ADMISSIBLE TEST FOR SHOULDER CONDITION IN LINE TRANSECT SAMPLING UNDER AN EXPONENTIAL MIXTURE MODEL

PIERO QUATTO

Department of Statistics

University of Milano-Bicocca

Via Bicocca degli Arcimboldi, 8

20126 Milano (MI), Italy

e-mail: piero.quatto@unimib.it

Abstract

Line transect sampling is a distance sampling method widely used for estimating wildlife population density. Since the usual approach assumes a model for the detection function, the estimate depends on the shape of such a function. In particular, the estimate is influenced by the so-called shoulder condition, which ensures that detection is nearly certain at small distances from the line transect. For instance, the half-normal model satisfies this condition, whereas the negative exponential model does not. So, testing whether the shoulder condition is consistent with the data represents a fundamental issue. The aim of this paper is to propose an admissible test for the shoulder condition in the exponential mixture model of the half-normal and the negative exponential. Critical value and p-value of the proposed test are calculated by means of asymptotic distribution theory.

1. Introduction

Distance sampling methodology provides an effective approach for estimating wildlife population density. This paper focus on the line transect design (Buckland et al. 2001), which assumes that.

- k not overlapping lines are randomly chosen within the study area.

- Animals of interest are uniformly distributed with respect to perpendicular distance from the lines.

- Along each of the selected lines, an observer measures the distance from the line to any animal detected.

- Animals on the lines are detected with certainty.

- Animals are detected at their initial location, prior to any movement.

- Distances are measured without errors.

- Detections are independent events.

Since the number of animals observed from each line may be relatively small in several contexts (as for instance in ornithology), sampled distances are pooled together to increase the sample size. If n is the total number of the animals detected from the k lines, let z_1, ..., z_n be the sample of pooled distances.

Let f be the probability density function (pdf) of observed distances and let g be the detection function, i.e. $g(z)$ is the conditional probability of detecting an animal, given that it is at distance z from the line. From above assumptions, z_1, ..., z_n turn out to be independent and identically distributed with pdf

$$f(z) = \frac{g(z)}{\int_0^\infty g(y)dy}. \tag{1}$$

Thus, the general formula for estimating the population density δ is given by

$$\hat{\delta} = \frac{n}{2l}\,\hat{f}(0),$$

where l is the sum of the length of the considered lines and $\hat{f}(0)$ is an estimator of f at 0 which satisfies the fundamental identity

$$f(0) = \frac{1}{\int_0^{+\infty} g(y)dy},$$

under the assumption that the detection on the line is certain, i.e. $g(0) = 1$. So, in order to estimate δ, the essential problem consists in estimating $f(0)$.

At first sight two popular families of detection functions (Zhang, 2001; Eidous, 2005) are considered

the half-normal family

$$g(z) = \exp\left(-\frac{z^2}{2\sigma^2}\right)(\sigma > 0) \tag{2}$$

and the negative exponential family

$$g(z) = \exp\left(-\frac{z}{\lambda}\right)(\lambda > 0). \tag{3}$$

The former satisfies the shape criterion

$$g'(0) = 0 \tag{4}$$

whereas the latter does not.

This property, also known as the shoulder condition, ensures that an animal is nearly certain to be detected if it is at small distance from the observer (Buckland et al., 2001; Buckland et al., 2004).

However, such a condition fails when detectability decreases sharply around the observation line. Eidous (2005) demonstrated that usual estimators of δ are extremely sensitive to departures from the shape criterion. Therefore, testing the shape criterion is a preliminary step for any attempt to estimate wildlife population density via line transect sampling (Mack, 1998; Eidous, 2005).

The aim of this paper is to propose an optimal procedure for testing the shoulder condition (4) in the exponential mixture of (2) and (3) defined by

$$g(z) = \exp\left(\eta z^2 + \theta z\right), \tag{5}$$

with $z > 0$ and

$$(\eta, \theta) \in\,] - \infty, 0] \times] - \infty, 0] - \{(0, 0)\}. \tag{6}$$

In particular, the paper provides an admissible test τ for the pair of hypotheses

$$H_0 : g'(0) = 0 \quad \text{vs} \quad H_1 : g'(0) < 0,$$

that is equivalent to

$$H_0 : \theta = 0 \quad \text{vs.} \quad H_1 : \theta < 0 \tag{7}$$

for the mixture model (5), since $g'(0) = \theta$. so, there cannot exist another test which is at least as powerful as τ against all alternative hypotheses in (7) and more powerful against some.

The admissible test τ is discussed in Section 2. In Section 3 critical value and p-value of τ are calculated through the asymptotic distribution of the test statistic under the null hypothesis.

2. Admissible Test

Because of (5), the distance pdf (1) belongs to the exponential family given by

$$f(z) = \frac{\exp(\eta z^2 + \theta z)}{\gamma(\eta, \theta)}, \tag{8}$$

with $z > 0$, (η, θ) satisfying (6) and

$$\gamma(\eta, \theta) = \int_0^{+\infty} \exp(\eta y^2 + \theta y) dy. \tag{9}$$

Given n independent random variables $Z_1, ..., Z_n$ with pdf (8), we consider the problem of testing (7).

It can be proved (see Appendix 1) that the uniformly most powerful unbiased (UMPU) test τ rejects the null hypothesis for large values of the statistic

$$\mathcal{Q}_n = \frac{\dfrac{1}{n} \sum_{j=i}^{n} Z_i^2}{\left(\dfrac{1}{n} \sum_{i=1}^{n} Z_i\right)^2}. \tag{10}$$

Hence, the critical region of the UMPU test τ can be written as

$$\mathcal{Q}_n \geq q_{n,\alpha}, \tag{11}$$

where α denotes the level of significance and $q_{n,\,\alpha}$ represents the corresponding critical value so that

$$p(\mathcal{Q}_n \geq q_{n,\,\alpha}|\,H_0) = \alpha.$$

Since τ is the UMPU level-α test, it is admissible in the sense that there cannot exist another level-α test which is at least as powerful as τ against all alternative hypotheses in (7) and more powerful against some (Lehmann and Romano, 2005).

It may be observed that (10) is equivalent to the test statistic proposed by Zhang (2001), although optimal properties was not considered in that paper.

3. Critical Value and *p*-Value

The asymptotic normal distribution under H_0

$$\sqrt{n}(\mathcal{Q}_n - \pi/2) \xrightarrow{\ d\ } N(0,\ \pi^2(\pi - 3)/2) \tag{12}$$

is derived from bivariate central limit theorem and delta method (see Appendix 2).

Therefore, for large n the approximate critical value is given by

$$q_{n,\,\alpha} \cong 1.5708 + 0.8359\, z_{1-\alpha}\,/\sqrt{n}, \tag{13}$$

where $z_{1-\alpha}$ is the $(1 - \alpha)$th quantile of the standard normal distribution. Asymptotic critical values given by (13) for a range of sample sizes and significance levels are reported in table 1.

Table 1. Asymptotic critical values $q_{n,\,\alpha}$ of the test.

α	0.01	0.05	0.10
$n = 30$	1.926	1.822	1.766
$n = 40$	1.878	1.788	1.740
$n = 50$	1.846	1.765	1.722
$n = 60$	1.822	1.748	1.709
$n = 100$	1.765	1.708	1.678

Furthermore, the approximate *p*-value is provided by

$$\hat{\alpha} \cong 1 - \Phi\left(\frac{q - 1.5708}{0.8359}\sqrt{n}\right), \tag{14}$$

where Φ is the cumulative distribution function of the standard normal distribution and q is the observed value of the test statistic (10).

Conclusions

For the problem of testing the shoulder condition in line transect sampling an admissible test τ is provided in a suitable exponential family. Hence, there cannot exist another test which is at least as powerful as τ against all considered alternatives and more powerful against some.

Finally, the limiting null distribution of the test statistic is derived and some asymptotic critical values are tabulated.

Appendix 1.

Let Z_1, ..., Z_n be n independent random variables with pdf belonging to the exponential family defined by (8).

Under the null hypothesis in (7), the distribution of the test statistic $\mathcal{Q}_n$ does not depend on the nuisance parameter η, since (10) can be written as

$$\mathcal{Q}_n = \frac{\dfrac{1}{n}\sum_{i=1}^{n} X_i^2}{\left(\dfrac{1}{n}\sum_{i=1}^{n} X_i\right)^2},$$

where

$$X_i = \sqrt{-2\eta}\, Z_i \quad (i = 1, ..., n;\ \eta < 0)$$

has pdf independent of η for the reason that (9) and (8) become respectively

$$\gamma(\eta,\, 0) = \int_0^{+\infty} \exp(\eta y^2)dy = \sqrt{-\frac{\pi}{4\eta}}$$

and

$$f(z) = \sqrt{-\frac{4\eta}{\pi}} \exp(\eta z^2) \ (z > 0).$$

Hence (by Corollary 5.1.1 of Lehmann and Romano, 2005) the statistic (10) is independent of

$$S_n = \sum_{i=1}^{n} Z_i^2$$

when $\theta = 0$. Moreover, (10) is decreasing in

$$T_n = \sum_{i=1}^{n} Z_i.$$

Therefore (by Theorem 5.1.1 of Lehmann and Romano, 2005) the UMPU test rejects H_0 for large values of $\mathcal{Q}_n$, as asserted in Section 2.

Appendix 2.

Let $Z_1, ..., Z_n$ be n independent random variables with pdf belonging to the exponential family defined by (8).

By bivariate central limit theorem (Lehmann, 2001),

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i - \mu_1, \ \frac{1}{n}\sum_{i=1}^{n} Z_i^2 - \mu_2\right) \xrightarrow{\ d\ } N(0, \Sigma),$$

where

$$\mu_1 = E(Z), \ \mu_2 = E(Z^2), \ 0 = (0, 0)$$

and

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

with

$$\sigma_1^2 = Var(Z), \ \sigma_2^2 = Var(Z^2), \ \sigma_{12} = Cov(Z, Z^2).$$

Then, by delta method (Lehmann, 2001),

$$\sqrt{n}(\mathcal{Q}_n - \mu) \xrightarrow{\ d\ } N(0,\ \nu^2),$$

where

$$\mu = \frac{\mu_2}{\mu_1^2}$$

and

$$V^2 = \frac{4\mu_2^2}{\mu_1^6}\,\sigma_1^2 - \frac{4\mu_2}{\mu_1^5}\,\sigma_{12} + \frac{1}{\mu_1^4}\,\sigma_2^2 = \frac{\mu_4 - \mu_2^2}{\mu_1^4} + 4\mu_2\,\frac{\mu_2^2 - \mu_1\mu_3}{\mu_1^6}$$

with

$$\mu_3 = E(Z^3),\ \mu_4 = E(Z^4).$$

So, under H_0

$$\mu = \pi/2$$

and

$$\nu^2 = \pi^2(\pi - 3)/2.$$

Hence (12) follows.

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