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Risk Measures on P(R) and Value At Risk with Probability/Loss function

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Abstract

We propose a generalization of the classical notion of the $V@R_\lambda$ that takes into account not only the probability of the losses, but the balance between such probability and the amount of the loss. This is obtained by defining a new class of law invariant risk measures based on an appropriate family of acceptance sets. The $V@R_\lambda$ and other known law invariant risk measures turn out to be special cases of our proposal. We further prove the dual representation of Risk Measures on $\mathcal{P}(\mathbb{R})$.

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1 Introduction

We introduce a new class of law invariant risk measures $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ that are directly defined on the set $\mathcal{P}(\mathbb{R})$ of probability measures on $\mathbb{R}$ and are monotone and quasi-convex on $\mathcal{P}(\mathbb{R})$.

As Cherny and Madan (2009) [4] pointed out, for a (*translation invariant)* coherent risk measure defined on random variables, all the positions can be spited in two classes: acceptable and not acceptable; in contrast, for an *acceptability index* there is a whole continuum of degrees of acceptability defined by a system $\{\mathcal{A}^m\}_{m\in\mathbb{R}}$ of sets. This formulation has been further investigated by Drapeau and Kupper (2010) [6] for the quasi convex case.

We adopt this approach and we build the maps Φ from a family $\{\mathcal{A}^m\}_{m\in\mathbb{R}}$ of acceptance sets of distribution functions by defining:

$$\Phi(P) := -\sup\{m \in \mathbb{R} \mid P \in \mathcal{A}^m\}.$$

In Section 3 we study the properties of such maps, we provide some specific examples and in particular we propose an interesting generalization of the classical notion of $V@R_\lambda$.

The key idea of our proposal - the definition of the $\Lambda V@R$ in Section 4 - arises from the consideration that in order to assess the risk of a financial position it is necessary to consider not only the probability λ of the loss, as in the case of the $V@R_\lambda$, but the dependence between such *probability* λ and the *amount* of the loss. In other terms, a risk prudent agent is willing to accept greater losses only with smaller probabilities. Hence, we replace the constant λ with a (increasing) function $\Lambda : \mathbb{R} \to [0,1]$ defined on losses, which we call *Probability/Loss function.* The balance between the probability and the amount of the losses is incorporated in the definition of the family of acceptance sets

$$\mathcal{A}^m := \{Q \in \mathcal{P}(\mathbb{R}) \mid Q(-\infty, x] \leq \Lambda(x),\ \forall x \leq m\},\ m \in \mathbb{R}.$$

If P_X is the distribution function of the random variable X, our new measure is defined by:

$$\Lambda V@R(P_X) := -\sup\{m \in \mathbb{R} \mid P(X \leq x) \leq \Lambda(x),\ \forall x \leq m\}.$$

As a consequence, the acceptance sets $\mathcal{A}^m$ are not obtained by the translation of $\mathcal{A}^0$ which implies that the map is not any more translation invariant. However, the similar property

$$\Lambda V@R(P_{X+\alpha}) = \Lambda^\alpha V@R(P_X) - \alpha,$$

where $\Lambda^\alpha(x) = \Lambda(x+\alpha)$, holds true and is discussed in Section 4.

The $V@R_\lambda$ and the worst case risk measure are special cases of the $\Lambda V@R$.

In Section 5 we address the dual representation of these maps. We choose to define the risk measures on the entire set $\mathcal{P}(\mathbb{R})$ and not only on its subset of probabilities having compact support. We endow $\mathcal{P}(\mathbb{R})$ with the $\sigma(\mathcal{P}(\mathbb{R}), C_b(\mathbb{R}))$ topology. The selection of this topology is also justified by the fact (see Proposition 5) that for monotone maps $\sigma(\mathcal{P}(\mathbb{R}), C_b(\mathbb{R})) - lsc$ is equivalent to continuity from below.

Except for $\Phi = +\infty$, we show that there are no *convex*, $\sigma(\mathcal{P}(\mathbb{R}), C_b(\mathbb{R})) - lsc$ translation invariant maps $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$. But there are many quasi-convex and $\sigma(\mathcal{P}(\mathbb{R}), C_b(\mathbb{R})) - lsc$ maps $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ that in addition are monotone and translation invariant, as for example the $V@R_\lambda$, the entropic risk measure and the worst case risk measure. This is another good motivation to adopt quasi convexity versus convexity.

Finally we provide the dual representation of quasi-convex, monotone and $\sigma(\mathcal{P}(\mathbb{R}), C_b(\mathbb{R})) - lsc$ maps $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ - *defined on the entire set* $\mathcal{P}(\mathbb{R})$ - and compute the dual representation of the risk measures associated to families of acceptance sets and consequently of the $\Lambda V@R$.

2 Law invariant Risk Measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $L^0 =: L^0(\Omega, \mathcal{F}, \mathbb{P})$ be the space of $\mathcal{F}$ measurable random variables that are $\mathbb{P}$ almost surely finite.
Any random variable $X \in L^0$ induces a probability measure P_X on $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ by $P_X(B) = \mathbb{P}(X^{-1}(B))$ for every Borel set $B \in \mathcal{B}_{\mathbb{R}}$. We refer to [1] Chapter 15 for a detailed study of the convex set $\mathcal{P} =: \mathcal{P}(\mathbb{R})$ of probability measures on $\mathbb{R}$. Here we just recall some basic notions: for any $X \in L^0$ we have $P_X \in \mathcal{P}$ so that we will associate to any random variable a unique element in $\mathcal{P}$. If $\mathbb{P}(X = x) = 1$ for some $x \in \mathbb{R}$ then P_X is the Dirach distribution δ_x that concentrates the mass in the point x.
A map $\rho : L \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, defined on given subset $L \subset L^0$, is law invariant if $X, Y \in L$ and $P_X = P_Y$ implies $\rho(X) = \rho(Y)$.

Therefore, when considering law invariant risk measures $\rho : L^0 \to \overline{\mathbb{R}}$ it is natural to shift the problem to the set $\mathcal{P}$ by defining the new map $\Phi : \mathcal{P} \to \overline{\mathbb{R}}$ as $\Phi(P_X) = \rho(X)$. This map Φ is well defined on the entire $\mathcal{P}$, since there exists a bi-injective relation between $\mathcal{P}$ and the quotient space $\frac{L^0}{\sim}$, where the equivalence is given by $X \sim_{\mathcal{D}} Y \Leftrightarrow P_X = P_Y$. However, $\mathcal{P}$ is only a convex set and the usual operations on $\mathcal{P}$ are not induced by those on L^0, namely $(P_X + P_Y)(A) = P_X(A) + P_Y(A) \neq P_{X+Y}(A)$, $A \in \mathcal{B}_{\mathbb{R}}$. Recall that the first order stochastic dominance on $\mathcal{P}$ is given by: $Q \preccurlyeq_{mon} P \Leftrightarrow F_P(x) \leq F_Q(x)$ for all $x \in \mathbb{R}$, where $F_P(x) = P(-\infty, x]$ and $F_Q(x) = Q(-\infty, x]$ are the distribution functions of $P, Q \in \mathcal{P}$. It will be more convenient to adopt on $\mathcal{P}(\mathbb{R})$ the opposite order relation:

$$P \preccurlyeq Q \quad \Leftrightarrow \quad Q \preccurlyeq_{mon} P \quad \Leftrightarrow \quad F_P(x) \leq F_Q(x) \quad \text{for all } x \in \mathbb{R}.$$

The financial intuition is natural: the risky position X has a lower level of risk with respect to $\preccurlyeq$ since its distribution $F_X(x)$ converges faster to zero as $x \to -\infty$ and slower to one as $x \to +\infty$. In this way F_X concentrates more probability on higher values of x. Notice that $X \geq Y$ $\mathbb{P}$-a.s. implies $P_X \preccurlyeq P_Y$ and this motivates the increasing (instead of the usual decreasing) monotonicity assumption in the following definition.

Definition 1 *A Risk Measure on $\mathcal{P}(\mathbb{R})$ is a map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ such that:*

(Mon) *Φ is monotone increasing: $P \preccurlyeq Q$ implies $\Phi(P) \leq \Phi(Q)$;*

(QCo) *Φ is quasi-convex: $\Phi(\lambda P + (1-\lambda)Q) \leq \Phi(P) \vee \Phi(Q)$, $\lambda \in [0, 1]$.*

Quasiconvexity can be equivalently reformulated in terms of sublevel sets: a map Φ is quasi-convex if for every $c \in \mathbb{R}$ the set $\mathcal{A}_c = \{P \in \mathcal{P} \mid \Phi(P) \leq c\}$ is convex. As recalled in [17] this notion of convexity is different from the one given for random variables (as in [8]) because it does not concern diversification of financial positions. A natural interpretation in terms of compound lotteries is the following: whenever two probability measures P and Q are acceptable at some level c and $\lambda \in [0, 1]$ is a probability, then the compound lottery $\lambda P + (1-\lambda)Q$,

which randomizes over P and Q, is also acceptable at the same level.
In terms of random variables (namely X, Y which induce P_X, P_Y) the randomized probability $\lambda P_X + (1-\lambda)P_Y$ will correspond to some random variable $Z \neq \lambda X + (1-\lambda)Y$ so that the diversification is realized at the level of distribution and not at the level of portfolio selection.

As suggested by [17], we define the translation operator T_m on the set $\mathcal{P}(\mathbb{R})$ by: $T_m P(-\infty, x] = P(-\infty, x-m]$, for every $m \in \mathbb{R}$. Equivalently, if P_X is the probability distribution of a random variable X we define the translation operator as $T_m P_X = P_{X+m}$, $m \in \mathbb{R}$. As a consequence we map the distribution $F_X(x)$ into $F_X(x-m)$. Notice that $T_m P \preccurlyeq P$ for any $m > 0$.

Definition 2 *If $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ is a risk measure on $\mathcal{P}$, we say that*

(TrI) *Φ is translation invariant if $\Phi(T_m P) = \Phi(P) - m$ for any $m \in \mathbb{R}$.*

Notice that (TrI) corresponds exactly to the notion of cash additivity for risk measures defined on a space of random variables as introduced in [2]. It is well known (see [5]) that for maps defined on random variables, quasiconvexity and cash additivity imply convexity. However, in the context of distributions (QCo) and (TrI) do not imply convexity of the map Φ, as can be shown with the simple examples of the $V@R$ and the worst case risk measure ρ_w (see the examples in Section 3.1).

The set $\mathcal{P}(\mathbb{R})$ spans the space $ca(\mathbb{R}) := \{\mu \text{ signed measure } \mid V_\mu < +\infty\}$ of all signed measures of bounded variations on $\mathbb{R}$. $ca(\mathbb{R})$ (or simply ca) endowed with the norm $V_\mu = \sup\{\sum_{i=1}^{n} |\mu(A_i)| \text{ s.t. } \{A_1, ..., A_n\} \text{ partition of } \mathbb{R}\}$ is a norm complete and an AL-space (see [1] paragraph 10.11).

Let $C_b(\mathbb{R})$ (or simply C_b) be the space of bounded continuous function $f : \mathbb{R} \to \mathbb{R}$. We endow $ca(\mathbb{R})$ with the weak* topology $\sigma(ca, C_b)$. The dual pairing $\langle \cdot, \cdot \rangle : C_b \times ca \to \mathbb{R}$ is given by $\langle f, \mu \rangle = \int f d\mu$ and the function $\mu \mapsto \int f d\mu$ ($\mu \in ca$) is $\sigma(ca, C_b)$ continuous. Notice that $\mathcal{P}$ is a $\sigma(ca, C_b)$-closed convex subset of ca (p. 507 in [1]) so that $\sigma(\mathcal{P}, C_b)$ is the relativization of $\sigma(ca, C_b)$ to $\mathcal{P}$ and any $\sigma(\mathcal{P}, C_b)$-closed subset of $\mathcal{P}$ is also $\sigma(ca, C_b)$-closed.

Even though $(ca, \sigma(ca, C_b))$ is not metrizable in general, its subset $\mathcal{P}$ is separable and metrizable (see [1], Th.15.12) and therefore when dealing with convergence in $\mathcal{P}$ we may work with sequences instead of nets.

For every real function F we denote by $\mathcal{C}(F)$ the set of points in which the function F is continuous.

Theorem 3 *([15] Theorem 2, p.314)) Suppose that P_n, $P \in \mathcal{P}$. Then $P_n \overset{\sigma(\mathcal{P}, C_b)}{\longrightarrow} P$ if and only if $F_{P_n}(x) \to F_P(x)$ for every $x \in \mathcal{C}(F_P)$.*

A sequence of probabilities $\{P_n\} \subset \mathcal{P}$ is increasing, denoted with $P_n \uparrow$, if $F_{P_n}(x) \leq F_{P_{n+1}}(x)$ for all $x \in \mathbb{R}$ and all n.

Definition 4 *Suppose that P_n, $P \in \mathcal{P}$. We say that $P_n \uparrow P$ whenever $P_n \uparrow$ and $F_{P_n}(x) \uparrow F_P(x)$ for every $x \in \mathcal{C}(F_P)$. We say that*

(CfB) Φ *is continuous from below if* $P_n \uparrow P$ *implies* $\Phi(P_n) \uparrow \Phi(P)$.

Proposition 5 *Let* $\Phi : \mathcal{P} \to \overline{\mathbb{R}}$ *be (Mon). Then the following are equivalent:*
Φ *is* $\sigma(\mathcal{P}, C_b)$*-lower semicontinuous*
Φ *is continuous from below.*

Proof. Let Φ be $\sigma(\mathcal{P}, C_b)$-lower semicontinuous and suppose that $P_n \uparrow P$. Then $F_{P_n}(x) \uparrow F_P(x)$ for every $x \in \mathcal{C}(F_P)$ and we deduce from Theorem 3 that $P_n \overset{\sigma(\mathcal{P},C_b)}{\longrightarrow} P$. (Mon) implies $\Phi(P_n) \uparrow$ and $k := \lim_n \Phi(P_n) \leq \Phi(P)$. The lower level set $A_k = \{Q \in \mathcal{P} \mid \Phi(Q) \leq k\}$ is $\sigma(\mathcal{P}, C_b)$ closed and, since $P_n \in A_k$, we also have $P \in A_k$, i.e. $\Phi(P) = k$, and Φ is continuous from below.

Conversely, suppose that Φ is continuous from below. As $\mathcal{P}$ is metrizable we may work with sequences instead of nets. For $k \in \mathbb{R}$ consider $A_k = \{P \in \mathcal{P} \mid \Phi(P) \leq k\}$ and a sequence $\{P_n\} \subseteq A_k$ such that $P_n \overset{\sigma(\mathcal{P},C_b)}{\longrightarrow} P \in \mathcal{P}$. We need to show that $P \in A_k$. Lemma 6 shows that each $F_{Q_n} := (\inf_{m \geq n} F_{P_m}) \wedge F_P$ is the distribution function of a probability measure $Q_n \in \mathcal{P}$. Notice that $F_{Q_n} \leq F_{P_n}$ and $Q_n \uparrow$. From $P_n \overset{\sigma(\mathcal{P},C_b)}{\longrightarrow} P$ and the definition of Q_n, we deduce that $F_{Q_n}(x) \uparrow F_P(x)$ for every $x \in \mathcal{C}(F_P)$ so that $Q_n \uparrow P$. From (Mon) and $Q_n \preccurlyeq P_n$, we get $\Phi(Q_n) \leq \Phi(P_n)$. From (CfB) then: $\Phi(P) = \lim_n \Phi(Q_n) \leq \liminf_n \Phi(P_n) \leq k$. Thus $P \in A_k$. ■

Lemma 6 *For every* $P_n \overset{\sigma(\mathcal{P},C_p)}{\longrightarrow} P$ *we have that*

$$F_{Q_n} := \inf_{m \geq n} F_{P_m} \wedge F_P,\ n \in \mathbb{N},$$

is a distribution function associated to a probability measure $Q_n \in \mathcal{P}$.

Proof. For each n, F_{Q_n} is increasing and $\lim_{x \to -\infty} F_{Q_n}(x) = 0$. Moreover for real valued maps right continuity and upper semicontinuity are equivalent. Since the inf-operator preserves upper semicontinuity we can conclude that F_{Q_n} is right continuous for every n. Now we have to show that for each n, $\lim_{x \to +\infty} F_{Q_n}(x) = 1$. By contradiction suppose that, for some n, $\lim_{x \to +\infty} F_{Q_n}(x) = \lambda < 1$. We can choose a sequence $\{x_k\}_k \subset \mathbb{R}$ with $x_k \in \mathcal{C}(F_P)$, $x_k \uparrow +\infty$. In particular $F_{Q_n}(x_k) \leq \lambda$ for all k and $F_P(x_k) > \lambda$ definitively, say for all $k \geq k_0$. We can observe that since $x_k \in \mathcal{C}(F_P)$ we have, for all $k \geq k_0$, $\inf_{m \geq n} F_{P_m}(x_k) < \lim_{m \to +\infty} F_{P_m}(x_k) = F_P(x_k)$. This means that the infimum is attained for some index $m(k) \in \mathbb{N}$, i.e. $\inf_{m \geq n} F_{P_m}(x_k) = F_{P_{m(k)}}(x_k)$, for all $k \geq k_0$. Since $P_{m(k)}(-\infty, x_k] = F_{P_{m(k)}}(x_k) \leq \lambda$ then $P_{m(k)}(x_k, +\infty) \geq 1 - \lambda$ for $k \geq k_0$. We have two possibilities. Either the set $\{m(k)\}_k$ is bounded or $\overline{\lim}_k m(k) = +\infty$. In the first case, we know that the number of $m(k)$'s is finite. Among these $m(k)$'s we can find at least one $\overline{m}$ and a subsequence $\{x_h\}_h$ of $\{x_k\}_k$ such that $x_h \uparrow +\infty$ and $P_{\overline{m}}(x_h, +\infty) \geq 1 - \lambda$ for every h. We then conclude that

$$\lim_{h \to +\infty} P_{\overline{m}}(x_h, +\infty) \geq 1 - \lambda$$

and this is a contradiction. If $\overline{\lim}_k m(k) = +\infty$, fix $\overline{k} \geq k_0$ such that $P(x_{\overline{k}}, +\infty) < 1 - \lambda$ and observe that for every $k > \overline{k}$

$$P_{m(k)}(x_{\overline{k}}, +\infty) \geq P_{m(k)}(x_k, +\infty) \geq 1 - \lambda.$$

Take a subsequence $\{m(h)\}_h$ of $\{m(k)\}_k$ such that $m(h) \uparrow +\infty$. Then:

$$\lim_{h\to\infty} \inf P_{m(h)}(x_{\overline{k}}, +\infty) \geq 1 - \lambda > P(x_{\overline{k}}, +\infty),$$

which contradicts the weak convergence $P_n \overset{\sigma(\mathcal{P},C_b)}{\longrightarrow} P$. ■

Example 7 (The certainty equivalent) *It is very simple to build risk measures on $\mathcal{P}(\mathbb{R})$. Take any continuous, bounded from below and strictly decreasing function $f : \mathbb{R} \to \mathbb{R}$. Then the map $\Phi_f : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ defined by:*

$$\Phi_f(P) := -f^{-1}\left(\int f dP\right)$$

is a Risk Measure on $\mathcal{P}(\mathbb{R})$. It is also easy to check that Φ_f is (CFB) and therefore $\sigma(\mathcal{P}, C_b)-$l.s.c. Notice that Proposition 22 will then imply that Φ_f can not be convex. By selecting the function $f(x) = e^{-x}$ we obtain $\Phi_f(P) = \ln\left(\int \exp(-x) dF_P(x)\right)$, which is in addition (TrI). Its associated risk measure $\rho : L^0 \to \mathbb{R} \cup \{+\infty\}$ defined on random variables, $\rho(X) = \Phi_f(P_X) = \ln\left(Ee^{-X}\right)$, is the Entropic Risk Measure. In Section 5 we will see more examples based on this construction.

3 A remarkable class of risk measures on $\mathcal{P}(\mathbb{R})$

Given a family $\{F_m\}_{m\in\mathbb{R}}$ of functions $F_m : \mathbb{R} \to [0, 1]$, we consider the associated sets of probability measures

$$\mathcal{A}^m := \{Q \in \mathcal{P} \mid F_Q \leq F_m\} \tag{1}$$

and the associated map $\Phi : \mathcal{P} \to \overline{\mathbb{R}}$ defined by

$$\Phi(P) := -\sup\{m \in \mathbb{R} \mid P \in \mathcal{A}^m\}. \tag{2}$$

We assume hereafter that for each $P \in \mathcal{P}$ there exists m such that $P \notin \mathcal{A}^m$ so that $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$.

Definition 8 *A monotone decreasing family of sets $\{\mathcal{A}^m\}_{m\in\mathbb{R}}$ contained in $\mathcal{P}$ is* left continuous *in m if*

$$\mathcal{A}^m =: \bigcap_{\varepsilon>0} \mathcal{A}^{m-\varepsilon}$$

In particular it is left continuous *if it is* left continuous *in m for every $m \in \mathbb{R}$.*

Lemma 9 *Let $\{F_m\}_{m\in\mathbb{R}}$ be a family of functions $F_m : \mathbb{R} \to [0,1]$ and $\mathcal{A}^m$ be the set defined in (1). Then:*

1. *If, for every $x \in \mathbb{R}$, $F_{\boldsymbol{\cdot}}(x)$ is decreasing (w.r.t. m) then the family $\{\mathcal{A}^m\}$ is monotone decreasing: $\mathcal{A}^m \subseteq \mathcal{A}^n$ for any level $m \geq n$,*
2. *For any m, $\mathcal{A}^m$ is convex and satisfies: $Q \preceq P \in \mathcal{A}^m \Rightarrow Q \in \mathcal{A}^m$*
3. *If, for every $m \in \mathbb{R}$, $F_m(x)$ is right continuous w.r.t. x then $\mathcal{A}^m$ is $\sigma(\mathcal{P}, C_b)-$closed,*
4. *Suppose that, for every $x \in \mathbb{R}$, $F_m(x)$ is decreasing w.r.t. m. If $F_m(x)$ is left continuous w.r.t. m, then the family $\{\mathcal{A}^m\}$ is left continuous.*
5. *Suppose that, for every $x \in \mathbb{R}$, $F_m(x)$ is decreasing w.r.t. m and that, for every $m \in \mathbb{R}$, $F_m(x)$ is right continuous and increasing w.r.t. x and $\lim_{x\to+\infty} F_m(x) = 1$. If the family $\{\mathcal{A}^m\}$ is left continuous in m then $F_m(x)$ is left continuous in m.*

Proof. 1. If $Q \in \mathcal{A}^m$ and $m \geq n$ then $F_Q \leq F_m \leq F_n$, i.e. $Q \in \mathcal{A}^n$.

2. Let $Q, P \in \mathcal{A}^m$ and $\lambda \in [0,1]$. Consider the convex combination $\lambda Q + (1-\lambda)P$ and notice that

$$F_{\lambda Q+(1-\lambda)P} \leq F_Q \vee F_P \leq F_m,$$

as $F_P \leq F_m$ and $F_Q \leq F_m$. Then $\lambda Q + (1-\lambda)P \in \mathcal{A}^m$.

3. Let $Q_n \in A^m$ and $Q \in \mathcal{P}$ satisfy $Q_n \overset{\sigma(\mathcal{P},C_b)}{\to} Q$. By Theorem 3 we know that $F_{Q_n}(x) \to F_Q(x)$ for every $x \in \mathcal{C}(F_Q)$. For each n, $F_{Q_n} \leq F_m$ and therefore $F_Q(x) \leq F_m(x)$ for every $x \in \mathcal{C}(F_Q)$. By contradiction, suppose that $Q \notin \mathcal{A}^m$. Then there exists $\bar{x} \notin \mathcal{C}(F_Q)$ such that $F_Q(\bar{x}) > F_m(\bar{x})$. By right continuity of F_Q for every $\varepsilon > 0$ we can find a right neighborhood $[\bar{x}, \bar{x} + \delta(\varepsilon))$ such that

$$|F_Q(x) - F_Q(\bar{x})| < \varepsilon \quad \forall\, x \in [\bar{x}, \bar{x} + \delta(\varepsilon))$$

and we may require that $\delta(\varepsilon) \downarrow 0$ if $\varepsilon \downarrow 0$.Notice that for each $\varepsilon > 0$ we can always choose an $x_\varepsilon \in (\bar{x}, \bar{x} + \delta(\varepsilon))$ such that $x_\varepsilon \in \mathcal{C}(F_Q)$. For such an x_ε we deduce that

$$F_m(\bar{x}) < F_Q(\bar{x}) < F_Q(x_\varepsilon) + \varepsilon \leq F_m(x_\varepsilon) + \varepsilon.$$

This leads to a contradiction since if $\varepsilon \downarrow 0$ we have that $x_\varepsilon \downarrow \bar{x}$ and thus by right continuity of F_m

$$F_m(\bar{x}) < F_Q(\bar{x}) \leq F_m(\bar{x}).$$

4. By assumption we know that $F_{m-\varepsilon}(x) \downarrow F_m(x)$ as $\varepsilon \downarrow 0$, for all $x \in \mathbb{R}$. By item 1, we know that $\mathcal{A}^m \subseteq \bigcap_{\varepsilon>0} \mathcal{A}^{m-\varepsilon}$. By contradiction we suppose that

$$\bigcap_{\varepsilon>0} \mathcal{A}^{m-\varepsilon} \supsetneqq \mathcal{A}^m,$$

so that there will exist $Q \in \mathcal{P}$ such that $F_Q \le F_{m-\varepsilon}$ for every $\varepsilon > 0$ but $F_Q(\overline{x}) > F_m(\overline{x})$ for some $\overline{x} \in \mathbb{R}$. Set $\delta = F_Q(\overline{x}) - F_m(\overline{x})$ so that $F_Q(\overline{x}) > F_m(\overline{x}) + \frac{\delta}{2}$. Since $F_{m-\varepsilon} \downarrow F_m$ we may find $\overline{\varepsilon} > 0$ such that $F_{m-\overline{\varepsilon}}(\overline{x}) - F_m(\overline{x}) < \frac{\delta}{2}$. Thus $F_Q(\overline{x}) \le F_{m-\varepsilon}(\overline{x}) < F_m(\overline{x}) + \frac{\delta}{2}$ and this is a contradiction.

5. Assume that $\mathcal{A}^{m-\varepsilon} \downarrow \mathcal{A}^m$. Define $F(x) := \lim_{\varepsilon \downarrow 0} F_{m-\varepsilon}(x) = \inf_{\varepsilon > 0} F_{m-\varepsilon}(x)$ for all $x \in \mathbb{R}$. Then $F : \mathbb{R} \to [0,1]$ is increasing, right continuous (since the inf preserves this property). Notice that for every $\varepsilon > 0$ we have $F_{m-\varepsilon} \ge F \ge F_m$ and then $\mathcal{A}^{m-\varepsilon} \supseteq \{Q \in \mathcal{P} \mid F_Q \le F\} \supseteq \mathcal{A}^m$ and $\lim_{x\to+\infty} F(x) = 1$. Necessarily we conclude $\{Q \in \mathcal{P} \mid F_Q \le F\} = \mathcal{A}^m$. By contradiction we suppose that $F(\overline{x}) > F_m(\overline{x})$ for some $\overline{x} \in \mathbb{R}$. Define $F_{\overline{Q}} : \mathbb{R} \to [0,1]$ by: $F_{\overline{Q}}(x) = F(x)\mathbf{1}_{[\overline{x},+\infty)}(x)$. The above properties of F guarantees that $F_{\overline{Q}}$ is a distribution function of a corresponding probability measure $\overline{Q} \in \mathcal{P}$, and since $F_{\overline{Q}} \le F$, we deduce $\overline{Q} \in \mathcal{A}^m$, but $F_{\overline{Q}}(\overline{x}) > F_m(\overline{x})$ and this is a contradiction. ∎

Lemma 10 *Let $\{F_m\}_{m\in\mathbb{R}}$ be a family of functions $F_m : \mathbb{R} \to [0,1]$ and Φ be the associated map defined in (2). Then:*

1. *The map Φ is (Mon) on $\mathcal{P}$.*
2. *If, for every $x \in \mathbb{R}$, $F_{\bullet}(x)$ is decreasing (w.r.t. m) then Φ is (QCo) on $\mathcal{P}$.*
3. *If, for every $x \in \mathbb{R}$, $F_{\bullet}(x)$ is left continuous and decreasing (w.r.t. m) and if, for every $m \in \mathbb{R}$, $F_m(\bullet)$ is right continuous (w.r.t. x) then*

$$A_m := \{Q \in \mathcal{P} \mid \Phi(Q) \le m\} = \mathcal{A}^{-m}, \ \forall m, \tag{3}$$

and Φ is $\sigma(\mathcal{P}, C_b)-$lower-semicontinuous.

Proof. 1. From $Q \preceq P$ we have $F_Q \le F_P$ and

$$\{m \in \mathbb{R} \mid F_P \le F_m\} \subseteq \{m \in \mathbb{R} \mid F_Q \le F_m\},$$

which implies $\Phi(Q) \le \Phi(P)$.

2. We show that $Q_1, Q_2 \in \mathcal{P}$, $\Phi(Q_1) \le n$ and $\Phi(Q_2) \le n$ imply that $\Phi(\lambda Q_1 + (1-\lambda)Q_2) \le n$, that is

$$\sup\left\{m \in \mathbb{R} \mid F_{\lambda Q_1 + (1-\lambda)Q_2} \le F_m\right\} \ge -n.$$

By definition of the supremum, $\forall \varepsilon > 0\ \exists m_i$ s.t. $F_{Q_i} \le F_{m_i}$ and $m_i > -\Phi(Q_i) - \varepsilon \ge -n - \varepsilon$. Then $F_{Q_i} \le F_{m_i} \le F_{-n-\varepsilon}$, as $\{F_m\}$ is a decreasing family. Therefore $\lambda F_{Q_1} + (1-\lambda)F_{Q_2} \le F_{-n-\varepsilon}$ and $-\Phi(\lambda Q_1 + (1-\lambda)Q_2\lambda) \ge -n - \varepsilon$. As this holds for any $\varepsilon > 0$, we conclude that Φ is quasi-convex.

3. The fact that $\mathcal{A}^{-m} \subseteq A_m$ follows directly from the definition of Φ, as if $Q \in \mathcal{A}^{-m}$

$$\Phi(Q) := -\sup\{n \in \mathbb{R} \mid Q \in \mathcal{A}^n\} = \inf\left\{n \in \mathbb{R} \mid Q \in \mathcal{A}^{-n}\right\} \le m.$$

We have to show that $A_m \subseteq \mathcal{A}^{-m}$. Let $Q \in A_m$. Since $\Phi(Q) \leq m$, for all $\varepsilon > 0$ there exists m_0 such that $m+\varepsilon > -m_0$ and $F_Q \leq F_{m_0}$. Since $F_{\cdot}(x)$ is decreasing (w.r.t. m) we have that $F_Q \leq F_{-m-\varepsilon}$, therefore $Q \in \mathcal{A}^{-m-\varepsilon}$ for any $\varepsilon > 0$. By the left continuity in m of $F_{\cdot}(x)$, we know that$\{\mathcal{A}^m\}$ is left continuous (Lemma 9, item 4) and so: $Q \in \bigcap_{\epsilon>0} \mathcal{A}^{-m-\varepsilon} = \mathcal{A}^{-m}$.

From the assumption that $F_m(\cdot)$ is right continuous (w.r.t. x) and Lemma 9 item 3, we already know that $\mathcal{A}^m$ is $\sigma(\mathcal{P}, C_b)-$closed, for any $m \in \mathbb{R}$, and therefore the lower level sets $A_m = \mathcal{A}^{-m}$ are $\sigma(\mathcal{P}, C_b)-$closed and Φ is $\sigma(\mathcal{P}, C_b)-$lower-semicontinuous. ■

Definition 11 *A family $\{F_m\}_{m\in\mathbb{R}}$ of functions $F_m : \mathbb{R} \to [0,1]$ is* feasible *if*

- *For any $P \in \mathcal{P}$ there exists m such that $P \notin \mathcal{A}^m$*
- *For every $m \in \mathbb{R}$, $F_m(\cdot)$ is right continuous (w.r.t. x)*
- *For every $x \in \mathbb{R}$, $F_{\cdot}(x)$ is decreasing and left continuous (w.r.t. m).*

From Lemmas 9 and 10 we immediately deduce:

Proposition 12 *Let $\{F_m\}_{m\in\mathbb{R}}$ be a feasible family. Then the associated family $\{\mathcal{A}^m\}_{m\in\mathbb{R}}$ is monotone decreasing and left continuous and each set $\mathcal{A}^m$ is convex and $\sigma(\mathcal{P}, C_b)-$closed. The associated map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ is well defined, (Mon), (Qco) and $\sigma(\mathcal{P}, C_b)-$l.s.c.*

Remark 13 *Let $\{F_m\}_{m\in\mathbb{R}}$ be a feasible family. If there exists an $\overline{m}$ such that $\lim_{x\to+\infty} F_{\overline{m}}(x) < 1$ then $\lim_{x\to+\infty} F_m(x) < 1$ for every $m \geq \overline{m}$ and then $\mathcal{A}^m = \emptyset$ for every $m \geq \overline{m}$. Obviously if an acceptability set is empty then it does not contribute to the computation of the risk measure defined in (2). For this reason we will always consider w.l.o.g. a class $\{F_m\}_{m\in\mathbb{R}}$ such that $\lim_{x\to+\infty} F_m(x) = 1$ for every m.*

3.1 Examples

As explained in the introduction, we define a family of risk measures employing a Probability/Loss function Λ. Fix the *right continuous* function $\Lambda : \mathbb{R} \to [0,1]$ and define the family $\{F_m\}_{m\in\mathbb{R}}$ of functions $F_m : \mathbb{R} \to [0,1]$ by

$$F_m(x) := \Lambda(x)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x). \tag{4}$$

It is easy to check that if $\sup_{x\in\mathbb{R}} \Lambda(x) < 1$ then the family $\{F_m\}_{m\in\mathbb{R}}$ is feasible and therefore, by Proposition 12, the associated map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ is well defined, (Mon), (Qco) and $\sigma(\mathcal{P}, C_b)-$l.s.c.

Example 14 *When $\sup_{x\in R} \Lambda(x) = 1$, Φ may take the value $-\infty$. The extreme case is when, in the definition of the family (4), the function Λ is equal to the constant one, $\Lambda(x) = 1$, and so: $\mathcal{A}^m = \mathcal{P}$ for all m and $\Phi = -\infty$.*

Example 15 ***Worst case risk measure:*** $\Lambda(x) = 0$**.**

Take in the definition of the family (4) the function Λ *to be equal to the constant zero*: $\Lambda(x) = 0$. *Then:*

$$\begin{aligned} F_m(x) &:= \mathbf{1}_{[m,+\infty)}(x) \\ \mathcal{A}^m &:= \{Q \in \mathcal{P} \mid F_Q \leq F_m\} = \{Q \in \mathcal{P} \mid Q \preccurlyeq \delta_m\} \\ \Phi_w(P) &:= -\sup\{m \mid P \in \mathcal{A}^m\} = -\sup\{m \mid P \preccurlyeq \delta_m\} = -\inf_{x\in\mathbb{R}}(F_P(x)) \end{aligned}$$

so that, if $X \in L^0$ *has distribution function* P_X*,*

$$\Phi_w(P_X) = -\sup\{m \in \mathbb{R} \mid P_X \preccurlyeq \delta_m\} = -ess\inf(X) := \rho_w(X)$$

coincide with the worst case risk measure ρ_w*. As the family* $\{F_m\}$ *is feasible,* $\Phi_w : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ *is (Mon), (Qco) and* $\sigma(\mathcal{P}, C_b)-$*l.s.c. In addition, it also satisfies (TrI).*

Even though $\rho_w : L^0 \to \mathbb{R} \cup \{\infty\}$ *is convex, as a map defined on random variables, the corresponding* $\Phi_w : \mathcal{P} \to \mathbb{R}\cup\{\infty\}$*, as a map defined on distribution functions, is not convex, but it is quasi-convex and concave. Indeed, let* $P \in \mathcal{P}$ *and, since* $F_P \geq 0$*, we set:*

$$-\Phi_w(P) = \inf(F_P) := \sup\{x \in \mathbb{R} : F_P(x) = 0\}.$$

If F_1*,* F_2 *are two distribution functions corresponding to* P_1*,* $P_2 \in \mathcal{P}$ *then for all* $\lambda \in (0,1)$ *we have:*

$$\inf(\lambda F_1 + (1-\lambda)F_2) = \min(\inf(F_1), \inf(F_2)) \leq \lambda \inf(F_1) + (1-\lambda)\inf(F_2)$$

and therefore, for all $\lambda \in [0,1]$

$$\min(\inf(F_1), \inf(F_2)) \leq \inf(\lambda F_1 + (1-\lambda)F_2) \leq \lambda \inf(F_1) + (1-\lambda)\inf(F_2).$$

Example 16 ***Value at Risk*** $V@R_\lambda$*:* $\Lambda(x) := \lambda \in (0,1)$.

Take in the definition of the family (4) the function Λ *to be equal to the constant* λ, $\Lambda(x) = \lambda \in (0,1)$*. Then*

$$\begin{aligned} F_m(x) &:= \lambda \mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x) \\ \mathcal{A}^m &:= \{Q \in \mathcal{P} \mid F_Q \leq F_m\} \\ \Phi_{V@R_\lambda}(P) &:= -\sup\{m \in \mathbb{R} \mid P \in \mathcal{A}^m\} \end{aligned}$$

If the random variable $X \in L^0$ *has distribution function* P_X *and* $q_X^+(\lambda) = \sup\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \leq \lambda\}$ *is the right continuous inverse of* P_X *then*

$$\begin{aligned} \Phi_{V@R_\lambda}(P_X) &= -\sup\{m \mid P_X \in \mathcal{A}^m\} \\ &= -\sup\{m \mid \mathbb{P}(X \leq x) \leq \lambda \ \forall x < m\} \\ &= -\sup\{m \mid \mathbb{P}(X \leq m) \leq \lambda\} \\ &= -q_X^+(\lambda) := V@R_\lambda(X) \end{aligned}$$

coincide with the Value At Risk of level $\lambda \in (0,1)$. As the family $\{F_m\}$ is feasible, $\Phi_{V@R_\lambda} : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ is (Mon), (Qco), $\sigma(\mathcal{P}, C_b)$-l.s.c. In addition, it also satisfies (TrI).

As well known, $V@R_\lambda : L^0 \to \mathbb{R} \cup \{\infty\}$ is not quasi-convex, as a map defined on random variables, even though the corresponding $\Phi_{V@R_\lambda} : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$, as a map defined on distribution functions, is quasi-convex (see [6] for a discussion on this issue).

Example 17 *Fix the family $\{\Lambda_m\}_{m\in\mathbb{R}}$ of functions $\Lambda_m : \mathbb{R} \to [0,1]$ such that for every $m \in \mathbb{R}$, $\Lambda_m(\cdot)$ is right continuous (w.r.t. x) and for every $x \in \mathbb{R}$, $\Lambda_\cdot(x)$ is decreasing and left continuous (w.r.t. m). Define the family $\{F_m\}_{m\in\mathbb{R}}$ of functions $F_m : \mathbb{R} \to [0,1]$ by*

$$F_m(x) := \Lambda_m(x)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x). \tag{5}$$

It is easy to check that if $\sup_{x\in\mathbb{R}} \Lambda_{m_0}(x) < 1$, for some $m_0 \in \mathbb{R}$, then the family $\{F_m\}_{m\in\mathbb{R}}$ is feasible and therefore the associated map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ is well defined, (Mon), (Qco), $\sigma(\mathcal{P}, C_b)$-l.s.c.

4 On the $\Lambda V@R_\lambda$

We now propose a generalization of the $V@R_\lambda$ which appears useful for possible application whenever an agent is facing some ambiguity on the parameter λ, namely λ is given by some uncertain value in a confidence interval $[\lambda^m, \lambda^M]$, with $0 \le \lambda^m \le \lambda^M \le 1$. The $V@R_\lambda$ corresponds to case $\lambda^m = \lambda^M$ and one typical value is $\lambda^M = 0,05$.

We will distinguish two possible classes of agents:

Risk prudent Agents Fix the *increasing* right continuous function $\Lambda : \mathbb{R} \to [0,1]$, choose as in (4)

$$F_m(x) = \Lambda(x)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x)$$

and set $\lambda^m := \inf \Lambda \ge 0$, $\lambda^M := \sup \Lambda \le 1$. As the function Λ is increasing, we are assigning to a lower loss a lower probability. In particular given two possible choices Λ_1, Λ_2 for two different agents, the condition $\Lambda_1 \le \Lambda_2$ means that the agent 1 is more risk prudent than agent 2.
Set, as in (1), $\mathcal{A}^m = \{Q \in \mathcal{P} \mid F_Q \le F_m\}$ and define as in (2)

$$\Lambda V@R(P) := -\sup\{m \in \mathbb{R} \mid P \in \mathcal{A}^m\}.$$

Thus, in case of a random variable X

$$\Lambda V@R(P_X) := -\sup\{m \in \mathbb{R} \mid \mathbb{P}(X \le x) \le \Lambda(x),\ \forall x \le m\}.$$

In particular it can be rewritten as

$$\Lambda V@R(P_X) = -\inf\{x \in \mathbb{R} \mid \mathbb{P}(X \le x) > \Lambda(x)\}.$$

If both F_X and Λ are continuous $\Lambda V@R$ corresponds to the smallest intersection between the two curves.

In this section, we assume that

$$\lambda^M < 1.$$

Besides its obvious financial motivation, this request implies that the corresponding family F_m is feasible and so $\Lambda V@R(P) > -\infty$ for all $P \in \mathcal{P}$.

The feasibility of the family $\{F_m\}$ implies that the $\Lambda V@R : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ is well defined, (Mon), (QCo) and (CfB) (or equivalently $\sigma(\mathcal{P}, C_b)$-lsc) map.

Example 18 *One possible simple choice of the function Λ is represented by the step function:*

$$\Lambda(x) = \lambda^m \mathbf{1}_{(-\infty,\bar{x})}(x) + \lambda^M \mathbf{1}_{[\bar{x},+\infty)}(x)$$

The idea is that with a probability of λ^M we are accepting to loose at most $\bar{x}$. In this case we observe that:

$$\Lambda V@R(P) = \begin{cases} V@R_{\lambda^M}(P) & \text{if } V@R_{\lambda^m}(P) \leq -\bar{x} \\ V@R_{\lambda^m}(P) & \text{if } V@R_{\lambda^m}(P) > -\bar{x}. \end{cases}$$

Even though the $\Lambda V@R$ is continuous from below (proposition 12 and 5), it may not be continuous from above, as this example shows. For instance take $\bar{x} = 0$ and P_{X_n} induced by a sequence of uniformly distributed random variables $X_n \sim U\left[-\lambda^m - \frac{1}{n}, 1 - \lambda^m - \frac{1}{n}\right]$. We have $P_{X_n} \downarrow P_{U[-\lambda^m,1-\lambda^m]}$ but $\Lambda V@R(P_{X_n}) = -\frac{1}{n}$ for every n and $\Lambda V@R(P_{U[-\lambda^m,1-\lambda^m]}) = \lambda^M - \lambda^m$.

Remark 19 *(i) If $\lambda^m = 0$ the domain of $\Lambda V@R(P)$ is not the entire convex set $\mathcal{P}$. We have two possible cases*

- *$supp(\Lambda) = [x^*, +\infty)$: in this case $\Lambda V@R(P) = -\inf supp(F_P)$ for every $P \in \mathcal{P}$ such that $supp(F_P) \supset supp(\Lambda)$.*
- *$supp(\Lambda) = (-\infty, +\infty)$: in this case*

$$\Lambda V@R(P) = +\infty \qquad \text{for all } P \text{ such that } \lim_{x \to -\infty} \frac{F_P(x)}{\Lambda(x)} > 1$$

$$\Lambda V@R(P) < +\infty \qquad \text{for all } P \text{ such that } \lim_{x \to -\infty} \frac{F_P(x)}{\Lambda(x)} < 1$$

In the case $\lim_{x \to -\infty} \frac{F_P(x)}{\Lambda(x)} = 1$ both the previous behaviors might occur.
(ii) In case that $\lambda^m > 0$ then $\Lambda V@R(P) < +\infty$ for all $P \in \mathcal{P}$, so that $\Lambda V@R$ is finite valued.

We can prove a further structural property which is the counterpart of (TrI) for the $\Lambda V@R$. Let $\alpha \in \mathbb{R}$ any cash amount

$$\begin{aligned}\Lambda V@R(P_{X+\alpha}) &= -\sup\{m \mid \mathbb{P}(X+\alpha \leq x) \leq \Lambda(x),\ \forall x \leq m\} \\ &= -\sup\{m \mid \mathbb{P}(X \leq x-\alpha) \leq \Lambda(x),\ \forall x \leq m\} \\ &= -\sup\{m \mid \mathbb{P}(X \leq y) \leq \Lambda(y+\alpha),\ \forall y \leq m-\alpha\} \\ &= -\sup\{m+\alpha \mid \mathbb{P}(X \leq y) \leq \Lambda(y+\alpha),\ \forall y \leq m\} \\ &= \Lambda^{\alpha} V@R(P_X) - \alpha\end{aligned}$$

where $\Lambda^{\alpha}(x) = \Lambda(x+\alpha)$. We may conclude that if we add a sure positive (resp. negative) amount α to a risky position X then the risk decreases (resp. increases) of the value $-\alpha$, constrained to a lower (resp. higher) level of risk prudence described by $\Lambda^{\alpha} \geq \Lambda$ (resp. $\Lambda^{\alpha} \leq \Lambda$). For an arbitrary $P \in \mathcal{P}$ this property can be written as

$$\Lambda V@R(T_{\alpha}P) = \Lambda^{\alpha} V@R(P) - \alpha, \quad \forall\, \alpha \in \mathbb{R},$$

where $T_{\alpha}P(-\infty, x] = P(-\infty, x-\alpha]$.

Risk Seeking Agents Fix the *decreasing* right continuous function $\Lambda : \mathbb{R} \to [0,1]$, with $\inf \Lambda < 1$. Similarly as above, we define

$$F_m(x) = \Lambda(x)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x)$$

and the (Mon), (QCo) and (CfB) map

$$\Lambda V@R(P) := -\sup\{m \in \mathbb{R} \mid F_P \leq F_m\} = -\sup\{m \in \mathbb{R} \mid \mathbb{P}(X \leq m) \leq \Lambda(m)\}.$$

In this case, for eventual huge losses we are allowing the highest level of probability. As in the previous example let $\alpha \in \mathbb{R}$ and notice that

$$\Lambda V@R(P_{X+\alpha}) = \Lambda^{\alpha} V@R(P_X) - \alpha.$$

where $\Lambda^{\alpha}(x) = \Lambda(x+\alpha)$. The property is exactly the same as in the former example but here the interpretation is slightly different. If we add a sure positive (resp. negative) amount α to a risky position X then the risk decreases (resp. increases) of the value $-\alpha$, constrained to a lower (resp. higher) level of risk seeking since $\Lambda^{\alpha} \leq \Lambda$ (resp. $\Lambda^{\alpha} \geq \Lambda$).

Remark 20 *For a decreasing Λ, there is a simpler formulation - which will be used in Section 5.3 - of the $\Lambda V@R$ that is obtained replacing in F_m the function Λ with the line $\Lambda(m)$ for all $x < m$. Let*

$$\tilde{F}_m(x) = \Lambda(m)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x).$$

This family is of the type (5) and is feasible, provided the function Λ is continuous. For a decreasing Λ, it is evident that

$$\Lambda V@R(P) = \Lambda\widetilde{V}@R(P) := -\sup\left\{m \in \mathbb{R} \mid F_P \leq \tilde{F}_m\right\},$$

as the function Λ lies above the line $\Lambda(m)$ for all $x \leq m$.

5 Quasi-convex Duality

In literature we also find several results about the dual representation of law invariant risk measures. Kusuoka [13] contributed to the coherent case, while Frittelli and Rosazza [10] extended this result to the convex case. Jouini, Schachermayer and Touzi (2006) [12], in the convex case, and Svindland (2010) [14] in the quasi-convex case, showed that every law invariant risk measure is already weakly lower semicontinuous. Recently, Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2010) [5] provided a robust dual representation for law invariant quasi-convex risk measures, which has been extended to the dynamic case in [9].

In Sections 5.1 and 5.2 we will treat the general case of maps defined on $\mathcal{P}$, while in Section 5.3 we specialize these results to show the dual representation of maps associated to feasible families.

5.1 Reasons of the failure of the convex duality for Translation Invariant maps on $\mathcal{P}$

It is well known that the classical convex duality provided by the Fenchel-Moreau theorem guarantees the representation of convex and lower semicontinuous functions and therefore is very useful for the dual representation of convex risk measures (see [11]). For any map $\Phi : \mathcal{P} \to \mathbb{R}\cup\{\infty\}$ let Φ^* be the convex conjugate:

$$\Phi^*(f) := \sup_{Q\in\mathcal{P}} \left\{ \int f dQ - \Phi(Q) \right\},\ f \in C_b.$$

Applying the fact that $\mathcal{P}$ is a $\sigma(ca, C_b)$ closed convex subset of ca one can easily check that the following version of Fenchel-Moreau Theorem holds true for maps defined on $\mathcal{P}$.

Proposition 21 (Fenchel-Moreau) *Suppose that $\Phi : \mathcal{P} \to \overline{\mathbb{R}}$ is $\sigma(\mathcal{P}, C_b)-$lsc and convex. If $Dom(\Phi) \neq \varnothing$ then $Dom(\Phi^*) \neq \varnothing$ and*

$$\Phi(Q) = \sup_{f\in C_b} \left\{ \int f dQ - \Phi^*(f) \right\}.$$

One trivial example of a proper $\sigma(\mathcal{P}, C_b)-$lsc and convex map on $\mathcal{P}$ is given by $Q \to \int f dQ$, for some $f \in C_b$. But this map does not satisfy the (TrI) property. Indeed, we show that in the setting of risk measures defined on $\mathcal{P}$, weakly lower semicontinuity and convexity are incompatible with translation invariance.

Proposition 22 *For any map $\Phi : \mathcal{P} \to \overline{\mathbb{R}}$, if there exists a sequence $\{Q_n\}_n \subseteq \mathcal{P}$ such that $\lim_n \Phi(Q_n) = -\infty$ then $Dom(\Phi^*) = \varnothing$. Thus the only $\sigma(\mathcal{P}, C_b)-$lsc, convex and (TrI) map $\Phi : \mathcal{P} \to \overline{\mathbb{R}}$ is $\Phi = +\infty$.*

Proof. For any $f \in C_b(\mathbb{R})$

$$\Phi^*(f) = \sup_{Q\in\mathcal{P}} \left\{ \int f dQ - \Phi(Q) \right\} \geq \int f d(Q_n) - \Phi(Q_n) \geq \inf_{x\in\mathbb{R}} f(x) - \Phi(Q_n) \uparrow \infty.$$

Observe that a translation invariant map satisfies $\lim_n \Phi(T_n Q) = \lim_n \{\Phi(Q) - n\} = -\infty$, for any $Q \in Dom(\Phi)$. The thesis follows from Proposition 21 and what just proved, replacing Q_n with $T_n Q$. ∎

5.2 Quasi-convex duality

As described in the Examples in Section 3, the $\Phi_{V@R_\lambda}$ and Φ_w are proper, $\sigma(ca, C_b)-$lcs, quasi-convex (Mon) and (TrI) maps $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$. Therefore, the negative result outlined in Proposition 22 for the convex case can not be true in the quasi-convex setting.

We recall that one of the main contribution to quasi-convex duality comes from the dual representation by Volle [16].

Here we replicate this result and provide the dual representation of a $\sigma(\mathcal{P}, C_b)$ lsc quasi-convex maps defined on the entire set $\mathcal{P}$. The main difference is that our map Φ is defined on a convex subset of ca and not a vector space. But since $\mathcal{P}$ is $\sigma(ca, C_b)$-closed, the first part of the proof will match very closely the one given by Volle. In order to achieve the dual representation of $\sigma(\mathcal{P}, C_b)$ lsc risk measures $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ we will impose the monotonicity assumption of Φ and deduce that in the dual representation the supremum can be restricted to the set

$$C_b^- = \{f \in C_b \mid f \text{ is decreasing}\}.$$

This is natural as the first order stochastic dominance implies (see Th. 2.70 [8]) that

$$C_b^- = \left\{ f \in C_b \mid Q, P \in \mathcal{P} \text{ and } Q \preceq P \Rightarrow \int f dQ \leq \int f dP \right\}. \tag{6}$$

Notice that differently from [6] the following proposition does not require the extension of the risk map to the entire space $ca(\mathbb{R})$.

Proposition 23 *(i) Any $\sigma(\mathcal{P}, C_b)-$lsc and quasi-convex functional $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ can be represented as*

$$\Phi(P) = \sup_{f\in C_b} R\left(\int f dP, f \right) \tag{7}$$

where $R : \mathbb{R} \times C_b \to \overline{\mathbb{R}}$ is defined by

$$R(t, f) := \inf_{Q\in\mathcal{P}} \left\{ \Phi(Q) \mid \int f dQ \geq t \right\}. \tag{8}$$

(ii) If in addition Φ is monotone then (7) holds with C_b replaced by C_b^-.

Proof. We will use the fact that $\sigma(\mathcal{P}, C_b)$ is the relativization of $\sigma(ca, C_b)$ to the set $\mathcal{P}$. In particular the lower level sets will be $\sigma(ca, C_b)$-closed.

(i) By definition, for any $f \in C_b(\mathbb{R})$, $R\left(\int f dP, f\right) \leq \Phi(P)$ and therefore

$$\sup_{f \in C_b} R\left(\int f dP, f\right) \leq \Phi(P), \quad P \in \mathcal{P}.$$

Fix any $P \in \mathcal{P}$ and take $\varepsilon \in \mathbb{R}$ such that $\varepsilon > 0$. Then P does not belong to the $\sigma(ca, C_b)$-closed convex set

$$\mathcal{C}_\varepsilon := \{Q \in \mathcal{P} : \Phi(Q) \leq \Phi(P) - \varepsilon\}$$

(if $\Phi(P) = +\infty$, replace the set $\mathcal{C}_\varepsilon$ with $\{Q \in \mathcal{P} : \Phi(Q) \leq M\}$, for any M). By the Hahn Banach theorem there exists a continuous linear functional that strongly separates P and $\mathcal{C}_\varepsilon$, i.e. there exists $\alpha \in \mathbb{R}$ and $f_\varepsilon \in C_b$ such that

$$\int f_\varepsilon dP > \alpha > \int f_\varepsilon dQ \quad \text{for all } Q \in \mathcal{C}_\varepsilon. \tag{9}$$

Hence:

$$\left\{Q \in \mathcal{P} : \int f_\varepsilon dP \leq \int f_\varepsilon dQ\right\} \subseteq (\mathcal{C}_\varepsilon)^C = \{Q \in \mathcal{P} : \Phi(Q) > \Phi(P) - \varepsilon\} \tag{10}$$

and

$$\begin{aligned}\Phi(P) &\geq \sup_{f \in C_b} R\left(\int f dP, f\right) \geq R\left(\int f_\varepsilon dP, f_\varepsilon\right) \\ &= \inf\left\{\Phi(Q) \mid Q \in \mathcal{P} \text{ such that } \int f_\varepsilon dP \leq \int f_\varepsilon dQ\right\} \\ &\geq \inf\{\Phi(Q) \mid Q \in \mathcal{P} \text{ satisfying } \Phi(Q) > \Phi(P) - \varepsilon\} \geq \Phi(P) - \varepsilon. \end{aligned} \tag{11}$$

(ii) We furthermore assume that Φ is monotone. As shown in (i), for every $\varepsilon > 0$ we find f_ε such that (9) holds true. We claim that there exists $g_\varepsilon \in C_b^-$ satisfying:

$$\int g_\varepsilon dP > \alpha > \int g_\varepsilon dQ \quad \text{for all } Q \in \mathcal{C}_\varepsilon. \tag{12}$$

and then the above argument (in equations (9)-(11)) implies the thesis.

We define the decreasing function

$$g_\varepsilon(x) =: \sup_{y \geq x} f_\varepsilon(y) \in C_b^-.$$

First case: suppose that $g_\varepsilon(x) = \sup_{x \in \mathbb{R}} f_\varepsilon(x) =: s$. In this case there exists a sequence of $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ such that $x_n \to +\infty$ and $f_\varepsilon(x_n) \to s$, as $n \to \infty$. Define

$$g_n(x) = s\mathbf{1}_{(-\infty, x_n]} + f_\varepsilon(x)\mathbf{1}_{(x_n, +\infty)}$$

and notice that $s \geq g_n \geq f_\varepsilon$ and $g_n \uparrow s$. For any $Q \in \mathcal{C}_\varepsilon$ we consider Q_n defined by $F_{Q_n}(x) = F_Q(x)\mathbf{1}_{[x_n,+\infty)}$. Since $Q_n \preccurlyeq Q$, monotonicity of Φ implies $Q_n \in \mathcal{C}_\varepsilon$. Notice that

$$\int g_n dQ - \int f_\varepsilon dQ_n = (s - f_\varepsilon(x_n))Q(-\infty, x_n] \overset{n\to+\infty}{\longrightarrow} 0, \text{ as } n \to \infty. \quad (13)$$

From equation (9) we have

$$s \geq \int f_\varepsilon dP > \alpha > \int f_\varepsilon dQ_n \quad \text{for all } n \in \mathbb{N}. \quad (14)$$

Letting $\delta = s - \alpha > 0$ we obtain $s > \int f_\varepsilon dQ_n + \frac{\delta}{2}$. From (13), there exists $\overline{n} \in \mathbb{N}$ such that $0 \leq \int g_n dQ - \int f_\varepsilon dQ_n < \frac{\delta}{4}$ for every $n \geq \overline{n}$. Therefore $\forall\, n \geq \overline{n}$

$$s > \int f_\varepsilon dQ_n + \frac{\delta}{2} > \int g_n dQ - \frac{\delta}{4} + \frac{\delta}{2} = \int g_n dQ + \frac{\delta}{4}$$

and this leads to a contradiction since $g_n \uparrow s$. So the first case is excluded.

Second case: suppose that $g_\varepsilon(x) < s$ for any $x > \overline{x}$. As the function $g_\varepsilon \in C_b^-$ is decreasing, there will exists at most a countable sequence of intervals $\{A_n\}_{n\geq 0}$ on which g_ε is constant. Set $A_0 = (-\infty, b_0)$, $A_n = [a_n, b_n) \subset \mathbb{R}$ for $n \geq 1$. W.l.o.g. we suppose that $A_n \cap A_m = \emptyset$ for all $n \neq m$ (else, we paste together the sets) and $a_n < a_{n+1}$ for every $n \geq 1$. We stress that $f_\varepsilon(x) = g_\varepsilon(x)$ on $D =: \bigcap_{n\geq 0} A_n^C$. For every $Q \in \mathcal{C}_\varepsilon$ we define the probability $\overline{Q}$ by its distribution function as

$$F_{\overline{Q}}(x) = F_Q(x)\mathbf{1}_D + \sum_{n\geq 1} F_Q(a_n)\mathbf{1}_{[a_n,b_n)}.$$

As before, $\overline{Q} \preccurlyeq Q$ and monotonicity of Φ implies $\overline{Q} \in \mathcal{C}_\varepsilon$. Moreover

$$\int g_\varepsilon dQ = \int_D f_\varepsilon dQ + f_\varepsilon(b_0)Q(A_0) + \sum_{n\geq 1} f_\varepsilon(a_n)Q(A_n) = \int f_\varepsilon d\overline{Q}.$$

From $g_\varepsilon \geq f_\varepsilon$ and equation (9) we deduce

$$\int g_\varepsilon dP \geq \int f_\varepsilon dP > \alpha > \int f_\varepsilon d\overline{Q} = \int g_\varepsilon dQ \quad \text{for all } Q \in \mathcal{C}_\varepsilon.$$

■

We reformulate the Proposition 23 and provide two dual representations of $\sigma(\mathcal{P}(\mathbb{R}), C_b)$-lsc Risk Measure $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$. The first one is given in terms of the dual function R used by [5]. The second one is obtained from Proposition 23 considering the left continuous version of R and rewriting it (see Lemma 25) in the formulation proposed by [6]. If $R : \mathbb{R} \times C_b(\mathbb{R}) \to \overline{\mathbb{R}}$, the left continuous version of $R(\cdot, f)$ is defined by:

$$R^-(t, f) := \sup\{R(s, f) \mid s < t\}. \quad (15)$$

Proposition 24 *Any $\sigma(\mathcal{P}(\mathbb{R}), C_b)$-lsc Risk Measure $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ can be represented as*

$$\Phi(P) = \sup_{f \in C_b^-} R\left(\int f dP, f\right) = \sup_{f \in C_b^-} R^-\left(\int f dP, f\right). \tag{16}$$

The function $R^-(t,f)$ can be written as

$$R^-(t,f) = \inf\{m \in \mathbb{R} \mid \gamma(m,f) \geq t\}, \tag{17}$$

where $\gamma : \mathbb{R} \times C_b(\mathbb{R}) \to \overline{\mathbb{R}}$ is given by:

$$\gamma(m,f) := \sup_{Q \in \mathcal{P}} \left\{ \int f dQ \mid \Phi(Q) \leq m \right\}, \; m \in \mathbb{R}. \tag{18}$$

Proof. Notice that $R(\cdot, f)$ is increasing and $R(t,f) \geq R^-(t,f)$. If $f \in C_b^-$ then $Q \preceq P \Rightarrow \int f dQ \leq \int f dP$. Therefore,

$$R^-\left(\int f dP, f\right) := \sup_{s < \int f dP} R(s,f) \geq \lim_{P_n \uparrow P} R(\int f dP_n, f).$$

From Proposition 23 (ii) we obtain:

$$\Phi(P) = \sup_{f \in C_b^-} R\left(\int f dP, f\right) \geq \sup_{f \in C_b^-} R^-\left(\int f dP, f\right) \geq \sup_{f \in C_b^-} \lim_{P_n \uparrow P} R(\int f dP_n, f)$$
$$= \lim_{P_n \uparrow P} \sup_{f \in C_b^-} R(\int f dP_n, f) = \lim_{P_n \uparrow P} \Phi(P_n) = \Phi(P).$$

by (CfB). This proves (16). The second statement follows from the Lemma 25. ■

Lemma 25 *Let Φ be any map $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ and $R : \mathbb{R} \times C_b(\mathbb{R}) \to \overline{\mathbb{R}}$ be defined in (8). The left continuous version of $R(\cdot, f)$ can be written as:*

$$R^-(t,f) := \sup\{R(s,f) \mid s < t\} = \inf\{m \in \mathbb{R} \mid \gamma(m,f) \geq t\}, \tag{19}$$

where $\gamma : \mathbb{R} \times C_b(\mathbb{R}) \to \overline{\mathbb{R}}$ is given in (18).

Proof. Let the RHS of equation (19) be denoted by

$$S(t,f) := \inf\{m \in \mathbb{R} \mid \gamma(m,f) \geq t\}, \; (t,f) \in \mathbb{R} \times C_b(\mathbb{R}),$$

and note that $S(\cdot, f)$ is the left inverse of the increasing function $\gamma(\cdot, f)$ and therefore $S(\cdot, f)$ is left continuous.
Step I. To prove that $R^-(t,f) \geq S(t,f)$ it is sufficient to show that for all $s < t$ we have:

$$R(s,f) \geq S(s,f), \tag{20}$$

Indeed, if (20) is true

$$R^-(t, f) = \sup_{s<t} R(s, f) \geq \sup_{s<t} S(s, f) = S(t, f),$$

as both R^- and S are left continuous in the first argument.
Writing explicitly the inequality (20)

$$\inf_{Q\in\mathcal{P}} \left\{ \Phi(Q) \mid \int f dQ \geq s \right\} \geq \inf \left\{ m \in \mathbb{R} \mid \gamma(m, f) \geq s \right\}$$

and letting $Q \in \mathcal{P}$ satisfying $\int f dQ \geq s$, we see that it is sufficient to show the existence of $m \in \mathbb{R}$ such that $\gamma(m, f) \geq s$ and $m \leq \Phi(Q)$. If $\Phi(Q) = -\infty$ then $\gamma(m, f) \geq s$ for any m and therefore $S(s, f) = R(s, f) = -\infty$.

Suppose now that $\infty > \Phi(Q) > -\infty$ and define $m := \Phi(Q)$. As $\int f dQ \geq s$ we have:

$$\gamma(m, f) := \sup_{Q\in\mathcal{P}} \left\{ \int f dQ \mid \Phi(Q) \leq m \right\} \geq s$$

Then $m \in \mathbb{R}$ satisfies the required conditions.

Step II : To obtain $R^-(t, f) := \sup_{s<t} R(s, f) \leq S(t, f)$ it is sufficient to prove that, for all $s < t$, $R(s, f) \leq S(t, f)$, that is

$$\inf_{Q\in\mathcal{P}} \left\{ \Phi(Q) \mid \int f dQ \geq s \right\} \leq \inf \left\{ m \in \mathbb{R} \mid \gamma(m, f) \geq t \right\}. \tag{21}$$

Fix any $s < t$ and consider any $m \in \mathbb{R}$ such that $\gamma(m, f) \geq t$. By the definition of γ, for all $\varepsilon > 0$ there exists $Q_\varepsilon \in \mathcal{P}$ such that $\Phi(Q_\varepsilon) \leq m$ and $\int f dQ_\varepsilon > t - \varepsilon$. Take ε such that $0 < \varepsilon < t - s$. Then $\int f dQ_\varepsilon \geq s$ and $\Phi(Q_\varepsilon) \leq m$ and (21) follows. ■

5.3 Computation of the dual function

The following proposition is useful to compute the dual function $R^-(t, f)$ for the examples considered in this paper.

Proposition 26 *Let $\{F_m\}_{m\in\mathbb{R}}$ be a feasible family and suppose in addition that, for every m, $F_m(x)$ is increasing in x and $\lim_{x\to+\infty} F_m(x) = 1$. The associated map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ defined in (2) is well defined, (Mon), (Qco) and $\sigma(\mathcal{P}, C_b)-$l.s.c. and the representation (16) holds true with R^- given in (17) and*

$$\gamma(m, f) = \int f dF_{-m} + F_{-m}(-\infty) f(-\infty). \tag{22}$$

Proof. From equations (1) and (3) we obtain:

$$\mathcal{A}^{-m} = \{Q \in \mathcal{P}(\mathbb{R}) \mid F_Q \leq F_{-m}\} = \{Q \in \mathcal{P} \mid \Phi(Q) \leq m\}$$

so that

$$\gamma(m, f) := \sup_{Q\in\mathcal{P}} \left\{ \int f dQ \mid \Phi(Q) \leq m \right\} = \sup_{Q\in\mathcal{P}} \left\{ \int f dQ \mid F_Q \leq F_{-m} \right\}.$$

Fix $m \in \mathbb{R}$, $f \in C_b^-$ and define the distribution function $F_{Q_n}(x) = F_{-m}(x)\mathbf{1}_{[-n,+\infty)}$ for every $n \in \mathbb{N}$. Obviously $F_{Q_n} \leq F_{-m}$, $Q_n \uparrow$ and, taking into account (6), $\int f dQ_n$ is increasing. For any $\varepsilon > 0$, let $Q^\varepsilon \in \mathcal{P}$ satisfy $F_{Q^\varepsilon} \leq F_{-m}$ and $\int f dQ^\varepsilon > \gamma(m,f) - \varepsilon$. Then: $F_{Q_n^\varepsilon}(x) := F_{Q^\varepsilon}(x)\mathbf{1}_{[-n,+\infty)} \uparrow F_{Q^\varepsilon}$, $F_{Q_n^\varepsilon} \leq F_{Q_n}$ and

$$\int f dQ_n \geq \int f dQ_n^\varepsilon \uparrow \int f dQ^\varepsilon > \gamma(m,f) - \varepsilon.$$

We deduce that $\int f dQ_n \uparrow \gamma(m,f)$ and, since

$$\int f dQ_n = \int_{-n}^{+\infty} f dF_{-m} + F_{-m}(-n)f(-n),$$

we obtain (22). ■

In the following examples $m \in \mathbb{R}$, $f \in C_b^-$ and f^l is the left inverse of f.

Example 27 *Computation of the dual function R^- for the $V@R$ and the worst case measure. The family $\{F_m\}_{m\in\mathbb{R}}$ is given by (see the Examples 15 and 16) $F_m = \lambda 1_{(-\infty,m)} + 1_{[m,+\infty)}$, for $\lambda \in [0,1)$. Hence we get from (22)*

$$\gamma(m,f) = (1-\lambda)f(-m) + \lambda f(-\infty).$$

If $\lambda > 0$, from (17) and (16)

$$\begin{aligned} R^-(t,f) &= -f^l\left(\frac{t - \lambda f(-\infty)}{1-\lambda}\right), \\ \Phi_{V@R_\lambda}(P) &= -\inf_{f \in C_b^-} f^l\left(\frac{\int f dP - \lambda f(-\infty)}{1-\lambda}\right) \end{aligned}$$

If $\lambda = 0$, $\gamma(m,f) = f(-m)$ and from (17), (16)

$$\begin{aligned} R^-(t,f) &= -f^l(t), \\ \Phi_w(P) &= -\inf_{f \in C_b^-} f^l\left(\int f dP\right) \end{aligned}$$

Example 28 *Computation of $\gamma(m,f)$ for the $\Lambda V@R$.*

As $F_m = \Lambda(x)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x)$, we compute from (22):

$$\gamma(m,f) = \int_{-\infty}^{-m} f d\Lambda + (1 - \Lambda(-m))f(-m) + \Lambda(-\infty)f(-\infty).$$

If Λ is decreasing we may use Remark 20 to derive a simpler formula for γ. Indeed, $\Lambda V@R(P) = \Lambda\widetilde{V}@R(P)$ where $\forall m \in \mathbb{R}$

$$\widetilde{F}_m(x) = \Lambda(m)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x)$$

and so

$$\gamma(m,f) = \Lambda(-m)f(-\infty) + (1 - \Lambda(-m))f(-m),$$

which is increasing in m.

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