_u\overline{q_v}) - (\omega_T-\omega)(Ju,v)_{L^2},$$

for all u, $v \in V$, where

$$\mathbb{T} = \begin{bmatrix} 0 & -1 \\ 2\sigma+1 & 0 \end{bmatrix}.$$

Hence, observing that $P^c\Psi = 0$, the equation (2.11) for f is equivalent to

$$\left(\frac{df}{dt}, v\right)_{L^2} = Q_{L_T}(f,v) + \left((\omega-\omega_T)Jf + \dot{\omega}\frac{dP^c}{d\omega}\psi + \dot{\gamma}P^cJ\chi, v\right)_{L^2} +$$

$$+\left(8\pi\sqrt{\lambda}\left(\frac{\sqrt{\omega}-\sqrt{\omega_T}}{4\pi}\mathbb{T}q_f + P^cJN(q_\chi)\right)G_\lambda, q_vG_\lambda\right)_{L^2},$$

for all $v \in V$.
Since our dispersive estimate holds only on the continuous spectral subspace, we need to prove that it is enough to estimate the symplectic projection of $\chi(t)$ onto that subspace. This is stated in the following lemma where we denote, with a slight abuse, as $\mathcal{R}(a)$ $\mathcal{R}(a,b)$ bounded continuous real valued functions vanishing as a, $b \to 0$, and

$$\mathcal{R}_1(\omega) = \mathcal{R}(\|\omega-\omega_0\|_{C^0([0,T])}).$$

Lemma 2.2. *If $|\omega-\omega_T|$ is small enough, then the function g can be estimated in terms of h as follows:*

$$\|g\|_{L^\infty_{w^{-1}}} \le \mathcal{R}_1(\omega)|\omega-\omega_T|\|h\|_{L^\infty_{w^{-1}}}.$$

The last lemma can be proved following the proof of Lemma 3.2 in [33].
As a consequence, one can apply the operator P_T^c to both sides of the equation for f and obtain

$$\left(\frac{dh}{dt}, v\right)_{L^2} = Q_{L_T}(h,v) + \left(P_T^c\left[(\omega-\omega_T)Jf + \dot{\omega}\frac{dP^c}{d\omega}\psi + \dot{\gamma}P^cJ\chi\right], v\right)_{L^2} + \tag{2.12}$$

$$+\left(8\pi\sqrt{\lambda}P_T^c\left(\frac{\sqrt{\omega}-\sqrt{\omega_T}}{4\pi}\mathbb{T}q_f + P^cJN(q_\chi)\right)G_\lambda, q_vG_\lambda\right)_{L^2},$$

for any $v \in V$.

2.2.2 Asymptotic expansion of dynamics

In order to prove the asymptotic stability of the ground state we need to show that for large times z and h are small. For this purpose, the goal of this section is to expand the inhomogeneous terms in the modulation equations.
In what follows we denote

$$(q,p) = q_1p_1 + q_2p_2, \qquad \forall p,q \in \mathbb{C}^2.$$

With an abuse of notation in what follows we denote by $q_\omega = \begin{pmatrix} \left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{1/(2\sigma)} \\ 0 \end{pmatrix}$ the charge of the function $\begin{pmatrix} \Phi_\omega \\ 0 \end{pmatrix}$.

As a preliminary step, we expand the nonlinear part of the equation (2.7) $N(q_\chi)$ as

$$N(q_\chi) = N_2(q_\chi) + N_3(q_\chi) + N_R(q_\chi), \tag{2.13}$$

where N_2 and N_3 are the quadratic and cubic terms in q_χ respectively, while N_R is the remainder. Exploiting the Taylor expansion of the function $F(t) = t^\sigma$ around $|q_\omega|^2$, one gets

$$\mathrm{Re}(N_2(q_\chi)\overline{q_v}) = \mathrm{Re}((\sigma|q_\omega|^{2(\sigma-1)}|q_\chi|^2 q_\omega + 2\sigma|q_\omega|^{2(\sigma-1)}(q_\omega, q_\chi)q_\chi + 2(\sigma-1)\sigma|q_\omega|^{2(\sigma-2)}(q_\omega, q_\chi)^2 q_\omega)\overline{q_v}),$$

and

$$\mathrm{Re}(N_3(q_\chi)\overline{q_v}) = \mathrm{Re}((\sigma|q_\omega|^{2(\sigma-1)}|q_\chi|^2 q_\chi + 2(\sigma-1)\sigma|q_\omega|^{2(\sigma-2)}(q_\omega, q_\chi)^2 q_\chi +$$

$$+2(\sigma-1)\sigma|q_\omega|^{2(\sigma-2)}(q_\omega, q_\chi)|q_\chi|^2 q_\omega + \frac{4}{3}(\sigma-2)(\sigma-1)\sigma|q_\omega|^{2(\sigma-3)}(q_\omega, q_\chi)^3 q_\omega)\overline{q_v}),$$

for any $q_v \in \mathbb{C}$. For later convenience, let us define the following symmetric forms

$$N_2(q_1, q_2) = \sigma|q_\omega|^{2(\sigma-1)}(q_1, q_2)q_\omega + \sigma|q_\omega|^{2(\sigma-1)}[(q_\omega, q_1)q_2 + (q_\omega, q_2)q_1] +$$

$$+2(\sigma-1)\sigma|q_\omega|^{2(\sigma-2)}(q_\omega, q_1)(q_\omega, q_2)q_\omega,$$

and

$$N_3(q_1, q_2, q_3) = \frac{1}{6}\sigma|q_\omega|^{2(\sigma-1)} \sum_{i,j,k=1}^{3} (q_i, q_j)q_k + \frac{1}{3}(\sigma-1)\sigma|q_\omega|^{2(\sigma-2)} \sum_{i,j,k=1}^{3} (q_\omega, q_i)(q_\omega, q_j)q_k +$$

$$+\frac{1}{3}(\sigma-1)\sigma|q_\omega|^{2(\sigma-2)} \sum_{i,j,k=1}^{3} (q_\omega, q_i)(q_j, q_k)q_\omega + \frac{4}{3}(\sigma-2)(\sigma-1)\sigma|q_\omega|^{2(\sigma-3)}(q_\omega, q_1)(q_\omega, q_2)(q_\omega, q_3)q_\omega.$$

In order to prove the asymptotic stability result, we shall prove in Section 2.4, the following asymptotics

$$\|f(t)\|_{L^\infty_{w^{-1}}} \sim t^{-1}, \qquad z(t) \sim t^{-\frac{1}{2}}, \qquad \|\psi(t)\|_V \sim t^{-\frac{1}{2}}, \tag{2.14}$$

as $t \to +\infty$.

Remark 2.3. As in [33], the first step in proving these expected asymptotics is to separate leading terms and remainders in the right hand sides of the modulation equations (2.8) - (2.10), (2.12). Basically, in the next subsections, we will expand the expression for $\dot{\omega}$, $\dot{\gamma}$, and $\dot{z}$ up to and including the terms of order $t^{-3/2}$, and for $\dot{h}$ up to and including t^{-1}.

Remark 2.4. Note that since the nonlinearity depends only on the charges the same holds for its Taylor expansion.

Equation for ω

Substituting the expansion for the nonlinear part N given in (2.13) in equation (2.8) and considering the asymptotics (2.14) one gets

$$\dot{\omega} = \frac{1}{\Delta}\operatorname{Re}((JN_2(q_\psi) + 2JN_2(q_\psi, q_f) + JN_3(q_\psi))q_\omega) + \frac{1}{\Delta^2}\left(\psi, \frac{d\Phi_\omega}{d\omega}\right)_{L^2}\operatorname{Re}(JN_2(q_\psi)q_\omega) + \Omega_R,$$

where $\Delta = \frac{1}{2}\frac{d}{d\omega}\|\Phi_\omega\|^2_{L^2}$ and the remainder Ω_R is estimated by

$$|\Omega_R| \le \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})^2.$$

Recalling that $\psi = z\Psi + \overline{z}\Psi^*$, one can rewrite the former equation for $\dot{\omega}$ as

(2.15) $\dot{\omega} = \Omega_{20}z^2 + \Omega_{11}z\overline{z} + \Omega_{02}\overline{z}^2 + \Omega_{30}z^3 + \Omega_{21}z^2\overline{z} + \Omega_{12}z\overline{z}^2 + \Omega_{03}\overline{z}^3 + z(q_f, \Omega'_{10}) + \overline{z}(q_f, \Omega'_{01}) + \Omega_R.$

Remark 2.5. Since the second component of the vector q_ω equals 0, one has

$$\Omega_{11} = 2\frac{q_\omega}{\Delta}Re(JN_2(q_\Psi)\overline{q_{\Psi^*}}) = 0.$$

This fact will turn out to be useful in writing the canonical form of the modulation equations.

Equation for γ

As in the previous subsection the equation for $\dot{\gamma}$ (2.9) can expanded as

$$\dot{\gamma} = \frac{1}{\Delta}\operatorname{Re}((JN_2(q_\psi) + 2JN_2(q_\psi, q_f) + JN_3(q_\psi))\overline{q_{J\frac{d\Phi_\omega}{d\omega}}}) + \frac{1}{\Delta^2}\left(\psi, J\frac{d^2\Phi_\omega}{d^2\omega}\right)_{L^2}\operatorname{Re}(JN_2(q_\psi)q_\omega) + \Gamma_R,$$

where the remainder Γ_R is estimated by

$$|\Gamma_R| \le \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})^2.$$

As before, the equation for $\dot{\gamma}$ shall be written in the form

(2.16) $\dot{\gamma} = \Gamma_{20}z^2 + \Gamma_{11}z\overline{z}\Gamma_{02}\overline{z}^2 + \Gamma_{30}z^3 + \Gamma_{21}z^2\overline{z} + \Gamma_{12}z\overline{z}^2 + \Gamma_{03}\overline{z}^3 + z(q_f, \Gamma'_{10}) + \overline{z}(q_f, \Gamma'_{01}) + \Gamma_R.$

Remark 2.6. In this case Γ_{11} does not vanish as in equation (2.15).

Equation for z

Exploiting the results of the previous subsections, equation (2.10) can be expanded as

$$\begin{aligned}\dot{z} - i\xi z &= \tfrac{2}{\kappa}\operatorname{Re}(JN_2(q_\psi)\overline{q_f}) + \tfrac{1}{\kappa}\operatorname{Re}((JN_2(q_\psi) + JN_3(q_\psi))\overline{q_{J\Psi}}) + \\ &-\tfrac{1}{\Delta\kappa}\left(\tfrac{d\psi}{d\omega}, J\Psi\right)_{L^2}\operatorname{Re}(JN_2(q_\psi)q_\omega) + \tfrac{1}{\Delta\kappa}(\psi, \Psi)_{L^2}\operatorname{Re}(JN_2(q_\psi)\overline{q_{J\frac{d\Phi_\omega}{d\omega}}}) + Z_R,\end{aligned}$$

where $\kappa = -(\Psi, J\Psi)_{L^2}$ and

$$|Z_R| \le \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})^2.$$

With the same notation as before, the previous equation can be written in the form

(2.17)

$$\dot{z} = i\xi z + Z_{20}z^2 + Z_{11}z\overline{z} + Z_{02}\overline{z}^2 + Z_{30}z^3 + Z_{21}z^2\overline{z} + Z_{12}z\overline{z}^2 + Z_{03}\overline{z}^3 + z\operatorname{Re}(q_f\overline{Z'_{10}}) + \overline{z}\operatorname{Re}(q_f\overline{Z'_{01}}) + Z_R,$$

and it turns out that

(2.18)

$$\begin{aligned}Z_{11} &= \tfrac{2}{\kappa}\operatorname{Re}(JN_2(q_\Psi, q_{\Psi^*})\overline{q_\Psi}), \quad Z_{20} = \tfrac{1}{\kappa}\operatorname{Re}(JN_2(q_\Psi)\overline{q_\Psi}), \quad Z_{02} = \tfrac{1}{\kappa}\operatorname{Re}(JN_2(q_{\Psi^*})\overline{q_\Psi}),\\ Z_{21} &= \tfrac{3}{\kappa}\operatorname{Re}(JN_3(q_{\Psi^*}, q_\Psi, q_\Psi)\overline{q_\Psi}) + \tfrac{1}{\Delta\kappa}\Big[\left(\tfrac{d\Psi^*}{d\omega}, j\Psi\right)_{L^2}\operatorname{Re}(JN_2(q_\Psi)\overline{q_{J\Phi_\omega}}) + \\ &\quad -(\Psi^*, \Psi)_{L^2}\operatorname{Re}(JN_2(q_\Psi)\overline{q_{\frac{d\Phi_\omega}{d\omega}}}) - 2\|\Psi\|^2_{L^2}\operatorname{Re}(JN_2(q_{\Psi^*}, q_\Psi)\overline{q_{\frac{d\Phi_\omega}{d\omega}}})\Big],\\ Z'_{10} &= 2\frac{JN_2(q_{\Psi^*}, q_\Psi)}{\overline{\kappa}}, \quad Z'_{01} = 2\frac{JN_2(q_\Psi)}{\overline{\kappa}}.\end{aligned}$$

Equation for h

In order to expand asymptotically the equation (2.12) for h, the following remark will be useful.

Remark 2.7. For any $f \in L^2(\mathbb{R}^3)$ the following holds

$$P_T^c P^c f = P_T^c(I - P^d)f = P_T^c(P_T^c + P_T^d - P^d)f = P_T^c f + P_T^c(P_T^d - P^d)f.$$

Let us denote

$$\rho(t) = \omega(t) - \omega_T + \dot{\gamma}(t),$$

then equation (2.12) can be rewritten as

$$\left(\frac{dh}{dt}, v\right)_{L^2} = Q_{L_T}(h, v) + (\rho P_T^c J h, v)_{L^2} + (8\pi P_T^c J N_2(q_\psi) G_\lambda, q_v G_\lambda)_{L^2} +$$

$$+ \left(P_T^c \left[\dot{\omega}\frac{dP^c}{d\omega}\psi + \dot{\gamma} P^c J \psi + \rho J g + \dot{\gamma}(P_T^d - P^d) J f\right], v\right)_{L^2} +$$

$$+ \left(8\pi\sqrt{\lambda} P_T^c \left(\frac{\sqrt{\omega} - \sqrt{\omega_T}}{4\pi}\mathbb{T} q_f + P^c J N(q_\chi) - J N_2(q_\psi)\right) G_\lambda, q_v G_\lambda\right)_{L^2},$$

for any $v \in V$.
Denote

$$H_R' = P_T^c \left[\dot{\omega}\frac{dP^c}{d\omega}\psi + \dot{\gamma} P^c J \psi + \rho J g + \dot{\gamma}(P_T^d - P^d) J f\right],$$

and

$$H_R'' = 8\pi\sqrt{\lambda} P_T^c \left(\frac{\sqrt{\omega} - \sqrt{\omega_T}}{4\pi}\mathbb{T} q_f + P^c J N(q_\chi) - J N_2(q_\psi)\right) G_\lambda.$$

The next lemma will justify what follows.

Lemma 2.8. *There exists a constant $C > 0$ such that for each $h \in X_T^c$ holds*

$$\left\|[P_T^c J - i(\Pi_T^+ - \Pi_T^-)]h\right\|_{L^1_w} \le C\|h\|_{L^\infty_{w^{-1}}}.$$

The proof is in Appendix 2.6.2 for any $t > 0$. Finally, let us define

$$L_M(t) = L_T + i\rho(t)(\Pi_T^+ - \Pi_T^-), \tag{2.19}$$

then the previous equation becomes

$$\left(\frac{dh}{dt}, v\right)_{L^2} = Q_{L_M}(h, v) + (8\pi P_T^c J N_2(q_\psi) G_\lambda, q_v G_\lambda)_{L^2} + (\widetilde{H}_R, v)_{L^2} + (H_R'', q_v G_\lambda)_{L^2}, \tag{2.20}$$

for any $v \in V$, where we have denoted

$$\widetilde{H}_R = H_R' + \rho[P_T^c J - i(\Pi_T^+ - \Pi_T^-)]h.$$

Finally, let us expand the second summand in the right hand side of (2.20), getting

$$\left(\frac{dh}{dt}, v\right)_{L^2} = Q_{L_M}(h, v) + (z^2 H_{20} + z\overline{z} H_{11} + \overline{z}^2 H_{02})\overline{q_v} + (\widetilde{H}_R, v)_{L^2} + (H_R'', q_v G_\lambda)_{L^2},$$

for any $v \in V$, where

$$\begin{aligned} H_{20} &= (8\pi\sqrt{\lambda}P_T^c JN_2(q_\Psi)G_\lambda, G_\lambda)_{L^2}, \\ H_{11} &= 2(8\pi\sqrt{\lambda}P_T^c JN_2(q_\Psi, q_{\Psi^*})G_\lambda, G_\lambda)_{L^2}, \\ H_{02} &= (8\pi\sqrt{\lambda}P_T^c JN_2(q_{\Psi^*})G_\lambda, G_\lambda)_{L^2}. \end{aligned}$$

Thanks to the estimates done for the other equations and Lemma 2.8, one can estimates the remainders in the following way:

$$\|H_R'\|_{L^1_w} \leq C\left(|z|(|\dot\omega| + |\dot\gamma|) + \mathcal{R}_1(\omega)(|\omega-\omega_T| + |\dot\gamma|\|f\|_{L^\infty_{w^{-1}}})\right) \leq$$

$$\leq \mathcal{R}_1(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})\left(|z|^3 + |z|\|f\|_{L^\infty_{w^{-1}}} + \|f\|^2_{L^\infty_{w^{-1}}} + |\omega-\omega_T|\|f\|_{L^\infty_{w^{-1}}}\right),$$

hence

$$\|\widetilde{\mathcal{H}}_R\|_{L^1_w} \leq \mathcal{R}_1(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})\left(|z|^3 + |z|\|f\|_{L^\infty_{w^{-1}}} + \|f\|^2_{L^\infty_{w^{-1}}} + |\omega-\omega_T|\|f\|_{L^\infty_{w^{-1}}}\right), \tag{2.21}$$

and

$$\|H_R''\|_{L^1_w} \leq \mathcal{R}_1(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})\left(|z|^3 + |z|\|f\|_{L^\infty_{w^{-1}}} + \|f\|^2_{L^\infty_{w^{-1}}} + |\omega-\omega_T|(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})\right). \tag{2.22}$$

Remark 2.9. In the same way one could directly expand the equation for the function f getting

$$\left(\frac{df}{dt}, v\right)_{L^2} = Q_L(f,v) + (z^2F_{20} + z\overline{z}F_{11} + \overline{z}^2F_{02})\overline{q_v} + (\widetilde{F}_R, v)_{L^2} + (F_R'', q_vG_\lambda)_{L^2}, \tag{2.23}$$

for any $v \in V$, where

$$\begin{aligned} F_{20} &= (8\pi\sqrt{\lambda}JN_2(q_\Psi)G_\lambda, G_\lambda)_{L^2}, \\ F_{11} &= 2(8\pi\sqrt{\lambda}JN_2(q_\Psi, q_{\Psi^*})G_\lambda, G_\lambda)_{L^2}, \\ F_{02} &= (8\pi\sqrt{\lambda}JN_2(q_{\Psi^*})G_\lambda, G_\lambda)_{L^2}. \end{aligned}$$

and

$$\widetilde{F}_R = \dot\omega\frac{dP^c}{d\omega}\psi + \dot\gamma P^cJ\psi + \dot\gamma(P_T^d - P^d)Jf,$$

$$F_R'' = 8\pi\sqrt{\lambda}\left(\frac{\sqrt{\omega}-\sqrt{\omega_T}}{4\pi}\mathbb{T}q_f + P^cJN(q_\chi) - JN_2(q_\psi)\right)G_\lambda.$$

Furthermore, the L^1_w norms of the remainders $\widetilde{F}_R$ and F_R'' can be estimated by the corresponding norms of the remainders $\widetilde{H}_R$ and H_R''.

2.3 Canonical form of the equations

In this section we would like to use the technique of normal coordinates in order to transform the modulation equations for ω, γ, z, and h to a simpler canonical form. We will also try to keep the estimates of the remainders as much close as possible to the original ones.

2.3.1 Canonical form of the equation for h

Our goal is to exploit a change of variable in such a way that the function h is mapped in a new function decaying in time at least as $t^{-3/2}$. For this purpose one could expand h as

$$h = h_1 + k + k_1, \tag{2.24}$$

where

$$k = a_{20}z^2 + a_{11}z\overline{z} + a_{02}\overline{z}^2,$$

with some coefficients $a_{ij} = a_{ij}(x, \omega)$ such that $a_{ij} = \overline{a_{ji}}$, and

$$k_1 = -\exp\left(\int_0^t L_M(s)ds\right) k(0).$$

Note that $h_1(0) = h(0)$, since $k_1(0) = -k(0)$.

Proposition 2.10. *There exist $a_{ij} \in L^\infty_{w^{-1}}(\mathbb{R}^3)$, for i, $j = 0, 1, 2$, such that the equation for h_1 has the form*

$$\left(\frac{dh_1}{dt}, v\right)_{L^2} = Q_{L_M}(h_1, v) + (\widehat{H}_R, v)_{L^2} + (H''_R, q_v G_\lambda)_{L^2}, \tag{2.25}$$

for all $v \in V$, where $\widehat{H}_R = \widetilde{H}_R + \overline{H}_R$ with

$$\overline{H}_R = -\left[\dot{\omega}\left(\frac{da_{20}}{d\omega}z^2 + \frac{da_{11}}{d\omega}z\overline{z} + \frac{da_{02}}{d\omega}\overline{z}^2\right) + (2a_{20}z + a_{11}\overline{z})(\dot{z} - i\xi_T z) + \right. \tag{2.26}$$
$$\left. + (a_{11}z + 2a_{20}\overline{z})(\dot{\overline{z}} + i\xi_T\overline{z}) - \rho(\Pi^+_T - \Pi^-_T)k\right].$$

Proof. The thesis is proved substituting (2.24) into (2.20) and equating the coefficients of the quadratic powers of z which leads to the system

$$\begin{cases} Q_{L_T}(a_{20}, v) + \mathrm{Re}(H_{20}\overline{q_v}) - (2i\xi_T a_{20}, v)_{L^2} = 0 \\ Q_{L_T}(a_{11}, v) + \mathrm{Re}(H_{11}\overline{q_v}) = 0 \\ Q_{L_T}(a_{02}, v) + \mathrm{Re}(H_{02}\overline{q_v}) + (2i\xi_T a_{02}, v)_{L^2} = 0 \end{cases}, \tag{2.27}$$

for all $v \in V$. The former system admits the solution

$$\begin{aligned} a_{11} &= -L_T^{-1}H_{11} \\ a_{20} &= -(L_T - 2i\xi_T - 0)^{-1}H_{20} \\ a_{02} &= \overline{a_{02}} = -(L_T + 2i\xi_T - 0)^{-1}H_{02} \end{aligned}$$

□

Remark 2.11. From the explicit structure of the remainder $\widehat{H}_R$ it follows that it still satisfies estimate (2.21).

We will need to apply the next lemma which can be proved as Proposition 2.3 in [33].

Lemma 2.12. *If $\sigma \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right)$ and $f \in V \cap L^1_w$, then there exists some constant $C > 0$ such that for any $t \geq 0$*

$$\|e^{-L_T t}(L_T + 2i\xi_T - 0)^{-1}P^c_T f\|_{L^\infty_{w^{-1}}} \leq C(1+t)^{-3/2}\|f\|_{L^1_w}.$$

Remark 2.13. Let us note that

$$h = P^c_T h = P^c_T h_1 + P^c_T k + P^c_T k_1,$$

hence, in order to estimate the decay of $\|h\|_{L^\infty_{w^{-1}}}$, it suffices to estimate the decay of

$$\|P^c_T h_1\|_{L^\infty_{w^{-1}}}, \quad \|P^c_T k\|_{L^\infty_{w^{-1}}}, \quad \text{and} \quad \|P^c_T k_1\|_{L^\infty_{w^{-1}}}.$$

2.3.2 Canonical form of the equation for ω

Since $\Omega_{11}=0$, we can exploit the method by Buslaev and Sulem in [11], Proposition 4.1 and get the following proposition.

Proposition 2.14. *There exist coefficients $b_{ij}=b_{ij}(\omega)$, with $i,\ j=0,\ 1,\ 2,\ 3$, and vector functions $b'_{ij}=b'_{ij}(x,\omega)$, with $i,\ j=0,\ 1$, such that function*

$$\omega_1=\omega+b_{20}z^2+b_{11}z\overline{z}+b_{02}\overline{z}^2+b_{30}z^3+b_{21}z^2\overline{z}+b_{12}z\overline{z}^2+b_{03}\overline{z}^3+$$

$$+z(f,b'_{10})_{L^2}+\overline{z}(f,b'_{01})_{L^2},$$

solves a differential equation of the form

$$\dot{\omega}_1=\widehat{\Omega}_R,$$

for some remainder $\widehat{\Omega}_R$.

Proof. Substituting the equations (2.15), (2.17), and (2.23) into the derivative with respect to time of the expression for ω_1 and equating the coefficients of z^2, $z\overline{z}$, $\overline{z}^2$, z, and $\overline{z}$ one gets the following system

$$\begin{cases}\Omega_{20}+2i\xi b_{20}=0\\ \Omega_{02}-2i\xi b_{02}=0\\ \Omega_{30}+3i\xi b_{30}+2Z_{20}b_{20}+\mathrm{Re}(F_{20}\overline{q_{b'_{10}}})=0\\ \Omega_{03}-3i\xi b_{03}+2Z_{02}b_{02}+\mathrm{Re}(F_{02}\overline{q_{b'_{01}}})=0\\ \Omega_{21}+i\xi b_{21}+2Z_{11}b_{20}+2Z_{20}b_{02}+\mathrm{Re}(F_{11}\overline{q_{b'_{10}}}+F_{20}\overline{q_{b'_{01}}})=0\\ \Omega_{12}-i\xi b_{12}+2Z_{11}b_{02}+2Z_{20}b_{20}+\mathrm{Re}(F_{11}\overline{q_{b'_{01}}}+F_{20}\overline{q_{b'_{10}}})=0\\ (q_f,\Omega'_{10})+i\xi(f,b'_{10})_{L^2}+Q_L(f,b'_{10})=0\\ (q_f,\Omega'_{01})+i\xi(f,b'_{01})_{L^2}+Q_L(f,b'_{01})=0\end{cases}.$$

The last two equations of this system can be solved in a way similar to the ones system (2.27), and the proof follows. ☐

Remark 2.15. From the proof of the previous proposition it also follows that the remainder $\widehat{\Omega}_R$ can be estimated as Ω_R, namely

$$|\widehat{\Omega}_R|\le\mathcal{R}(\omega,|z|+\|f\|_{L^\infty_{w^{-1}}})(|z|^2+\|f\|_{L^\infty_{w^{-1}}})^2.$$

In the next lemma we prove a uniform bound for $|\omega_T-\omega|$ on the interval $[0,T]$. For later convenience let us denote

$$\mathcal{R}_2(\omega,|z|+\|f\|_{L^\infty_{w^{-1}}})=\mathcal{R}\left(\max_{0\le t\le T}|\omega_T-\omega|,\max_{0\le t\le T}(|z|+\|f\|_{L^\infty_{w^{-1}}})\right).$$

Remark 2.16. Let us note that $|\omega|\le|\omega_0|+|\omega_0-\omega_T|+|\omega-\omega_T|$, then

$$\max_{0\le t\le T}\mathcal{R}(\omega,|z|+\|f\|_{L^\infty_{w^{-1}}})=\mathcal{R}\left(\max_{0\le t\le T}|\omega_T-\omega|,\max_{0\le t\le T}(|z|+\|f\|_{L^\infty_{w^{-1}}})\right).$$

The next lemma can be proved as in Section 3.5 of [33].

Lemma 2.17. *For any $t\in[0,T]$ we have*

$$|\omega_T-\omega|\le\mathcal{R}_2(\omega,|z|+\|f\|_{L^\infty_{w^{-1}}})\left[\int_t^T(|z(\tau)|+\|f(\tau)\|_{L^\infty_{w^{-1}}})^2d\tau+\right.$$

$$\left.+(|z_T|+\|f_T\|_{L^\infty_{w^{-1}}})^2+(|z|+\|f\|_{L^\infty_{w^{-1}}})^2\right].$$

2.3.3 Canonical form of the equation for γ

Equations (2.16) for γ and (2.15) for ω differ just because in general $\Gamma_{11} \neq 0$. But we can perform the same change of variable in the previous subsection, namely

$$\gamma_1 = \gamma + d_{20}z^2 + d_{02}\overline{z}^2 + d_{30}z^3 + d_{21}z^2\overline{z} + d_{12}z\overline{z}^2 + d_{03}\overline{z}^3 + z(f, d'_{10})_{L^2} + \overline{z}(f, d'_{01})_{L^2},$$

for some suitable coefficients $d_{ij} = d_{ij}(\omega)$, with i, $j = 0$, 1, 2, 3, and vector functions $d'_{ij} = d'_{ij}(x, \omega)$, with i, $j = 0$, 1. Then the function γ_1 solves the differential equation

$$\dot{\gamma}_1 = \Gamma_{11}(\omega)z\overline{z} + \widehat{\Gamma}_R,$$

for some remainder $\widehat{\Gamma}_R$, which can be estimated as Γ_R, i.e.

$$|\widehat{\Gamma}_R| \leq \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})^2.$$

2.3.4 Canonical form of the equation for z

Exploiting the change of variable (2.24) used to obtain the canonical form of equation (2.20) for h, one can prove the following proposition.

Proposition 2.18. *There exist coefficients $c_{ij} = c_{ij}(\omega)$, with i, $j = 0$, 1, 2, 3, such that function*

$$z_1 = z + c_{20}z^2 + c_{11}z\overline{z} + c_{02}\overline{z}^2 + c_{30}z^3 + c_{21}z^2\overline{z} + c_{03}\overline{z}^3,$$

solves a differential equation of the form

$$\dot{z}_1 = i\xi z_1 + iK|z_1|^2 z_1 + \widehat{Z}_R, \tag{2.28}$$

where

$$iK = Z_{21} + Z'_{21} + \frac{i}{\xi}Z_{20}Z_{11} - \frac{i}{\xi}Z_{11}^2 - \frac{2i}{3\xi}Z_{02}^2,$$

with the coefficient Z_{ij}, i, $j = 0, 1, 3$, defined in (2.18)*, and*

$$\widetilde{Z}_R = (g + P^c_T h_1 + P^c_T k_1, Z'_{10})z + (g + P^c_T h_1 + P^c_T k_1, Z'_{01})\overline{z} + Z_R.$$

The proof is a matter of calculation, but we give it explicitly to stress the role of the functions a_{ij}, i, $j = 0$, 1, 2.

Proof. Substituting (2.24) in the equation (2.17) the differential equation for z becomes

$$\dot{z} = i\xi z + Z_{20}z^2 + Z_{11}z\overline{z} + Z_{02}\overline{z}^2 + Z_{30}z^3 + Z_{21}z^2\overline{z} + Z_{12}z\overline{z}^2 + Z_{03}\overline{z}^3 + \tag{2.29}$$

$$+ Z'_{30}z^3 + Z'_{21}z^2\overline{z} + Z'_{12}z\overline{z}^2 + Z'_{03}\overline{z}^3 + \widetilde{Z}_R,$$

where

$$\begin{aligned}
Z'_{30} &= \mathrm{Re}(q_{a_{20}}\overline{Z'_{10}}),\\
Z'_{03} &= \mathrm{Re}(q_{a_{02}}\overline{Z'_{01}}),\\
Z'_{21} &= \mathrm{Re}(q_{a_{11}}\overline{Z'_{10}}) + \mathrm{Re}(q_{a_{20}}\overline{Z'_{01}}),\\
Z'_{12} &= \mathrm{Re}(q_{a_{11}}\overline{Z'_{01}}) + \mathrm{Re}(q_{a_{02}}\overline{Z'_{10}}),
\end{aligned}$$

and the remainder $\widetilde{Z}_R$ is as in the statement of the proposition.

Inserting equation (2.29) into the time derivative of the expression for z_1 and equating the coefficients of z^2, $z\overline{z}$, $\overline{z}^2$, z^3, $z\overline{z}^2$, and $\overline{z}^3$ one obtains the system

$$\begin{cases} i\xi c_{20} + Z_{20} = 0 \\ -i\xi c_{11} + Z_{11} = 0 \\ -3i\xi c_{02} + Z_{02} = 0 \\ 2i\xi c_{30} + Z_{30} + Z'_{30} + 2c_{20}Z_{20} + c_{11}Z_{20} = 0 \\ Z_{12} + Z'_{12} + 2c_{20}Z_{20} + c_{11}(Z_{11} + Z_{02}) + 2c_{02}Z_{11} - 2i\xi c_{12} = 0 \\ -4i\xi c_{03} + Z_{03} + Z'_{03} + c_{11}Z_{02} = 0 \end{cases}$$

The theorem follows from the fact the the above system is solvable and in particular

$$c_{20} = \frac{i}{\xi}Z_{20}, \quad c_{11} = -\frac{i}{\xi}Z_{11}, \quad \text{and} \quad c_{02} = \frac{i}{3\xi}Z_{02}.$$

□

Remark 2.19. For later convenience let us note that, since Z_{21}, Z_{20}, Z_{11}, and Z_{02} are purely imaginary, one has

$$\mathrm{Re}(iK) = \mathrm{Re}(Z'_{21}).$$

Moreover, we need the following lemma.

Lemma 2.20. *There exists* $\sigma^* \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right]$ *such that if* $\sigma \in \left(\frac{1}{\sqrt{2}}, \sigma^*\right)$, *then*

$$\mathrm{Re}(Z'_{21}) < 0,$$

$\forall\omega$ *belonging to an open neighbourhood of* ω_0.

Proof. First of all recall that $\xi_T = 2\sigma\sqrt{1-\sigma^2}\omega_T$, then one can compute

$$\kappa = -(\Psi, J\Psi)_{L^2} = \frac{i}{4\pi\sqrt{\omega_T}}\left(\frac{1}{\sqrt{1-2\sigma\sqrt{1-\sigma^2}}} - \frac{(\sqrt{1-\sigma^2}-1)^2}{\sigma^2}\frac{1}{\sqrt{1+2\sigma\sqrt{1-\sigma^2}}}\right) = \tag{2.30}$$

$$= \frac{i}{4\pi\sqrt{\omega_T}}\frac{\sigma^2\sqrt{1+2\sigma\sqrt{1-\sigma^2}} - (\sqrt{1-\sigma^2}-1)^2\sqrt{1-2\sigma\sqrt{1-\sigma^2}}}{\sigma^2(2\sigma^2-1)}.$$

Since κ is purely imaginary with positive imaginary part and $L_T^{-1}2P_T^cJ$ is self-adjoint, for the first summand in the expression for $\mathrm{Re}(Z'_{21})$ one gets

$$\mathrm{Re}(q_{a_{11}}\overline{Z'_{10}}) = -2\,\mathrm{Re}\left(\frac{q_{L_T^{-1}2P_T^cJN_2(q_\Psi,q_{\Psi^*})}\overline{JN_2(q_\Psi,q_{\Psi^*})}}{\kappa}\right) = 0.$$

Hence,

$$\mathrm{Re}(Z'_{21}) = -2\,\mathrm{Re}\left(\frac{q_{a_{20}}\overline{JN_2(q_\Psi)}}{\kappa}\right).$$

By direct computations one has

$$a_{20}(x) = (L_T - (2i\xi_T + 0))^{-1}H_{20} = A\frac{e^{-\sqrt{\omega_T+2\xi_T}|x|}}{4\pi|x|}\begin{pmatrix}1\\-i\end{pmatrix} + C\frac{e^{-i\sqrt{-\omega_T+2\xi_T}|x|}}{4\pi|x|}\begin{pmatrix}1\\i\end{pmatrix},$$

with

$$\begin{array}{l} A=-\frac{4\pi}{d}[((2\sigma+1)\sqrt{\omega_T}-i\sqrt{-\omega_T+2\xi_T})(H_{20})_1+(i\sqrt{\omega_T}+\sqrt{-\omega_T+2\xi_T})(H_{20})_2] \\ C=\frac{4\pi}{d}[((2\sigma+1)\sqrt{\omega_T}-\sqrt{\omega_T+2\xi_T})(H_{20})_1-(i\sqrt{\omega_T}-i\sqrt{\omega_T+2\xi_T})(H_{20})_2] \end{array},$$

where $d=2i(2\sigma+1)\omega_T+2(\sigma+1)\sqrt{\omega_T}\sqrt{-\omega_T+2\xi_T}-2i(\sigma+1)\sqrt{\omega_T}\sqrt{\omega_T+2\xi_T}-2\sqrt{\omega_T+2\xi_T}\sqrt{-\omega_T+2\xi_T}$. From which follows

$$q_{a_{20}}=\frac{4\pi}{d}\left[\begin{pmatrix} (i\sqrt{-\omega_T+2\xi_T}-\sqrt{\omega_T+2\xi_T})(H_{20})_1 \\ ((2\sigma+1)\sqrt{\omega_T}-\sqrt{\omega_T+2\xi_T}-\sqrt{-\omega_T+2\xi_T})(H_{20})_1 \end{pmatrix}+\right.$$

$$\left.+\begin{pmatrix} -i(2\sqrt{\omega_T}+\sqrt{\omega_T+2\xi_T}+i\sqrt{-\omega_T+2\xi_T})(H_{20})_2 \\ (-\sqrt{\omega_T+2\xi_T}+i\sqrt{-\omega_T+2\xi_T})(H_{20})_2 \end{pmatrix}\right].$$

Hence

(2.31)

$$\begin{array}{rl} \mathrm{Re}((q_{a_{20}})_1)= & \frac{16\pi}{|d|^2}[i(H_{20})_1(-(\sigma+1)\sqrt{\omega_T}\sqrt{\omega_T+2\xi_T}\sqrt{-\omega_T+2\xi_T}+((\sigma+1)\omega_T+\xi_T)\sqrt{-\omega_T+2\xi_T})+ \\ & (H_{20})_2(-(2(\sigma+1)\xi_T+(2\sigma+1)\omega_T)\sqrt{\omega_T}+(\xi_T+(2\sigma+1)\omega_T)\sqrt{\omega_T+2\xi_T})] \\ \mathrm{Im}((q_{a_{20}})_2)= & \frac{16\pi}{|d|^2}[i(H_{20})_1((2(\sigma+1)^2\omega_T+\xi_T)\sqrt{-\omega_T+2\xi_T}-(3\sigma+2)\sqrt{\omega_T}\sqrt{\omega_T+2\xi_T}\sqrt{-\omega_T+2\xi_T})+ \\ & +(H_{20})_2((\sigma+1)\omega_T^{3/2}+((\sigma+1)\omega_T-\xi_T)\sqrt{\omega_T+2\xi_T})]. \end{array}$$

Moreover, by (2.13) one gets

(2.32)
$$JN_2(q_\Psi)=\begin{pmatrix} -2\sigma|q_{\omega_T}|^{2\sigma-1}(q_\Psi)_1(q_\Psi)_2 \\ \sigma|q_{\omega_T}|^{2\sigma-1}(3(q_\Psi)_1^2+(q_\Psi)_2^2)+2\sigma(\sigma-1)|q_{\omega_T}|^{2\sigma-1}(q_\Psi)_1^2 \end{pmatrix}=$$

$$=\begin{pmatrix} -2i\sigma|q_{\omega_T}|^{2\sigma-1}\left(1-\frac{\sqrt{1-\sigma^2}-1}{\sigma}\right)\left(1+\frac{\sqrt{1-\sigma^2}-1}{\sigma}\right) \\ 2\sigma|q_{\omega_T}|^{2\sigma-1}\left(1-\frac{\sqrt{1-\sigma^2}-1}{\sigma}\right) \end{pmatrix},$$

which implies

(2.33)
$$H_{20}=(8\pi\sqrt{\omega_T}P_T^cJN_2(q_\Psi)G_{\omega_T},G_{\omega_T})_{L^2}=$$

$$=JN_2(q_\Psi)-\frac{(JN_2(q_\Psi))_1|q_{\omega_T}|}{16\pi\Delta\omega^{3/2}}\left(\frac{1}{\sigma}-1\right)\begin{pmatrix}1\\0\end{pmatrix}+$$

$$+\frac{\sqrt{\omega_T}}{\kappa}\begin{pmatrix} -(JN_2(q_\Psi))_2\left(\frac{1}{\sqrt{\omega_T-\xi_T}+\sqrt{\omega_T}}-\frac{\sqrt{1-\sigma^2}-1}{\sigma}\frac{1}{\sqrt{\omega_T+\xi_T}+\sqrt{\omega_T}}\right)^2 \\ (JN_2(q_\Psi))_1\left(\frac{1}{\sqrt{\omega_T-\xi_T}+\sqrt{\omega_T}}+\frac{\sqrt{1-\sigma^2}-1}{\sigma}\frac{1}{\sqrt{\omega_T+\xi_T}+\sqrt{\omega_T}}\right)^2 \end{pmatrix}.$$

Let us notice that (2.32) and (2.33) imply

$$\begin{array}{rl} i(H_{20})_1 i(JN_2(q_\Psi))_1= & -\frac{1}{2\sigma-1}(JN_2(q_\Psi))_1^2+ \\ & +\frac{\sqrt{\omega_T}}{4\pi i\kappa}\left(\frac{1}{\sqrt{\omega_T-\xi_T}+\sqrt{\omega_T}}-\frac{\sqrt{1-\sigma^2}-1}{\sigma}\frac{1}{\sqrt{\omega_T+\xi_T}+\sqrt{\omega_T}}\right)^2 i(JN_2(q_\Psi))_1(JN_2(q_\Psi))_2 \\ (H_{20})_2 i(JN_2(q_\Psi))_1= & (JN_2(q_\Psi))_2 i(JN_2(q_\Psi))_1+ \\ & -\frac{\sqrt{\omega_T}}{4\pi i\kappa}\left(\frac{1}{\sqrt{\omega_T-\xi_T}+\sqrt{\omega_T}}+\frac{\sqrt{1-\sigma^2}-1}{\sigma}\frac{1}{\sqrt{\omega_T+\xi_T}+\sqrt{\omega_T}}\right)^2(JN_2(q_\Psi))_1^2 \\ i(H_{20})_1(JN_2(q_\Psi))_2= & \frac{1}{2\sigma-1}i(JN_2(q_\Psi))_1(JN_2(q_\Psi))_2+ \\ & +\frac{\sqrt{\omega_T}}{4\pi i\kappa}\left(\frac{1}{\sqrt{\omega_T-\xi_T}+\sqrt{\omega_T}}-\frac{\sqrt{1-\sigma^2}-1}{\sigma}\frac{1}{\sqrt{\omega_T+\xi_T}+\sqrt{\omega_T}}\right)^2(JN_2(q_\Psi))_2^2 \\ (H_{20})_2(JN_2(q_\Psi))_2= & (JN_2(q_\Psi))_2^2+ \\ & +\frac{\sqrt{\omega_T}}{4\pi i\kappa}\left(\frac{1}{\sqrt{\omega_T-\xi_T}+\sqrt{\omega_T}}+\frac{\sqrt{1-\sigma^2}-1}{\sigma}\frac{1}{\sqrt{\omega_T+\xi_T}+\sqrt{\omega_T}}\right)^2 i(JN_2(q_\Psi))_1(JN_2(q_\Psi))_2, \end{array}$$

then by (2.30) and (2.31) it follows

$$\mathrm{Re}(Z'_{21}) = -2\,\mathrm{Re}\left(\frac{q_{a_{20}}\overline{JN_2(q_\Psi)}}{\kappa}\right) = \frac{2}{i\kappa}(\mathrm{Re}((q_{a_{20}})_1)i(JN_2(q_\Psi))_1 + \mathrm{Im}((q_{a_{20}})_2)(JN_2(q_\Psi))_2) =$$

$$= \frac{128\pi\omega_T^{3/2}|q_{\omega_T}|^{4\sigma-2}}{i\kappa|d|^2}\sigma^2\left(1-\frac{\sqrt{1-\sigma^2}-1}{\sigma}\right)^2 f(\sigma),$$

with

$$f(\sigma) = \Bigg(\left(2(1+\sigma)^2 + 2\sigma\sqrt{1-\sigma^2}\right)\sqrt{-1+4\sigma\sqrt{1-\sigma^2}} - (2+3\sigma)\sqrt{-1+4\sigma\sqrt{1-\sigma^2}}\sqrt{1+4\sigma\sqrt{1-\sigma^2}} +$$

$$+\left(1+\frac{-1+\sqrt{1-\sigma^2}}{\sigma}\right)\left((-1-\sigma)\sqrt{-1+16\sigma^2-16\sigma^4} + \left(1+\sigma+2\sigma\sqrt{1-\sigma^2}\right)\sqrt{-1+4\sigma\sqrt{1-\sigma^2}}\right)\Bigg)\cdot$$

$$\cdot\left(\frac{1}{-1+2\sigma} - \frac{\left(\frac{1}{1+\sqrt{1-2\sigma\sqrt{1-\sigma^2}}} - \frac{-1+\sqrt{1-\sigma^2}}{\sigma+\sigma\sqrt{1+2\sigma\sqrt{1-\sigma^2}}}\right)^2}{\frac{1}{\sqrt{1-2\sigma\sqrt{1-\sigma^2}}} - \frac{\left(-1+\sqrt{1-\sigma^2}\right)^2}{\sigma^2\sqrt{1+2\sigma\sqrt{1-\sigma^2}}}}\right) +$$

$$+\Bigg(1+\sigma+\left(1+\sigma-2\sigma\sqrt{1-\sigma^2}\right)\sqrt{1+4\sigma\sqrt{1-\sigma^2}} +$$

$$+\left(1+\frac{-1+\sqrt{1-\sigma^2}}{\sigma}\right)\left(-1-2\sigma+\left(-4\sigma-4\sigma^2\right)\sqrt{1-\sigma^2} + \left(1+2\sigma+2\sigma\sqrt{1-\sigma^2}\right)\sqrt{1+4\sigma\sqrt{1-\sigma^2}}\right)\Bigg)\cdot$$

$$\cdot\left(1 - \frac{\left(\frac{1}{1+\sqrt{1-2\sigma\sqrt{1-\sigma^2}}} + \frac{-1+\sqrt{1-\sigma^2}}{\sigma+\sigma\sqrt{1+2\sigma\sqrt{1-\sigma^2}}}\right)^2}{\frac{1}{\sqrt{1-2\sigma\sqrt{1-\sigma^2}}} - \frac{\left(-1+\sqrt{1-\sigma^2}\right)^2}{\sigma^2\sqrt{1+2\sigma\sqrt{1-\sigma^2}}}}\right).$$

Notice that one has $f(\sigma) \to \widetilde{f} > 0$, $d \to \widetilde{d} \neq 0$, and $i\kappa \to -\infty$ as $\sigma \to 1/\sqrt{2}$; this implies

$$\lim_{\sigma\to 1/\sqrt{2}} \mathrm{Re}(Z'_{21}) = \frac{128\sqrt{2}\omega_T^{3/2}|q_{\omega_T}|^{2\sqrt{2}-2}}{\pi|\widetilde{d}|^2}\widetilde{f}\lim_{\sigma\to 1/\sqrt{2}}\frac{1}{i\kappa} = 0^-.$$

Hence there is a neighborhood of $\frac{1}{\sqrt{2}}$ where $\mathrm{Re}(Z'_{21})$ is strictly negative. A *Mathematica* plot of the function $f(\sigma)$ in the range $\left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right)$ is given in figure 2.19.
Summing up, one can conclude that there exists $\sigma^* \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right]$ such that $\mathrm{Re}(Z'_{21}) < 0$ for $\sigma \in \left(\frac{1}{\sqrt{2}}, \sigma^*\right)$. □

Remark 2.21. The following reformulation on the equation for z_1 will turn out to be useful. First of all, if we denote $K_T = K(\omega_T)$, then the ordinary differential equation for z_1 becomes

$$\dot{z_1} = i\xi z_1 + iK_T|z_1|^2 z_1 + \widehat{\widehat{Z}}_R,$$

for some remainder $\widehat{\widehat{Z}}_R$.

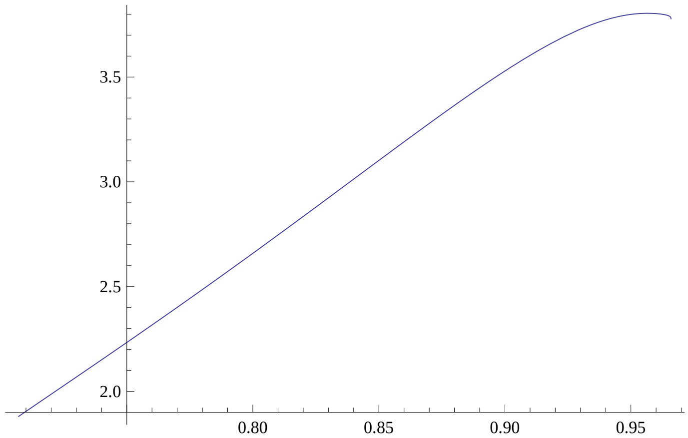

Figure 2.1: $f(\sigma)$

Secondly, let us notice that z_1 is oscillating while $y = |z_1|^2$ decreases at infinity. Hence, it is easier to deal with the variable y, which satisfies the equation

$$\dot{y} = 2\,\mathrm{Re}(iK_T)y^2 + Y_R, \tag{2.34}$$

where Y_R is some suitable remainder.

Remark 2.22. From Lemma 2.2 we have

$$|(g + P_T^c h_1 + P_T^c k_1, Z'_{10})| \le \mathcal{R}(\omega)(\|g\|_{L^\infty_{w^{-1}}} + \|P_T^c h_1\|_{L^\infty_{w^{-1}}} + \|P_T^c k_1\|_{L^\infty_{w^{-1}}}) \le$$
$$\le \mathcal{R}_1(\omega)(|\omega_T - \omega|\|h\|_{L^\infty_{w^{-1}}} + \|P_T^c h_1\|_{L^\infty_{w^{-1}}} + \|P_T^c k_1\|_{L^\infty_{w^{-1}}}),$$

hence

$$|Y_R| = |\widehat{\widehat{Z}}_R||z| = |\widetilde{Z}_R + i(K - K_T)|z_1|^2 z_1||z| \le$$
$$\le \mathcal{R}_1(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})|z|[(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})^2 + |z||\omega_T - \omega|(|z|^2 + \|h\|_{L^\infty_{w^{-1}}}) +$$
$$+ |z|\|P_T^c k_1\|_{L^\infty_{w^{-1}}} + |z|\|P_T^c h_1\|_{L^\infty_{w^{-1}}}].$$

2.4 Majorants

In this section we exploit the so-called majorant method to prove large time asymptotic for the solutions of the modulation equations. Preliminary, we need some assumptions on the initial conditions.

2.4.1 Initial conditions

Let us fix some $\epsilon > 0$ to be chosen subsequently to control uniformly estimates. Then we assume that

$$\begin{aligned} &|z(0)| \le \epsilon^{1/2} \\ &\|f(0)\|_{L^1_w} \le c\epsilon^{1/2}, \end{aligned} \tag{2.35}$$

where $c > 0$ is some positive constant.

From the definition of z_1 one has

$$z_1 - z = \mathcal{R}(\omega)|z|^2.$$

Then the following estimate holds

$$y(0) = |z_1(0)|^2 \leq |z(0)|^2 + \mathcal{R}(\omega, |z(0)|)|z(0)|^3 \leq \epsilon + \mathcal{R}(\omega, |z(0)|)\epsilon^{3/2}.$$

We also want an estimate for the initial datum of the function $h(t)$, for this purpose recall that $h = f + (P^d - P^d_T)f$. Hence,

$$\|h(0)\|_{L^1_w} \leq \|f(0)\|_{L^1_w} + \|(P^d - P^d_T)f(0)\|_{L^1_w} \leq c\epsilon^{3/2} + \mathcal{R}_1(\omega)|\omega_T - \omega|\|f(0)\|_{L^\infty_{w^{-1}}},$$

for some constant $c > 0$.
Thanks to the former estimates, one can prove the following lemma.

Lemma 2.23. *Let us assume conditions* (2.35) *on the initial data. Then*

$$\|P^c_T k_1\|_{L^\infty_{w^{-1}}} \leq c\frac{|z(0)|^2}{(1+t)^{3/2}} \leq \frac{c\epsilon}{(1+t)^{3/2}},$$

for all $t \geq 0$.

Proof. Let us denote $\zeta = \int_0^t \rho(\tau)d\tau$.
From the definition of the exponential and the idempotency of the projections one gets

$$e^{i\zeta\Pi^\pm_T} = \Pi^\pm_T e^{i\zeta} + \Pi^\mp_T + P^d_T.$$

Then it follows

$$e^{i\zeta(\Pi^+_T - \Pi^-_T)} = (\Pi^+_T e^{i\zeta} + \Pi^-_T + P^d_T)(\Pi^-_T e^{-i\zeta} + \Pi^+_T + P^d_T) = \Pi^+_T e^{i\zeta} + \Pi^-_T e^{-i\zeta} + P^d_T.$$

The lemma follows from the fact that L_T commutes with the projectors $\Pi^\pm_T$, the definition (2.19) of the operator L_M and the decay of the evolution of the functions $P^c_T a_{ij}$, i, $j = 0, 1, 2$, stated in Lemma 2.12, namely

$$\|P^c_T k_1\|_{L^\infty_{w^{-1}}} = \left\| e^{\int_0^t L_M(\tau)d\tau} P^c_T k(0) \right\|_{L^\infty_{w^{-1}}} =$$

$$= \|e^{L_T t} P^c_T (e^{i\zeta}\Pi^+_T + e^{-i\zeta}\Pi^-_T + P^d_T)(a_{20}z^2(0) + a_{11}z(0)\overline{z(0)} + a_{02}\overline{z(0)}^2)\|_{L^\infty_{w^{-1}}} \leq$$

$$\leq c\frac{|z(0)|^2}{(1+t)^{3/2}} \leq \frac{c\epsilon}{(1+t)^{3/2}}.$$

□

2.4.2 Definition of the majorants

We are now in the position to define the majorants:

$$M_0(T) = \max_{0 \leq t \leq T} |\omega_T - \omega| \left(\frac{\epsilon}{1+\epsilon t}\right)^{-1} \tag{2.36}$$

$$M_1(T) = \max_{0 \leq t \leq T} |z(t)| \left(\frac{\epsilon}{1+\epsilon t}\right)^{-1/2} \tag{2.37}$$

$$M_2(T) = \max_{0 \leq t \leq T} \|P^c_T h_1(t)\|_{L^\infty_{w^{-1}}} \left(\frac{\epsilon}{1+\epsilon t}\right)^{-3/2} \tag{2.38}$$

.
We denote

$$M = (M_0, M_1, M_2). \tag{2.39}$$

Remark 2.24. From the estimates on g, k_1 and the definitions of the majorants follows

$$\|f\|_{L^\infty_{w^{-1}}} = \|g + P_T^c h_1 + P_T^c k + P_T^c k_1\|_{L^\infty_{w^{-1}}} \leq$$

$$\leq \mathcal{R}_1(\omega)\left(|\omega_T - \omega| + |z|^2 + \frac{\epsilon}{(1+t)^{3/2}}\|P_T^c h_1\|_{L^\infty_{w^{-1}}}\right) \leq$$

$$\leq \frac{\epsilon}{1+\epsilon t}\mathcal{R}_1(\omega)(M_1^2 + \epsilon^{1/2}M_2).$$

From the assumptions (2.35) on the initial data one obtains

$$y(0) \leq \epsilon + \mathcal{R}(\epsilon^{1/2}M)\epsilon^{3/2} \leq \epsilon(1 + \mathcal{R}(\epsilon^{1/2}M)\epsilon^{1/2}),$$

$$\|h(0)\|_{L^1_w} \leq c\epsilon^{3/2}\mathcal{R}(\epsilon^{1/2}M)\epsilon^2 M_0(1 + M_1^2 + \epsilon^{1/2}M_2).$$

2.4.3 The equation for y

We want to study the asymptotic behavior of the solution of equation (2.34) for the variable y introduced in Remark 2.18. To do that we need the following lemma which is the analogous of Lemma 4.1 in [33].

Lemma 2.25. *The remainder Y_R in equation* (2.34) *satisfies the estimate*

$$|Y_R| \leq \mathcal{R}(\epsilon^{1/2}M)\frac{\epsilon^{5/2}}{(1+\epsilon t)^2\sqrt{\epsilon t}}(1+|M|)^5.$$

Hence, equation (2.34) is of the form

$$\dot{y} = 2\operatorname{Re}(iK_T)y^2 + Y_R, \tag{2.40}$$

with

$$\begin{aligned} &\operatorname{Re}(iK_T) < 0,\\ &y(0) \leq \epsilon y_0,\\ &|Y_R| \leq \overline{Y}\tfrac{\epsilon^{5/2}}{(1+\epsilon t)^2\sqrt{\epsilon t}}, \end{aligned}$$

where y_0 and $\overline{Y} > 0$ are some constants. Then we can apply Proposition 5.6 in [11] and get the next lemma.

Lemma 2.26. *Assuming the initial condition and the source term of equation* (2.34) *as above, the solution $y(t)$ is bonded as follows for any $t > 0$*

$$\left|y(t) - \frac{y(0)}{1 + 2\operatorname{Im}(K_T)y_0 t}\right| \leq c\overline{Y}\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2},$$

where $c = c(y_0, \operatorname{Im}(K_T))$.

2.4.4 The equation for $P_T^c h_1$

As a first step let us estimate the remianders in the equation (2.25) for h_1. This is done in the next two lemmas.

Lemma 2.27. *The remainders* $\widetilde{H}_R$ *and* H''_R *can be estimated as*

$$\|P_T^c\widetilde{H}_R\|_{L^1_w} \le \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}((1+M_1)^3+\epsilon^{1/2}(1+|M|)^4),$$

and

$$\|P_T^c H''_R\|_{L^1_w} \le \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}((1+M_1)^3+\epsilon^{1/2}(1+|M|)^4).$$

Proof. From the estimate (2.21) on $\widetilde{H}_R$ one has

$$\|P_T^c\widetilde{H}_R\|_{L^1_w} \le \mathcal{R}_2(\omega,|z|+\|f\|_{L^\infty_{w^{-1}}})[|z|^3+(|z|+|\omega_T-\omega|)(|z|^2+\|P_T^c k_1\|_{L^\infty_{w^{-1}}}+$$

$$+\|P_T^c h_1\|_{L^\infty_{w^{-1}}})+(|z|^2+\|P_T^c k_1\|_{L^\infty_{w^{-1}}}+\|P_T^c h_1\|_{L^\infty_{w^{-1}}})^2]\le$$

$$\le \mathcal{R}(\epsilon^{1/2}M)\left[\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}M_1^3+\right.$$

$$+\left(\left(\frac{\epsilon}{1+\epsilon t}\right)^{1/2}M_1+\frac{\epsilon}{1+\epsilon t}M_0\right)\left(\frac{\epsilon}{1+\epsilon t}M_1^2+\frac{\epsilon}{(1+t)^{3/2}}+\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}M_2\right)+$$

$$\left.+\left(\frac{\epsilon}{1+\epsilon t}M_1^2+\frac{\epsilon}{(1+t)^{3/2}}+\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}M_2\right)\right]\le$$

$$\le \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}((1+M_1)^3+\epsilon^{1/2}(1+|M|)^4).$$

The bound for H''_R follows in the same way from the estimate (2.22). □

In the next lemma we get a estimate the evolution under the linear operator L_T of the remainder $P_T^c\overline{H}_R$.

Lemma 2.28. *For any* $t,\ s\ge 0$ *the following estimate holds*

$$\|e^{L_T t}P_T^c\overline{H}_R(s)\|_{L^\infty_{w^{-1}}}(1+t)^{3/2}\le \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2}(M_1^3+\epsilon^{1/2}(1+|M|)^3).$$

Proof. From the analytic expression (2.26) of $\overline{H}_R$ and the estimates of the evolution of the functions a_{20}, a_{11}, and a_{02} stated in Lemma 2.12, one has

$$\|e^{L_T t}P_T^c\overline{H}_R(s)\|_{L^\infty_{w^{-1}}}(1+t)^{3/2}\le$$

$$\le \mathcal{R}_2(\omega,|z|+\|f\|_{L^\infty_{w^{-1}}})|z|[|z||\omega_T-\omega|+(|z|+\|k_1\|_{L^\infty_{w^{-1}}}+\|h_1\|_{L^\infty_{w^{-1}}})^2]\le$$

$$\le \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon s}\right)^{1/2}M_1\left[\left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2}M_0M_1+\right.$$

$$+\left(\left(\frac{\epsilon}{1+\epsilon s}\right)^{1/2} M_1 + \frac{\epsilon}{(1+s)^{3/2}} + \left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2} M_2\right)^2\Bigg] \leq$$

$$\leq \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2}(M_1^3+\epsilon^{1/2}(1+|M|)^3).$$

□

From the two previous lemmas we can get the following result.

Lemma 2.29. *Let us consider the equation for* $P_T^c h_1$

$$\left(\frac{dP_T^c h_1}{dt}, v\right)_{L^2} = Q_{L_M}(P_T^c h_1, v) + (P_T^c \widehat{H}_R, v)_{L^2} + (P_T^c H_R'', q_v G_\lambda)_{L^2},$$

with initial condition and source terms satisfying

$$\|h_1(0)\|_{L^1_w} \leq \epsilon^{3/2} h_0,$$

$$\widehat{H}_R = \widetilde{H}_R + H_R'',$$

such that

$$\|P_T^c \widetilde{H}_R\|_{L^1_w} \leq \overline{H}_1 \left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2},$$

$$\|P_T^c H_R''\|_{L^1_w} \leq \overline{H}_2 \left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2},$$

$$\|e^{L_T t} P_T^c \overline{H}_R(s)\|_{L^\infty_{w^{-1}}}(1+t)^{3/2} \leq \overline{H}_3 \left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2}(M_1^3+\epsilon^{1/2}(1+|M|)^3).$$

for some positive constant h_0, $\overline{H}_1$, $\overline{H}_2$ *and* $\overline{H}_3$. *Then its solution is bounded as follows*

$$\|P_T^c h_1\|_{L^\infty_{w^{-1}}} \leq c\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}(h_0+\overline{H}_1+\overline{H}_2+\overline{H}_3),$$

where $c = c(\omega_T) > 0$.

Proof. By the Duhamel representation (see Section 1.6.2) one has

$$(P_T^c h_1, v)_{L^2} = \left(e^{\int_0^t L_M(\tau)d\tau} h_1(0) + \int_0^t e^{\int_s^t L_M(\tau)d\tau} P_T^c \widehat{H}_R(s) ds, v\right)_{L^2} +$$

$$+\left(\int_0^t e^{\int_s^t L_M(\tau)d\tau} P_T^c H_R''(s) ds, q_v G_\lambda\right)_{L^2},$$

for all $v \in V$.
Then from the dispersive estimate in Theorem 1.16 and the estimates on the remainders proved above in the duality paring defined by the inner product L^2, one has

$$\|P_T^c h_1\|_{L^\infty_{w^{-1}}} = \sup_{0\neq v\in L^1_w} \frac{(P_T^c h_1, v)_{L^2}}{\|v\|_{V\cap L^1_w}} \leq$$

$$\leq c(\omega_T)\left(\frac{1}{(1+t)^{3/2}}\|h_1(0)\|_{L^1_w} + \int_0^t \frac{1}{(1+t-s)^{3/2}}(\|P_T^c \widetilde{H}_R(s)\|_{L^1_w} + \|P_T^c H_R''(s)\|_{L^1_w})ds+\right.$$

$$+\int_0^t \|e^{L_T(t-s)}P_T^c\overline{H}_R(s)\|_{L^\infty_{w^{-1}}} ds\Big) \leq$$

$$\leq c(\omega_T)\left(\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2} h_0 + \int_0^t \frac{1}{(1+t-s)^{3/2}}\left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2} ds(\overline{H}_1+\overline{H}_2+\overline{H}_3)\right).$$

The lemma follows from the fact that

$$\int_0^t \frac{1}{(1+t-s)^{3/2}}\left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2} ds \leq c\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2},$$

for some constant $c>0$. □

2.4.5 Uniform bounds for the majorants

To prove that the majorants are uniformly bounded, the following lemma will be useful.

Lemma 2.30. *For any $T>0$ the majorants M_0, M_1, and M_2 satisfy the following inequalities*

$$\begin{aligned}
&M_0(T) \leq \mathcal{R}(\epsilon^{1/2}M)[(1+M_1)^4+\epsilon(1+|M|)^2],\\
&(M_1(T))^2 \leq \mathcal{R}(\epsilon^{1/2}M)[1+\epsilon^{1/2}(1+|M|)^5],\\
&M_2(T) \leq \mathcal{R}(\epsilon^{1/2}M)[(1+M_1)^3+\epsilon^{1/2}(1+|M|)^4].
\end{aligned}$$

Proof. It follows form Lemma 2.26 and 2.29 as Lemma 4.6 in [33], but we give the proof for sake of completeness.
`Step 1.` Let us begin noting that

$$|z|^2+\|f\|_{L^\infty_{w^{-1}}} \leq \mathcal{R}_2(\omega,|z|+\|f\|_{L^\infty_{w^{-1}}})(|z|^2+\|P_T^c k_1\|_{L^\infty_{w^{-1}}}+\|P_T^c h_1\|_{L^\infty_{w^{-1}}}) \leq$$

$$\leq \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{(1+t)^{3/2}}+\frac{\epsilon}{1+\epsilon t}M_1^2+\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}M_2\right) \leq$$

$$\leq \mathcal{R}(\epsilon^{1/2}M)\frac{\epsilon}{1+\epsilon t}(1+M_1^2+\epsilon^{1/2}M_2).$$

Then by the definition of M_0 and the bound on $|\omega_T-\omega|$:

$$M_0(T) \leq \max_{0\leq t\leq T}\left[\left(\frac{\epsilon}{1+\epsilon t}\right)^{-1}\mathcal{R}(\epsilon^{1/2}M)\left(\int_t^T\left(\frac{\epsilon}{1+\epsilon\tau}\right)^2(1+M_1(\tau)^2+\right.\right.$$

$$\left.\left.+\epsilon^{1/2}M_2(\tau))^2 d\tau+\left(\frac{\epsilon}{1+\epsilon t}\right)^2(1+M_1^2+\epsilon^{1/2}M_2)^2\right)\right] \leq$$

$$\leq \mathcal{R}(\epsilon^{1/2}M)[(1+M_1)^4+\epsilon(1+|M|)^2].$$

`Step 2.` Since $y=|z_1|^2$, we can exploit the inequality proved in Lemma 2.26, the fact that $\overline{Y}=\mathcal{R}(\epsilon^{1/2}M)(1+|M|)^5$ and $y(0)\leq\epsilon y_0$, one gets

$$y \leq \mathcal{R}(\epsilon^{1/2}M)\left[\frac{\epsilon}{1+\epsilon t}+\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}(1+|M|)^5\right].$$

From which follows

$$|z|^2 \leq y+\mathcal{R}(\omega)|z|^3 \leq$$

$$\leq \mathcal{R}(\epsilon^{1/2}M)\left[\frac{\epsilon}{1+\epsilon t}+\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}(1+|M|)^5+\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}M_1^3\right] \leq \mathcal{R}(\epsilon^{1/2}M)[1+\epsilon^{1/2}(1+|M|)^5].$$

`Step 3.` Recall that

$$\|h(0)\|_{L^1_w} \leq c\epsilon^{3/2}\mathcal{R}(\epsilon^{1/2}M)\epsilon^2 M_0(1+M_1^2+\epsilon^{1/2}M_2),$$

$$\overline{H}_1 = \mathcal{R}(\epsilon^{1/2}M)((1+M_1)^3+\epsilon^{1/2}(1+|M|)^4),$$

$$\overline{H}_2 = \mathcal{R}(\epsilon^{1/2}M)((1+M_1)^3+\epsilon^{1/2}(1+|M|)^4),$$

$$\overline{H}_3 = \mathcal{R}(\epsilon^{1/2}M)(M_1^3+\epsilon^{1/2}(1+|M|)^3).$$

Hence from Lemma 2.29 follows

$$\|P_T^c h_1\|_{L^\infty_{w^{-1}}} \leq \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}((1+M_1)^3+\epsilon^{1/2}(1+|M|)^4),$$

which implies the inequality for M_2. □

We are now in the position to prove the uniform boundedness of the majorants.

Proposition 2.31. *If $\epsilon > 0$ is sufficiently small, there exist a positive constant $\overline{M}$ independent of T and ϵ such that*

$$|M(T)| \leq \overline{M},$$

for all $T > 0$.

Proof. From the previous lemma follows

$$|M|^2 \leq \mathcal{R}(\epsilon^{1/2}M)[(1+M_1)^8+\epsilon^{1/2}(1+|M|)^8] \leq \mathcal{R}(\epsilon^{1/2}M)(1+\epsilon^{1/2}F(M)),$$

where in the last inequality we have replaced the estimate for M_1^2, and $F(M)$ is a suitable polynomial function.
Furthermore, $M(0)$ is small and $M(T)$ is a continuous function. Hence it follows that $|M|$ is bounded independent of $\epsilon \ll 1$. □

The last proposition gives a summary of the behavior of the functions $\omega(t)$, $z(t)$, $P_T^c h_1(t)$, and $f(t)$.

Corollary 2.32. *There exists a finite limit ω_∞ for the function $\omega(t)$ as $t \to +\infty$. Moreover the following holds for all $t > 0$*

$$\begin{aligned}
&|\omega_\infty - \omega(t)| \leq \overline{M}\tfrac{\epsilon}{1+\epsilon t},\\
&|z(t)| \leq \overline{M}\left(\tfrac{\epsilon}{1+\epsilon t}\right)^{1/2},\\
&\|P_T^c h_1(t)\|_{L^\infty_{w^{-1}}} \leq \overline{M}\left(\tfrac{\epsilon}{1+\epsilon t}\right)^{3/2},\\
&\|f(t)\|_{L^\infty_{w^{-1}}} \leq \overline{M}\tfrac{\epsilon}{1+\epsilon t}.
\end{aligned}$$

2.5 Large time behavior of the solution and scattering asymptotics

2.5.1 Large time behavior of the solution of equation (2.2)

The results of the previous section lead us to the following theorem.

Theorem 2.33. *Let $u(t)$ be a solution of equation (2.2) with initial datum $u_0 \in V \cap L^1_w$ of the form*

$$u_0(x) = e^{i\theta_0}\Phi_{\omega_0}(x) + z_0\Psi(x) + \overline{z}_0\Psi^*(x) + f_0(x),$$

where $\theta_0 \in \mathbb{R}$, $\omega_0 > 0$, $z_0 \in \mathbb{C}$ with

$$|z(0)| \le \epsilon^{1/2}, \qquad \|f_0\|_{L^1_w} \le c\epsilon^{3/2},$$

for some ϵ, $c > 0$. Then, provided ϵ is small enough, there exist $\omega(t)$, $\gamma(t)$, $z(t) \in C^1([0,+\infty))$ solutions of the modulation equations (2.8)-(2.10), and two constants ω_∞, $\overline{M} > 0$ such that $\omega_\infty = \lim_{t\to+\infty} \omega(t)$ and for all $t \ge 0$

$$u(t,x) = e^{i(\int_0^t \omega(s)ds + \gamma(t))}\left(\Phi_{\omega(t)}(x) + z(t)\Psi(t,x) + \overline{z(t)}\Psi^*(t,x) + f(t,x)\right),$$

where

$$|\omega_\infty - \omega(t)| \le \overline{M}\frac{\epsilon}{1+\epsilon t}, \quad |z(t)| \le \overline{M}\left(\frac{\epsilon}{1+\epsilon t}\right)^{1/2}, \quad \|f(t)\|_{L^\infty_{w^{-1}}} \le \overline{M}\frac{\epsilon}{1+\epsilon t}.$$

Proof. Let us recall that the decomposition of the function f as

$$f = g + h_1 + k + k_1$$

depends on the quantity $\omega(T)$. On the other hand Corollary 2.32 claims that the function $\omega(t)$ converges to some $\omega_\infty > 0$ as $t \to +\infty$.
As a consequence, one can reformulate the decomposition by choosing $T = +\infty$. Moreover, all the estimates obtained before for finite T can be extended to $T = +\infty$ without modification. Hence the theorem. □

The next goal is to construct precise asymptotic expressions for $\omega(t)$, $\gamma(t)$, and $z(t)$. For later convenience let us define (recall that ξ depends explicitly on ω, see (2.3); and similarly for K, see (2.28) and subsequent, and γ)

$$\xi_\infty = \xi(\omega_\infty),$$
$$\gamma_\infty = \gamma(\omega_\infty),$$
$$K_\infty = K(\omega_\infty).$$

Lemma 2.34. *Under the assumption of Theorem 2.33 the functions $\omega(t)$, $\gamma(t)$, and $z(t)$ have the following asymptotic behavior as $t \to +\infty$:*

$$\omega(t) = \omega_\infty + \frac{q_1}{1+\epsilon k_\infty t} + \frac{q_2}{1+\epsilon k_\infty t}\cos(2\xi_\infty t + a_1\log(1+\epsilon k_\infty t) + a_2) + O(t^{-3/2}),$$

$$\gamma(t) = \gamma_\infty + b_1\log(1+\epsilon k_\infty t) + O(t^{-1}),$$

$$z(t) = z_\infty \frac{e^{i\int_0^t \xi(\tau)d\tau}}{(1+\epsilon k_\infty t)^{\frac{1-i\delta}{2}}} + O(t^{-1}),$$

where

$$z_\infty = z_1(0) + \int_0^{+\infty} e^{-i\int_0^s \xi(\tau)d\tau}(1+\epsilon k_\infty s)^{\frac{1-i\delta}{2}} Z_1(s)ds,$$

$\epsilon k_\infty = 2\,\mathrm{Im}(K_\infty)y_0$, $\delta = \frac{\mathrm{Re}(K_\infty)}{\mathrm{Im}(K_\infty)}$, *and q_1, q_2, a_1, a_2, b_1 are constants.*

Proof. We will prove just the asymptotics for $z(t)$; the formulas for $\omega(t)$ and $\gamma(t)$ can be deduced as in Sections 6.1 and 6.2 of [11].
In order to do that let us recall the equation for $z_1(t)$ can be written as

$$\dot{z_1} = i\xi z_1 + iK_\infty |z_1|^2 z_1 + \widehat{\widehat{Z}}_R,$$

moreover Remark 2.22 and the inequalities satisfied by the majorants in Lemma 2.30 justify the following estimates on $\widehat{\widehat{Z}}_R$

$$|\widehat{\widehat{Z}}_R| \leq \mathcal{R}_1(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})[(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})^2 + |z||\omega_T - \omega|(|z|^2 + \|h\|_{L^\infty_{w^{-1}}}) +$$
$$+ |z|\|P^c_T k_1\|_{L^\infty_{w^{-1}}} + |z|\|P^c_T h_1\|_{L^\infty_{w^{-1}}}] \leq$$
$$\leq \mathcal{R}(\epsilon^{1/2} M) \frac{\epsilon^2}{(1+\epsilon t)^{3/2}\sqrt{\epsilon t}} (1 + \overline{M}^4) = O(t^{-2}),$$

as $t \to +\infty$. On the other hand, Lemma 2.26 implies

$$y(t) = \frac{y(0)}{1 + 2\,\mathrm{Im}(K_\infty) y(0) t} + O(t^{-3/2}), \qquad \text{as } t \to +\infty.$$

Let us note that $|z_1|$ satisfies the same bound of $|z|$, namely

$$|z_1| \leq \overline{M} \left(\frac{\epsilon}{1+\epsilon t} \right)^{1/2},$$

then the equation for $z_1(t)$ can be rewritten in the formulas

$$\dot{z_1} = i\xi z_1 + iK_\infty \frac{y(0)}{1 + 2\,\mathrm{Im}(K_\infty) y(0) t} z_1 + Z_1,$$

where $Z_1 = O(t^{-2})$ as $t \to +\infty$.
Since $y(0) = \epsilon y_0$, one has $\epsilon K_\infty y_0 = \frac{i}{2} \epsilon k_\infty (1 - i\delta)$ and the equation for $z_1(t)$ becomes

$$\dot{z_1} = \left(i\xi - \frac{i}{2} \epsilon k_\infty (1 - i\delta) \frac{1}{1 + \epsilon k_\infty t} \right) z_1 + Z_1.$$

Hence, one gets

$$z_1(t) = \frac{e^{i\int_0^t \xi(\tau) d\tau}}{(1+\epsilon k_\infty t)^{\frac{1-i\delta}{2}}} \left(z_1(0) + \int_0^s e^{-i\int_0^t \xi(\tau) d\tau} (1 + \epsilon k_\infty s)^{\frac{1-i\delta}{2}} ds \right) = z_\infty \frac{e^{i\int_0^t \xi(\tau) d\tau}}{(1+\epsilon k_\infty t)^{\frac{1-i\delta}{2}}} + z_R,$$

where z_∞ is as in the statement of the lemma and

$$z_R = -\int_t^{+\infty} e^{i\int_s^t \xi(\tau) d\tau} \left(\frac{1+\epsilon k_\infty s}{1+\epsilon k_\infty t} \right)^{\frac{1-i\delta}{2}} Z_1(s) ds.$$

The bound on Z_1 implies $z_R = O(t^{-1})$. Therefore $z(t)$ has the asymptotic behavior as $t \to +\infty$ stated in the lemma because

$$z(t) = z_1(t) + O(t^{-1}) = z_\infty \frac{e^{i\int_0^t \xi(\tau) d\tau}}{(1+\epsilon k_\infty t)^{\frac{1-i\delta}{2}}} + O(t^{-1}).$$

□

2.5.2 Scattering asymptotics

Let us make the following ansatz

$$u(t,x) = s(t,x) + \zeta(t,x) + f(t,x),$$

where

$$s(t,x) = e^{i\Theta(t)}\Phi_{\omega(t)}(x),$$

is the modulated soliton and

$$\zeta(t,x) = e^{i\Theta(t)}[(z(t)+\overline{z}(t))\Psi_1(x) + i(z(t)-\overline{z}(t))\Psi_2(x)]$$

is the fluctuating component. Recall that the functions Φ_ω, Ψ_1 and Ψ_2 satisfy

$$\omega\Phi_\omega = -H_\alpha \Phi_\omega,$$

$$\omega\Psi_1 = -i\xi\Psi_2 - H_{\alpha_1}\Psi_1,$$

$$\omega\Psi_2 = i\xi\Psi_1 - H_{\alpha_2}\Psi_2.$$

Therefore from equation (2.2) one gets

$$\left(i\frac{df}{dt}, v\right)_{L^2} = Q_0(f,v) - \nu(|q_u|^{2\sigma}q_u - |q_s|^{2\sigma}q_s - \alpha_1 q_{(z+\overline{z})\Psi_1} - \alpha_2 q_{(z-\overline{z})\Psi_2})\overline{q_v} +$$

$$+(\dot{\gamma}(s+\zeta) - i\dot{\omega}\frac{d}{d\omega}(s+\zeta) - ie^{i\Theta}[(\dot{z}-i\xi z)(\Psi_1 + i\Psi_2) + (\dot{\overline{z}} - i\xi\overline{z})(\Psi_1 - i\Psi_2)], v)_{L^2},$$

for all $v \in V$, where Q_0 is the quadratic form of the free Laplacian. Hence, as in [1], the solution $f(t)$ can be formally expressed as

$$f(t,x) = U_t * f_0(x) + i\int_0^t U_{t-\tau}(x) q_f(\tau)d\tau - i\int_0^t U_{t-\tau} * G(\tau)d\tau,$$

where we have denoted

$$G(t) = \dot{\gamma}(t)(s(t)+\zeta(t)) - i\dot{\omega}(t)\frac{d}{d\omega}(s(t)+\zeta(t)) +$$

$$-ie^{i\Theta(t)}[(\dot{z}(t) - i\xi z(t))(\Psi_1(t) + i\Psi_2(t)) + (\dot{\overline{z}}(t) - i\xi\overline{z}(t))(\Psi_1(t) - i\Psi_2(t))]$$

and $U_t(x) = \frac{e^{i\frac{|x|^2}{4t}}}{(4\pi i t)^{3/2}}$ is the propagator of the free Laplacian in $\mathbb{R}^3$.
In order to prove the asymptotic stability result we need the two following lemmas.

Lemma 2.35. *If the assumptions of Theorem 2.33 hold true, then*

$$\int_0^t U_{t-\tau}(x) q_f(\tau)d\tau = U_t * \int_0^{+\infty} U_{-\tau}(x) q_f(\tau)d\tau - \int_t^{+\infty} U_{t-\tau}(x) q_f(\tau)d\tau = U_t * \phi_0 + r_0,$$

where $\phi_0 \in L^2(\mathbb{R}^3)$ *and* $r_0 = O(t^{-1/4})$ *as* $t \to +\infty$ *in* $L^2(\mathbb{R}^3)$.

Proof. One can proceed as it is done in the case $\sigma \in (0, 1/\sqrt{2})$ (see the proof of Theorem 1.29): since $\phi_0(x) = \frac{1}{(4\pi i)^{3/2}}\widetilde{\phi}_0\left(\frac{r^2}{4}\right)$, for some function $\widetilde{\phi}_0 : \mathbb{R}^+ \to \mathbb{C}$, one gets

$$\|\phi_0\|^2_{L^2} = \frac{1}{(4\pi)^2}\int_0^{+\infty}\left|\widetilde{\phi}_0\left(\frac{r^2}{4}\right)\right|^2 r^2 dr = \frac{1}{(2\pi)^2}\int_0^{+\infty}|\widetilde{\phi}_0(y)|^2\sqrt{y}dy.$$

Hence $\phi_0 \in L^2(\mathbb{R}^3)$ if and only if $\widetilde{\phi}_0 \in L^2(\mathbb{R}^+, \sqrt{y}dy)$. On the other hand, one can make the change of variables $u = \frac{1}{\tau}$ in the integral function $\widetilde{\phi}_0$ and get

$$\widetilde{\phi}_0(y) = \int_0^{+\infty} e^{-iyu}\frac{1}{u}q_f\left(\frac{1}{u}\right)\sqrt{u}du,$$

then $\widehat{\widetilde{\phi}_0} = \frac{1}{u}q_f\left(\frac{1}{u}\right)$. Moreover, by corollary 2.32 one has

$$\|\widehat{\widetilde{\phi}_0}\|^2_{L^2} = \int_0^{+\infty}\frac{1}{u^2}\left|q_f\left(\frac{1}{u}\right)\right|^2\sqrt{u}du \le C\int_0^{+\infty}\frac{\sqrt{u}}{(u+\epsilon)^2}du \le C,$$

for some constant $C > 0$, hence the Plancherel identity implies

$$\widetilde{\phi}_0 \in L^2(\mathbb{R}^+, \sqrt{y}dy).$$

In the same way, for any $t > 0$ the following holds

$$\|r_0\|^2_{L^2} = \frac{1}{(2\pi)^2}\left\|\frac{1}{u}q_f\left(t + \frac{1}{u}\right)\right\|^2_{L^2(\mathbb{R}^+, \sqrt{u}du)} \le C\frac{1}{\sqrt{1+\epsilon t}},$$

for some constant $C > 0$ independent of t. Which concludes the proof. □

The analogous result for the integral function $\int_0^t U_{t-\tau} * G(\tau)d\tau$ requires different tools.

Lemma 2.36. *Assume that the assumptions of Theorem 2.33 hold true, then*

$$\int_0^t U_{t-\tau} * G(\tau)d\tau = U_t * \int_0^{+\infty} U_{-\tau} * G(\tau)d\tau - U_t * \int_t^{+\infty} U_{-\tau} * G(\tau)d\tau = U_t * \phi_1 + r_1,$$

where $\phi_1 \in L^2(\mathbb{R}^3)$ and $r_1 = O(t^{-1/2})$ as $t \to +\infty$ in $L^2(\mathbb{R}^3)$.

Proof. We exploit the idea used in [33] to prove Lemma 5.5.

`Step 1: restriction to the leading terms.`

From the expansions (2.15), (2.16) and (2.17) for $\dot{\omega}(t)$, $\dot{\gamma}(t)$ and $\dot{z}(t) - i\xi z(t)$ follow that the function $G(t)$ is made by a quadratic part consisting in the terms multiplied $e^{i\Theta(t)}z_\infty^2$, $e^{i\Theta(t)}\overline{z_\infty}^2$ or $e^{i\Theta(t)}|z_\infty|^2$, with

$$z_\infty = \frac{e^{i\xi_\infty t}}{\sqrt{1+\epsilon k_\infty t}},$$

which are of order t^{-1} and a remainder of order $t^{-3/2}$. The convergence and the decay of the remainder is trivial from the unitarity of U_t. Furthermore, from the analytic definition of G it follows that it is a complex linear combination of functions of the form

$$Q(x) = e^{-\sqrt{\alpha}|x|^2}, \qquad \alpha = \omega_\infty, \omega_\infty + \nu_\infty, \omega_\infty - \nu_\infty.$$

Hence it suffices to prove the lemma for the functions $\Pi(t)Q(x)$, where $\Pi(t)$ is one between $e^{i\Theta(t)}z_\infty^2$, $e^{i\Theta(t)}\overline{z_\infty}^2$ and $e^{i\Theta(t)}|z_\infty|^2$.

`Step 2: decomposition of` $U_t * Q$.
Let us note that we can rewrite the convolution product as follows

$$U_t * Q = \frac{e^{i\frac{|x|^2}{4t}}}{(4\pi i t)^{3/2}} \int_{\mathbb{R}^3} e^{-i\frac{(x,y)}{2t}} Q(y)dy + \frac{e^{i\frac{|x|^2}{4t}}}{(4\pi i t)^{3/2}} \int_{\mathbb{R}^3} e^{-i\frac{(x,y)}{2t}} (e^{i\frac{|y|^2}{4t}} - 1)Q(y)dy =$$

$$= \frac{e^{i\frac{|x|^2}{4t}}}{(2it)^{3/2}} \widehat{Q}\left(\frac{x}{2t}\right) + \frac{e^{i\frac{|x|^2}{4t}}}{(2it)^{3/2}} \widehat{Q_t}\left(\frac{x}{2t}\right), \tag{2.41}$$

where $Q_t(y) = (e^{i\frac{|y|^2}{4t}} - 1)Q(y)$.
Since $|e^{i\theta} - 1| \leq \theta$ and the function $G(y)$ is exponentially decaying as $|y| \to +\infty$, the L^2 norm of the second term of (2.41) can be estimated in the following way for any $t > 1$

$$\frac{1}{(2t)^{3/2}} \left\| \widehat{Q_t}\left(\frac{\cdot}{2t}\right) \right\|_{L^2} = \|\widehat{Q_t}(\cdot)\|_{L^2} \leq \frac{1}{4t} \left(\int_{\mathbb{R}^3} |y|^4 |Q(y)|^2 dy \right)^{1/2} \leq \frac{C}{t},$$

for some constant $C > 0$. Hence, recalling that $\Pi(\tau) \leq (1 + \epsilon k_\infty \tau)^{-1}$, we obtain

$$\int_0^{+\infty} \Pi(\tau) U_\tau * Q_t d\tau \in L^2(\mathbb{R}^3),$$

and

$$\int_t^{+\infty} \Pi(\tau) U_\tau * Q_t d\tau = O(t^{-1}),$$

as $t \to +\infty$ in $L^2(\mathbb{R}^3)$.
`Step 3: Analysis of the first term in` (2.41) `in a particular case.`
Let us first show how to treat the terms with the phase $\Theta(t)$ replaced by $\omega_\infty t$.
Note that

$$\widehat{Q}(x) = \frac{1}{\alpha + |x|^2},$$

Hence, in the case of the summands with $|z_\infty|^2$ it suffices to prove the integrability of the function

$$I(x) = \int_0^\infty e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\sqrt{\tau}}{(1 + \epsilon k_\infty \tau)(|x|^2 + 4\alpha\tau^2)} d\tau =$$

$$= A(x) \int_0^\infty e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \left(\frac{\sqrt{\tau}}{(1 + \epsilon k_\infty \tau)} - \frac{4\alpha}{\epsilon k_\infty} \frac{\tau\sqrt{\tau}}{(|x|^2 + 4\alpha\tau^2)} \right) d\tau +$$

$$+ \frac{4\alpha}{\epsilon^2 k_\infty^2} A(x) \int_0^\infty e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\sqrt{\tau}}{(|x|^2 + 4\alpha\tau^2)} d\tau = I_1(x) + I_2(x),$$

and the decay of

$$I_t(x) = \int_t^\infty e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\sqrt{\tau}}{(1 + \epsilon k_\infty \tau)(|x|^2 + 4\alpha\tau^2)} d\tau =$$

$$= A(x) \int_t^\infty e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \left(\frac{\sqrt{\tau}}{(1 + \epsilon k_\infty \tau)} - \frac{4\alpha}{\epsilon k_\infty} \frac{\tau\sqrt{\tau}}{(|x|^2 + 4\alpha\tau^2)} \right) d\tau +$$

$$+ \frac{4\alpha}{\epsilon^2 k_\infty^2} A(x) \int_t^\infty e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\sqrt{\tau}}{(|x|^2 + 4\alpha\tau^2)} d\tau = I_{1,t}(x) + I_{2,t}(x),$$

where $A(x) = \frac{\epsilon^2 k_\infty^2}{4\alpha + \epsilon^2 k_\infty^2 |x|^2}$.
For the function $I_2(x)$ one has

$$|I_2(x)| \leq \frac{4\alpha}{\epsilon^2 k_\infty^2} A(x) \int_0^\infty \frac{\sqrt{\tau}}{(|x|^2 + 4\alpha\tau^2)} d\tau = C \frac{A(x)}{\sqrt{|x|}} \in L^2(\mathbb{R}^3).$$

With the same estimate it is trivial to prove

$$I_{2,t}(x) = O(t^{-1/2})$$

as $t \to +\infty$, in $L^2(\mathbb{R}^3)$.
In order to treat I_1 note that

$$\frac{\sqrt{\tau}}{(1+\epsilon k_\infty t)} - \frac{4\alpha}{\epsilon k_\infty} \frac{\tau\sqrt{\tau}}{(|x|^2 + 4\alpha\tau^2)} = -\frac{1}{\epsilon k_\infty \sqrt{\tau}(1+\epsilon k_\infty \tau)} + \frac{|x|^2}{\epsilon k_\infty \sqrt{\tau}(|x|^2 + 4\alpha\tau^2)}.$$

Since $\frac{1}{\epsilon k_\infty \sqrt{\tau}(1+\epsilon k_\infty \tau)} = O(t^{-3/2})$ as $t \to +\infty$, is integrable on $(0, +\infty)$ and $A(x) \in L^2(\mathbb{R}^3)$ one has to prove

$$\frac{|x|^2 A(x)}{\epsilon k_\infty} \int_0^{+\infty} e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{1}{\sqrt{\tau}(|x|^2 + 4\alpha\tau^2)} d\tau =$$

$$= A(x) \int_0^{+\infty} e^{i\omega_\infty (\tau - \frac{|x|^2}{4\omega_\infty \tau})} \frac{1}{\sqrt{\tau}} d\tau - 4\alpha A(x) \int_0^{+\infty} e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\tau^{3/2}}{(|x|^2 + 4\alpha\tau^2)} d\tau \in L^2(\mathbb{R}^3).$$

From formulas 3.871.3 and 3.871.4 in [27] one has

$$A(x) \int_0^{+\infty} e^{i\omega_\infty (\tau - \frac{|x|^2}{4\omega_\infty \tau})} \frac{1}{\sqrt{\tau}} d\tau = \frac{e^{i\pi/4}}{\sqrt{\pi\omega_\infty}} A(x) |x|^{3/2} e^{-\sqrt{\omega_\infty}|x|} \in L^2(\mathbb{R}^3).$$

It remains to handle with the second integral in the former sum which can be done integrating by parts in the following way

$$\left| A(x) \int_0^{+\infty} e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\tau^{3/2}}{(|x|^2 + 4\alpha\tau^2)} d\tau \right| =$$

$$= 4A(x) \left| \int_0^{+\infty} e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{d}{d\tau} \left[\frac{\tau^{7/2}}{(|x|^2 + 4\alpha\tau^2)(|x|^2 + 4\omega_\infty \tau^2)} \right] d\tau \right| \leq$$

$$\leq C A(x) \int_0^{+\infty} \frac{\tau^{5/2}}{(|x|^2 + 4\min\{\alpha, \omega_\infty\}\tau^2)^2} d\tau \leq C \frac{A(x)}{\sqrt{|x|}} \in L^2(\mathbb{R}^3).$$

Then we are done.
In order to estimate the decay of $I_{1,t}$ it suffices to study the decay of

$$\frac{|x|^2 A(x)}{\epsilon k_\infty} \int_t^{+\infty} e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{1}{\sqrt{\tau}(|x|^2 + 4\alpha\tau^2)} d\tau =$$

$$= A(x) \int_t^{+\infty} e^{i\omega_\infty (\tau - \frac{|x|^2}{4\omega_\infty \tau})} \frac{1}{\sqrt{\tau}} d\tau - 4\alpha A(x) \int_t^{+\infty} e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\tau^{3/2}}{(|x|^2 + 4\alpha\tau^2)} d\tau,$$

which can be done integrating by parts as before. Let us do that for the second term (the computation for the first one are analogous and simpler):

$$\left| A(x) \int_t^{+\infty} e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\tau^{3/2}}{(|x|^2 + 4\alpha\tau^2)} d\tau \right| =$$

$$= 4A(x)\left|\int_t^{+\infty} e^{i(\omega_\infty\tau-\frac{|x|^2}{4\tau})}\frac{d}{d\tau}\left[\frac{\tau^{7/2}}{(|x|^2+4\alpha\tau^2)(|x|^2+4\omega_\infty\tau^2)}\right]d\tau\right| \leq$$

$$\leq CA(x)\left[t^{-1/2}+\int_t^{+\infty}\frac{\tau^{5/2}}{(|x|^2+4\min\{\alpha,\omega_\infty\}\tau^2)^2}d\tau\right] \leq$$

$$\leq CA(x)\left(1+\frac{1}{\sqrt{|x|}}\right)t^{-1/2}.$$

The case of the summands with z_∞^2 is analogous, while the case of $\overline{z_\infty}^2$ is more difficult because $|x|^2+4(\omega_\infty-2\xi_\infty)\tau^2=0$ for

$$\tau = t^* = \frac{|x|}{2\sqrt{2\xi_\infty-\omega_\infty}}.$$

Let $g:\mathbb{R}^+\to\mathbb{R}^+$ be a continuous function with the properties:

$$0<g(t^*)<t^* \quad \forall t^*>0, \qquad \text{and} \qquad A(x)g(t^*)\in L^2(\mathbb{R}^3).$$

It follows that $g(t^*)=O(t^*)=O(|x|)$ as $|x|\to+\infty$. Hence, one can represent $(0,+\infty)=(0,t^*-g(t^*)]\cup(t^*-g(t^*),t^*+g(t^*)]\cup(t^*+g(t^*),+\infty)$. Integrating by parts once more one has

$$\left|A(x)\int_0^{t^*-g(t^*)} e^{i((\omega_\infty-2\xi_\infty)\tau-\frac{|x|^2}{4\tau})}\frac{t^{3/2}}{|x|^2+4\alpha\tau^2}d\tau\right| \leq$$

$$\leq CA(x)\left((t^*-g(t^*))^{-1/2}+\int_0^{t^*-g(t^*)}\frac{t^{5/2}}{(|x|^2+4\alpha\tau^2)||x|^2+4(\omega_\infty-2\xi_\infty)\tau^2|}d\tau+\right.$$

$$+\int_0^{t^*-g(t^*)}\frac{t^{9/2}}{(|x|^2+4\alpha\tau^2)^2||x|^2+4(\omega_\infty-2\xi_\infty)\tau^2|}d\tau+$$

$$\left.+\int_0^{t^*-g(t^*)}\frac{t^{9/2}}{(|x|^2+4\alpha\tau^2)||x|^2+4(\omega_\infty-2\xi_\infty)\tau^2|^2}d\tau\right) \leq$$

$$\leq CA(x)((t^*-g(t^*))^{-1/2}+(t^*-g(t^*))^{3/8})\in L^2(\mathbb{R}^3),$$

where the last inequality follows from formula 3.194.1 in [27]. In the same way (exploiting formula 3.194.2 instead of 3.194.1 in [27]), one has

$$\left|A(x)\int_{t^*+g(t^*)}^{\infty} e^{i((\omega_\infty-2\xi_\infty)\tau-\frac{|x|^2}{4\tau})}\frac{t^{3/2}}{|x|^2+4\alpha\tau^2}d\tau\right| \leq$$

$$\leq CA(x)((t^*+g(t^*))^{-1/8}+(t^*+g(t^*))^{-9/8}+(t^*-g(t^*))^{-1/2})\in L^2(\mathbb{R}^3).$$

Finally,

$$\left|A(x)\int_{t^*-g(t^*)}^{t^*+g(t^*)} e^{i((\omega_\infty-2\xi_\infty)\tau-\frac{|x|^2}{4\tau})}\frac{t^{3/2}}{|x|^2+4\alpha\tau^2}d\tau\right| \leq$$

$$\leq CA(x)\int_{t^*-g(t^*)}^{t^*+g(t^*)}\frac{1}{\sqrt{\tau}}d\tau \leq C\frac{A(x)g(t^*)}{\sqrt{t^*-g(t^*)}}\in L^2(\mathbb{R}^3).$$

Summing up, the integrability of the integral function

$$\int_0^{+\infty}\Pi(\tau)U_\tau * Q d\tau$$

is achieved. It is left to study the decay of

$$A(x)\int_t^{+\infty} e^{i((\omega_\infty-2\xi_\infty)\tau-\frac{|x|^2}{4\tau})}\frac{t^{3/2}}{|x|^2+4\alpha\tau^2}d\tau.$$

First of all, let us note that integrating by parts one obtains

$$\left|A(x)\int_{(0,t^*-g(t^*)]\cap[t,+\infty)} e^{i((\omega_\infty-2\xi_\infty)\tau-\frac{|x|^2}{4\tau})}\frac{t^{3/2}}{|x|^2+4\alpha\tau^2}d\tau\right|\leq$$

$$\leq CA(x)\int_t^{t^*-g(t^*)}\left|\frac{d}{d\tau}\frac{t^{7/2}}{(|x|^2+4\alpha\tau^2)(|x|^2+4(\omega_\infty-2\xi_\infty)\tau^2)}\right|d\tau\leq$$

$$\leq CA(x)\left(t^{-1/2}+\int_t^{t^*-g(t^*)}\frac{\sqrt{\tau}}{|x|^2+4\alpha\tau^2}d\tau+\int_t^{t^*-g(t^*)}\frac{\sqrt{\tau}}{||x|^2+4(\omega_\infty-2\xi_\infty)\tau^2|}d\tau+\right.$$

$$\left.+\int_t^{t^*-g(t^*)}\frac{\tau^{9/2}}{(|x|^2+4\alpha\tau^2)||x|^2+4(\omega_\infty-2\xi_\infty)\tau^2|^2}d\tau\right).$$

The three integrals in the last inequality can be estimated in the following way:

(i) $\int_t^{t^*-g(t^*)}\frac{\sqrt{\tau}}{|x|^2+4\alpha\tau^2}d\tau\leq C\int_t^{+\infty}\tau^{-3/2}d\tau\leq Ct^{-1/2}$;

(ii) $\int_t^{t^*-g(t^*)}\frac{\sqrt{\tau}}{|x|^2+4(2\xi_\infty-\omega_\infty)\tau^2}d\tau=\int_t^{t^*-g(t^*)}\frac{\sqrt{\tau}}{(|x|+2\sqrt{2\xi_\infty-\omega_\infty}\tau)||x|-2\sqrt{2\xi_\infty-\omega_\infty}\tau|}d\tau$
$\leq Ct^{-1/2}\int_0^{t^*-g(t^*)}\frac{1}{||x|-2\sqrt{2\xi_\infty-\omega_\infty}\tau|}d\tau\leq Ct^{-1/2}$;

(iii) $\int_t^{t^*-g(t^*)}\frac{\tau^{9/2}}{(|x|^2+4\alpha\tau^2)||x|^2+4(\omega_\infty-2\xi_\infty)\tau^2|^2}d\tau\leq Ct^{-1/2}\int_0^{t^*-g(t^*)}\frac{\tau}{||x|-2\sqrt{\omega_\infty-2\xi_\infty}\tau|^2}d\tau$
$\leq Ct^{-1/2}(1+\ln||x|-2\sqrt{\omega_\infty-2\xi_\infty}(t^*-g(t^*))|)$.

Hence, since $A(x)\ln||x|-2\sqrt{\omega_\infty-2\xi_\infty}(t^*-g(t^*))|\in L^2(\mathbb{R}^3)$, one can conclude

$$A(x)\int_{(0,t^*-g(t^*)]\cap[t,+\infty)} e^{i((\omega_\infty-2\xi_\infty)\tau-\frac{|x|^2}{4\tau})}\frac{t^{3/2}}{|x|^2+4\alpha\tau^2}d\tau=O(t^{-1/2})$$

as $t\to+\infty$, in $L^2(\mathbb{R}^3)$.
Let us now observe that

$$\left|A(x)\int_{(t^*-g(t^*),+\infty)\cap[t,+\infty)} e^{i((\omega_\infty-2\xi_\infty)\tau-\frac{|x|^2}{4\tau})}\frac{t^{3/2}}{|x|^2+4\alpha\tau^2}d\tau\right|\leq$$

$$\leq CA(x)\int_t^{+\infty}\left|\frac{d}{d\tau}\frac{t^{7/2}}{(|x|^2+4\alpha\tau^2)(|x|^2+4(\omega_\infty-2\xi_\infty)\tau^2)}\right|d\tau\leq$$

$$\leq B(x)A(x)\left(t^{-1/2}+\int_t^{+\infty}\frac{\sqrt{\tau}}{|x|^2+4\alpha\tau^2}d\tau\right)\leq CB(x)A(x)t^{-1/2}\in L^2(\mathbb{R}^3),$$

where $B:\mathbb{R}^3\to\mathbb{R}^+$ is a continuous bounded function.
Finally,

$$\left|A(x)\int_{(t^*-g(t^*),(t^*+g(t^*)]\cap[t,+\infty)} e^{i((\omega_\infty-2\xi_\infty)\tau-\frac{|x|^2}{4\tau})}\frac{t^{3/2}}{|x|^2+4\alpha\tau^2}d\tau\right|\leq$$

$$\leq CA(x)\int_t^{t^*+g(t^*)}\frac{1}{\sqrt{\tau}}d\tau \leq CA(x)g(t^*)t^{-1/2}\in L^2(\mathbb{R}^3).$$

Summing up, thanks to the unitarity of U_t, we proved

$$U_t * \int_t^{+\infty}\Pi(\tau)U_\tau * Qd\tau = O(t^{-1/2}),$$

as $t\to+\infty$, in $L^2(\mathbb{R}^3)$.

`Step 4: conclusion of the proof.`

The conclusions of the previous step hold true if the phase $\omega_\infty t$ is replaced by $\Theta(t)$. In fact, the estimates which involve the integral of the absolute value are totally unaffected by change of phase, then it is only left to adjust the argument involving integration by parts. This can be done integrating by parts exactly as before, which leaves a factor $e^{i(\Theta(t)-\omega_\infty t)}$ in the integrand. Then, the boundary terms can be treated in the same way because $|e^{i(\Theta(t)-\omega_\infty t)}|=1$. Finally, the extra contribution to the integrand can be estimated as it is done for the summand arising from differentiation of $t^{7/2}$ since $|\dot{\Theta}(t)-\omega_\infty|\leq\frac{C}{1+\epsilon k_\infty t}$ for all $t>0$, where C is a positive constant. ☐

Summing up, we have proved the following asymptotic stability result.

Theorem 2.37. *Let $\sigma\in\left(\frac{1}{\sqrt{2}},\sigma^*\right)$, for a certain $\sigma^*\in\left(\frac{1}{\sqrt{2}},\frac{\sqrt{3}+1}{2\sqrt{2}}\right]$ and $u(t)\in C(\mathbb{R}^+,V)$ be a solution of equation* (2.2) *with*

$$u(0)=u_0=e^{i\omega_0 t+\gamma_0}\Phi_{\omega_0}+e^{i\omega_0 t+\gamma_0}[(z_0+\overline{z_0})\Psi_1+i(z_0-\overline{z_0})\Psi_2]+f_0\in V\cap L^1_w(\mathbb{R}^3),$$

for some $\omega_0>0$, γ_0, $z_0\in\mathbb{R}$ and $f_0\in L^2(\mathbb{R}^3)\cap L^1_w(\mathbb{R}^3)$. Furthermore, assume that the initial datum u_0 is close to a solitary wave, i.e.

$$|z_0|\leq\epsilon^{1/2}\qquad \textit{and}\qquad \|f_0\|_{L^1_w}\leq c\epsilon^{3/2},$$

where c, $\epsilon>0$.
Then, if ϵ is sufficiently small, the solution $u(t)$ can be asymptotically decomposed as follows

$$u(t)=e^{i\omega_\infty t+ib_1\log(1+\epsilon k_\infty t)}\Phi_{\omega_\infty}+U_t*\phi_\infty+r_\infty(t),\quad \textit{as } t\to+\infty,$$

where ω_∞, $\epsilon k_\infty>0$, $b_1\in\mathbb{R}$ and ϕ_∞, $r_\infty(t)\in L^2(\mathbb{R}^3)$ with

$$\|r_\infty(t)\|_{L^2}=O(t^{-1/4})\quad \textit{as } t\to+\infty,$$

in $L^2(\mathbb{R}^3)$.

Remark 2.38. Numerical evidences (see Lemma 2.20) suggest $\sigma^*=\frac{\sqrt{3}+1}{2\sqrt{2}}\simeq 0,96$.

2.6 Appendices

2.6.1 Eigenfunctions associated to $\pm i\xi$ and generalized eigenfunctions

The eigenfunctions associated to $\pm i\xi$

Here we want to describe the eigenspaces associated to the simple purely imaginary eigenvalues $\pm i\xi=\pm i2\sigma\sqrt{1-\sigma^2}\omega$.
Let us start with the eigenvalue $i\xi$. The following proposition holds true.

Proposition 2.39. *The eigenspace associated to $i\xi$ is spanned by*

$$\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix} = \frac{e^{-\sqrt{\omega-\xi}|x|}}{4\pi|x|} \begin{pmatrix} 1 \\ i \end{pmatrix} - \frac{\sqrt{1-\sigma^2}-1}{\sigma} \frac{e^{-\sqrt{\omega+\xi}|x|}}{4\pi|x|} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Proof. In order to prove the proposition we need to solve the equation

$$L\Psi = i\xi\Psi$$

in $D(L)$. For $x \neq 0$, the previous equation is equivalent to the system

$$\begin{cases} (-\triangle+\omega)^2\Psi_1 - \xi^2\Psi_1 = 0 \\ \Psi_2 = \frac{i}{\xi}(-\triangle+\omega)\Psi_1 \end{cases},$$

from which follows that Ψ_1 must belong to $L^2(\mathbb{R}^3)$ and solve the equation

$$(-\triangle+\omega-\xi)(-\triangle+\omega+\xi)\Psi_1 = 0.$$

Hence, the solutions in $L^2(\mathbb{R}^3)$ are of the form

$$\begin{cases} \Psi_1(x) = A\frac{e^{-\sqrt{\omega-\xi}|x|}}{4\pi|x|} + B\frac{e^{-\sqrt{\omega+\xi}|x|}}{4\pi|x|} \\ \Psi_2(x) = iA\frac{e^{-\sqrt{\omega-\xi}|x|}}{4\pi|x|} - iB\frac{e^{-\sqrt{\omega+\xi}|x|}}{4\pi|x|} \end{cases},$$

for any A, $B \in \mathbb{C}$.
It is left to look for A, $B \in \mathbb{C}$ such that $\Psi_i \in D(L_i)$ for $i = 1, 2$, i.e.

$$\begin{cases} -\frac{\sqrt{\omega-\xi}}{4\pi}A - \frac{\sqrt{\omega+\xi}}{4\pi}B = -(2\sigma+1)\frac{\sqrt{\omega}}{4\pi}(A+B) \\ -i\frac{\sqrt{\omega-\xi}}{4\pi}A + i\frac{\sqrt{\omega+\xi}}{4\pi}B = -\frac{\sqrt{\omega}}{4\pi}(iA - iB) \end{cases}.$$

Exploiting the fact that $\xi = 2\sigma\sqrt{1-\sigma^2}\omega$ one can show that the two equations of the previous system are linearly dependent and

$$B = -\frac{\sqrt{1-\sigma^2}+1}{\sigma}A.$$

The thesis follows by setting $A = 1$. □

Let us note that in the previous proof we have chosen the constant in such a way that $\Psi_1(x) \in \mathbb{R}$ and $\Psi_2(x) \in i\mathbb{R}$ for any $x \in \mathbb{R}^3 \setminus \{0\}$. This fact will be used to prove the next proposition.

Proposition 2.40. *The eigenspace associated to $-i\xi$ is spanned by*

$$\Psi^* = \begin{pmatrix} \Psi_1 \\ -\Psi_2 \end{pmatrix}.$$

Proof. In the previous proposition we proved that

$$\begin{cases} L_2\Psi_2 = i\xi\Psi_1 \\ -L_1\Psi_1 = i\xi\Psi_2 \end{cases},$$

with Ψ_1 real and Ψ_2 purely imaginary.
Taking the conjugate of both equations and recalling that the operators L_i, $i = 1, 2$ act on the real and imaginary parts separately, one has

$$\begin{cases} L_2(-\Psi_2) = -i\xi\Psi_1 \\ -L_1\Psi_1 = -i\xi(-\Psi_2) \end{cases},$$

which is equivalent to

$$L\Psi^* = -i\xi\Psi^*,$$

because the operators L_i, $i = 1, 2$ are linear. The proof is complete. □

The generalized eigenfunctions

Our goal is to compute the generalized eigenfunctions associated to the continuous spectrum. In order to do that, we treat the two branches $\mathcal{C}_+$ and $\mathcal{C}_-$ of the continuous spectrum separately.

Proposition 2.41. *The generalized eigenfunctions associated to* $\mathcal{C}_+$ *are*

$$\Psi_+(x) = A\frac{e^{-\sqrt{\omega+\eta}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ -i \end{pmatrix} + C\frac{e^{-i\sqrt{\eta-\omega}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix} + D\frac{e^{i\sqrt{\eta-\omega}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix},$$

for any $\eta \in [\omega, +\infty)$ *and* $D \in \mathbb{C}$, *with*

$$\begin{aligned} A &= \tfrac{\sigma\sqrt{\omega}}{\sqrt{\omega+\eta}-(\sigma+1)\sqrt{\omega}}(C+D), \\ C &= \tfrac{(2\sigma+1)\omega+(\sigma+1)\sqrt{\omega}(i\sqrt{\eta-\omega}-\sqrt{\eta+\omega})-i\sqrt{\eta^2-\omega^2}}{-(2\sigma+1)\omega+(\sigma+1)\sqrt{\omega}(i\sqrt{\eta-\omega}+\sqrt{\eta+\omega})-i\sqrt{\eta^2-\omega^2}}D. \end{aligned}$$

Proof. For any $\eta \in [\omega, +\infty)$, we need to solve the system

$$L\Psi_+ = i\eta\Psi_+,$$

where $\Psi_+ \in L^\infty(\mathbb{R}^3)$ does not necessary belongs to $L^2(\mathbb{R}^3)$. As in the computation for the eigenfunction at $\pm i\xi$, if $x \neq 0$ the former equation is equivalent to the system

$$\begin{cases} (-\triangle+\omega-\xi)(-\triangle+\omega+\xi)(\Psi_+)_1 = 0 \\ (\Psi_+)_2 = \frac{i}{\xi}(-\triangle+\omega)(\Psi_+)_1 \end{cases},$$

which leads to

$$\Psi_+(x) = A\frac{e^{-\sqrt{\omega+\eta}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ -i \end{pmatrix} + B\frac{e^{\sqrt{\omega+\eta}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix} + C\frac{e^{-i\sqrt{\eta-\omega}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix} + D\frac{e^{i\sqrt{\eta-\omega}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix},$$

for some $A, B, C, D \in \mathbb{C}$. Since we require $\Psi_+ \in L^\infty(\mathbb{R}^3)$, we get $B = 0$. Moreover, the boundary conditions in the domain of the operators L_1 and L_2 must be satisfied by $(\Psi_+)_1$ and $(\Psi_+)_2$ respectively. Then A, C, and D solve the system

$$\begin{cases} -\frac{\sqrt{\omega+\eta}}{4\pi}A - i\frac{\sqrt{\eta+\omega}}{4\pi}C + i\frac{\sqrt{\eta+\omega}}{4\pi}D = -\frac{(2\sigma+1)\sqrt{\omega}}{4\pi}(A+C+D) \\ i\frac{\sqrt{\omega+\eta}}{4\pi}A + \frac{\sqrt{\eta+\omega}}{4\pi}C - \frac{\sqrt{\eta+\omega}}{4\pi}D = -\frac{\sqrt{\omega}}{4\pi}(-iA+iC+iD) \end{cases},$$

which concludes the proof. □

In the same way, one can prove the analogous result about $\mathcal{C}_-$.

Proposition 2.42. *The generalized eigenfunctions associated to* $\mathcal{C}_-$ *are*

$$\Psi_-(x) = A\frac{e^{-\sqrt{\omega-\eta}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix} + C\frac{e^{-i\sqrt{-(\eta+\omega)}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ -i \end{pmatrix} + D\frac{e^{i\sqrt{-(\eta+\omega)}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ -i \end{pmatrix},$$

for any $\eta \in (-\infty, -\omega]$, *where* $D \in \mathbb{C}$ *and*

$$\begin{aligned} A &= \tfrac{\sigma\sqrt{\omega}}{\sqrt{\omega-\eta}-(\sigma+1)\sqrt{\omega}}(C+D), \\ C &= \tfrac{(2\sigma+1)\omega+(\sigma+1)\sqrt{\omega}(i\sqrt{-(\eta+\omega)}-\sqrt{\omega-\eta})-i\sqrt{\eta^2-\omega^2}}{-(2\sigma+1)\omega+(\sigma+1)\sqrt{\omega}(i\sqrt{-(\eta+\omega)}+\sqrt{\eta+\omega})-i\sqrt{\eta^2-\omega^2}}D. \end{aligned}$$

2.6.2 Proof of Lemma 2.8

In this appendix we prove Lemma 2.8 whose statement is recalled for the reader's convenience.

Lemma 2.43. *There exists a constant $C > 0$ such that for each $h \in X^c$ holds*

$$\left\|[P^c J - i(\Pi^+ - \Pi^-)]h\right\|_{L^1_w} \leq C\|h\|_{L^\infty_{w^{-1}}}.$$

Proof. From the definitions of the operators P^c and $\Pi^\pm$ one gets

$$P^c J - i(\Pi^+ - \Pi^-) = \Pi^+(J - iI) + \Pi^-(J + iI) =$$

$$= \frac{1}{2\pi i}\left[\int_{\mathcal{C}^+}(R(\lambda+0) - R(\lambda-0))(J - iI)d\lambda + \int_{\mathcal{C}^-}(R(\lambda+0) - R(\lambda-0))(J + iI)d\lambda\right].$$

We will estimate just the first integral because the second one can be handled in the same way. Exploiting the explicit form of the resolvent (1.10) it follows that

$$R(\lambda)(J - iI) = (i\lambda\mathcal{G}_{\lambda^2} * + \Gamma_{\lambda^2} *)\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} + \frac{4\pi}{D(\lambda^2)}\begin{bmatrix} \Lambda_1 + \Sigma_2 & i(\Lambda_1 + \Sigma_2) \\ -i(\Lambda_2 + \Sigma_1) & \Lambda_2 + \Sigma_1 \end{bmatrix} =$$

$$= R_*(\lambda)(J - iI) + R_m(\lambda)(J - iI),$$

where R_* and R_m correspond to the convolution term of the resolvent and the multiplicative term. Note that

$$i\lambda\mathcal{G}_{\lambda^2}(x-y) + \Gamma_{\lambda^2}(x-y) = 2G_{\omega - i\lambda}(x-y) = \frac{e^{i\sqrt{-\omega+i\lambda}|x-y|}}{2\pi|x-y|}$$

is continuous on $\mathcal{C}_+$. Hence, the integral on $\mathcal{C}_+$ of the convolution addends vanishes.
Let us now consider the multiplicative addends in the integral on $\mathcal{C}_+$. From the explicit formulas for Λ_1 and Σ_2 given in Proposition 1.10 one can compute

$$(\Lambda_1+\Sigma_2)(x,y) = 8\pi(\alpha_2-\alpha_1)G_{\omega-i\lambda}(y)G_{\omega+i\lambda}(x)+[8\pi(\alpha_2+\alpha_1)-4i\sqrt{-\omega-i\lambda}]G_{\omega-i\lambda}(y)G_{\omega-i\lambda}(x) =$$

$$= 4\sigma\sqrt{\omega}\frac{e^{i\sqrt{-\omega+i\lambda}|y|}}{4\pi|y|}\frac{e^{i\sqrt{-\omega-i\lambda}|x|}}{4\pi|x|} - [4(\sigma+1)\sqrt{\omega} - 4i\sqrt{-\omega-i\lambda}]\frac{e^{i\sqrt{-\omega+i\lambda}(|x|+|y|)}}{(4\pi)^2|x||y|}.$$

Denote

$$D_\pm(\lambda^2) = D((\lambda \pm 0)^2).$$

Then it follows

$$\int_{\mathcal{C}^+}[(R_m(\lambda+0) - R_m(\lambda-0))(J - iI)]_{1,1}d\lambda =$$

$$= \int_{\mathcal{C}^+}\frac{\sigma\sqrt{\omega}e^{i\sqrt{-\omega+i\lambda}|y|}e^{-i\sqrt{-\omega-i\lambda}|x|} + ((\sigma+1)\sqrt{\omega} - i\sqrt{-\omega-i\lambda})e^{i\sqrt{-\omega+i\lambda}(|x|+|y|)}}{\pi|x||y|D_+(\lambda^2)}d\lambda +$$

$$- \int_{\mathcal{C}^+}\frac{\sigma\sqrt{\omega}e^{i\sqrt{-\omega+i\lambda}|y|}e^{i\sqrt{-\omega-i\lambda}|x|} + ((\sigma+1)\sqrt{\omega} + i\sqrt{-\omega-i\lambda})e^{i\sqrt{-\omega+i\lambda}(|x|+|y|)}}{\pi|x||y|D_-(\lambda^2)}d\lambda.$$

If we compute the change of variable $k = \sqrt{-\omega - i\lambda}$ in the first integral of the last equality, and $k = -\sqrt{-\omega - i\lambda}$ in the second one, then one has

$$\left|\int_{\mathcal{C}^+}[(R_m(\lambda+0) - R_m(\lambda-0))(J - iI)]_{1,1}d\lambda\right| =$$

$$= \left| \frac{4i}{\pi|x||y|} \left(\int_{-\infty}^{+\infty} \sigma\sqrt{\omega}k \frac{e^{-\sqrt{k^2+2\omega}|y|}}{D(k)} e^{-ik|x|} dk + \int_{-\infty}^{+\infty} 2ik^2 \frac{e^{-\sqrt{k^2+2\omega}(|x|+|y|)}}{D(k)} dk \right) \right| \leq$$

$$\leq C \frac{e^{-\sqrt{2\omega}|y|}}{|y||x|} \min\left\{ \frac{1}{|x|}, e^{-\sqrt{2\omega}|x|} \right\} \leq C \frac{e^{-\sqrt{2\omega}|y|}}{|y|} \frac{e^{-\sqrt{2\omega}|x|}}{|x|},$$

where the first inequality is obtained integrating by parts both integrals.
The integral of the other three elements of the matrix operator $(R_m(\lambda+0) - R_m(\lambda-0))(J - iI)$ can be estimated in the same way and this implies the statement of the lemma. □

Part II

Nondispersive vanishing and blow up at infinity for the energy critical nonlinear Schrödinger equation in $\mathbb{R}^3$

Chapter 3

Nondispersive vanishing and blow up at infinity for the energy critical nonlinear Schrödinger equation in $\mathbb{R}^3$

3.1 Introduction

3.1.1 Setting of the problem and statement of the result

In this chapter we consider the energy critical focusing nonlinear Schrödinger equation

$$\begin{aligned} i\frac{du}{dt} &= -\Delta u - |u|^4 u, \quad x \in \mathbb{R}^3, \\ u(0) &= u_0 \in \dot{H}^1(\mathbb{R}^3). \end{aligned} \tag{3.1}$$

The Cauchy problem (3.1) is locally well posed, which means that for any initial datum $u_0 \in \dot{H}^1(\mathbb{R}^3)$ there exists a unique solution u defined on a maximal interval of definition $I = (T_-, T_+)$ such that $u \in C(I, \dot{H}^1(\mathbb{R}^3)) \cap L^{10}(\mathcal{I} \times \mathbb{R}^3)$ for any compact interval $\mathcal{I} \subset I$. If $T_+ < +\infty$ (or $T_- > -\infty$), then $\|u\|_{L^{10}((0,T_+)\times\mathbb{R}^3)} = +\infty$ (respectively $\|u\|_{L^{10}((T_-,0)\times\mathbb{R}^3)} = +\infty$), and one says that the solution blows up in finite time. Moreover, the solutions during their life span satisfy conservation of energy:

$$E(u(t)) \equiv \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t,x)|^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |u(t,x)|^6 \, dx = E(u_0). \tag{3.2}$$

The problem is energy critical in the sense that (3.1) as well as (3.2) are invariant with respect to the scaling $u(t,x) \to \lambda^{1/2} u(\lambda x, \lambda^2 t)$, $\lambda \in \mathbb{R}_+$. For $\dot{H}^1$ small data one has global existence and scattering. In the case of large data blow up may occur. Indeed, the classical virial identity

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^3} |x|^2 |u(t,x)|^2 dx = 8E(u) - \frac{16}{3}\int_{\mathbb{R}^3} |u(t,x)|^6 dx$$

shows that if $xu_0 \in L^2(\mathbb{R}^3)$ and $E(u_0) < 0$, the solution has to break down in finite time. Furthermore, Equation (3.1) admits an explicit stationary solution (ground state):

$$W(x) = (1 + \frac{1}{3}|x|^2)^{-1/2}, \quad \Delta W + W^5 = 0,$$

so that scattering cannot always occur even for solutions that exist globally in time.

The ground state W is known to play an important role in the dynamics of (3.1). It was proved by Kenig and Merle [31] that $E(W)$ is an energy threshold for the dynamics in the following sense. If u_0 is radial and $E(u_0) < E(W)$ then

(i) the solution of (3.1) is global and scatters to zero as a free wave in both directions, provided $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$;

(ii) the solution blows up in finite time in both direction, provided $u_0 \in L^2$ and $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$.

The behavior of radial solutions with critical energy $E(u_0) = E(W)$ was classified by Duyckaerts and Merle in [19]. In this case, in addition to the finite time blow up and scattering to zero, one has the existence of solutions that converge as $t \to \infty$ to a rescaled ground state. In the case of energy slightly greater than $E(W)$ the dynamics is expected to be richer and to include the solutions that as $t \to \infty$ behave like $e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x)$ with fairly general $\alpha(t)$ and $\lambda(t)$. For a closely related model of the critical wave equation, the existence of this type of solutions with $\lambda(t) \to \infty$ (blow up at infinity) and $\lambda(t) \to 0$, $t\lambda(t) \to \infty$ (non-dispersive vanishing) was recently proved by Donninger and Krieger [17]. Our objective in this chapter is to obtain an analogous result for NLS (3.1). More precisely, we prove the following.

Theorem 3.1. *There exists $\beta_0 > 0$ such that for any ν, $\alpha_0 \in \mathbb{R}$ with $|\nu| + |\alpha_0| \leq \beta_0$ and any $\delta > 0$ there exist $T > 0$ and a radial solution $u \in C([T, +\infty), \dot{H}^1 \cap \dot{H}^2)$ to (3.1) of the form:*

$$u(t, x) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + \zeta(t, x), \tag{3.3}$$

where $\lambda(t) = t^\nu$, $\alpha(t) = \alpha_0 \ln t$, and $\zeta(t)$ verifies:

$$\begin{aligned} &\|\zeta(t)\|_{\dot{H}^1\cap\dot{H}^2} \leq \delta, \\ &\|\zeta(t)\|_{L^\infty} \leq Ct^{-\frac{1+\nu}{2}}, \\ &\| < \lambda(t)x >^{-1} \zeta(t)\|_{L^\infty} \leq Ct^{-1-\frac{3}{2}\nu}, \end{aligned} \tag{3.4}$$

for all $t \geq T$. The constants C here and below are independent of ν, α_0 and δ.
Furthermore, there exists $\zeta^* \in \dot{H}^s$, $\forall s > \frac{1}{2} - \nu$, such that, as $t \to +\infty$, $\zeta(t) - e^{it\Delta}\zeta^* \to 0$ in $\dot{H}^1 \cap \dot{H}^2$.

In order to prove the main Theorem 3.1, in Section 2 we construct (Proposition 3.2) a sufficiently good approximate solution of (3.1) very much in the spirit of [17], [35], [39]. In Section 3 we build up an exact solution by solving the problem for the small remainder with zero initial data at infinity, the main technical tool of the construction being some suitable energy type estimates for the linearized evolution. These estimates are proved in Section 4.

3.2 Approximate solutions

In this section we prove the following result.

Proposition 3.2. *For any ν and α_0 sufficiently small and any $0 < \delta \leq 1$ there exists a radial approximate solution $u^{ap} \in C^\infty(\mathbb{R}^3, \mathbb{R}_+^*)$ of (3.1) such that the following holds for $t \geq T$ with some $T = T(\nu, \alpha_0, \delta) > 0$.*

(i) u^{ap} has the form: $u^{ap}(t,x) = e^{i\alpha(t)}\lambda^{1/2}(t)(W(\lambda(t)x)+\chi^{ap}(t,\lambda(t)x))$, where $\chi^{ap}(t,y)$, $y=\lambda(t)x$, verifies

$$\|\chi^{ap}(t)\|_{\dot H^k} \le C\delta^{\nu+k-1/2}t^{-\nu(k-1)}, \quad k=1,2, \tag{3.5}$$

$$\|\chi^{ap}(t)\|_{L^\infty} \le Ct^{-(1+2\nu)/2}, \tag{3.6}$$

$$\||y|^{-1}\chi^{ap}(t)\|_{L^\infty} + \|\nabla\chi^{ap}(t)\|_{L^\infty} \le Ct^{-1-2\nu}, \tag{3.7}$$

$$\||y|^{-2}\chi^{ap}(t)\|_{L^\infty} + \||y|^{-1}\nabla_y\chi^{ap}(t)\|_{L^\infty} \le C(|\nu|+|\alpha_0|)t^{-1-2\nu}, \tag{3.8}$$

$$\|\nabla^2\chi^{ap}(t)\|_{L^\infty} \le C(|\nu|+|\alpha_0|)t^{-1-2\nu}. \tag{3.9}$$

Furthermore, there exists $\zeta^* \in \dot H^s$, for any $s > \frac{1}{2}-\nu$, such that, as $t\to+\infty$, $e^{i\alpha(t)}\lambda^{1/2}(t)\chi^{ap}(t,\lambda(t)\cdot) - e^{it\Delta}\zeta^* \to 0$ in $\dot H^1 \cap \dot H^2$.
(ii) The corresponding error $R = -i\frac{du^{ap}}{dt} - \Delta u^{ap} - |u^{ap}|^4u^{ap}$ satisfies

$$\|R(t)\|_{\dot H^k} \le t^{-(2+\frac{1}{8})(1+2\nu)+\nu(k+1)}, \quad k=0,1,2. \tag{3.10}$$

The construction of $u^{ap}(t)$ will be achieved by considering separately the three regions that correspond to three different space scales: the inner region with the scale $t^\nu|x| \lesssim 1$, the self-similar region where $|x| = O(t^{1/2})$, and, finally, the remote region where $|x| = O(t)$. In the inner region the solution will be constructed as a perturbation of the profile $e^{i\alpha_0\ln t}t^{\nu/2}W(t^\nu x)$. The self-similar and remote regions are the regions where the solution is small and is described essentially by the linear equation $i\frac{du}{dt} = -\triangle u$. In the self-similar region the profile of the solution will be determined uniquely by the matching conditions coming out from the inner region, while in the remote region the profile remains essentially a free parameter of the construction, only the limiting behavior at the origin is prescribed by the matching procedure.

3.2.1 The inner region

We start by considering the inner region $0 \le t^\nu|x| \le 10t^{1/2+\nu-\epsilon_1}$ with $0<\epsilon_1<1/2+\nu$ to be fixed later. Writing $u(t,x)$ as $u(t,x) = e^{i\alpha(t)}\lambda^{1/2}(t)\psi(t,\rho)$, $\rho = \lambda(t)|x|$, we get from (3.1)

$$it^{-2\nu}\frac{d\psi}{dt} - \alpha_0t^{-(1+2\nu)}\psi + i\nu t^{-(1+2\nu)}(\frac{1}{2}+\rho\partial_\rho)\psi = -\triangle\psi - |\psi|^4\psi. \tag{3.11}$$

Write $\psi(t,\rho) = W(\rho)+\chi(t,\rho)$. Then $\vec\chi(t) = \binom{\chi(t)}{\bar\chi(t)}$ solves

$$it^{-2\nu}\frac{d\vec\chi}{dt} = H\vec\chi + \mathcal{N}(\chi), \tag{3.12}$$

where

$$H = -\triangle\sigma_3 - 3W^4\sigma_3 - 2W^4\sigma_3\sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathcal{N}(\chi) = \begin{pmatrix} N(\chi) \\ -\overline{N(\chi)} \end{pmatrix}, \quad N(\chi) = N_0 + N_1(\chi) + N_2(\chi),$$

$$N_0 = \alpha_0t^{-(1+2\nu)}W - i\nu t^{-(1+2\nu)}W_1, \quad W_1(\rho) = (\frac{1}{2}+\rho\partial_\rho)W(\rho)$$

$$N_1(\chi) = \alpha_0t^{-(1+2\nu)}\chi - i\nu t^{-(1+2\nu)}(\frac{1}{2}+\rho\partial_\rho)\chi,$$

$$N_2(\chi) = -|W+\chi|^4(W+\chi) + W^5 + 3W^4\chi + 2W^4\overline{\chi}.$$

We look for a solution to (3.12) of the form

$$\chi(t,\rho)=\sum_{k=1}^{\infty}t^{-k(1+2\nu)}\chi_k(\rho). \tag{3.13}$$

Substituting (3.13) into (3.12) and identifying the terms with the same powers of t we get the following system for $\{\chi_k\}_{k\geq 1}$:

$$H\vec{\chi}_k=\mathcal{D}_k, \quad k\geq 1, \tag{3.14}$$

where $\mathcal{D}_k=\begin{pmatrix} D_k \\ -\overline{D_k}\end{pmatrix}$,

$$\begin{aligned} D_1 &= -\alpha_0 W + i\nu W_1,\\ D_k &= D_k^{(1)}+D_k^{(2)}, \quad k\geq 2, \end{aligned}$$

$D_k^{(1)}$ and $D_k^{(2)}$ being contributions of $it^{-2\nu}\frac{d\chi}{dt}-N_1(\chi)$ and $-N_2(\chi)$ respectively:

$$\begin{aligned} D_k^{(1)} &= -i(1+2\nu)(k-1)\chi_{k-1}-\alpha_0\nu\chi_{k-1}+i\nu(\frac{1}{2}+\rho\partial_\rho)\chi_{k-1},\\ N_2(\chi) &= -\sum_{k=2}^{\infty}t^{-k(1+2\nu)}D_k^{(2)}(\rho). \end{aligned}$$

Note that D_k depends on χ_p, $1\leq p\leq k-1$ only:

$$D_k=D_k(\rho;\chi_p, 1\leq p\leq k-1).$$

We subject (3.14) to zero initial conditions at 0: $\chi_k(0)=\partial_\rho\chi_k(0)=0$.

Lemma 3.3. *System* (3.14) *has a unique solution* $\{\chi_k\}_{k\geq 1}$ *verifying:*
i) for any $k\geq 1$, χ_k *is a* C^∞ *function that has an even Taylor expansion at* $\rho=0$ *that starts at order* $2k$*;*
ii) as $\rho\to+\infty$, χ_k, $k\geq 1$, *has the following asymptotic expansion*

$$\chi_k(\rho)=\sum_{l=0}^{k}\sum_{j\leq 2k-2l-1}\alpha_{l,j}^{(k)}(\ln\rho)^l\rho^j, \tag{3.15}$$

with some coefficients $\alpha_{l,j}^{(k)}$ *verifying* $\alpha_{k,2m}^{(k)}=0$ *for all* k,m*. The asymptotic expansion* (3.15) *can be differentiated any number of times with respect to* ρ*.*

Proof. It will be convenient for us to rewrite (3.14) as

$$L_+v_k^+=G_k^+, \quad L_-v_k^-=G_k^-, \quad k\geq 1, \tag{3.16}$$

where

$$\begin{aligned} v_k^+ &= \operatorname{Re}\chi_k, & v_k^- &= \operatorname{Im}\chi_k,\\ G_k^+ &= \operatorname{Re}D_k, & G_k^- &= \operatorname{Im}D_k,\\ L_+ &= -\triangle-5W^4, & L_- &= -\triangle-W^4. \end{aligned}$$

For $k=1$ (3.16) gives

$$L_+v_1^+=-\alpha_0 W, \quad L_-v_1^-=\nu W_1. \tag{3.17}$$

The homogeneous equation $L_\pm f = 0$ has two explicit solutions $\Phi_\pm$, $\Theta_\pm$ given by

$$\Phi_-(\rho) = W(\rho), \quad \Theta_-(\rho) = \left(1+\frac{\rho^2}{3}\right)^{-1/2}\left(\frac{\rho}{3}-\frac{1}{\rho}\right),$$
(3.18)
$$\Phi_+(\rho) = W_1(\rho), \quad \Theta_+(\rho) = -2\left(1+\frac{\rho^2}{3}\right)^{-3/2}\left(\frac{1}{\rho}-2\rho+\frac{\rho^3}{9}\right).$$

Therefore, solving (3.17) with zero initial conditions at the origin we obtain

$$\begin{aligned} v_1^+(\rho) &= \alpha_0 \textstyle\int_0^\rho s^2(\Theta_+(\rho)\Phi_+(s) - \Theta_+(s)\Phi_+(\rho))W(s)ds \\ v_1^-(\rho) &= -\nu \textstyle\int_0^\rho s^2(\Theta_-(\rho)\Phi_-(s) - \Theta_-(s)\Phi_-(\rho))W_1(s)ds. \end{aligned} \tag{3.19}$$

Since W, W_1 are C^∞ even functions, v_1^+ and v_1^- are also C^∞ functions with even Taylor expansion at $\rho = 0$ that starts at order 2. Furthermore, the asymptotic expansions of v_1^+ and v_1^- as $\rho \to +\infty$ can be obtained directly from (3.19). As claimed, one has

$$v_1^+(\rho) + iv_1^-(\rho) = \sum_{j\le 1} \alpha_{0,j}^{(1)}\rho^j + \sum_{j\le 0}\alpha_{1,j}^{(1)}\rho^{2j-1}\ln\rho, \qquad \text{as } \rho \to +\infty.$$

We next proceed by induction. Let us consider $k > 1$ and assume that we have found χ_i, $i = 1, \cdots, k-1$, that verify i), ii).Then one can easily check that D_k is an even C^∞ function with a Taylor series at 0 starting at order $2(k-1)$ and as $\rho \to +\infty$, D_k admits an asymptotic expansion of the form

$$D_k(\rho) = \sum_{l=0}^{k-1}\sum_{j\le 2k-2l-3} d_{j,l}^{(k)}(\ln\rho)^l\rho^j + (\ln\rho)^k\sum_{j\le -5} d_{j,k}^{(k)}\rho^j,$$

where $d_{-2,k-1}^{(k)} = 0$ and $d_{2m,k}^{(k)} = 0$, $\forall m$. Therefore, solving $L_\pm v_k^\pm = G_k^\pm$ with zero conditions at $\rho = 0$ we get a C^∞ even solution $v_k^\pm$ which is $O(\rho^{2k})$ at the origin. Finally, the asymptotic expansion at infinity follows directly from the representation

$$v_k^\pm(\rho) = -\int_0^\rho s^2(\Theta_\pm(\rho)\Phi_\pm(s) - \Theta_\pm(s)\Phi_\pm(\rho))G_k^\pm(s)ds.$$

□

Remark 3.4. Clearly, for any k, χ_k is a polynomial with respect to α_0 and ν of the form

$$\chi_k = \sum_{1\le m+n\le k} \alpha_0^m \nu^n \chi_{m,n}^k(\rho),$$

where the coefficients $\chi_{m,n}^k$ are C^∞ functions of ρ with an even Taylor expansion at 0 that starts at order $2k$. As $\rho \to +\infty$, $\chi_{m,n}^k$, admits an asymptotic expansion of the form (3.15).

For any $N \ge 2$, define

$$\chi^{(N)}(t,\rho) = \sum_{k=1}^N t^{-k(1+2\nu)}\chi_k(\rho).$$

It follows from our construction that $\chi^{(N)}$ verifies,

$$\left|\rho^{-k}\partial_\rho^l(-it^{-2\nu}\frac{d\vec{\chi}^{(N)}}{dt} + H\vec{\chi}^{(N)} + \mathcal{N}(\chi^{(N)}))\right| \le C_{N,l,k}t^{-(N+1)(1+2\nu)} < \rho >^{2N-1-l-k}, \tag{3.20}$$

for any $k,l\in\mathbb{N}$, $k+l\le 2N$, $0\le\rho\le 10t^{\frac12+\nu-\epsilon_1}$, $t\ge 1$.
Fix $N=27$, $\epsilon_1=\frac{1+2\nu}{27}$[1] and set

$$\psi_{in}^{ap}=W+\chi_{in}^{ap},\quad \chi_{in}^{ap}=\chi^{(27)},$$
$$\mathcal{R}_{in}=-it^{-2\nu}\frac{d\psi_{in}^{ap}}{dt}-\triangle\psi_{in}^{ap}+\alpha_0t^{-1-2\nu}\psi_{in}^{ap}-i\nu t^{-1-2\nu}(\frac12+\rho\partial_\rho)\psi_{in}^{ap}-|\psi_{in}^{ap}|^4\psi_{in}^{ap}.$$

As a direct consequence of Lemma 3.3 and estimate (3.20) we obtain the following result.

Lemma 3.5. *For any $\alpha_0\in\mathbb{R}$ and any $\nu>-\frac12$ there exists $T=T(\alpha_0,\nu)>0$ such that for $t\ge T$ the following holds.*
(i) The profile $\chi_{in}^{ap}(t)$ verifies

$$\|\chi_{in}^{ap}\|_{L^\infty(0\le\rho\le 10t^{\frac12+\nu-\epsilon_1})}\le C(|\nu|+|\alpha_0|)t^{-\frac12-\nu}, \tag{3.21}$$

$$\|\rho^{-k}\partial_\rho^l\chi_{in}^{ap}\|_{L^\infty(0\le\rho\le 10t^{\frac12+\nu-\epsilon_1})}\le C(|\nu|+|\alpha_0|)t^{-1-2\nu},\quad 1\le k+l\le 2, \tag{3.22}$$

$$\|\rho^{-k}\partial_\rho^l\chi_{in}^{ap}\|_{L^2(\rho^2d\rho,0\le\rho\le 10t^{\frac12+\nu-\epsilon_1})}\le C(|\nu|+|\alpha_0|)t^{-(\frac12+\nu)(k+l-\frac12)},\ k+l\le 2. \tag{3.23}$$

(ii) The error $\mathcal{R}_{in}(t)$ admits the estimate

$$\left\|\rho^{-k}\partial_\rho^l\mathcal{R}_{in}(t)\right\|_{L^2(\rho^2d\rho,0\le\rho\le 10t^{\frac12+\nu-\epsilon_1})}\le t^{-3(1+2\nu)/4-\varepsilon_1(2N+1/2)},\quad k+l\le 2. \tag{3.24}$$

3.2.2 The self-similar region

We next consider the self-similar region $\frac{1}{10}t^{-\varepsilon_1}\le|x|t^{-1/2}\le 10t^{\varepsilon_2}$, where $0<\varepsilon_2<1/2$ to be fixed later. Write $u(t,x)=e^{i\alpha_0\ln t}t^{-1/4}w(t,y)$, $y=t^{-1/2}|x|$. Then, $w(t)$ solves

$$it\frac{dw}{dt}=(\mathcal{L}+\alpha_0)w-|w|^4w, \tag{3.25}$$

where $\mathcal{L}=-\triangle+\frac{i}{2}\left(\frac12+y\partial_y\right)$.
Note that in the limit $\rho\to+\infty$, $y\to 0$ one has, at least, formally

$$\begin{aligned}&t^{\nu/2}(W(\rho)+\sum_{k\ge1}t^{-k(1+2\nu)}\chi_k(\rho))=\\&t^{-1/4}\sum_{n\ge0}\sum_{0\le l\le\frac n2}t^{-\frac14(2n+1)(1+2\nu)}(\ln y+(\frac12+\nu)\ln t)^l\sum_{k\ge l}\alpha_{l,2k-n-1}^{(k)}y^{2k-n-1},\end{aligned} \tag{3.26}$$

where $\alpha_{l,j}^{(k)}$, $k\ne0$, are given by Lemma 3.3 and $\alpha_{l,j}^{(0)}$ come from the expansion of $W(\rho)$ as $\rho\to\infty$:

$$W(\rho)=\sum_{j\le0}\alpha_{0,j}^{(0)}\rho^j,\quad \alpha_{0,2m}^{(0)}=0\ \ \forall m\in\mathbb{Z}.$$

Equation (3.26) suggests the following ansatz for w:

$$w(t,y)=\sum_{n\ge0}\sum_{0\le l\le\frac n2}t^{-\frac14(2n+1)(1+2\nu)}(\ln y+(\frac12+\nu)\ln t)^lA_{n,l}(y). \tag{3.27}$$

[1]This choice has no specific meaning here. To produce an approximate solution with an error verifying (3.10) it is sufficient to require $(2N+3)\varepsilon_1>3(1+2\nu)/2$, $0<\varepsilon_1<\frac{1+2\nu}{20}$, see (3.24) and (3.45), (3.46).

As it will become clear later, to prove Proposition 3.2, it is sufficient to consider only three first terms of expansion (3.27). Therefore, we look for an approximate solution of the form

$$w_{ss}^{ap}(t,y) = t^{-(1+2\nu)/4}A_{0,0}(y) + t^{-3(1+2\nu)/4}A_{1,0}(y) + t^{-5(1+2\nu)/4}\big(A_{2,0}(y) + (\ln y + (\frac{1}{2}+\nu)\ln t)A_{2,1}(y)\big).$$

Substituting this ansatz into the expression $-it\frac{w}{dt} + (\mathcal{L}+\alpha_0)w - |w|^4 w$ one gets

(3.28)
$$-it\frac{dw_{ss}^{ap}}{dt} + (\mathcal{L}+\alpha_0)w_{ss}^{ap} - |w_{ss}^{ap}|^4 w_{ss}^{ap} = t^{-(1+2\nu)/4}S_{0,0}(y) + t^{-3(1+2\nu)/4}S_{1,0}(y) + t^{-5(1+2\nu)/4}(S_{0,0}(y) + (\ln y + (\frac{1}{2}+\nu)\ln t)S_{2,1}(y)) + S(t,y),$$

where

$$S_{n,0}(y) = (\mathcal{L}+\mu_n)A_{n,0}(y), \quad n = 0,1,$$
$$S_{2,1}(y) = (\mathcal{L}+\mu_2)A_{2,1}(y),$$
$$S_{2,0}(y) = (\mathcal{L}+\mu_2)A_{2,0}(y) - i\nu A_{2,1}(y) - \tfrac{2}{y}\partial_y A_{2,1}(y) - \tfrac{A_{2,1}(y)}{y^2} - |A_{0,0}(y)|^4 A_{0,0}(y),$$
$$S(t,y) = -|w_{ss}^{ap}(t,y)|^4 w_{ss}^{ap}(t,y) + t^{-5(1+2\nu)/4}|A_{0,0}(y)|^4 A_{0,0}(y).$$

Here $\mu_n = \alpha_0 + \frac{i}{4}(2n+1)(1+2\nu)$.
We require that $S_{n,l} = 0$, $n = 0,1,2$, $l = 0,1$, which means that the corresponding $A_{n,l}$ have to solve

(3.29)
$$\begin{cases} (\mathcal{L}+\mu_n)A_{n,0} = 0, \quad n = 0,1, \\ (\mathcal{L}+\mu_2)A_{2,1} = 0, \\ (\mathcal{L}+\mu_2)A_{2,0} = i\nu A_{2,1} + \frac{2}{y}\partial_y A_{2,1} + \frac{A_{2,1}}{y^2} + |A_{0,0}|^4 A_{0,0} \end{cases}.$$

In addition, in order to have the matching with the inner region, $A_{n,l}$ have to satisfy

(3.30)
$$A_{n,l}(y) = \sum_{k\geq l} \alpha_{l,2k-n-1}^{(k)} y^{2k-n-1}, \quad y \to 0.$$

Lemma 3.6. *There exists a unique solution of* (3.29) *that as* $y \to 0$ *admits an asymptotis expansion of the form*

(3.31)
$$A_{n,l}(y) = \sum_{k\geq l} d_{n,k,l} y^{2k-n-1},$$

with $d_{0,0,0} = \alpha_{0,-1}^{(0)}$, $d_{1,1,0} = \alpha_{0,0}^{(1)}$ *and* $d_{2,1,0} = \alpha_{0,-1}^{(1)}$.

Proof. First of all note that the equation $(\mathcal{L}+\mu)f = 0$ has a basis of solutions $e_1(y,\mu)$, $e_2(y,\mu)$ such that:
(i) $e_1(y,\mu) = \frac{1}{y} + (\mu - \frac{i}{4})\tilde{e}_1(y,\mu)$, where $\tilde{e}_1$ is an entire function of y and μ, odd with respect to y;
(ii) e_2 is a entire function of y and μ, even with respect to y, and as $y \to 0$, $e_2(y,\mu) = 1 + O(y^2)$.
Two first equations of (3.29) together with (3.31) give

(3.32)
$$A_{0,0}(y) = \alpha_{0,-1}^{(0)} e_1(y,\mu_0), \quad A_{1,0}(y) = \alpha_{0,0}^{(1)} e_2(y,\mu_1).$$

We next consider the remaining equations of (3.29). Equation $(\mathcal{L}+\mu_2)A_{2,1}(y) = 0$ and (3.31) yield $A_{2,1}(y) = c_0 e_1(y,\mu_2)$, with some constant c_0. Then, for $A_{2,0}$ we have $(\mathcal{L}+\mu_2)A_{2,0} = F$, where

$$F = c_0(i\nu + \frac{2}{y}\partial_y + y^{-2})e_1(y,\mu_2) + |A_{0,0}|^4 A_{0,0}.$$

As $y \to 0$, F has an asymptotic expansion of the form

$$F(y) = \sum_{i \geq -2} \kappa_i y^{2i-1},$$

with some coefficients κ_i, κ_{-2} and $\kappa_{-1} + c_0$ being independent of c_0.
Write $A_{2,0}(y) = -\frac{\kappa_{-2}}{6y^3} + \widetilde{A}_{2,0}(y)$. Then $\widetilde{A}_{2,0}$ solves

$$(\mathcal{L} + \mu_2)\widetilde{A}_{2,0} = \widetilde{F}, \tag{3.33}$$

where $\widetilde{F} = F + \frac{\kappa_{-2}}{6}(\mathcal{L} + \mu_2)\frac{1}{y^3}$ has the following asymptotics as $y \to 0$:

$$\widetilde{F}(y) = \sum_{i \geq -1} \tilde{\kappa}_i y^{2i-1}, \quad \tilde{\kappa}_{-1} = \tilde{\kappa}^0_{-1} + c_0,$$

with κ^0_{-1} independent of c_0. Take $c_0 = -\kappa^0_{-1}$. Then Equation (3.33) has a unique solution of the form

$$\widetilde{A}_{2,0}(y) = \alpha^{(1)}_{0,-1} e_1(y, \mu_2) + \text{a } C^\infty \text{odd function}.$$

□

Remark 3.7. By uniqueness, $A_{n,l}$ given by Lemma 3.6 verify matching conditions (3.30). Note also that all $A_{n,l}$ are entire functions of α_0 and ν.

We next study the behavior of $A_{n,l}$ as $y \to +\infty$. To this purpose notice that for any $\mu \in \mathbb{C}$, equation $(\mathcal{L} + \mu)f = 0$ has a basis of solutions $f_1(y, \mu)$, $f_2(y, \mu)$ such that yf_1, yf_2 are smooth functions in both variables and as $y \to +\infty$ one has

$$f_1(y, \mu) = y^{-1/2+2i\mu}(1 + O(y^{-2})), \;\; f_2(y, \mu) = e^{i\frac{y^2}{4}} y^{-5/2-2i\mu}(1 + O(y^{-2})). \tag{3.34}$$

These asymptotics are uniform in μ on compact subsets of $\mathbb{C}$ and can be differentiated any number of times with respect to y.
Decomposing $A_{1,0}$, $A_{2,0}$, $A_{2,1}$ in the basis f_1, f_2 one gets

$$\begin{aligned} A_{n,0}(y) &= d^n_1 f_1(y, \mu_n) + d^n_2 f_2(y, \mu_n), \quad n = 0, 1, \\ A_{2,1}(y) &= d^2_1 f_1(y, \mu_2) + d^2_2 f_2(y, \mu_2), \end{aligned} \tag{3.35}$$

with some coefficients d^n_j, $j = 1, 2$, $n = 0, 1, 2$. As a consequence, as $y \to +\infty$, one has

$$\begin{aligned} A_{0,0}(y) &= d^0_1 y^{2i\alpha_0 - 1 - \nu}(1 + O(y^{-2})) + d^0_2 e^{iy^2/4} y^{-2i\alpha_0 - 2 + \nu}(1 + O(y^{-2})), \\ A_{1,0}(y) &= d^1_1 y^{2i\alpha_0 - 2 - 3\nu}(1 + O(y^{-2})) + d^1_2 e^{iy^2/4} y^{-2i\alpha_0 - 1 + 3\nu}(1 + O(y^{-2})), \\ A_{2,1}(y) &= d^2_1 y^{2i\alpha_0 - 3 - 5\nu}(1 + O(y^{-2})) + d^2_2 e^{iy^2/4} y^{-2i\alpha_0 + 5\nu}(1 + O(y^{-2})). \end{aligned} \tag{3.36}$$

Asymptotics (3.36) can be differentiated any number of times with respect to y.
Let us now consider $A_{2,0}$ and write it as

$$A_{2,0}(y) = 2d^2_1 \nu \ln y f_1(y, \mu_2) - 2(\nu + 1)d^2_2 \ln y f_2(y, \mu_2) + \widehat{A}_{2,0}(y). \tag{3.37}$$

Then $\widehat{A}_{2,0}(y)$ solves

$$(\mathcal{L} + \mu_2)\widehat{A}_{2,0} = G, \tag{3.38}$$

with $G = d_2^2 G_1 + G_2$, where

$$G_1 = -d_2^2(1+2\nu)(2y^{-1}\partial_y + y^{-2} - i)f_2(y,\mu_2),$$
$$G_2 = |A_{0,0}|^4 A_{0,0} + d_1^2(1+2\nu)(2y^{-1}\partial_y + y^{-2})f_1(y,\mu_2).$$

It follows from the asymptotics (3.34), (3.36) that G_j, $j=1,2$, has the following behavior as $y \to +\infty$,

$$G_1(y) = e^{iy^2/4} y^{-2i\alpha_0} G_{1,1}(y), \quad G_2(y) = \sum_{m=-2}^{3} e^{imy^2/4} y^{-2i\alpha_0\nu(2m-1)} G_{2,m}(y),$$
$$\partial_y^l G_{1,1}(y) = O(y^{-2+5\nu-l}),$$
$$\partial_y^l G_{2,m}(y) = O(y^{-5-5\nu-|m|(1-2\nu)-l}), \quad -2 \le m \le 3,$$

for any $l \ge 0$, provided ν is sufficiently small.
Integrating (3.38) one gets

$$\widehat{A}_{2,0}(y) = \lambda_1 f_1(y,\mu_2) + \lambda_2 f_2(y,\mu_2) + d_2^2 g_1(y) + g_2(y). \tag{3.39}$$

Here λ_i, $i=1,2$, is a constant and g_i, $i=1,2$, is the solution of $(\mathcal{L}+\mu_2)g_i = G_i$, with the following behavior as $y \to +\infty$:

$$\begin{aligned} &g_1(y) = e^{iy^2/4} y^{-2i\alpha_0} g_{1,1}(y),\\ &g_2(y) = \textstyle\sum_{m=-2}^{3} e^{imy^2/4} y^{-2i\alpha_0\nu(2m-1)} g_{2,m}(y),\\ &\partial_y^l g_{1,1}(y) = O(y^{-2+5\nu-l}),\\ &\partial_y^l g_{2,m}(y) = O(y^{-5-5\nu-m(1-2\nu)-l}), \quad m=0,1\\ &\partial_y^l g_{2,m}(y) = O(y^{-7-5\nu-|m|(1-2\nu)-l}), \quad m=-2,-1,2,3, \end{aligned} \tag{3.40}$$

for any $l \ge 0$.
Denote

$$\psi_{ss}^{ap}(t,\rho) = t^{-(1+2\nu)/4} w_{ss}^{ap}(t, t^{-(1+2\nu)/2}\rho),$$
$$\chi_{ss}^{ap}(t,\rho) = \psi_{ss}^{ap}(t,\rho) - W(\rho),$$
$$\mathcal{R}_{ss}(t,\rho) = t^{-5(1+2\nu)/4} S(t^{-(1+2\nu)/2}\rho, t).$$

The next lemma is a direct consequence of (3.30), (3.34), (3.36), (3.37), (3.39) and (3.40).

Lemma 3.8. *For any $\alpha_0, \nu \in \mathbb{R}$ sufficiently small there exists $T(\alpha_0,\nu) > 0$ such that for $t \ge T(\alpha_0,\nu)$ the following holds.*
(i) $\chi_{ss}^{ap}(t)$ verifies

$$\|\chi_{ss}^{ap}(t)\|_{L^\infty(\frac{1}{10}t^{\frac12+\nu-\epsilon_1} \le \rho \le 10 t^{\frac12+\nu+\epsilon_2})} \le C t^{-\frac12-\nu}, \tag{3.41}$$

$$\|\rho^{-k}\partial_\rho^l \chi_{ss}^{ap}(t)\|_{L^\infty(\frac{1}{10}t^{\frac12+\nu-\epsilon_1} \le \rho \le 10 t^{\frac12+\nu+\epsilon_2})} \le C t^{-1-2\nu}, \quad k+l=1, \tag{3.42}$$

$$\|\rho^{-k}\partial_\rho^l \chi_{ss}^{ap}(t)\|_{L^\infty(\frac{1}{10}t^{\frac12+\nu-\epsilon_1} \le \rho \le 10 t^{\frac12+\nu+\epsilon_2})} \le C(|\alpha_0|+|\nu|) t^{-1-2\nu}, \quad k+l=2, \tag{3.43}$$

$$\|\rho^{-k}\partial_\rho^l \chi_{ss}^{ap}(t)\|_{L^2(\rho^2 d\rho, \frac{1}{10}t^{\frac12+\nu-\epsilon_1} \le \rho \le 10 t^{\frac12+\nu+\epsilon_2})} \le C t^{-(1+2\nu)(1-2\varepsilon_2)/4}, \quad 1 \le k+l \le 2, \tag{3.44}$$

(ii) The error $\mathcal{R}_{ss}(t)$ admits the estimate

$$\|\rho^{-k}\partial_\rho^l \mathcal{R}_{ss}(t)\|_{L^2(\rho^2 d\rho, \frac{1}{10}t^{\frac12+\nu-\epsilon_1} \le \rho \le 10 t^{\frac12+\nu+\epsilon_2})} \le C t^{-(2+\frac14)(1+2\nu)+5\varepsilon_1/2}, \quad 0 \le k+l \le 2. \tag{3.45}$$

(iii) The difference $\psi_{in}^{ap}(\rho,t)-\psi_{ss}^{ap}(t,\rho)$ *verifies*

$$|\partial_\rho^l(\psi_{in}^{ap}(t)-\psi_{ss}^{ap}(t))|\le C\rho^{-2-l}t^{-(1+2\nu)}(\ln t+t^{3(1+2\nu)/2-(2N+3)\varepsilon_1}). \tag{3.46}$$

for any $l\ge 0$ *and* $\frac{1}{10}t^{\frac12+\nu-\epsilon_1}\le\rho\le 10t^{\frac12+\nu-\epsilon_1}$.

3.2.3 The remote region

We next consider the remote region $|x|\ge\frac{1}{10}t^{1/2+\varepsilon_2}$. In this region we take as an approximate solution to (3.1) the following radial profile:

$$u_{out}^{ap}(t,x)=v_1(t,x)+v_2(t,x)+v_3(t,x),$$

where

$$\begin{aligned}&v_1(t,x)=e^{i\alpha_0\ln t}[d_1^0t^{-(1+\nu)/2}f_1(y,\mu_0)+d_1^1t^{-(2+3\nu)/2}f_1(y,\mu_1)],\quad y=t^{-1/2}|x|,\\&v_2(t,x)=\Theta_\delta\left(\tfrac{x}{t}\right)e^{i\alpha_0\ln t}\left[d_2^0t^{-(1+\nu)/2}f_2(y,\mu_0)+d_2^1t^{-(2+3\nu)/2}f_2(y,\mu_1)+\right.\\&\left.+t^{-(3+5\nu)/2}\left(d_2^2g_1(y)-\left(d_2^2(2\nu+1)\ln\left(\tfrac{|x|}{t}\right)-\lambda_2\right)f_2(y,\mu_2)\right)\right],\end{aligned}$$

$\Theta_\delta(\xi)=\Theta(\frac{\xi}{\delta})$, $\Theta\in C_0^\infty(\mathbb{R}^3)$ is radial, $\Theta(\xi)=\begin{cases}1 & \text{if } |\xi|\le 1\\ 0 & \text{if } |\xi|\ge 2\end{cases}$.
Finally, $v_3(t,x)$ is given by

$$v_3(t,x)=\frac{e^{i\frac{|x|^2}{4t}}}{t^{5/2}}\hat{v}_3\left(\frac{x}{t}\right),\quad \hat{v}_3=-iz\Delta\Theta_\delta-2i\nabla z\cdot\nabla\Theta_\delta,$$

where

$$z(\xi)=d_2^0|\xi|^{-2i\alpha_0-2+\nu}+d_2^1|\xi|^{-2i\alpha_0-1+3\nu}-(d_2^2(2\nu+1)\ln|\xi|-\lambda_2)|\xi|^{-2i\alpha_0+5\nu}.$$

It follows from the asymptotics (3.34) that for $t\ge T$ with some $T=T(\delta)>0$ and any $l\ge 0$, one has

$$\begin{aligned}&|\nabla^l v_1(t,x)|\le C_l|x|^{-l-1-\nu},\quad \frac{1}{10}t^{1/2+\varepsilon_2}\le|x|,\\&|\nabla^l v_2(t,x)|\le\frac{C_l}{t^{3/2}}\left|\frac{x}{t}\right|^{l-2+\nu},\quad\frac{1}{10}t^{1/2+\varepsilon_2}\le|x|\le 2\delta t.\end{aligned}\tag{3.47}$$

Furthermore, v_2 can be written as

$$\begin{aligned}&v_2(t,x)=v_{2,0}(t,x)+v_{2,1}(t,x),\\&v_{2,0}(t,x)=\frac{e^{i\frac{|x|^2}{4t}}}{t^{3/2}}\Theta_\delta\left(\frac{x}{t}\right)z\left(\frac{x}{t}\right),\quad v_{2,1}(t,x)=\frac{e^{i\frac{|x|^2}{4t}}}{t^{3/2}}\Theta_\delta\left(\frac{x}{t}\right)\hat{v}_{2,1}(t,x),\end{aligned}\tag{3.48}$$

with $\hat{v}_{2,1}$ verifying, for any $l\ge 0$,

$$|\nabla^l\hat{v}_{2,1}(t,x)|\le C_lt^{3-\nu}|x|^{-l-4+\nu},\quad\frac{1}{10}t^{1/2+\varepsilon_2}\le|x|\le 2\delta t.\tag{3.49}$$

We next address v_3. One has

$$\begin{aligned}&\|\nabla^l v_3(t)\|_{L^\infty(|x|\ge\frac{1}{10}t^{1/2+\varepsilon_2})}\le C_lt^{-5/2}\delta^{-4+l+\nu},\\&\|\nabla^l v_3(t)\|_{L^2(|x|\ge\frac{1}{10}t^{1/2+\varepsilon_2})}\le C_lt^{-1}\delta^{-5/2+l+\nu},\end{aligned}\tag{3.50}$$

for any $l \geq 0$ and $t \geq T(\delta)$.
As a direct consequence of estimates (3.47), (3.49), (3.50), one obtains

$$
\begin{aligned}
&\|u_{out}^{ap}(t)\|_{L^\infty(|x|\geq \frac{1}{10}t^{\frac{1}{2}+\varepsilon_2})} \leq Ct^{-(\frac{1}{2}+\varepsilon_2)(1+\nu)},\\
&\||x|^{-1}u_{out}^{ap}(t)\|_{L^\infty(|x|\geq \frac{1}{10}t^{\frac{1}{2}+\varepsilon_2})} \leq Ct^{-5/4},\\
&\|\nabla^l u_{out}^{ap}(t)\|_{L^\infty(|x|\geq \frac{1}{10}t^{\frac{1}{2}+\varepsilon_2})} \leq Ct^{-5/4}, \quad l=1,2,\\
&\|\nabla^l u_{out}^{ap}(t)\|_{L^2(|x|\geq \frac{1}{10}t^{\frac{1}{2}+\varepsilon_2})} \leq C\delta^{\nu+l-1/2}, \quad l=1,2,\\
&\|\nabla^l (u_{out}^{ap}(t)-v_{2,0}(t))\|_{L^2(|x|\geq \frac{1}{10}t^{\frac{1}{2}+\varepsilon_2})} \leq Ct^{-\frac{1}{2}(\frac{1}{2}+\varepsilon_2)(1+2\nu)}, \quad l=1,2,\\
&\||x|^{-1}(u_{out}^{ap}(t)-v_{2,0}(t))\|_{L^2(|x|\geq \frac{1}{10}t^{\frac{1}{2}+\varepsilon_2})} \leq Ct^{-\frac{1}{2}(\frac{1}{2}+\varepsilon_2)(1+2\nu)/2},
\end{aligned}
\tag{3.51}
$$

provided $\frac{3}{8} \leq \varepsilon_2 < \frac{1}{2}$, ν is sufficiently small and $t \geq T(\delta)$.
Denote

$$u_{ss}^{ap}(t,x) = e^{i\alpha_0 \ln t} t^{-1/4} w_{ss}^{ap}(t, t^{-1/2}|x|),$$

and consider the difference $u_{ss}^{ap}(t,x) - u_{out}^{ap}(t,x)$. For $\frac{1}{10}t^{1/2+\varepsilon_2} \leq |x| \leq 10t^{1/2+\varepsilon_2}$ one has

$$u_{ss}^{ap}(t,x) - u_{out}^{ap}(t,x) = e^{i\alpha_0 \ln t} t^{-(3+5\nu)/2}((d_1^2(1+2\nu)\ln|x| + \lambda_1)f_1(y,\nu_2) + g_2(y)), \tag{3.52}$$

which together with (3.34) and (3.40) implies that

$$|\nabla^l(u_{out}^{ap} - u_{ss}^{ap})| \leq C_l(|\ln t| t^{-(\frac{1}{2}+\varepsilon_2)(3+5\nu+l)} + t^{-(\frac{1}{2}+\varepsilon_2)(3+5\nu+1)}), \tag{3.53}$$

for any $l \geq 0$ and $\frac{1}{10}t^{1/2+\varepsilon_2} \leq |x| \leq 10t^{1/2+\varepsilon_2}$, provided $\frac{3}{8} \leq \varepsilon_2 < \frac{1}{2}$ and ν is sufficiently small.
We next analyze the error $R_{out}(t) = -i\frac{du_{out}^{ap}}{dt}(t) - \triangle u_{out}^{ap}(t) - |u_{out}^{ap}(t)|^4 u_{out}^{ap}(t)$. It has the form

$$
\begin{aligned}
R_{out}(t,x) = &-\frac{e^{i\frac{|x|^2}{4t}}}{t^{9/2}}\left[t\hat{v}_{2,1}(t,x)\Delta\Theta_\delta\left(\frac{x}{t}\right) + 2t^2\nabla\hat{v}_{2,1}(t,x)\cdot\nabla\Theta_\delta\left(\frac{x}{t}\right)\right.\\
&\left.+\Delta\hat{v}_3\left(\frac{x}{t}\right)\right] - |u_{out}^{ap}|^4 u_{out}^{ap}.
\end{aligned}
\tag{3.54}
$$

Combined with (3.47), (3.49), (3.50), representation (3.54) gives for $\frac{3}{8} \leq \varepsilon_2 < \frac{1}{2}$ and ν sufficiently small,

$$\|\nabla^l R_{out}(t)\|_{L^2(|x|\geq \frac{1}{10}t^{1/2+\varepsilon_2})} \leq Ct^{-\frac{9}{4}(1+2\nu)}, \quad t \geq T(\delta), \quad l=0,1,2. \tag{3.55}$$

3.2.4 Proof of Proposition 3.2

We are now in position to conclude the proof of Proposition 3.2. Fix ε_2 such that $\frac{3}{8} \leq \varepsilon_2 < \frac{1}{2}$ and consider the radial profile $u^{ap}(t,x)$ defined by

$$
\begin{aligned}
u^{ap}(t,x) = &\Theta(t^{-1/2+\varepsilon_1}x)u_{in}^{ap}(t,x) + (1-\Theta(t^{-1/2+\varepsilon_1}x))\Theta(t^{-1/2-\varepsilon_2}x)u_{ss}^{ap}(t,x)\\
&+ (1-\Theta(t^{-1/2-\varepsilon_2}x))u_{out}^{ap}(t,x), \quad x \in \mathbb{R}^3,
\end{aligned}
$$

where $u_{in}^{ap}(t,x) = e^{i\alpha_0 \ln t} t^{\nu/2}\psi_{in}^{ap}(t, t^\nu|x|)$. Write u^{ap} as $u^{ap}(t,x) = e^{i\alpha_0 \ln t} t^{\nu/2}(W(y) + \chi^{ap}(t,y))$, $y = t^\nu x$. By Lemma 3.5 (estimates (3.21), (3.22)), Lemma 3.8 (estimates (3.41), (3.42), (3.43)) and (3.51) one has

$$\|\chi^{ap}(t)\|_{L^\infty} \le Ct^{-(1+2\nu)/2} \tag{3.56}$$
$$\||y|^{-1}\chi^{ap}(t)\|_{L^\infty} + \|\nabla\chi^{ap}(t)\|_{L^\infty} \le Ct^{-1-2\nu}, \tag{3.57}$$
$$\||y|^{-2}\chi^{ap}(t)\|_{L^\infty} + \||y|^{-1}\nabla_y\chi^{ap}(t)\|_{L^\infty} \le C(|\nu|+|\alpha_0|)t^{-1-2\nu}, \tag{3.58}$$
$$\|\nabla^2\chi^{ap}(t)\|_{L^\infty} \le C(|\nu|+|\alpha_0|)t^{-1-2\nu}. \tag{3.59}$$

All the estimates stated in this subsection are valid for ν sufficiently small and $t \ge T(\alpha_0,\nu,\delta)$. Furthermore, it follows from Lemma 3.5 (estimate (3.23)), Lemma 3.8 (estimate (3.43)) and two last inequalities in (3.51) that

$$\begin{aligned} \|\nabla^l\chi^{ap}(t)\|_{L^2(|y|\le 10t^{1/2+\nu+\varepsilon_2})} &\le Ct^{-(1+2\nu)(1-2\varepsilon_2)/4}, \quad l=1,2,\\ \|\nabla^l(\chi^{ap}(t)-\chi_0^{ap}(t))\|_{L^2(|y|\ge t^{1/2+\nu+\varepsilon_2})} &\le Ct^{-(1+2\nu)/4}, \quad l=1,2, \end{aligned} \tag{3.60}$$

where $\chi_0^{ap}(t,y) = e^{-i\alpha_0\ln t}t^{-\nu/2}v_{2,0}(t,t^{-\nu}y)$.
Inequalities (3.60) imply, in particular,

$$\|\nabla^l\chi^{ap}(t)\|_{L^2(\mathbb{R}^3)} \le Ct^{-\nu(l-1)}\delta^{\nu+l-1/2}, \quad l=1,2.$$

Moreover, introducing $\zeta^*(x) = \pi^{-3/2}e^{3i\pi/4}\int_{\mathbb{R}^3} d\xi e^{ix\cdot\xi}\Theta_\delta(2\xi)z(2\xi)$ and observing that $\zeta^* \in \dot{H}^s(\mathbb{R}^3)$ for any $s > 1/2-\nu$, and $\|\nabla^l(v_{2,0}-e^{i\Delta t}\zeta^*)\|_{L^2(|x|\ge t^\gamma)} \to 0$ as $t\to+\infty$ for any $\gamma > \frac{1-2\nu}{3-2\nu}$ and any $l \ge 1$, one obtains that

$$e^{i\alpha(t)}\chi^{ap}(t,\lambda(t)\cdot) - e^{it\Delta}\zeta^* \to 0 \ \text{ in } \dot{H}^1\cap\dot{H}^2 \ \text{ as } t\to+\infty.$$

This concludes the proof of the first part of Proposition 3.2.
We next consider the error $R = -i\frac{du^{ap}}{dt} - \Delta u^{ap} - |u^{ap}|^4u^{ap}$. It has the form

$$R = E_1+E_2+E_3+E_4.$$

where

$$\begin{aligned} E_1 =& i(\frac{1}{2}-\varepsilon_1)t^{-1}(u_{in}^{ap}(t,x)-u_{ss}^{ap}(t,x))\tilde{\Theta}(t^{-1/2+\varepsilon_1}x)\\ &-2t^{-1/2+\varepsilon_1}(\nabla u_{in}^{ap}(t,x)-\nabla u_{ss}^{ap}(t,x))\cdot\nabla\Theta(t^{-1/2+\varepsilon_1}x)\\ &-t^{-1+2\varepsilon_1}(u_{in}^{ap}(t,x)-u_{ss}^{ap}(t,x))\Delta\Theta(t^{-1/2+\varepsilon_1}x),\\ E_2 =& i(\frac{1}{2}+\varepsilon_2)t^{-1}(u_{ss}^{ap}(t,x)-u_{out}^{ap}(t,x))\tilde{\Theta}(t^{-1/2-\varepsilon_2}x)\\ &-2t^{-1/2-\varepsilon_2}(\nabla u_{ss}^{ap}(t,x)-\nabla u_{out}^{ap}(t,x))\cdot\nabla\Theta(t^{-1/2-\varepsilon_2}x)\\ &-t^{-1-2\varepsilon_2}(u_{ss}^{ap}(t,x)-u_{out}^{ap}(t,x))\Delta\Theta(t^{-1/2-\varepsilon_2}x),\\ &\tilde{\Theta}(\xi) = \xi\cdot\nabla\Theta(\xi), \end{aligned}$$

and E_3, E_4 are given by

$$\begin{aligned} E_3 =& \Theta(t^{-1/2+\varepsilon_1}x)R_{in}(t,x) + (1-\Theta(t^{-1/2+\varepsilon_1}x))\Theta(t^{-1/2-\varepsilon_2}x)R_{ss}(t,x)\\ &+(1-\Theta(t^{-1/2-\varepsilon_2}x))R_{out}(t,x),\\ E_4 =& \Theta(t^{-1/2+\varepsilon_1}x)(|u_{in}^{ap}|^4u_{in}^{ap} - |u^{ap}|^4u^{ap})\\ &+(1-\Theta(t^{-1/2+\varepsilon_1}x))\Theta(t^{-1/2-\varepsilon_2}x)(|u_{ss}^{ap}|^4u_{ss}^{ap} - |u^{ap}|^4u^{ap})\\ &+(1-\Theta(t^{-1/2-\varepsilon_2}x))(|u_{out}^{ap}|^4u_{out}^{ap} - |u^{ap}|^4u^{ap}). \end{aligned}$$

Here

$$R_{in}(t,x) = e^{i\alpha_0 \ln t} t^{5\nu/2} \mathcal{R}_{in}(t, t^\nu|x|), \quad R_{ss}(t,x) = e^{i\alpha_0 \ln t} t^{5\nu/2} \mathcal{R}_{ss}(t, t^\nu|x|).$$

First we adress E_1. By lemma 3.8 (iii) we have

$$\|E_1\|_{H^2} \le Ct^{-9(1+2\nu)/4+\nu+5\varepsilon_1/2} \ln t \le Ct^{-(2+\frac{3}{20})(1+2\nu)}. \tag{3.61}$$

Similarly, from (3.53) we get for E_2:

$$\|E_2\|_{H^2} \le Ct^{-1-(\frac{1}{2}+\varepsilon_2)(\frac{3}{2}+5\nu)} \ln t \le Ct^{-(2+\frac{1}{4})(1+2\nu)}. \tag{3.62}$$

Next, we consider E_3. From Lemma 3.5 (ii) , Lemma 3.8 (ii) and (3.55) it is apparent that

$$\|E_3\|_{H^2} \le Ct^{-\frac{9}{4}(1+2\nu)+5\varepsilon_1/2} \le Ct^{-(2+\frac{3}{20})(1+2\nu)}. \tag{3.63}$$

Finally, applying Lemma 3.5 (estimates (3.21), (3.22)), Lemma 3.8 (estimates (3.41), (3.42),(3.43),(3.46)) and (3.51), (3.53), it is not difficult to check that

$$\|E_4\|_{H^2} \le Ct^{-3(1+2\nu)}. \tag{3.64}$$

Combining (3.61), (3.62), (3.63), (3.64), we get (3.10), which concludes the proof of Proposition 3.2.

3.3 Construction of an exact solution

We are now in position to prove Theorem 3.1. Consider (3.1) and write $u(t,x) = e^{i\alpha_0 \ln t} t^{\nu/2} \Psi(\tau, y)$, where $y = t^\nu x$ and $\tau = \frac{t^{1+2\nu}}{1+2\nu}$. Further decomposing Ψ as

$$\Psi(\tau, y) = \Psi^{ap}(\tau, y) + f(\tau, y), \quad \Psi^{ap}(\tau, y) = e^{-i\alpha_0 \ln t} t^{-\nu/2} u^{ap}(t,x),$$

where u^{ap} is the approximate solution of (3.1) given by Proposition (3.2), we get the following equation for the remainder f

$$i\frac{d\vec{f}}{d\tau} = \mathcal{H}(\tau)\vec{f} + \mathcal{F}(f) + r, \quad \vec{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix}, \tag{3.65}$$

where

$$\begin{aligned}
&\mathcal{H}(\tau) = H + \tau^{-1} l, \\
&H = -\triangle\sigma_3 - 3W^4\sigma_3 - 2W^4\sigma_3\sigma_1, \quad l = \frac{\alpha_0}{2\nu+1}\sigma_3 - i\frac{\nu}{2\nu+1}(\frac{1}{2} + y\cdot\nabla), \\
&\mathcal{F}(f) = \begin{pmatrix} F(f) \\ -\overline{F(f)} \end{pmatrix}, \quad F(f) = F_1(f) + F_2(f) \\
&F_1(f) = \mathcal{V}_1(\tau) f + \mathcal{V}_2(\tau)\overline{f}, \\
&\mathcal{V}_1(\tau) = 3(W^4 - |\Psi^{ap}(\tau)|^4), \quad \mathcal{V}_2(\tau) = 2(W^4 - (\Psi^{ap}(\tau))^2 |\Psi^{ap}(\tau)|^2), \\
&F_2(f) = -|\Psi^{ap} + f|^4(\Psi^{ap} + f) + |\Psi^{ap}|^4\Psi^{ap} + 3|\Psi^{ap}|^4 f + 2(\Psi^{ap})^2|\Psi^{ap}|^2\overline{f}, \\
&r = \begin{pmatrix} \mathtt{r} \\ -\overline{\mathtt{r}} \end{pmatrix}, \quad \mathtt{r}(\tau, y) = t^{-5\nu/2} e^{-i\alpha_0 \ln t} R(t,x).
\end{aligned}$$

R being the error given by Proposition 3.2. Note that by Proposition 3.2 one has

$$\|\mathcal{V}_i(\tau)\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C(|\alpha_0| + |\nu|)\tau^{-1}, \quad i = 1, 2, \tag{3.66}$$

$$\|\Psi^{ap}(\tau)\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C, \tag{3.67}$$

$$\|r(\tau)\|_{H^2(\mathbb{R}^3)} \leq C\tau^{-2-\frac{1}{8}}, \tag{3.68}$$

for any $\tau \geq \tau_0$ with some $\tau_0 > 0$.
Our intention is to solve (3.65) with zero condition at $\tau = +\infty$ by a fix point argument. To carry out this analysis we will need some energy type estimates for the linearized equation $i\frac{d\vec{f}}{d\tau} = \mathcal{H}(\tau)\vec{f}$. The required estimates are collected in the next subsection, their proofs being removed to Section 4.

3.3.1 Linear estimates

We start by recalling some basic spectral properties of the operator H (a more detailed discussion and the proofs can be found, for example, in [19]). Since we are considering only radial solutions, we will view H as an operator on $L^2_{rad}(\mathbb{R}^3; \mathbb{C}^2)$ with domain $D(H) = H^2_{rad}(\mathbb{R}^3; \mathbb{C}^3)$. H satisfies the relations

$$\sigma_3 H \sigma_3 = H^*, \quad \sigma_1 H \sigma_1 = -H.$$

The essential spectrum of H fills up the real axis. The discrete spectrum of H consists of two simple purely imaginary eigenvalues $i\lambda_0$, $-i\lambda_0$, $\lambda_0 > 0$. The corresponding eigenfunctions ζ_+, ζ_- are in $\mathcal{S}(\mathbb{R}^3)$ and can be chosen in such a way that $\zeta_- = \sigma_1\zeta_+ = \bar{\zeta}_+$. Notice also that $HW\binom{1}{-1} = HW_1\binom{1}{1} = 0$. which means that H has a resonance at zero.
Consider the projection of the linearized equation $i\frac{d\vec{f}}{d\tau} = \mathcal{H}(\tau)\vec{f}$ onto the essential spectrum of H:

$$i\frac{d\vec{f}}{d\tau} = P\mathcal{H}(\tau)P\vec{f}. \tag{3.69}$$

Here P is the spectral projection of H onto the essential spectrum given by

$$P = I - P_+ - P_-, \quad P_\pm = \frac{< \cdot, \sigma_3\zeta_\mp >}{< \zeta_\pm, \sigma_3\zeta_\mp >}\zeta_\pm,$$

$< \cdot, \cdot >$ is the scalar product in $L^2(\mathbb{R}^3, \mathbb{C}^2)$.
Let $U(\tau, s)$ be the propagator associated to Equation (3.69). In Section 4 we prove the following results.

Proposition 3.9. *There exists a constant $C > 0$ such that*

$$\|U(\tau, s)f\|_{H^2} \leq C\left(\frac{s}{\tau}\right)^{C(|\alpha_1|+|\nu_1|)}\|f\|_{H^2},$$

for any $s \geq \tau > 0$ and any $f \in H^2_{rad}$. Here $\alpha_1 = \frac{\alpha_0}{1+2\nu}$, $\nu_1 = \frac{\nu}{1+2\nu}$.

3.3.2 Contraction argument

We now transforme (3.65) into a fix point problem. Rewrite (3.65) in the following integral form

$$f(\tau) = J(f)(\tau), \tag{3.70}$$

where

$$\begin{aligned}
J(f)(\tau) &= J_0(f)(\tau) + J_+(f)(\tau) + J_-(f)(\tau),\\
J_0(f)(\tau) &= i\int_\tau^{+\infty} dsU(\tau,s)P(\mathcal{F}_1(f(s)) + r(s)),\\
J_+(f)(\tau) &= i\int_\tau^{+\infty} dse^{\lambda_0(\tau-s)}P_+(\mathcal{F}_2(f(s)) + r(s)),\\
J_-(f)(\tau) &= -i\int_{\tau_1}^{\tau} dse^{-\lambda_0(\tau-s)}P_-(\mathcal{F}_2(f(s)) + r(s)),\\
\mathcal{F}_1(f) &= \mathcal{F}(f) + s^{-1}l(P_+ + P_-)\vec{f},\\
\mathcal{F}_2(f) &= \mathcal{F}(f) + s^{-1}l\vec{f},
\end{aligned}$$

$\tau_1 \geq \max\{\tau_0, 1\}$ to be fixed later (slightly abusing notation we identify in (3.70) $\mathbb{C}^2$ vectors of the form $\binom{f}{\bar{f}}$ with their first component f).
Our intention is to view J as a mapping in the space $C([\tau_1, +\infty), H^2_{rad})$ equipped with the norm

$$|||f||| = \sup_{\tau\geq\tau_1} \|f(\tau)\|_{H^2}\tau^{1+1/16}$$

and to show that J is contraction of the unite ball $|||f||| \leq 1$ into itself provided $|\alpha_0| + |\nu|$ is sufficiently small and τ_1 is chosen sufficiently large. Indeed, by (3.67), (3.66) one has, for any $f, g \in H^2$ with $\|f\|_{H^2} \leq 1$, $\|g\|_{H^2} \leq 1$,

$$\|\mathcal{F}_1(f) - \mathcal{F}_1(g)\|_{H^2} \leq C(\|f\|_{H^2} + \|g\|_{H^2} + (|\alpha_0| + |\nu|)\tau^{-1})\|f - g\|_{H^2},$$

$$\|P_\pm(\mathcal{F}_2(f) - \mathcal{F}_2(g))\| \leq C(\|f\|_{H^2} + \|g\|_{H^2} + (|\alpha_0| + |\nu|)\tau^{-1})\|f - g\|_{H^2},$$

which together with (3.68) and Proposition 3.9 gives

$$|||J(f)||| \leq \frac{1}{2} + C\tau_1^{-1/16}, \quad |||J(f) - J(g)||| \leq (\frac{1}{2} + C\tau_1^{-1/16})|||f - g|||,$$

for any $f, g \in \{|||h||| \leq 1\}$, provided $|\alpha_0| + |\nu|$ is sufficientlt small. This means that for τ_1 sufficiently large, J is a contraction of the unit ball $|||f||| \leq 1$ into itself and consequently, has a unique fixe point f that satisfies

$$\|f(\tau)\|_{H^2} \leq \tau^{-1-1/16}, \quad \forall\tau \geq \tau_1,$$

which together with Proposition 3.2 gives Theorem 3.1.

3.4 Linearized evolution

In this section we prove Proposition 3.9. The proof will be achieved by combining the results of [19] with a careful spectral analysis of the operator H around zero energy. More precisely, in subsection 1 we consider the operator H as before, restricted to the subspace of radial functions, and construct a basis of Jost solutions for the equation $H\zeta = E\zeta$. In subsection 2 we study the spectral decomposition of H near $E = 0$. Finally, in subsection 3 we prove Proposition 3.9 by combining the results of the previous two subsections with the coercivity properties of H established in [19].

3.4.1 Solutions to the equation $H\zeta = E\zeta$.

In this subsection we construct a basis of Jost solutions of the equation $H\zeta = E\zeta$, $E \in \mathbb{R}$. Since the subject is completely standard we will only briefly sketch the proofs (see also [9], [34] for a closely related construction in the context of energy subcritical NLS). Recall that

$$H = -(\partial_\rho^2 + 2\rho^{-1}\partial_\rho)\sigma_3 + V(\rho), \quad V = \begin{pmatrix} V_1 & V_2 \\ -V_2 & -V_1 \end{pmatrix},$$
$$V_1(\rho) = -3W^4(\rho), \quad V_2(\rho) = -2W^4(\rho), \quad W(\rho) = (1+\rho^2/3)^{-1/2}.$$

We emphasize that $V(\rho)$ is a smooth function of ρ that decays as ρ^{-4} as $\rho \to \infty$. Since $\sigma_1 H = -H\sigma_1$ it suffices to consider the case $E \geq 0$, so we write $E = k^2$, $k \geq 0$. It will be convenient for us to remove the first derivative in H. In order to do that set $f = \rho\zeta$, then one gets

$$\tilde{H} f = E f, \quad \tilde{H} = -\partial_\rho^2 \sigma_3 + V(\rho). \tag{3.71}$$

We will consider the operator $\tilde{H}$ on $\mathbb{R}$, to recover the original radial $\mathbb{R}^3$ problem it suffices to restrict $\tilde{H}$ to the subspace of odd functions.
We start by constructing the most rapidly decaying solution to (3.71).

Lemma 3.10. *For all $k \geq 0$ there exists a real solution $f_3(\rho,k)$ of the equation*

$$\tilde{H} f = k^2 f, \tag{3.72}$$

such that $f_3(\rho,k) = e^{-k\rho}\chi_3(\rho,k)$, where χ_3 is C^∞ function of $(\rho,k) \in \mathbb{R}\times\mathbb{R}_+^*$ verifying $\chi_3(\rho,k) = \binom{0}{1} + a(\rho,k)$,

$$\begin{aligned} &|\partial_\rho^l \partial_k^m a(\rho,k)| \leq C_l < \rho >^{-2-l+m} (1 + k < \rho >)^{-1-m}, m = 0,1, \\ &|\partial_\rho^l \partial_k^2 a(\rho,k)| \leq C_l < \rho >^{-l} (1 + k < \rho >)^{-3} \ln\left(\frac{1}{k < \rho >} + 2\right), \end{aligned} \tag{3.73}$$

for all $\rho \geq 0$, $k > 0$ and $l \geq 0$.

Proof. One writes the following integral equation for χ_3:

$$\chi_3(\rho,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_\rho^{+\infty} K(\rho - s, k)\sigma_3 V(s)\chi_3(s,k)ds,$$

$$K(\xi,k) = \begin{pmatrix} \frac{\sin k\xi}{k} & 0 \\ 0 & \frac{\sinh k\xi}{k} \end{pmatrix} e^{k\xi}.$$

The statement of the lemma follows then from the estimate

$$|\partial_k^l K(\xi,k)| \leq C_l \frac{|\xi|^{l+1}}{< k\xi >^{l+1}}, \quad \xi \leq 0, \ k \geq 0, \ l \geq 0$$

and the decay properties of V:

$$|\partial_\rho^l V(\rho)| \leq C_l < \rho >^{-4-l}, \quad \rho \in \mathbb{R}, \quad l \geq 0,$$

by standard Volterra iterations. ☐

We next construct the oscillating solutions to Equation (3.72).

Lemma 3.11. *For all $k \geq 0$ there exists a solution $f_1(\rho, k)$ of Equation (3.72) such that f_1 is a smooth function of $(\rho, k) \in \mathbb{R} \times \mathbb{R}_+^*$ of the form $f_1(\rho, k) = e^{ik\rho}(\binom{1}{0} + b(\rho, k))$, where b verifies*

$$
\begin{aligned}
&|b(\rho, k)| \leq C(<\rho>^{-2} + ke^{-k\rho}),\\
&|\partial_\rho b(\rho, k)| \leq C(<\rho>^{-3} + k^2 e^{-k\rho}),\\
&|\partial_k b(\rho, k)| \leq C(<\rho>^{-1} + <k\rho> e^{-k\rho}),\\
&|\partial^2_{\rho k} b(\rho, k)| \leq C(<\rho>^{-2} + k <k\rho> e^{-k\rho}),
\end{aligned}
\tag{3.74}
$$

for all $\rho \geq 0$, $0 \leq k \lesssim 1$. In addition, one has

$$
|\partial_k^2 b(\rho, k)| \leq C \ln\left(\frac{1}{k} + 1\right),
$$

for all $0 \leq \rho \lesssim 1$, $0 < k \lesssim 1$.

Proof. To construct f_1 we will reduce the order of the system (3.72) by means of the substitution $f_1 = z_0 f_3 + z_1 \binom{1}{0}$. Further setting $z_2 = z_0' f_{3,2}$, $f_3 = \binom{f_{3,1}}{f_{3,2}}$, we get that $z = \binom{z_1}{z_2}$ solves

$$
\begin{aligned}
&- z_1'' - k^2 z_1 + V_{11} z_1 + V_{12} z_2 = 0,\\
&- z_2' + k z_2 + V_{21} z_1 + V_{22} z_2 = 0.
\end{aligned}
\tag{3.75}
$$

Here

$$
\begin{aligned}
&V_{11} = V_1 - V_2 \frac{f_{3,1}}{f_{3,2}}, \quad V_{12} = \frac{2}{f_{3,2}^2}(f_{3,1} f_{3,2}' - f_{3,1}' f_{3,2}),\\
&V_{21} = V_2, \quad V_{22} = -\frac{1}{f_{3,2}}(f_{3,2}' + k f_{3,2}).
\end{aligned}
$$

By Lemma 3.10, there exists $R > 0$ independent of k, such that the functions $V_{ij}(\rho, k)$, $i, j = 1, 2$ are smooth in both variables for $k > 0$ and $\rho \geq R$ and verify for all $l \geq 0$, $\rho \geq R$, $k > 0$,

$$
\begin{aligned}
&|\partial_\rho^l V_{j1}(\rho, k)| \leq C_l <\rho>^{-4-l}, \quad j = 1, 2,\\
&|\partial_\rho^l \partial_k V_{11}(\rho, k)| \leq C_l <\rho>^{-5-l} <k\rho>^{-2},\\
&|\partial_\rho^l \partial_k^2 V_{11}(\rho, k)| \leq C_l <\rho>^{-4-l} <k\rho>^{-3} \ln\left(\frac{1}{k\rho} + 2\right),\\
&|\partial_\rho^l \partial_k^m V_{j2}(\rho, k)| \leq C_l <\rho>^{-3-l+m} <k\rho>^{-1-m}, \quad j = 1, 2, \quad m = 0, 1,\\
&|\partial_\rho^l \partial_k^2 V_{22}(\rho, k)| \leq C_l <\rho>^{-1-l} <k\rho>^{-3} \ln\left(\frac{1}{k\rho} + 2\right),
\end{aligned}
\tag{3.76}
$$

Writing for z the following integral equation

$$
z(\rho, k) = e^{ik\rho}\binom{1}{0} - \int_\rho^\infty \begin{pmatrix} \frac{\sin k(\rho - s)}{k} & 0 \\ 0 & e^{-k(s-\rho)} \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} z(s, k) ds,
$$

and taking into account (3.76), one proves easily the existence of a smooth solution satisfying

$$
\begin{aligned}
&|\partial_\rho^l \partial_k^m (e^{-ik\rho} z_1 - 1)| + <\rho> |\partial_\rho^l \partial_k^m (e^{-ik\rho} z_2)| \leq C_l <\rho>^{-2-l+m} <k\rho>^{-1-m}, \quad m = 0, 1,\\
&|\partial_\rho^n \partial_k^2 (e^{-ik\rho} z_1 - 1)| + |\partial_\rho^n \partial_k^2 (e^{-ik\rho} z_2)| \leq C \ln\left(\frac{1}{k\rho} + 2\right), \quad n = 0, 1,
\end{aligned}
\tag{3.77}
$$

for all $\rho \geq R$, $k > 0$, $l \geq 0$.
To reconstruct f_1, we set

$$z_0(\rho, k) = \int_R^\rho \frac{z_2(s,k)}{f_{3,2}(s,k)} ds - \int_R^{+\infty} \frac{z_2(s,0)}{f_{3,2}(s,0)} ds.$$

Then, for $\rho \geq R$, the statement of Lemma 3.11 follows directly from (3.77) and Lemma 3.10. To cover the case $x \leq R$ one can invoke the Cauchy problem with initial data at $\rho = R$. $\square$

Note that since $k^2 \in \mathbb{R}$, $f_2(\cdot, k) = \overline{f_1(\cdot, k)}$ is also a solution of (3.72).

Remark 3.12. Recall that the equation $\tilde{H} f = 0$ has a basis of explicit solutions $\rho\Phi_\pm(\rho)\binom{1}{\pm 1}$, $\rho\Theta_\pm(\rho)\binom{1}{\pm 1}$, with $\Phi_\pm$, $\Theta_\pm$ given by (3.18). Comparing the behavior of $\rho\Phi_\pm$, $\rho\Theta_\pm$, with the asymptotics of $f_1(\rho, 0)$, $f_3(\rho, 0)$, one gets

$$f_1(\rho,0) = \frac{1}{2}\rho(\xi_0(\rho) + \xi_1(\rho)), \quad f_3(\rho,0) = \frac{1}{2}\rho(\xi_1(\rho) - \xi_0(\rho)), \tag{3.78}$$

where $\xi_0 = \frac{1}{\sqrt{3}} W \binom{1}{-1}$, $\xi_1 = -\frac{2}{\sqrt{3}} W_1 \binom{1}{1}$.

Next, we construct an exponentially growing solution at $+\infty$.

Lemma 3.13. *For any $k > 0$, there exists a solution $f_4(\rho, k)$ to (3.72) such that $f_4 = e^{k\rho}\chi_4$ with χ_4 verifying*

$$\partial_\rho^l(\chi_4(\rho,k) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = O_k(\rho^{-3-l}), \quad \rho \to +\infty.$$

Proof. We construct f_4 by means of the following integral equation:

$$\begin{aligned} \chi_4(\rho,k) =& \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_\rho^{+\infty} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2k} \end{pmatrix} V\chi_4(s,k) ds \\ &+ \int_{R_1}^\rho \begin{pmatrix} \frac{e^{k(s-\rho)}\sin k(\rho - s)}{k} & 0 \\ 0 & \frac{e^{2k(s-\rho)}}{2k} \end{pmatrix} V\chi_4(s,k) ds. \end{aligned} \tag{3.79}$$

For $k > 0$ and R_1 sufficiently large (depending on k), the operator generating (3.79) is small on the space of bounded continuous functions. Therefore, (3.79) has a solution χ_4 verifying $|\chi_4(\rho,k)| \leq C$, $\rho \geq R_1$. Iterating this bound one gets that $\chi_4(\rho,k) - \binom{0}{1} = O_k(\rho^{-3})$ as $\rho \to \infty$. Finally, the estimates for the derivatives can be obtained differentiating (3.79). $\square$

We now briefly describe some properties of the solutions f_j, $j = 1, \ldots, 4$ that we will need later. Recall that the Wronskian $w(f,g) = \langle f', g\rangle_{\mathbb{R}^2} - \langle f, g'\rangle_{\mathbb{R}^2}$ does not depend on ρ if f and g are solutions of (3.71).
The estimates of Lemmas 3.10, 3.11, 3.13 lead to the relations:

$$w(f_1,f_2) = 2ik, \;\; w(f_1,f_3) = w(f_2,f_3) = 0, \;\; w(f_3,f_4) = -2k, \quad k > 0, \tag{3.80}$$

the three first relations being valid for $k = 0$ as well. Notice also that by Lemmas 3.10, 3.11, $\partial_k f_1(\rho, 0)$, $\partial_k f_3(\rho, 0)$, are solutions of the equation $\tilde{H} f = 0$ verifying for $\rho \geq 0$,

$$|\partial_k f_1(\rho,0) - \begin{pmatrix} i\rho \\ 0 \end{pmatrix}| \leq C, \quad |\partial^2_{k\rho} f_1(\rho,0) - \begin{pmatrix} i \\ 0 \end{pmatrix}| \leq \frac{C}{<\rho>^2},$$
$$|\partial_k f_3(\rho,0) + \begin{pmatrix} 0 \\ \rho \end{pmatrix}| \leq \frac{C}{<\rho>}, \quad |\partial^2_{k\rho} \zeta_3(\rho,0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}| \leq \frac{C}{<\rho>^2},$$

As a consequence, one has

$$w(\partial_k f_1|_{k=0}, f_1|_{k=0}) = i, \quad w(\partial_k f_1|_{k=0}, f_3|_{k=0}) = 0,$$
$$w(\partial_k f_3|_{k=0}, f_1|_{k=0}) = 0, \quad w(\partial_k f_3|_{k=0}, f_3|_{k=0}) = -1. \tag{3.81}$$

In addition to scalar Wronskian we will use matrix Wronskians. If F, G are 2×2 matrix solutions of (3.72), their matrix Wronskian

$$W(F,G) = F^{t\prime}G - F^tG'$$

is independent of ρ.
Set $g_j(\rho,k) = f_j(-\rho,k)$, $j = 1,\dots,4$. Since the potential V is even, g_j, $j = 1,\dots,4$ are again solutions of (3.72) which have the same asymptotic behavior as $\rho \to -\infty$ as f_j as $\rho \to +\infty$. Consider the matrix solutions F, G, defined by

$$F = (f_1, f_3), \quad G = (g_1, g_3).$$

Denote $D(k) = W(F,G)$. It follows from Lemmas 3.10, 3.11 that D is smooth for $k > 0$ and admits the estimate

$$|\partial_k^2 D(k)| \le C \ln\left(\frac{1}{k} + 1\right), \quad 0 < k \lesssim 1. \tag{3.82}$$

In addition, by (3.78), (3.80), (3.81), one has

$$D(0) = 0, \quad \partial_k D(0) = \begin{pmatrix} -2i & 0 \\ 0 & 2 \end{pmatrix}. \tag{3.83}$$

3.4.2 Scattering solutions and the distorted Fourier transform in a vicinity of zero energy

Set

$$\mathcal{F}(\rho,k) = F(\rho,k)s(k), \tag{3.84}$$

where $s(k) = D^{t-1}(k)\binom{2ik}{0}$. By (3.82), (3.83), $s = \binom{s_1}{s_2}$ is a smooth function of k for $0 < k < k_0$ (k_0 sufficiently small), continuous up to $k = 0$, verifying

$$s_1(0) = -1, \quad s_2(0) = 0,$$
$$|\partial_k s(k)| \le C|\ln k|, \quad 0 < k \le k_0. \tag{3.85}$$

By construction, one has

$$w(\mathcal{F}, g_1) = 2ik, \quad w(\mathcal{F}, g_3) = 0,$$

for any $0 \le k < k_0$. As a consequence,

$$\mathcal{F}(\rho,k) = r_1(k)g_1(\rho,k) + g_2(\rho,k) + r_2(k)g_3(\rho,k), \quad 0 \le k < k_0, \tag{3.86}$$

with some coefficients $r_1(k)$, $r_2(k)$ that, by (3.78), (3.85), verify

$$r_1(0) = r_2(0) = 0. \tag{3.87}$$

Computing the Wronskians $w(\mathcal{F}, \bar{\mathcal{F}})$ and $w(\mathcal{F}, \bar{\mathcal{G}})$, where $\mathcal{G}(\rho,k) = \mathcal{F}(-\rho,k)$, one gets

$$|s_1(k)|^2 + |r_1(k)|^2 = 1, \quad r_1(k)\overline{s_1(k)} + \overline{r_1(k)}s_1(k) = 0, \quad 0 \le k < k_0.$$

One can write the following Wronskian representation for r_1:

$$r_1(k) = s_1(k)\frac{w(g_2, f_1)}{2ik} + s_2(k)\frac{w(g_2, f_3)}{2ik}, \quad k \neq 0. \tag{3.88}$$

Using (3.85) and the relations

$$w(g_2, f_3)|_{k=0} = w(g_2, f_1)|_{k=0} = \partial_k w(g_2, f_1)|_{k=0},$$

one easily deduces from (3.88) that r_1 is smooth for $0 < k < k_0$, continuous up to $k = 0$, and verifies

$$|\partial_k r_1(k)| \leq C|\ln k|, \quad 0 < k < k_0, \tag{3.89}$$

which in its turn, implies that r_2 is smooth for $0 < k < k_0$, continuous up to $k = 0$ and admits a similar estimate:

$$|\partial_k r_2(k)| \leq C|\ln k|, \quad 0 < k < k_0. \tag{3.90}$$

Introduce the following odd solution of (3.72):

$$e(\rho, k) = \mathcal{F}(-\rho, k) - \mathcal{F}(\rho, k).$$

By (3.84), (3.86),

$$e = a_1 f_1 + f_2 + a_2 f_3, \quad a_j = r_j - s_j, \;\; j = 1, 2. \tag{3.91}$$

It follows from (3.85), (3.87), (3.89), (3.90) that

$$a_1(0) = 1, \;\; a_2(0) = 0, \tag{3.92}$$

and

$$|\partial_k a_j| \leq C|\ln k|, \quad 0 < k < k_0, \;\; j = 1, 2, \tag{3.93}$$

which together with Lemmas 3.10, 3.11 implies the following result.

Lemma 3.14. *One has:*
(i) $e(\rho, k) = e_0(\rho, k) + e_1(\rho, k)$, *where* $e_0(\rho, k) = a_1(k)e^{ik\rho}\binom{1}{0} + e^{-ik\rho}\binom{1}{0}$ *and the remainder* $e_1(\rho, k)$ *admits the estimates*

$$\begin{aligned} &|e_1(\rho, k)| \leq C(<\rho>^{-2} + k|\ln k|e^{-k\rho}), \quad \rho \geq 0, \\ &|\partial_k e_1(\rho, k)| \leq C|\ln k|(<\rho>^{-1} + e^{-k\rho/2}), \quad \rho \geq 0, \\ &\|e_1(\cdot, k)\|_{L^2(\mathbb{R}_+)} \leq C, \\ &\|\rho e_1(\cdot, k)\|_{L^2(\mathbb{R}_+)} + \|\partial_k e_1(\cdot, k)\|_{L^2(\mathbb{R}_+)} \leq Ck^{-1/2}|\ln k|, \end{aligned} \tag{3.94}$$

for any $0 < k \leq k_0$.
(ii) $(\rho\partial_\rho - k\partial_k)e(\rho, k) = e^{ik\rho}\binom{1}{0}k\partial_k a_1(k) + e_2(\rho, k)$, *with* $e_2(\rho, k)$ *verifying*

$$\begin{aligned} &|e_2(\rho, k)| \leq C(<\rho>^{-1} + k|\ln k|e^{-k\rho/2}), \quad \rho \geq 0, \\ &\|e_2(\cdot, k)\|_{L^2(\mathbb{R}_+)} \leq C, \end{aligned} \tag{3.95}$$

for any $0 < k \leq k_0$.

For $0<\kappa<k_0$, introduce the operators $\mathbb{E}_\kappa : L^2(\mathbb{R}_+,\mathbb{C}^2)\to L^2(\mathbb{R}^3,\mathbb{C}^2)$,

$$(\mathbb{E}_\kappa\Phi)(y)=\frac{1}{2^{3/2}\pi}\int_{\mathbb{R}_+}dk\theta_\kappa(k)\mathcal{E}(y,k)\Phi(k),\quad \Phi\in L^2(\mathbb{R}_+,\mathbb{C}^2),$$

where $\mathcal{E}(y,k)$ is a 2×2 matrix given by

$$\mathcal{E}(y,k)=\rho^{-1}(e(\rho,k),\sigma_1\overline{e(\rho,k)}),\quad \rho=|y|,$$

$\theta_\kappa(k)=\theta(\kappa^{-1}k)$, θ is a C^∞ even function verifying $\theta(k)=\begin{cases}1 & \text{if } |k|\le 1/4\\ 0 & \text{if } |k|\ge 1/2\end{cases}$.
Since $e(\rho,k)$ is a solution of the equation $\tilde{H}e=k^2e$, one has $H\mathbb{E}_\kappa=\mathbb{E}_\kappa k^2\sigma_3$.
By Lemma 3.14 (i), the operators $\mathbb{E}_\kappa$ are bounded uniformly with respect to $\kappa\le k_0$. The action of the adjoint operators $\mathbb{E}_\kappa^* : L^2(\mathbb{R}^3,\mathbb{C}^2)\to L^2(\mathbb{R}_+,\mathbb{C}^2)$ is given by

$$(\mathbb{E}_\kappa^* u)(k)=\frac{1}{2^{3/2}\pi}\theta_\kappa(k)\int_{\mathbb{R}^3}dy\mathcal{E}^*(y,k)u(y),\quad u\in L^2(\mathbb{R}^3,\mathbb{C}^2).$$

Clearly,

$$\mathbb{E}_\kappa^*\sigma_3\zeta_\pm=0 \tag{3.96}$$

for any $0<\kappa\le k_0$.
The following relation is a standard consequence of the asymptotics given by Lemma 3.14 (i),

$$\mathbb{E}_{\kappa_2}^*\sigma_3\mathbb{E}_{\kappa_1}\sigma_3=\theta_{\kappa_1}(k)\theta_{\kappa_2}(k), \tag{3.97}$$

for any $0<\kappa_1,\kappa_2\le k_0$.

Remark 3.15. Notice that because of the presence of the cut off function θ_κ, $\mathbb{E}_\kappa$ is bounded as an operator from $L^2([0,k_0])$ to $H^m(\mathbb{R}^3)$ for any $m\ge 0$, uniformly in $\kappa\le k_0$.

We next introduce quasi-resonant functions $h_\kappa(y)$, $0<\kappa\le k_0$, by setting

$$h_\kappa=\sqrt{2}\mathbb{E}_\kappa\begin{pmatrix}1\\0\end{pmatrix}.$$

Lemma 3.16. *For any $0<\kappa\le k_0$, $h_\kappa\in <y>^{-1}L^2(\mathbb{R}^3)$ and as $\kappa\to 0$, one has*

$$\|h_\kappa\|_{L^2(\mathbb{R}^3)}=O(\kappa^{1/2}),\quad \|yh_\kappa\|_{L^2(\mathbb{R}^3)}=O(\kappa^{-1/2}), \tag{3.98}$$

$$<h_\kappa,\sigma_3(\xi_0+\xi_1)>=4\pi+O(\kappa^{1/2}\ln\kappa),\quad <h_\kappa,\sigma_3(\xi_1-\xi_0)>=O(\kappa^{1/2}\ln\kappa). \tag{3.99}$$

Proof. Applying Lemma 3.14 (i), we decompose h_κ as follows:

$$\begin{aligned}
h_\kappa(y)&=h_{\kappa,0}(y)+h_{\kappa,1}(y)+h_{\kappa,2}(y),\\
h_{\kappa,0}(y)&=\frac{1}{2\pi\rho}\kappa\hat{\theta}(\kappa\rho)\begin{pmatrix}1\\0\end{pmatrix},\\
h_{\kappa,1}(y)&=\frac{1}{2\pi\rho}\int_{\mathbb{R}_+}dke^{ik\rho}(a_1(k)-1)\theta_\kappa(k)\begin{pmatrix}1\\0\end{pmatrix},\\
h_{\kappa,2}(y)&=\frac{1}{2\pi\rho}\int_{\mathbb{R}_+}dk\theta_\kappa(k)e_1(\rho,k),
\end{aligned} \tag{3.100}$$

where $\hat{\theta}(\rho) = \int_{\mathbb{R}} e^{ik\rho}\theta(k)dk$, $\rho = |y|$.
Clearly, $h_{\kappa,0} \in < y >^{-1} L^2(\mathbb{R}^3)$ and one has

$$\|h_{\kappa,0}\|_{L^2(\mathbb{R}^3)} \le C\kappa^{1/2}, \quad \|yh_{\kappa,0}\|_{L^2(\mathbb{R}^3)} \le C\kappa^{-1/2}. \tag{3.101}$$

Consider $h_{\kappa,i}$, $i = 1, 2$. It follows from (3.92), (3.93), (3.94) that

$$\|h_{\kappa,i}\|_{L^2(\mathbb{R}^3)} \le C\kappa, \ \|yh_{\kappa,i}\|_{L^2(\mathbb{R}^3)} \le C\kappa^{1/2}|\ln\kappa|, \quad i = 1, 2, \tag{3.102}$$

which together with (3.101) leads to the estimates

$$\|h_\kappa\|_{L^2(\mathbb{R}^3)} \le C\kappa^{1/2}, \quad \|yh_\kappa\|_{L^2(\mathbb{R}^3)} \le C\kappa^{-1/2}. \tag{3.103}$$

We next compute $< h_\kappa, \sigma_3(\xi_1 \pm \xi_0) >$. By (3.101), (3.102), we have

$$\begin{aligned} &< h_\kappa, \sigma_3(\xi_1 \pm \xi_0) >=< h_{\kappa,0}, \sigma_3(\xi_1 \pm \xi_0) > +O(\kappa^{1/2}\ln\kappa), \\ &< h_{\kappa,0}, \sigma_3(\xi_1 - \xi_0) >= O(\kappa), \\ &< h_{\kappa,0}, \sigma_3(\xi_1 + \xi_0) >= 2\kappa \int_{\mathbb{R}} d\rho\hat{\theta}(\kappa\rho) + O(\kappa) = 4\pi + O(\kappa), \end{aligned} \tag{3.104}$$

which gives (3.99). □

3.4.3 Proof of Proposition 3.9

We start by deriving some coercivity bounds for the operator H.

Lemma 3.17. *There exists κ_0, $0 < \kappa_0 \le k_0$, and $C > 0$ such that*

$$\langle Hf, \sigma_3 f\rangle \ge C\kappa\|\nabla f\|^2_{L^2(\mathbb{R}^3)}, \tag{3.105}$$

for any $0 < \kappa \le \kappa_0$ and any $f \in \dot{H}^1_{rad}(\mathbb{R}^3, \mathbb{C}^2)$ verifying

$$\langle f, \sigma_3\zeta_-\rangle = \langle f, \sigma_3\zeta_+\rangle = \langle f, \sigma_3 h_\kappa\rangle = \left\langle f, \sigma_3\sigma_1\bar{h}_\kappa\right\rangle = 0. \tag{3.106}$$

Remark 3.18. Notice that since $\zeta_\pm, h_\kappa \in < y >^{-1} L^2(\mathbb{R}^3)$ the scalar products that appear in (3.106) are well defined for any $f \in \dot{H}^1$.

Proof. The proof of Lemma 3.17 is based on the following result which is due to Duyckaerts and Merle:

Lemma 3.19. *There exists $c_0 > 0$ such that*

$$\langle Hf, \sigma_3 f\rangle \ge c_0\|\nabla f\|^2_{L^2(\mathbb{R}^3)},$$

for any $f \in \dot{H}^1_{rad}(\mathbb{R}^3, \mathbb{C}^2)$ verifying

$$\langle f, \sigma_3\zeta_-\rangle = \langle f, \sigma_3\zeta_+\rangle = \langle f, \Delta\xi_0\rangle = \langle f, \Delta\xi_1\rangle = 0,$$

see [19] for the proof.
Let $f \in \dot{H}^1_{rad}$ such that (3.106) holds. One can write f as

$$f = \alpha_0\xi_0 + \alpha_1\xi_1 + g,$$

where

$$\alpha_j = -\frac{\langle f, \Delta\xi_j\rangle}{\|\nabla\xi_j\|_{L^2(\mathbb{R}^3)}}{}^2, \quad j = 0, 1,$$

and $g \in \dot{H}^1_{rad}$ verifies

$$\langle g, \sigma_3\zeta_-\rangle = \langle g, \sigma_3\zeta_+\rangle = \langle g, \Delta\xi_0\rangle = \langle g, \Delta\xi_1\rangle = 0.$$

Therefore, by Lemma 3.19,

$$\langle Hf, \sigma_3 f\rangle = \langle Hg, \sigma_3 g\rangle \geq c_0\|\nabla g\|^2_{L^2(\mathbb{R}^3)}. \tag{3.107}$$

Furthermore, since f verifies (3.106), one has

$$A(\kappa)\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} = \begin{pmatrix}\langle g, \sigma_3 h_\kappa\rangle\\ \langle g, \sigma_3\sigma_1\bar{h}_\kappa\rangle\end{pmatrix},$$

where

$$A(\kappa) = -\begin{pmatrix}\langle\xi_0, \sigma_3 h_\kappa\rangle & \langle\xi_1, \sigma_3 h_\kappa\rangle\\ \langle h_\kappa, \sigma_3\xi_0\rangle & -\langle h_\kappa, \sigma_3\xi_1\rangle\end{pmatrix}.$$

By (3.99),

$$A(\kappa) = -2\pi\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} + O(\kappa^{1/2}\ln\kappa), \quad \kappa \to 0.$$

Therefore, for κ sufficiently small, one has

$$|\alpha_1| + |\alpha_2| \leq C\|\nabla g\|_{L^2(\mathbb{R}^3)}\| < y > h_\kappa\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{-1/2}\|\nabla g\|_{L^2}.$$

As a consequence,

$$\|\nabla f\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{-1/2}\|\nabla g\|_{L^2}.$$

Combining this inequality with (3.107) we get (3.105). □

Next, we prove

Lemma 3.20. *There exists κ_1, $0 < \kappa_1 \leq k_0$, and $C > 0$ such that for any $0 < \kappa \leq \kappa_1$ one has*

$$\|f\|_{H^1(\mathbb{R}^3)} \leq \frac{C}{\kappa}\|\nabla f\|_{L^2(\mathbb{R}^3)},$$

for all $f \in H^1_{rad}(\mathbb{R}^3)$ verifying $\mathbb{E}^*_\kappa f = 0$.

Proof. By (3.92), (3.93) and Lemma 3.14 (i), $\mathbb{E}^*_\kappa f$ can be written as

$$(\mathbb{E}^*_\kappa f)(k) = \Phi_0(k) + \Phi_r(k),$$

where

$$\Phi_0(k) = \frac{1}{2^{3/2}\pi}\theta_\kappa(k)\check{f}(k),$$

$\check{f}(k) = 2\int_{\mathbb{R}^3} dy\frac{\cos k|y|}{|y|}f(y)$, and the remainder Φ_r satisfies

$$\|\Phi_r\|_{L^2(\mathbb{R}_+)} \leq C\kappa^{1/2}\|f\|_{L^2(\mathbb{R}^3)}.$$

Therefore, $\mathbb{E}^*_\kappa f = 0$ implies

$$\|\check{f}\|_{L^2(0,\kappa/4)} \leq C\kappa^{1/2}\|f\|_{L^2(\mathbb{R}^3)}. \tag{3.108}$$

Notice also that for any $f \in H^1_{rad}$ and any $0 < \kappa \le 1$ one has

$$\|f\|_{H^1(\mathbb{R}^3)} \le C(\|\check{f}\|_{L^2(0,\kappa/4)} + \kappa^{-1}\|\nabla f\|_{L^2(\mathbb{R}^3)}).$$

Combining this inequality with (3.108), we get

$$\|f\|_{H^1(\mathbb{R}^3)} \le \frac{C}{\kappa}\|\nabla f\|_{L^2(\mathbb{R}^3)},$$

provided κ is sufficiently small. □

We finally combine Lemmas 3.17, 3.20 to derive the following result which will be in the heart of the proof of Proposition 3.9

Lemma 3.21. *There exists* κ_2, $0 < \kappa_2 \le k_0$, *and* $C > 0$ *such that for any* $0 < \kappa \le \kappa_2$ *one has*

$$\langle Hf, \sigma_3 f\rangle \ge C\kappa^3\|f\|^2_{H^1} - \frac{\kappa}{C}\|\mathbb{E}^*_\kappa \sigma_3 f\|^2_{L^2(\mathbb{R}_+)}, \tag{3.109}$$

for any $f \in H^1_{rad}(\mathbb{R}^3, \mathbb{C}^2)$ *verifying* $\langle f, \sigma_3\zeta_\pm\rangle = 0$.

Proof. Write $f = f_1 + f_2$, where $f_1 = \mathbb{E}_\kappa\sigma_3\mathbb{E}^*_\kappa\sigma_3 f$ and $f_2 = f - f_1$. One clearly has

$$\|f_1\|_{H^1(\mathbb{R}^3)} \le C\|\mathbb{E}^*_\kappa\sigma_3 f\|_{L^2(\mathbb{R}_+)}, \quad \|Hf_1\|_{L^2(\mathbb{R}^3)} \le C\kappa^2\|\mathbb{E}^*_\kappa\sigma_3 f\|_{L^2(\mathbb{R}_+)}, \tag{3.110}$$

for any $0 < \kappa \le k_0$.
Consider f_2. It follows from (3.96), (3.97) that for any $\kappa' \le \kappa/2$,

- $\langle f_2, \sigma_3\zeta_\pm\rangle = 0$;
- $\mathbb{E}^*_{\kappa'}\sigma_3 f_2 = 0$;
- $\langle f_2, \sigma_3 h_{\kappa'}\rangle = \langle f_2, \sigma_3\sigma_1\bar{h}_{\kappa'}\rangle = 0$.

Hence, by Lemmas 3.17, 3.20, one has

$$\langle Hf_2, \sigma_3 f_2\rangle \ge C\kappa^3\|f_2\|^2_{H^1(\mathbb{R}^3)}, \tag{3.111}$$

provided κ is sufficiently small.
Combining (3.110), (3.111) one gets (3.109).

□

We are now in the position to prove Proposition 3.9. Consider the equation

$$\begin{aligned} &i\frac{du}{d\tau} = P\mathcal{H}(\tau)Pu, \\ &u(s) = f, \end{aligned} \tag{3.112}$$

where

$$\mathcal{H}(\tau) = H + \tau^{-1}l, \quad l = \alpha_1\sigma_3 - i\nu_1(\frac{1}{2} + y\cdot\nabla),$$

$\alpha_1, \nu_1 \in \mathbb{R}$, $s > 0$ and $f \in \mathcal{S}(\mathbb{R}^3)$ verifying $\langle f, \sigma_3\zeta_\pm\rangle = 0$.
Fix κ such $0 < \kappa \le \kappa_2$ and consider the functional $G_1(\tau) = \langle Hu, \sigma_3 u\rangle + c_0\|\mathbb{E}^*_\kappa\sigma_3 u\|^2_{L^2(\mathbb{R}_+)}$. Clearly,

$$G_1(\tau) \le C\|u(\tau)\|^2_{H^1(\mathbb{R}^3)}. \tag{3.113}$$

Moreover, since $\langle u(\tau), \sigma_3\zeta_\pm\rangle = 0$, choosing c_0 sufficiently large we get:

$$G_1(\tau) \geq c_1\|u(\tau)\|^2_{H^1(\mathbb{R}^3)}. \tag{3.114}$$

We next compute the derivative $\frac{d}{d\tau}G_1$. One has

$$i\frac{d}{d\tau}\langle Hu, \sigma_3 u\rangle = \frac{2i}{\tau}\,\mathrm{Im} < lu, \sigma_3 Hu >,$$

which implies

$$\left|\frac{d}{d\tau}\langle Hu, \sigma_3 u\rangle\right| \leq \frac{C}{\tau}(|\alpha_1| + |\nu_1|)\|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^3)}. \tag{3.115}$$

Next, we address $\|\mathbb{E}^*_\kappa \sigma_3 u\|^2_{L^2(\mathbb{R}^3)}$. Denote $\Phi(\tau) = \mathbb{E}^*_\kappa\sigma_3 u(\tau)$. Then $\Phi(\tau, k)$ solves

$$i\Phi_\tau = k^2\sigma_3\Phi + \frac{1}{\tau}Y, \tag{3.116}$$

where

$$Y = \mathbb{E}^*_\kappa\sigma_3 lu.$$

Integrating by parts and applying Lemma 3.14 (ii), one can rewrite Y in the form

$$Y(\tau, k) = Y_0(\tau, k) + Y_1(\tau, k),$$

where

$$Y_0(\tau, k) = i\nu_1 k\partial_k\Phi(\tau, k),$$

and $Y_1(\tau, k)$ admits the estimate

$$\|Y_1(\tau)\|_{L^2(\mathbb{R}_+)} \leq C(|\alpha_1| + |\nu_1|)\|u(\tau)\|_{L^2(\mathbb{R}^3)}.$$

Therefore, (3.116) gives

$$\left|\frac{d}{d\tau}\|\Phi(\tau)\|^2_{L^2(\mathbb{R}_+)}\right| \leq \frac{C}{\tau}(|\alpha_1| + |\nu_1|)\|u(\tau)\|^2_{L^2(\mathbb{R}^3)}.$$

Combining this inequality with (3.116) and taking into account (3.114) one gets

$$\left|\frac{d}{d\tau}G_1(\tau)\right| \leq \frac{C}{\tau}(|\alpha_1| + |\nu_1|)\|u(\tau)\|^2_{H^1(\mathbb{R}^3)} \leq \frac{C}{\tau}(|\alpha_1| + |\nu_1|)G_1(\tau). \tag{3.117}$$

Integrating we obtain

$$G_1(\tau) \leq C\left(\frac{s}{\tau}\right)^{C(|\alpha_1|+|\nu_1|)} G_1(s), \quad 0 < \tau \leq s,$$

which by (3.113), (3.114), leads to the bound

$$\|U(\tau, s)f\|_{H^1(\mathbb{R}^3)} \leq C\left(\frac{s}{\tau}\right)^{C(|\alpha_1|+|\nu_1|)} \|f\|_{H^1(\mathbb{R}^3)}, \tag{3.118}$$

for any $0 < \tau \leq s$ and any $f \in H^1_{rad}$. To control the higher regularity, consider the functional $G_2(\tau) = \left\langle H^2 u, \sigma_3 Hu\right\rangle + c_2 G_1(\tau)$. One has

$$C^{-1}\|u\|^2_{H^3(\mathbb{R}^3)} \leq G_2 \leq C\|u\|^2_{H^3(\mathbb{R}^3)},$$

provided c_2 is chosen sufficiently large.
Computing the derivative $\frac{d}{d\tau}\left\langle H^2 u(\tau), \sigma_3 H u(\tau)\right\rangle$ and taking into account (3.117) we get

$$\left|\frac{d}{d\tau}G_2(\tau)\right| \leq \frac{C}{\tau}(|\alpha_1| + |\nu_1|)\|u(\tau)\|^2_{H^3(\mathbb{R}^3)} \leq \frac{C}{\tau}(|\alpha_1| + |\nu_1|)G_2(\tau). \tag{3.119}$$

which implies

$$\|U(\tau, s)f\|_{H^3(\mathbb{R}^3)} \leq C\left(\frac{s}{\tau}\right)^{C(|\alpha_1|+|\nu_1|)}\|f\|_{H^3(\mathbb{R}^3)}, \tag{3.120}$$

for any $0 < \tau \leq s$.
The H^2 bounded stated in Proposition 3.9 follows from (3.118), (3.120) by interpolation.

Acknowledgements

First of all I would like to thank my Supervisors Dott. R. Adami and Dott. D. Noja at Univeristà degli Studi di Milano-Bicocca, and Prof. G. Perelman at Université Paris-Est for all the fruitful discussions and their patience they had in this years. In particular, I am grateful to Prof. Perelman for having accept the *cotutelle* project during my second year in the Ph.D program, Dott. Noja for having supported me in such international adventure and Dott. Adami for having had the idea to begin that all.
I thank Prof. A. Komech and Prof. J. Krieger that kindly accepted to be Referees of this thesis which I hope they found not completely boring.
I also want to thank the member of the Defence Committee: Prof. Hajer Bahouri, Prof. Dario Bambusi, Dott. Diego Noja, and Prof. Galina Perelman.
I am very grateful for the kindness, the niceness as well as the helpfulness of all the people of the "Dipartimento di Matematica e Applicazioni" in Milan and the "Laboratoire LAMA" in Paris. In particular, I wish to turn my thought to all the Ph.D students which are really too many to be all named.
Finally, I thank all the professors (some of them are great friends as well) that I met and encouraged me in this long, difficult, and beautiful way. They also are a lot hence I hope none will get offended if I do not write down all their names. Together with them, I would like to thank all my family and my friends that often not only had to bear that strange and exotic thing named Math but also supported and still support me in finding and walking along my way.

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