# ON THE BEHAVIOR AT COLLISIONS OF SOLUTIONS TO SCHRÖDINGER EQUATIONS WITH MANY-PARTICLE AND CYLINDRICAL POTENTIALS

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Abstract. The asymptotic behavior of solutions to Schrödinger equations with singular homogeneous potentials is investigated. Through an Almgren type monotonicity formula and separation of variables, we describe the exact asymptotics near the singularity of solutions to at most critical semilinear elliptic equations with cylindrical and quantum multi-body singular potentials. Furthermore, by an iterative Brezis-Kato procedure, pointwise upper estimate are derived.

## 1. Introduction

The purpose of the present paper is to describe the behavior of solutions to a class of Schrödinger equations with singular homogeneous potentials including cylindrical and quantum multi-body ones.

The interaction between $M$ particles of coordinates $y^1, \dots, y^M$ in $\mathbb{R}^k$ is described in classical mechanics by potentials of the form

$$V(y^1, \dots, y^M) = \sum_{\substack{j,m=1 \\ j<m}}^{M} V_{j,m}(y^j - y^m)$$

where $V_{j,m}(y) \to 0$ as $|y| \to +\infty$, see [28]. From the mathematical point of view, a particular interest arises in the case of inverse square potentials $V_{j,m}(y) = \frac{\lambda_j \lambda_m}{|y|^2}$, since they have the same order of homogeneity as the laplacian thus making the corresponding Schrödinger operator invariant by scaling. Schrödinger equations with the resulting $M$-body potential

$$V(y^1, \dots, y^M) = \sum_{\substack{j,m=1 \\ j<m}}^{M} \frac{\lambda_j \lambda_m}{|y^j - y^m|^2}, \quad \lambda_j, \lambda_m \in \mathbb{R}, \tag{1}$$

have been studied by several authors; we mention in particular [27] where many-particle Hardy inequalities are proved and [12] where the existence of ground state solutions for semilinear Schrödinger equations with potentials of type (1) is investigated. It is worth pointing out that hamiltonians with singular potentials having the same homogeneity as the operator arise in relativistic quantum mechanics, see [31].

There is a natural relation between 2-particle potentials (1) and cylindrical potentials, whose singular set is some $k$-codimensional subspace of the configuration space. Indeed, in the special

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case $M = 2$, after the change of variables in $\mathbb{R}^{2k}$

$$z^1 = \frac{1}{\sqrt{2}}(y^1 - y^2), \quad z^2 = \frac{1}{\sqrt{2}}(y^1 + y^2), \tag{2}$$

the potential $V(y^1, y^2) = \frac{\lambda_1 \lambda_2}{|y_1 - y_2|^2}$ takes the form

$$\frac{\lambda_1 \lambda_2}{2|z^1|^2}. \tag{3}$$

Elliptic equations with cylindrical inverse square potentials arise in several fields of applications, e.g. in the search for solitary waves with no vanishing angular momentum of nonlinear evolution equations of Schrödinger and Klein-Gordon type, see [3]. In the recent literature, many papers have been devoted to the study of semilinear elliptic equations with cylindrical potentials; we mention among others [3, 4, 5, 32, 36]. We point out that cylindrical type (and a fortiori many-particle) potentials give rise to substantially major difficulties with respect to the case of an isolated singularity, because in the cylindrical/many-particle case separation of variables (radial and angular) does not actually "eliminate" the singularity, being the angular part of the operator also singular.

We consider both linear and semilinear Schrödinger equations with singular homogeneous potentials belonging to a class including as particular cases both (1) and (3). For every $3 \leqslant k \leqslant N$, let us define the sets

$$\mathcal{A}_k := \big\{ J \subseteq \{1, 2, \dots, N\} \text{ such that } \#J = k \big\}$$

and

$$\mathcal{B}_k := \{ (J_1, J_2) \in \mathcal{A}_k \times \mathcal{A}_k \text{ such that } J_1 \cap J_2 = \emptyset \text{ and } J_1 < J_2 \}$$

where $\#J$ stands for the cardinality of $J$ and $J_1 < J_2$ stands for the "alphabetic ordering" for multi-indices (see the list of notations at the end of this section).

In the sequel, for every $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ and $J \in \mathcal{A}_k$, we denote as $x_J$ the $k$-uple $(x_i)_{i \in J}$ so that $|x_J|^2 = \sum_{i \in J} x_i^2$. In a similar way, for any $x \in \mathbb{R}^N \setminus \{0\}$ and $J \in \mathcal{A}_k$ we write $\theta_J = \frac{x_J}{|x|}$. Moreover we denote

$$\begin{aligned} \Sigma := &\{ (\theta_1, \dots, \theta_N) \in \mathbb{S}^{N-1} : \theta_J = 0 \text{ for some } J \in \mathcal{A}_k \} \\ &\cup \{ (\theta_1, \dots, \theta_N) \in \mathbb{S}^{N-1} : \theta_{J_1} = \theta_{J_2} \text{ for some } (J_1, J_2) \in \mathcal{B}_k \} \end{aligned} \tag{4}$$

and

$$\widetilde{\Sigma} = \{ x \in \mathbb{R}^N \setminus \{0\} : x/|x| \in \Sigma \} \cup \{0\}. \tag{5}$$

The potentials we are going to consider are of the type

$$V(x) = \sum_{J \in \mathcal{A}_k} \frac{\alpha_J}{|x_J|^2} + \sum_{(J_1, J_2) \in \mathcal{B}_k} \frac{\alpha_{J_1 J_2}}{|x_{J_1} - x_{J_2}|^2}, \quad \text{for all } x \in \mathbb{R}^N \setminus \widetilde{\Sigma}, \tag{6}$$

where $\alpha_J, \alpha_{J_1 J_2} \in \mathbb{R}$. We notice that $\mathcal{B}_k$ is empty whenever $k > \frac{N}{2}$; in such a case we consider potentials $V$ with only the cylindrical part, i.e. with only the first summation at right hand side of (6).

Letting, for all $\theta \in \mathbb{S}^{N-1} \setminus \Sigma$,

$$a(\theta) = \sum_{J \in \mathcal{A}_k} \frac{\alpha_J}{|\theta_J|^2} + \sum_{(J_1, J_2) \in \mathcal{B}_k} \frac{\alpha_{J_1 J_2}}{|\theta_{J_1} - \theta_{J_2}|^2} \not\equiv 0, \tag{7}$$

we can write the potential $V$ in (6) as

$$V(x) = \frac{a(\frac{x}{|x|})}{|x|^2}$$

and the associated hamiltonian as

$$\mathcal{L}_a = -\Delta - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2}.$$

As a natural setting to study the properties of operators $\mathcal{L}_a$, we introduce the functional space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ defined as the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the Dirichlet norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \bigg(\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx\bigg)^{1/2}.$$

The potential $V$ in (6) satisfies a Hardy type inequality. Indeed, it was proved in [33] (see also [5] and [39]) that the following Hardy's inequality for cylindrically singular potentials holds:

$$\left(\frac{k-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x_J|^2}\,dx \leqslant \int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx \tag{8}$$

for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $J \in \mathcal{A}_k$, being the constant $\big(\frac{k-2}{2}\big)^2$ optimal. Using a change of variables of type (2), from (8) it follows the "two-particle Hardy inequality":

$$\frac{(k-2)^2}{2} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x_{J_1} - x_{J_2}|^2}\,dx \leqslant \int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx \tag{9}$$

for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $(J_1, J_2) \in \mathcal{B}_k$, being the constant $\frac{(k-2)^2}{2}$ optimal. From (8) and (9) we deduce that the potential $V$ in (6) satisfies the following "many-particle Hardy inequality":

$$\left(\frac{k-2}{2}\right)^2 \int_{\mathbb{R}^N} V(x)|u(x)|^2\,dx \leqslant \bigg(\sum_{J\in\mathcal{A}_k} \alpha_J^+ + \sum_{(J_1,J_2)\in\mathcal{B}_k} \alpha^+_{J_1J_2}\bigg) \int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx \tag{10}$$

for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, where $\alpha_J^+ = \max\{\alpha_J, 0\}$ and $\alpha^+_{J_1J_2} = \max\{\alpha_{J_1J_2}, 0\}$. We refer to [27] for a deep analysis of many-particle Hardy inequalities and related best constants.

In order to discuss the positivity properties of the Schrödinger operator $\mathcal{L}_a$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we consider the best constant in the Hardy-type inequality (10), i.e.

$$\Lambda(a) := \sup_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |x|^{-2}a(x/|x|)\,u^2(x)\,dx}{\displaystyle\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx}. \tag{11}$$

By (10), $\Lambda(a) \leqslant \frac{4}{(k-2)^2}(\sum_{J\in\mathcal{A}_k} \alpha_J^+ + \sum_{(J_1,J_2)\in\mathcal{B}_k} \alpha^+_{J_1J_2})$. It is easy to verify that the quadratic form associated to $\mathcal{L}_a$ is positive definite in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ if and only if

$$\Lambda(a) < 1. \tag{12}$$

The relation between the value $\Lambda(a)$ and the first eigenvalue of the angular component of the operator on the unit $(N-1)$-dimensional sphere $\mathbb{S}^{N-1}$ is discussed in Lemma 2.3. More precisely, Lemma 2.3 ensures that the quadratic form associated to $\mathcal{L}_a$ is positive definite if and only if

$$\mu_1(a) > -\left(\frac{N-2}{2}\right)^2,$$

where $\mu_1(a)$ is the first eigenvalue of the operator $L_a := -\Delta_{\mathbb{S}^{N-1}} - a$ on the sphere $\mathbb{S}^{N-1}$. The spectrum of the angular operator $L_a$ is discrete and consists in a nondecreasing sequence of eigenvalues

$$\mu_1(a) \leqslant \mu_2(a) \leqslant \cdots \leqslant \mu_k(a) \leqslant \cdots$$

diverging to $+\infty$, see Lemma 2.2.

We study nonlinear equations obtained as perturbations of the operator $\mathcal{L}_a$ in a bounded domain $\Omega \subset \mathbb{R}^N$ containing the origin. More precisely, we deal with semilinear equations of the type

$$\mathcal{L}_a u = h(x)\, u + f(x,u), \quad \text{in } \Omega\,. \tag{13}$$

We assume that the linear perturbing potential $h$ is negligible with respect to the potential $V$ near the collision singular set $\widetilde{\Sigma}$ defined in (5), in the sense that there exist $C_h > 0$ and $\varepsilon > 0$ such that, for a.e. $x \in \Omega \setminus \widetilde{\Sigma}$,

**(H)** $h \in W^{1,\infty}_{\rm loc}(\Omega \setminus \widetilde{\Sigma})$ and $\displaystyle |h(x)| + |\nabla h(x)\cdot x| \leqslant C_h\bigg(\sum_{J\in\mathcal{A}_k} |x_J|^{-2+\varepsilon} + \sum_{(J_1,J_2)\in\mathcal{B}_k} |x_{J_1} - x_{J_2}|^{-2+\varepsilon}\bigg).$

We notice that it is not restrictive to assume $\varepsilon \in (0,1)$ in (**H**).

As far as the nonlinear perturbation is concerned, we assume that $f$ satisfies

**(F)** $$\begin{cases} f \in C^0(\Omega\times\mathbb{R}), \quad F \in C^1(\Omega\times\mathbb{R}), \quad s \mapsto f(x,s) \in C^1(\mathbb{R}) \text{ for a.e. } x \in \Omega, \\ |f(x,s)s| + |f'_s(x,s)s^2| + |\nabla_x F(x,s)\cdot x| \leqslant C_f(|s|^2 + |s|^{2^*}) \quad \text{for a.e. } x\in\Omega \text{ and all } s \in \mathbb{R}, \end{cases}$$

where $F(x,s) = \int_0^s f(x,t)\,dt$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $C_f > 0$ is a constant independent of $x \in \Omega$ and $s \in \mathbb{R}$, $\nabla_x F$ denotes the gradient of $F$ with respect to the $x$ variable, and $f'_s(x,s) = \frac{\partial f}{\partial s}(x,s)$.

We say that a function $u \in H^1(\Omega)$ is a $H^1(\Omega)$-weak solution to (13) if, for all $w \in H^1_0(\Omega)$,

$$\mathcal{Q}^\Omega_a(u,w) = \int_\Omega h(x)u(x)w(x)\,dx + \int_\Omega f(x,u(x))w(x)\,dx,$$

where $\mathcal{Q}^\Omega_a : H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{Q}^\Omega_a(u,w) := \int_\Omega \nabla u(x)\cdot\nabla w(x)\,dx - \int_\Omega \frac{a(x/|x|)}{|x|^2}\,u(x)w(x)\,dx.$$

Schrödinger equations with inverse square homogeneous singular potentials can be regarded as critical from the mathematical point of view, as they do not belong to the Kato class. A rich literature deals with such critical equations, both in the case of one isolated pole, see e.g. [16, 24, 25, 29, 40, 42], and in that of multiple singularities, see [7, 14, 15, 19, 23]. The analysis of fundamental spectral properties such as essential self-adjointness and positivity carried out in [19, 21] for Schrödinger operators with isolated inverse square singularities, highlighted how the asymptotic behavior of solutions to associated elliptic equations near the singularity plays a crucial role. A precise evaluation of the asymptotics of solutions turned out to be an important tool also to establish existence of ground states for nonlinear Schrödinger equations with multi-singular Hardy potentials (see [23]) and of solutions to nonlinear systems of Schrödinger equations with

Hardy potentials [1]. A first result about the study of the asymptotic behavior of solutions near isolated singularities is contained in [22], where Hölder continuity of solutions to degenerate elliptic equations with singular weights has been established thus allowing the evaluation of the exact asymptotic behavior of solutions to Schrödinger equations with Hardy potentials near the pole. An extension to the case of Schrödinger equations with dipole-type potentials (namely purely angular multiples of inverse square potentials) has been obtained in [20] by separation of variables and comparison principles, and later generalized to Schrödinger equations with singular homogeneous electromagnetic potentials of Aharonov-Bohm type [17] by the Almgren monotonicity formula. Comparison and maximum principles play a crucial role also in [37], where the existence of the limit at the singularity of any quotient of two positive solutions to Fuchsian type elliptic equations is proved. We mention that related asymptotic expansions near singularities were obtained in [34, 35] for elliptic equations on manifolds with conical singularities by Mellin transform methods (see also [30]); we refer to [18] for a comparison between such results and asymptotics via Almgren monotonicity methods. It is also worth citing [9], where some asymptotic formulas are heuristically obtained for the three-body one-dimensional problem.

Due to the presence of multiple collisions, one should expect that solutions to equations (13) behave singularly at the origin: our purpose is to describe the rate and the shape of the singularity of solutions, by relating them to the eigenvalues and the eigenfunctions of the angular operator $L_a$ on the sphere $\mathbb{S}^{N-1}$.

The following theorem provides a classification of the behavior of any solution $u$ to (13) near the singularity based on the limit as $r \to 0^+$ of the *Almgren's frequency* function (see [2, 26])

$$\mathcal{N}_{u,h,f}(r) = \frac{r \int_{B_r} \left( |\nabla u(x)|^2 - \frac{a(x/|x|)}{|x|^2} u^2(x) - h(x)u^2(x) - f(x,u(x)) \right) dx}{\int_{\partial B_r} |u(x)|^2 \, dS}, \tag{14}$$

where, for any $r > 0$, $B_r$ denotes the ball $\{x \in \mathbb{R}^N : |x| < r\}$.

**Theorem 1.1.** *Let $u \not\equiv 0$ be a nontrivial weak $H^1(\Omega)$-solution to (13) in a bounded open set $\Omega \subset \mathbb{R}^N$ containing 0, $N \geqslant k \geqslant 3$, with a satisfying (7) and (12), h satisfying* (**H**), *and f satisfying* (**F**). *Then, letting $\mathcal{N}_{u,h,f}(r)$ as in* (14), *there exists $k_0 \in \mathbb{N}$, $k_0 \geqslant 1$, such that*

$$\lim_{r \to 0^+} \mathcal{N}_{u,h,f}(r) = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(a)}. \tag{15}$$

*Furthermore, if $\gamma$ denotes the limit in (15), $m \geqslant 1$ is the multiplicity of the eigenvalue $\mu_{k_0}(a)$ and $\{\psi_i : j_0 \leqslant i \leqslant j_0 + m - 1\}$ ($j_0 \leqslant k_0 \leqslant j_0 + m - 1$) is an $L^2(\mathbb{S}^{N-1})$-orthonormal basis for the eigenspace associated to $\mu_{k_0}(a)$, then*

$$\lambda^{-\gamma} u(\lambda x) \to |x|^\gamma \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i\left(\frac{x}{|x|}\right) \qquad \text{in } H^1(B_1) \quad \text{as } \lambda \to 0^+ \tag{16}$$

*where*

$$\beta_i = \int_{\mathbb{S}^{N-1}} \left[ R^{-\gamma} u(R\theta) + \int_0^R \frac{h(s\theta)u(s\theta) + f\big(s\theta, u(s\theta)\big)}{2\gamma + N - 2} \left( s^{1-\gamma} - \frac{s^{\gamma+N-1}}{R^{2\gamma+N-2}} \right) ds \right] \psi_i(\theta) \, dS(\theta), \tag{17}$$

*for all $R > 0$ such that $\overline{B_R} = \{x \in \mathbb{R}^N : |x| \leqslant R\} \subset \Omega$ and $(\beta_{j_0}, \beta_{j_0+1}, \dots, \beta_{j_0+m-1}) \neq (0, 0, \dots, 0)$.*

Due to the homogeneity of the potentials, Schrödinger operators $\mathcal{L}_a$ are invariant by the Kelvin transform,

$$\tilde{u}(x) = |x|^{-(N-2)} u\bigg(\frac{x}{|x|^2}\bigg),$$

which is an isomorphism of $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Indeed, if $u \in H^1(\Omega)$ weakly solves (13) in a bounded open set $\Omega$ containing 0, then its Kelvin's transform $\tilde{u}$ weakly solves (13) with $h$ replaced by $|x|^{-4}h(\frac{x}{|x|^2})$ and $f(x,\cdot)$ replaced by $|x|^{-N-2}f\big(\frac{x}{|x|^2}, |x|^{N-2}\cdot\big)$ in the external domain $\widetilde{\Omega} = \big\{x \in \mathbb{R}^N : x/|x|^2 \in \Omega\big\}$. Therefore, under suitable decay conditions on $h$ at $\infty$ and proper subcriticality assumptions on $f$, the asymptotic behavior at infinity of solutions to (13) in external domains can be easily deduced from Theorem 1.1 and the Kelvin transform (see [17, Theorems 1.4 and 1.6]).

A major breakthrough in the description of the singularity of solutions at zero can be done by evaluating the behavior of eigenfunctions $\psi_i$; indeed such eigenfunctions solve an elliptic equation on $\mathbb{S}^{N-1}$ exhibiting itself a potential which is singular on $\Sigma$. After a stereographic projection of $\mathbb{S}^{N-1}$ onto $\mathbb{R}^{N-1}$, the equation satisfied by each $\psi_i$ takes a form which is similar to (13) in a lowered dimension with a potential whose singular set is $(N-1-k)-$dimensional and to which we can apply the above theorem to deduce a precise asymptotics in terms of eigenvalues and eigenfunctions of an operator on $\mathbb{S}^{N-2}$; the procedure can be iterated $(N-k)-$times until we come to an equation with a potential with isolated singularities whose corresponding angular operator is no more singular. A detailed analysis of the asymptotic behavior of eigenfunctions is performed in section 7.

A pointwise upper estimate on the behavior of solutions can be derived by a Brezis-Kato type iteration argument, see [8]. More precisely, we can estimate the solutions by terms of the first eigenvalue and eigenfunction of the angular potential $\hat{a}$ obtained by summing up only the positive contributions of $a$, i.e.

$$\hat{a}(\theta) = \sum_{J\in\mathcal{A}_k} \frac{\alpha_J^+}{|\theta_J|^2} + \sum_{(J_1,J_2)\in\mathcal{B}_k} \frac{\alpha^+_{J_1\,J_2}}{|\theta_{J_1}-\theta_{J_2}|^2}. \tag{18}$$

Under the assumption

$$\Lambda(\hat{a}) = \sup_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-2}\hat{a}(x/|x|)\, u^2(x)\,dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx} < 1, \tag{19}$$

by Lemma 2.3 the number

$$\hat{\sigma} = -\frac{N-2}{2} + \sqrt{\bigg(\frac{N-2}{2}\bigg)^2 + \mu_1(\hat{a})} \tag{20}$$

is well defined. We denote as $\hat{\psi}_1 \in H^1(\mathbb{S}^{N-1})$, $\|\hat{\psi}_1\|_{L^2(\mathbb{S}^{N-1})} = 1$, the first positive $L^2-$normalized eigenfuntion of the eigenvalue problem $L_a\psi = \mu_1(\hat{a})\psi$ in $\mathbb{S}^{N-1}$.

**Theorem 1.2.** *Let $u$ be a weak $H^1(\Omega)$-solution to (13) in a bounded open set $\Omega \subset \mathbb{R}^N$ containing 0, $N \geqslant k \geqslant 3$, with $a$ satisfying (7) and $\hat{a}$ as in (18) satisfying (19). If $h$ satisfies* (**H**) *and $f$ satisfies* (**F**)*, then for any $\Omega' \Subset \Omega$ there exists $C > 0$ such that*

$$|u(x)| \leqslant C|x|^{\hat{\sigma}}\hat{\psi}_1\Big(\frac{x}{|x|}\Big) \quad \textit{for a.e. } x \in \Omega'.$$

In particular, if all $\alpha_J, \alpha_{J_1J_2}$ are positive, then $\hat{a} \equiv a$ and the above theorem ensures that all solutions are pointwise bounded by $|x|^\sigma \psi_1(x/|x|)$ where $\sigma = -\frac{N-2}{2} + \big[(\frac{N-2}{2})^2 + \mu_1(a)\big]^{1/2}$. On the other hand, if all $\alpha_J, \alpha_{J_1J_2}$ are negative, then $\hat{a} \equiv 0$ and the above theorem implies that all solutions are bounded.

The paper is organized as follows. In section 2 we prove some Hardy-type inequalities with singular potentials of type (6) and discuss the relation between the positivity of the quadratic form associated to $\mathcal{L}_a$ and the first eigenvalue of the angular operator on the sphere $\mathbb{S}^{N-1}$. In section 3 we derive a Pohozaev-type identity for solutions to (13) through a suitable approximating procedure which allows getting rid of the singularity of the angular potential. In Section 4 we deduce a Brezis-Kato estimate to prove an a-priori super-critical summability of solutions to (13) which allows us to include the critical growth case in the Almgren type monotonicity formula which is obtained in Section 5 and which is used in section 6 together with a blow-up method to prove Theorem 1.1. Section 7 is devoted to the study of the asymptotic behavior of the eigenfunctions of the angular operator. Section 8 contains some Brezis-Kato estimates in weighted Sobolev spaces which allow proving Theorem 1.2. A final appendix contains a Pohozaev-type identity for semilinear elliptic equations with an anisotropic inverse-square potential with a bounded angular coefficient.

**Notation.** We list below some notation used throughout the paper.

- For all $r > 0$, $B_r$ denotes the ball $\{x \in \mathbb{R}^N : |x| < r\}$ in $\mathbb{R}^N$ with center at 0 and radius $r$.
- For all $r > 0$, $\overline{B_r} = \{x \in \mathbb{R}^N : |x| \leqslant r\}$ denotes the closure of $B_r$.
- $dS$ denotes the volume element on the spheres $\partial B_r$, $r > 0$.
- If $J_1 = \{j_{1,1}, \dots, j_{1,k}\}$ and $J_2 = \{j_{2,1}, \dots, j_{2,k}\}$ are two multi-indices of $k$ elements, by $J_1 < J_2$ we mean that there exists $n \in \{1, \dots, k\}$ such that $j_{1,i} = j_{2,i}$ for any $1 \leqslant i \leqslant n-1$ and $j_{1,n} < j_{2,n}$.
- For all $t \in \mathbb{R}$, $t^+ = t_+ := \max\{t, 0\}$ (respectively $t^- = t_- := \max\{-t, 0\}$) denotes the positive (respectively negative) part of $t$.
- $S = \inf_{v \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \|\nabla v\|^2_{L^2} \|v\|^{-2}_{L^{2^*}}$ denotes the best constant in the classical Sobolev's embedding.

## 2. Hardy type inequalities

The following Hardy's inequality on the unit sphere holds.

**Lemma 2.1.** *Let $a$ as in (7). For every $\psi \in H^1(\mathbb{S}^{N-1})$ there holds*

$$\left(\frac{k-2}{2}\right)^2 \int_{\mathbb{S}^{N-1}} a(\theta)|\psi(\theta)|^2\,dS \leqslant \bigg(\sum_{J\in\mathcal{A}_k} \alpha_J^+ + \sum_{(J_1,J_2)\in\mathcal{B}_k} \alpha^+_{J_1\,J_2}\bigg)\bigg[\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}}\psi(\theta)|^2\,dS + \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{S}^{N-1}} |\psi(\theta)|^2\,dS\bigg].$$

Proof. Let $\psi \in H^1(\mathbb{S}^{N-1})$ and $\phi \in C^\infty_{\rm c}(0,+\infty)$. Rewriting inequality (10) for $u(x)=\phi(r)\psi(\theta)$, $r=|x|$, $\theta=\frac{x}{|x|}$, we obtain that

$$\begin{aligned}\Big(\frac{k-2}{2}\Big)^2\bigg(\int_0^{+\infty}\frac{r^{N-1}}{r^2}\,\phi^2(r)\,dr\bigg)\bigg(\int_{\mathbb{S}^{N-1}}a(\theta)|\psi(\theta)|^2\,dS\bigg)&\\ \leqslant\bigg(\sum_{J\in\mathcal{A}_k}\alpha_J^+ + \sum_{(J_1,J_2)\in\mathcal{B}_k}\alpha^+_{J_1\,J_2}\bigg)\bigg(\int_0^{+\infty} r^{N-1}|\phi'(r)|^2\,dr\bigg)\bigg(\int_{\mathbb{S}^{N-1}}|\psi(\theta)|^2\,dS\bigg)&\\ +\bigg(\sum_{J\in\mathcal{A}_k}\alpha_J^+ + \sum_{(J_1,J_2)\in\mathcal{B}_k}\alpha^+_{J_1\,J_2}\bigg)\bigg(\int_0^{+\infty}\frac{r^{N-1}}{r^2}\,\phi^2(r)\,dr\bigg)\bigg(\int_{\mathbb{S}^{N-1}}|\nabla_{\mathbb{S}^{N-1}}\psi(\theta)|^2\,dS\bigg),&\end{aligned}$$

and hence, by optimality of the classical Hardy constant,

$$\begin{aligned}\Big(\frac{k-2}{2}\Big)^2\bigg(\int_{\mathbb{S}^{N-1}}a(\theta)|\psi(\theta)|^2\,dS\bigg)&\\ \leqslant\bigg(\sum_{J\in\mathcal{A}_k}\alpha_J^+ + \sum_{(J_1,J_2)\in\mathcal{B}_k}\alpha^+_{J_1\,J_2}\bigg)\bigg[\bigg(\int_{\mathbb{S}^{N-1}}|\psi(\theta)|^2\,dS\bigg)\inf_{\phi\in C^\infty_{\rm c}(0,+\infty)}\frac{\int_0^{+\infty}r^{N-1}|\phi'(r)|^2\,dr}{\int_0^{+\infty}r^{N-3}\,\phi^2(r)\,dr}&\\ +\int_{\mathbb{S}^{N-1}}|\nabla_{\mathbb{S}^{N-1}}\psi(\theta)|^2\,dS\bigg]&\\ =\bigg(\sum_{J\in\mathcal{A}_k}\alpha_J^+ + \sum_{(J_1,J_2)\in\mathcal{B}_k}\alpha^+_{J_1\,J_2}\bigg)\bigg[\Big(\frac{N-2}{2}\Big)^2\int_{\mathbb{S}^{N-1}}|\psi(\theta)|^2\,dS+\int_{\mathbb{S}^{N-1}}|\nabla_{\mathbb{S}^{N-1}}\psi(\theta)|^2\,dS\bigg].&\end{aligned}$$

The proof is thereby complete. □

Let us consider the following class of angular potentials

$$\mathcal{F}:=\bigg\{f\in L^\infty_{\rm loc}(\mathbb{S}^{N-1}\setminus\Sigma):\frac{|f(\theta)|}{\sum_{J\in\mathcal{A}_k}|\theta_J|^{-2}+\sum_{(J_1,J_2)\in\mathcal{B}_k}|\theta_{J_1}-\theta_{J_2}|^{-2}}\in L^\infty(\mathbb{S}^{N-1})\bigg\}. \tag{21}$$

From Lemma 2.1 we have that, for every $f\in\mathcal{F}$, the supremum

$$\Lambda(f):=\sup_{\psi\in H^1(\mathbb{S}^{N-1})\setminus\{0\}}\frac{\int_{\mathbb{S}^{N-1}}f(\theta)\,\psi^2(\theta)\,dS(\theta)}{\int_{\mathbb{S}^{N-1}}|\nabla_{\mathbb{S}^{N-1}}\psi(\theta)|^2\,dS(\theta)+\big(\frac{N-2}{2}\big)^2\int_{\mathbb{S}^{N-1}}\psi^2(\theta)\,dS(\theta)} \tag{22}$$

is finite. On the other hand, arguing as in the proof of [42, Lemma 1.1], we can easily verify that

$$\Lambda(f)=\sup_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\displaystyle\int_{\mathbb{R}^N}|x|^{-2}f(x/|x|)\,u^2(x)\,dx}{\displaystyle\int_{\mathbb{R}^N}|\nabla u(x)|^2\,dx}. \tag{23}$$

Furthermore, it is easy to verify that

$$\Lambda(f)\geqslant 0$$

and

$$\Lambda(f)=0\quad\text{if and only if}\quad f\leqslant 0 \text{ a.e. in } \mathbb{S}^{N-1}.$$

For every $f\in\mathcal{F}$ satisfying $\Lambda(f)<1$, we can perform a complete spectral analysis of the angular Schrödinger operator $-\Delta_{\mathbb{S}^{N-1}}-f$ on the sphere.

**Lemma 2.2.** *Let $f \in \mathcal{F}$ satisfying $\Lambda(f) < 1$. Then the spectrum of the operator*

$$L_f := -\Delta_{\mathbb{S}^{N-1}} - f$$

*on $\mathbb{S}^{N-1}$ consists in a diverging sequence $\mu_1(f) \leqslant \mu_2(f) \leqslant \cdots \leqslant \mu_k(f) \leqslant \cdots$ of real eigenvalues with finite multiplicity the first of which admits the variational characterization*

$$\mu_1(f) = \min_{\psi \in H^1(\mathbb{S}^{N-1}) \setminus \{0\}} \frac{\int_{\mathbb{S}^{N-1}} \left[ \left| \nabla_{\mathbb{S}^{N-1}} \psi(\theta) \right|^2 - f(\theta) |\psi(\theta)|^2 \right] dS(\theta)}{\int_{\mathbb{S}^{N-1}} |\psi(\theta)|^2 \, dS(\theta)}. \tag{24}$$

*Moreover $\mu_1(f)$ is simple and its associated eigenfunctions do not change sign in $\mathbb{S}^{N-1}$.*

PROOF. By Lemma 2.1 and assumption $\Lambda(f) < 1$, the operator $T : L^2(\mathbb{S}^{N-1}) \to L^2(\mathbb{S}^{N-1})$ defined as

$$Th = u \quad \text{if and only if} \quad -\Delta_{\mathbb{S}^{N-1}} u - fu + \left(\tfrac{N-2}{2}\right)^2 u = h$$

is well-defined, symmetric, and compact. The conclusion follows from classical spectral theory. In particular, we point out that the simplicity of the first eigenvalue follows from the fact that, since $k > 1$, the singular set $\Sigma$ does not disconnect the sphere. □

For all $f \in \mathcal{F}$, let us consider the quadratic form associated to the Schrödinger operator $\mathcal{L}_f$, i.e.

$$Q_f(u) := \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^N} \frac{f(x/|x|) \, u^2(x)}{|x|^2} \, dx.$$

The problem of positivity of $Q_f$ is solved in the following lemma.

**Lemma 2.3.** *Let $f \in \mathcal{F}$. The following conditions are equivalent:*

i) $Q_f$ *is positive definite, i.e.* $\displaystyle \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{Q_f(u)}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx} > 0;$

ii) $\Lambda(f) < 1$;

iii) $\mu_1(f) > -\left(\frac{N-2}{2}\right)^2$ *where $\mu_1(f)$ is defined in (24).*

PROOF. The equivalence between i) and ii) follows from the definition of $\Lambda(f)$, see (23). On the other hand, arguing as in [42, Proposition 1.3 and Lemma 1.1] (see also [17, Lemmas 1.1 and 2.1]) one can obtain equivalence between i) and iii). □

Henceforward, we shall assume that (12) holds, so that the quadratic form associated to the operator $\mathcal{L}_a$ is positive definite.

**Example 2.4.** Let us consider cylindrical potentials, i.e. the particular case in which

$$\alpha_J = \begin{cases} \alpha, & \text{if } J = \bar{J} = \{1, 2, \dots, k\}, \\ 0, & \text{if } J \neq \{1, 2, \dots, k\}, \end{cases} \qquad \text{for some } \alpha \in \mathbb{R} \tag{25}$$

and

$$\alpha_{J_1 \, J_2} = 0 \qquad \text{for any } (J_1, J_2) \in \mathcal{B}_k, \tag{26}$$

so that $a(\theta) = \alpha / |\theta_{\bar{J}}|^2$. Then, from the optimality of the constant $\left(\frac{k-2}{2}\right)^2$ in (8), it follows that $\Lambda(a) = \alpha^+ \left(\frac{2}{k-2}\right)^2$ and (12) reads as $\alpha < \left(\frac{k-2}{2}\right)^2$. Moreover there holds

$$\mu_1(a) = -\frac{(k-2)(N-k)}{2} - \alpha + (N-k) \sqrt{\left(\frac{k-2}{2}\right)^2 - \alpha}. \tag{27}$$

In order to verify (27), let us set

$$\gamma' = -\frac{k-2}{2} + \sqrt{\left(\frac{k-2}{2}\right)^2 - \alpha} \tag{28}$$

and consider the function $u(x) = |x_{\bar{J}}|^{\gamma'} = \left(\sum_{i=1}^{k} x_i^2\right)^{\gamma'/2} \in H^1_{\rm loc}(\mathbb{R}^N)$. Then $u$ solves the equation

$$-\Delta u(x) - \frac{\alpha}{|x_{\bar{J}}|^2}\, u(x) = 0 \quad \text{in } \{x \in \mathbb{R}^N : x_{\bar{J}} \neq 0\}. \tag{29}$$

The function $u$ may be rewritten as $u(x) = |x|^{\gamma'}\psi\big(\frac{x}{|x|}\big)$ once we define $\psi(\theta) = |\theta_{\bar{J}}|^{\gamma'}$ for any $\theta \in \mathbb{S}^{N-1} \setminus \Sigma$. Since $u$ solves (29), we obtain

$$-\gamma'(\gamma'+N-2)r^{\gamma'-2}\psi(\theta) - r^{\gamma'-2}\Delta_{\mathbb{S}^{N-1}}\psi(\theta) = r^{\gamma'-2}a(\theta)\psi(\theta), \quad \text{for any } r>0 \text{ and } \theta \in \mathbb{S}^{N-1}\setminus\Sigma.$$

This yields

$$-\Delta_{\mathbb{S}^{N-1}}\psi(\theta) - a(\theta)\psi(\theta) = \gamma'(\gamma'+N-2)\psi(\theta), \quad \text{in } \mathbb{S}^{N-1}.$$

This shows that $\psi$ is a positive eigenfunction of the operator $L_a$ and hence by Lemma 2.2 the corresponding eigenvalue must coincide with $\mu_1(a)$, i.e. $\gamma'(\gamma'+N-2) = \mu_1(a)$. (27) follows by (28).

**Example 2.5.** Let us also consider two-body potentials, i.e. the case in which $N \geqslant 2k$,

$$\alpha_J = 0 \qquad \text{for any } J \in \mathcal{A}_k$$

and

$$\alpha_{J_1\,J_2} = \begin{cases} \alpha, & \text{if } J_1 = \bar{J}_1 = \{1,2,\dots,k\} \text{ and } J_2 = \bar{J}_2 = \{k+1,k+2,\dots,2k\}, \\ 0, & \text{if } (J_1,J_2) \neq (\bar{J}_1,\bar{J}_2), \end{cases}$$

so that $a(\theta) = \alpha/|\theta_{\bar{J}_1} - \theta_{\bar{J}_2}|^2$. The optimality of the constant $\frac{(k-2)^2}{2}$ in inequality (9) implies that $\Lambda(a) = \alpha^+\frac{2}{(k-2)^2}$ and condition (12) reads as $\alpha < \frac{(k-2)^2}{2}$. Moreover we have

$$\mu_1(a) = -\frac{(k-2)(N-k)}{2} - \frac{\alpha}{2} + (N-k)\sqrt{\left(\frac{k-2}{2}\right)^2 - \frac{\alpha}{2}}. \tag{30}$$

In order to prove (30) we put

$$\gamma'' = -\frac{k-2}{2} + \sqrt{\left(\frac{k-2}{2}\right)^2 - \frac{\alpha}{2}} \tag{31}$$

and we define $u(x) = |x_{\bar{J}_1} - x_{\bar{J}_2}|^{\gamma''} \in H^1_{\rm loc}(\mathbb{R}^N)$. Then $u$ solves the equation

$$-\Delta u(x) - \frac{\alpha}{|x_{\bar{J}_1} - x_{\bar{J}_2}|^2}u(x) = 0 \quad \text{in } \{x \in \mathbb{R}^N : x_{\bar{J}_1} \neq x_{\bar{J}_2}\}. \tag{32}$$

Proceeding as in Example 2.4, by (31) and (32) we conclude that $\psi(\theta) = |\theta_{\bar{J}_1} - \theta_{\bar{J}_2}|^{\gamma''}$ is an eigenfunction of $\mu_1(a)$ and that $\mu_1(a)$ is given by (30).

We extend to singular potentials of the form (6) the Hardy type inequality with boundary terms proved by Wang and Zhu in [43].

**Lemma 2.6.** *Let $a$ be as in (7) and assume that (12) holds. Then*

$$\int_{B_r}\left(|\nabla u(x)|^2-\frac{a(\frac{x}{|x|})}{|x|^2}|u(x)|^2\right)dx+\frac{N-2}{2r}\int_{\partial B_r}|u(x)|^2\,dS \geqslant\left(\mu_1(a)+\left(\frac{N-2}{2}\right)^2\right)\int_{B_r}\frac{|u(x)|^2}{|x|^2}\,dx \tag{33}$$

*for all $r>0$ and $u\in H^1(B_r)$.*

Proof. By scaling, it is enough to prove the inequality for $r=1$. Let $u\in C^\infty(B_1)\cap H^1(B_1)$ with $0\notin\operatorname{supp}u$. Passing to polar coordinates, we have that

$$\begin{aligned}\int_{B_1}\left(|\nabla u(x)|^2-\frac{a(\frac{x}{|x|})}{|x|^2}|u(x)|^2\right)dx+\frac{N-2}{2}\int_{\partial B_1}|u(x)|^2\,dS \\ =\int_{\mathbb{S}^{N-1}}\left(\int_0^1 r^{N-1}|\partial_r u(r,\theta)|^2\,dr\right)dS(\theta)+\frac{N-2}{2}\int_{\mathbb{S}^{N-1}}|u(1,\theta)|^2\,dS(\theta) \\ +\int_0^1\frac{r^{N-1}}{r^2}\left(\int_{\mathbb{S}^{N-1}}\left[|\nabla_{\mathbb{S}^{N-1}}u(r,\theta)|^2-a(\theta)|u(r,\theta)|^2\right]\,dS(\theta)\right)dr.\end{aligned} \tag{34}$$

For all $\theta\in\mathbb{S}^{N-1}$, let $\varphi_\theta\in C^\infty(0,1)$ be defined by $\varphi_\theta(r)=u(r,\theta)$, and $\widetilde{\varphi}_\theta\in C^\infty(B_1)$ be the radially symmetric function given by $\widetilde{\varphi}_\theta(x)=\varphi_\theta(|x|)$. We notice that $0\notin\operatorname{supp}\widetilde{\varphi}_\theta$. The Hardy inequality with boundary term proved in [43] yields

$$\begin{aligned}&\int_{\mathbb{S}^{N-1}}\left(\int_0^1 r^{N-1}|\partial_r u(r,\theta)|^2\,dr+\frac{N-2}{2}\,|u(1,\theta)|^2\right)dS(\theta) \\ &=\int_{\mathbb{S}^{N-1}}\left(\int_0^1 r^{N-1}|\varphi_\theta'(r)|^2\,dr+\frac{N-2}{2}\,|\varphi_\theta(1)|^2\right)dS(\theta) \\ &=\frac{1}{\omega_{N-1}}\int_{\mathbb{S}^{N-1}}\left(\int_{B_1}|\nabla\widetilde{\varphi}_\theta(x)|^2\,dx+\frac{N-2}{2}\int_{\partial B_1}|\widetilde{\varphi}_\theta(x)|^2\,dS\right)dS(\theta) \\ &\geqslant\frac{1}{\omega_{N-1}}\left(\frac{N-2}{2}\right)^2\int_{\mathbb{S}^{N-1}}\left(\int_{B_1}\frac{|\widetilde{\varphi}_\theta(x)|^2}{|x|^2}\,dx\right)dS(\theta) \\ &=\left(\frac{N-2}{2}\right)^2\int_{\mathbb{S}^{N-1}}\left(\int_0^1\frac{r^{N-1}}{r^2}|u(r,\theta)|^2\,dr\right)dS(\theta)=\left(\frac{N-2}{2}\right)^2\int_{B_1}\frac{|u(x)|^2}{|x|^2}\,dx.\end{aligned} \tag{35}$$

where $\omega_{N-1}$ denotes the volume of the unit sphere $\mathbb{S}^{N-1}$, i.e. $\omega_{N-1}=\int_{\mathbb{S}^{N-1}}dS(\theta)$. On the other hand, from the definition of $\mu_1(a)$ it follows that, for every $r\in(0,1)$,

$$\int_{\mathbb{S}^{N-1}}\left[|\nabla_{\mathbb{S}^{N-1}}u(r,\theta)|^2-a(\theta)|u(r,\theta)|^2\right]dS(\theta)\geqslant\mu_1(a)\int_{\mathbb{S}^{N-1}}|u(r,\theta)|^2dS(\theta). \tag{36}$$

From (34), (35), and (36), we deduce that

$$\int_{B_1}\left(|\nabla u(x)|^2-\frac{a(\frac{x}{|x|})}{|x|^2}|u(x)|^2\right)dx+\frac{N-2}{2}\int_{\partial B_1}|u(x)|^2\,dS\geqslant\left[\left(\frac{N-2}{2}\right)^2+\mu_1(a)\right]\int_{B_1}\frac{|u(x)|^2}{|x|^2}\,dx$$

for all $u\in C^\infty(B_1)\cap H^1(B_1)$ with $0\notin\operatorname{supp}u$, which, by density, yields the stated inequality for all $H^1(B_r)$-functions for $r=1$. □

**Corollary 2.7.** *For all $r > 0$ and $u \in H^1(B_r)$, there holds*

$$\int_{B_r} |\nabla u(x)|^2\,dx + \frac{N-2}{2r}\int_{\partial B_r} |u(x)|^2\,dS \geqslant \left(\frac{k-2}{2}\right)^2 \int_{B_r} \frac{|u(x)|^2}{|x_J|^2}\,dx \tag{37}$$

*for any $J \in \mathcal{A}_k$ and*

$$\int_{B_r} |\nabla u(x)|^2\,dx + \frac{N-2}{2r}\int_{\partial B_r} |u(x)|^2\,dS \geqslant \frac{(k-2)^2}{2} \int_{B_r} \frac{|u(x)|^2}{|x_{J_1} - x_{J_2}|^2}\,dx \tag{38}$$

*for any $(J_1, J_2) \in \mathcal{B}_k$.*

Proof. Let $r > 0$ and $u \in H^1(B_r)$. Choosing $a$ as in the Example 2.4 with $\alpha < \left(\frac{k-2}{2}\right)^2$, from Lemma 2.6, it follows that

$$\int_{B_r} \left(|\nabla u(x)|^2 - \frac{\alpha}{|x_J|^2}|u(x)|^2\right) dx + \frac{N-2}{2r}\int_{\partial B_r} |u(x)|^2\,dS \geqslant 0$$

hence

$$\alpha \int_{B_r} \frac{|u(x)|^2}{|x_J|^2}\,dx \leqslant \int_{B_r} |\nabla u(x)|^2\,dx + \frac{N-2}{2r}\int_{\partial B_r} |u(x)|^2\,dS.$$

Letting $\alpha \to \left(\frac{k-2}{2}\right)^2$, (37) follows. To prove (38), we may choose $a$ as in Example 2.5 and proceed as in the proof of (37). □

**Corollary 2.8.** *Let $a$ be as in (7) and assume that (12) holds. Then, for all $r > 0$, $u \in H^1(B_r)$, $J \in \mathcal{A}_k$ and $(J_1, J_2) \in \mathcal{B}_k$, there holds*

$$\int_{B_r} |\nabla u(x)|^2\,dx - \int_{B_r} \frac{a(\frac{x}{|x|})}{|x|^2}|u(x)|^2\,dx + \Lambda(a)\frac{N-2}{2r}\int_{\partial B_r} |u(x)|^2\,dS \geqslant (1-\Lambda(a))\int_{B_r} |\nabla u(x)|^2\,dx\,, \tag{39}$$

$$\int_{B_r} |\nabla u(x)|^2\,dx - \int_{B_r} \frac{a(\frac{x}{|x|})}{|x|^2}|u(x)|^2\,dx + \frac{N-2}{2r}\int_{\partial B_r} |u(x)|^2\,dS \geqslant (1-\Lambda(a))\left(\frac{k-2}{2}\right)^2 \int_{B_r} \frac{|u(x)|^2}{|x_J|^2}\,dx, \tag{40}$$

*and*

$$\int_{B_r} |\nabla u(x)|^2\,dx - \int_{B_r} \frac{a(\frac{x}{|x|})}{|x|^2}|u(x)|^2\,dx + \frac{N-2}{2r}\int_{\partial B_r} |u(x)|^2\,dS \geqslant (1-\Lambda(a))\frac{(k-2)^2}{2}\int_{B_r} \frac{|u(x)|^2}{|x_{J_1} - x_{J_2}|^2}\,dx \tag{41}$$

*with $\Lambda(a)$ as in (23).*

PROOF. By scaling, it is enough to prove the inequalities for $r = 1$. Let $u \in C^\infty(B_1) \cap H^1(B_1)$ with $0 \notin \operatorname{supp} u$. Passing in polar coordinates we obtain

$$\begin{aligned}(42)\quad \int_{B_1} \Big(|\nabla u(x)|^2 - \frac{a(\frac{x}{|x|})}{|x|^2}|u(x)|^2\Big)\,dx + \Lambda(a)\frac{N-2}{2}\int_{\partial B_1}|u(x)|^2\,dS \\ = \int_{\mathbb{S}^{N-1}}\Big(\int_0^1 r^{N-1}|\partial_r u(r,\theta)|^2\,dr\Big)\,dS(\theta) + \Lambda(a)\frac{N-2}{2}\int_{\mathbb{S}^{N-1}}|u(1,\theta)|^2\,dS(\theta) \\ + \int_0^1 \frac{r^{N-1}}{r^2}\Big(\int_{\mathbb{S}^{N-1}}\big[|\nabla_{\mathbb{S}^{N-1}}u(r,\theta)|^2 - a(\theta)|u(r,\theta)|^2\big]\,dS(\theta)\Big)\,dr.\end{aligned}$$

By (22) and (12) we have

$$\begin{aligned}&\int_{\mathbb{S}^{N-1}}\Big[\big|\nabla_{\mathbb{S}^{N-1}}u(r,\theta)\big|^2 - a(\theta)|u(r,\theta)|^2\Big]\,dS(\theta) \\ &\quad\geqslant (1-\Lambda(a))\int_{\mathbb{S}^{N-1}}\big|\nabla_{\mathbb{S}^{N-1}}u(r,\theta)\big|^2\,dS(\theta) - \Lambda(a)\Big(\frac{N-2}{2}\Big)^2\int_{\mathbb{S}^{N-1}}|u(r,\theta)|^2\,dS(\theta)\end{aligned}$$

which inserted into (42) gives

$$\begin{aligned}&\int_{B_1}\Big(|\nabla u(x)|^2 - \frac{a(\frac{x}{|x|})}{|x|^2}|u(x)|^2\Big)\,dx + \Lambda(a)\frac{N-2}{2}\int_{\partial B_1}|u(x)|^2\,dS \geqslant (1-\Lambda(a))\int_{B_1}|\nabla u(x)|^2\,dx \\ &+ \Lambda(a)\bigg[\int_{\mathbb{S}^{N-1}}\Big(\int_0^1 r^{N-1}|\partial_r u(r,\theta)|^2\,dr + \frac{N-2}{2}|u(1,\theta)|^2\Big)\,dS(\theta) - \Big(\frac{N-2}{2}\Big)^2\int_{B_1}\frac{|u(x)|^2}{|x|^2}\,dx\bigg].\end{aligned}$$

Now, inequality (39) follows immediately from (35).

From (39) and (37) we obtain

$$\begin{aligned}&\int_{B_1}\Big(|\nabla u(x)|^2 - \frac{a(\frac{x}{|x|})}{|x|^2}|u(x)|^2\Big)\,dx + \frac{N-2}{2}\int_{\partial B_1}|u(x)|^2\,dS \\ &\geqslant (1-\Lambda(a))\Big(\int_{B_1}|\nabla u(x)|^2\,dx + \frac{N-2}{2}\int_{\partial B_1}|u(x)|^2\,dS\Big) \geqslant (1-\Lambda(a))\Big(\frac{k-2}{2}\Big)^2\int_{B_1}\frac{|u(x)|^2}{|x_J|^2}\,dx\end{aligned}$$

for all $J \in \mathcal{A}_k$ and for all $u \in C^\infty(B_1) \cap H^1(B_1)$ with $0 \notin \operatorname{supp} u$.

On the other hand by (39) and (38) we obtain

$$\begin{aligned}&\int_{B_1}\Big(|\nabla u(x)|^2 - \frac{a(\frac{x}{|x|})}{|x|^2}|u(x)|^2\Big)\,dx + \frac{N-2}{2}\int_{\partial B_1}|u(x)|^2\,dS \\ &\quad\geqslant (1-\Lambda(a))\Big(\int_{B_1}|\nabla u(x)|^2\,dx + \frac{N-2}{2}\int_{\partial B_1}|u(x)|^2\,dS\Big) \\ &\quad\geqslant (1-\Lambda(a))\frac{(k-2)^2}{2}\int_{B_1}\frac{|u(x)|^2}{|x_{J_1} - x_{J_2}|^2}\,dx\end{aligned}$$

for all $(J_1, J_2) \in \mathcal{B}_k$ and for all $u \in C^\infty(B_1) \cap H^1(B_1)$ with $0 \notin \operatorname{supp} u$.

By density the stated inequalities follow for any $u \in H^1(B_1)$. □

From (33) and (39), we can derive a Hardy-Sobolev type inequality which takes into account the boundary terms; to this aim, the following lemma is needed.

**Lemma 2.9.** *Let $\widetilde{S}_N > 0$ be the best constant of the Sobolev embedding $H^1(B_1) \subset L^{2^*}(B_1)$, i.e.*

$$\widetilde{S}_N := \inf_{v \in H^1(B_1)\setminus\{0\}} \frac{\int_{B_1} \left(|\nabla u(x)|^2 + |u(x)|^2\right) dx}{\left(\int_{B_1} |u(x)|^{2^*} dx\right)^{2/2^*}}. \tag{43}$$

*Then, for every $r > 0$ and $u \in H^1(B_r)$, there holds*

$$\int_{B_r} \left(|\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2}\right) dx \geqslant \widetilde{S}_N \left(\int_{B_r} |u(x)|^{2^*} dx\right)^{2/2^*}. \tag{44}$$

Proof. Inequality (44) follows simply by scaling from the definition of $\widetilde{S}_N$. □

The following boundary Hardy-Sobolev inequality holds true.

**Corollary 2.10.** *Let $a$ be as in (7) and assume that (12) holds. Then, for all $r > 0$ and $u \in H^1(B_r)$, there holds*

$$\begin{aligned}\int_{B_r} |\nabla u(x)|^2 dx - \int_{B_r} \frac{a(\frac{x}{|x|})}{|x|^2} |u(x)|^2 dx + \frac{1+\Lambda(a)}{2} \frac{N-2}{2r} \int_{\partial B_r} |u(x)|^2 dS \\ \geqslant \frac{\widetilde{S}_N}{2} \min\left\{1 - \Lambda(a), \mu_1(a) + \left(\frac{N-2}{2}\right)^2\right\} \left(\int_{B_r} |u(x)|^{2^*} dx\right)^{2/2^*},\end{aligned} \tag{45}$$

*where $\widetilde{S}_N$ is defined in (43).*

Proof. Inequality (45) follows simply by summing up (33) and (39) and using Lemma 2.9. □

## 3. A Pohozaev-type identity

In order to approximate $L_a := -\Delta_{\mathbb{S}^{N-1}} - a$ with operators with bounded coefficients, for all $\lambda \in \mathbb{R}$, we define

$$a_\lambda(\theta) := \begin{cases} \displaystyle\sum_{J \in \mathcal{A}_k} \frac{\alpha_J}{|\theta_J|^2 + \lambda} + \sum_{(J_1, J_2) \in \mathcal{B}_k} \frac{\alpha_{J_1\, J_2}}{|\theta_{J_1} - \theta_{J_2}|^2 + \lambda} & \text{if } \lambda > 0 \\ a(\theta) & \text{if } \lambda \leqslant 0 \end{cases} \tag{46}$$

in such a way that $a_\lambda \in L^\infty(\mathbb{S}^{N-1})$ for any $\lambda > 0$. We notice that $a_\lambda \in \mathcal{F}$ for any $\lambda \in \mathbb{R}$.

Since we are interested in the asymptotics of solutions at 0, we focus our attention on a ball $B_{r_0}$ which is sufficiently small to ensure positivity of the quadratic forms associated to equation (13) and to some proper approximations of (13) in $B_{r_0}$. Let $u$ be a solution of (13), with the perturbation potential $h$ satisfying (**H**) and the nonlinear term $f$ satisfying (**F**). If condition (12) holds, there exists $r_0 > 0$ such that

$$\begin{aligned} B_{r_0} \subseteq \Omega \quad \text{and} \quad \Lambda(a) + C_h r_0^{\varepsilon} \binom{N}{k} \left(\frac{2}{k-2}\right)^2 \left(1 + \binom{N-k}{k}\right) \\ + C_f S^{-1} \left[(\omega_{N-1}/N)^{\frac{2}{N}} r_0^2 + \|u\|^{2^*-2}_{L^{2^*}(B_{r_0})}\right] < 1,\end{aligned} \tag{47}$$

with $a$ as (7), $\Lambda(a)$ as in (22) and $\binom{N-k}{k} = 0$ whenever $N < 2k$.

**Lemma 3.1.** *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded open set such that $0 \in \Omega$, and let a satisfy (7) and (12). Suppose that h satisfies* (**H**), *f satisfies* (**F**), *u is a $H^1(\Omega)$-weak solution to (13) in $\Omega$, and $r_0 > 0$ is as in (47). Then there exists $\bar{\lambda} > 0$ such that, for every $\lambda \in (0, \bar{\lambda})$, the Dirichlet boundary value problem*

$$\begin{cases} -\Delta v(x) - \dfrac{a_\lambda\big(\frac{x}{|x|}\big)}{|x|^2}\, v(x) = h_\lambda(x)v(x) + f(x, v(x)), & \text{in } B_{r_0}, \\ v\big|_{\partial B_{r_0}} = u\big|_{\partial B_{r_0}}, & \text{on } \partial B_{r_0}, \end{cases} \tag{48}$$

*with*

$$h_\lambda(x) = \begin{cases} \min\{1/\lambda, \max\{-1/\lambda, h(x)\}\}, & \text{if } \lambda > 0, \\ h(x), & \text{if } \lambda \leqslant 0, \end{cases}$$

*admits a weak solution $u_\lambda \in H^1(B_{r_0})$ such that*

$$u_\lambda \to u \quad \text{in } H^1(B_{r_0}) \quad \text{as } \lambda \to 0^+.$$

Proof. Let $\tilde{v}$ be the unique $H^1(B_{r_0})$-weak solution to the problem

$$\begin{cases} -\Delta \tilde{v} - \dfrac{a\big(\frac{x}{|x|}\big)}{|x|^2}\, \tilde{v}(x) = h(x)\tilde{v}, & \text{in } B_{r_0}, \\ \tilde{v} = u\big|_{\partial B_{r_0}}, & \text{on } \partial B_{r_0}. \end{cases} \tag{49}$$

The existence and uniqueness of such a $\tilde{v}$ can be proven by introducing the continuous bilinear form $\mathcal{Q} : H^1_0(B_{r_0}) \times H^1_0(B_{r_0}) \to \mathbb{R}$

$$\mathcal{Q}(w_1, w_2) := \int_{B_{r_0}} \bigg[\nabla w_1(x) \cdot \nabla w_2(x) - \bigg(\frac{a\big(\frac{x}{|x|}\big)}{|x|^2} + h(x)\bigg) w_1(x) w_2(x)\bigg]\, dx,$$

and the continuous functional $\Psi \in H^{-1}(B_{r_0})$

$${}_{H^{-1}(B_{r_0})}\big\langle \Psi, w \big\rangle_{H^1_0(B_{r_0})} = -\int_{B_{r_0}} \nabla u(x)\cdot\nabla w(x) dx + \int_{B_{r_0}} \frac{a\big(\frac{x}{|x|}\big)}{|x|^2} u(x)w(x)dx + \int_{B_{r_0}} h(x)u(x)w(x)dx\,.$$

By (**H**), (8), (9), and (11), we have

$$\begin{aligned} \mathcal{Q}(w, w) &= \int_{B_{r_0}} \bigg(|\nabla w(x)|^2 - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2} w^2(x) - h(x)w^2(x)\bigg)\, dx \\ &\geqslant \int_{B_{r_0}} \bigg(|\nabla w(x)|^2 - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2} w^2(x) - C_h\bigg(\sum_{J\in\mathcal{A}_k} |x_J|^{-2+\varepsilon} + \sum_{(J_1,J_2)\in\mathcal{B}_k} |x_{J_1} - x_{J_2}|^{-2+\varepsilon}\bigg) w^2(x)\bigg)\, dx \\ &\geqslant \bigg[1 - \Lambda(a) - C_h r_0^\varepsilon \binom{N}{k} \left(\frac{2}{k-2}\right)^2 \bigg(1 + \binom{N-k}{k}\bigg)\bigg] \int_{B_{r_0}} |\nabla w(x)|^2\, dx \end{aligned} \tag{50}$$

for all $w \in H^1_0(B_{r_0})$. By (50), (12) and (47) it follows that the bilinear form $\mathcal{Q}$ is coercive. The Lax-Milgram lemma yields existence and uniqueness of a solution $v \in H^1_0(B_{r_0})$ of the variational problem

$$\mathcal{Q}(v, w) = {}_{H^{-1}(B_{r_0})}\big\langle \Psi, w \big\rangle_{H^1_0(B_{r_0})} \qquad \text{for any } w \in H^1_0(B_{r_0}).$$

Then the function $\tilde v := v + u$ is the unique solution of (49).

Let us now define the map $\Phi : \mathbb{R} \times H^1_0(B_{r_0}) \to H^{-1}(B_{r_0})$ as

$$\Phi(\lambda, w) = -\Delta w - \frac{a_\lambda\big(\frac{x}{|x|}\big)}{|x|^2}\, w - h_\lambda(x) w - f(x, \tilde v + w) + \left( \frac{a\big(\frac{x}{|x|}\big)}{|x|^2} + h(x) - \frac{a_\lambda\big(\frac{x}{|x|}\big)}{|x|^2} - h_\lambda(x) \right) \tilde v.$$

By (7), (8), (9), (**H**) and (**F**), the function $\Phi$ is continuous and its first variation with respect to the $w$ variable

$$\Phi'_w : \mathbb{R} \times H^1_0(B_{r_0}) \to \mathcal{L}(H^1_0(B_{r_0}), H^{-1}(B_{r_0}))$$

is also continuous. We claim that

$$\Phi(0, u - \tilde v) = 0 \text{ in } H^{-1}(B_{r_0}) \quad \text{and} \quad \Phi'_w(0, u - \tilde v) \in \mathcal{L}\big(H^1_0(B_{r_0}), H^{-1}(B_{r_0})\big) \text{ is an isomorphism.}$$

The first claim is an immediate consequence of the definition of $u$ and $\tilde v$. Let us prove the second one. By (**F**), (11), and Hölder and Sobolev inequalities, for every $w \in H^1_0(B_{r_0})$ we obtain

$$\begin{aligned}
&{}_{H^{-1}(B_{r_0})}\Big\langle \Phi'_w(0, u - \tilde v) w, w \Big\rangle_{H^1_0(B_{r_0})} \\
&= \int_{B_{r_0}} |\nabla w(x)|^2\, dx - \int_{B_{r_0}} \frac{a\big(\frac{x}{|x|}\big)}{|x|^2} w^2(x)\, dx - \int_{B_{r_0}} h(x) w^2(x)\, dx - \int_{B_{r_0}} f'_s(x, u(x)) w^2(x)\, dx \\
&\geqslant \int_{B_{r_0}} |\nabla w(x)|^2\, dx - \int_{B_{r_0}} \frac{a\big(\frac{x}{|x|}\big)}{|x|^2} w^2(x)\, dx - \int_{B_{r_0}} h(x) w^2(x)\, dx \\
&\qquad - C_f \int_{B_{r_0}} \big(1 + |u(x)|^{2^*-2}\big) w^2(x)\, dx \\
&\geqslant (1 - \Lambda(a)) \int_{B_{r_0}} |\nabla w(x)|^2\, dx \\
&\qquad - C_h r_0^{\varepsilon} \binom{N}{k} \left( \frac{2}{k-2} \right)^2 \left( 1 + \binom{N-k}{k} \right) \int_{B_{r_0}} |\nabla w(x)|^2\, dx \\
&\qquad - C_f S^{-1} \Big[ (\omega_{N-1}/N)^{\frac{2}{N}} r_0^2 + \|u\|^{2^*-2}_{L^{2^*}(B_{r_0})} \Big] \int_{B_{r_0}} |\nabla w(x)|^2\, dx\,.
\end{aligned}$$

The above estimate, together with (47), shows that the quadratic form $w \mapsto \langle \Phi'_w(0, u - \tilde v) w, w\rangle$ is positive definite over $H^1_0(B_{r_0})$. Then the Lax-Milgram lemma applied to the continuous and coercive bilinear form $(w_1, w_2) \mapsto {}_{H^{-1}(B_{r_0})}\big\langle \Phi'_w(0, u - \tilde v) w_1, w_2 \big\rangle_{H^1_0(B_{r_0})}$ ensures that the operator $\Phi'_w(0, u - \tilde v) \in \mathcal{L}(H^1_0(B_{r_0}), H^{-1}(B_{r_0}))$ is an isomorphism and hence our second claim is proved.

We are now in position to apply the Implicit Function Theorem to the map $\Phi$, thus showing the existence of $\bar\lambda > 0$, $\rho > 0$, and of a continuous function

$$g : (-\bar\lambda, \bar\lambda) \to B(u - \tilde v, \rho)$$

with $B(u - \tilde v, \rho) = \{w \in H^1_0(B_{r_0}) : \|w - u + \tilde v\|_{H^1_0(B_{r_0})} < \rho\}$, such that $\Phi(\lambda, g(\lambda)) = 0$ for all $\lambda \in (-\bar\lambda, \bar\lambda)$ and, if $(\lambda, w) \in (-\bar\lambda, \bar\lambda) \times B(u - \tilde v, \rho)$ and $\Phi(\lambda, w) = 0$, then $w = g(\lambda)$. The function $u_\lambda := g(\lambda) + \tilde v$ solves (48) for any $\lambda \in (0, \bar\lambda)$. Moreover, by the continuity of $g$ over the interval $(-\bar\lambda, \bar\lambda)$ and the fact that $g(0) = u - \tilde v$, $u_\lambda - u = g(\lambda) - u + \tilde v \to 0$ in $H^1_0(B_{r_0})$ as $\lambda \to 0^+$. This proves that $u_\lambda \to u$ in $H^1(B_{r_0})$ as $\lambda \to 0^+$. □

**Remark 3.2.** We notice that, if $f \in L^1(\Omega)$ for some $\Omega \subset \mathbb{R}^N$ bounded open set such that $0 \in \Omega$, then, for every $r > 0$ such that $B_r \subseteq \Omega$,

$$\int_{B_r} |f(x)|\, dx = \int_0^r \left( \int_{\partial B_s} |f|\, dS \right) ds < +\infty,$$

and hence the function $s \mapsto \int_{\partial B_s} |f|\, dS$ belongs to $L^1(0,r)$ and is the weak derivative of the $W^{1,1}(0,r)$-function $s \to \int_{B_s} |f(x)|\, dx$. In particular, for every $u \in H^1(\Omega)$ and every $J \in \mathcal{A}_k$, $(J_1, J_2) \in \mathcal{B}_k$, the $L^1(0,r)$-function

$$s \mapsto \int_{\partial B_s} |\nabla u(x)|^2\, dS, \quad \text{respectively } s \mapsto \int_{\partial B_s} \frac{u^2(x)}{|x_J|^2}\, dS, \quad s \mapsto \int_{\partial B_s} \frac{u^2(x)}{|x_{J_1} - x_{J_2}|^2}\, dS,$$

is the weak derivative of the $W^{1,1}(0,r)$-function

$$s \to \int_{B_s} |\nabla u(x)|^2\, dx, \quad \text{respectively } s \mapsto \int_{B_s} \frac{u^2(x)}{|x_J|^2}\, dx, \quad s \mapsto \int_{B_s} \frac{u^2(x)}{|x_{J_1} - x_{J_2}|^2}\, dx.$$

Solutions to (13) satisfy the following Pohozaev-type identity.

**Theorem 3.3.** *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded open set such that $0 \in \Omega$. Let $a$ satisfy (7), (12), and $u$ be a $H^1(\Omega)$-weak solution to (13) in $\Omega$ with $h$ satisfying* (**H**) *and $f$ satisfying* (**F**). *Then*

$$\begin{aligned}(51)\quad & -\frac{N-2}{2} \int_{B_r} \left[ |\nabla u(x)|^2 - \frac{a(\frac{x}{|x|})}{|x|^2} u^2(x) \right] dx + \frac{r}{2} \int_{\partial B_r} \left[ |\nabla u(x)|^2 - \frac{a(\frac{x}{|x|})}{|x|^2} u^2(x) \right] dS \\ &= r \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu} \right|^2 dS - \frac{1}{2} \int_{B_r} (\nabla h(x) \cdot x) u^2(x)\, dx - \frac{N}{2} \int_{B_r} h(x)\, u^2(x)\, dx + \frac{r}{2} \int_{\partial B_r} h(x) u^2(x)\, dS \\ &\quad + r \int_{\partial B_r} F(x, u(x))\, dS - \int_{B_r} [\nabla_x F(x, u(x)) \cdot x + N F(x, u(x))]\, dx\end{aligned}$$

*and*

$$\begin{aligned}(52)\quad & \int_{B_r} \left( |\nabla u(x)|^2 - \frac{a(\frac{x}{|x|})}{|x|^2} u^2(x) \right) dx \\ &= \int_{\partial B_r} u \frac{\partial u}{\partial \nu}\, dS + \int_{B_r} h(x) u^2(x)\, dx + \int_{B_r} f(x, u(x)) u(x)\, dx,\end{aligned}$$

*for a.e. $r \in (0, r_0)$, where $r_0 > 0$ satisfies (47) and $\nu = \nu(x)$ is the unit outer normal vector $\nu(x) = \frac{x}{|x|}$.*

Proof. Let $a_\lambda$ as in (46), $r_0$ as in (47), and $u_\lambda$, $h_\lambda$ as in Lemma 3.1. Since $a_\lambda$ and $h_\lambda$ are bounded for every $\lambda>0$ the following Pohozaev identity

$$-\frac{N-2}{2}\int_{B_r}\bigg[|\nabla u_\lambda(x)|^2-\frac{a_\lambda(\frac{x}{|x|})}{|x|^2}u_\lambda^2(x)\bigg]dx+\frac{r}{2}\int_{\partial B_r}\bigg[|\nabla u_\lambda(x)|^2-\frac{a_\lambda(\frac{x}{|x|})}{|x|^2}u_\lambda^2(x)\bigg]dS$$
$$=r\int_{\partial B_r}\bigg|\frac{\partial u_\lambda}{\partial\nu}\bigg|^2dS+\int_{B_r}h_\lambda(x)u_\lambda(x)\left(x\cdot\nabla u_\lambda(x)\right)\big)\,dx$$
$$+r\int_{\partial B_r}F(x,u_\lambda(x))\,dx-\int_{B_r}[\nabla_xF(x,u_\lambda(x))\cdot x+NF(x,u_\lambda(x))]\,dx \tag{53}$$

holds for all $r\in(0,r_0)$, see Proposition A.1. Furthermore, testing (48) with $u_\lambda$, integrating by parts, and using the regularity of $u_\lambda$ outside the origin, we obtain that

$$\int_{B_r}\bigg(|\nabla u_\lambda(x)|^2-\frac{a_\lambda(\frac{x}{|x|})}{|x|^2}u_\lambda^2(x)\bigg)dx$$
$$=\int_{\partial B_r}u_\lambda\frac{\partial u_\lambda}{\partial\nu}\,dS+\int_{B_r}h_\lambda(x)u_\lambda^2(x)\,dx+\int_{B_r}f(x,u_\lambda(x))u_\lambda(x)\,dx \tag{54}$$

for all $r\in(0,r_0)$.

From the convergence of $u_\lambda$ to $u$ in $H^1(B_{r_0})$ as $\lambda\to0^+$ proved in Lemma 3.1, inequalities (37–38), and the Dominated Convergence Theorem, it follows that

$$\frac{a_\lambda(\frac{x}{|x|})}{|x|^2}u_\lambda^2-\frac{a(\frac{x}{|x|})}{|x|^2}u^2=\frac{a_\lambda(\frac{x}{|x|})}{|x|^2}(u_\lambda+u)(u_\lambda-u)+\frac{a_\lambda(\frac{x}{|x|})-a(\frac{x}{|x|})}{|x|^2}u^2\to0$$

in $L^1(B_{r_0})$ as $\lambda\to0^+$, i.e.

$$\lim_{\lambda\to0^+}\int_{B_{r_0}}\bigg|\frac{a_\lambda(\frac{x}{|x|})}{|x|^2}u_\lambda^2(x)-\frac{a(\frac{x}{|x|})}{|x|^2}u^2(x)\bigg|dx$$
$$=\lim_{\lambda\to0^+}\int_0^{r_0}\bigg[\int_{\partial B_s}\bigg|\frac{a_\lambda(\frac{x}{|x|})}{|x|^2}u_\lambda^2(x)-\frac{a(\frac{x}{|x|})}{|x|^2}u^2(x)\bigg|dS\bigg]ds=0. \tag{55}$$

From (55) we deduce that

$$\int_{B_r}\frac{a_\lambda(\frac{x}{|x|})}{|x|^2}u_\lambda^2(x)\,dx\to\int_{B_r}\frac{a(\frac{x}{|x|})}{|x|^2}u^2(x)\,dx\quad\text{as }\lambda\to0^+\quad\text{for all }r\in(0,r_0).$$

and, along a sequence $\lambda_n\to0^+$,

$$\int_{\partial B_r}\frac{a_{\lambda_n}(\frac{x}{|x|})}{|x|^2}u_{\lambda_n}^2\,dS\to\int_{\partial B_r}\frac{a(\frac{x}{|x|})}{|x|^2}u^2\,dS\quad\text{as }n\to+\infty\quad\text{for a.e. }r\in(0,r_0). \tag{56}$$

On the other hand, from

$$\lim_{\lambda\to0^+}\int_{B_{r_0}}|\nabla(u_\lambda-u)(x)|^2\,dx=\lim_{\lambda\to0^+}\int_0^{r_0}\bigg[\int_{\partial B_s}|\nabla(u_\lambda-u)|^2\,dS\bigg]ds=0,$$

we deduce that, along a sequence converging monotonically to zero still denoted by $\lambda_n$,

$$\int_{\partial B_r}|\nabla u_{\lambda_n}|^2dS\to\int_{\partial B_r}|\nabla u|^2dS\quad\text{as }n\to+\infty\quad\text{for a.e. }r\in(0,r_0) \tag{57}$$

and

$$\int_{\partial B_r} \left|\frac{\partial u_{\lambda_n}}{\partial \nu}\right|^2 dS \to \int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}\right|^2 dS \quad \text{as } n\to+\infty \quad \text{for a.e. } r\in(0,r_0). \tag{58}$$

Let us fix $\lambda > 0$ and $r \in (0, r_0)$. Since

$$\int_0^r \left[\int_{\partial B_s} h_\lambda(x)|u_\lambda(x)|^2\,dS\right] ds = \int_{B_r} h_\lambda(x)|u_\lambda(x)|^2\,dx < +\infty,$$

there exists a sequence $\{\delta_k\}_{k\in\mathbb{N}} \subset (0,r)$ such that $\lim_{k\to+\infty}\delta_k = 0$ and

$$\delta_k \int_{\partial B_{\delta_k}} h_\lambda(x)|u_\lambda(x)|^2\,dS \longrightarrow 0 \quad \text{as } k\to+\infty. \tag{59}$$

Recalling that $u_\lambda \in C^{1,\tau}_{\rm loc}(B_{r_0}\setminus\{0\})$ for some $\tau\in(0,1)$, integration by parts yields

$$\begin{aligned}\int_{B_r\setminus B_{\delta_k}} h_\lambda(x)u_\lambda(x)\,(x\cdot\nabla u_\lambda(x))\,dx &= -\frac12\int_{B_r\setminus B_{\delta_k}} (\nabla h_\lambda(x)\cdot x)u_\lambda^2(x)\,dx\\ &- \frac N2\int_{B_r\setminus B_{\delta_k}} h_\lambda(x)u_\lambda^2(x)\,dx + \frac r2\int_{\partial B_r} h_\lambda(x)u_\lambda^2(x)\,dS - \frac{\delta_k}{2}\int_{\partial B_{\delta_k}} h_\lambda(x)u_\lambda^2(x)\,dS.\end{aligned}$$

Letting $k\to+\infty$, by (59) and (**H**), we obtain

$$\begin{aligned}\int_{B_r} h_\lambda(x)u_\lambda(x)\,(x\cdot\nabla u_\lambda(x))\,dx &= -\frac12\int_{B_r} (\nabla h_\lambda(x)\cdot x)u_\lambda^2(x)\,dx\\ &- \frac N2\int_{B_r} h_\lambda(x)u_\lambda^2(x)\,dx + \frac r2\int_{\partial B_r} h_\lambda(x)u_\lambda^2(x)\,dS.\end{aligned}$$

Arguing as above, using (**H**) we can prove that

$$\lim_{\lambda\to0^+}\int_{B_r} (\nabla h_\lambda(x)\cdot x)u_\lambda^2(x)\,dx = \int_{B_r} (\nabla h(x)\cdot x)u^2(x)\,dx \quad \text{for all } r\in(0,r_0),$$

$$\lim_{\lambda\to0^+}\int_{B_r} h_\lambda(x)u_\lambda^2(x)\,dx = \int_{B_r} h(x)u^2(x)\,dx \quad \text{for all } r\in(0,r_0),$$

and, along a sequence $\lambda_n\to0^+$,

$$\lim_{n\to+\infty}\int_{\partial B_r} h_{\lambda_n}(x)u_{\lambda_n}^2(x)\,dS = \int_{\partial B_r} h(x)u^2(x)\,dS \quad \text{for a.e. } r\in(0,r_0). \tag{60}$$

It remains to study the convergence of the terms in (53) and (54) related to the nonlinearity $f$. By (**F**), convergence of $u_\lambda$ to $u$ in $H^1(B_{r_0})$, and the Dominated Convergence Theorem, we have that

$$\lim_{\lambda\to0^+}\int_{B_r} [\nabla_x F(x,u_\lambda(x))\cdot x + NF(x,u_\lambda(x))]\,dx = \int_{B_r} [\nabla_x F(x,u(x))\cdot x + NF(x,u(x))]\,dx\,,$$

$$\lim_{\lambda\to0^+}\int_{B_r} f(x,u_\lambda(x))u_\lambda(x)\,dx = \int_{B_r} f(x,u(x))u(x)\,dx,$$

for all $r\in(0,r_0)$, and along a sequence $\lambda_n\to0^+$,

$$r\int_{\partial B_r} F(x,u_{\lambda_n}(x))\,dx \to r\int_{\partial B_r} F(x,u(x))\,dx \quad \text{as } n\to+\infty \tag{61}$$

for a.e. $r\in(0,r_0)$.

Therefore, we can pass to the limit in (53) and in (54) along a sequence $\lambda_n \to 0^+$ such that (56), (57), (58), (60), and (61) hold true, thus obtaining (51) and (52). □

## 4. A Brezis-Kato type estimate

Throughout this section, we let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded open set such that $0 \in \Omega$, $a$ satisfy (7), (12), $h$ satisfy (**H**), and $V \in L^1_{\rm loc}(\Omega)$ satisfy the form-bounded condition

$$\sup_{v \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |V(x)| v^2(x)\,dx}{\|v\|^2_{H^1(\Omega)}} < +\infty,$$

see [33]. The above condition (which is in particular satisfied by $L^{N/2}$ and $L^{N/2,\infty}$ functions, potentials of the form (6), etc.) in particular implies that for every $u \in H^1(\Omega)$, $Vu \in H^{-1}(\Omega)$. We assume that $u \in H^1(\Omega)$ is a weak solution to

$$-\Delta u(x) - \frac{a(\frac{x}{|x|})}{|x|^2} u(x) = h(x)u(x) + V(x)u(x), \quad \text{in } \Omega. \tag{62}$$

In the spirit of [40, Theorem 2.3], we prove the following Brezis-Kato type result.

**Proposition 4.1.** *If $V_+ \in L^{N/2}(\Omega)$, letting*

$$q_{\rm lim} := \begin{cases} \frac{2^*}{2} \min\left\{\frac{4}{\Lambda(a)} - 2, 2^*\right\}, & \text{if} \quad \Lambda(a) > 0, \\ \frac{(2^*)^2}{2}, & \text{if} \quad \Lambda(a) = 0, \end{cases}$$

*then for every $1 \leqslant q < q_{\rm lim}$ there exists $r_q > 0$ depending on $q, N, k, a, h$ such that $B_{r_q} \subset \Omega$ and $u \in L^q(B_{r_q})$.*

Proof. For any $2 < \tau < \frac{2}{2^*} q_{\rm lim}$ define $C(\tau) := \frac{4}{\tau + 2}$ and let $\ell_\tau > 0$ be large enough so that

$$\bigg( \int_{V_+(x) \geqslant \ell_\tau} V_+^{\frac{N}{2}}(x)\,dx \bigg)^{\frac{2}{N}} < \frac{S(C(\tau) - \Lambda(a))}{2}. \tag{63}$$

Let $r > 0$ be such that $B_r \subset \Omega$. For any $w \in H^1_0(B_r)$, by Hölder and Sobolev inequalities and (63), we have

$$\begin{aligned} \int_\Omega V(x)|w(x)|^2\,dx &\leqslant \int_{B_r \cap \{V_+(x) \leqslant \ell_\tau\}} V_+(x)|w(x)|^2\,dx + \int_{B_r \cap \{V_+(x) \geqslant \ell_\tau\}} V_+(x)|w(x)|^2\,dx \\ &\leqslant \ell_\tau \int_{B_r} |w(x)|^2\,dx + \bigg( \int_{V_+(x) \geqslant \ell_\tau} V_+^{\frac{N}{2}}(x)\,dx \bigg)^{\frac{2}{N}} \bigg( \int_{B_r} |w(x)|^{2^*}\,dx \bigg)^{\frac{2}{2^*}} \\ &\leqslant \ell_\tau \int_{B_r} |w(x)|^2\,dx + \frac{C(\tau) - \Lambda(a)}{2} \int_{B_r} |\nabla w(x)|^2\,dx. \end{aligned} \tag{64}$$

Let $\eta \in C^\infty_c(B_r)$ be such that $\eta \equiv 1$ in $B_{r/2}$ and define $v(x) := \eta(x)u(x) \in H^1_0(B_r)$. Then $v$ is a $H^1(\Omega)$-weak solution of the equation

$$-\Delta v(x) - \frac{a(\frac{x}{|x|})}{|x|^2} v(x) = h(x)v(x) + V(x)v(x) + g(x), \quad \text{in } \Omega, \tag{65}$$

where $g(x) = -u(x)\Delta\eta(x) - 2\nabla u(x)\cdot\nabla\eta(x) \in L^2(B_r)$. For any $n\in\mathbb{N}$, $n\geqslant 1$, let us define the function $v^n := \min\{|v|,n\}$. Testing (65) with $(v^n)^{\tau-2}v \in H^1_0(B_r)$ we obtain

$$\int_{B_r}(v^n(x))^{\tau-2}|\nabla v(x)|^2\,dx + (\tau-2)\int_{B_r}(v^n(x))^{\tau-3}|v(x)||\nabla v(x)|^2\chi_{\{|v(x)|<n\}}(x)\,dx - \int_{B_r}\frac{a(\frac{x}{|x|})}{|x|^2}(v^n(x))^{\tau-2}v^2(x)\,dx = \int_{B_r}h(x)(v^n(x))^{\tau-2}v^2(x)\,dx + \int_{B_r}V(x)(v^n(x))^{\tau-2}v^2(x)\,dx + \int_{B_r}g(x)(v^n(x))^{\tau-2}v(x)\,dx\ . \tag{66}$$

Since

$$|\nabla((v^n)^{\frac{\tau}{2}-1}v)|^2 = (v^n)^{\tau-2}|\nabla v|^2 + \frac{(\tau-2)(\tau+2)}{4}(v^n)^{\tau-3}|v||\nabla v|^2\chi_{\{|v(x)|<n\}}\ ,$$

then by (66), (11), (**H**), and (64) with $w=(v^n)^{\frac{\tau}{2}-1}v$ we obtain

$$\begin{aligned}C(\tau)\int_{B_r}|\nabla((v^n(x))^{\frac{\tau}{2}-1}v(x))|^2\,dx &\leqslant \int_{B_r}\frac{a(\frac{x}{|x|})}{|x|^2}((v^n(x))^{\frac{\tau}{2}-1}v(x))^2\,dx + \int_{B_r}h(x)((v^n(x))^{\frac{\tau}{2}-1}v(x))^2\,dx \\ &\quad + \int_{B_r}V(x)((v^n(x))^{\frac{\tau}{2}-1}v(x))^2\,dx + \int_{B_r}g(x)(v^n(x))^{\tau-2}v(x)\,dx \\ &\leqslant \left[\Lambda(a) + C_h r^{\varepsilon}\binom{N}{k}\left(\frac{2}{k-2}\right)^2\left(1+\binom{N-k}{k}\right) + \frac{C(\tau)-\Lambda(a)}{2}\right]\int_{B_r}|\nabla((v^n(x))^{\frac{\tau}{2}-1}v(x))|^2\,dx \\ &\quad + \ell_\tau\int_{B_r}(v^n(x))^{\tau-2}(v(x))^2\,dx + \int_{B_r}|g(x)|(v^n(x))^{\tau-2}|v(x)|\,dx\ .\end{aligned} \tag{67}$$

Let us consider the last term in the right hand side of (67). Since $g\in L^2(B_r)$, then by Hölder inequality

$$\begin{aligned}\int_{B_r}|g(x)|(v^n(x))^{\tau-2}|v(x)|\,dx &\leqslant \|g\|_{L^2(\Omega)}\left(\int_{B_r}(v^n(x))^{2\tau-4}|v(x)|^2\,dx\right)^{\frac12} \\ &= \|g\|_{L^2(\Omega)}\left(\int_{B_r}(v^n(x))^{\frac{2(\tau-1)(\tau-2)}{\tau}}(v^n(x))^{\frac{2(\tau-2)}{\tau}}|v(x)|^2\,dx\right)^{\frac12} \\ &\leqslant \|g\|_{L^2(\Omega)}\left(\int_{B_r}|(v^n(x))^{\frac{\tau}{2}-1}v(x)|^{\frac{4(\tau-1)}{\tau}}\,dx\right)^{\frac12}\end{aligned}$$

and, since $\frac{4(\tau-1)}{\tau} < 2^*$ for any $\tau < \frac{2}{2^*}q_{\lim}$, by Hölder inequality, Sobolev embedding, and Young inequality, we obtain

$$
\begin{aligned}
(68)\quad & \int_{B_r} |g(x)|(v^n(x))^{\tau-2}|v(x)|\,dx \\
&\leqslant \|g\|_{L^2(\Omega)}\left(\frac{\omega_{N-1}}{N}\right)^{\frac12-\frac{2(\tau-1)}{2^*\tau}} r^{\frac N2-\frac{2N(\tau-1)}{2^*\tau}}\left(\int_{B_r}|(v^n(x))^{\frac\tau2-1}v(x)|^{2^*}\,dx\right)^{\frac{2(\tau-1)}{2^*\tau}} \\
&\leqslant \|g\|_{L^2(\Omega)}\left(\frac{\omega_{N-1}}{N}\right)^{\frac12-\frac{2(\tau-1)}{2^*\tau}} r^{\frac N2-\frac{(N-2)(\tau-1)}{\tau}}S^{-\frac{\tau-1}{\tau}}\left(\int_{B_r}|\nabla((v^n(x))^{\frac\tau2-1}v(x))|^2\,dx\right)^{\frac{\tau-1}{\tau}} \\
&\leqslant \frac1\tau\|g\|^\tau_{L^2(\Omega)}+\frac{\tau-1}{\tau}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{\tau}{2(\tau-1)}-\frac{2}{2^*}} r^{\frac{N\tau}{2(\tau-1)}-N+2}S^{-1}\int_{B_r}|\nabla((v^n(x))^{\frac\tau2-1}v(x))|^2\,dx.
\end{aligned}
$$

Inserting (68) into (67) we obtain

$$
\begin{aligned}
&\Bigg[\frac{C(\tau)-\Lambda(a)}{2}-C_h r^\varepsilon\binom{N}{k}\left(\frac{2}{k-2}\right)^2\left(1+\binom{N-k}{k}\right) \\
&\quad-\frac{\tau-1}{\tau}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{\tau}{2(\tau-1)}-\frac{2}{2^*}} r^{\frac{N\tau}{2(\tau-1)}-N+2}S^{-1}\Bigg]\int_{B_r}|\nabla((v^n(x))^{\frac\tau2-1}v(x))|^2\,dx \\
&\leqslant \frac1\tau\|g\|^\tau_{L^2(\Omega)}+\ell_\tau\int_{B_r}(v^n(x))^{\tau-2}(v(x))^2\,dx
\end{aligned}
$$

and by Sobolev embedding we also have

$$
\begin{aligned}
(69)\quad & S\Bigg[\frac{C(\tau)-\Lambda(a)}{2}-C_h r^\varepsilon\binom{N}{k}\left(\frac{2}{k-2}\right)^2\left(1+\binom{N-k}{k}\right) \\
&\quad-\frac{\tau-1}{\tau}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{\tau}{2(\tau-1)}-\frac{2}{2^*}} r^{\frac{N\tau}{2(\tau-1)}-N+2}S^{-1}\Bigg]\left(\int_{B_r}(v^n(x))^{\frac{2^*}{2}\tau-2^*}|v(x)|^{2^*}\,dx\right)^{\frac{2}{2^*}} \\
&\leqslant \frac1\tau\|g\|^\tau_{L^2(\Omega)}+\ell_\tau\int_{B_r}(v^n(x))^{\tau-2}(v(x))^2\,dx\ .
\end{aligned}
$$

Since $\tau < \frac{2}{2^*}q_{\lim}$ then $C(\tau)-\Lambda(a)$ is positive and $\frac{N\tau}{2(\tau-1)}-N+2$ is also positive. Hence we may fix $r$ small enough in such a way that the left hand side of (69) becomes positive. Since $v \in L^\tau(B_r)$, letting $n \to +\infty$, the right hand side of (69) remains bounded and hence by Fatou Lemma we infer that $v \in L^{\frac{2^*}{2}\tau}(B_r)$. Since $\eta \equiv 1$ in $B_{r/2}$ we may conclude that $u \in L^{\frac{2^*}{2}\tau}(B_{r/2})$. This completes the proof of the lemma. □

## 5. The Almgren type frequency function

Let $u$ be a weak $H^1(\Omega)$-solution to equation (13) in a bounded domain $\Omega \subset \mathbb{R}^N$ containing the origin with $a$ satisfying (7) and (12), $h$ satisfying (**H**) and $f$ satisfying (**F**).

By (**F**) and Sobolev embedding, we infer that the function

$$
V(x) := \begin{cases} \frac{f(x,u(x))}{u(x)}, & \text{if } u(x) \neq 0, \\ 0, & \text{if } u(x) = 0, \end{cases}
$$

belongs to $L^{N/2}(\Omega)$ and hence we may apply Proposition 4.1 to the function $u$. Therefore, throughout this section, we may fix

$$2^* < q < q_{\lim} \tag{70}$$

and $r_q$ as in Proposition 4.1 in such a way that $u \in L^q(B_{r_q})$.

By Remark 3.2, the function

$$H(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} |u|^2 \, dS \tag{71}$$

belongs to $L^1_{\rm loc}(0,R)$ for every $R>0$ such that $B_R \subseteq \Omega$. It is also easy to verify that

$$H(r) = \int_{\mathbb{S}^{N-1}} |u(r\theta)|^2 \, dS(\theta) \quad \text{for a.e. } r \in (0,R). \tag{72}$$

Further regularity of $H$ is established in the following lemma.

**Lemma 5.1.** *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded open set such that $0 \in \Omega$ and let $u$ be a weak $H^1(\Omega)$-solution to equation (13) in $\Omega$ with $a$ satisfying* (7) *and* (12)*, $h$ satisfying* (**H**)*, and $f$ satisfying* (**F**)*. If $H$ is the function defined in (71) and $R>0$ is such that $B_R \subseteq \Omega$, then $H \in W^{1,1}_{\rm loc}(0,R)$ and*

$$H'(r) = \frac{2}{r^{N-1}} \int_{\partial B_r} u \frac{\partial u}{\partial \nu} \, dS \tag{73}$$

*in a distributional sense and for a.e. $r \in (0,R)$.*

Proof. Since $u, \frac{\partial u}{\partial \nu} \in L^2(B_R)$, by Remark 3.2, we have that

$$r \mapsto \frac{2}{r^{N-1}} \int_{\partial B_r} u \frac{\partial u}{\partial \nu} \, dS \in L^1_{\rm loc}(0,R).$$

If $0<s<r<R$, by Fubini's Theorem we obtain

$$\begin{aligned} \int_s^r \frac{2}{t^{N-1}} \left( \int_{\partial B_t} u \frac{\partial u}{\partial \nu} \, dS \right) dt &= \int_s^r \left( \int_{\mathbb{S}^{N-1}} 2u(t\theta) \frac{\partial u}{\partial \nu}(t\theta) \, dS(\theta) \right) dt \\ &= \int_{\mathbb{S}^{N-1}} \left( \int_s^r 2u(t\theta) \frac{\partial u}{\partial \nu}(t\theta) \, dt \right) dS(\theta). \end{aligned}$$

From classical Brezis-Kato [8] estimates, standard bootstrap, and elliptic regularity theory, it follows that $u \in C^{1,\tau}_{\rm loc}(\Omega \setminus \widetilde{\Sigma})$ for some $\tau \in (0,1)$. Hence, for every $\theta \in \mathbb{S}^{N-1} \setminus \Sigma$, and consequently for a.e. $\theta \in \mathbb{S}^{N-1}$, $\frac{\partial u}{\partial \nu}(t\theta) = \frac{d}{dt} u(t\theta)$ for every $t \in (s,r)$. Therefore, in view of (72), we deduce that

$$\begin{aligned} \int_s^r \frac{2}{t^{N-1}} \left( \int_{\partial B_t} u \frac{\partial u}{\partial \nu} \, dS \right) dt &= \int_{\mathbb{S}^{N-1}} \left( \int_s^r \frac{d}{dt} |u(t\theta)|^2 \, dt \right) dS(\theta) \\ &= \int_{\mathbb{S}^{N-1}} \left( |u(r\theta)|^2 - |u(s\theta)|^2 \right) dS(\theta) = H(r) - H(s) \end{aligned}$$

thus proving that $H \in W^{1,1}_{\rm loc}(0,R)$ and that its weak derivative is given by (73). □

Now we show that, if $u \not\equiv 0$, $H(r)$ does not vanish for every $r \in (0, r_0)$.

**Lemma 5.2.** *Let $\Omega\subset\mathbb{R}^N$, $N\geqslant 3$, be a bounded open set such that $0\in\Omega$, $a$ satisfy (7) and (12), and $u\not\equiv 0$ be a weak $H^1(\Omega)$-solution to (13) in $\Omega$, with $h$ verifying* (**H**) *and $f$ as in* (**F**)*. Then $H(r)>0$ for any $r\in(0,r_0)$, where $H=H(r)$ is defined by (71) and $r_0>0$ satisfies (47).*

Proof. Suppose by contradiction that there exists $R\in(0,r_0)$ such that $H(R)=0$. Then $u=0$ a.e. on $\partial B_R$ and thus $u\in H^1_0(B_R)$. Multiplying both sides of (13) by $u$ and integrating by parts over $B_R$ we obtain

$$\int_{B_R}|\nabla u(x)|^2dx-\int_{B_R}\frac{a(\frac{x}{|x|})}{|x|^2}|u(x)|^2\,dx=\int_{B_R}h(x)\,|u(x)|^2\,dx+\int_{B_R}f(x,u(x))u(x)\,dx.$$

Proceeding as in (50) and using (**F**), Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}0&=\int_{B_R}\left(|\nabla u(x)|^2-\frac{a(\frac{x}{|x|})}{|x|^2}u^2(x)-h(x)u^2(x)-f(x,u(x))u(x)\right)dx\\&\geqslant\left[1-\Lambda(a)-C_hr_0^{\varepsilon}\binom{N}{k}\left(\frac{2}{k-2}\right)^2\left(1+\binom{N-k}{k}\right)\right]\int_{B_R}|\nabla u(x)|^2\,dx\\&\quad-C_fS^{-1}\left[(\omega_{N-1}/N)^{\frac{2}{N}}r_0^2+\|u\|^{2^*-2}_{L^{2^*}(B_{r_0})}\right]\int_{B_R}|\nabla u(x)|^2\,dx,\end{aligned}$$

which, together with (47), implies $u\equiv 0$ in $B_R$. Since $u\equiv 0$ in a neighborhood of the origin, we may apply, away from the singular set $\widetilde{\Sigma}$ (which has zero measure), classical unique continuation principles for second order elliptic equations with locally bounded coefficients (see e.g. [44]) to conclude that $u=0$ a.e. in $\Omega$, a contradiction. □

We also consider the function $D:(0,r_0)\to\mathbb{R}$ defined as

$$D(r)=\frac{1}{r^{N-2}}\int_{B_r}\left(|\nabla u(x)|^2-\frac{a(\frac{x}{|x|})}{|x|^2}|u(x)|^2-h(x)\,|u(x)|^2-f(x,u(x))u(x)\right)dx,\tag{74}$$

where $r_0$ is defined in (47). The regularity of the function $D$ is established in the following lemma.

**Lemma 5.3.** *Let $\Omega\subset\mathbb{R}^N$, $N\geqslant 3$, be a bounded open set such that $0\in\Omega$. Let $a$ satisfy (7) and (12), and $u$ be a weak $H^1(\Omega)$-solution to (13), with $h$ satisfying* (**H**) *and $f$ satisfying* (**F**)*. Then the function $D$ defined in (74) belongs to $W^{1,1}_{\rm loc}(0,r_0)$ and*

$$\begin{aligned}D'(r)=&\frac{2}{r^{N-1}}\left[r\int_{\partial B_r}\left|\frac{\partial u}{\partial\nu}\right|^2dS-\frac12\int_{B_r}(\nabla h(x)\cdot x)|u(x)|^2\,dx-\int_{B_r}h(x)|u(x)|^2\,dx\right]\\&+r^{1-N}\int_{B_r}\big((N-2)f(x,u(x))u(x)-2NF(x,u(x))-2\nabla_xF(x,u(x))\cdot x\big)\,dx\\&+r^{2-N}\int_{\partial B_r}\big(2F(x,u(x))-f(x,u(x))u(x)\big)\,dS\end{aligned}\tag{75}$$

*in a distributional sense and for a.e. $r\in(0,r_0)$.*

Proof. For any $r\in(0,r_0)$ let

$$I(r)=\int_{B_r}\left(|\nabla u(x)|^2-\frac{a\big(\frac{x}{|x|}\big)}{|x|^2}|u(x)|^2-h(x)\,|u(x)|^2-f(x,u(x))u(x)\right)dx.\tag{76}$$

From Remark 3.2, we deduce that $I \in W^{1,1}(0, r_0)$ and

$$I'(r) = \int_{\partial B_r} \left( |\nabla u(x)|^2 - \frac{a(\frac{x}{|x|})}{|x|^2} |u(x)|^2 - h(x)|u(x)|^2 - f(x, u(x))u(x) \right) dS \tag{77}$$

for a.e. $r \in (0, r_0)$ and in the distributional sense. Therefore $D \in W^{1,1}_{\mathrm{loc}}(0, r_0)$ and, plugging (51), (76), and (77) into

$$D'(r) = r^{1-N}[-(N-2)I(r) + rI'(r)],$$

we obtain (75) for a.e. $r \in (0, r_0)$ and in the distributional sense. □

By virtue of Lemma 5.2, if $u$ is a weak $H^1(\Omega)$-solution to (13), $u \not\equiv 0$, the *Almgren type frequency function*

$$\mathcal{N}(r) = \mathcal{N}_{u,h,f}(r) = \frac{D(r)}{H(r)} \tag{78}$$

is well defined in $(0, r_0)$. Collecting Lemmas 5.1 and 5.3, we compute the derivative of $\mathcal{N}$.

**Lemma 5.4.** *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded open set such that $0 \in \Omega$, $a$ satisfy (7) and (12), and $u \not\equiv 0$ be a weak $H^1(\Omega)$-solution to (13), with $h$ satisfying* (**H**) *and $f$ satisfying* (**F**)*. Then the function $\mathcal{N}$ defined in (78) belongs to $W^{1,1}_{\mathrm{loc}}(0, r_0)$ and*

$$\mathcal{N}'(r) = \nu_1(r) + \nu_2(r) \tag{79}$$

*in a distributional sense and for a.e. $r \in (0, r_0)$, where*

$$\nu_1(r) = \frac{2r\left[ \left( \int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}\right|^2 dS \right) \cdot \left( \int_{\partial B_r} |u|^2 dS \right) - \left( \int_{\partial B_r} u \frac{\partial u}{\partial \nu} dS \right)^2 \right]}{\left( \int_{\partial B_r} |u|^2 dS \right)^2} \tag{80}$$

*and*

$$\nu_2(r) = -\frac{\int_{B_r} (2h(x) + \nabla h(x) \cdot x)|u(x)|^2 \, dx}{\int_{\partial B_r} |u|^2 \, dS} + \frac{r \int_{\partial B_r} \big(2F(x, u(x)) - f(x, u(x))u(x)\big) \, dS}{\int_{\partial B_r} |u|^2 \, dS} \tag{81}$$
$$+ \frac{\int_{B_r} \big((N-2)f(x, u(x))u(x) - 2NF(x, u(x)) - 2\nabla_x F(x, u(x)) \cdot x\big) \, dx}{\int_{\partial B_r} |u|^2 \, dS}.$$

Proof. From Lemmas 5.1, 5.2, and 5.3, it follows that $\mathcal{N} \in W^{1,1}_{\mathrm{loc}}(0, r_0)$. From (52), (74), and (73) we infer

$$D(r) = \frac{1}{2} r H'(r) \tag{82}$$

for a.e. $r \in (0, r_0)$. From (82) we have that

$$\mathcal{N}'(r) = \frac{D'(r)H(r) - D(r)H'(r)}{(H(r))^2} = \frac{D'(r)H(r) - \frac{1}{2}r(H'(r))^2}{(H(r))^2}$$

and the proof of the lemma easily follows from (73) and (75). □

We now prove that $\mathcal{N}(r)$ admits a finite limit as $r \to 0^+$. To this aim, the following estimate plays a crucial role.

**Lemma 5.5.** *Under the same assumptions as in Lemma 5.4, there exist $\tilde r \in (0, \min\{r_0, r_q\})$ and a positive constant $\overline{C} = \overline{C}(N, k, a, h, f, u) > 0$ depending on $N$, $k$, $a$, $h$, $f$, $u$ but independent of $r$ such that*

$$\int_{B_r}\bigg(|\nabla u(x)|^2 - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2}|u(x)|^2 - h(x)\,|u(x)|^2 - f(x,u(x))u(x)\bigg)dx \tag{83}$$
$$\geqslant -\frac{N-2}{2r}\int_{\partial B_r}|u(x)|^2 dS$$
$$+\overline{C}\bigg(\sum_{J\in\mathcal{A}_k}\int_{B_r}\frac{|u(x)|^2}{|x_J|^2}\,dx + \sum_{(J_1,J_2)\in\mathcal{B}_k}\int_{B_r}\frac{|u(x)|^2}{|x_{J_1}-x_{J_2}|^2} + \bigg(\int_{B_r}|u(x)|^{2^*}dx\bigg)^{\frac{2}{2^*}}\bigg)$$

*and*

$$\mathcal{N}(r) > -\frac{N-2}{2} \tag{84}$$

*for every $r \in (0, \tilde r)$.*

Proof. By (40), (41), and (45), we have that

$$\int_{B_r}\bigg(|\nabla u(x)|^2 - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2}|u(x)|^2 - h(x)\,|u(x)|^2 - f(x,u(x))u(x)\bigg)dx$$
$$\geqslant -\frac{N-2}{2r}\,\frac{\binom{N}{k}\big(1+\binom{N-k}{k}\big)+\frac{1+\Lambda(a)}{2}}{\binom{N}{k}\big(1+\binom{N-k}{k}\big)+1}\int_{\partial B_r}|u(x)|^2 dS$$
$$+\frac{\big(\frac{k-2}{2}\big)^2}{\binom{N}{k}\big(1+\binom{N-k}{k}\big)+1}\bigg[1-\Lambda(a) - C_h r^{\varepsilon}\,\frac{\binom{N}{k}\big(1+\binom{N-k}{k}\big)+1}{(\frac{k-2}{2})^2}\bigg]\times$$
$$\times\bigg(\sum_{J\in\mathcal{A}_k}\int_{B_r}\frac{|u(x)|^2}{|x_J|^2}\,dx + \sum_{(J_1,J_2)\in\mathcal{B}_k}\int_{B_r}\frac{|u(x)|^2}{|x_{J_1}-x_{J_2}|^2}\bigg)$$
$$+\bigg[\frac{1}{2}\widetilde{S}_N\,\frac{\min\big\{1-\Lambda(a), \mu_1(a)+\big(\frac{N-2}{2}\big)^2\big\}}{\binom{N}{k}\big(1+\binom{N-k}{k}\big)+1}$$
$$- C_f\bigg(\Big(\frac{\omega_{N-1}}{N}\Big)^{\frac{2}{N}} r^2 + \|u\|^{2^*-2}_{L^{2^*}(B_r)}\bigg)\bigg]\bigg(\int_{B_r}|u(x)|^{2^*}dx\bigg)^{2/2^*}$$

for every $r \in (0, r_0)$. Since $\Lambda(a) < 1$, from the above estimate it follows that we can choose $\tilde r \in (0, r_0)$ sufficiently small such that estimate (83) holds for $r \in (0, \tilde r)$ for some positive constant $\overline{C} = \overline{C}(N, k, a, h, f, u) > 0$. Estimate (83), together with (71) and (74), yields (84). □

**Lemma 5.6.** *Under the same assumptions as in Lemma 5.4, let $\tilde r$ be as in Lemma 5.5 and $\nu_2$ as in (81). Then there exist a positive constant $C_1 > 0$ depending on $N, q, C_f, C_h, \overline{C}, \tilde r, \|u\|_{L^q(B_{\tilde r})}$ and* *a function* $g \in L^1(0,\tilde r)$, $g \geqslant 0$ *a.e. in* $(0,\tilde r)$, *such that*

$$|\nu_2(r)| \leqslant C_1 \left[\mathcal{N}(r) + \frac{N}{2}\right] \left(r^{-1+\varepsilon} + r^{-1+\frac{2(q-2^*)}{q}} + g(r)\right)$$

*for a.e.* $r \in (0,\tilde r)$ *and*

$$\int_0^r g(s)\,ds \leqslant \frac{\|u\|_{L^{2^*}(\Omega)}^{2^*(1-\alpha)}}{1-\alpha}\, r^{\frac{N(q-2^*)}{q}(\alpha-\frac{2}{2^*})}$$

*for all* $r \in (0,\tilde r)$ *and for some* $\alpha$ *satisfying* $\frac{2}{2^*} < \alpha < 1$.

PROOF. From (**H**) and (83) we deduce that

$$\left|\int_{B_r}(2h(x)+\nabla h(x)\cdot x)|u(x)|^2\,dx\right| \leqslant 2C_h r^{\varepsilon}\bigg(\sum_{J\in\mathcal{A}_k}\int_{B_r}\frac{|u(x)|^2}{|x_J|^2}\,dx + \sum_{(J_1,J_2)\in\mathcal{B}_k}\int_{B_r}\frac{|u(x)|^2}{|x_{J_1}-x_{J_2}|^2}\bigg)$$
$$\leqslant 2C_h\overline{C}^{-1}\, r^{\varepsilon+N-2}\left[D(r)+\tfrac{N-2}{2}H(r)\right],$$

and, therefore, for any $r \in (0,\tilde r)$, we have that

$$\left|\frac{\int_{B_r}(2h(x)+\nabla h(x)\cdot x)|u(x)|^2\,dx}{\int_{\partial B_r}|u|^2\,dS}\right| \leqslant 2C_h\overline{C}^{-1}\, r^{-1+\varepsilon}\,\frac{D(r)+\frac{N-2}{2}H(r)}{H(r)} \tag{85}$$
$$= 2C_h\overline{C}^{-1}\, r^{-1+\varepsilon}\left[\mathcal{N}(r)+\frac{N-2}{2}\right].$$

By (**F**), Hölder's inequality, and (83), for some constant const $=$ const$\,(N, C_f) > 0$ depending on $N, C_f$, and for all $r \in (0,\tilde r)$, there holds

$$\left|\int_{B_r}\big((N-2)f(x,u(x))u(x) - 2NF(x,u(x)) - 2\nabla_x F(x,u(x))\cdot x\big)\,dx\right|$$
$$\leqslant \text{const}\int_{B_r}\big(|u(x)|^2+|u(x)|^{2^*}\big)\,dx$$
$$\leqslant \text{const}\left(\Big(\frac{\omega_{N-1}}{N}\Big)^{\frac{2}{N}} r^2 + \|u\|_{L^{2^*}(B_r)}^{2^*-2}\right)\left(\int_{B_r}|u(x)|^{2^*}dx\right)^{2/2^*}$$
$$\leqslant \text{const}\left(\Big(\frac{\omega_{N-1}}{N}\Big)^{\frac{2}{N}} r^2 + \Big(\frac{\omega_{N-1}}{N}\Big)^{\frac{2(q-2^*)}{Nq}} r^{\frac{2(q-2^*)}{q}}\|u\|_{L^q(B_{\tilde r})}^{2^*-2}\right)$$
$$\times\,\overline{C}^{-1} r^{N-2}\left[D(r)+\tfrac{N-2}{2}H(r)\right]$$

and hence

$$\left|\frac{\int_{B_r}\big((N-2)f(x,u(x))u(x) - 2NF(x,u(x)) - 2\nabla_x F(x,u(x))\cdot x\big)\,dx}{\int_{\partial B_r}|u|^2\,dS}\right| \tag{86}$$
$$\leqslant \text{const}\,\overline{C}^{-1}\left(\Big(\frac{\omega_{N-1}}{N}\Big)^{\frac{2}{N}}\tilde r^{\frac{22^*}{q}} + \Big(\frac{\omega_{N-1}}{N}\Big)^{\frac{2(q-2^*)}{Nq}}\|u\|_{L^q(B_{\tilde r})}^{2^*-2}\right)$$
$$\times\, r^{-1+\frac{2(q-2^*)}{q}}\left[\mathcal{N}(r)+\tfrac{N-2}{2}\right].$$

Let us fix $\frac{2}{2^*} < \alpha < 1$. Then, by Hölder's inequality and (83),

$$\begin{aligned}(87)\qquad \left(\int_{B_r} |u(x)|^{2^*}\,dx\right)^{\alpha} &= \left(\int_{B_r} |u(x)|^{2^*}\,dx\right)^{\alpha-\frac{2}{2^*}} \left(\int_{B_r} |u(x)|^{2^*}\,dx\right)^{\frac{2}{2^*}} \\ &\leqslant \left(\frac{\omega_{N-1}}{N}\right)^{\frac{q-2^*}{q}(\alpha-\frac{2}{2^*})} r^{\frac{N(q-2^*)}{q}(\alpha-\frac{2}{2^*})} \|u\|^{2^*(\alpha-\frac{2}{2^*})}_{L^q(B_{\tilde r})} \overline{C}^{-1} r^{N-2}\left[D(r)+\frac{N-2}{2}H(r)\right] \\ &= \overline{C}^{-1}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{\beta}{N}} r^{-1+\beta} \|u\|^{2^*(\alpha-\frac{2}{2^*})}_{L^q(B_{\tilde r})}\left[\mathcal{N}(r)+\frac{N-2}{2}\right]\left(\int_{\partial B_r}|u|^2\,dS\right)\end{aligned}$$

for all $r \in (0,\tilde r)$, where $\beta = \frac{N(q-2^*)}{q}(\alpha - \frac{2}{2^*}) > 0$. From (**F**), (87), and (84), there exists some const $=$ const $(N,q,C_f) > 0$ depending on $N,q,C_f$ such that, for all $r\in(0,\tilde r)$,

$$(88)\qquad \left|\frac{r\int_{\partial B_r}\big(2F(x,u(x))-f(x,u(x))u(x)\big)\,dx}{\int_{\partial B_r}|u|^2\,dS}\right| \leqslant \text{const}\, r\left(1+\frac{\int_{\partial B_r}|u|^{2^*}\,dS}{\int_{\partial B_r}|u|^2\,dS}\right)$$

$$\leqslant \text{const}\, r\left[\mathcal{N}(r)+\frac{N}{2}\right] + \frac{\text{const}}{\overline{C}}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{\beta}{N}}\|u\|^{2^*(\alpha-\frac{2}{2^*})}_{L^q(B_{\tilde r})}\left[\mathcal{N}(r)+\frac{N-2}{2}\right]\frac{r^\beta\int_{\partial B_r}|u|^{2^*}\,dS}{\left(\int_{B_r}|u(x)|^{2^*}\,dx\right)^{\alpha}}.$$

By a direct calculation, we have that

$$(89)\qquad \frac{r^\beta\int_{\partial B_r}|u|^{2^*}dS}{\left(\int_{B_r}|u(x)|^{2^*}dx\right)^{\alpha}} = \frac{1}{1-\alpha}\left[\frac{d}{dr}\left(r^\beta\left(\int_{B_r}|u(x)|^{2^*}dx\right)^{1-\alpha}\right) - \beta\, r^{-1+\beta}\left(\int_{B_r}|u(x)|^{2^*}dx\right)^{1-\alpha}\right]$$

in the distributional sense and for a.e. $r\in(0,\tilde r)$. Since

$$\lim_{r\to 0^+} r^\beta\left(\int_{B_r}|u(x)|^{2^*}dx\right)^{1-\alpha} = 0$$

we deduce that the function

$$r \mapsto \frac{d}{dr}\left(r^\beta\left(\int_{B_r}|u(x)|^{2^*}dx\right)^{1-\alpha}\right)$$

is integrable over $(0,\tilde r)$. Being

$$r^{-1+\beta}\left(\int_{B_r}|u(x)|^{2^*}dx\right)^{1-\alpha} = o(r^{-1+\beta})$$

as $r\to 0^+$, we have that also the function

$$r \mapsto r^{-1+\beta}\left(\int_{B_r}|u(x)|^{2^*}dx\right)^{1-\alpha}$$

is integrable over $(0,\tilde r)$. Therefore, by (89), we deduce that

$$(90)\qquad g(r) := \frac{r^\beta\int_{\partial B_r}|u|^{2^*}dS}{\left(\int_{B_r}|u(x)|^{2^*}dx\right)^{\alpha}} \in L^1(0,\tilde r)$$

and

$$0 \leqslant \int_0^r g(s)\,ds \leqslant \frac{\|u\|_{L^{2^*}(\Omega)}^{2^*(1-\alpha)}}{1-\alpha}\, r^{\beta} \tag{91}$$

for all $r \in (0, \tilde{r})$. Collecting (85), (86), (88), (90), and (91), we obtain the stated estimate. □

**Lemma 5.7.** *Under the same assumptions as in Lemma 5.4, let $\tilde{r}$ be as in Lemma 5.5 and $\mathcal{N}$ as in (78). Then there exist a positive constant $C_2 > 0$ depending on $N$, $q$, $C_f$, $C_h$, $\overline{C}$, $\tilde{r}$, $\|u\|_{L^q(B_{\tilde{r}})}$, $\mathcal{N}(\tilde{r})$, $\varepsilon$ such that*

$$\mathcal{N}(r) \leqslant C_2 \tag{92}$$

*for all $r \in (0, \tilde{r})$.*

Proof. By Lemma 5.4, Schwarz's inequality, and Lemma 5.6, we obtain

$$\left(\mathcal{N} + \frac{N}{2}\right)'(r) \geqslant \nu_2(r) \geqslant -C_1 \left[\mathcal{N}(r) + \frac{N}{2}\right] \left(r^{-1+\varepsilon} + r^{-1+\frac{2(q-2^*)}{q}} + g(r)\right) \tag{93}$$

for a.e. $r \in (0, \tilde{r})$. After integration over $(r, \tilde{r})$ it follows that

$$\mathcal{N}(r) \leqslant -\frac{N}{2} + \left(\mathcal{N}(\tilde{r}) + \frac{N}{2}\right) \exp\left(C_1\left(\frac{\tilde{r}^{\varepsilon}}{\varepsilon} + \frac{q}{2(q-2^*)}\tilde{r}^{\frac{2(q-2^*)}{q}} + \int_0^{\tilde{r}} g(s)\,ds\right)\right)$$

for any $r \in (0, \tilde{r})$, thus proving estimate (92). □

**Lemma 5.8.** *Under the same assumptions as in Lemma 5.4, the limit*

$$\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$$

*exists and is finite.*

Proof. By Lemmas 5.6 and 5.7, the function $\nu_2$ defined in (81) belongs to $L^1(0, \tilde{r})$. Hence, by Lemma 5.4 and Schwarz's inequality, $\mathcal{N}'$ is the sum of a nonnegative function and of a $L^1$-function on $(0, \tilde{r})$. Therefore

$$\mathcal{N}(r) = \mathcal{N}(\tilde{r}) - \int_r^{\tilde{r}} \mathcal{N}'(s)\,ds$$

admits a limit as $r \to 0^+$ which is necessarily finite in view of (84) and (92). □

A first consequence of the above analysis on the Almgren's frequency function is the following estimate of $H(r)$.

**Lemma 5.9.** *Under the same assumptions as in Lemma 5.4, let $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ be as in Lemma 5.8 and $\tilde{r}$ as in Lemma 5.5. Then there exists a constant $K_1 > 0$ such that*

$$H(r) \leqslant K_1 r^{2\gamma} \quad \textit{for all } r \in (0, \tilde{r}). \tag{94}$$

*On the other hand for any $\sigma > 0$ there exists a constant $K_2(\sigma) > 0$ depending on $\sigma$ such that*

$$H(r) \geqslant K_2(\sigma)\, r^{2\gamma+\sigma} \quad \textit{for all } r \in (0, \tilde{r}). \tag{95}$$

PROOF. By Lemma 5.8, $\mathcal{N}' \in L^1(0,\tilde r)$ and, by Lemma 5.7, $\mathcal{N}$ is bounded, then from (93) and (91) it follows that

$$\mathcal{N}(r) - \gamma = \int_0^r \mathcal{N}'(s)\,ds \geqslant -C_3 r^{\delta} \tag{96}$$

for some constant $C_3 > 0$ and all $r \in (0,\tilde r)$, where

$$\delta = \min\left\{\varepsilon, \frac{N(q-2^*)}{q}\left(\alpha - \frac{2}{2^*}\right), \frac{2(q-2^*)}{q}\right\} \tag{97}$$

with $\alpha$ as in Lemma 5.6.

Therefore by (82) and (96) we deduce that, for $r \in (0,\tilde r)$,

$$\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} \geqslant \frac{2\gamma}{r} - 2C_3 r^{-1+\delta},$$

which, after integration over the interval $(r,\tilde r)$, yields (94).

Let us prove (95). Since $\gamma = \lim_{r\to 0^+}\mathcal{N}(r)$, for any $\sigma > 0$ there exists $r_\sigma > 0$ such that $\mathcal{N}(r) < \gamma + \sigma/2$ for any $r \in (0, r_\sigma)$ and hence

$$\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} < \frac{2\gamma+\sigma}{r} \quad \text{for all } r \in (0,r_\sigma).$$

Integrating over the interval $(r, r_\sigma)$ and by continuity of $H$ outside 0, we obtain (95) for some constant $K_2(\sigma)$ depending on $\sigma$. □

## 6. The blow-up argument

Throughout this section we let $u$ be a weak $H^1(\Omega)$-solution to equation (13) in a bounded domain $\Omega \subset \mathbb{R}^N$ containing the origin with $a$ satisfying (7), (12), $h$ satisfying (**H**), and $f$ satisfying (**F**). Let $H$ and $D$ be the functions defined in (71) and (74) and $\tilde r$ be as in Lemma 5.5.

**Lemma 6.1.** *For $\lambda \in (0,\tilde r)$, let*

$$w^\lambda(x) = \frac{u(\lambda x)}{\sqrt{H(\lambda)}}. \tag{98}$$

*Then there exists $\bar r \in (0,\tilde r)$ depending on $N$, $k$, $a$, $h$, $f$, $\varepsilon$, and $\|u\|_{L^q(B_{\tilde r})}$ such that the set $\{w^\lambda\}_{\lambda\in(0,\bar r)}$ is bounded in $H^1(B_1)$.*

PROOF. From (72) it follows that $\int_{\partial B_1} |w^\lambda|^2 dS = 1$. Moreover, by scaling and (92),

$$\int_{B_1} |\nabla w^\lambda(x)|^2 dx - \int_{B_1} \frac{a(\frac{x}{|x|})}{|x|^2}|w^\lambda(x)|^2\,dx - \lambda^2\int_{B_1} h(\lambda x)|w^\lambda(x)|^2\,dx \\ - \frac{\lambda^2}{\sqrt{H(\lambda)}}\int_{B_1} f(\lambda x, \sqrt{H(\lambda)}w^\lambda(x))w^\lambda(x)\,dx = \mathcal{N}(\lambda) \leqslant C_2 \tag{99}$$

for every $\lambda \in (0, \tilde r)$. By (39) applied to $w^\lambda$ we have that

$$\int_{B_1} |\nabla w^\lambda(x)|^2\,dx - \int_{B_1} \frac{a(\frac{x}{|x|})}{|x|^2}|w^\lambda(x)|^2\,dx + \Lambda(a)\frac{N-2}{2}\int_{\partial B_1}|w^\lambda|^2\,dS \geqslant (1-\Lambda(a))\int_{B_1}|\nabla w^\lambda(x)|^2\,dx. \tag{100}$$

Moreover by Corollary 2.7 we have

$$\begin{aligned}\left|\lambda^2\int_{B_1} h(\lambda x)|w^\lambda(x)|^2\,dx\right| &\leqslant C_h\lambda^\varepsilon\bigg(\sum_{J\in\mathcal{A}_k}\int_{B_1}\frac{|w^\lambda(x)|^2}{|x_J|^2}\,dx + \sum_{(J_1,J_2)\in\mathcal{B}_k}\int_{B_1}\frac{|w^\lambda(x)|^2}{|x_{J_1}-x_{J_2}|^2}\,dx\bigg)\\ &\leqslant C_h\lambda^\varepsilon\binom{N}{k}\bigg(1+\binom{N-k}{k}\bigg)\bigg[\Big(\frac{2}{k-2}\Big)^2\int_{B_1}|\nabla w^\lambda(x)|^2\,dx + \frac{2(N-2)}{(k-2)^2}\int_{\partial B_1}|w^\lambda|^2\,dS\bigg].\end{aligned} \tag{101}$$

From (**F**), Hölder's inequality, Lemma 2.9, and Lemma 2.6,

$$\begin{aligned}&\frac{\lambda^2}{\sqrt{H(\lambda)}}\bigg|\int_{B_1} f(\lambda x, \sqrt{H(\lambda)}w^\lambda(x))w^\lambda(x)\,dx\bigg|\\ &\leqslant C_f\lambda^2\int_{B_1}|w^\lambda(x)|^2\,dx + C_f\lambda^2(H(\lambda))^{\frac{2^*}{2}-1}\int_{B_1}|w^\lambda(x)|^{2^*}\,dx\\ &\leqslant C_f\bigg(\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2}{N}}\lambda^2 + \lambda^2(H(\lambda))^{\frac{2^*}{2}-1}\Big(\int_{B_1}|w^\lambda(x)|^{2^*}\,dx\Big)^{2/N}\bigg)\Big(\int_{B_1}|w^\lambda(x)|^{2^*}\,dx\Big)^{2/2^*}\\ &\leqslant C_f\widetilde S_N^{-1}\bigg(\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2}{N}}\lambda^2 + \Big(\int_{B_\lambda}|u(x)|^{2^*}\,dx\Big)^{2/N}\bigg)\bigg(\int_{B_1}\Big(|\nabla w^\lambda(x)|^2 + \frac{|w^\lambda(x)|^2}{|x|^2}\Big)\,dx\bigg)\\ &\leqslant \frac{C_f}{\widetilde S_N}\bigg(\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2}{N}}\lambda^2 + \|u\|^{2^*-2}_{L^q(B_{\tilde r})}\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2(q-2^*)}{qN}}\lambda^{\frac{2(q-2^*)}{q}}\bigg)\bigg(\int_{B_1}\Big(|\nabla w^\lambda(x)|^2 + \frac{|w^\lambda(x)|^2}{|x|^2}\Big)\,dx\bigg)\\ &\leqslant \frac{C_f((N-2)^2+4)}{\widetilde S_N(N-2)^2}\bigg(\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2}{N}}\lambda^2 + \|u\|^{2^*-2}_{L^q(B_{\tilde r})}\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2(q-2^*)}{qN}}\lambda^{\frac{2(q-2^*)}{q}}\bigg)\bigg(\int_{B_1}|\nabla w^\lambda(x)|^2\,dx\bigg)\\ &\quad + \frac{2C_f}{\widetilde S_N(N-2)}\bigg(\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2}{N}}\lambda^2 + \|u\|^{2^*-2}_{L^q(B_{\tilde r})}\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2(q-2^*)}{qN}}\lambda^{\frac{2(q-2^*)}{q}}\bigg).\end{aligned} \tag{102}$$

From (99–102), we deduce that

$$\begin{aligned}&\bigg[1-\Lambda(a) - C_h\lambda^\varepsilon\binom{N}{k}\bigg(1+\binom{N-k}{k}\bigg)\Big(\frac{2}{k-2}\Big)^2\\ &\quad - \frac{C_f((N-2)^2+4)}{\widetilde S_N(N-2)^2}\bigg(\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2}{N}}\lambda^2 + \|u\|^{2^*-2}_{L^q(B_{\tilde r})}\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2(q-2^*)}{qN}}\lambda^{\frac{2(q-2^*)}{q}}\bigg)\bigg]\int_{B_1}|\nabla w^\lambda(x)|^2\,dx\\ &\leqslant C_2 + \Lambda(a)\frac{N-2}{2} + C_h\lambda^\varepsilon\binom{N}{k}\bigg(1+\binom{N-k}{k}\bigg)\frac{2(N-2)}{(k-2)^2}\\ &\quad + \frac{2C_f}{\widetilde S_N(N-2)}\bigg(\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2}{N}}\lambda^2 + \|u\|^{2^*-2}_{L^q(B_{\tilde r})}\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2(q-2^*)}{qN}}\lambda^{\frac{2(q-2^*)}{q}}\bigg)\end{aligned}$$

for every $\lambda \in (0, \tilde{r})$, which implies that $\{w^\lambda\}_{\lambda \in (0,\bar{r})}$ is bounded in $H^1(B_1)$ if $\bar{r}$ is chosen sufficiently small. □

In the next lemma we prove a *doubling* type result.

**Lemma 6.2.** *There exists $C_4 > 0$ such that*

$$\frac{1}{C_4} H(\lambda) \leqslant H(R\lambda) \leqslant C_4 H(\lambda) \quad \text{for any } \lambda \in (0, \tilde{r}/2) \text{ and } R \in [1, 2]\,, \tag{103}$$

$$\int_{B_R} |\nabla w^\lambda(x)|^2 dx \leqslant 2^{N-2} C_4 \int_{B_1} |\nabla w^{R\lambda}(x)|^2 dx \quad \text{for any } \lambda \in (0, \tilde{r}/2) \text{ and } R \in [1, 2]\,, \tag{104}$$

*and*

$$\int_{B_R} |w^\lambda(x)|^2 dx \leqslant 2^N C_4 \int_{B_1} |w^{R\lambda}(x)|^2 dx \quad \text{for any } \lambda \in (0, \tilde{r}/2) \text{ and } R \in [1, 2]\,, \tag{105}$$

*where $w^\lambda$ is defined in (98).*

Proof. By (84), (92), and (82), it follows that

$$-\frac{N-2}{r} \leqslant \frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} \leqslant \frac{2\,C_2}{r} \qquad \text{for any } r \in (0, \tilde{r}).$$

Let $R \in (1, 2]$. For any $\lambda < \tilde{r}/R$, integration over $(\lambda, R\lambda)$ and the fact that $R \leqslant 2$ yield

$$2^{2-N} H(\lambda) \leqslant H(R\lambda) \leqslant 4^{C_2} H(\lambda) \qquad \text{for any } \lambda \in (0, \tilde{r}/R).$$

Since the above chain of inequalities trivially holds also for $R = 1$, the proof of (103) is complete with $C_4 = \max\{4^{C_2}, 2^{N-2}\}$. By scaling and (103), we obtain that, for any $\lambda \in (0, \tilde{r}/2)$ and $R \in [1, 2]$,

$$\begin{aligned}\int_{B_R} |\nabla w^\lambda(x)|^2 dx &= \frac{\lambda^{2-N}}{H(\lambda)} \int_{B_{R\lambda}} |\nabla u(x)|^2 dx \\ &= R^{N-2} \frac{H(R\lambda)}{H(\lambda)} \int_{B_1} |\nabla w^{R\lambda}(x)|^2 dx \leqslant R^{N-2} C_4 \int_{B_1} |\nabla w^{R\lambda}(x)|^2 dx,\end{aligned}$$

thus providing (104). In a similar way, (105) follows from (103) by scaling. □

**Lemma 6.3.** *For every $\lambda \in (0, \tilde{r})$, let $w^\lambda$ as in (98). Then there exist $M > 0$ and $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ there exists $R_\lambda \in [1, 2]$ such that*

$$\int_{\partial B_{R_\lambda}} |\nabla w^\lambda|^2 dS \leqslant M \int_{B_{R_\lambda}} |\nabla w^\lambda(x)|^2 dx.$$

Proof. We recall that, by Lemma 6.1, the set $\{w^\lambda\}_{\lambda \in (0,\bar{r})}$ is bounded in $H^1(B_1)$. Moreover by Lemma 6.2, we have that the set $\{w^\lambda\}_{\lambda \in (0,\bar{r}/2)}$ is bounded in $H^1(B_2)$ and hence

$$\limsup_{\lambda \to 0^+} \int_{B_2} |\nabla w^\lambda(x)|^2 dx < +\infty. \tag{106}$$

Let us denote, for every $\lambda \in (0, \bar{r}/2)$,

$$f_\lambda(r) = \int_{B_r} |\nabla w^\lambda(x)|^2 \, dx.$$

The function $f_\lambda$ is absolutely continuous in $[0,2]$ and its distributional derivative is given by

$$f'_\lambda(r) = \int_{\partial B_r} |\nabla w^\lambda|^2 dS \quad \text{for a.e. } r \in (0,2).$$

Suppose by contradiction that for any $M > 0$ there exists a sequence $\lambda_n \to 0^+$ such that

$$\int_{\partial B_r} |\nabla w^{\lambda_n}|^2 dS > M \int_{B_r} |\nabla w^{\lambda_n}(x)|^2 dx \quad \text{for all } r \in [1,2]\,, \tag{107}$$

which may be rewritten as

$$f'_{\lambda_n}(r) > M f_{\lambda_n}(r) \quad \text{for a.e. } r \in [1,2] \text{ and for any } n \in \mathbb{N}. \tag{108}$$

Integrating (108) over $[1,2]$ we obtain

$$f_{\lambda_n}(2) > e^M f_{\lambda_n}(1) \qquad \text{for any } n \in \mathbb{N}.$$

Letting $n \to +\infty$ we obtain

$$\limsup_{n\to+\infty} f_{\lambda_n}(1) \leqslant e^{-M} \cdot \limsup_{n\to+\infty} f_{\lambda_n}(2).$$

This implies

$$\liminf_{\lambda\to0^+} f_\lambda(1) \leqslant e^{-M} \cdot \limsup_{\lambda\to0^+} f_\lambda(2) \qquad \text{for any } M > 0.$$

Using (106) and letting $M \to +\infty$ we infer

$$\liminf_{\lambda\to0^+} \int_{B_1} |\nabla w^\lambda(x)|^2 dx = \liminf_{\lambda\to0^+} f_\lambda(1) = 0.$$

Therefore, there exists a sequence $\widetilde{\lambda}_n \to 0$ such that

$$\lim_{n\to+\infty} \int_{B_1} |\nabla w^{\widetilde{\lambda}_n}(x)|^2 dx = 0 \tag{109}$$

and, up to a subsequence still denoted by $\widetilde{\lambda}_n$, we may suppose that $w^{\widetilde{\lambda}_n} \rightharpoonup w$ in $H^1(B_1)$ for some $w \in H^1(B_1)$. Notice that, for any $\lambda \in (0,\tilde r)$, $\int_{\partial B_1} |w^\lambda|^2 dS = 1$ and hence by compactness of the trace map from $H^1(B_1)$ into $L^2(\partial B_1)$, it follows that $\int_{\partial B_1} |w|^2 dS = 1$. Moreover, by weak lower semicontinuity and (109), we also have

$$\int_{B_1} |\nabla w(x)|^2 dx \leqslant \lim_{n\to+\infty} \int_{B_1} |\nabla w^{\widetilde{\lambda}_n}(x)|^2 dx = 0$$

from which it follows that $w \equiv \text{const}$ in $B_1$. On the other hand, for every $\lambda \in (0,\tilde r)$,

$$-\Delta w^\lambda(x) - \frac{a(\frac{x}{|x|})}{|x|^2} w^\lambda(x) = \lambda^2 h(\lambda x)\, w^\lambda(x) + \frac{\lambda^2}{\sqrt{H(\lambda)}} f(\lambda x, \sqrt{H(\lambda)} w^\lambda(x)) \quad \text{in } B_{\tilde r/\lambda}. \tag{110}$$

For every $\phi \in H^1_0(B_1)$, by (**F**) and Hölder's inequality,

$$\begin{aligned}(111)\quad &\frac{\lambda^2}{\sqrt{H(\lambda)}}\left|\int_{B_1} f(\lambda x, \sqrt{H(\lambda)}w^\lambda(x))\phi(x)\,dx\right| \\ &\leqslant C_f\lambda^2\int_{B_1}|w^\lambda(x)||\phi(x)|\,dx + C_f\lambda^2\int_{B_1}|u(\lambda x)|^{2^*-2}|w^\lambda(x)||\phi(x)|\,dx \\ &\leqslant C_f\lambda^2\|w^\lambda\|_{H^1(B_1)}\|\phi\|_{H^1(B_1)} + C_f\big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2(q-2^*)}{qN}}\lambda^{\frac{2(q-2^*)}{q}}\|w^\lambda\|_{L^{2^*}(B_1)}\|\phi\|_{L^{2^*}(B_1)}\|u\|^{2^*-2}_{L^q(B_\lambda)} \\ &= o(1) \quad \text{as } \lambda\to 0^+\end{aligned}$$

and, by (**H**) and Corollary 2.7,

$$\begin{aligned}(112)\quad &\lambda^2\left|\int_{B_1} h(\lambda x)\,w^\lambda(x)\phi(x)\,dx\right| \\ &\leqslant C_h\lambda^\varepsilon\binom{N}{k}\left(1+\binom{N-k}{k}\right)\left(\frac{2}{k-2}\right)^2\left(\int_{B_1}|\nabla w^\lambda(x)|^2\,dx + \frac{N-2}{2}\right)^{1/2}\left(\int_{B_1}|\nabla\phi(x)|^2\,dx\right)^{1/2} \\ &= o(1) \quad \text{as } \lambda\to 0^+.\end{aligned}$$

From (111), (112), and weak convergence $w^{\widetilde{\lambda}_n} \rightharpoonup w$ in $H^1(B_1)$, we can pass to the limit in (110) along the sequence $\widetilde{\lambda}_n$ and obtain that $w$ is a $H^1(B_1)$-weak solution to the equation

$$-\Delta w(x) - \frac{a(\frac{x}{|x|})}{|x|^2}w(x) = 0 \quad \text{in} \quad B_1.$$

Since $w$ is constant in $B_1$, this implies $w\equiv 0$ in $B_1$ which contradicts $\int_{\partial B_1}|w|^2dS = 1$. □

**Lemma 6.4.** *Let $w^\lambda$ and $R_\lambda$ be as in the statement of Lemma 6.3. Then there exists $\overline{M}>0$ such that*

$$\int_{\partial B_1}|\nabla w^{\lambda R_\lambda}|^2dS \leqslant \overline{M} \qquad \text{for any } 0<\lambda<\min\Big\{\lambda_0, \frac{\bar r}{2}\Big\}.$$

Proof. We have

$$\begin{aligned}\int_{\partial B_1}|\nabla w^{\lambda R_\lambda}|^2dS =& \frac{(\lambda R_\lambda)^2}{H(\lambda R_\lambda)}\int_{\partial B_1}|\nabla u(\lambda R_\lambda x)|^2dS(x) = \frac{\lambda^2 R_\lambda^{3-N}}{H(\lambda R_\lambda)}\int_{\partial B_{R_\lambda}}|\nabla u(\lambda x)|^2dS(x) \\ =& \frac{R_\lambda^{3-N}H(\lambda)}{H(\lambda R_\lambda)}\frac{\lambda^2}{H(\lambda)}\int_{\partial B_{R_\lambda}}|\nabla u(\lambda x)|^2dS(x) = \frac{R_\lambda^{3-N}H(\lambda)}{H(\lambda R_\lambda)}\int_{\partial B_{R_\lambda}}|\nabla w^\lambda|^2dS\end{aligned}$$

and, by (103–104), Lemma 6.3, Lemma 6.1, and the fact that $1\leqslant R_\lambda\leqslant 2$, we finally obtain

$$\int_{\partial B_1}|\nabla w^{\lambda R_\lambda}|^2dS \leqslant C_4M\int_{B_{R_\lambda}}|\nabla w^\lambda(x)|^2dx \leqslant 2^{N-2}C_4^2M\int_{B_1}|\nabla w^{\lambda R_\lambda}(x)|^2dx \leqslant \overline{M} < +\infty$$

for any $0<\lambda<\min\big\{\lambda_0, \frac{\bar r}{2}\big\}$, thus completing the proof. □

**Lemma 6.5.** *Let $u$ be a weak $H^1(\Omega)$-solution to (13), $u \not\equiv 0$, in a bounded open set $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, with $a$ satisfying (7) and (12), $h$ satisfying* (**H**)*, and $f$ satisfying* (**F**)*. Let $\gamma$ be as in Lemma 5.8. Then*

(i) *there exists $k_0 \in \mathbb{N}$ such that $\gamma = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(a)}$;*

(ii) *for every sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and an eigenfunction $\psi$ of the operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ associated to the eigenvalue $\mu_{k_0}(a)$ such that $\|\psi\|_{L^2(\mathbb{S}^{N-1})} = 1$ and*

$$\frac{u(\lambda_{n_k}x)}{\sqrt{H(\lambda_{n_k})}} \to |x|^\gamma \psi\Big(\frac{x}{|x|}\Big)$$

*strongly in $H^1(B_1)$.*

Proof. Let $\lambda_n \to 0^+$ and consider the sequence $w^{\lambda_n R_{\lambda_n}}$ as in (98) and $R_\lambda$ as in Lemma 6.3. By Lemmas 6.1 and 6.2, we have that the set $\{w^{\lambda R_\lambda}\}_{\lambda\in(0,\bar r/4)}$ is bounded in $H^1(B_2)$. Then there exists a subsequence $w^{\lambda_{n_k} R_{\lambda_{n_k}}}$ such that $w^{\lambda_{n_k} R_{\lambda_{n_k}}} \rightharpoonup w$ in $H^1(B_2)$ for some function $w \in H^1(B_2)$. Due to compactness of the trace map from $H^1(B_1)$ into $L^2(\partial B_1)$, we obtain that $\int_{\partial B_1} |w|^2 dS = 1$. In particular $w \not\equiv 0$. Furthermore, weak convergence and (111–112) allow passing to the weak limit in the equation

$$-\Delta w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x) - \frac{a(\frac{x}{|x|})}{|x|^2} w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x) = \lambda_{n_k}^2 R_{\lambda_{n_k}}^2 h(\lambda_{n_k} R_{\lambda_{n_k}} x)\, w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x) + \frac{\lambda_{n_k}^2 R_{\lambda_{n_k}}^2}{\sqrt{H(\lambda_{n_k} R_{\lambda_{n_k}})}} f\Big(\lambda_{n_k} R_{\lambda_{n_k}} x, \sqrt{H(\lambda_{n_k} R_{\lambda_{n_k}})}\, w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x)\Big) \tag{113}$$

which holds in a weak sense in $B_{\bar r/(\lambda_{n_k} R_{\lambda_{n_k}})} \supset B_2$ thus yielding

$$-\Delta w(x) - \frac{a(\frac{x}{|x|})}{|x|^2} w(x) = 0 \quad \text{in } B_2. \tag{114}$$

From Lemma 6.4 and density in $H^1(B_1)$ of $C^\infty(\overline{B}_1)$-functions whose support is compactly included in $\overline{B}_1 \setminus \widetilde{\Sigma}$ with $\widetilde{\Sigma}$ defined in (5), it follows that, for all $\phi \in H^1(B_1)$,

$$\begin{aligned}\int_{B_1} \Big(\nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x) \cdot \nabla\phi(x) - \frac{a(\frac{x}{|x|})}{|x|^2} w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x)\phi(x)\Big)\, dx \\ = \lambda_{n_k}^2 R_{\lambda_{n_k}}^2 \int_{B_1} h(\lambda_{n_k} R_{\lambda_{n_k}} x)\, w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x)\phi(x)\, dx + \int_{\partial B_1} \frac{\partial w^{\lambda_{n_k} R_{\lambda_{n_k}}}}{\partial\nu}\,\phi\, dS \\ + \frac{\lambda_{n_k}^2 R_{\lambda_{n_k}}^2}{\sqrt{H(\lambda_{n_k} R_{\lambda_{n_k}})}} \int_{B_1} f\Big(\lambda_{n_k} R_{\lambda_{n_k}} x, \sqrt{H(\lambda_{n_k} R_{\lambda_{n_k}})}\, w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x)\Big)\phi(x)\, dx.\end{aligned} \tag{115}$$

We notice that from (84) it follows that $\gamma \geqslant -\frac{N-2}{2}$. Then, by (**F**) and (94),

$$\frac{\lambda^2}{\sqrt{H(\lambda)}} \left| \frac{f\big(\lambda x, \sqrt{H(\lambda)}\, w^\lambda(x)\big)}{w^\lambda(x)} \right| \leqslant C_f \frac{\lambda^2}{\sqrt{H(\lambda)}} \Big(\sqrt{H(\lambda)} + (H(\lambda))^{\frac{2^*-1}{2}} |w^\lambda(x)|^{2^*-2}\Big)$$

for all $\lambda \in (0, \tilde r)$. Hence, if $s = \frac{q}{2^*-2} > N/2$ with $q$ as in (70), from (98) and Proposition 4.1, we obtain that

$$\left\| \frac{\lambda^2}{\sqrt{H(\lambda)}} \frac{f\big(\lambda x, \sqrt{H(\lambda)}\, w^\lambda(x)\big)}{w^\lambda(x)} \right\|_{L^s(B_2)} \leqslant \text{const}\left(1 + \lambda^2 (H(\lambda))^{\frac{2^*-2}{2}} \left(\int_{B_2} |w^\lambda(x)|^{(2^*-2)s} dx\right)^{1/s}\right)$$

$$= \text{const}\left(1 + \lambda^{2-\frac{N}{s}} \left(\int_{B_{2\lambda}} |u(x)|^q dx\right)^{1/s}\right) = O(1)$$

as $\lambda \to 0^+$. Therefore from classical Brezis-Kato [8] estimates (see also Theorem 8.6 part i)), classical bootstrap and elliptic regularity theory, there holds

$$w^{\lambda_{n_k} R_{\lambda_{n_k}}} \to w \quad \text{in } C^{1,\tau}_{\rm loc}(B_2 \setminus \widetilde\Sigma),$$

for any $\tau \in (0,1)$, which in particular yields

$$\frac{\partial w^{\lambda_{n_k} R_{\lambda_{n_k}}}}{\partial \nu} \to \frac{\partial w}{\partial \nu} \quad \text{in } C^{0,\tau}_{\rm loc}(\partial B_1 \setminus \Sigma) \quad \text{and a.e. in } \partial B_1. \tag{116}$$

From (116) and Lemma 6.4, it follows that

$$\frac{\partial w^{\lambda_{n_k} R_{\lambda_{n_k}}}}{\partial \nu} \rightharpoonup \frac{\partial w}{\partial \nu} \quad \text{weakly in } L^2(\partial B_1). \tag{117}$$

Passing to limit in (115) and using (117) and (111–112), we obtain that

$$\int_{B_1} \left(\nabla w(x) \cdot \nabla \phi(x) - \frac{a(\frac{x}{|x|})}{|x|^2} w(x)\phi(x)\right) dx = \int_{\partial B_1} \frac{\partial w}{\partial \nu}\, \phi \, dS. \tag{118}$$

Subtracting (118) from (115), choosing $\phi = w^{\lambda_{n_k} R_{\lambda_{n_k}}} - w$, and arguing as in (111–112) and Corollary 2.8, we obtain that

$$w^{\lambda_{n_k} R_{\lambda_{n_k}}} \to w \quad \text{in } H^1(B_1). \tag{119}$$

For every $k \in \mathbb{N}$ and $r \in (0,1)$, let us define

$$D_k(r) = \frac{1}{r^{N-2}} \int_{B_r} \bigg[ \Big|\nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x)\Big|^2 - \frac{a(\frac{x}{|x|})}{|x|^2} |w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x)|^2 - \lambda_{n_k}^2 R_{\lambda_{n_k}}^2 h(\lambda_{n_k} R_{\lambda_{n_k}} x) |w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x)|^2$$
$$- \frac{\lambda_{n_k}^2 R_{\lambda_{n_k}}^2}{\sqrt{H(\lambda_{n_k} R_{\lambda_{n_k}})}} f\Big(\lambda_{n_k} R_{\lambda_{n_k}} x, \sqrt{H(\lambda_{n_k} R_{\lambda_{n_k}})} w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x)\Big) w^{\lambda_{n_k} R_{\lambda_{n_k}}}(x)\bigg] dx$$

and

$$H_k(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} |w^{\lambda_{n_k} R_{\lambda_{n_k}}}|^2 \, dS.$$

We also define

$$D_w(r) = \frac{1}{r^{N-2}} \int_{B_r} \left[ |\nabla w(x)|^2 - \frac{a(\frac{x}{|x|})}{|x|^2} |w(x)|^2 \right] dx \quad \text{for all } r \in (0,1) \tag{120}$$

and

$$H_w(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} |w|^2 \, dS \quad \text{for all } r \in (0,1). \tag{121}$$

Using a change of variables, one sees that

$$\mathcal{N}_k(r) := \frac{D_k(r)}{H_k(r)} = \frac{D(\lambda_{n_k} R_{\lambda_{n_k}} r)}{H(\lambda_{n_k} R_{\lambda_{n_k}} r)} = \mathcal{N}(\lambda_{n_k} R_{\lambda_{n_k}} r) \quad \text{for all } r \in (0,1). \tag{122}$$

From (101), (102), and (119), it follows that, for any fixed $r \in (0,1)$,

$$D_k(r) \to D_w(r). \tag{123}$$

On the other hand, by compactness of the trace embedding $H^1(B_r) \hookrightarrow L^2(\partial B_r)$, we also have, for any fixed $r \in (0,1)$,

$$H_k(r) \to H_w(r). \tag{124}$$

From Lemma 2.6 and classical unique continuation principle for second order elliptic equations with locally bounded coefficients (see e.g. [44]) applied away from the singular set $\widetilde{\Sigma}$, it follows that $D_w(r) > -\frac{N-2}{2} H_w(r)$ for all $r \in (0,1)$. Therefore, if for some $r \in (0,1)$, $H_w(r) = 0$ then $D_w(r) > 0$; passing to the limit in (122) and using (123)-(124) this should give a contradiction with Lemma 5.8. Hence $H_w(r) > 0$ for all $r \in (0,1)$. Thus the function

$$\mathcal{N}_w(r) := \frac{D_w(r)}{H_w(r)}$$

is well defined for $r \in (0,1)$. This, together with (122), (123), (124), and Lemma 5.8, shows that

$$\mathcal{N}_w(r) = \lim_{k\to\infty} \mathcal{N}(\lambda_{n_k} R_{\lambda_{n_k}} r) = \gamma \tag{125}$$

for all $r \in (0,1)$. Therefore $\mathcal{N}_w$ is constant in $(0,1)$ and hence $\mathcal{N}_w'(r) = 0$ for any $r \in (0,1)$. Hence, by (114) and Lemma 5.4 with $h \equiv 0$, $f \equiv 0$, we obtain

$$\left( \int_{\partial B_r} \left| \frac{\partial w}{\partial \nu} \right|^2 dS \right) \cdot \left( \int_{\partial B_r} |w|^2 dS \right) - \left( \int_{\partial B_r} w \frac{\partial w}{\partial \nu} \, dS \right)^2 = 0 \quad \text{for a.e. } r \in (0,1).$$

This shows that $w$ and $\frac{\partial w}{\partial \nu}$ have the same direction as vectors in $L^2(\partial B_r)$ and hence there exists $\eta = \eta(r)$ such that

$$\frac{\partial w}{\partial \nu}(r,\theta) = \eta(r) w(r,\theta) \quad \text{for a.e. } r \in (0,1),\, \theta \in \mathbb{S}^{N-1}. \tag{126}$$

Testing the above identity with $w(r,\theta)$, we have that necessarily $\eta(r) = \frac{H_w'(r)}{2H_w(r)}$ implying that $\eta \in L^1_{\rm loc}(0,1)$. Moreover, since $w \in C^1_{\rm loc}(B_2 \setminus \widetilde{\Sigma})$, identity (126) also holds, for all $\theta \in \mathbb{S}^{N-1} \setminus \Sigma$, in the sense of absolutely continuous functions with respect to $r$ and, after integration, we obtain

$$w(r,\theta) = e^{\int_1^r \eta(s) ds} w(1,\theta) = \varphi(r)\psi(\theta) \quad \text{for all } r \in (0,1),\ \theta \in \mathbb{S}^{N-1} \setminus \Sigma, \tag{127}$$

where $\varphi(r) = e^{\int_1^r \eta(s) ds}$ and $\psi(\theta) = w(1,\theta)$. Since

$$-\Delta w - \frac{a(\frac{x}{|x|})}{|x|^2} w = -\frac{\partial^2 w}{\partial r^2} - \frac{N-1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} L_a w,$$

then (127) yields

$$\left(-\varphi''(r) - \frac{N-1}{r}\varphi'(r)\right)\psi(\theta) + \frac{\varphi(r)}{r^2}L_a\psi(\theta) = 0.$$

Taking $r$ fixed we deduce that $\psi$ is an eigenfunction of the operator $L_a$. If $\mu_{k_0}(a)$ is the corresponding eigenvalue then $\varphi(r)$ solves the equation

$$-\varphi''(r) - \frac{N-1}{r}\varphi(r) + \frac{\mu_{k_0}(a)}{r^2}\varphi(r) = 0$$

and hence $\varphi(r)$ is of the form

$$\varphi(r) = c_1 r^{\sigma^+_{k_0}} + c_2 r^{\sigma^-_{k_0}}$$

for some $c_1, c_2 \in \mathbb{R}$, where

$$\sigma^+_{k_0} = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(a)} \quad \text{and} \quad \sigma^-_{k_0} = -\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(a)}.$$

Since the function $\frac{1}{|x_J|}\big(|x|^{\sigma^-_{k_0}}\psi(\frac{x}{|x|})\big) \notin L^2(B_1)$ for any $J \in \mathcal{A}_k$ and hence $|x|^{\sigma^-_{k_0}}\psi(\frac{x}{|x|}) \notin H^1(B_1)$, then $c_2 = 0$ and $\varphi(r) = c_1 r^{\sigma^+_{k_0}}$. Since $\varphi(1) = 1$, we obtain that $c_1 = 1$ and then

$$w(r,\theta) = r^{\sigma^+_{k_0}}\psi(\theta), \quad \text{for all } r \in (0,1) \text{ and } \theta \in \mathbb{S}^{N-1} \setminus \Sigma. \tag{128}$$

Consider now the sequence $w^{\lambda_{n_k}}$. Up to a further subsequence still denoted by $w^{\lambda_{n_k}}$, we may suppose that $w^{\lambda_{n_k}} \rightharpoonup \overline{w}$ for some $\overline{w} \in H^1(B_1)$ and that $R_{\lambda_{n_k}} \to \overline{R}$ for some $\overline{R} \in [1,2]$.

Strong convergence of $w^{\lambda_{n_k} R_{\lambda_{n_k}}}$ in $H^1(B_1)$ implies that, up to a subsequence, both $w^{\lambda_{n_k} R_{\lambda_{n_k}}}$ and $|\nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}|$ are dominated by a $L^2(B_1)$-function uniformly with respect to $k$. Moreover by (103), up to a subsequence we may assume that the limit

$$l := \lim_{k\to+\infty} \frac{H(\lambda_{n_k} R_{\lambda_{n_k}})}{H(\lambda_{n_k})}$$

exists and is finite. Then, by the Dominated Convergence Theorem, we have

$$\lim_{k\to+\infty}\int_{B_1} w^{\lambda_{n_k}}(x)v(x)\,dx = \lim_{k\to+\infty} R^N_{\lambda_{n_k}}\int_{B_{1/R_{\lambda_{n_k}}}} w^{\lambda_{n_k}}(R_{\lambda_{n_k}}x)v(R_{\lambda_{n_k}}x)\,dx$$
$$= \lim_{k\to+\infty} R^N_{\lambda_{n_k}}\sqrt{\frac{H(\lambda_{n_k}R_{\lambda_{n_k}})}{H(\lambda_{n_k})}}\int_{B_1}\chi_{B_{1/R_{\lambda_{n_k}}}}(x)\,w^{\lambda_{n_k}R_{\lambda_{n_k}}}(x)v(R_{\lambda_{n_k}}x)\,dx$$
$$= \overline{R}^N\sqrt{l}\int_{B_1}\chi_{B_{1/\overline{R}}}(x)w(x)v(\overline{R}x)\,dx = \overline{R}^N\sqrt{l}\int_{B_{1/\overline{R}}} w(x)v(\overline{R}x)\,dx = \sqrt{l}\int_{B_1} w(x/\overline{R})v(x)\,dx$$

for any $v \in C^\infty(\mathbb{R}^N)$ with $\operatorname{supp} v \subset B_1$. By a density argument, it follows that the previous convergence also holds for all $v \in L^2(B_1)$. This proves that $w^{\lambda_{n_k}} \rightharpoonup \sqrt{l}\,w(\cdot/\overline{R})$ in $L^2(B_1)$ (actually weakly in $H^1(B_1)$) and in particular $\overline{w} = \sqrt{l}\, w(\cdot/\overline{R})$. Moreover

$$\lim_{k\to+\infty}\int_{B_1}|\nabla w^{\lambda_{n_k}}(x)|^2\,dx = \lim_{k\to+\infty} R^N_{\lambda_{n_k}}\int_{B_{1/R_{\lambda_{n_k}}}}|\nabla w^{\lambda_{n_k}}(R_{\lambda_{n_k}}x)|^2dx$$
$$= \lim_{k\to+\infty} R^{N-2}_{\lambda_{n_k}}\frac{H(\lambda_{n_k}R_{\lambda_{n_k}})}{H(\lambda_{n_k})}\int_{B_1}\chi_{B_{1/R_{\lambda_{n_k}}}}|\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}(x)|^2\,dx$$
$$= \overline{R}^{N-2}l\int_{B_1}\chi_{B_{1/\overline{R}}}(x)|\nabla w(x)|^2\,dx = \overline{R}^{N-2}l\int_{B_{1/\overline{R}}}|\nabla w(x)|^2\,dx = \int_{B_1}|\sqrt{l}\,\nabla(w(x/\overline{R}))|^2\,dx.$$

This shows that $w^{\lambda_{n_k}} \to \overline{w} = \sqrt{l}\, w(\cdot/\overline{R})$ strongly in $H^1(B_1)$. Furthermore, by (128) and the fact that $\int_{\partial B_1}|\overline{w}|^2dS = \int_{\partial B_1}|w|^2dS = 1$, we deduce that $\overline{w} = w$.

It remains to prove part (i). By (128) and $\int_{\mathbb{S}^{N-1}}|\psi(\theta)|^2dS = 1$ we have that

$$\int_{B_r}\left(|\nabla w(x)|^2 - \frac{a(\frac{x}{|x|})}{|x|^2}|w(x)|^2\right)dx = (\sigma^+_{k_0})^2\int_0^r s^{N-1+2(\sigma^+_{k_0}-1)}\,ds$$
$$+\left(\int_0^r s^{N-1+2(\sigma^+_{k_0}-1)}\,ds\right)\left(\int_{\mathbb{S}^{N-1}}\big(|\nabla_{\mathbb{S}^{N-1}}\psi(\theta)|^2 - a(\theta)|\psi(\theta)|^2\big)dS\right)$$
$$= \frac{(\sigma^+_{k_0})^2 + \mu_{k_0}(a)}{N+2(\sigma^+_{k_0}-1)}r^{N+2(\sigma^+_{k_0}-1)} = \sigma^+_{k_0}r^{N+2(\sigma^+_{k_0}-1)}$$

and

$$\int_{\partial B_r}|w(x)|^2dS = r^{N-1}\int_{\mathbb{S}^{N-1}}|w(r\theta)|^2dS = r^{N-1+2\sigma^+_{k_0}}\int_{\mathbb{S}^{N-1}}|\psi(\theta)|^2dS = r^{N-1+2\sigma^+_{k_0}}.$$

Therefore, by (120), (121), and (125), it follows

$$\gamma = \mathcal{N}_w(r) = \frac{D_w(r)}{H_w(r)} = \frac{r\int_{B_r}\big(|\nabla w(x)|^2 - \frac{a(x/|x|)}{|x|^2}|w(x)|^2\big)\,dx}{\int_{\partial B_r}|w|^2dS} = \sigma^+_{k_0}.$$

This completes the proof of the lemma. □

Let us now describe the behavior of $H(r)$ as $r \to 0^+$.

**Lemma 6.6.** *Under the same assumptions as in Lemma 5.4 and letting* $\gamma := \lim_{r\to 0^+}\mathcal{N}(r) \in \mathbb{R}$ *as in Lemma 5.8, the limit*

$$\lim_{r\to 0^+} r^{-2\gamma}H(r)$$

*exists and it is finite.*

Proof. In view of (94) it is sufficient to prove that the limit exists. By (71), (82), and Lemma 5.8 we have

$$\frac{d}{dr}\frac{H(r)}{r^{2\gamma}} = -2\gamma r^{-2\gamma-1}H(r) + r^{-2\gamma}H'(r) = 2r^{-2\gamma-1}(D(r)-\gamma H(r)) = 2r^{-2\gamma-1}H(r)\int_0^r\mathcal{N}'(s)ds.$$

Let $\nu_1$ and $\nu_2$ be as in (80) and (81). After integration over $(r,\tilde{r})$,

$$\frac{H(\tilde{r})}{\tilde{r}^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} = \int_r^{\tilde{r}}2s^{-2\gamma-1}H(s)\left(\int_0^s\nu_1(t)dt\right)ds + \int_r^{\tilde{r}}2s^{-2\gamma-1}H(s)\left(\int_0^s\nu_2(t)dt\right)ds. \tag{129}$$

By Schwarz's inequality we have that $\nu_1(t)\geqslant 0$ and hence

$$\lim_{r\to 0^+}\int_r^{\tilde r} 2s^{-2\gamma-1}H(s)\left(\int_0^s \nu_1(t)dt\right)ds$$

exists. On the other hand, by (94), Lemma 5.6, and (92), we deduce that

$$\left|s^{-2\gamma-1}H(s)\left(\int_0^s \nu_2(t)dt\right)\right|\leqslant K_1C_1\Big(C_2+\frac{N}{2}\Big)s^{-1}\int_0^s\Big(t^{-1+\varepsilon}+t^{-1+\frac{2(q-2^*)}{q}}+g(t)\Big)dt$$
$$\leqslant K_1C_1\Big(C_2+\frac{N}{2}\Big)s^{-1}\bigg(\frac{s^\varepsilon}{\varepsilon}+\frac{q}{2(q-2^*)}s^{\frac{2(q-2^*)}{q}}+\frac{\|u\|^{2^*(1-\alpha)}_{L^{2^*}(\Omega)}}{1-\alpha}s^{\frac{N(q-2^*)}{q}(\alpha-\frac{2}{2^*})}\bigg)$$

for all $s\in(0,\widetilde r)$, which proves that $s^{-2\gamma-1}H(s)\left(\int_0^s\nu_2(t)dt\right)\in L^1(0,\widetilde r)$. We may conclude that both terms in the right hand side of (129) admit a limit as $r\to 0^+$ thus completing the proof of the lemma. □

The next step of our asymptotic analysis relies on the proof that $\lim_{r\to 0^+} r^{-2\gamma}H(r)$ is indeed strictly positive. In the sequel we denote by $\psi_i$ a $L^2$-normalized eigenfunction of the operator $L_a=-\Delta_{\mathbb{S}^{N-1}}-a$ associated to the $i$-th eigenvalue $\mu_i(a)$, i.e.

$$\begin{cases} L_a\psi_i(\theta)=\mu_i(a)\,\psi_i(\theta), & \text{in } \mathbb{S}^{N-1},\\ \int_{\mathbb{S}^{N-1}}|\psi_i(\theta)|^2\,dS(\theta)=1.\end{cases}\tag{130}$$

Moreover, we choose the $\psi_i$'s in such a way that the set $\{\psi_i\}_{i\in\mathbb{N}}$ forms an orthonormal basis of $L^2(\mathbb{S}^{N-1})$.

Let $u$ be a nontrivial weak $H^1(\Omega)$-solution to (13). From Lemma 6.5, we deduce that, under assumptions (7), (12), and (**H**–**F**), there exist $j_0,m\in\mathbb{N}$, $j_0,m\geqslant 1$ such that $m$ is the multiplicity of the eigenvalue $\mu_{j_0}(a)=\mu_{j_0+1}(a)=\cdots=\mu_{j_0+m-1}(a)$ and

$$\gamma=\lim_{r\to 0^+}\mathcal{N}(r)=-\frac{N-2}{2}+\sqrt{\left(\frac{N-2}{2}\right)^2+\mu_i(a)},\quad i=j_0,\dots,j_0+m-1.\tag{131}$$

Let $\mathcal{E}_0$ be the eigenspace of the operator $L_a$ associated to the eigenvalue $\mu_{j_0}(a)$, so that the set $\{\psi_i\}_{i=j_0,\dots,j_0+m-1}$ is an orthonormal basis of $\mathcal{E}_0$.

**Lemma 6.7.** *Let $\Omega\subset\mathbb{R}^N$, $N\geqslant 3$, be a bounded open set such that $0\in\Omega$, $a$ such that (7) and (12) hold, and $h,f$ as in (**H**–**F**). If $u$ is a weak $H^1(\Omega)$-solution to (13), then*

$$\sup_{\substack{i=j_0,\dots,j_0+m-1,\ J\in\mathcal{A}_k,\\ (J_1,J_2)\in\mathcal{B}_k,\ \lambda\in(0,\bar r)}}\frac{\int_{B_\lambda}\Big(\frac{|u(x)|}{|x_J|^{2-\varepsilon}}+\frac{|u(x)|}{|x_{J_1}-x_{J_2}|^{2-\varepsilon}}+|f(x,u(x))|\Big)\big|\psi_i(\frac{x}{|x|})\big|\,dx}{\lambda^{N-2+\delta+\gamma}}<+\infty,$$

*where $\bar r$ is as in Lemma 6.1 and $\delta>0$ is defined in (97).*

Proof. From Lemma 6.1 and Corollary 2.7, it follows that, for some positive constant $C_5$ independent of $\lambda$, $J$, $(J_1,J_2)$, and $i$,

$$\int_{B_1}\Big(|x_J|^{-2+\varepsilon}+|x_{J_1}-x_{J_2}|^{-2+\varepsilon}\Big)|w^\lambda(x)|\big|\psi_i\big(\tfrac{x}{|x|}\big)\big|\,dx\leqslant C_5$$

for all $i = j_0, \dots, j_0 + m - 1$, $J \in \mathcal{A}_k$, $(J_1, J_2) \in \mathcal{B}_k$, and $\lambda \in (0, \bar{r})$, where $w^\lambda$ is defined in (98). Moreover, arguing as in (111), by (97), we have as $\lambda \to 0^+$

$$\frac{\lambda^2}{\sqrt{H(\lambda)}} \int_{B_1} |f(\lambda x, u(\lambda x))| \left|\psi_i\left(\tfrac{x}{|x|}\right)\right| dx \leqslant C_6 \lambda^{\frac{2(q-2^*)}{q}} = O(\lambda^\delta)$$

where $C_6 > 0$ is a positive constant. The conclusion follows from (94) and a change of variable. □

**Lemma 6.8.** *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded open set such that $0 \in \Omega$, a satisfy (7) and (12), and $u \not\equiv 0$ be a weak $H^1(\Omega)$-solution to (13), with $h, f$ satisfying* (**H**–**F**). *Let $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ be as in Lemma 5.8 and $j_0, m \in \mathbb{N}$ as in (131), i.e. $m$ is the multiplicity of the eigenvalue $\mu_{j_0}(a) = \mu_{j_0+1}(a) = \cdots = \mu_{j_0+m-1}(a)$ and (131) holds for all $i = j_0, \dots, j_0 + m - 1$. Then the function $\varphi_i$ defined as*

$$\varphi_i(\lambda) := \int_{\mathbb{S}^{N-1}} u(\lambda\,\theta)\psi_i(\theta)\, dS(\theta), \quad \textit{with } \psi_i \textit{ as in (130)}, \tag{132}$$

*satisfies, as $\lambda \to 0^+$ ,*

$$\varphi_i(\lambda) = \lambda^\gamma \Bigg( R^{-\gamma}\varphi_i(R) + \frac{2 - N - \gamma}{2 - N - 2\gamma} \int_\lambda^R s^{-N+1-\gamma}\Upsilon_i(s) ds \\ - \frac{\gamma R^{-N+2-2\gamma}}{2 - N - 2\gamma} \int_0^R s^{\gamma-1}\Upsilon_i(s)\, ds\Bigg) + O(\lambda^{\gamma+\delta}) \tag{133}$$

*for every $i \in \{j_0, \dots, j_0 + m - 1\}$ and $R > 0$ such that $\overline{B_R} \subset \Omega$, where $\delta$ is defined in (97) and*

$$\Upsilon_i(\lambda) := \int_{B_\lambda} \Big(h(x)u(x) + f(x, u(x))\Big)\psi_i\left(\frac{x}{|x|}\right) dx. \tag{134}$$

Proof. Let $R > 0$ be such that $\overline{B_R} \subset \Omega$. For any $\lambda \in (0, R)$, we expand $\theta \mapsto u(\lambda\theta) \in L^2(\mathbb{S}^{N-1})$ in Fourier series with respect to the orthonormal basis $\{\psi_i\}$ of $L^2(\mathbb{S}^{N-1})$ defined in (130), i.e.

$$u(\lambda\,\theta) = \sum_{i=1}^\infty \varphi_i(\lambda)\psi_i(\theta) \quad \text{in } L^2(\mathbb{S}^{N-1}), \tag{135}$$

with $\varphi_i$ is defined in (132). On the other hand, $\int_{B_R} (|x_J|^{-2+\varepsilon} + |x_{J_1} - x_{J_2}|^{-2+\varepsilon})u^2(x)\, dx < +\infty$ for all $J \in \mathcal{A}_k$ and $(J_1, J_2) \in \mathcal{B}_k$ by Corollary 2.7, hence $\int_{\mathbb{S}^{N-1}} (|\theta_J|^{-2+\varepsilon} + |\theta_{J_1} - \theta_{J_2}|^{-2+\varepsilon})u^2(\lambda\theta)dS(\theta)$ is finite for all $J \in \mathcal{A}_k$, $(J_1, J_2) \in \mathcal{B}_k$, and a.e. $\lambda \in (0, R)$, which, together with Lemma 2.1, implies that $\theta \mapsto h(\lambda\theta)u(\lambda\theta) \in H^{-1}(\mathbb{S}^{N-1})$ for a.e. $\lambda \in (0, R)$. Moreover, by (**F**), it is also easy to verify that $\theta \mapsto f(\lambda\theta, u(\lambda\theta)) \in H^{-1}(\mathbb{S}^{N-1})$ for a.e. $\lambda \in (0, R)$. Therefore, we may write

$$h(\lambda\,\theta)u(\lambda\,\theta) + f(\lambda\,\theta, u(\lambda\,\theta)) = \sum_{i=1}^\infty \zeta_i(\lambda)\psi_i(\theta) \quad \text{in } H^{-1}(\mathbb{S}^{N-1}) \text{ for a.e. } \lambda \in (0, R) \tag{136}$$

where

$$\begin{aligned}\zeta_i(\lambda) &= {}_{H^{-1}(\mathbb{S}^{N-1})}\big\langle h(\lambda\cdot)u(\lambda\cdot) + f(\lambda\cdot, u(\lambda\cdot)), \psi_i\big\rangle_{H^1(\mathbb{S}^{N-1})} \\ &= \int_{\mathbb{S}^{N-1}} \Big(h(\lambda\,\theta)u(\lambda\,\theta) + f(\lambda\,\theta, u(\lambda\,\theta))\Big)\psi_i(\theta)\, dS(\theta).\end{aligned} \tag{137}$$

We notice that, in view of Remark 3.2, $\zeta_i \in L^1_{\rm loc}(0,R)$ and

$$\lambda^{N-1}\zeta_i(\lambda) = \Upsilon_i'(\lambda) \quad \text{a.e. in } (0,R), \tag{138}$$

where $\Upsilon_i$ is defined in (134). Since $u$ solves (13), by (130) we obtain that, for any $i \in \mathbb{N}$, $\varphi_i$ solves

$$-\varphi_i''(\lambda) - \frac{N-1}{\lambda}\varphi_i'(\lambda) + \frac{\mu_i(a)}{\lambda^2}\varphi_i(\lambda) = \zeta_i(\lambda) \quad \text{in the sense of distributions in } (0,R),$$

which can be also written as

$$-\Big(\lambda^{N-1+2\sigma_i}\big(\lambda^{-\sigma_i}\varphi_i(\lambda)\big)'\Big)' = \lambda^{N-1+\sigma_i}\zeta_i(\lambda) \quad \text{in the sense of distributions in } (0,R),$$

where

$$\sigma_i = -\frac{N-2}{2} + \sqrt{\Big(\frac{N-2}{2}\Big)^2 + \mu_i(a)}.$$

Integrating by parts the right hand side and taking into account (138), we obtain that there exists $c_i \in \mathbb{R}$ such that

$$\big(\lambda^{-\sigma_i}\varphi_i(\lambda)\big)' = -\lambda^{-N+1-\sigma_i}\Upsilon_i(\lambda) - \sigma_i\lambda^{-N+1-2\sigma_i}\bigg(c_i + \int_\lambda^R s^{\sigma_i-1}\Upsilon_i(s)ds\bigg)$$

in the sense of distributions in $(0,R)$, thus implying that $\varphi_i \in W^{2,1}_{\rm loc}(0,R)$. A further integration yields

$$\begin{aligned}
\varphi_i(\lambda) &= \lambda^{\sigma_i}\bigg(R^{-\sigma_i}\varphi_i(R) + \int_\lambda^R s^{-N+1-\sigma_i}\Upsilon_i(s)ds\bigg) \\
&\qquad + \sigma_i\lambda^{\sigma_i}\int_\lambda^R s^{-N+1-2\sigma_i}\bigg(c_i + \int_s^R t^{\sigma_i-1}\Upsilon_i(t)dt\bigg)ds \\
&= \lambda^{\sigma_i}\bigg(R^{-\sigma_i}\varphi_i(R) + \int_\lambda^R s^{-N+1-\sigma_i}\Upsilon_i(s)ds + \frac{\sigma_i c_i R^{-N+2-2\sigma_i}}{2-N-2\sigma_i}\bigg) - \frac{\sigma_i c_i \lambda^{-N+2-\sigma_i}}{2-N-2\sigma_i} \\
&\qquad + \frac{\sigma_i\lambda^{\sigma_i}}{2-N-2\sigma_i}\int_\lambda^R t^{-N+1-\sigma_i}\Upsilon_i(t)\,dt - \frac{\sigma_i\lambda^{-N+2-\sigma_i}}{2-N-2\sigma_i}\int_\lambda^R t^{\sigma_i-1}\Upsilon_i(t)\,dt \\
&= \lambda^{\sigma_i}\bigg(R^{-\sigma_i}\varphi_i(R) + \frac{2-N-\sigma_i}{2-N-2\sigma_i}\int_\lambda^R s^{-N+1-\sigma_i}\Upsilon_i(s)ds + \frac{\sigma_i c_i R^{-N+2-2\sigma_i}}{2-N-2\sigma_i}\bigg) \\
&\qquad + \frac{\sigma_i\lambda^{-N+2-\sigma_i}}{N-2+2\sigma_i}\bigg(c_i + \int_\lambda^R t^{\sigma_i-1}\Upsilon_i(t)\,dt\bigg).
\end{aligned} \tag{139}$$

Let $j_0, m \in \mathbb{N}$ be as in (131), i.e. $m$ is the multiplicity of the eigenvalue

$$\mu_{j_0}(a) = \mu_{j_0+1}(a) = \cdots = \mu_{j_0+m-1}(a)$$

and

$$\gamma = \lim_{r\to 0^+}\mathcal{N}(r) = \sigma_i, \quad i = j_0,\dots,j_0+m-1, \tag{140}$$

see Lemma 6.5. The Parseval identity yields

$$H(\lambda) = \int_{\mathbb{S}^{N-1}} |u(\lambda\theta)|^2 \, dS(\theta) = \sum_{i=1}^{\infty} |\varphi_i(\lambda)|^2, \quad \text{for all } 0 < \lambda \leqslant R. \tag{141}$$

From Lemma 6.7, it follows that

$$\Upsilon_i(\lambda) = O(\lambda^{N-2+\delta+\sigma_i}) \quad \text{for every } i \in \{j_0, \dots, j_0+m-1\} \quad \text{as } \lambda \to 0^+. \tag{142}$$

From (142), it follows that

$$s \mapsto s^{-N+1-\sigma_i} \Upsilon_i(s) \in L^1(0,R) \quad \text{for every } i \in \{j_0, \dots, j_0+m-1\} \tag{143}$$

which yields

$$\lambda^{\sigma_i}\left(R^{-\sigma_i}\varphi_i(R) + \frac{2-N-\sigma_i}{2-N-2\sigma_i}\int_\lambda^R s^{-N+1-\sigma_i}\Upsilon_i(s)ds + \frac{\sigma_i c_i R^{-N+2-2\sigma_i}}{2-N-2\sigma_i}\right) = O(\lambda^{\sigma_i}) = o(\lambda^{-N+2-\sigma_i}) \tag{144}$$

for all $i \in \{j_0, \dots, j_0+m-1\}$ as $\lambda \to 0^+$. From (142), it also follows that

$$t \mapsto t^{\sigma_i - 1}\Upsilon_i(t) \in L^1(0,R) \quad \text{for every } i \in \{j_0, \dots, j_0+m-1\}.$$

From $\frac{u}{|x|} \in L^2(B_R)$, we deduce that

$$\int_0^R r^{N-3}\varphi_i^2(r)\, dr < +\infty.$$

Then, since $\int_0^R r^{N-3}(r^{-N+2-\sigma_i})^2\, dr = +\infty$, from (139) and (144) it follows that

$$c_i + \int_0^R t^{\sigma_i-1}\Upsilon_i(t)\, dt = 0$$

and hence

$$\varphi_i(\lambda) = \lambda^{\sigma_i}\left(R^{-\sigma_i}\varphi_i(R) + \frac{2-N-\sigma_i}{2-N-2\sigma_i}\int_\lambda^R s^{-N+1-\sigma_i}\Upsilon_i(s)ds - \frac{\sigma_i R^{-N+2-2\sigma_i}}{2-N-2\sigma_i}\int_0^R t^{\sigma_i-1}\Upsilon_i(t)\, dt\right)$$
$$- \frac{\sigma_i \lambda^{-N+2-\sigma_i}}{N-2+2\sigma_i}\int_0^\lambda t^{\sigma_i-1}\Upsilon_i(t)\, dt.$$

On the other hand, from (142) it follows that

$$\lambda^{-N+2-\sigma_i}\int_0^\lambda t^{\sigma_i-1}\Upsilon_i(t)\, dt = O(\lambda^{\sigma_i+\delta}) \quad \text{as } \lambda \to 0^+$$

for all $i \in \{j_0, \dots, j_0+m-1\}$, thus completing the proof. □

**Lemma 6.9.** *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded open set such that $0 \in \Omega$, $a$ satisfy (7) and (12), and $u \not\equiv 0$ be a weak $H^1(\Omega)$-solution to (13), with $h, f$ satisfying* (**H–F**). *Then*

$$\lim_{r\to 0^+} r^{-2\gamma}H(r) > 0.$$

Proof. Let $R > 0$ be such that $\overline{B_R} \subset \Omega$ and $j_0, m \in \mathbb{N}$ as in (131). We argue by contradiction and assume that $\lim_{\lambda\to 0^+} \lambda^{-2\gamma} H(\lambda) = 0$. Then, letting $\varphi_i$ as in (132), (141) implies that

$$\lim_{\lambda\to 0^+} \lambda^{-\gamma}\varphi_i(\lambda) = 0 \quad \text{for all } i \in \{j_0, \dots, j_0 + m - 1\}. \tag{145}$$

From Lemma 6.8, (143), and (145), we deduce that

$$R^{-\gamma}\varphi_i(R) - \frac{\gamma R^{-N+2-2\gamma}}{2-N-2\gamma}\int_0^R s^{\gamma-1}\Upsilon_i(s)\,ds = -\frac{2-N-\gamma}{2-N-2\gamma}\int_0^R s^{-N+1-\gamma}\Upsilon_i(s)ds$$

for all $i \in \{j_0, \dots, j_0 + m - 1\}$. Hence (133) can be rewritten as

$$\varphi_i(\lambda) = -\frac{2-N-\gamma}{2-N-2\gamma}\,\lambda^{\gamma}\int_0^\lambda s^{-N+1-\gamma}\Upsilon_i(s)ds + O(\lambda^{\gamma+\delta}) \quad \text{as } \lambda \to 0^+ \tag{146}$$

for all $i \in \{j_0, \dots, j_0 + m - 1\}$. From (146), (140), and (142), we infer the estimate

$$\varphi_i(\lambda) = O(\lambda^{\gamma+\delta}) \quad \text{as } \lambda \to 0^+, \quad \text{for every } i \in \{j_0, \dots, j_0 + m - 1\},$$

namely, setting $u^\lambda(\theta) = u(\lambda\theta)$,

$$(u^\lambda, \psi_i)_{L^2(\mathbb{S}^{N-1})} = O(\lambda^{\gamma+\delta}) \quad \text{as } \lambda \to 0^+, \quad \text{for every } i \in \{j_0, \dots, j_0 + m - 1\},$$

and hence

$$(u^\lambda, \psi)_{L^2(\mathbb{S}^{N-1})} = O(\lambda^{\gamma+\delta}) \quad \text{as } \lambda \to 0^+,$$

for every $\psi \in \mathcal{E}_0$, being $\mathcal{E}_0$ the eigenspace of the operator $L_a$ associated to the eigenvalue $\mu_{j_0}(a)$. Let $w^\lambda(\theta) = (H(\lambda))^{-1/2}u(\lambda\theta)$. From (95), there exists $C(\delta) > 0$ such that $\sqrt{H(\lambda)} \geqslant C(\delta)\lambda^{\gamma+\frac{\delta}{2}}$ for $\lambda$ small, and therefore

$$(w^\lambda, \psi)_{L^2(\mathbb{S}^{N-1})} = O(\lambda^{\delta/2}) = o(1), \quad \text{as } \lambda \to 0^+ \tag{147}$$

for every $\psi \in \mathcal{E}_0$. From Lemma 6.5, for every sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_j}\}_{j\in\mathbb{N}}$ and an eigenfunction $\widetilde{\psi} \in \mathcal{E}_0$ such that

$$\int_{\mathbb{S}^{N-1}} |\widetilde{\psi}(\theta)|^2 dS = 1 \quad \text{and} \quad w^{\lambda_{n_j}} \to \widetilde{\psi} \quad \text{in } L^2(\mathbb{S}^{N-1}). \tag{148}$$

From (147) and (148), we infer that

$$0 = \lim_{j\to+\infty}(w^{\lambda_{n_j}}, \widetilde{\psi})_{L^2(\mathbb{S}^{N-1})} = \|\widetilde{\psi}\|^2_{L^2(\mathbb{S}^{N-1})} = 1,$$

thus reaching a contradiction. □

Combining Lemma 6.5 with Lemma 6.9, we can now prove Theorem 1.1.

**Proof of Theorem 1.1.** Identity (15) follows immediately from Lemma 6.5. As in the statement of the theorem, let $m$ be the multiplicity of the eigenvalue $\mu_{k_0}(a)$ found in Lemma 6.5, $j_0 \in \mathbb{N}\setminus\{0\}$, such that $j_0 \leqslant k_0 \leqslant j_0 + m - 1$, $\mu_{j_0}(a) = \mu_{j_0+1}(a) = \cdots = \mu_{j_0+m-1}(a)$, and $\gamma = \lim_{r\to 0^+}\mathcal{N}(r)$.

In order to prove (16), let $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence such that $\lambda_n \to 0^+$ as $n \to +\infty$. Then by Lemmas 6.5, 6.6, and 6.9, there exist a subsequence $\lambda_{n_j}$ and $\beta_{j_0}, \dots, \beta_{j_0+m-1} \in \mathbb{R}$ such that

$$\lambda_{n_j}^{-\gamma}u(\lambda_{n_j}x) \to |x|^\gamma \sum_{i=j_0}^{j_0+m-1}\beta_i\psi_i\Big(\frac{x}{|x|}\Big) \quad \text{in } H^1(B_1) \quad \text{as } j \to +\infty \tag{149}$$

and $(\beta_{j_0}, \beta_{j_0+1}, \dots, \beta_{j_0+m-1}) \neq (0, 0, \dots, 0)$, which implies

$$\lambda_{n_j}^{-\gamma} u(\lambda_{n_j}\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in } L^2(\mathbb{S}^{N-1}) \quad \text{as } j \to +\infty. \tag{150}$$

We now prove that the $\beta_i$'s depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_j}\}_{j\in\mathbb{N}}$. Let us fix $R > 0$ such that $\overline{B_R} \subset \Omega$. Defining $\varphi_i$ as in (132), from (150) it follows that, for any $i = j_0, \dots, j_0 + m - 1$,

$$\lambda_{n_j}^{-\gamma} \varphi_i(\lambda_{n_j}) = \int_{\mathbb{S}^{N-1}} \frac{u(\lambda_{n_j}\theta)}{\lambda_{n_j}^{\gamma}} \psi_i(\theta)\, dS(\theta) \to \sum_{\ell=j_0}^{j_0+m-1} \beta_\ell \int_{\mathbb{S}^{N-1}} \psi_\ell(\theta)\psi_i(\theta)\, dS(\theta) = \beta_i \tag{151}$$

as $j \to +\infty$. On the other hand, from Lemma 6.8, it follows that, for any $i = j_0, \dots, j_0 + m - 1$,

$$\lambda^{-\gamma}\varphi_i(\lambda) \ \to R^{-\gamma}\varphi_i(R) + \frac{2-N-\gamma}{2-N-2\gamma} \int_0^R s^{-N+1-\gamma}\Upsilon_i(s)\, ds - \frac{\gamma R^{-N+2-2\gamma}}{2-N-2\gamma} \int_0^R s^{\gamma-1}\Upsilon_i(s)\, ds$$

as $\lambda \to 0^+$, with $\Upsilon_i$ as in (134), and therefore from (151) we deduce that

$$\beta_i = R^{-\gamma}\varphi_i(R) + \frac{2-N-\gamma}{2-N-2\gamma} \int_0^R s^{-N+1-\gamma}\Upsilon_i(s)\, ds - \frac{\gamma R^{-N+2-2\gamma}}{2-N-2\gamma} \int_0^R s^{\gamma-1}\Upsilon_i(s)\, ds,$$

for any $i = j_0, \dots, j_0 + m - 1$. Integration by parts and (138) allow rewriting the above formula as in (17). In particular the $\beta_i$'s depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$, thus implying that the convergence in (149) actually holds as $\lambda \to 0^+$ and proving the theorem. □

## 7. Asymptotic behavior of eigenfunctions

We describe the asymptotic behavior of eigenfunctions of the operator $L_a = -\Delta_{\mathbb{S}^{N-1}} - a$ near the singular set of the function $a$. Actually, for simplicity we study the asymptotic behavior of eigenfunctions near the south pole as an application of Theorem 1.1 after a stereographic projection of $\mathbb{S}^{N-1}$ onto $\mathbb{R}^{N-1}$ with respect to the "north pole".

Throughout this section we assume that $3 \leqslant k \leqslant N - 1$ and that $a$ satisfies (7) and (12). Note that if $k = N$ then $a$ is constant and hence the eigenfunctions of $L_a$ are smooth.

By Lemma 2.2 the spectrum of $L_a$ consists of a diverging sequence of eigenvalues $\mu_1(a) < \mu_2(a) \leqslant \dots \leqslant \mu_n(a) \leqslant \dots$ each of them having finite multiplicity.

Let $\mu_i(a)$ be an eigenvalue of $L_a$ and let $\psi \in H^1(\mathbb{S}^{N-1})$ be a corresponding eigenfunction, i.e.

$$-\Delta_{\mathbb{S}^{N-1}}\psi(\theta) - a(\theta)\psi(\theta) = \mu_i(a)\psi(\theta) \qquad \text{in } \mathbb{S}^{N-1}. \tag{152}$$

Let $\Pi : \mathbb{S}^{N-1} \setminus \{e_N\} \to \mathbb{R}^{N-1}$ be the standard stereographic projection with respect to the "north pole". Here by $e_N$, we denote the vector $(0, 0, \dots, 0, 1) \in \mathbb{R}^N$.

Let $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$ be the function given by

$$\phi(y) = \frac{4}{\left(|y|^2 + 1\right)^2} \qquad \text{for all } y \in \mathbb{R}^{N-1}. \tag{153}$$

If $\theta \in \mathbb{S}^{N-1} \setminus \{e_N\}$ and $x, z \in T_\theta\mathbb{S}^{N-1}$ (by $T_\theta\mathbb{S}^{N-1}$ we mean the tangent space to $\mathbb{S}^{N-1}$ at $\theta$), then

$$(x, z)_{T_\theta\mathbb{S}^{N-1}} = \phi(\Pi(\theta))\ (d\Pi(\theta)[x], d\Pi(\theta)[z])_{\mathbb{R}^{N-1}}$$

where the vector space $T_{\Pi(\theta)}\mathbb{R}^{N-1}$ is identified with $\mathbb{R}^{N-1}$. In the following lemma the equation satisfied by the projection of $\psi$ is deduced.

**Lemma 7.1.** *Let $3 \leqslant k \leqslant N-1$, $a$ satisfy (7) and (12), and let $\Pi$ and $\phi$ be respectively the stereographic projection with respect to the north pole and the function defined in (153). Let $\mu_i(a)$ be an eigenvalue of the operator $L_a$ and let $\psi \in H^1(\mathbb{S}^{N-1})$ be a corresponding eigenfunction. Then the function*

$$\widetilde{\psi}(y) := (\phi(y))^{\frac{N-3}{4}}\psi(\Pi^{-1}(y)) \tag{154}$$

*belongs to $\mathcal{D}^{1,2}(\mathbb{R}^{N-1})$ and weakly solves*

$$-\Delta\widetilde{\psi}(y) - \frac{b(\frac{y}{|y|})}{|y|^2}\,\widetilde{\psi}(y) = \widetilde{h}(y)\widetilde{\psi}(y) \tag{155}$$

*where $b$ and $\widetilde{h}$ are defined by*

$$b(\omega) = \sum_{J\in\mathcal{A}_k, N\notin J}\frac{\alpha_J}{|\omega_J|^2} + \sum_{(J_1,J_2)\in\mathcal{B}_k, N\notin J_1\cup J_2}\frac{\alpha_{J_1J_2}}{|\omega_{J_1}-\omega_{J_2}|^2}, \quad \text{for any } \omega\in\mathbb{S}^{N-2}\setminus\Sigma_1, \tag{156}$$

*where*

$$\begin{aligned}\Sigma_1 :=&\{(\omega_1,\dots,\omega_{N-1})\in\mathbb{S}^{N-2} : \omega_J = 0 \text{ for some } J\in\mathcal{A}_k^{N-1}\}\\ &\cup\{(\omega_1,\dots,\omega_{N-1})\in\mathbb{S}^{N-2} : \omega_{J_1}=\omega_{J_2} \text{ for some } (J_1,J_2)\in\mathcal{B}_k^{N-1}\},\end{aligned}$$

$$\begin{aligned}\mathcal{A}_k^{N-1}&=\big\{J\subseteq\{1,2,\dots,N-1\} : \#J = k\big\},\\ \mathcal{B}_k^{N-1}&=\{(J_1,J_2)\in\mathcal{A}_k^{N-1}\times\mathcal{A}_k^{N-1}\colon J_1\cap J_2=\emptyset, J_1<J_2\},\end{aligned}$$

*and*

$$\begin{aligned}\widetilde{h}(y) = \phi(y)\bigg(\frac{(N-3)(N-1)}{4}+\mu_i(a)\bigg) + \sum_{J\in\mathcal{A}_k, N\in J}\frac{4\alpha_J}{4|y_{J'}|^2+(|y|^2-1)^2}\\ + \sum_{(J_1,J_2)\in\mathcal{B}_k, N\in J_1\setminus J_2}\frac{4\alpha_{J_1J_2}}{4|y_{J_1'}-y_{J_2'}|^2+(|y|^2-1-2y_{m_k})^2}\\ + \sum_{(J_1,J_2)\in\mathcal{B}_k, N\in J_2\setminus J_1}\frac{4\alpha_{J_1J_2}}{4|y_{J_1'}-y_{J_2'}|^2+(|y|^2-1-2y_{n_k})^2}\,.\end{aligned} \tag{157}$$

*for a.e. $y\in\mathbb{R}^{N-1}$, where for any $(J_1,J_2)\in\mathcal{B}_k$, $n_k = \max J_1$, $m_k = \max J_2$, and for any $J = \{n_1,\dots,n_k\}\in\mathcal{A}_k$, $n_1<n_2<\dots<n_k$, we denote $J' = J\setminus\{n_k\}\in\mathcal{A}_{k-1}$.*

Proof. The conformal laplacian on $\mathbb{S}^{N-1}$ is given by

$$-\Delta_{\mathbb{S}^{N-1}} + \frac{(N-3)(N-1)}{4},$$

while, since $\mathbb{R}^{N-1}$ has zero scalar curvature, the conformal laplacian in $\mathbb{R}^{N-1}$ coincides with the usual Laplace operator. Then for any function $\eta\in C^2(\mathbb{S}^{N-1}\setminus\{e_N\})$ we have

$$-\Delta_{\mathbb{S}^{N-1}}\eta(\theta) + \frac{(N-3)(N-1)}{4}\eta(\theta) = -\phi^{-\frac{N+1}{4}}\Delta(\phi^{\frac{N-3}{4}}\cdot(\eta\circ\Pi^{-1}))\Big|_{\Pi(\theta)} \tag{158}$$

for every $\theta \in \mathbb{S}^{N-1} \setminus \{e_N\}$. For the definition of the conformal Laplacian and for a proof of (158) see [13, §3] or [6, (1.2.27)].

We claim that the function $\widetilde{\psi}$ defined in (154) belongs to $\mathcal{D}^{1,2}(\mathbb{R}^{N-1})$ and weakly solves

$$-\Delta\widetilde{\psi}(y) - \phi(y)a(\Pi^{-1}(y))\widetilde{\psi}(y) = \left(\frac{(N-3)(N-1)}{4} + \mu_i(a)\right)\phi(y)\widetilde{\psi}(y) \quad \text{in } \mathbb{R}^{N-1}, \tag{159}$$

i.e.

$$\int_{\mathbb{R}^{N-1}} \nabla\widetilde{\psi}(y)\cdot\nabla v(y)\,dy - \int_{\mathbb{R}^{N-1}} \phi(y)a(\Pi^{-1}(y))\widetilde{\psi}(y)v(y)\,dy$$
$$= \left(\frac{(N-3)(N-1)}{4} + \mu_i(a)\right)\int_{\mathbb{R}^{N-1}} \phi(y)\widetilde{\psi}(y)v(y)\,dy \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^{N-1}).$$

Indeed by (158), integration by parts, and the change of variables $y := \Pi(\theta) \in \mathbb{R}^{N-1}$, for any $v_1, v_2 \in C_c^\infty(\mathbb{R}^{N-1})$ we have

$$\int_{\mathbb{R}^{N-1}} \nabla v_1(y)\cdot\nabla v_2(y)\,dy = \int_{\mathbb{S}^{N-1}}\Big(\nabla_{\mathbb{S}^{N-1}}w_1(\theta)\cdot\nabla_{\mathbb{S}^{N-1}}w_2(\theta) + \tfrac{(N-3)(N-1)}{4}w_1(\theta)w_2(\theta)\Big)\,dS(\theta) \tag{160}$$

with $w_j(\theta) = \phi(\Pi(\theta))^{-\frac{N-3}{4}}v_j(\Pi(\theta))$, $j = 1, 2$. Moreover

$$\int_{\mathbb{R}^{N-1}} v_1(y)v_2(y)\phi(y)\,dy = \int_{\mathbb{S}^{N-1}} w_1(\theta)w_2(\theta)\,dS(\theta) \tag{161}$$

and

$$\int_{\mathbb{R}^{N-1}} a(\Pi^{-1}(y))v_1(y)v_2(y)\phi(y)\,dy = \int_{\mathbb{S}^{N-1}} a(\theta)w_1(\theta)w_2(\theta)\,dS(\theta) \tag{162}$$

with $w_1, w_2$ as above. By density, (160–162) actually hold for any $v_1, v_2 \in \mathcal{D}^{1,2}(\mathbb{R}^{N-1})$ and hence the claim follows.

We now write the function $a(\Pi^{-1}(y))$ in a more explicit way. We recall that the function $\Pi^{-1} : \mathbb{R}^{N-1} \to \mathbb{S}^{N-1}$ is given by

$$\Pi^{-1}(y) = \frac{2}{|y|^2+1}\,y + \frac{|y|^2-1}{|y|^2+1}\,e_N$$

where we identified $\mathbb{R}^{N-1}$ with the subspace of $\mathbb{R}^N$ of all $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ such that $x_N = 0$. Therefore for any $\theta \in \mathbb{S}^{N-1} \setminus \{e_N\}$, if $y = (y_1, \dots, y_{N-1}) = \Pi(\theta)$, we have

$$\theta_J = \begin{cases} \frac{2}{|y|^2+1}\,y_J = \Big(\frac{2}{|y|^2+1}\,y_{J'}, \frac{2}{|y|^2+1}\,y_{n_k}\Big), & \text{if } N \notin J, \\ \Big(\frac{2}{|y|^2+1}\,y_{J'}, \frac{|y|^2-1}{|y|^2+1}\Big), & \text{if } N \in J\,. \end{cases}$$

Hence, for every $J \in \mathcal{A}_k$,

$$|\theta_J|^2 = \begin{cases} \frac{4}{(|y|^2+1)^2}\,|y_J|^2, & \text{if } N \notin J, \\ \frac{4|y_{J'}|^2+(|y|^2-1)^2}{(|y|^2+1)^2}, & \text{if } N \in J, \end{cases}$$

and, for every $(J_1,J_2)\in\mathcal{B}_k$, with $J_1=\{n_1,\dots,n_k\}$ and $J_2=\{m_1,\dots,m_k\}$,

$$|\theta_{J_1}-\theta_{J_2}|^2=\begin{cases}\frac{4}{(|y|^2+1)^2}\,|y_{J_1}-y_{J_2}|^2, & \text{if } N\notin J_1\cup J_2,\\ \frac{4|y_{J_1'}-y_{J_2'}|^2+(|y|^2-1-2y_{m_k})^2}{(|y|^2+1)^2}, & \text{if } N\in J_1\setminus J_2,\\ \frac{4|y_{J_1'}-y_{J_2'}|^2+(|y|^2-1-2y_{n_k})^2}{(|y|^2+1)^2}, & \text{if } N\in J_2\setminus J_1\,.\end{cases}\tag{163}$$

By (163) we obtain

$$\begin{aligned}\phi(y)a(\Pi^{-1}(y))=&\sum_{J\in\mathcal{A}_k,N\notin J}\frac{\alpha_J}{|y_J|^2}+\sum_{(J_1,J_2)\in\mathcal{B}_k,N\notin J_1\cup J_2}\frac{\alpha_{J_1J_2}}{|y_{J_1}-y_{J_2}|^2}\\&+\sum_{J\in\mathcal{A}_k,N\in J}\frac{4\alpha_J}{4|y_{J'}|^2+(|y|^2-1)^2}+\sum_{(J_1,J_2)\in\mathcal{B}_k,N\in J_1\setminus J_2}\frac{4\alpha_{J_1J_2}}{4|y_{J_1'}-y_{J_2'}|^2+(|y|^2-1-2y_{m_k})^2}\\&+\sum_{(J_1,J_2)\in\mathcal{B}_k,N\in J_2\setminus J_1}\frac{4\alpha_{J_1J_2}}{4|y_{J_1'}-y_{J_2'}|^2+(|y|^2-1-2y_{n_k})^2}\,.\end{aligned}\tag{164}$$

The conclusion follows from (163) and (164). □

According with (11) we introduce the number

$$\Lambda(b):=\sup_{v\in\mathcal{D}^{1,2}(\mathbb{R}^{N-1})\setminus\{0\}}\frac{\int_{\mathbb{R}^{N-1}}|y|^{-2}b(y/|y|)\,v^2(y)\,dy}{\int_{\mathbb{R}^{N-1}}|\nabla v(y)|^2\,dy}\,.\tag{165}$$

**Lemma 7.2.** *Let $3\leqslant k\leqslant N-1$, $a$ satisfy (7) and (12), and let $b$ be the corresponding function defined in* (156)*. Then $\Lambda(b)<1$ with $\Lambda(b)$ as in (165).*

Proof. Let $v\in C_c^\infty(\mathbb{R}^{N-1})$ and let $w(\theta)=\phi(\Pi(\theta))^{-\frac{N-3}{4}}v(\Pi(\theta))$. Then by (160-162) and (22) we have

$$\begin{aligned}&\int_{\mathbb{R}^{N-1}}|\nabla v(y)|^2dy-\frac{(N-3)(N-1)}{4}\int_{\mathbb{R}^{N-1}}\phi(y)|v(y)|^2dy-\int_{\mathbb{R}^{N-1}}\phi(y)a(\Pi^{-1}(y))|v(y)|^2dy\\&\quad=\int_{\mathbb{S}^{N-1}}|\nabla_{\mathbb{S}^{N-1}}w(\theta)|^2dS(\theta)-\int_{\mathbb{S}^{N-1}}a(\theta)|w(\theta)|^2dS(\theta)\\&\quad\geqslant(1-\Lambda(a))\int_{\mathbb{S}^{N-1}}|\nabla_{\mathbb{S}^{N-1}}w(\theta)|^2dS(\theta)-\Lambda(a)\left(\frac{N-2}{2}\right)^2\int_{\mathbb{S}^{N-1}}|w(\theta)|^2dS(\theta)\\&\quad=(1-\Lambda(a))\left(\int_{\mathbb{R}^{N-1}}|\nabla v(y)|^2dy-\frac{(N-3)(N-1)}{4}\int_{\mathbb{R}^{N-1}}\phi(y)|v(y)|^2dy\right)\\&\qquad-\Lambda(a)\left(\frac{N-2}{2}\right)^2\int_{\mathbb{R}^{N-1}}\phi(y)|v(y)|^2dy\end{aligned}$$

and, in turn,

$$\Lambda(a)\int_{\mathbb{R}^{N-1}}|\nabla v(y)|^2dy+\Lambda(a)\left[\left(\frac{N-2}{2}\right)^2-\frac{(N-3)(N-1)}{4}\right]\int_{\mathbb{R}^{N-1}}\phi(y)|v(y)|^2dy \tag{166}$$
$$\geqslant\int_{\mathbb{R}^{N-1}}\phi(y)a(\Pi^{-1}(y))|v(y)|^2dy=\int_{\mathbb{R}^{N-1}}\frac{b(y/|y|)}{|y|^2}|v(y)|^2dy+\int_{\mathbb{R}^{N-1}}R(y)|v(y)|^2dy$$

where $R(y)=\phi(y)a(\Pi^{-1}(y))-\frac{b(y/|y|)}{|y|^2}$ is bounded in a sufficiently small neighborhood of 0 by (164). On the other hand if we define, for any $\delta>0$, $B_\delta\subset\mathbb{R}^{N-1}$ to be the open ball of radius $\delta$ centered at the origin and

$$C(\delta):=\sup_{\substack{v\in C^\infty(\mathbb{R}^{N-1})\setminus\{0\}\\ \operatorname{supp}v\subset B_\delta}}\frac{\int_{\mathbb{R}^{N-1}}\left[\left(\Lambda(a)\left(\frac{N-2}{2}\right)^2-\frac{(N-3)(N-1)}{4}\right)\phi(y)-R(y)\right]|v(y)|^2dy}{\int_{\mathbb{R}^{N-1}}|\nabla v(y)|^2dy}, \tag{167}$$

then $C(\delta)\to0$ as $\delta\to0^+$. Therefore, by (166), (167), and (12), we deduce that there exists $\delta_1>0$ such that for any $\delta\in(0,\delta_1)$

$$\sup_{v\in C^\infty_c(\mathbb{R}^{N-1})\setminus\{0\}}\frac{\int_{\mathbb{R}^{N-1}}\frac{b(\frac{y}{|y|})}{|y|^2}|v(y)|^2dy}{\int_{\mathbb{R}^{N-1}}|\nabla v(y)|^2dy}=\sup_{\substack{v\in C^\infty(\mathbb{R}^{N-1})\setminus\{0\}\\ \operatorname{supp}v\subset B_\delta}}\frac{\int_{\mathbb{R}^{N-1}}\frac{b(\frac{y}{|y|})}{|y|^2}|v(y)|^2dy}{\int_{\mathbb{R}^{N-1}}|\nabla v(y)|^2dy}\leqslant\Lambda(a)+C(\delta)<1.$$

The conclusion follows by density. □

By Lemmas 7.2 and 2.2, we deduce that the spectrum of the operator $L_b:=-\Delta_{\mathbb{S}^{N-2}}-b$ on $\mathbb{S}^{N-2}$ consists of real eigenvalues with finite multiplicity $\mu_1(b)<\mu_2(b)\leqslant\ldots\leqslant\mu_k(b)\leqslant\ldots$.

Let $\widetilde{h}$ be the function defined in (157) and, according with (14), for any nontrivial $\mathcal{D}^{1,2}(\mathbb{R}^{N-1})$-solution $v$ of the equation

$$-\Delta v(y)-\frac{b(\frac{y}{|y|})}{|y|^2}v(y)=\widetilde{h}(y)v(y),$$

we define the corresponding Almgren's frequency function by

$$\mathcal{N}_{v,\widetilde{h},0}(r)=\frac{r\int_{B_r}\left(|\nabla v(y)|^2-\frac{b(y/|y|)}{|y|^2}v^2(y)-\widetilde{h}(y)v^2(y)\right)dy}{\int_{\partial B_r}|v(y)|^2\,dS}. \tag{168}$$

We are ready to prove the following asymptotic description of eigenfunctions.

**Proposition 7.3.** *Let $3\leqslant k\leqslant N-1$, let $a$ satisfy (7), (12), and let $b$ and $\widetilde{h}$ be respectively defined in (156) and (157). Let $\mu_i(a)$ be an eigenvalue of the operator $L_a$ and let $\psi\in H^1(\mathbb{S}^{N-1})\setminus\{0\}$ be an associated eigenfunction. Let $\widetilde{\psi}\in\mathcal{D}^{1,2}(\mathbb{R}^{N-1})$ be the corresponding function defined in (154). Then there exists $\widetilde{k}_0\in\mathbb{N}$, $\widetilde{k}_0\geqslant1$, such that*

$$\lim_{r\to0^+}\mathcal{N}_{\widetilde{\psi},\widetilde{h},0}(r)=-\frac{N-3}{2}+\sqrt{\left(\frac{N-3}{2}\right)^2+\mu_{\widetilde{k}_0}(b)}. \tag{169}$$

*Furthermore, if $\widetilde{\gamma}$ denotes the limit in (169), $\widetilde{m}\geqslant1$ is the multiplicity of the eigenvalue $\mu_{\widetilde{k}_0}(b)$ and $\{\eta_j:\widetilde{\ell}_0\leqslant j\leqslant\widetilde{\ell}_0+\widetilde{m}-1\}$ ($\widetilde{\ell}_0\leqslant\widetilde{k}_0\leqslant\widetilde{\ell}_0+\widetilde{m}-1$) is an $L^2(\mathbb{S}^{N-2})$-orthonormal basis for the* *eigenspace associated to* $\mu_{\widetilde{k}_0}(b)$*, then*

$$\lambda^{-\widetilde{\gamma}}\psi(\Pi^{-1}(\lambda\Pi(\theta))) \to 4^{-\frac{N-3}{4}}|\Pi(\theta)|^{\widetilde{\gamma}} \sum_{j=\widetilde{\ell}_0}^{\widetilde{\ell}_0+\widetilde{m}-1} \widetilde{\beta}_j \eta_j\left(\frac{\Pi(\theta)}{|\Pi(\theta)|}\right) \quad \text{in } H^1_{\rm loc}(\mathbb{S}^{N-1}\setminus\{e_N\}) \quad \text{as } \lambda\to 0^+,$$

*where*

$$\widetilde{\beta}_j = \int_{\mathbb{S}^{N-2}}\left[R^{-\widetilde{\gamma}}\widetilde{\psi}(R\omega) + \int_0^R \frac{\widetilde{h}(s\omega)\widetilde{\psi}(s\omega)}{2\widetilde{\gamma}+N-3}\left(s^{1-\widetilde{\gamma}} - \frac{s^{\widetilde{\gamma}+N-2}}{R^{2\widetilde{\gamma}+N-3}}\right)ds\right]\eta_j(\omega)\,dS(\omega),$$

*for all* $R\in(0,\overline{R})$ *for some* $\overline{R}>0$*, and* $(\widetilde{\beta}_{\widetilde{\ell}_0}, \widetilde{\beta}_{\widetilde{\ell}_0+1}, \dots, \widetilde{\beta}_{\widetilde{\ell}_0+\widetilde{m}-1}) \neq (0,0,\dots,0)$.

Proof. Since $\psi$ is a solution of (152), then, by Lemma 7.1, $\widetilde{\psi}$ solves (155). By Lemma 7.2, $\Lambda(b)<1$ i.e. the function $b$ satisfies the positivity condition required in Theorem 1.1. Moreover by (157), the function $\widetilde{h}\in C^1(\overline{B}_\delta)$ for some $\delta>0$ small enough. Hence we may apply Theorem 1.1 to the function $\widetilde{\psi}$ to conclude. □

## 8. Pointwise estimates

Let $\hat{\sigma}$ as in (20) and $\hat{\psi}_1\in H^1(\mathbb{S}^{N-1})$, $\|\hat{\psi}_1\|_{L^2(\mathbb{S}^{N-1})}=1$, be the first positive eigenfuntion of the eigenvalue problem $L_a\psi=\mu_1(\hat{a})\psi$ in $\mathbb{S}^{N-1}$. The following lemma holds true.

**Lemma 8.1.** *If* $\hat{a}$ *satisfies (18) and (19) , then*

$$\mu_1(\hat{a})\leqslant 0, \quad \hat{\sigma}\leqslant 0, \quad \text{and} \quad \inf_{\mathbb{S}^{N-1}}\hat{\psi}_1>0.$$

Proof. The fact that $\mu_1(\hat{a})\leqslant 0$ follows easily by taking a constant function in the Rayleigh quotient minimized by $\mu_1(\hat{a})$ (see (24)). Moreover, there exists $\delta>0$ such that, letting

$$\Sigma_\delta := \left(\bigcup_{\substack{J\in\mathcal{A}_k\\ \alpha_J>0}}\{(\theta_1,\dots,\theta_N)\in\mathbb{S}^{N-1}:|\theta_J|<\delta\}\right)\cup\left(\bigcup_{\substack{(J_1,J_2)\in\mathcal{B}_k\\ \alpha_{J_1J_2}>0}}\{(\theta_1,\dots,\theta_N)\in\mathbb{S}^{N-1}:|\theta_{J_1}-\theta_{J_2}|<\delta\}\right)$$

there holds $\hat{a}(\theta)+\mu_1(\hat{a})>0$ in $\Sigma_\delta$. By classical elliptic regularity theory and maximum principles applied to the equation satisfied by $\hat{\psi}_1$ in $\mathbb{S}^{N-1}\setminus\Sigma_{\delta/2}$, we have that $\min_{\mathbb{S}^{N-1}\setminus\Sigma_{\delta/2}}\hat{\psi}_1>0$ and $\min_{\partial\Sigma_\delta}\hat{\psi}_1>0$. Moreover, testing

$$\begin{cases} -\Delta_{\mathbb{S}^{N-1}}(\hat{\psi}_1-\min_{\partial\Sigma_\delta}\hat{\psi}_1) = (\mu_1(\hat{a})+\hat{a}(\theta))\hat{\psi}_1\geqslant 0, & \text{in } \Sigma_\delta,\\ \hat{\psi}_1-\min_{\partial\Sigma_\delta}\hat{\psi}_1\geqslant 0 & \text{on } \partial\Sigma_\delta,\end{cases}$$

with $-(\hat{\psi}_1-\min_{\partial\Sigma_\delta}\hat{\psi}_1)^-$ we obtain that $\hat{\psi}_1\geqslant\min_{\partial\Sigma_\delta}\hat{\psi}_1$ in $\Sigma_\delta$. □

Let us introduce the weight function

$$\rho(x)=|x|^{\hat{\sigma}}\hat{\psi}_1\left(\frac{x}{|x|}\right) \qquad \text{for all } x\in\mathbb{R}^N\setminus\widetilde{\Sigma}. \tag{170}$$

From Lemma 8.1, under assumptions (18) and (19), there holds

$$d=d(\operatorname{diam}\Omega,N,\hat{a}):=\sup_{\Omega\setminus\widetilde{\Sigma}}\rho^{2-2^*}\in(0,+\infty). \tag{171}$$

We notice that $\rho \in H^1_{\rm loc}(\mathbb{R}^N)$ and introduce the weighted space $\mathcal{D}^{1,2}_\rho(\mathbb{R}^N)$ as the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the norm

$$\|v\|_{\mathcal{D}^{1,2}_\rho(\mathbb{R}^N)} := \bigg(\int_{\mathbb{R}^N} \rho^2(x)\big|\nabla v(x)\big|^2 dx\bigg)^{1/2} \tag{172}$$

and, similarly, $\mathcal{D}^{1,2}_\rho(\Omega)$ as the completion of $C^\infty_c(\Omega)$ with respect to (172).

**Lemma 8.2.** *$C^\infty_c(\mathbb{R}^N \setminus \widetilde{\Sigma})$ is dense in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.*

Proof. By density of $C^\infty_c(\mathbb{R}^N)$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, it is enough to prove, for every $J \in \mathcal{A}_k$ and $(J_1, J_2) \in \mathcal{B}_k$, the density of $C^\infty_c(\mathbb{R}^N \setminus \{x_J = 0\})$ and of $C^\infty_c(\mathbb{R}^N \setminus \{x_{J_1} = x_{J_2}\})$ in $C^\infty_c(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$. Let $\phi \in C^\infty(0,\infty)$ such that $\phi(t) = 0$ for all $t \in (0,1)$ and $\phi(t) = 1$ for all $t \in (2,\infty)$. If $J \in \mathcal{A}_k$ and $u \in C^\infty_c(\mathbb{R}^N)$, let $u_n(x) = \phi(n|x_J|)u(x) \in C^\infty_c(\mathbb{R}^N \setminus \{x_J = 0\})$. Since

$$\nabla u_n(x) - \nabla u(x) = \nabla u(x)\big(\phi(n|x_J|) - 1\big) + nu(x)\phi'(n|x_J|)\frac{x_J}{|x_J|},$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u(x)|^2 \big(\phi(n|x_J|) - 1\big)^2 dx = 0$$

by the Dominated Convergence Theorem, and

$$n^2 \int_{\mathbb{R}^N} u^2(x)(\phi'(n|x_J|))^2 dx = n^{2-k} \int_{\mathbb{R}^N} u^2\Big(y_1, \dots, \frac{y_J}{n}, \dots, y_N\Big)(\phi'(|y_J|))^2 dy = O(n^{2-k})$$

as $n \to +\infty$, we conclude that $u_n \to u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, thus proving the density of $C^\infty_c(\mathbb{R}^N \setminus \{x_J = 0\})$ in $C^\infty_c(\mathbb{R}^N)$ and hence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The density of $C^\infty_c(\mathbb{R}^N \setminus \{x_{J_1} = x_{J_2}\})$ can be proven in a similar way. □

**Lemma 8.3.** *If $\hat{a}$ satisfy (18) and (19), then*

(i) *$C^\infty_c(\mathbb{R}^N \setminus \widetilde{\Sigma})$ is dense in $\mathcal{D}^{1,2}_\rho(\mathbb{R}^N)$;*
(ii) *$v \in \mathcal{D}^{1,2}_\rho(\mathbb{R}^N)$ if and only if $\rho v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$;*
(iii) *for all $v \in \mathcal{D}^{1,2}_\rho(\mathbb{R}^N)$*

$$\int_{\mathbb{R}^N} \rho^2(x)|\nabla v(x)|^2 dx = \int_{\mathbb{R}^N} \bigg(|\nabla(\rho v)(x)|^2 dx - \frac{\hat{a}(\frac{x}{|x|})}{|x|^2}(\rho v)^2(x)\bigg) dx \tag{173}$$

Proof. We first prove that (173) holds for all $v \in C^\infty_c(\mathbb{R}^N \setminus \widetilde{\Sigma})$. Indeed, by direct computation $\rho$ solves

$$-\Delta\rho(x) - \frac{\hat{a}(\frac{x}{|x|})}{|x|^2}\,\rho(x) = 0 \qquad \text{in } \mathbb{R}^N \setminus \widetilde{\Sigma}. \tag{174}$$

Let $v \in C^\infty_c(\mathbb{R}^N \setminus \widetilde{\Sigma})$ and put $u = \rho v$ so that $u \in C^\infty_c(\mathbb{R}^N) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$. Then, testing (174) with $\rho v^2$ we obtain

$$\int_{\mathbb{R}^N} \nabla\rho(x)\nabla(\rho(x)v^2(x))\,dx - \int_{\mathbb{R}^N} \frac{\hat{a}(\frac{x}{|x|})}{|x|^2}\,\rho^2(x)v^2(x)\,dx = 0. \tag{175}$$

Moreover

$$\nabla\rho\nabla(\rho v^2) = v^2|\nabla\rho|^2 + 2\rho v\nabla\rho\nabla v \tag{176}$$

and

$$|\nabla u|^2 = v^2|\nabla\rho|^2 + 2v\rho\nabla\rho\nabla v + \rho^2|\nabla v|^2. \tag{177}$$

By (175)-(177) we then have

$$\begin{aligned} Q_{\hat a}(\rho v) &= \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^N} \frac{\hat a(\frac{x}{|x|})}{|x|^2} u^2(x)\,dx \\ &= \int_{\mathbb{R}^N} \rho^2(x)|\nabla v(x)|^2 dx + \int_{\mathbb{R}^N} \nabla\rho(x)\nabla(\rho(x)v^2(x))\,dx - \int_{\mathbb{R}^N} \frac{\hat a(\frac{x}{|x|})}{|x|^2}\rho^2(x)v^2(x)\,dx \\ &= \int_{\mathbb{R}^N} \rho^2(x)|\nabla v(x)|^2 dx, \quad \text{for all } v \in C_c^\infty(\mathbb{R}^N \setminus \widetilde\Sigma). \end{aligned} \tag{178}$$

To prove (i), by density of $C_c^\infty(\mathbb{R}^N)$ in $\mathcal{D}^{1,2}_\rho(\mathbb{R}^N)$, it is enough to prove the density of $C_c^\infty(\mathbb{R}^N\setminus\widetilde\Sigma)$ in $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_{\mathcal{D}^{1,2}_\rho(\mathbb{R}^N)}$. Let $v \in C_c^\infty(\mathbb{R}^N)$. It is easy to verify that $u = \rho v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, hence, by Lemma 8.2, there exists a sequence $\{u_n\}_n \subset C_c^\infty(\mathbb{R}^N\setminus\widetilde\Sigma)$ such that $u_n \to u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Letting $v_n = \frac{u_n}{\rho}$, we have that $v_n \in C_c^\infty(\mathbb{R}^N\setminus\widetilde\Sigma)$ and, by (178),

$$\int_{\mathbb{R}^N} \rho^2(x)|\nabla v_n(x) - \nabla v_m(x)|^2 dx = Q_{\hat a}(u_n - u_m).$$

Therefore, since $u_n$ is a Cauchy sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and, by (19) and Lemma 2.3, $(Q_{\hat a}(u))^{1/2}$ is an equivalent norm in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we conclude that $v_n$ is a Cauchy sequence in $\mathcal{D}^{1,2}_\rho(\mathbb{R}^N)$ and hence converges to some $\tilde v \in \mathcal{D}^{1,2}_\rho(\mathbb{R}^N)$. Since $v_n \to v$ a.e. in $\mathbb{R}^N$, we deduce that $\tilde v = v$ and then $v_n \to v$ in $\mathcal{D}^{1,2}_\rho(\mathbb{R}^N)$. The proof of (i) is thereby complete.

To prove (ii-iii), let $v \in \mathcal{D}^{1,2}_\rho(\mathbb{R}^N)$. By (i), there exists a sequence $\{v_n\}_n \subset C_c^\infty(\mathbb{R}^N\setminus\widetilde\Sigma)$ such that $v_n \to v$ in $\mathcal{D}^{1,2}_\rho(\mathbb{R}^N)$. Letting $u_n = v_n\rho \in C_c^\infty(\mathbb{R}^N\setminus\widetilde\Sigma)$, by (178) we have that

$$\int_{\mathbb{R}^N} \rho^2(x)|\nabla v_n(x)|^2 dx = \int_{\mathbb{R}^N} |\nabla u_n(x)|^2 dx - \int_{\mathbb{R}^N} \frac{\hat a(\frac{x}{|x|})}{|x|^2} u_n^2(x)\,dx \tag{179}$$

and $\|v_n - v_m\|^2_{\mathcal{D}^{1,2}_\rho(\mathbb{R}^N)} = Q_{\hat a}(u_n - u_m)$. Therefore, since $v_n$ is a Cauchy sequence in $\mathcal{D}^{1,2}_\rho(\mathbb{R}^N)$ and $(Q_{\hat a}(u))^{1/2}$ is an equivalent norm in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we infer that $u_n$ is a Cauchy sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and hence converges to some $u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Since $u_n = \rho v_n \to \rho v$ a.e. in $\mathbb{R}^N$, we deduce that $\rho v = u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover, we can pass to the limit in (179), thus obtaining (173) and proving (iii). In a similar way, one can prove that if $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ then $\frac{u}{\rho} \in \mathcal{D}^{1,2}_\rho(\mathbb{R}^N)$, thus completing the proof of (ii). □

Thanks to Lemma 2.3, (19), and the standard Sobolev inequality, the number

$$S(\hat a) = \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{Q_{\hat a}(u)}{\left(\int_{\mathbb{R}^N}|u(x)|^{2^*}dx\right)^{2/2^*}}$$

is strictly positive and provides the best constant in the following weighted Sobolev inequality.

**Lemma 8.4.** *Let $N \geqslant k \geqslant 3$ and let $\hat a$ satisfy (18) and (19). Then*

$$\int_{\mathbb{R}^N} \rho^2(x)|\nabla v(x)|^2\,dx \geqslant S(\hat a)\left(\int_{\mathbb{R}^N} \rho^{2^*}(x)|v(x)|^{2^*}\,dx\right)^{2/2^*} \tag{180}$$

*for all* $v \in \mathcal{D}^{1,2}_{\rho}(\mathbb{R}^N)$.

PROOF. It follows from Lemma 8.3 and the change $u = \rho v$. □

We also define the weighted Sobolev space $H^1_\rho(\Omega)$ as the completion of $V_\rho(\Omega)$ with respect to the norm

$$\|v\|_{H^1_\rho(\Omega)} := \left(\int_\Omega \rho^2(x)|\nabla v(x)|^2\,dx + \int_\Omega \rho^2(x)v^2(x)\,dx\right)^{1/2}$$

where $V_\rho(\Omega)$ denotes the space of all functions $v \in C^\infty(\Omega)\cap H^1(\Omega)$ such that

$$\overline{\{x\in\Omega : v(x)\neq 0\}}^{\,\Omega} \subset \Omega\setminus\widetilde{\Sigma}.$$

For any $q \geqslant 1$, we also denote as $L^q(\rho^{2^*},\Omega)$ the weighted $L^q$-space endowed with the norm

$$\|u\|_{L^q(\rho^{2^*},\Omega)} := \left(\int_\Omega \rho^{2^*}(x)|u(x)|^q\,dx\right)^{1/q}.$$

**Lemma 8.5.** *Let $N \geqslant k \geqslant 3$, $\Omega\subset\mathbb{R}^N$ be a bounded open set such that $0\in\Omega$, $\hat a$ satisfy (18) and (19), and $h$ satisfy* (**H**). *Let $V\in L^1_{\rm loc}(\Omega\setminus\widetilde{\Sigma})$ such that*

$$\sup_{v\in H^1_\rho(\Omega)\setminus\{0\}} \frac{\int_\Omega \rho^{2^*}(x)|V(x)|v^2(x)\,dx}{\|v\|^2_{H^1_\rho(\Omega)}} < +\infty, \tag{181}$$

*and $v\in H^1_\rho(\Omega)\cap L^q(\rho^{2^*},\Omega)$, $q>2$, be a weak solution to*

$$-\mathrm{div}(\rho^2(x)\nabla v(x)) = \big(\rho^2(x)h(x) + \rho^{2^*}(x)V(x)\big)v(x). \tag{182}$$

*If*

$$V_+ \in L^s(\rho^{2^*},\Omega) \qquad \text{for some } s > N/2, \tag{183}$$

*then for any $\Omega'\Subset\Omega$ such that $0\in\Omega'$, $v\in L^{\frac{2^*q}{2}}(\rho^{2^*},\Omega')$ and*

$$\|v\|_{L^{\frac{2^*q}{2}}(\rho^{2^*},\Omega')} \leqslant S(\hat a)^{-\frac1q}\|v\|_{L^q(\rho^{2^*},\Omega)}\left(\frac{20}{C(q)}\frac{d}{(\mathrm{dist}(\Omega',\partial\Omega))^2} + \frac{4(q-2)d}{(\mathrm{dist}(\Omega',\partial\Omega))^2} + \frac{4\ell_q}{C(q)}\right)^{\frac1q}, \tag{184}$$

*where $C(q) := \min\left\{\frac14,\frac{4}{q+4}\right\}$, $d$ is as in (171), and*

$$\ell_q = \max\left\{\left(\frac{\max\{16,q+4\}}{S(\hat a)}\|V_+\|^{2s/N}_{L^s(\rho^{2^*},\Omega)}\right)^{\frac{N}{2s-N}},\ \frac{dC_h^{2/\varepsilon}\left(\frac{2}{k-2}\right)^{\frac{2(2-\varepsilon)}{\varepsilon}}\binom{N}{k}^{2/\varepsilon}\left(1+\binom{N-k}{k}\right)^{2/\varepsilon}}{(1-\Lambda(\hat a))^{\frac{2-\varepsilon}{\varepsilon}}}(\max\{16,q+4\})^{\frac{2-\varepsilon}{\varepsilon}}\right\}. \tag{185}$$

Proof. Let $w \in \mathcal{D}^{1,2}_\rho(\Omega)$. Then by Lemma 8.4 we have

$$
\begin{aligned}
&\int_\Omega \rho^{2^*}(x)V_+(x)|w(x)|^2\,dx \\
&\leqslant \ell_q \int\limits_{V_+(x)\leqslant \ell_q} \rho^{2^*}(x)|w(x)|^2\,dx + \int\limits_{V_+(x)\geqslant \ell_q} \rho^{2^*-2}(x)V_+(x)\rho^2(x)|w(x)|^2\,dx \\
&\leqslant \ell_q \int_\Omega \rho^{2^*}(x)|w(x)|^2\,dx + \left(\int_\Omega \rho^{2^*}(x)|w(x)|^{2^*}\,dx\right)^{\frac{2}{2^*}} \left(\int\limits_{V_+(x)\geqslant \ell_q} \rho^{2^*}(x)V_+^{\frac{N}{2}}(x)\,dx\right)^{\frac{2}{N}} \\
&\leqslant \ell_q \int_\Omega \rho^{2^*}(x)|w(x)|^2\,dx + \frac{1}{S(\hat a)}\left(\int_\Omega \rho^2(x)\big|\nabla w(x)\big|^2\,dx\right)\left(\int\limits_{V_+(x)\geqslant \ell_q} \rho^{2^*}(x)V_+^{\frac{N}{2}}(x)\,dx\right)^{\frac{2}{N}}.
\end{aligned}
\tag{186}
$$

Next, Hölder inequality and the definition of $\ell_q$ yield

$$
\begin{aligned}
\int\limits_{V_+(x)\geqslant \ell_q} \rho^{2^*}(x)V_+^{\frac{N}{2}}(x)\,dx &\leqslant \left(\int_\Omega \rho^{2^*}(x)V_+^s(x)\,dx\right)^{\frac{N}{2s}}\left(\int\limits_{V_+(x)\geqslant \ell_q} \rho^{2^*}(x)\,dx\right)^{\frac{2s-N}{2s}} \\
&\leqslant \left(\int_\Omega \rho^{2^*}(x)V_+^s(x)\,dx\right)^{\frac{N}{2s}}\left(\int\limits_{V_+(x)\geqslant \ell_q} \left(\frac{V_+(x)}{\ell_q}\right)^s\rho^{2^*}(x)\,dx\right)^{\frac{2s-N}{2s}} \\
&\leqslant \|V_+\|^s_{L^s(\rho^{2^*},\Omega)}\,\ell_q^{-s+\frac{N}{2}} \leqslant \left(\min\left\{\frac{S(\hat a)}{16},\frac{S(\hat a)}{q+4}\right\}\right)^{\frac{N}{2}},
\end{aligned}
\tag{187}
$$

Inserting (187) into (186) we obtain for any $w \in \mathcal{D}^{1,2}_\rho(\Omega)$

$$
\begin{aligned}
&\int_\Omega \rho^{2^*}(x)V_+(x)|w(x)|^2\,dx \\
&\qquad\leqslant \ell_q \int_\Omega \rho^{2^*}(x)|w(x)|^2\,dx + \frac{1}{2}\min\left\{\frac{1}{8},\frac{2}{q+4}\right\}\int_\Omega \rho^2(x)|\nabla w(x)|^2 dx.
\end{aligned}
\tag{188}
$$

On the other hand, letting $\delta_q = \Big(C_h d\binom{N}{k}\big(1+\binom{N-k}{k}\big)\Big)^{1/(2-\varepsilon)}\ell_q^{-1/(2-\varepsilon)}$, from (**H**), (8), (9), (23), (173), and (171), for every $w \in \mathcal{D}^{1,2}_\rho(\Omega)$ we can estimate

$$\begin{aligned}
(189)\quad & \int_\Omega \rho^2(x)|h(x)||w(x)|^2\,dx \\
&\leqslant C_h\bigg[\delta_q^\varepsilon\bigg(\sum_{J\in\mathcal{A}_k}\int_{|x_J|\leqslant\delta_q}\frac{\rho^2(x)w^2(x)}{|x_J|^2}\,dx + \sum_{(J_1,J_2)\in\mathcal{B}_k}\int_{|x_{J_1}-x_{J_2}|\leqslant\delta_q}\frac{\rho^2(x)w^2(x)}{|x_{J_1}-x_{J_2}|^2}\,dx\bigg) \\
&\quad + \delta_q^{-2+\varepsilon} d\bigg(\sum_{J\in\mathcal{A}_k}\int_{|x_J|\geqslant\delta_q}\rho^{2^*}(x)w^2(x)\,dx + \sum_{(J_1,J_2)\in\mathcal{B}_k}\int_{|x_{J_1}-x_{J_2}|\geqslant\delta_q}\rho^{2^*}(x)w^2(x)\,dx\bigg)\bigg] \\
&\leqslant C_h\tbinom{N}{k}\Big(1+\tbinom{N-k}{k}\Big)\bigg(\delta_q^\varepsilon\Big(\tfrac{2}{k-2}\Big)^2(1-\Lambda(\hat a))^{-1}\int_\Omega\rho^2(x)|\nabla w(x)|^2dx \\
&\quad + \delta_q^{-2+\varepsilon} d\int_\Omega\rho^{2^*}(x)w^2(x)\,dx\bigg) \\
&\leqslant \ell_q\int_\Omega\rho^{2^*}(x)|w(x)|^2\,dx + \frac12\min\left\{\frac18,\frac{2}{q+4}\right\}\int_\Omega\rho^2(x)|\nabla w(x)|^2dx.
\end{aligned}$$

Summing up (188) and (189), we obtain

$$\begin{aligned}
(190)\quad & \int_\Omega\big(\rho^{2^*}(x)V_+(x)+|h(x)|\rho^2(x)\big)|w(x)|^2\,dx \\
&\qquad\leqslant 2\ell_q\int_\Omega\rho^{2^*}(x)|w(x)|^2\,dx + \min\left\{\frac18,\frac{2}{q+4}\right\}\int_\Omega\rho^2(x)|\nabla w(x)|^2dx
\end{aligned}$$

for all $w\in\mathcal{D}^{1,2}_\rho(\Omega)$. As in [17, 20] we define $v^n := \min\{n,|v|\}\in H^1_\rho(\Omega)$ and we introduce a cut-off function $\eta\in C^\infty_c(\Omega)$ satisfying

$$\eta\equiv 1 \text{ in } \Omega' \qquad\text{and}\qquad |\nabla\eta|\leqslant\frac{2}{\operatorname{dist}(\Omega',\partial\Omega)}.$$

Testing (182) with $\eta^2(v^n)^{q-2}v\in\mathcal{D}^{1,2}_\rho(\Omega)$ we obtain

$$\begin{aligned}
&(q-2)\int_\Omega\rho^2(x)\eta^2(x)(v^n(x))^{q-2}\chi_{\{y\in\Omega:|v(y)|<n\}}(x)|\nabla|v|(x)|^2\,dx \\
&\qquad+\int_\Omega\rho^2(x)\eta^2(x)(v^n(x))^{q-2}|\nabla v(x)|^2\,dx \\
&=\int_\Omega(\rho^{2^*}(x)V(x)+\rho^2(x)h(x))\eta^2(x)|v(x)|^2(v^n(x))^{q-2}\,dx \\
&\qquad-2\int_\Omega\rho^2(x)\eta(x)(v^n(x))^{q-2}v(x)\nabla v(x)\cdot\nabla\eta(x)\,dx \\
&\leqslant\int_\Omega(\rho^{2^*}(x)V_+(x)+\rho^2(x)|h(x)|)\eta^2(x)|v(x)|^2(v^n(x))^{q-2}\,dx \\
&\qquad+2\int_\Omega\rho^2(x)|\nabla\eta(x)|^2(v^n(x))^{q-2}|v(x)|^2\,dx+\frac12\int_\Omega\rho^2(x)\eta^2(x)(v^n(x))^{q-2}|\nabla v(x)|^2\,dx
\end{aligned}$$

and hence

$$(q-2)\int_\Omega \rho^2(x)\eta^2(x)(v^n(x))^{q-2}|\nabla v^n(x)|^2\,dx+\frac12\int_\Omega \rho^2(x)\eta^2(x)(v^n(x))^{q-2}|\nabla v(x)|^2\,dx \tag{191}$$
$$\leqslant \int_\Omega \big(\rho^{2^*}(x)V_+(x)+\rho^2(x)|h(x)|\big)\eta^2(x)|v(x)|^2(v^n(x))^{q-2}\,dx$$
$$+2\int_\Omega \rho^2(x)|\nabla\eta(x)|^2(v^n(x))^{q-2}|v(x)|^2\,dx.$$

By direct computation we also have

$$\Big|\nabla\Big((v^n)^{\frac{q-2}{2}}\eta v\Big)\Big|^2\leqslant \frac{(q+4)(q-2)}{4}(v^n)^{q-2}\eta^2|\nabla v^n|^2$$
$$+2\eta^2(v^n)^{q-2}|\nabla v|^2+2|\nabla\eta|^2(v^n)^{q-2}|v|^2+\frac{q-2}{2}(v^n)^q|\nabla\eta|^2$$

and hence by (191) we obtain

$$C(q)\int_\Omega \rho^2(x)\Big|\nabla\Big((v^n)^{\frac{q-2}{2}}\eta v\Big)\Big|^2\,dx \tag{192}$$
$$\leqslant \int_\Omega \big(\rho^{2^*}(x)V_+(x)+\rho^2(x)|h(x)|\big)\eta^2(x)|v(x)|^2(v^n(x))^{q-2}\,dx$$
$$+2(C(q)+1)\int_\Omega \rho^2(x)(v^n(x))^{q-2}|v(x)|^2|\nabla\eta(x)|^2\,dx+C(q)\frac{q-2}{2}\int_\Omega\rho^2(x)(v^n(x))^q|\nabla\eta(x)|^2\,dx.$$

Applying estimate (190) to the function $w=\eta(v^n)^{\frac{q-2}{2}}v$, by (192) we have

$$\frac{C(q)}{2}\int_\Omega \rho^2(x)\Big|\nabla\Big((v^n)^{\frac{q-2}{2}}\eta v\Big)\Big|^2\,dx\leqslant 2\ell_q\int_\Omega\rho^{2^*}(x)\eta^2(x)(v^n(x))^{q-2}|v(x)|^2\,dx$$
$$+2(C(q)+1)\int_\Omega \rho^2(x)(v^n(x))^{q-2}|v(x)|^2|\nabla\eta(x)|^2\,dx+C(q)\frac{q-2}{2}\int_\Omega\rho^2(x)(v^n(x))^q|\nabla\eta(x)|^2\,dx.$$

and this with Lemma 8.4 and (171) implies

$$\left(\int_\Omega \rho^{2^*}(x)|v^n(x)|^{2^*\frac{q-2}{2}}|v(x)|^{2^*}\eta^{2^*}(x)\,dx\right)^{\frac{2}{2^*}}\leqslant\frac{4\ell_q}{C(q)S(\hat a)}\int_\Omega\rho^{2^*}(x)\eta^2(x)(v^n(x))^{q-2}|v(x)|^2\,dx$$
$$+\frac{4(C(q)+1)d}{C(q)S(\hat a)}\int_\Omega\rho^{2^*}(x)(v^n(x))^{q-2}|v(x)|^2|\nabla\eta(x)|^2\,dx+\frac{(q-2)d}{S(\hat a)}\int_\Omega\rho^{2^*}(x)(v^n(x))^q|\nabla\eta(x)|^2\,dx.$$

The proof of the lemma then follows letting $n\to+\infty$. □

**Theorem 8.6.** *Let $N\geqslant k\geqslant 3$, $\Omega\subset\mathbb{R}^N$ be a bounded open set such that $0\in\Omega$, $\hat a$ satisfy (18) and (19), $h$ as in* (**H**), *and $V\in L^1_{\rm loc}(\Omega\setminus\widetilde\Sigma)$ verify (181).*

i) *If $V_+\in L^s(\rho^{2^*},\Omega)$ for some $s>N/2$, then for any $\Omega'\Subset\Omega$ there exists a positive constant*

$$C_\infty=C_\infty(N,k,\hat a,h,\|V_+\|_{L^s(\rho^{2^*},\Omega)},{\rm dist}(\Omega',\partial\Omega),{\rm diam}(\Omega))$$

*depending only on $N,k,\hat a,h,\|V_+\|_{L^s(\rho^{2^*},\Omega)},{\rm dist}(\Omega',\partial\Omega)$ and ${\rm diam}(\Omega)$, such that for every solution $u\in H^1(\Omega)$ of*

$$-\Delta u(x)-\frac{\hat a(\frac{x}{|x|})}{|x|^2}u(x)=\big(h(x)+\rho^{2^*-2}(x)V(x)\big)u(x)\quad\text{in }\Omega, \tag{193}$$

*there holds* $\rho^{-1}u \in L^\infty(\Omega')$ *and*

$$\|\rho^{-1}u\|_{L^\infty(\Omega')} \leqslant C_\infty \|u\|_{L^{2^*}(\Omega)}.$$

ii) *If* $V_+ \in L^{N/2}(\rho^{2^*}, \Omega)$*, then for any* $\Omega' \Subset \Omega$ *and for any* $s \geqslant 1$ *there exists a positive constant*

$$C_s = C_s(N, k, \hat{a}, h, V, s, \mathrm{dist}(\Omega', \partial\Omega), \mathrm{diam}(\Omega))$$

*depending only on* $N, k, \hat{a}, h, V, s, \mathrm{dist}(\Omega', \partial\Omega)$, $\mathrm{diam}(\Omega)$*, such that every solution* $u \in H^1(\Omega)$ *to* (193) *satisfies* $\rho^{-1}u \in L^s(\rho^{2^*}, \Omega')$ *and*

$$\|\rho^{-1}u\|_{L^s(\rho^{2^*}, \Omega')} \leqslant C_s \|u\|_{L^{2^*}(\Omega)}.$$

Proof. Let $u \in H^1(\Omega)$ be a weak solution of (193), $\Omega' \Subset \Omega$, and $R > 0$ such that

$$\Omega' \Subset \Omega' + B(0, 2R) \Subset \Omega.$$

We claim that the function $v(x) := \rho^{-1}(x)u(x)$ belongs to $H^1_\rho(\Omega' + B(0, 2R))$. Indeed, arguing as in Lemma 8.2, we can prove that $V_\rho(\Omega)$ is dense in $H^1(\Omega)$, hence there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset V_\rho(\Omega)$ such that $u_n \to u$ in $H^1(\Omega)$. If $\eta \in C^\infty_c(\Omega)$ is a cut-off function such that $\eta \equiv 1$ in $\Omega' + B(0, 2R)$, from (173) it follows that

$$\int_\Omega |\nabla(\eta(x)(u_n(x) - u_m(x)))|^2 dx - \int_\Omega \frac{\hat{a}(\frac{x}{|x|})}{|x|^2} \eta^2(x)(u_n(x) - u_m(x))^2\, dx$$
$$= \int_\Omega \rho^2(x) |\nabla(\eta(x)\rho^{-1}(x)(u_n(x) - u_m(x)))|^2 dx.$$

This shows that $\{\rho^{-1}u_n\}$ is a Cauchy sequence in $H^1_\rho(\Omega' + B(0, 2R))$ which then converges to $v(x) = \rho^{-1}(x)u(x)$. In particular $v \in H^1_\rho(\Omega' + B(0, 2R))$.

By direct computation one also sees that $v$ is a weak solution of (182). By Lemma 8.5, proceeding exactly as in the proofs of [17, Theorem 9.3] and [20, Theorem 1.2], we arrive to the conclusion. □

**Remark 8.7.** The statement of Theorem 9.4 in our previous paper [17] should be corrected as in the statement of Theorem 1.2. The missing point in Theorem 9.4. as it was stated in [17] relies in the fact that the constant $\widetilde{C}_\infty$ such that $\||x|^{-\sigma}u\|_{L^\infty(\Omega')} \leqslant \widetilde{C}_\infty \|u\|_{L^{2^*}(\Omega)}$ depends on $u$, more precisely on the distribution function of $f(x, u)/u$.

In a similar way, the statements of Theorems 9.3 and 10.4 should be corrected as in Theorem 8.6 above, i.e. the constant $C_s$ (respectively $C_{s,2}$) appearing in the statement (ii) of Theorem 9.3 (respectively 10.4) depends on $(\Re(V))_+$ (more precisely on its distribution function) and not only on its $L^{N/2}(\rho^{2^*}, \Omega)$-norm (respectively $L^s(\rho^p, \Omega)$-norm) as incorrectly stated in [17].

Anyway, the proofs of Theorems 9.3 and 9.4 contained in [17] are correct and lead to analogous conclusion as those stated in Theorems 1.2 and 8.6 of the present paper. Moreover all the proofs and statements in the rest of the paper [17] are not affected by these corrections.

**Proof of Theorem 1.2.** Let us define

$$V(x) := \begin{cases} \rho^{2-2^*}(x)\Big(\frac{f(x,u(x))}{u(x)} - \sum_{J\in\mathcal{A}_k} \frac{\alpha_J^-}{|\theta_J|^2} - \sum_{(J_1,J_2)\in\mathcal{B}_k} \frac{\alpha^-_{J_1\,J_2}}{|\theta_{J_1} - \theta_{J_2}|^2}\Big), & \text{if } u(x) \neq 0,\\ \rho^{2-2^*}(x)\Big(-\sum_{J\in\mathcal{A}_k} \frac{\alpha_J^-}{|\theta_J|^2} - \sum_{(J_1,J_2)\in\mathcal{B}_k} \frac{\alpha^-_{J_1\,J_2}}{|\theta_{J_1} - \theta_{J_2}|^2}\Big), & \text{if } u(x) = 0,\end{cases}$$

where $\alpha_J^- = \max\{-\alpha_J, 0\}$ and $\alpha_{J_1J_2}^- = \max\{-\alpha_{J_1J_2}, 0\}$. By (**F**) and the Sobolev embedding $H^1(\Omega) \subset L^{2^*}(\Omega)$, we have that $V^+ \in L^{N/2}(\rho^{2^*}, \Omega)$ and $u$ weakly solves

$$-\Delta u(x) - \frac{\hat{a}(\frac{x}{|x|})}{|x|^2} u(x) = \big(h(x) + \rho^{2^*-2}(x) V(x)\big) u(x) \quad \text{in } \Omega.$$

From part ii) of Theorem 8.6, it follows that $\rho^{-1} u \in L^s(\rho^{2^*}, \Omega')$ for any $\Omega' \Subset \Omega$ and for any $s \geqslant 1$. By (**F**) we deduce that $V^+ \in L^s(\rho^{2^*}, \Omega')$ for all $s \geqslant \frac{N-2}{4}$ and in particular for some $s > N/2$. The proof of the theorem follows now by part i) of Theorem 8.6. □

## Appendix

To prove Theorem 3.3 we used, for the approximating problems, a Pohozaev-type identity (see (53)), whose proof is quite classical (see e.g. [38, 41]) and requires just few adaptations due to the presence of a singularity. For the sake of completeness we give below a proof.

**Proposition A.1.** *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded open set such that $0 \in \Omega$. Let $b \in L^\infty(\mathbb{S}^{N-1})$, $h \in L^\infty(\Omega)$, and let $f$ satisfy* (**F**). *Denote by $\nu = \nu(x)$ the unit outer normal vector $\nu(x) = \frac{x}{|x|}$. If $u$ is a $H^1(\Omega)$-weak solution to $\mathcal{L}_b u = h(x)\, u + f(x, u)$ in $\Omega$ and $r_0 >$ is such that $B_{r_0} \subseteq \Omega$, then for a.e. $r \in (0, r_0)$*

$$\begin{aligned}(194)\qquad -\frac{N-2}{2} \int_{B_r} \left[ |\nabla u(x)|^2 - \frac{b(\frac{x}{|x|})}{|x|^2} u^2(x) \right] dx + \frac{r}{2} \int_{\partial B_r} \left[ |\nabla u(x)|^2 - \frac{b(\frac{x}{|x|})}{|x|^2} u^2(x) \right] dS \\ = r \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu} \right|^2 dS + \int_{B_r} h(x) u(x) \left( x \cdot \nabla u(x) \right)\big)\, dx \\ + r \int_{\partial B_r} F(x, u(x))\, dS - \int_{B_r} \left[ \nabla_x F(x, u(x)) \cdot x + N F(x, u(x)) \right] dx\ .\end{aligned}$$

Proof. By classical Brezis-Kato [8] estimates, bootstrap, and elliptic regularity theory, (**F**) and the boundedness of the coefficients $b, h$ imply that $u \in H^2_{\rm loc}(\Omega \setminus \{0\}) \cap C^{1,\alpha}_{\rm loc}(\Omega \setminus \{0\})$ for any $\alpha \in (0, 1)$. Therefore by (**F**) and Hardy inequality, we have

$$\begin{aligned}\int_0^r \left[ \int_{\partial B_s} \left( |\nabla u(x)|^2 + \frac{u^2(x)}{|x|^2} + \left| \frac{\partial u}{\partial \nu}(x) \right|^2 + |F(x, u(x))| \right) dS \right] ds \\ = \int_{B_r} \left( |\nabla u(x)|^2 + \frac{u^2(x)}{|x|^2} + \left| \frac{\partial u}{\partial \nu}(x) \right|^2 + |F(x, u(x))| \right) dx < +\infty\end{aligned}$$

and hence there exists a decreasing sequence $\{\delta_n\} \subset (0, r)$ such that $\lim_{n \to +\infty} \delta_n = 0$ and

$$(195)\qquad \delta_n \int_{\partial B_{\delta_n}} \left( |\nabla u(x)|^2 + \frac{u^2(x)}{|x|^2} + \left| \frac{\partial u}{\partial \nu}(x) \right|^2 + |F(x, u(x))| \right) dS \longrightarrow 0 \quad \text{as } n \to +\infty\,.$$

Multiplying equation $\mathcal{L}_b u = h(x)\,u + f(x,u)$ by $x\cdot\nabla u(x)$ and integrating over $B_r\setminus B_{\delta_n}$, it follows that

$$\int_{B_r\setminus B_{\delta_n}} \nabla u(x)\cdot\nabla(x\cdot\nabla u(x))\,dx - \int_{B_r\setminus B_{\delta_n}} \frac{b(\frac{x}{|x|})}{|x|^2} u(x)(x\cdot\nabla u(x))\,dx$$
$$= r\int_{\partial B_r}\left|\frac{\partial u}{\partial\nu}\right|^2 dS - \delta_n\int_{\partial B_{\delta_n}}\left|\frac{\partial u}{\partial\nu}\right|^2 dS + \int_{B_r\setminus B_{\delta_n}} h(x)u(x)(x\cdot\nabla u(x))\,dx$$
$$+\int_{B_r\setminus B_{\delta_n}} f(x,u(x))(x\cdot\nabla u(x))\,dx\,. \tag{196}$$

Standard integration by parts shows that

$$\int_{B_r\setminus B_{\delta_n}} \nabla u(x)\cdot\nabla(x\cdot\nabla u(x))\,dx$$
$$= -\frac{N-2}{2}\int_{B_r\setminus B_{\delta_n}}|\nabla u(x)|^2\,dx + \frac{r}{2}\int_{\partial B_r}|\nabla u(x)|^2\,dS - \frac{\delta_n}{2}\int_{\partial\delta_n}|\nabla u(x)|^2\,dS\,. \tag{197}$$

Passing in radial coordinates $r=|x|$, $\theta=\frac{x}{|x|}$ and observing that $\partial_r u(r,\theta)=\nabla u(r\theta)\cdot\theta$, we obtain

$$\int_{B_r\setminus B_{\delta_n}} \frac{b(\frac{x}{|x|})}{|x|^2} u(x)(x\cdot\nabla u(x))\,dx = \int_{\mathbb{S}^{N-1}} b(\theta)\left(\int_{\delta_n}^r s^{N-2}u(s\theta)\partial_s u(s\theta)\,ds\right)dS(\theta)$$
$$= \int_{\mathbb{S}^{N-1}} b(\theta)\bigg(r^{N-2}u^2(r\theta) - \delta_n^{N-2}u^2(\delta_n\theta)$$
$$-(N-2)\int_{\delta_n}^r s^{N-3}u^2(s\theta)\,ds - \int_{\delta_n}^r s^{N-2}u(s\theta)\partial_s u(s\theta)\,ds\bigg)dS(\theta)$$
$$= r\int_{\partial B_r}\frac{b(\frac{x}{|x|})}{|x|^2}u^2(x)\,dS - \delta_n\int_{\partial B_{\delta_n}}\frac{b(\frac{x}{|x|})}{|x|^2}u^2(x)\,dS$$
$$-(N-2)\int_{B_r\setminus B_{\delta_n}}\frac{b(\frac{x}{|x|})}{|x|^2}u^2(x)\,dx - \int_{B_r\setminus B_{\delta_n}}\frac{b(\frac{x}{|x|})}{|x|^2}u(x)\,(x\cdot\nabla u(x))\,dx\ ,$$

which yields

$$\int_{B_r\setminus B_{\delta_n}} \frac{b(\frac{x}{|x|})}{|x|^2} u(x)\,(x\cdot\nabla u(x))\,dx$$
$$= -\frac{N-2}{2}\int_{B_r\setminus B_{\delta_n}}\frac{b(\frac{x}{|x|})}{|x|^2}u^2(x)\,dx + \frac{r}{2}\int_{\partial B_r}\frac{b(\frac{x}{|x|})}{|x|^2}u^2(x)\,dS - \frac{\delta_n}{2}\int_{\partial B_{\delta_n}}\frac{b(\frac{x}{|x|})}{|x|^2}u^2(x)\,dS\,. \tag{198}$$

By (**F**) and the fact that $u\in C^{1,\alpha}_{\rm loc}(\Omega\setminus\{0\})$ we obtain

$$r\int_{\partial B_r}F(x,u(x))\,dS - \delta_n\int_{\partial B_{\delta_n}}F(x,u(x))\,dS = \int_{B_r\setminus B_{\delta_n}}\operatorname{div}\big(F(x,u(x))x\big)\,dx$$
$$= \int_{B_r\setminus B_{\delta_n}}\left[\nabla_x F(x,u(x))\cdot x + NF(x,u(x))\right]dx + \int_{B_r\setminus B_{\delta_n}} f(x,u(x))(\nabla u(x)\cdot x)\,dx \tag{199}$$

Letting $n\to+\infty$, (194) follows by (195–199). □

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