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HECKE ALGEBRAS ASSOCIATED TO COXETER GROUPS

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“ Things should be made as simple as possible but no simpler.

A. EINSTEIN

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Introduction

The main subject of this thesis is the study of cohomological properties of Hecke algebras associated to arbitrary Coxeter groups. Hecke algebras occur in several areas of mathematics: representation theory, which is the main motivation of the present work, topology and knot theory, harmonic analysis, number theory.

In particular, Iwahori–Hecke algebras (i.e., Hecke algebras attached to Weyl groups) and their linear representations play a decisive rôle in the character theory of finite groups of Lie type (cf. [CR87, Ch. 8]).

Motivation

A Hecke algebra \mathcal{H} is defined abstractly as an associative R -algebra with a presentation involving

- a Coxeter system (W, S) and
- a set of parameters \mathbf{q} lying in the commutative ring R .

The Coxeter system (W, S) determines the main properties of \mathcal{H} . Indeed, if \mathbf{q} is the constant parameter 1, one has an isomorphism between the Hecke algebra and the R -group algebra of the Coxeter group; by Tits’ work (cf. [Bou07c, Ch. IV Ex. 27]), if the ring is the field of complex numbers and the Hecke algebra is of spherical type (i.e., the associated Coxeter group is finite), this isomorphism is generic, i.e., it holds for all values of the parameter in an open set around $1 \in \mathbb{C}$. In this context one may think that a Hecke algebra $\mathcal{H}(W, S, R, \mathbf{q})$ is a “deformation” of the group algebra $R[W]$.

Coxeter groups are finitely generated and have a canonical length function ℓ . Using this function one defines the Poincaré series of a Coxeter group to be the formal power series

$$p_{(W,S)}(t) = \sum_{w \in W} t^{\ell(w)} \in \mathbb{Z}[[t]].$$

From this definition, it is not obvious how to compute Poincaré series explicitly.

- If W is finite, a closed formula for $p_{(W,S)}(t)$ involving the *degrees* is provided by the Chevalley–Shephard–Todd theorem (cf. [ST54], [Che55]).
- If W is *affine* (cf. Definition 3.24), then there is another closed formula, due to R. Bott (cf. [Ste68]).
- Moreover, if W is infinite, there is a well-known recursive formula (cf. [Ser71], [Bou07c]) for the Poincaré series. That is, the following equality holds in the ring of formal power series:

$$(\dagger) \quad \frac{1}{p_{(W,S)}(t)} = \sum_{I \subsetneq S} (-1)^{|S|-|I|-1} \frac{1}{p_{(W_I, I)}(t)},$$

where (W_I, I) is the parabolic Coxeter subsystem generated by a proper subset I of S .

The alternating sum in the recursive expression (\dagger) suggests that $p_{(W,S)}(t)$ might have a suitable interpretation as an Euler characteristic.

To study cohomological properties, such as Euler characteristic, one needs resolutions, that is, acyclic complexes with prescribed homology (in degree 0).

Complexes for Coxeter groups arise in a topological/geometric context, in several ways.

- The Coxeter complex $\Sigma(W, S)$ associated to the nontrivial poset of parabolic cosets of W . It is a W -CW-complex, and it is contractible if, and only if, W is infinite. Cell stabilizers are identified with conjugates of parabolic subgroups. In general, parabolic subgroups might happen to be infinite.
- By Tits' reflection representation, each Coxeter group (W, S) is a \mathbb{R} -linear group of finite degree, and there is a canonical geometric object, the Tits' open cone \mathcal{C} , on which W acts. The cone \mathcal{C} is a model for the universal space for proper group actions $\underline{E}W$, although, in general, the action is not cocompact.
- The Davis–Moussong complex, which is a model for $\underline{E}W$, with cocompact action (cf. [Mou88], [Dav08]).

For the computation of Euler characteristic, the Coxeter complex would be quite useful, however, it is only defined for Coxeter groups.

Since Hecke algebras are “deformations” of group algebras of Coxeter groups, it seems reasonable to try to “deform” the Coxeter complex into a complex suitable for our problem. During this process the geometry disappears in favour of algebra, and all we can do is to define a suitable complex of \mathcal{H} -modules which, nonetheless, suffices for our purposes.

The Hecke–Coxeter chain complex C_\bullet for a Hecke algebra \mathcal{H} is a bounded complex of left \mathcal{H} -modules, which constitutes a (non necessarily projective) resolution of the trivial module (cf. Theorem 5.30). Under suitable conditions on the Poincaré series of the finite parabolic subgroup of the Coxeter system (W, S) , the Hecke algebra \mathcal{H} is *Euler* (cf. Definition 1.23 and Theorem 5.36). This in particular allows one to define the Euler characteristic $\chi_{\mathcal{H}}$ of the Hecke algebra.

From this viewpoint, in particular, the Euler characteristic allows one to give an interpretation of the Poincaré series of a Coxeter group in terms of cohomological data of a Hecke algebra, i.e., the alternating sum formula (4.2) can be written as

$$(\ddagger) \quad p_{(W,S)}(q) \chi_{\mathcal{H}} = 1,$$

(cf. Theorem 5.38). Expressions similar to (\ddagger) occur in the context of Koszul algebras (cf. [PP05]); it would be interesting to investigate if the two phenomena are connected in some more general framework.

A second motivation of interest of the present thesis is a result, Theorem 4.16, in the theory of Coxeter groups. It is essentially a result in the theory of (Coxeter) graphs which may be applied to give a new proof of a very interesting result.

For Coxeter groups, it provides a characterization in terms of finiteness conditions of group theoretical properties such as amenability and the existence of free subgroups of rank at least 2 (cf. [dlH87]). This can be thought as a “Tits' alternative”-like theorem.

Outline

The first chapter establishes the homological algebra which will be useful later.

In particular, the notion of *Euler R -algebra* \mathbf{A} is given, and for such algebras the Hattori–Stallings rank is defined. It is an element of the R -module $\mathbf{A}/[\mathbf{A}, \mathbf{A}]$ which generalizes to the non-commutative case the notion of rank. Moreover, for

Euler algebras a *canonical trace function* applied to the rank element allows one to define the Euler characteristic of \mathbf{A} .

Chapter 2 recalls the notion of group cohomology and focuses on how projective resolutions of the constant object \mathbb{Q} arise from topology. The universal space for actions with prescribed stabilizers $E_x(-)$ is introduced: this was of great inspiration for defining the Hecke–Coxeter chain complex in Chapter 5 §5.

In the second part of this chapter, growth series of finitely generated groups are discussed as well as amenability.

The subject of Chapter 3 is the class of Coxeter groups. Finite Coxeter groups play a decisive rôle in Lie theory and in many other mathematical areas. They have been subject of intense study from many viewpoints.

Coxeter groups are linear groups with a very strong combinatorial structure, and they admit a distinguished poset of subgroups (parabolic subgroups) which are Coxeter groups themselves. If all the maximal parabolic subgroups are finite, the Coxeter group is of *cocompact type*.

Thus, one can define the Coxeter complex $\Sigma(W, S)$, which is, for cocompact types, a CW-model of the \underline{EW} , and hence it provides a finite, projective resolution of the trivial $\mathbb{Q}[W]$ -module.

Under mild conditions on the ring R , the R -group algebras of Coxeter groups are Euler and one might associate an Euler characteristic to them, which coincides with the usual Euler characteristics of the group W .

The first part of Chapter 4 deals with Poincaré series of Coxeter groups. In the second part we describe (cf. Theorem 4.16) the class \mathcal{M} of minimal non-spherical, non-affine, Coxeter systems and, for such, we determine the Poincaré series and the growth.

As an application, the above two steps let us give another proof of an old result (cf. [dIH87]): Theorem 4.22. This is a purely group-theoretic fact, which can be interpreted as a kind of Tits’ alternative for Coxeter groups:

- Coxeter systems with all irreducible subsystems of spherical or affine type are amenable, of polynomial growth, and do not have subgroups isomorphic to the free group F_2 on two letters.
- Coxeter systems not falling in the previous case are not amenable, have exponential growth and contain subgroups isomorphic to F_2 .

Chapter 5 is the final and most important chapter of this thesis and has several purposes.

- Various contexts where Hecke algebras naturally arise are shown.
- A general definition of a Hecke algebra over a ring R is given.
- Some combinatorics of a Hecke algebra is studied, stressing the importance of parabolic subalgebras.
- The trivial and sign module are defined and the modules induced from the trivial module of a parabolic subalgebra are studied: the invertibility of Poincaré series of the parabolic subalgebras determines the projectivity and finite generation.
- A complex of left \mathcal{H} -modules is defined in a canonical way, which is a \mathcal{H} -module theoretic analogue of the Coxeter complex of (W, S) .
- If W is infinite, it is shown that this complex is a resolution of the trivial module (cf. Theorem 5.30).
- Under suitable conditions on the Poincaré series of all the finite parabolic subalgebras, projectivity is proved for the algebras of cocompact type and, in general, the **FP**-property of \mathcal{H} is established.

- Some further structure of Hecke algebras is shown, so that Hecke algebras are Euler and admit an Euler characteristic.
- Finally, the Euler characteristic of a Hecke algebra is computed through the Hecke–Coxeter chain complex. For suitable choices of the ring R , it coincides with the inverse of the Poincaré series of (W, S) (cf. Theorem 5.38).

Several appendices are also included.

Appendix A recalls no(ta)tions of graph theory.

Appendix B contains the lists of Coxeter graphs together with important informations such as the degrees and the convergence radii of the Poincaré series.

Appendix C contains few complementary facts about complex power series which are needed in the thesis.

Finally, Appendix D contains information about the computations I made, mainly with GAP and CHEVIE.

A word about notation

Notation should be consistent, light, and possibly self-explanatory, conventions should be widely accepted. Since this is not a trivial task, it might not be achieved at the time, and hence a summary of notation follows.

Symbol : Meaning

\mathbb{N}	: The set $\{1, 2, 3, \dots\}$
\mathbb{N}_0	: The set $\mathbb{N} \cup \{0\}$
$\mathbb{Z}_{\geq k}$: The set $\{k + n \mid n \in \mathbb{N}_0\}$, with $k \in \mathbb{Z}$
$\mathcal{P}(X)$: The power set of a set X
$\frac{1}{\infty}$: $= 0$
$X \sqcup Y$: Disjoint union (i.e., coproduct) of sets X and Y
$P(x)$ a. e.	: The property P holds almost everywhere, i. e. for all but finitely many x 's
$P(n)$ for $n \gg 0$: The property P holds for large enough n
\mathfrak{V}	: The set of vertices of a graph
\mathfrak{E}	: The set of edges of a graph
$\text{Obj}(\mathcal{C})$: The class of objects of \mathcal{C}
$X \in \mathcal{C}$: $X \in \text{Obj}(\mathcal{C})$
$\text{Mor}(\mathcal{C})$: The class of morphisms of \mathcal{C}
$\mathcal{C}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)$: The class of \mathcal{C} -morphisms between $X, Y \in \text{Obj}(\mathcal{C})$
$\text{Hom}_A(X, Y)$: $\text{Hom}_{A\text{-Mod}}(X, Y)$
$X = \text{id}_X$: Identity morphism in $\mathcal{C}(X, X)$
0	: The zero object of a category \mathcal{C} , in particular the zero module for the category $A\text{-Mod}$
$0 \in \mathcal{C}(X, Y)$: The unique morphism $X \rightarrow 0 \rightarrow Y$ in \mathcal{C}
$\text{Iso}(X)$: The group of isometries of a metric space, in particular of a Riemannian manifold
$G \curvearrowright X$: The left action of a group G on a set X
$\text{Stab}_G(X)$: For $G \curvearrowright X$ and $Y \subseteq X$, the subgroup $\{g \in G \mid g.x \in Y \forall x \in Y\}$
G_x	: For $G \curvearrowright X$ and $x \in X$, the isotropy subgroup $\text{Stab}_G(\{x\})$
$\mathcal{V}P$: Virtually P , where P is a property of groups
\mathbf{C}_n	: The cyclic group of order n
\mathbf{D}_n	: The dihedral group of order $2n$
$\text{Cl}(G)$: The set of conjugacy classes of the group G .

Representation theory and cohomology

“Cohomology is representation theory.

J. L. ALPERIN, [Alp87]

The purpose of this chapter is to give the definition of an Euler algebra. The idea is to “enrich” an R -algebra \mathbf{A} with some further structure (cf. Definition 1.23) so that an Euler characteristic $\chi_{\mathbf{A}}$ may be associated to \mathbf{A} .

This definition is, essentially, a weakening of the axioms of a Hopf algebra: in particular, group algebras fit in the picture and the process of defining the Euler characteristic mimics the usual definition of the R -Euler characteristic of a group (cf. [Bro82]).

The first three sections of the chapter summarize (and fix the notation for) the homological algebra and representation theory needed later.

1. R -algebras, modules

In this thesis, rings will be unital and associative. In particular, if R and S are rings and $f: R \rightarrow S$ is a morphism of rings, then $f(1_R) = 1_S$.

1.1. R -algebras. All along this thesis, the symbol R will denote a commutative “base ring”, and the word “algebra” has to be intended as “algebra over a commutative base ring”. If \mathbf{A} is an algebra the opposite algebra is denoted \mathbf{A}^{op} .

Modules over an R -algebra have the underlying structure of R -modules, and, for an R -algebra \mathbf{A} , the commutator subgroup

$$[\mathbf{A}, \mathbf{A}] = \{ ab - ba \mid a, b \in \mathbf{A} \}$$

is an R -submodule of \mathbf{A} . Then let $\underline{\mathbf{A}} = \mathbf{A}/[\mathbf{A}, \mathbf{A}]$ be the quotient R -module.

1.2. \mathbf{A} -modules. If $M, N \in \mathbf{A}\text{-Mod}$ are left \mathbf{A} -modules, let $\text{Hom}_{\mathbf{A}}(M, N)$ be the R -module of \mathbf{A} -morphisms and let $\text{Hom}(M, N)$ be the R -module of R -morphisms. Analogous notation will be used for the tensor product: if $M \in \text{Mod-}\mathbf{A}$ is a right \mathbf{A} -module (i.e., a left \mathbf{A}^{op} -module) and $N \in \mathbf{A}\text{-Mod}$ is a left \mathbf{A} -module, then $M \otimes_{\mathbf{A}} N$ denotes the tensor product over \mathbf{A} , while the unadorned version $M \otimes N$ denotes the product over the base ring R . For a left \mathbf{A} -module M , set $M^* = \text{Hom}_{\mathbf{A}}(M, \mathbf{A})$.

Let \mathbf{A}, \mathbf{B} be R -algebras. Let M be a left \mathbf{A} -module, let N be an (\mathbf{A}, \mathbf{B}) -bimodule and let L be a left \mathbf{B} -module. Then $\text{Hom}_{\mathbf{A}}(M, N)$ naturally has the structure of right \mathbf{B} -module via

$$(f.b)(m) = f(m).b \quad \text{for } f \in \text{Hom}_{\mathbf{A}}(M, N), b \in \mathbf{B}, m \in M$$

and $N \otimes_{\mathbf{B}} L$ is a left \mathbf{A} -module via $a.(n \otimes \ell) = (a.n) \otimes \ell$.

The notions of free, projective, flat, finitely generated, etc., modules are defined as usual. The regular left \mathbf{A} -module is denoted ${}^{\text{reg}}\mathbf{A}$.

1.3. Hom- \otimes adjunction. It follows that the R -modules $\text{Hom}_{\mathbf{A}}(M, N) \otimes_{\mathbf{B}} L$ and $\text{Hom}_{\mathbf{A}}(M, N \otimes_{\mathbf{B}} L)$ are defined and there is a *canonical morphism* of R -modules, expressing the adjunction between $\text{Hom}_{\mathbf{A}}(-, -)$ and $- \otimes_{\mathbf{B}} -$:

$$(1.1) \quad \begin{aligned} \gamma_{M,N,L}: \text{Hom}_{\mathbf{A}}(M, N) \otimes_{\mathbf{B}} L &\rightarrow \text{Hom}_{\mathbf{A}}(M, N \otimes_{\mathbf{B}} L), \\ \gamma(f \otimes \ell)(m) &= f(m) \otimes \ell. \end{aligned}$$

In particular, if $\mathbf{A} = \mathbf{B}$ and $N = \mathbf{A}$ one deduces that there are morphisms

$$(1.2) \quad \gamma_{M,L}: M^* \otimes_{\mathbf{A}} L \rightarrow \text{Hom}_{\mathbf{A}}(M, L)$$

and, if further $L = M$,

$$(1.3) \quad \gamma_M: M^* \otimes_{\mathbf{A}} M \rightarrow \text{End}_{\mathbf{A}}(M).$$

Finally, the *evaluation map* is needed:

$$\begin{aligned} \text{ev}_M: M^* \otimes_{\mathbf{A}} M &\rightarrow \underline{\mathbf{A}}, \\ \text{ev}_M(f \otimes m) &= f(m) + [\mathbf{A}, \mathbf{A}]. \end{aligned}$$

1.1. REMARK. Note that, since \mathbf{A} is not, in general, commutative, one needs to take the evaluation modulo the commutator submodule. Indeed, one has $f.a \otimes m = f \otimes a.m$, then must have $\text{ev}_M(f.a \otimes m) = \text{ev}_M(f \otimes a.m)$, while

$$(f.a)(m) = f(m)a, \quad \text{and} \quad f(a.m) = af(m).$$

The class of finitely generated, projective left \mathbf{A} -modules is denoted $\text{proj}(\mathbf{A})$. The fact that the morphism γ_P given in (1.3) is an isomorphism if, and only if, $P \in \text{proj}(\mathbf{A})$ will be used repeatedly (cf. [Bou07b, Ch. 2 §4]).

1.4. Restriction and induction. For an R -algebra \mathbf{A} with a subalgebra $\mathbf{B} \leq \mathbf{A}$ let

$$\text{res}_{\mathbf{B}}^{\mathbf{A}}(-): \mathbf{A}\text{-Mod} \rightarrow \mathbf{B}\text{-Mod}$$

and

$$\text{ind}_{\mathbf{B}}^{\mathbf{A}}(-) = \mathbf{A} \otimes_{\mathbf{B}} (-): \mathbf{B}\text{-Mod} \rightarrow \mathbf{A}\text{-Mod}$$

be, respectively, the restriction and induction functors.

If \mathbf{A} is flat as right \mathbf{B} -module, then $\text{ind}_{\mathbf{B}}^{\mathbf{A}}(-)$ is exact and maps projective (resp. finitely generated, projective) left \mathbf{B} -modules to projective (resp. finitely generated, projective) left \mathbf{A} -modules (cf. [Wei94, Prop. 2.3.10]).

A left \mathbf{A} -module M is called *one-dimensional* if, and only if, $\text{res}_{\mathbf{B}}^{\mathbf{A}} M \simeq {}^{\text{reg}} R$.

2. Chain complexes

This section sets the notation for (co)chain complex of modules and related objects. In particular, sign conventions are established.

Let \mathbf{A} be an R -algebra, then a chain complex $(M_{\bullet}, \partial_{\bullet})$ is a sequence of left \mathbf{A} -modules M_k and boundary morphisms $\partial_k \in \text{Hom}_{\mathbf{A}}(M_k, M_{k-1})$ for all $k \in \mathbb{Z}$, such that

$$\partial_{k-1} \circ \partial_k: M_k \rightarrow M_{k-2}$$

is the zero map.

The category consisting of complexes of left \mathbf{A} -modules and chain maps is called the *chain complex category* of \mathbf{A} , and denoted $\text{Kom}_{\bullet}(\mathbf{A})$.

Other related notions, such as: cochain complexes, chain maps, subcomplex, quotient, cycle submodules $Z_{\bullet}(M) = \ker \partial_{\bullet}$, boundary submodules $B_{\bullet}(M) = \text{im } \partial_{\bullet}$, chain homology $H_{\bullet}: \text{Kom}_{\bullet}(\mathbf{A}) \rightarrow \text{Kom}_{\bullet}(\mathbf{A})$, homotopy equivalence and chain homotopy (both denoted \simeq), quasi-isomorphism, shift operator (denoted $_{-}[p]$), cone $\text{Cone}(f_{\bullet})$ and cylinder $\text{Cyl}(f_{\bullet})$ of a chain map will be used exactly as defined in [Bou07a, Ch. X §2].

2.1. Finiteness conditions and other properties. Some further nomenclature about chain complexes follows.

1.2. DEFINITION. Let M_\bullet be a chain complex in $\mathbf{A}\text{-Mod}$. Then

- (1) a complex M_\bullet is *bounded from below* (resp. *from above*) if $M_k = 0$ for $k \ll 0$ (resp. $k \gg 0$);
- (2) M_\bullet is called *bounded* if $M_k = 0$ for almost every $k \in \mathbb{Z}$ or, equivalently, if it is bounded both from below and from above;
- (3) a non-zero complex with $M_k = 0$ for all $k < 0$ complex M_\bullet has length $\sup\{k \in \mathbb{Z} \mid M_k \neq 0\}$;
- (4) a complex M_\bullet is *concentrated at k* if $M_\ell = 0$ for all $\ell \neq k$;
- (5) a complex M_\bullet is exact in degree k if $H_k(M_\bullet) = 0$;
- (6) a complex M_\bullet is called *exact* if $H_\bullet(M_\bullet) = 0$;
- (7) a complex whose homology is concentrated in degree 0 is called *acyclic*.

Typical nice properties for a module (e.g. freeness, projectivity, finite generation) extend to complexes in the following way.

1.3. DEFINITION (“Nice properties” for complexes). Suppose $M_\bullet \in \text{Kom}_\bullet(\mathbf{A})$ is a complex of \mathbf{A} -modules. Then M_\bullet is called

- (1) *free* if M_k is a free \mathbf{A} -module for all k ;
- (2) *projective* if M_k is a projective \mathbf{A} -module for all k ;
- (3) *finite* if it is both bounded and M_k is finitely generated for all k .

2.2. Shift, embedding and truncation. For clarity we just recall the following convention about signs. If M_\bullet is a chain complex and p an integer, define the shifted complex $M[p]_\bullet$ as follows: $M[p]_k = M_{k+p}$ and $\partial_k^{M[p]} = (-1)^p \partial_{k+p}^M$. If M_\bullet, N_\bullet are chain complexes, if $f_\bullet: M_\bullet \rightarrow N_\bullet$, let

$$f[p]_k = f_{p+k}: M[p]_k = M_{p+k} \rightarrow N[p]_k = N_{p+k}.$$

There is an obvious *embedding functor* which is an inclusion of categories

$$-\llbracket 0 \rrbracket: \mathbf{A}\text{-Mod} \rightarrow \text{Kom}_\bullet(\mathbf{A}),$$

and identifies modules with complexes concentrated at 0.

On the other hand, one may wish to “truncate” a complex: let $M_\bullet \in \text{Kom}_\bullet(\mathbf{A})$, and let $p \in \mathbb{Z}$. Then define a new chain complex $M_\bullet^{\geq p}$ as follows.

$$M_k^{\geq p} = \begin{cases} M_k & \text{if } k \geq p \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \left(\partial_k^{\geq p}: M_k^{\geq p} \rightarrow M_{k-1}^{\geq p} \right) = \begin{cases} \partial_k & \text{if } k > p \\ 0 & \text{otherwise.} \end{cases}$$

It is called the *above- p truncation* of M_\bullet . It is a subcomplex of M_\bullet , bounded from below, represented as follows.

$$M: \quad \dots \longrightarrow M_{p+2} \xrightarrow{\partial_{p+2}} M_{p+1} \xrightarrow{\partial_{p+1}} M_p \xrightarrow{\partial_p} M_{p-1} \xrightarrow{\partial_{p-1}} M_{p-1} \longrightarrow \dots$$

$$M^{\geq p}: \quad \dots \longrightarrow M_{p+2} \xrightarrow{\partial_{p+2}} M_{p+1} \xrightarrow{\partial_{p+1}} M_p \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

1.4. LEMMA. Let M_\bullet be a chain complex, then

$$H_k(M_\bullet^{\geq p}) = \begin{cases} H_k(M_\bullet) & \text{if } k > p \\ \text{cok } \partial_{p+1} & \text{if } k = p \\ 0 & \text{if } k < p. \end{cases}$$

In particular if M_\bullet is exact in degrees greater than p , the truncation above p of M_\bullet has homology concentrated at p .

PROOF. For $k \neq p$ the statement is obvious, for $k = p$ notice that $Z_p(M^{\geq p}) = \ker \partial_p^{\geq p} = M_p$ and $B_p(M^{\geq p}) = \text{im } \partial_{p+1}$. ■

2.3. Hom and \otimes complexes. The definitions given (for modules) in §1.2 can be extended to chain complexes. In order to fix the notation, the complete details follow.

1.5. DEFINITION (\otimes -complex). If $(M_\bullet, \partial_\bullet^M)$ is a complex of right \mathbf{A} -modules and $(N_\bullet, \partial_\bullet^N)$ is a complex of left \mathbf{A} -modules, then their tensor product is defined as the graded R -module $(M \otimes_{\mathbf{A}} N)_\bullet$ whose component in degree k is

$$(M \otimes_{\mathbf{A}} N)_k = \prod_{a+b=k} M_a \otimes_{\mathbf{A}} N_b$$

together with boundary maps

$$\partial_{a+b}^{M \otimes N}(m \otimes n) = (\partial_a^M m) \otimes n + (-1)^a m \otimes (\partial_b^N n)$$

if $m \otimes n$ is a monomial with $m \in M_a$ and $n \in N_b$. It is easy to check (cf. [Bou07a, Ch. X §4.1]) that $\partial_{k-1}^{M \otimes N} \circ \partial_k^{M \otimes N} = 0$, hence the tensor product is actually a chain complex.

Note that, if N_\bullet is concentrated in zero, then the tensor complex is the same as the complex obtained tensoring degree-wise each module M_a with N_0 , while if M_\bullet is concentrated in 0, the tensor complex has the same modules but different boundary morphisms (actually, only a sign $(-1)^a$ appears, thus the complexes are quasi-isomorphic).

1.6. DEFINITION (Hom-complex). Let \mathbf{A}, \mathbf{B} be R -algebras, and let $(M_\bullet, \partial_\bullet^M)$ be a complex of left \mathbf{A} -modules and $(N_\bullet, \partial_\bullet^N)$ be a complex of (\mathbf{A}, \mathbf{B}) -bimodules. Then the Hom-complex is defined as the complex of right \mathbf{B} -modules whose component in degree k is

$$\underline{\text{Hom}}_{\mathbf{A}}^k(M, N) = \underline{\text{Hom}}_{\mathbf{A}}^k(M, N) = \prod_{a-b=k} \text{Hom}_{\mathbf{A}}(M_a, N_b).$$

If $f = (f_{a,b}: M_a \rightarrow N_b)_{a-b=k}$ let the boundary map be defined as

$$(1.4) \quad (\delta^k(f))_{c,d} = \partial_{c-k}^N \circ f_{c,c-k} - (-1)^k f_{c-1,c-1-k} \circ \partial_c^M,$$

for $c-d = k+1$, hence $\delta^k f \in \underline{\text{Hom}}_{\mathbf{A}}^{k+1}(M, N)$.

One has that $\underline{\text{Hom}}_{\mathbf{A}}^k(M, N)$ is a chain complex (cf. [Bou07a, Ch. X §5.1]).

As in the case of modules, the dual complex is defined, and satisfies analogous properties.

1.7. DEFINITION (Dual complex). Let M_\bullet be a complex of left \mathbf{A} -modules, then the *dual complex* is the complex of right \mathbf{A} -modules

$$M_\bullet^\otimes = \underline{\text{Hom}}_{\mathbf{A}}^{\mathbf{A}}(M_\bullet, \mathbf{A}[[0]]) = \prod_k \text{Hom}_{\mathbf{A}}(M_{-k}, \mathbf{A}).$$

One checks that, if $f_k: M_{-k} \rightarrow \mathbf{A}$, the boundary $\delta_k f_k \in \underline{\text{Hom}}_{\mathbf{A}}^{\mathbf{A}}(M_\bullet, \mathbf{A}[[0]])$ is then given by

$$\delta_k^{M^\otimes}(f_k) = (-1)^{k+1} f_k \circ \partial_{-k+1}^M.$$

As a consequence of the chosen sign conventions, the natural transformation of bifunctors

$$(1.5) \quad \underline{\gamma}_{M,N}: -1^\otimes \otimes_{\mathbf{A}} -2 \rightarrow \underline{\text{Hom}}_{\mathbf{A}}(-1, -2)$$

carries a non-trivial sign: explicitly, one has

$$(1.6) \quad \underline{\gamma}(f \otimes n)(m) = (-1)^{ab} f(m)n,$$

for $(f: M_{-a} \rightarrow \mathbf{A}) \in \underline{\text{Hom}}_{\mathbf{A}}(M, \mathbf{A}[[0]])$, $m \in M_{-a}$, $n \in N_b$.

Moreover one has, for all \mathbf{A} -complex M ,

$$(1.7) \quad \gamma_M: M^{\otimes} \otimes_{\mathbf{A}} M \rightarrow \underline{\text{End}}_{\mathbf{A}}(M).$$

If P_{\bullet} is a finite, projective complex, then P_{\bullet}^{\otimes} is finite and projective and $\gamma_{P_{\bullet}}$ defined in (1.7) is an isomorphism of R -modules.

Moreover there is a morphism of chain complexes, the *standard evaluation*:

$$\begin{aligned} \underline{\text{ev}}_M: M^{\otimes} \otimes_{\mathbf{A}} M &\rightarrow \underline{\mathbf{A}}[[0]] \\ m_{-k}^* \otimes m_{\ell} &\mapsto \begin{cases} m_{-k}^*(m_{\ell}) + [\mathbf{A}, \mathbf{A}] & \text{if } k = \ell \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

3. Resolutions and cohomology

In order to define (and compute) cohomology, one wishes to associate a complex X_{\bullet} to a module M (e.g., using the embedding functor) such that the homology of the complex can be identified to the module one started with.

3.1. Resolutions and finiteness conditions. What one usually looks for is some complex whose modules have “nicer properties”, that is, one pays the price of passing from a single module to a complex for having the advantage of dealing with better-behaved modules. Under this point of view, of course, the embedding functor is non-interesting.

1.8. DEFINITION (Resolution). Let $(M_{\bullet}, \partial_{\bullet}^M) \in \text{Kom}_{\bullet}^+(\mathbf{A})$ a bounded-below chain complex of \mathbf{A} -modules, then a resolution of M_{\bullet} is 3-tuple $(X_{\bullet}, \partial_{\bullet}^X, \varepsilon_{\bullet})$ consisting of a bounded-below complex $(X_{\bullet}, \partial_{\bullet}^X)$, and a quasi-isomorphism $\varepsilon_{\bullet}: X_{\bullet} \rightarrow M_{\bullet}$.

1.9. REMARK. (1) Under the identification of M with $M[[0]]$, the above Definition agrees with the usual definition of resolution, namely the truncation $X_{\bullet}^{\geq 0}$ of an exact complex X_{\bullet} of the form

$$\dots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow X_{-1} = M \rightarrow 0.$$

(2) A resolution of a module M is a positive, acyclic complex whose homology is isomorphic to $M[[0]]$, i.e. the unique (possibly) nonzero is in degree 0 and isomorphic to M .

In particular, the notions of Definition 1.3 apply to resolutions, e.g. one can have free, projective, finite, etc., resolutions.

Thus, the *projective dimension* of a left \mathbf{A} -module M is the minimal length of a bounded projective resolution, or infinity if no bounded projective exists for M :

$$\text{pd}_{\mathbf{A}}(M) = \min(\{\text{lenght}(R_{\bullet}) \mid R_{\bullet} \rightarrow M \text{ bounded}\} \cup \{\infty\}).$$

Moreover, M is of type **FP** if, and only if, it admits a finite and projective resolution.

For each module there are many resolutions may exist, but it is well-known that, upon restricting to the (non-empty) class of projective resolutions, there exist essentially (i.e., up to homotopy equivalence) just one (cf. [Bou07a, Ch. X §3]).

3.2. Cohomology and $\underline{\text{Ext}}_0(-, -)$. The functors $\underline{\text{Ext}}_{\mathbf{A}}^{\bullet}(-, -)$ and $\text{Tor}_{\mathbf{A}}^{\bullet}(-, -)$, and hence (co)homology are defined in the usual way (cf. [Ben98, §2.4] or [Wei94, §§2.5, 2.6]). Hypercohomology is an extension of the notion of Ext- and Tor-functors for the categories of chain complexes.

Suppose M_{\bullet}, N_{\bullet} are bounded-below chain complexes of left \mathbf{A} -modules and suppose $P_{\bullet} \rightarrow M_{\bullet}$ is a resolution of M_{\bullet} . Then define

$$(1.8) \quad \underline{\text{Ext}}_{\mathbf{A}}^{\bullet}(M, N) = H_{\bullet}(\underline{\text{Hom}}_{\mathbf{A}}^{\bullet}(P, N))$$

or, passing to cochain complexes,

$$(1.8') \quad \underline{\text{Ext}}_{\mathbf{A}}^{\bullet}(M, N) = H^{\bullet}(\underline{\text{Hom}}_{\mathbf{A}}^{\bullet}(P, N)).$$

As one might expect, $\underline{\text{Ext}}$ -functors have a concrete interpretation.

1.10. PROPOSITION. *The R -module $\underline{\text{Ext}}_0^{\mathbf{A}}(M, N)$ represents the homotopy classes of chain maps from M to N .*

1.11. NOTATION. The homotopy class containing the chain map $f: M \rightarrow N$ will be denoted $[f] \in \underline{\text{Ext}}_0^{\mathbf{A}}(M, N)$.

PROOF. Let $f = \sum f_{i,i} \in \underline{\text{Hom}}_{\mathbf{A}}^0(M, N)$, where $f_{i,j}: M_i \rightarrow N_j$. Compute the boundary in the $\underline{\text{Hom}}$ -complex following (1.4)

$$(\delta^0 f)_{c,c-1} = \partial_c^N f_{c,c} - f_{c-1,c-1} \partial_c^M.$$

Then if $f \in \ker \delta^0$ one has that, for all $c \in \mathbb{Z}$, $\partial_c^N f_{c,c} = f_{c-1,c-1} \partial_c^M$, i.e. f is a chain map of degree zero.

Suppose now that $f \in \text{im } \delta^{-1}$, i.e. $f = \delta^{-1}g$ for a suitable $g = \sum g_{i-1,i} \in \underline{\text{Hom}}^{-1}(M, N)$. Again by (1.4), one has

$$f_{j,j} = (\delta^{-1}g)_{j,j} = \partial_{j+1}^N g_{j,j+1} + g_{j-1,j} \partial_j^M,$$

and then $\{\sigma_k = g_{k,k+1}: M_k \rightarrow N_{k+1} \mid k \in \mathbb{Z}\}$ is a chain homotopy contraction, and hence $f \approx 0$. \blacksquare

It is well-known that the definitions (1.8), (1.8') do not depend on the projective resolution used to compute them.

Moreover, there is a canonical isomorphism $\text{Ext}_{\mathbf{A}}^{\bullet}(M, N) \simeq \underline{\text{Ext}}_{\mathbf{A}}^{\bullet}(M[[0]], N[[0]])$.

4. Traces

For the purposes of this thesis a short compendium of linear algebra over \mathbf{A} -algebras is needed. In particular, an extension of the notion of a ‘‘trace’’ will be useful in the sequel. The discussion follows [Bas79] and [Sta65] in order to introduce the *Hattori-Stallings* trace for endomorphisms of projective modules and, in general, for modules of type \mathbf{FP} .

1.12. DEFINITION (Trace). Let R be a commutative ring, let \mathbf{A} be an associative R -algebra and let Θ be an R -module.

A Θ -valued trace of \mathbf{A} -modules is a collection consisting of R -linear morphisms

$$\tau_{M/\mathbf{A}}: \text{End}_{\mathbf{A}}(M) \rightarrow \Theta,$$

for M in some class \mathcal{X} of left \mathbf{A} -modules, such that

(1) if $\varphi, \psi \in \text{End}_{\mathbf{A}}(M)$, then

$$\tau_{M/\mathbf{A}}(\varphi + \psi) = \tau_{M/\mathbf{A}}(\varphi) + \tau_{M/\mathbf{A}}(\psi);$$

(2) if $\varphi: M \rightarrow N$ and $\psi: N \rightarrow M$ are morphisms of \mathbf{A} -modules, then

$$\tau_{M/\mathbf{A}}(\psi\varphi) = \tau_{N/\mathbf{A}}(\varphi\psi);$$

(3) if the diagram of \mathbf{A} -modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & f' \downarrow & & f \downarrow & & f'' \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

is commutative with exact rows, then $\tau_{M'/\mathbf{A}}(f') + \tau_{M''/\mathbf{A}}(f'') = \tau_{M/\mathbf{A}}(f)$.

Suppose that the class of modules \mathcal{X} , on which $\tau_{_/\mathbf{A}}(-)$ is defined, contains all finitely generated projective left \mathbf{A} -modules, then the trace can be extended canonically to a broader class of modules, namely to the class \mathbf{FP} ; thus, the following lemma is needed.

1.13. LEMMA. *Let P_\bullet be a finite, projective complex of \mathbf{A} -modules, and let $f_\bullet \approx g_\bullet: P_\bullet \rightarrow P_\bullet$ be chain homotopic chain maps.*

Suppose $\tau_{-/A}: \text{End}_{\mathbf{A}}(-) \rightarrow \Theta$ is a trace, then

$$\sum_{k \in \mathbb{Z}} (-1)^k \tau_{P_k/A}(f_k) = \sum_{k \in \mathbb{Z}} (-1)^k \tau_{P_k/A}(g_k).$$

PROOF. For short, put $\tau_{P_k/A} = \tau_k$. For a suitable σ_\bullet , write $f_k - g_k = \partial_{k+1}\sigma_k + \sigma_{k-1}\partial_k$. By Definition 1.12(1) and (2),

$$\begin{aligned} \tau_k(f_k) &= \tau_k(g_k) + \tau_k(\partial_{k+1}\sigma_k) + \tau_k(\sigma_{k-1}\partial_k) \\ &= \tau_k(g_k) + \tau_k(\partial_{k+1}\sigma_k) + \tau_{k-1}(\partial_k\sigma_{k-1}). \end{aligned}$$

Then, since

$$\sum_k (-1)^k \tau_k(\partial_{k+1}\sigma_k) + \tau_{k-1}(\partial_k\sigma_{k-1}) = 0,$$

the alternating sum formula gives

$$\begin{aligned} \sum_k (-1)^k \tau_k(f_k) &= \sum_k (-1)^k \tau_k(g_k) + \sum_k (-1)^k \tau_k(\partial_{k+1}\sigma_k) + \tau_{k-1}(\partial_k\sigma_{k-1}) \\ &= \sum_k (-1)^k \tau_k(g_k), \end{aligned}$$

as claimed. ■

4.1. Hattori–Stallings trace and rank. If $P \in \text{proj}(\mathbf{A})$, then the canonical map γ_P admits an inverse, and hence $\text{ev}_P \circ \gamma_P^{-1}: \text{End}_{\mathbf{A}}(P) \rightarrow \underline{\mathbf{A}}$ is defined. One immediately sees that it is a trace.

1.14. LEMMA. *For any $P \in \text{proj}(\mathbf{A})$ and any $f \in \text{End}_{\mathbf{A}}(P)$, let*

$$\text{tr}_{P/A}(f) = \text{ev}_P \circ \gamma_P^{-1}(f).$$

Then $\text{tr}_{-/A}(-)$ is an $\underline{\mathbf{A}}$ -valued trace.

PROOF. Additivity is immediate. Commutativity can be proved, using matrices: suppose $P, Q \in \text{proj}(\mathbf{A})$ and $\alpha: P \rightarrow Q, \beta: Q \rightarrow P$ are maps. Fix a (projective) basis $\{p_i\}_{i \in I}$ of P , with dual basis $\{p_i^*\}$ and a basis $\{q_j\}_{j \in J}$ of Q with dual basis $\{q_j^*\}$. Then one can write uniquely

$$\alpha = \sum_{i,j} \alpha_{i,j} p_i^*(-) q_j \quad \text{and} \quad \beta = \sum_{j,i} \beta_{j,i} q_j^*(-) p_i.$$

By elementary linear algebra one can write

$$\alpha\beta = \sum_{j,j'} \sum_i \alpha_{i,j} \beta_{j',i} q_j^*(-) q_{j'}$$

and

$$\beta\alpha = \sum_{i,i'} \sum_j \beta_{j,i} \alpha_{i',j} p_i^*(-) p_{i'},$$

hence,

$$\text{tr}_{Q/A}(\alpha\beta) = \sum_{i,j} \alpha_{i,j} \beta_{j,i} + [\mathbf{A}, \mathbf{A}] = \sum_{j,i} \beta_{j,i} \alpha_{i,j} + [\mathbf{A}, \mathbf{A}] = \text{tr}_{P/A}(\beta\alpha),$$

as claimed. For the third condition, just remark that any short exact sequence of projective modules splits, then $M \simeq M' \oplus M''$, and f can be written in blocks with f' and f'' in the diagonal. ■

Since projective resolutions are homotopy equivalent and by Lemma 1.13, one can give the following definition.

1.15. DEFINITION (Hattori–Stallings trace). Let M be a left \mathbf{A} -module of type \mathbf{FP} , and let P_\bullet be a projective resolution of M and let f_\bullet be a lifting on f along P_\bullet .

The *Hattori–Stallings* trace of an endomorphism $f \in \text{End}_{\mathbf{A}}(M)$ is

$$\text{tr}_M(f) = H_0(\text{ev}_P \circ (\gamma_P)^{-1})([f_\bullet]) \in \underline{\mathbf{A}}.$$

In particular one has

$$\text{tr}: \underline{\text{Ext}}_0^{\mathbf{A}}(M, M) \rightarrow \underline{\mathbf{A}}.$$

This trace has the properties summarized as follows.

1.16. PROPOSITION. Let $P = (P_\bullet, \partial_\bullet^P)$ be a finite, projective complex of left \mathbf{A} -modules, and let $[f], [g] \in \underline{\text{Ext}}_0^{\mathbf{A}}(P, P)$, $f = \sum_{k \in \mathbb{Z}} f_k$, be homotopy classes of chain maps of degree 0. Then

- (1) $\text{tr}_P([f]) = \sum_{k \in \mathbb{Z}} (-1)^k \text{tr}_{P_k}(f_k)$;
- (2) $\text{tr}_P([f] \circ [g]) = \text{tr}_P([g] \circ [f])$.
- (3) Let $Q = (Q_\bullet, \partial_\bullet^Q)$ be another finite, projective complex of left \mathbf{A} -modules which is homotopy equivalent to P , i.e., there exist chain maps $\varphi: P \rightarrow Q$, $\psi: Q \rightarrow P$, which composites are homotopy equivalent to the respective identity maps. Let $[h] \in \underline{\text{Ext}}_0^{\mathbf{A}}(Q, Q)$ such that $[\varphi] \circ [f] = [h] \circ [\varphi]$. Then $\text{tr}_P([f]) = \text{tr}_Q([h])$.

PROOF. Part (1) is a direct consequence of (1.6), and (2) follows from (1) and Lemma 1.14.

The left hand side quadrangle in the diagram

$$(1.9) \quad \begin{array}{ccccc} \underline{\text{Hom}}_{\mathbf{A}}(P, P) & \xleftarrow{\gamma} & P^{\otimes} \otimes_{\mathbf{A}} P & \xrightarrow{\text{ev}_P} & \underline{\mathbf{A}}[[0]] \\ \varphi \circ \circ \psi \downarrow & & \downarrow \psi^{\otimes} \otimes \varphi & & \parallel \\ \underline{\text{Hom}}_{\mathbf{A}}(Q, Q) & \xleftarrow{\gamma} & Q^{\otimes} \otimes_{\mathbf{A}} Q & \xrightarrow{\text{ev}_Q} & \underline{\mathbf{A}}[[0]] \end{array}$$

commutes, and the right hand side quadrangle commutes up to homotopy equivalence. This yields claim (3). \blacksquare

1.17. DEFINITION (Hattori–Stallings rank). The trace computed on the identity of M is the *Hattori–Stallings rank* of the module M .

$$r_M = \text{tr}_{M/\mathbf{A}}(\text{id}_M).$$

Under certain circumstances, the computation of the Hattori–Stallings rank is very easy.

1.18. LEMMA. Let $f \in \text{End}_{\mathbf{A}}(\mathbf{A}^n)$ be an endomorphism of the finitely generated, free, left \mathbf{A} -module \mathbf{A}^n , then

$$\text{tr}_{\mathbf{A}^n/\mathbf{A}}(f) = \text{tr} \varphi + [\mathbf{A}, \mathbf{A}],$$

where $\varphi = \varphi_{i,j}$ is the matrix representing f over the standard bases $\{e_i\}$ of \mathbf{A}^n and and e_j^* of $(\mathbf{A}^n)^*$.

Let P be a finitely generated, projective left \mathbf{A} -module, then it has the form $P = \text{im}(\pi)$, for some idempotent matrix $\pi = \pi^2 \in \text{Mat}_n(\mathbf{A})$, and

$$r_P = \text{tr}(\pi) + [\mathbf{A}, \mathbf{A}].$$

PROOF. One has $e_j^*(e_i) = \delta_{i,j}$ and

$$f = \sum_{i,j} \varphi_{i,j} (e_j^*(-) e_i) = \sum_{i,j} \gamma_{\mathbf{A}^n}(\varphi_{i,j} e_j^* \otimes e_i),$$

then $\text{tr}_{\mathbf{A}^n/\mathbf{A}}(f) = \sum_{i,j} \varphi_{i,j} e_j^*(e_i) + [\mathbf{A}, \mathbf{A}] = \sum_i \varphi_{i,i} + [\mathbf{A}, \mathbf{A}] = \text{tr} \varphi + [\mathbf{A}, \mathbf{A}]$, which proves the first statement.

The existence of π , as in the second statement, is a well-known characterization of finitely generated projective modules. Let $i: P \rightarrow A^n$ be the inclusion, and let $p: A^n \rightarrow P$ be the multiplication by π from the left.

Then by Definition 1.12(2) and the previous fact, one has

$$r_P = \operatorname{tr}_{P/A}(\operatorname{id}_P) = \operatorname{tr}_{P/A}(p \circ i) = \operatorname{tr}_{A^n/A}(i \circ p) = \operatorname{tr}(\pi) + [A, A]. \quad \blacksquare$$

4.2. The homotopy category. The homotopy and derived categories of a category of modules are the categories which model the behaviour of resolutions.

Let A be an R -algebra, let $\operatorname{Kom}_\bullet(A)$ be its chain complex category. Define the associated *homotopy category* $H(A)$ with objects the chain complexes of left A -modules and morphisms the homotopy classes of chain maps:

$$\operatorname{Obj}(H(A)) = \operatorname{Obj}(\operatorname{Kom}_\bullet(A)) \quad \text{and} \quad \operatorname{Mor}(H(A)) = \operatorname{Mor}(\operatorname{Kom}_\bullet(A)) / \approx.$$

Let $K(A)$ denote the additive category the objects of which are finite, projective chain complexes of left A -modules. Since morphisms are given by the homotopy classes of chain maps of degree 0, by Proposition 1.10 one has $\operatorname{Hom}_{K(A)}(P, Q) = \operatorname{Ext}_0^A(P, Q)$.

A *triangle* in $K(A)$ is a sequence $T_A = (A \rightarrow A' \rightarrow A'' \rightarrow A[-1])$ of chain complexes and homotopy classes of chain maps. A morphism of triangles T_A, T_B is a triple of maps $(f, f', f''): T_A \rightarrow T_B$ such that

$$\begin{array}{ccccccc} A & \xrightarrow{x} & A' & \xrightarrow{x'} & A'' & \xrightarrow{\omega} & A[-1] \\ f \downarrow & & f' \downarrow & & f'' \downarrow & & f[-1] \downarrow \\ B & \xrightarrow{y} & B' & \xrightarrow{y'} & B'' & \xrightarrow{\tau} & B[-1] \end{array}$$

is commutative in $K(A)$, i.e. it commutes in $\operatorname{Kom}_\bullet(A)$ up to homotopy.

In particular, $K(A)$ is a *triangulated category* (cf. [GM99], [Wei94, Chap. 10]) and distinguished triangles are triangles isomorphic to a triangle of the form

$$(1.10) \quad A \longrightarrow \operatorname{Cyl}(f) \longrightarrow \operatorname{Cone}(f) \longrightarrow A[-1],$$

for some $f: A \rightarrow B$ (cf. [Bou07a, Ch. X §2.6]).

Thus, if

$$(1.11) \quad A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

is a distinguished triangle in $K(A)$, one has $r_B = r_A + r_C$.

1.19. PROPOSITION. *Let $C = (C_\bullet, \partial_\bullet^C)$ be a chain complex of left A -modules concentrated in non-negative degrees with the following properties:*

- (1) C is acyclic,
- (2) C is bounded,
- (3) C_k is of type **FP** for all $k \in \mathbb{Z}$.

Then $H_0(C)$ is of type **FP**, and one has

$$(1.12) \quad r_{H_0(C)} = \sum_{k \geq 0} (-1)^k r_{C_k} \in \underline{A}.$$

PROOF. Let $\ell(C) = \min\{n \geq 0 \mid C_{n+j} = 0 \text{ for all } j \geq 0\}$ denote the length of C . We proceed by induction on $\ell(C)$. For $\ell(C) = 1$, there is nothing to prove. Suppose the claim holds for chain complexes D , $\ell(D) \leq \ell - 1$, satisfying the hypothesis (a)–(c), and let C be a complex satisfying (a)–(c) with $\ell(C) = \ell$. Let C^\wedge be the chain complex coinciding with C in all degrees $k \in \mathbb{Z} \setminus \{0\}$ and satisfying $C_0^\wedge = 0$. Then $C^\wedge[-1]$ satisfies (a)–(c) and $\ell(C^\wedge[-1]) \leq \ell - 1$. Then by induction, $M =$

$H_1(C^\wedge) = H_0(C^\wedge[-1])$ is of type **FP**, and $r_M = \sum_{k \geq 1} (-1)^{k+1} r_{C_k}$. By construction, one has a short exact sequence of left \mathbf{A} -modules $0 \rightarrow M \xrightarrow{\alpha} C_0 \rightarrow H_0(C) \rightarrow 0$. Let $(P_\bullet, \partial_\bullet^P, \varepsilon_M)$ be a finite, projective resolution of M , and let $(Q_\bullet, \partial_\bullet^Q, \varepsilon_{C_0})$ be a finite, projective resolution of C_0 . By the comparison theorem in homological algebra, there exists a chain map $\alpha_\bullet: P_\bullet \rightarrow Q_\bullet$ inducing α . Let $\text{Cone}(\alpha_\bullet)$ denote the mapping cone of α_\bullet . Then $(\text{Cone}(\alpha_\bullet), \tilde{\partial}_\bullet, \varepsilon_*)$ is a finite, projective resolution of $H_0(C)$, i.e., $H_0(C)$ is of type **FP**. Then, there is a distinguished triangle

$$P_\bullet \rightarrow Q_\bullet \rightarrow \text{Cone}(\alpha_\bullet) \rightarrow P[-1]_\bullet,$$

hence by the remark following (1.11) one has

$$(1.13) \quad r_{H_0(C)} = r_{\text{Cone}(\alpha_\bullet)} = r_Q - r_P = r_{C_0} - r_M.$$

This yields the inductive step and the claim. \blacksquare

5. Euler algebras and Euler characteristic

The purpose of the present section is to introduce some further structure on an algebra \mathbf{A} , so that one might associate to it an Euler characteristics (cf. §5.2).

1.20. DEFINITION (Algebra with antipode). Let \mathbf{A} be an associative R -algebra. An *antipode*, or *antipodal map*

$$-\natural: \mathbf{A} \rightarrow \mathbf{A}^{\text{op}}$$

is an isomorphism of R -algebras satisfying $-\natural^2 = \text{id}_{\mathbf{A}}$.

1.21. DEFINITION (Augmented algebra). Let \mathbf{A} be an associative R -algebra with antipode $-\natural$. Suppose there exists a linear character $\lambda \in \text{Hom}_{R\text{-Alg}}(\mathbf{A}, R)$ which defines a 1-dimensional left \mathbf{A} -module R_λ with left \mathbf{A} -action $a.r = \lambda(a)r$, and such that $\lambda(a^\natural) = \lambda(a)$ for all $a \in \mathbf{A}$.

Then the 3-tuple $(\mathbf{A}, -\natural, \lambda)$ is called an *augmented R -algebra* with antipode.

For arbitrary R -algebras, one may give the following definition.

1.22. DEFINITION (Trace function). For an associative R -algebra \mathbf{A} , a morphism $\tilde{\mu}: \mathbf{A} \rightarrow R$ is called a *trace function* if, and only if, $\tilde{\mu}(ab) = \tilde{\mu}(ba)$ for all $a, b \in \mathbf{A}$.

A trace function $\tilde{\mu}$ factors through the projection $\mathbf{A} \rightarrow \underline{\mathbf{A}} = \mathbf{A}/[\mathbf{A}, \mathbf{A}]$ and determines hence a map $\mu \in \text{Hom}_R(\underline{\mathbf{A}}, R)$; by abuse of language μ will be called a trace function as well.

For an augmented R -algebra with antipode $(\mathbf{A}, -\natural, \lambda)$ one may define \mathbf{A} to have some finiteness condition if, and only if, the trivial \mathbf{A} -module R_λ has that finiteness condition. In particular, one says that \mathbf{A} is of type **FP** if, and only if, the module R_λ is of type **FP**.

Finally, one may give the following definition, which will be central in the present thesis.

1.23. DEFINITION (Euler algebras). Assume that $(\mathbf{A}, -\natural, \lambda)$ is an augmented algebra of type **FP**, with an antipode, and such that $\underline{\mathbf{A}}$, considered as R -module, is free. Suppose moreover there exists a free R -basis $\mathcal{B} \subseteq \underline{\mathbf{A}}$ satisfying

- (1) $1 \in \mathcal{B}$;
- (2) $\mathcal{B}^\natural = \mathcal{B}$;
- (3) the symmetric R -bilinear form

$$(1.14) \quad \langle -, - \rangle: \mathbf{A} \times \mathbf{A} \rightarrow R, \quad \langle a, b \rangle = \delta_{a,b} \lambda(a), \quad a, b \in \mathcal{B},$$

where $\delta_{\cdot, \cdot}$ denotes Kronecker's δ -function, satisfies

$$(1.15) \quad \langle ab, c \rangle = \langle b, a^\natural c \rangle \quad \text{for all } a, b, c \in \mathbf{A}.$$

Then, let

$$\tilde{\mu}: \mathbf{A} \rightarrow R, \quad \tilde{\mu}(a) = \langle 1, a \rangle$$

and let $\mu: \mathbf{A} \rightarrow R$ the induced map (cf. Definition 1.22).

The 5-tuple $(\mathbf{A}, -^{\natural}, \lambda, \mathcal{B}, \mu)$ is called an *Euler R -algebra*.

1.24. LEMMA. *If $\mathbf{A} = (\mathbf{A}, -^{\natural}, \lambda, \mathcal{B})$ is an associative, augmented R -algebra with an antipode and a distinguished basis, then the map $\tilde{\mu}$ defined as in (1.14) is a trace function.*

PROOF. By definition, one has for all $a, b \in \mathbf{A}$ that $\langle a^{\natural}, b^{\natural} \rangle = \langle a, b \rangle$. Hence

$$(1.16) \quad \tilde{\mu}(ab - ba) = \langle 1, ab \rangle - \langle 1, ba \rangle = \langle a^{\natural}, b \rangle - \langle b^{\natural}, a \rangle = 0.$$

for all $a, b \in \mathbf{A}$. Then $\mu: \mathbf{A} \rightarrow R$ is the induced trace function. \blacksquare

5.1. Hattori–Stallings trace and Euler subalgebras. One needs to know the behaviour of the Hattori–Stallings trace with respect to subalgebras, assuming some compatibility conditions on the structures of associative, augmented algebras with antipode.

1.25. NOTATION. If $\mathbf{B} \leq \mathbf{A}$ is an inclusion of R -algebras, then $[\mathbf{B}, \mathbf{B}] \subseteq [\mathbf{A}, \mathbf{A}]$, hence there is an induced map of R -modules

$$(1.17) \quad \mathrm{tr}_{\mathbf{B}/\mathbf{A}}: \underline{\mathbf{B}} \rightarrow \underline{\mathbf{A}}, \quad b + [\mathbf{B}, \mathbf{B}] \mapsto b + [\mathbf{A}, \mathbf{A}]$$

1.26. DEFINITION. Let $(\mathbf{A}, -^{\natural}, \lambda, \mathcal{B}, \mu)$ be an Euler algebra. Let $\mathbf{B} \subseteq \mathbf{A}$ be an R -subalgebra of \mathbf{A} such that

- \mathbf{A} is a flat right \mathbf{B} -module;
- $\mathbf{B}^{\natural} = \mathbf{B}$;
- $\mathcal{B} \cap \mathbf{B}$ is a free R -basis of \mathbf{B} .

Then \mathbf{B} will be called an *Euler subalgebra* of \mathbf{A} .

Then, one has the following elementary fact.

1.27. LEMMA. *The 5-tuple $\mathbf{B} = (\mathbf{B}, -^{\natural}|_{\mathbf{B}}, \lambda|_{\mathbf{B}}, \mathcal{B} \cap \mathbf{B}, \mu|_{\mathbf{B}})$ is an Euler algebra.*

Call $\mu_{\mathbf{B}}$ its canonical trace $\mu_{\mathbf{B}}: \underline{\mathbf{B}} \rightarrow R$, then there is a commutative diagram

$$(1.18) \quad \begin{array}{ccc} \underline{\mathbf{B}} & \xrightarrow{\mathrm{tr}_{\mathbf{B}/\mathbf{A}}} & \underline{\mathbf{A}} \\ \mu_{\mathbf{B}} \searrow & & \swarrow \mu_{\mathbf{A}} \\ & R & \end{array}$$

1.28. PROPOSITION. *Let M be a left \mathbf{B} -module of type **FP**. Then $\mathbf{ind}_{\mathbf{B}}^{\mathbf{A}}(M)$ is of type **FP**, and one has*

$$(1.19) \quad r_{\mathbf{ind}_{\mathbf{A}}^{\mathbf{B}}(M)} = \mathrm{tr}_{\mathbf{B}/\mathbf{A}}(r_M).$$

PROOF. Induction $\mathbf{ind}_{\mathbf{B}}^{\mathbf{A}} = \mathbf{A} \otimes_{\mathbf{B}} -$ is a covariant additive right-exact functor mapping finitely generated projective left \mathbf{B} -modules to finitely generated projective left \mathbf{A} -modules. Moreover, if \mathbf{A} is a flat right \mathbf{B} -module, then $\mathbf{ind}_{\mathbf{B}}^{\mathbf{A}}$ is exact. Let P be a finitely generated left \mathbf{B} -module, and let $Q = \mathbf{ind}_{\mathbf{B}}^{\mathbf{A}}(P)$. Then one has a canonical map $\iota: P \rightarrow Q$, $\iota(p) = 1 \otimes p$, which is a homomorphism of left \mathbf{B} -modules. As induction is left adjoint to restriction, every map $f \in \mathrm{End}_{\mathbf{B}}(P)$ induces a map $\iota \circ f = (\iota \circ f)_* \in \mathrm{End}_{\mathbf{A}}(Q)$.

Let $P^* = \text{Hom}_{\mathbf{B}}(P, \mathbf{B})$ and $Q^* = \text{Hom}_{\mathbf{A}}(Q, \mathbf{A})$. Then for $f \in \text{Hom}_{\mathbf{B}}(P, \mathbf{B})$ one has an induced map $\iota_*(f) = (j \circ f)_* \in Q^*$ making the diagram

$$(1.20) \quad \begin{array}{ccccc} \text{End}_{\mathbf{B}}(P) & \xleftarrow{\gamma_P} & P^* \otimes_{\mathbf{B}} P & \xrightarrow{\text{ev}_P} & \mathbf{B} \\ \iota_* \downarrow & & \downarrow \iota_* \otimes \iota & & \downarrow \text{tr}_{\mathbf{B}/\mathbf{A}} \\ \text{End}_{\mathbf{A}}(Q) & \xleftarrow{\gamma_Q} & Q^* \otimes_{\mathbf{A}} Q & \xrightarrow{\text{ev}_Q} & \mathbf{A} \end{array}$$

commute. ■

1.29. COROLLARY. *One has, for a \mathbf{B} -module M of type \mathbf{FP} ,*

$$\mu_{\mathbf{A}}(r_{\text{ind}_{\mathbf{B}}^{\mathbf{A}}(M)}) = \mu_{\mathbf{A}}(\text{tr}_{\mathbf{B}/\mathbf{A}}(r_M)) = \mu_{\mathbf{B}}(r_M).$$

5.2. The Euler characteristic. The machinery developed in this chapter finally allows the definition of the Euler characteristic of an Euler algebra: this is the main technical tool we had to introduce in order to prove the main results of Chapter 5 §8.

1.30. DEFINITION (Euler characteristics). Let $\mathbf{A} = (\mathbf{A}, -^{\natural}, \lambda, \mathcal{B}, \mu)$ be a Euler R -algebra. Then

$$(1.21) \quad \chi_{\mathbf{A}} = \mu(r_{R\lambda}).$$

will be called the *Euler characteristic* of \mathbf{A} .

1.31. EXAMPLE. Later in the discussion, Proposition 2.2 will show that group algebras are actually Euler algebras, and hence Definition 1.30 applies.

Let R be a commutative ring, let G be a group of type \mathbf{FP} over R . The Euler characteristics (as defined above) $\chi_{R[G]}$ then coincides with the usual R -Euler characteristics of G (cf. [Bro82] or [Bas76]).

Facts from group theory

Representation theory is, *par excellence*, representation theory of groups. This is one of the most fundamental and active areas of research in pure mathematics.

1. Groups and group rings

Group representations are conveniently thought of as modules over group algebras. In the present work, unless otherwise specified, groups are to be intended *discrete groups*.

If R is a commutative ring and G is a group, the R -group algebra of G is denoted $R[G]$.

The following lemma focuses on some features of group algebras which are shared with Hecke algebras (cf. Chapter 5).

2.1. LEMMA. *Let $R[G]$ be a group algebra. Then*

- (1) *any left G -module (M, \cdot) becomes a right G module (M, \circ) via $g \circ m = m \cdot g^{-1}$;*
- (2) *for $H \leq G$ a subgroup, there is an inclusion of R -algebras $R[H] \leq R[G]$, the corresponding induction and restriction functors are simply denoted $\mathbf{ind}_H^G: R[H]\text{-Mod} \rightarrow R[G]\text{-Mod}$ and $\mathbf{res}_H^G: R[G]\text{-Mod} \rightarrow R[H]\text{-Mod}$;*
- (3) *the right H -module $\mathbf{res}_H^G R[G]^{\text{reg}}$ is free, and a set of representatives of the right coset space G/H is a basis.*

One immediately has the following result.

2.2. PROPOSITION. *Let G be a group and let R be a commutative ring. Then $R[G]$ is canonically an associative, augmented R -algebra with antipode. Moreover if $R[G]$ is of type **FP**, then it is canonically an Euler algebra.*

PROOF. The R -algebra $R[G]$ is associative and one canonically puts

- (1) the set G as a distinguished free R -basis of $R[G]$ containing $1_{R[G]} = 1_R 1_G$;
- (2) the map $_{\natural}: R[G]^{\text{op}} \rightarrow R[G]$ given by $\sum_{g \in G} r_g g \mapsto \sum_{g \in G} r_g g^{-1}$ as distinguished involution, then $G^{\natural} = G$ and $1^{\natural} = 1$;
- (3) $\lambda(g) = 1$ for all $g \in G$, and hence
- (4) $\tilde{\mu}\left(\sum_{g \in G} r_g g\right) = r_1$.

Thus, all the conditions of Definition 1.23 are fulfilled. ■

2.3. REMARK. The above proposition simply restates, from an R -algebra-theoretic point of view, the condition that G is of type **FP** over R (cf. [Bro82, Ch. VIII §6]).

2. Cohomology of groups

2.4. REMARK. The import of the involution $_{\natural}$ is converting left modules into right ones and viceversa. In particular the tensor product of left G -modules M and N is defined as the quotient $M \otimes_R N / \langle g^{\natural} m \otimes n - m \otimes gn \rangle$.

Let $R[G]$ be the group algebra of a group G and let M be a left $R[G]$ -module with a resolution $Q_\bullet \rightarrow M$, then define

$$H_\bullet(G, M) = \mathrm{Tor}_\bullet^{R[G]}(R, M) \simeq H_\bullet(R[[0]] \otimes_{R[G]} Q_\bullet).$$

In particular, if R is the trivial $R[G]$ -module, i.e., the 1-dimensional module with action $g.r = r$ for all $g \in G$, and P_\bullet is a projective resolution of R over $R[G]$, then $H_\bullet(G, R) = H_\bullet(P)$.

It follows that any computation related to (co)homology, one needs projective resolutions. In many cases resolutions are produced using methods from algebraic topology (cf. [Bro82] or [Geo08, Ch. 8]).

Finiteness conditions for group algebras often reduce to the notion of *cohomological dimension* (over R), i.e., $\mathrm{cd}_R(G) = \mathrm{pd}_{R[G]}(R)$. When $R = \mathbb{Z}$ one simply writes $\mathrm{cd}(G) = \mathrm{cd}_{\mathbb{Z}}(G)$, and this is an upper bound for the cohomological dimension over any commutative ring R .

2.1. Universal space for proper group actions and resolutions. The purpose of this section is to give a brief overview of common ways to produce resolutions of the trivial $R[G]$ -module. The situation, in general (e.g., for any commutative ring R) is highly non-trivial, but the present methods gave strong motivation to the work (cf. Chapter 5 §5).

A CW-complex X (cf. [Hat02, Ch. 0 and Ch. 2 §2]), together with a cellular action $G \curvearrowright X$ is called a G -CW-complex.

Let G be a group, and let X be a G -CW-complex such that

- X is non-empty, say $x_0 \in X$,
- $\pi_1(X, x_0) \simeq G$,
- $\pi_k(X, x_0) = 0$ for $k > 1$.

Then X is a model of the Eilenberg–MacLane space $K(G, 1)$ (also denoted $B(G)$).

For any group, there is a standard construction of a $B(G)$ starting with a bouquet of 1-spheres indexed by a system of generators, and adding cells “along the relations” to kill higher degree homotopy.

The universal cover \bar{X} of a $B(G)$ -space is a (model of a) *classifying space* for G and it is denoted $E(G)$. It is contractible and there is a cellular G -action $G \curvearrowright \bar{X}$ given by the “deck transformations”. The action is free, i.e., point stabilizers are trivial.

If X and Y are $B(G)$ ’s, then the two spaces have the same cellular* homology, and the cellular complex arising from a $E(G)$ -space is a free $\mathbb{Z}[G]$ -resolution of the trivial module \mathbb{Z} .

A group having a $B(G)$ which is a finite G -CW-complex is called a group of type **F**. If G is of type **F**, then the mentioned resolution is a finite, free $\mathbb{Z}[G]$ -resolution of \mathbb{Z} . This cannot occur, e.g., when G has non-trivial torsion or infinite cohomological dimension.

The problem of having a non-bounded resolution can be avoided using a more general construction (cf. [tD87, Ch. I §6] or [Mis03, §2]) of a G -CW-complex allowing point stabilizers to be in some predefined class of groups.

Loosely speaking, and under suitable conditions, the situation is the following:

- + the G -CW-complex is finite-dimensional, thus the cellular complex it determines is a bounded resolution of the trivial module;
- if there are non-trivial stabilizers the G -action is not free, and in particular the cellular complex is no more a free resolution of the trivial module;

*Cellular homology is the homology functor suitable for CW-complexes, and it is isomorphic to singular homology.

+ under further conditions on the class of point stabilizers, it may happen that the resolution is projective (and bounded) nonetheless.

If G is a group, a collection \mathfrak{X} of subgroups of G such that.

- $\{1\} \in \mathfrak{X}$,
- if $H \leq H' \leq G$ and $H' \in \mathfrak{X}$, then $H \in \mathfrak{X}$,
- if $H \in \mathfrak{X}$ and $g \in G$, then $H^g \in \mathfrak{X}$.

is called a *class of subgroups of G* .

The “trivial” collection \mathfrak{T} containing only the trivial subgroups is a class of groups. The collection \mathfrak{F} consisting of all finite subgroups of G is a class of subgroups.

With respect to a class \mathfrak{X} of subgroups of G , one may define (cf. [tD87, Ch. I §6]) a *classifying space $E_{\mathfrak{X}}(G)$ for the family \mathfrak{X}* .

A G -CW-complex X which is a model for $E_{\mathfrak{X}}(G)$ is characterized, up to G -homotopy equivalence, by the following properties:

- $G_x \in \mathfrak{X}$ for all $x \in X$
- The fixed point subspace X^{G_x} is contractible for all $x \in X$.

A model X of $E_{\mathfrak{F}}(G)$ is called a *universal space for proper group actions* and it is denoted $\underline{E}(G)$. One may have a finite-dimensional $\underline{E}(G)$ also when G has non-trivial torsion.

Finally, the cellular complex (with rational coefficient) of a $\underline{E}(G)$ -space is a projective resolution of the trivial $\mathbb{Q}[G]$ -module \mathbb{Q} (cf. [Mis01]).

In particular, if there is a finite-dimensional model of $\underline{E}(G)$, then there is a bounded, projective $\mathbb{Q}[G]$ -resolution of \mathbb{Q} .

2.5. REMARK. In Chapter 3 §5 it will be shown that, for a class of Coxeter systems (W, S) , there is a canonical, *finite-dimensional* model $\Sigma(W, S)$ of the $\underline{E}W$.

The associated chain complex will constitute a projective resolution of the trivial module, consisting of permutation modules.

3. Finitely generated groups: growth series

Let $\text{Cay}(G, X)$ be the Cayley graph of the finitely generated group G with respect to the finite generating set X and let ℓ be the associated length function (cf. Appendix A §1).

For all $n \in \mathbb{N}_0$ let $a_n = |\mathcal{S}(n)| = |\{g \in G \mid \ell(g) = n\}|$, then $a_n < \infty$ for all n and one may define the classical growth series (cf. [GdlH97]) as follows

$$p_{(G, X)}(t) = \sum_{n \geq 0} a_n t^n \in \mathbb{C}[[t]].$$

To simplify the discussion define moreover $b_n = |\{g \in G \mid \ell(g) \leq n\}| = \sum_{k=0}^n a_k$ and

$$\bar{p}_{(G, X)}(t) = \sum_{n \geq 0} b_n t^n \in \mathbb{C}[[t]];$$

it is the cumulative series of $p_{(G, X)}(t)$ (cf. Appendix C) and the coefficients b_n 's of this series satisfy

- (1) $b_0 = 1$ and $b_1 = |X| + 1$,
- (2) $b_n \geq b_{n-1}$ for all $n \geq 1$,
- (3) $b_{n+m} \leq b_n b_m$ for all $n, m \in \mathbb{N}_0$, since any product of two words of length not exceeding, respectively, n and m has length not exceeding $n + m$.

By Lemma C.3 the sequence $\sqrt[n]{b_n}$ admits a limit, hence the convergence radius $\bar{\rho}$ of $\bar{p}(t)$ is computed by

$$1/\bar{\rho} = \lim_{n \rightarrow \infty} \sqrt[n]{b_n}.$$

By (3) one checks that $|X| + 1 = b_1 \geq \sqrt[n]{b_n}$, hence $\bar{\rho} \geq 1/(|X| + 1) > 0$ and the series $\bar{p}_{(G,X)}(t)$ converges in a non-empty open disk. By Lemma C.1 one further has that the series $p_{(G,X)}(t)$ converges in an open disk of radius $\rho \geq 1/(|X| + 1)$ to a complex analytic function $\varphi_{(G,X)}$.

The Poincaré series of a finitely generated group is a power series with non-negative integer coefficients. Thus, Lemma C.2 reads:

2.6. PROPOSITION. *Let G be a finitely generated group with a finite generating set X . Let $p_{(G,X)}(t)$ be its Poincaré series.*

- (1) *If G is finite, then $p_{(G,X)}(t)$ is a polynomial;*
- (2) *if G is infinite, then $p_{(G,X)}(t)$ converges in an open disk of \mathbb{C} of radius $\rho \in (0, 1]$ to a complex analytic function.*

For a finitely generated group there is a well-defined notion of growth.

2.7. DEFINITION. Let (G, X) be a finitely generated group with finite generating set X , and cumulative growth function

$$\bar{p}(t) = \bar{p}_{(G,X)}(t) = \sum_{k \in \mathbb{N}_0} b_k t^k.$$

- (1) *If there exists a polynomial $\beta(k) \in \mathbb{Z}[t]$ such that $\beta(k) \geq b_k$ for all k , then (G, X) is of *polynomial growth*.*
- (2) *If there exist constants $c, b > 1$ such that $b_k \geq cb^k$ for all k , then (G, X) is of *exponential growth*.*

It is well-known that being of polynomial growth (resp., of exponential growth) is independent of the chosen (finite) generating system.

2.8. REMARK. An active area of research in years 1970-'80 was to determine whether there could possibly exist finitely generated groups of *intermediate* growth. R. Grigorchuk gave a positive answer in [Gri84].

On the other hand, it was known that no *linear* group could have intermediate growth (cf. [Tit72, Corollary 5]).

2.9. LEMMA. *Let G be a finitely generated linear group. For any finite generating set X let $\bar{p}_{(G,X)}(t)$ be the cumulative growth function and let $\rho_{(G,X)}$ be its convergence radius. Then*

- (1) *$\rho \geq 1$ if, and only if, G has polynomial growth;*
- (2) *$\rho < 1$ if, and only if, G has exponential growth.*

PROOF. In view of the previous Remark 2.8, it suffices to compute the convergence radius of a group of polynomial growth with bounding (from above) polynomial $\beta(k)$:

$$\frac{1}{\rho} = \limsup_k \sqrt[k]{b_k} \leq \limsup_k \sqrt[k]{\beta(k)} = 1,$$

and the convergence radius of a group of exponential growth with bounding (from below) exponential function cb^k

$$\frac{1}{\rho} = \limsup_k \sqrt[k]{b_k} \geq \limsup_k \sqrt[k]{cb^k} = b > 1.$$

This proves the claim. ■

One further elementary fact will be needed in the sequel.

2.10. LEMMA. *Let G be a finitely generated group with finite generating set X and Poincaré series $p_{(G,X)}(t)$ with convergence radius ρ_G . Any quotient $G' = G/N$ is finitely generated with a finite generating set $X' = \{xN \mid x \in X\}$; thus denote the*

canonical length function by ℓ' . If the Poincaré series $p_{(G',X')}(t)$ of the quotient has convergence radius $\rho_{G'}$, then

$$\rho_G \leq \rho_{G'}.$$

PROOF. If G is finite, every quotient is finite and both convergence radii are infinite. If G is infinite and G' is finite, then one has $\rho_G \leq 1 < \infty = \rho_{G'}$, as claimed.

Then suppose G and G' are infinite groups: by Lemma C.2(2) one can compute convergence radii through the cumulative growth function.

Let $b_n = |\ell^{-1}(\{0, \dots, n\})|$ and $b'_n = |(\ell')^{-1}(\{0, \dots, n\})|$.

Let $gN \in G'$ be such that $\ell'(gN) \leq n$, then there is a reduced expression $gN = (x_1N)(x_2N) \dots (x_tN) = x_1x_2 \dots x_tN$, for $t \leq n$, $x_i \in X$. Consider another $hN \in G'$, $\ell'(hN) \leq n$ and $hN = (y'_1N)(y'_2N) \dots (y'_rN)$ a reduced expression; then $hN \neq gN$ implies $y_1y_2 \dots y_r \neq x_1x_2 \dots x_t$.

Moreover $\ell(x_1x_2 \dots x_t) \leq n$. Thus, for every $gN \in G'$, fix a reduced expression $gN = (x_1N)(x_2N) \dots (x_tN)$, and define the map

$$\begin{array}{ccc} (\ell')^{-1}(\{0, \dots, n\}) & \rightarrow & \ell^{-1}(\{0, \dots, n\}) \\ gN & \mapsto & x_1x_2 \dots x_t \end{array};$$

it is an injective map of finite sets, hence $b'_n \leq b_n$ for all $n \geq 0$.

Then the statement follows from Lemma C.2(3). ■

An extremely important result about groups of polynomial growth is the following; it will be fundamental in Chapter 4 to prove the amenability (cf. §4) of some Coxeter groups.

2.11. THEOREM ([Gro81]). *Let G be a finitely generated group of polynomial growth. Then G is virtually nilpotent, i.e., it has a finite-index nilpotent subgroup.*

4. Amenability

Amenable groups are a very significant class of groups which play a decisive rôle in harmonic analysis, measure theory, geometric group theory, as well as in abstract group theory.

This section has the only purpose of collecting a few results about amenability, growth and lattices which will be useful later. They are available, e.g., in [BdlHV08, App. G] or [CSC10, Ch. 4]

2.12. DEFINITION (Amenable group). Let G be a locally compact topological group, with Haar measure μ . A linear function $m: \ell^\infty(G, \mu) \rightarrow \mathbb{R}$ is called a *left-invariant mean* if, and only if, the following hold:

- the operator norm $\|m\| = 1$,
- $m(f) \geq 0$ if $f(g) \geq 0$ for all $g \in G$,
- $m(1_G) = 1$ where 1_G is the constant function on G with value $1 \in \mathbb{R}$.
- $m(f \circ \lambda_g) = m(f)$ for all $g \in G$, where $\lambda_g(x) = gx$ is the left translation operator on G .

Then, G is called *amenable* if, and only if, it admits a left-invariant mean m .

2.13. REMARK. If the group G is discrete, the above definition takes a much simpler form. A discrete group G is amenable if, and only if, there is a function $m: 2^G \rightarrow \mathbb{R}$ such that the following conditions hold.

- $m(G) = 1$,
- $m(X_1 \sqcup X_2 \sqcup \dots \sqcup X_k) = m(X_1) + m(X_2) + \dots + m(X_k)$ for disjoint subsets $X_i \subseteq G$,
- $m(gX) = m(X)$ for all $g \in G$ and all $X \subseteq G$.

2.14. PROPOSITION. *The following facts are well-known.*

- (1) *Finite, abelian, nilpotent, or soluble groups are amenable.*
- (2) *The free group on 2 generators is not amenable.*
- (3) *A subgroup of an amenable group is amenable.*
- (4) *If G is virtually amenable, then G is amenable.*
- (5) *Let G be a locally compact group, with a lattice $H \leq G$ (i.e., H is a discrete subgroup of G such that the fundamental domain $H \backslash G$ of the action $H \curvearrowright G$ has finite Haar measure). Then G is amenable if, and only if H is amenable.*
- (6) *If G is a group of polynomial growth, then G is amenable.*
- (7) *If G is a semisimple, non compact Lie group, then G is not amenable. In particular, the indefinite orthogonal group $O(p, q)$, $p, q > 0$ is not amenable.*

PROOF. Finite and soluble groups are amenable by [CSC10, Prop. 4.4.6, and Thm. 4.6.3], thus (1) follows.

Statement (2) is [CSC10, Thm. 4.4.7], while (3) is [CSC10, Prop. 4.5.1]

Part (4) is [CSC10, Prop. 4.5.8] and statement (5) is [BdlHV08, Cor. G.3.8].

By Gromov's Theorem 2.11, if G has polynomial growth, then it is virtually nilpotent, thus virtually amenable by (1) and hence amenable by (4). This proves (6).

Finally, (7) is [BdlHV08, Ex. G.3.6(ii)]. ■

2.15. COROLLARY. *Let G be a linear group. Then the following are equivalent.*

- (1) *G is amenable*
- (2) *G does not contain a subgroup isomorphic to the free group on two generators.*

PROOF. One only needs to prove that (2) implies (1): thus assume G is linear without free non abelian subgroups. By Tits' alternative [Tit72, Thm. 1], then G is virtually soluble and, hence, amenable. ■

CHAPTER 3

Coxeter groups

“ Pochi sanno che, ascoltando all'incontrario il death metal, scopriranno dei messaggi che, ascoltati all'incontràr, sono uguali al death metal che avevi primi di girarli.

ELIO E LE STORIE TESE

Coxeter groups are an ubiquitous class of groups appearing in Lie theory, geometry, topology, arithmetic and several other areas of pure and applied mathematics.

The basic idea behind Coxeter groups is to capture the behaviour of mirror reflections of the space. The first section is a fast survey of various situations where groups generated by “reflections” appear. The main goal is to provide a geometric insight for Coxeter groups, which are the subject of §2ff.

1. Examples of Coxeter groups

1.1. Geometric reflection groups. For each $n > 1$ there is (up to isometry) exactly one n -dimensional, connected and simply connected, Riemannian manifold with constant sectional curvature κ : it is any of an n -sphere, an n -affine plane or an n -hyperbolic plane.

They have, respectively, positive, zero and negative curvature; up to rescaling the metric, suppose $\kappa \in \{1, 0, -1\}$. It is well known that \mathbb{S}^n , \mathbb{E}^n and \mathbb{H}^n admit the following embeddings:

$$\begin{aligned}\mathbb{S}^n &= \left\{ x \in \mathbb{R}^{n+1} \mid \sum_1^{n+1} x_i^2 = 1 \right\} \subseteq \mathbb{R}^{n+1}, \\ \mathbb{E}^n &= \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} = 1 \} \subseteq \mathbb{R}^{n+1}, \\ \mathbb{H}^n &= \left\{ x \in \mathbb{R}^{n+1} \mid \sum_1^n x_i^2 - x_{n+1}^2 = -1, x_{n+1} > 0 \right\} \subseteq \mathbb{R}^{n,1},\end{aligned}$$

where \mathbb{R}^{n+1} has the standard scalar product $(-, -)$ and $\mathbb{R}^{n,1}$ is again \mathbb{R}^{n+1} as vector space, but is given the “Minkowski product” defined by $(x, y)_{n,1} = \sum_1^n x_i y_i - x_{n+1} y_{n+1}$.

These manifolds, any of which will be denoted \mathbb{X}^n , admit closed, totally geodesic submanifolds of codimension 1, that is, *hyperplanes*, which are given by the intersection of \mathbb{X}^n with a linear hyperplane in \mathbb{R}^{n+1} .

A hyperplane $H \subseteq \mathbb{X}^n$ separates \mathbb{X}^n in two connected components, and a reflection across H is a non-trivial isometry of \mathbb{X}^n into itself fixing H pointwise. Explicitly one has the following characterizations.

- If $\mathbb{X}^n = \mathbb{S}^n$, then a hyperplane H is a great $(n-1)$ -sphere determined by the linear hyperplane \overline{H} , and a reflection across H is given by

$$\rho_H(x) = x - 2(x, \nu)\nu,$$

where ν is a unit normal vector to \overline{H} .

- If $\mathbb{X}^n = \mathbb{E}^n$, then a hyperplane H is an affine subspace determined by the linear hyperplane \overline{H} , and a reflection across H is given by

$$\rho_H(x) = x - 2(x - h, \nu)\nu,$$

where ν is a unit normal vector to \overline{H} and $h \in \overline{H}$.

- If $\mathbb{X}^n = \mathbb{H}^n$, a hyperplane is determined by a “timelike” linear hyperplane \overline{H} , and a reflection across H is given by

$$\rho_H(x) = x - 2(\nu, x)_{n,1}\nu,$$

where ν is a “spacelike”, unit, normal vector to \overline{H} .

It is easy to show that ρ_H is independent of both ν and h , hence it only depends on the hyperplane.

In all cases H is the fixed-point submanifold of the isometry ρ_H (this *a fortiori* proves that H is closed, totally geodesic), and it is isometric to \mathbb{X}^{n-1} . The subspace H separates \mathbb{X}^n and the closure of each connected component of $\mathbb{X}^n \setminus H$ is called a *half space* or *root*.

Suppose $\mathcal{L} = \{L_i \mid i \in \{1, \dots, k\}\}$ is a finite collection of half-spaces of \mathbb{X}^n such that their intersection $P_{\mathcal{L}} = \bigcap_{i=1}^k L_i$ has non-empty interior. Then $P_{\mathcal{L}}$ is the *convex polytope* determined by \mathcal{L} ; it is *spherical*, *affine* or *hyperbolic* according to whether \mathbb{X}^n is \mathbb{S}^n , \mathbb{E}^n or \mathbb{H}^n , respectively. We further assume that, for all $i \in \{1, \dots, k\}$, one has $L_i \not\supseteq \bigcap_{j \neq i} L_j$: under this mild hypothesis, the polytope P and the collection \mathcal{L} determine each other uniquely.

In \mathbb{X}^n , if L_1 and L_2 are half-spaces bounded by hyperplanes $H_1 = \text{bdr } L_1$ and $H_2 = \text{bdr } L_2$, such that $H_1 \cap H_2 \neq \emptyset$, with normal vectors ν_1 and ν_2 (either both pointing inwards or both outwards), one says that the *dihedral angle* they determine is $\vartheta(H_1, H_2) = \pi - \arccos((\nu_1, \nu_2))$.

Associated to a polytope $P = P_{\mathcal{L}}$ there is a *geometric reflection group* $W_{\mathcal{L}}$. It is the subgroup of $\text{Iso}(\mathbb{X}^n)$ generated by the reflections across the hyperplanes bounding P , i.e.

$$W_{\mathcal{L}} = \langle \rho_H \mid H = \text{bdr } L, L \in \mathcal{L} \rangle \leq \text{Iso}(\mathbb{X}^n),$$

with the fixed action $W_{\mathcal{L}} \curvearrowright \mathbb{X}^n$.

A particular place in the theory of geometric reflection group is played by the groups associated to polytopes with angles which are integral submultiples of π .

Under mild hypotheses, one has the following (cf. [Dav08, Ch. 6]) characterization.

Let $n > 1$ and let \mathcal{L} be a collection of half-spaces in \mathbb{X}^n , with bounding hyperplanes $\{H_1, \dots, H_k\}$. Suppose that

- (1) $\mathcal{L} = \{L_1, \dots, L_k\}$ and, for $i \neq j$, $L_i \not\supseteq L_j$,
- (2) if $H_i \cap H_j \neq \emptyset$, the dihedral angle they form is $\vartheta(H_i, H_j) = \pi/m_{i,j}$, for $m_{i,j} \in \mathbb{Z}_{\geq 2}$.

Let $P_{\mathcal{L}}$ be the corresponding convex polytope and $W_{\mathcal{L}}$ be the group of isometries generated by the reflection across the hyperplanes bounding $P_{\mathcal{L}}$.

Then

- (1) The action $W_{\mathcal{L}} \curvearrowright \mathbb{X}^n$ is properly discontinuous,
- (2) the polytope $P_{\mathcal{L}}$ is a fundamental domain for the action,
- (3) $W_{\mathcal{L}}$ is a discrete subgroup of $\text{Iso}(\mathbb{X}^n)$,

- (4) the group $W_{\mathcal{L}}$ is isomorphic to the Coxeter group with Coxeter matrix $(m_{i,j})$, cf. Definition 3.19.

1.2. Weyl groups in Lie theory. This section is purely motivational and aimed to present one of the most relevant contexts where Coxeter groups appear (cf. [Hum78]).

Let \mathfrak{L} be a finite dimensional semisimple Lie algebra over the complex field. One has the Cartan (or root space) decomposition

$$\mathfrak{L} = H \oplus \bigoplus_{\alpha \in \Phi} L_{\alpha},$$

where H is a maximal toral subalgebra (i.e., an algebra of semisimple elements) and the L_{α} are nonzero generalized eigenspaces for $\alpha \in \Phi \subseteq H^* = \text{Hom}(H, \mathbb{C})$. The (non-degenerate) Killing form provides the duality $H \leftrightarrow H^*$ and the identification between Φ and a finite subset of a euclidean space V .

Thus, to any pair (\mathfrak{L}, H) as above, one associates a pair (V, Φ) . One proves that the pair (V, Φ) is a *root system* (cf. [Bou07c, Ch. VI]), and the latter determine Weyl groups.

Weyl groups are Coxeter groups satisfying finiteness and integrality conditions and are easily classified through their Coxeter graphs. They are the groups associated to the graphs of types A_n to G_2 , cf. Table B.1.

Finally, the classification is sharp, in the sense that to each root system Φ one may associate a complex, semisimple Lie algebra $\mathfrak{L}(\Phi)$, $\mathfrak{L}(\Phi) \not\cong \mathfrak{L}(\Phi')$ for all $\Phi \not\cong \Phi'$.

1.3. Disclaimer: complex reflection groups. The case of complex reflection groups (cf. [BMR95], [BMR98], [Ari95]) is outside (and quite transversal to) the scope of this thesis.

For V a finite-dimensional complex vector space, an element $\rho \in \text{GL}(V)$ is a (*pseudo*-)reflection if

- the fixed point subspace $\ker(\rho - \text{id}_V)$ is a hyperplane (i.e. its codimension is 1),
- $\rho^m = 1$ for some $m \in \mathbb{N}$.

A *complex reflection group* is a finite subgroup of $\text{GL}(V)$, generated by pseudo-reflections.

Any Coxeter group is a group of reflection of a finite-dimensional real vector space W and hence of a complex vector space $\mathbb{C} \otimes W$. Then any finite Coxeter group is a complex reflection group.

The converse statement is easily seen to be false. The minimal counterexample is the complex reflection group $\mathbf{C}_3 \simeq \langle \zeta \rangle \leq \text{GL}_1(\mathbb{C}) \simeq \mathbb{C}^{\times}$, where ζ is any primitive 3rd root of 1, which is not a Coxeter group by Proposition 3.11(3).

1.4. Interplay. The above situations can be treated under the unifying framework of *Coxeter groups*.

Many mathematicians worked on (or used) the theory and, from different perspectives, much information is known about these objects, which has often taken the book form, e.g. [Bou07c], [Hum90], and, more recently, [BB05] and [Dav08].

Lie theory: The basic and fundamental reference is [Bou07c]: finite-dimensional complex semisimple Lie algebras are classified. In this context also see [Hum78] and [Car89].

Representation theory: Coxeter groups are the fundamental ingredient of the theory of *buildings*, which were invented by Tits (cf. [Tit74]) to describe algebraic groups over local fields.

Combinatorics: Kazhdan and Lusztig defined (cf. [KL79]) a “cell representation” for Hecke algebras and Coxeter groups, through the so-called KL-polynomials. The study of the geometry and combinatorics (cf. [BB05]) of such polynomials is one major field of investigation.

Group theory: One section of [Ser71] is devoted to Coxeter groups, since in this class of groups computations are feasible and extremely interesting.

Geometric group theory: A quite recent (i.e., written much after [Gro87]) book [Dav08] considers a variety of aspects of Coxeter groups, with emphasis on infinite ones, including the condition for Gromov hyperbolicity of a Coxeter group [Mou88].

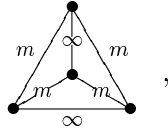
Arithmetic: In the study of the arithmetic properties (e.g. the covolume) of lattices in hyperbolic spaces, Coxeter subgroups of the group of hyperbolic isometries play a fundamental rôle (cf. [JRKT99]).

3.1. REMARK. Since the sources are extremely diverse, different authors have differences of language, which are sometimes mutually incompatible. This in particular happens for the word “hyperbolic”, for which at least three different usages are present in the literature.

In this thesis the word “hyperbolic” (cf. Definition 3.24) is used accordingly to [Bou07c, Ex.s 12–14, Ch. V], but it is worth pointing out the following.

- Authors working mainly in group theory, geometric group theory and arithmetic call *hyperbolic Coxeter group* any subgroup $W \leq \text{Iso}(\mathbb{H}^n)$, with the property that W is a Coxeter groups (for some set of “generating reflections, cf. §1.1).

For instance, let W be the group generated by the four reflections s_1, s_2, s_3, s_4 of \mathbb{H}^2 across the sides of a hyperbolic quadrangle Q as depicted in Figure 3.1. The angle between any two incident hyperplanes is $2\pi/m$, with $m \geq 5$. Then W is a group of isometries of \mathbb{H}^2 and moreover $(W, \{s_1, s_2, s_3, s_4\})$, is the Coxeter group with diagram



which is not hyperbolic (for the Definition 3.24).

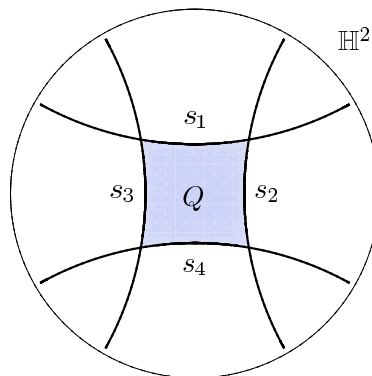
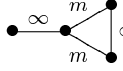


FIGURE 3.1. A hyperbolic quadrangle in (the conformal Poincaré disk model of) \mathbb{H}^2 . The angles between incident hyperplanes is $2\pi/m$, $m \geq 5$ and the volume of the shaded fundamental domain Q is $\pi \left(1 - \frac{4}{m}\right)$, by Gauß–Bonnet Theorem.

Another closely related example is e.g. , for $m \geq 3$

(cf. [AB08, §10.3.3]).

- Other authors use the wording “hyperbolic Coxeter group” to indicate that a Coxeter group is Gromov-hyperbolic.

The two notions do not agree; indeed Moussong’s Theorem (cf. [Mou88, Th. 17.1]) implies that there exist hyperbolic (non-cocompact) Coxeter groups which are not Gromov-hyperbolic, e.g. the Coxeter group with Coxeter diagram HNC1, cf. Table B.4.

All these incoherences in terminology obviously have a deep impact in the readability of the various classification results available (cf. [Vin81], [Lan50], [Vin67], [Vin85]).

2. Abstract Coxeter groups

This section contains no essentially new results, it has the purpose of fixing notation and conveniently state the facts which will be needed later.

2.1. Definition. All the information about a Coxeter group is encoded in a particular kind of graph.

3.2. DEFINITION (Coxeter graph). A Coxeter graph is a finite* combinatorial graph (cf. Appendix A) $\Gamma = (S, \mathfrak{E}, m)$, with labelling function

$$m: \mathfrak{E} \rightarrow \mathbb{N}_{\geq 3} \cup \{\infty\}.$$

Several connected Coxeter graphs are depicted in Appendix B §2.

To a Coxeter graph is associated a Coxeter system or, by an abuse of language, a Coxeter group.

3.3. DEFINITION (Coxeter system). Let $\Gamma = (S, \mathfrak{E}, m)$ be a Coxeter graph. Let $W = W(\Gamma)$ be the group with the following presentation:

$$(3.1) \quad W = \left\langle S \left| \begin{array}{ll} s^2 & \text{for all } s \in S \\ (sr)^{m(\{s,r\})} & \text{for } \{s,r\} \in \mathfrak{E}, \text{ with } m(\{s,r\}) < \infty \\ sr sr & \text{for } \{s,r\} \notin \mathfrak{E} \end{array} \right. \right\rangle.$$

The pair (W, S) is the Coxeter system associated to Γ .

If the Coxeter graph Γ is connected, the corresponding Coxeter system $W(\Gamma)$ is called *irreducible*.

3.4. REMARK. The relations appearing in (3.1) of the form s^2 are called *quadratic* relations.

For x, y elements of any monoid, for $m \in \mathbb{N}_0$, let $\pi_m(x, y)$ be the m -term product, defined as follows:

$$\pi_m(x, y) = \begin{cases} (xy)^{m/2} & \text{if } 2 \mid m \\ (xy)^{(m-1)/2} x & \text{if } 2 \nmid m \end{cases}.$$

Once assumed the quadratic relations hold, one can replace the relations of the form $(sr)^m$ with relations of the form $\pi_m(s, r) = \pi_m(r, s)$, and the latter are called *braid relations*. This is related to the fact that Coxeter groups of type A_n are homomorphic images of the braid group on $n + 1$ strands (cf. [KT08]).

*In principle, the finiteness assumption on the set S could be dropped at the level of the definition of a Coxeter group. Yet, doing so would make necessary to suppose, in most statements, that $|S| < \infty$ (cf. Lemma 3.22).

3.5. REMARK. Let $\Gamma_\infty(n)$ be the complete graph on n vertices with $m_{s,t} = \infty$ for all vertices $s \neq t$. It is a Coxeter graph and the associated Coxeter group

$$W_\infty(n) = W(\Gamma_\infty(n)) = \langle s_1, \dots, s_n \mid s_i^2, \text{ for all } i \in \{1, \dots, n\} \rangle$$

is the free product of n copies of \mathbf{C}_2 .

Moreover, every Coxeter group (W, S) with $|S| = n$ is a quotient of $W_\infty(n)$.

3.6. REMARK. Later in the discussion it is possible that the phrase “ (W, S) is a Coxeter system” will be shortened to “ W is a Coxeter group”. It is then essential to remark that a Coxeter group always comes equipped with a chosen generating set S .

Indeed, it is possible that two different Coxeter systems (with generating sets of different cardinalities) give rise to isomorphic *abstract* groups. This is the case, e.g., of the dihedral group of order 12: it is both given by the type G_2 and by $A_2 \times A_1$.

3.7. REMARK. In few occasions, we will need a neat distinction between the words in S , which are elements $w \in F(S)$ of the free group on the set S , and the elements of W , which are the corresponding images $\bar{w} \in W$ under the projection modulo the relation subgroup.

3.8. DEFINITION. Let (W, S) be a Coxeter system and let $w = s_{i_1}^{\varepsilon_1} s_{i_2}^{\varepsilon_2} \dots s_{i_n}^{\varepsilon_n} \in F(S)$ be a word in S , then its length is $\text{length}(w) = \sum |\varepsilon_i|$.

The Coxeter length of $\bar{w} \in W$ is defined as $\ell(\bar{w}) = \min\{\text{length}(u) \mid u \in F(S), \bar{u} = \bar{w}\}$, and thus there is a function

$$\ell: W \rightarrow \mathbb{N}_0.$$

If $u \in F(S)$ is such that $\text{length}(u) = \ell(\bar{u})$, then u is called a *reduced word* (or *reduced expression*) for \bar{u} .

Reduced expressions exist but are, in general, not unique. The word problem for Coxeter group was solved as follows.

3.9. THEOREM. For $w \in F(S)$ a word in S , two kinds of elementary M-operations are defined:

- (M1) delete a subword of type ss for $s \in S$;
- (M2) for $\{s, r\} \in \mathfrak{E}$ and $m = m(\{s, r\})$, replace a subword of type $\pi_m(s, r)$ with a subword of type $\pi_m(r, s)$.

One says that w is M-reduced if, and only if, for all elementary M-operation μ one has $\text{length}(\mu(w)) \geq \text{length}(w)$.

Then a word w is M-reduced if, and only if, it is a reduced word in the sense of Definition 3.8.

PROOF. Cf. Tits' [Tit69]. ■

3.10. PROPOSITION. Let (W, S) be a Coxeter system with length function $\ell: W \rightarrow \mathbb{N}_0$. Then, for $w, u \in W$ and $s \in S$, one has

- (1) $\ell(w) = 0$ if, and only if, $w = 1$, and $\ell(w) = 1$ if, and only if, $w \in S$,
- (2) $\ell(w) = \ell(w^{-1})$,
- (3) $|\ell(w) - \ell(u)| \leq \ell(wu) \leq \ell(w) + \ell(u)$,
- (4) $\ell(ws), \ell(sw) \in \{\ell(w) + 1, \ell(w) - 1\}$,
- (5) The function ℓ is bounded if, and only if, W is finite.

3.11. PROPOSITION. Let (W, S) be a finite Coxeter system, with length function ℓ . Then there exists an element $w_0 \in W$ of maximal length with the following properties.

- (1) For all $u \in W$, $\ell(w_0u) = \ell(uw_0) = \ell(w_0) - \ell(u)$, and in particular $(w_0)^2 = 1$.
- (2) If (W, S) is a Weyl group (cf. §1.2), then $\ell(w_0) = |\Phi^+|$
- (3) If $S \neq \emptyset$, the map of sets $w \mapsto w_0w$ is an involution without fixed points, hence every finite nontrivial Coxeter group has even cardinality.

Moreover, an element w_0 satisfying any of the conditions (1) or (2) is a longest element in W . In particular, w_0 is unique.

3.12. PROPOSITION. Let Γ_1 and Γ_2 be Coxeter graphs, then the sum $\Gamma_1 \sqcup \Gamma_2$ is a Coxeter graph and there is a canonical isomorphism

$$\varphi: W(\Gamma_1) \times W(\Gamma_2) \xrightarrow{\sim} W(\Gamma_1 \sqcup \Gamma_2).$$

Moreover, for $w_1 \in W(\Gamma_1)$ and $w_2 \in W(\Gamma_2)$, one has

$$\ell(\varphi(w_1, w_2)) = \ell(w_1) + \ell(w_2).$$

3.13. PROPOSITION. Let (W, S) be the Coxeter system associated to Γ , and let $s, t \in S$. Then s and t are conjugate in W if, and only if, they lie in the same connected component of Γ_{odd} .

PROOF. Cf. [BB05, Ch. 1, Ex. 16]. ■

2.2. Parabolic structure. Coxeter groups have the remarkable property of having a distinguished poset of subgroups, in correspondence with the subgraphs of the Coxeter graph. Moreover, each such a subgroup is a Coxeter group, as follows from the simple observation that any subgraph of a Coxeter graph is a Coxeter graph.

Let $\Gamma = (S, \mathfrak{E}, m)$ be a Coxeter graph and let $\pi_\Gamma: 2^S \rightarrow \mathcal{P}(\Gamma)$ be the functor defined in (A.1). Then any subgraph in the image of π_Γ is a Coxeter graph.

By Matsumoto's theorem (cf. [GP00, Thm. 1.2.2]) and since **Monoids** is a full subcategory of **Grp**, there is a morphism of groups $W(\Gamma') \rightarrow W(\Gamma'')$ whenever $\Gamma' \leq \Gamma''$. Then there is a functor

$$(3.2) \quad W(-): \mathcal{P}(\Gamma) \rightarrow \mathbf{Grp}.$$

3.14. LEMMA. Let $\Gamma = (S, \mathfrak{E}, m)$ be a Coxeter graph and let $I \subseteq S$. The Coxeter group $W(\pi_\Gamma(I))$ and the subgroup of $\langle I \rangle \leq W(\Gamma)$ are both defined, and there is a canonical isomorphism

$$(3.3) \quad \langle I \rangle \simeq W(\pi_\Gamma(I)),$$

through which we identify the two groups, both denoted W_I . Subgroups of the form W_I for some $I \subseteq S$ are named standard parabolic subgroups.

The parabolic structure of the Coxeter pair (W, S) is the functor

$$W_- = W(-) \circ \pi_\Gamma: 2^S \rightarrow \mathbf{Grp}.$$

In particular $W_S = W$.

3.15. PROPOSITION. If (W, S) is a Coxeter system and $I \subseteq S$, let ℓ and ℓ_I be, respectively, the length function of W and W_I , then

$$\ell_I = \ell|_{W_I}.$$

3.16. LEMMA. Let (W, S) be the Coxeter system associated to Γ , and let $I \subseteq S$ be a connected component. Then $W \simeq W_I \times W_{S \setminus I}$.

3.17. DEFINITION. Let (W, S) be a Coxeter system, let $I \subseteq S$ and $w \in W$. Then define the following sets

$$(3.4) \quad W^I = \{w \in W \mid \ell(ws) > \ell(w) \forall s \in I\} \subseteq W,$$

$$(3.5) \quad {}^I W = \{w \in W \mid \ell(sw) > \ell(w) \forall s \in I\} \subseteq W,$$

$$(3.6) \quad A^p(w) = \{s \in S \mid \ell(ws) > \ell(w)\} \subseteq S.$$

3.18. PROPOSITION. Let (W, S) be a Coxeter system, let $w \in W$ and let $I \subseteq S$.

- (1) There exist a unique element $w_I \in W_I$ and a unique element $w^I \in W^I$ such that $w = w^I w_I$. Moreover, $\ell(w) = \ell(w^I) + \ell(w_I)$.
- (2) There exist a unique element ${}_I w \in W_I$ and a unique element ${}^I w \in {}^I W$ such that $w = {}_I w {}^I w$. Moreover, $\ell(w) = \ell({}_I w) + \ell({}^I w)$.
- (3) W^I and ${}^I W$ are sets of coset representatives, distinguished in the sense that the decomposition is length-additive.
- (4) The element $w^I \in W^I$ is the unique shortest element in wW_I .
- (5) Let $y \in W^I$ and $u \in W_I$. Then $(yu)^I = y$, $(yu)_I = u$, and $\ell(yu) = \ell(y) + \ell(u)$.
- (6) For $s \in S$ one has $W = \{s\}W \sqcup s(\{s\}W)$, where \sqcup denotes disjoint union.
- (7) Let $I \subseteq J \subseteq S$. Then $W^J \subseteq W^I$. In particular, $W^S = \{1\}$ and $W^\emptyset = W$.
- (8) $A^p(w) = \bigcup_{w \in W^I} I = \max\{I \mid w \in W^I\}$.
- (9) The element w is contained in W^I if, and only if, $I \subseteq A^p(w)$. In particular, $\ell(w^I) \leq \ell(w)$ if, and only if $I \not\subseteq A^p(w)$.

PROOF. A proof of (1) to (4) can be found e.g., in [GP00, §2.1]. Statements (5) to (9) are immediate consequences of the previous facts and definitions. ■

2.3. Tits' representation theorem. The present section recalls one of the most important results about Coxeter groups. Essentially, what Tits proved is that every Coxeter group is a **Errata:** lattice **Corrige:** discrete subgroup in a real orthogonal group $O(\mathbb{R}^{|S|}, B)$ with respect to a scalar product defined by a suitable matrix B .

3.19. DEFINITION (Cartan and Coxeter matrices). Let $\Gamma = (S, \mathfrak{E}, m)$ be a Coxeter graph, then let C be the matrix with rows and columns indexed by S and with entries

$$C_{s,r} = \begin{cases} m(\{s, r\}) & \text{if } \{s, r\} \in \mathfrak{E} \\ 2 & \text{if } \{s, r\} \notin \mathfrak{E}, s \neq r, \\ 1 & \text{if } s = r. \end{cases}$$

Moreover let B be the matrix with entries

$$B_{s,r} = \begin{cases} -\cos\left(\frac{\pi}{C_{s,r}}\right) & \text{if } C_{s,r} \neq \infty, \\ -1 & \text{if } C_{s,r} = \infty. \end{cases}$$

The matrix C is the Coxeter matrix and B is the Cartan matrix of (W, S) . If $I \subseteq S$ and (W_I, I) is a sub-Coxeter system, then C_I and B_I are defined in the obvious way.

The condition of irreducibility of the Coxeter system (cf. Definition 3.3) has a linear-algebra counterpart. I.e., (W, S) is irreducible if, and only if, the Cartan matrix is *indecomposable*, that is, there is no non-trivial partition $I_1 \sqcup I_2 = S$ of S such that $B_{s,t} = 0$ for all $s \in I_1$ and $t \in I_2$.

For a Coxeter system (W, S) with Cartan matrix B , one has the following facts (cf. [Hum90, §2.7]).

- (1) B is symmetric with 1's on the diagonal.

- (2) By the Perron–Frobenius theorem for non-negative matrices one deduces that the smallest eigenvalue of B (considered as $B \in \text{End}_{\mathbb{R}}(\mathbb{R}^{|S|})$) is simple.
- (3) In particular, if B is indecomposable and positive semidefinite, then the Cartan matrix B_I of every proper Coxeter subsystem (W_I, I) is positive-definite.

3.20. THEOREM (Tits’ reflection representation). *Let (W, S) be the Coxeter system with Cartan matrix B . Let $V = \mathbb{R}^S$ be an euclidean space of dimension $n = |S|$ with basis $\{e_s \mid s \in S\}$ and scalar product $\langle -, - \rangle_B$ defined by the Cartan matrix B . Then let $\rho: W \rightarrow \text{GL}(V)$ be the morphism defined by*

$$\rho_s(x) = x - 2 \frac{\langle x, e_s \rangle_B}{\langle e_s, e_s \rangle_B} e_s, \quad s \in S,$$

and $\rho_w = \rho_{s_1} \circ \rho_{s_2} \circ \dots \circ \rho_{s_n}$, where $w = s_1 s_2 \dots s_n$ is a reduced expression. Then

- (1) the representation ρ is well-defined and it is a monomorphism;
- (2) $\text{im } \rho$ is discrete in $\text{GL}(V)$;
- (3) one has $\langle \rho_w(x), \rho_w(y) \rangle_B = \langle x, y \rangle_B$ for all $w \in W$ and all $x, y \in V$, hence there is a corestriction $\rho: W \rightarrow \text{O}(\mathbb{R}^S, B)$ to the orthogonal group leaving the symmetric, bilinear form $\langle -, - \rangle_B$ invariant.

PROOF. Cf. [Bou07c, Ch. V §4] ■

3.21. COROLLARY. *For a Coxeter group (W, S) the following hold.*

- (1) The groups W is linear and finitely generated.
- (2) The group W is virtually torsion free.
- (3) The scalar product defined by B is positive definite if, and only if, W is finite.
- (4) If $s, r \in S$ then the order of $sr \in W$ is exactly $m(s, r)$. Moreover, every generator $s \in S$ has non-trivial image $\bar{s} \in W$.
- (5) The Coxeter length function ℓ agrees with the usual length function defined for finitely and symmetrically generated groups.
- (6) The growth of (W, S) is either polynomial or exponential.

PROOF. Tits’ representation implies (1), thus (2) follows by Selberg’s lemma (cf. [Sel60, Lemma 8]). The third statement is the core of the classification of finite Coxeter groups and is proved, e.g. in [Bou07c, Ch.V §4.8]. The solution (Theorem 3.9) of the word problem for Coxeter groups implies (4). Statement (5) is trivial (cf. [GdlH97, §1]). Moreover, (6) follows from [Tit72, Corollary 5]. ■

Tits’ theorem in particular implies the following result.

3.22. LEMMA. *Let $\Gamma = (S, \mathfrak{E}, m)$ be a combinatorial, possibly infinite, graph with labelling function $m: \mathfrak{E} \rightarrow \mathbb{N}_{>3}$ and let (W, S) consist of the group W with presentation (3.1), and the set S . The following facts hold.*

- (1) If $|S| < \infty$, then (W, S) is actually a Coxeter system as in Definition 3.3 and W is finitely generated as abstract group.
- (2) If $|S| = \infty$, i.e., the “Coxeter-type” presentation is infinitely generated, then W is not finitely generated as abstract group. Thus, one may call such a group an infinitely-generated Coxeter group without ambiguity.

PROOF OF (2). Suppose S is infinite, and suppose that there exists a finite subset $R \subseteq W$ such that $\langle R \rangle = W$. Then, for each $r \in R$ choose an S -word

$$\prod_{k=1}^{L(r)} s_k(r) = r.$$

Let T be the finite union $T = \bigcup_{r \in R} \bigcup_{k \in \{1, \dots, L(r)\}} \{s_k(r)\}$. Then

- $T \subsetneq S$ the inclusion being proper since T is finite and S is infinite,
- $R \subseteq \langle T \rangle$ and hence $\langle T \rangle = W$.

On the other hand, a proper subset T of S cannot generate the whole W , by Corollary 3.21(4), a contradiction. ■

3. Euler-ness of the group algebra

The purpose of the present section is to prove that the parabolic structure of the R -group algebra of a Coxeter group is Euler (cf. Chapter 1 §5).

3.23. PROPOSITION. *Let (W, S) be a Coxeter system, and let R be a commutative ring. Then, if W is of type **FP** over R (e.g., by Maschke's Theorem, this happens when W is finite and $|W| \cdot 1_R \in R^\times$) the group algebra $R[W]$ is Euler, and the parabolic subalgebras $R[W_I]$ are Euler subalgebras.*

PROOF. By Proposition 2.2, group algebras are canonically Euler whenever they are of type **FP**. Thus, one just has to prove the statement about parabolic subalgebras: let $I \subseteq S$. One has $_^\natural$ defined by $w^\natural = w^{-1}$, the restriction of the antipode of $R[W]$. The linear character is the canonical augmentation ε given by $\varepsilon(w) = 1$ for all $w \in W_I$. The basis is $\mathcal{B}_I = W_I = W \cap R[W_I]^\dagger$ for all parabolic subgroup. The R -module $\underline{R[W]}$ is the free R -module having as basis the set of conjugacy classes $[w]$ of W . The canonical trace function $\mu: \underline{R[W]} \rightarrow R$ is given by $\mu\left(\sum_{C \in \text{Cl}(W)} r_C C\right) = r_{[1]}$. ■

4. Finiteness of Coxeter groups: a hierarchy

In this section, some combinatorial finiteness conditions are recalled, together with a geometric interpretation given essentially by Tits' geometric representation Theorem 3.20.

3.24. DEFINITION. Let (W, S) be a Coxeter system, with Cartan matrix B (cf. Definition 3.19).

Then (W, S) is called

- *spherical* if B is a positive-definite matrix;
- *affine* if B is positive-semidefinite but not positive-definite;
- *hyperbolic cocompact* if B is non-degenerate but not positive-definite, and each proper Cartan submatrix B_I is positive-definite.
- *hyperbolic* if B is non-degenerate but not positive-definite, and each proper Cartan submatrix B_I is positive-semidefinite.

The following facts are well-known.

3.25. LEMMA. *Let (W, S) be a Coxeter system with Coxeter graph Γ . Then*

- (1) *(W, S) is spherical if, and only if, W is finite.*
- (2) *(W, S) is spherical if, and only if, Γ is a finite sum of connected graphs of spherical type, cf. Table B.1;*
- (3) *if (W, S) is affine, then Γ is connected and it is one of the graphs of Table B.2;*
- (4) *if (W, S) has the property that every proper parabolic subsystem (W_I, I) is spherical, then (W, S) is spherical, affine or hyperbolic cocompact;*
- (5) *if (W, S) is hyperbolic or hyperbolic cocompact, then Γ is connected.*

[†]Normally, one would use the Hopf algebra structure to detect a copy of W inside its group algebra (cf. [Ben98, §3.1]), but this is not the case, since for the purposes of this work one does not need the multiplicative structure of the basis elements.

PROOF. Statements (1), (2) and (3) follow from [Bou07c, Ch. V, §4]. The Perron–Frobenius Theorem (cf. [LT85, §15.3]) implies (4) and, in turn, (5). ■

Moreover, in the spirit of Kropholler’s ideas (cf. [Kro93]), a notion of hierarchy of Coxeter groups may be given as follows.

- Let $\mathbf{Cox}_{-1} = \emptyset$ and let \mathbf{Cox}_0 be the class of finite Coxeter groups.
- For $k \geq 1$, let $(W, S) \in \mathbf{Cox}_k$ if, and only if, every proper parabolic subgroup is in \mathbf{Cox}_{k-1} .
- $\mathbf{Cox}_k \supseteq \mathbf{Cox}_{k-1}$ for all k , and $\bigcup_k \mathbf{Cox}_k$ is the whole class of Coxeter groups.
- The complexity $\Upsilon(W, S)$ is defined to be the unique k for which $(W, S) \in \mathbf{Cox}_k \setminus \mathbf{Cox}_{k-1}$.

3.26. LEMMA. *Let (W, S) be a Coxeter system with Coxeter graph Γ .*

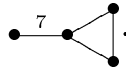
- (1) *The complexity $\Upsilon(W, S) = 0$ if, and only if, (W, S) is spherical.*
- (2) *The complexity $\Upsilon(W, S) = 1$ if, and only if, (W, S) is affine or hyperbolic cocompact.*
- (3) *If (W, S) is hyperbolic non-cocompact, then $\Upsilon(W, S) = 2$.*
- (4) *If $\Upsilon(W, S) = 2$, exactly one of the following must occur:*
 - (a) *Γ is of hyperbolic non cocompact type (and hence connected);*
 - (b) *Γ is connected but not hyperbolic;*
 - (c) *$\Gamma = \Gamma' \sqcup A_1$, with Γ' of affine or hyperbolic cocompact type.*
- (5) *If $|S| = n > 0$, then $\Upsilon(W, S) \leq n - 1$.*

PROOF. Statements (1) to (4a) are reformulations of well-known facts. Suppose then that $\Upsilon(W, S) = 2$ but (W, S) is not hyperbolic. Suppose further that the associated Coxeter diagram Γ is not connected. Then write $\Gamma = \Gamma_1 \sqcup \Gamma_2 \sqcup \Gamma_k$, for $k \geq 2$ and Γ_i connected non-empty.

Since the sum of spherical diagrams is spherical, then there must exist a Γ_i of infinite type, suppose Γ_1 . If $\Gamma_2 \sqcup \dots \sqcup \Gamma_k$ has at least two vertices, then one might remove two vertices from Γ and obtain a graph having Γ_1 as subgraph, a contradiction with $\Upsilon(W, S) = 2$. Thus $k = 2$ and Γ_2 is A_1 . Finally, Γ_1 must be of complexity 1 and hence it is either affine or hyperbolic cocompact.

The statement (5) is easily proved by induction. ■

3.27. PROBLEM. The class described in (4b) is non-empty: it contains, e.g. the diagram



It would be very interesting to completely classify the connected graphs of complexity 2 which are not hyperbolic.

5. The Coxeter complex

Let (W, S) be a Coxeter system and let $\Sigma = (\Sigma, \leq)$ be the poset of proper parabolic left cosets

$$\Sigma = \{wW_I \mid I \subsetneq S, w \in W\},$$

partially ordered by *reversed* inclusion, i.e., $wW_I \leq uW_J$ if, and only if, $wW_I \supseteq uW_J$ or, equivalently, if $J \subseteq I$ and $u^{-1}w \in W_J$. Thus, maximal cells are cosets of the form $\{w\}$, for some $w \in W$.

The group W acts on Σ from the left in the obvious way $w.uW_I = (wu)W_I$, and hence

$$(3.7) \quad \text{Stab}_W(wW_I) = {}^wW_I.$$

Then, one has the following:

3.28. LEMMA. *For an irreducible Coxeter system (W, S) which is of hyperbolic co-compact, affine or spherical type, the Coxeter complex $\Sigma(W, S)$ is a model of a $\underline{E}(W)$. Thus, in particular, there is a projective $\mathbb{Q}[W]$ -resolution of the trivial module \mathbb{Q} of length $|S| - 1$.*

PROOF. Every proper subsystem (W_I, I) , $I \subsetneq S$, and hence any point stabilizer, by (3.7), is in the class \mathfrak{F} of finite subgroups of W . Then the statement follows from the discussion in Chapter 2 §2.1. \blacksquare

More detailed information can be found in [AB08, Ch. 3].

The Coxeter complex provided a fundamental source of inspiration for [TW]: indeed the chain complex of $\mathbb{Q}[W]$ -modules associated to the Coxeter complex $\Sigma(W, S)$ can be written in the form

$$\bigoplus_{\substack{I \subsetneq S \\ |I|=1}} \mathbb{Q}[W/W_I] \longrightarrow \bigoplus_{\substack{I \subsetneq S \\ |I|=2}} \mathbb{Q}[W/W_I] \longrightarrow \dots \longrightarrow \bigoplus_{\substack{I \subsetneq S \\ |I|=|S|-1}} \mathbb{Q}[W/W_I] \longrightarrow \mathbb{Q},$$

where the modules $\mathbb{Q}[W/W_I]$ are permutation modules and, by (3.7),

$$\bigoplus_{\substack{I \subsetneq S \\ |I|=1}} \mathbf{ind}_I^S \mathbb{Q} \longrightarrow \bigoplus_{\substack{I \subsetneq S \\ |I|=2}} \mathbf{ind}_I^S \mathbb{Q} \longrightarrow \dots \longrightarrow \bigoplus_{\substack{I \subsetneq S \\ |I|=|S|-1}} \mathbf{ind}_I^S \mathbb{Q} \longrightarrow \mathbf{ind}_S^S \mathbb{Q},$$

where $\mathbf{ind}_I^S(_)$ denotes the induction functor from $\mathbb{Q}[W_I]$ -Mod to $\mathbb{Q}[W]$ -Mod.

It is well-known that, for an irreducible Coxeter system (W, S) , the complex $\Sigma(W, S)$ is contractible if, and only if, W is infinite.

Then, one deduces that the complex (3.8) has trivial homology when W is infinite, and has homology isomorphic to $\text{sgn}[n - 1]$, where sgn is the sign representation and $n = |S|$, when W is finite.

Thus, for non-spherical Coxeter systems, the Coxeter complex provides a resolution of the trivial $\mathbb{Q}[W]$ -module \mathbb{Q} : it is enough to consider the truncation above 0 of the chain complex (3.8).

5.1. Examples. In this section some (geometric realizations) of Coxeter complexes are shown.


3.29. EXAMPLE. Consider the Coxeter system of type A_2 , with Coxeter group

$$W = \langle r, s \mid s^2, r^2, (sr)^3 \rangle.$$

There are three proper parabolic subgroups, namely $U = \langle r \rangle \simeq \mathbf{C}_2$, $V = \langle s \rangle \simeq \mathbf{C}_2$, and the trivial one $\{1\}$.

The left cosets of the trivial subgroup being the singletons of the elements, one has that the left V -cosets are V , $rV = \{r, rs\}$ and $srV = \{sr, srs\}$; similarly, the left U -cosets are U , $sU = \{s, sr\}$, $rsU = \{rs, rsr\}$.

Since the (unique) braid relation implies that $srs = rsr$, one obtains the complex appearing in Figure 3.2. Its geometric realization is homomorphic to a 1-sphere.

3.30. EXAMPLE. Similarly, one can consider the type \tilde{A}_2 with diagram . The Coxeter group is infinite and the Coxeter complex (cf. Figure 3.3) is an affine 2-space, hence it is contractible and it has the same homology of a point.

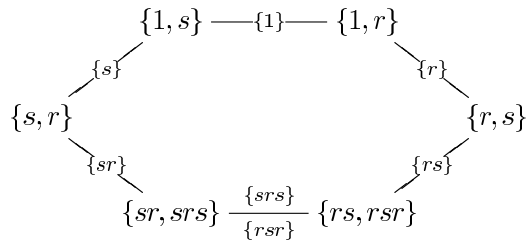


FIGURE 3.2. The Coxeter complex of type A_2 .

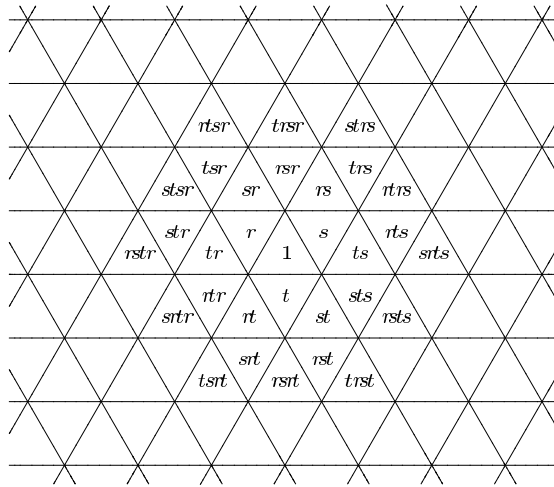


FIGURE 3.3. The Coxeter complex of type \tilde{A}_2 . A maximal cell $\{w\}$ is labelled with the element w .

3.31. EXAMPLE. Let W be the Coxeter group generated by the three reflections s_1, s_2, s_3 of \mathbb{H}^2 across the sides of a maximal hyperbolic triangle, T as depicted in Figure 3.4. Then W is a group of isometries of \mathbb{H}^2 and moreover $(W, \{s_1, s_2, s_3\})$, is the Coxeter group with diagram $\Gamma_\infty(3)$.

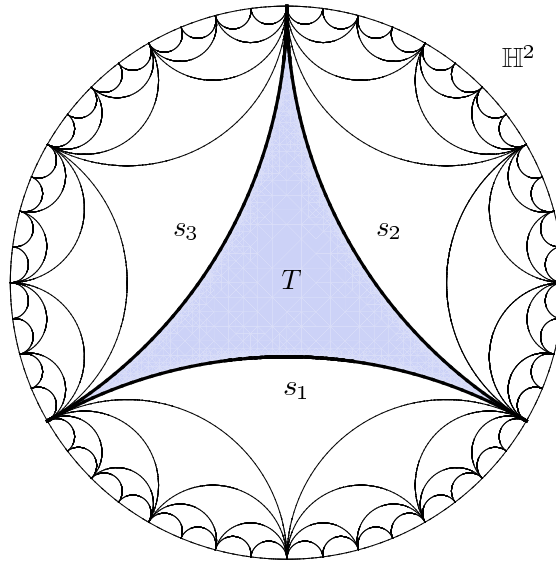


FIGURE 3.4. A hyperbolic triangle in (the conformal Poincaré disk model of) \mathbb{H}^2 with vertices on the sphere at infinity.

Poincaré series and an alternative for Coxeter groups

The main result of this chapter, Theorem 4.16, classifies the minimal non-spherical, non-affine Coxeter systems. This, together with the computations described in Appendix D and further results (cf. Lemma 4.19), allows one to know the convergence radii of their growth series, thus one may determine the exponential growth.

As an application, a new proof of a result (cf. Theorem 4.22) of P. De La Harpe is given. This result is analogous to Tits' alternative (cf. [Tit72]).

1. Poincaré series

Poincaré series are a central object of this chapter. It is important to remark that $|S| < \infty$.

4.1. DEFINITION (Poincaré series). Let (W, S) be a Coxeter system and let $\ell: W \rightarrow \mathbb{N}_0$ be its length function. Then let

$$(4.1) \quad p_{(W,S)}(t) = \sum_{w \in W} t^{\ell(w)} \in \mathbb{Z}[[t]].$$

If (W, S) has Coxeter graph Γ , it will be useful to introduce the notation $p_{\Gamma}(t) = p_{(W,S)}(t)$ and $\rho_{\Gamma} = \rho_{(W,S)}$.

The following facts are well-known.

4.2. PROPOSITION. *For a Coxeter pair (W, S) one has the following.*

- (1) *The Poincaré series is a rational function.*
- (2) *Let $p_{(W,S)}(t) \in \mathbb{C}[[t]]$, and let φ be the complex rational function the series converges to. If $D = \{z \in \mathbb{C} \mid |z| < r\}$ is an open disk centered in zero, and if no poles of φ is contained in D , then $p_{(W,S)}(t)$ converges pointwise to φ .*
- (3) *The Poincaré series is a polynomial if, and only if, W is finite.*
- (4) *If (W, S) is spherical, and ev_1 is the map "evaluation of a polynomial at 1", then*

$$\text{ev}_1(p_{(W,S)}(t)) = |W|.$$

- (5) *If $W = W_1 \times W_2$, i.e., the graph of (W, S) can be decomposed as the sum $\Gamma = \Gamma_1 \sqcup \Gamma_2$, then*

$$p_{\Gamma}(t) = p_{\Gamma_1}(t)p_{\Gamma_2}(t) \quad \text{and} \quad \rho_{\Gamma} = \min\{\rho_{\Gamma_1}, \rho_{\Gamma_2}\}.$$

- (6) *If $I \subseteq S$, then $p_{(W,I)}(t) \mid p_{(W,S)}(t)$ and $\rho_{(W,S)} \leq \rho_{(W,I)}$.*

PROOF. Statements (1) to (4) are proven, e.g., in [Bou07c, Ch. IV]. The first part of (5) follows from Proposition 3.12, and the statement about the convergence radii is an consequence of, e.g., Lemma 2.10.

Finally, by Proposition 3.18(1), one has

$$p_{(W,S)}(t) = \left(\sum_{w \in W^I} t^{\ell(w)} \right) p_{(W_I, I)}(t),$$

which proves (6). ■

4.3. REMARK. The example of Remark 3.6 shows that the Poincaré series depends on the Coxeter system, but, in general, it is not invariant under group isomorphism: indeed

$$p_{G_2}(t) = \frac{1-t^6}{1-t} \frac{1-t^2}{1-t} \neq \frac{1-t^3}{1-t} \frac{1-t^2}{1-t} \frac{1-t^2}{1-t} = p_{A_2}(t) p_{A_1}(t) = p_{A_2 \times A_1}(t).$$

For spherical Coxeter systems, the Poincaré series is actually a polynomial, which is computed with the aid of Proposition 4.2(5) and the following fundamental result of factorization.

4.4. PROPOSITION (Chevalley–Shephard–Todd). *If (W, S) is spherical and $|S| = n$, then there exist natural numbers d_1, \dots, d_n (the degrees, cf. Table B.1) such that*

$$p_{(W,S)}(t) = \prod_{i=1}^n \frac{1-t^{d_i}}{1-t}.$$

PROOF. Cf. [Hum90, Ch. 3]. ■

For infinite Coxeter groups, it is not immediate from the very definition (4.1), how to compute the Poincaré series concretely.

In this direction, the Bott’s Theorem 4.5 provides a closed expression for affine Coxeter systems, while a recursive formula is well-known for any infinite Coxeter group (cf. Proposition 4.6).

4.5. PROPOSITION (Bott). *Let Γ be a finite, crystallographic Coxeter diagram with degrees d_i , and let $\tilde{\Gamma}$ be the associated affine Coxeter diagram (cf. Table B.2).*

Then,

$$p_{\tilde{\Gamma}}(t) = p_{\Gamma}(t) \prod_{d_i \text{ degrees}} \frac{1}{1-t^{d_i-1}}.$$

PROOF. Cf. [Ste68, §3]. ■

4.6. PROPOSITION. *Let (W, S) be an infinite Coxeter system. Then*

$$(4.2) \quad \frac{1}{p_{(W,S)}(t)} = \sum_{I \subsetneq S} (-1)^{|S|-|I|-1} \frac{1}{p_{(W_I, I)}(t)}.$$

PROOF. Cf. [Bou07c, Ex. 26 f), Ch. IV §1]. ■

In the case of Coxeter systems, one may give results about the convergence radius of the Poincaré series which are much sharper than Proposition 2.6.

4.7. PROPOSITION. *Let (W, S) be a Coxeter system with Poincaré series $p_{(W,S)}(t)$ and convergence radius $\rho_W = \rho_{(W,S)}$ (cf. Chapter 2 §3).*

(1) *If W is finite, then $p_{(W,S)}(t)$ is a polynomial, hence it converges everywhere and $\rho_W = \infty$.*

(2) *If W is infinite, then $|S| \geq 2$ and $\frac{1}{|S|-1} \leq \rho_W \leq 1$.*

Moreover, the bounds in (2) are sharp.

PROOF. The statement (1) is trivial. For (2), let $|S| = n \geq 2$. Since $p_{(W,S)}(t) \rightarrow |W|$ as $t \rightarrow 1$, the point $1 \in \mathbb{C}$ is a pole when W is infinite.

In view of Remark 3.5 and Lemma 2.10, it suffices to prove that the convergence radius ρ_n of $W_\infty(n)$ is $\frac{1}{n-1}$.

The Poincaré series $p_n(t)$ of the groups of type $W_\infty(n)$ can be easily computed. Indeed, any element \bar{w} of length k is obtained *uniquely* as the concatenation $\bar{w} = \bar{u}s_k$ of a word $u = s_1s_2 \dots s_{k-1}$ of length $k-1$ with a single generator s_k , different from the last letter s_{k-1} . Thus, for $k > 0$, there exist $n(n-1)^{k-1}$ elements in W of length k and hence

$$p_n(t) = 1 + \sum_{k=1}^{\infty} n(n-1)^{k-1}t^k = \frac{1+t}{1-(n-1)t},$$

which is a rational function, holomorphic in a disk of radius $1/(n-1)$, as claimed.

The bounds are attained, on one side by the groups $W_\infty(n)$ and on the other side, by the affine Coxeter systems (cf. Proposition 4.5). ■

2. Minimal non-spherical, non-affine Coxeter systems

The purpose of the present section is to provide the list \mathcal{M} of minimal (irreducible) non-spherical, non-affine Coxeter systems.

This list will turn out to coincide with Lannér's list (cf. [Lan50]) of hyperbolic Coxeter groups with simplicial fundamental domain, and it will be used to give an alternative proof to a Theorem by P. De La Harpe (cf. [dlH87, Corollaire]).

The class of all Coxeter graphs is partially ordered by the relation $\Gamma' \leq \Gamma$ (cf. Definition A.4). One may define the following notions in order to identify graphs which only differ by a bijection of the vertices.

4.8. DEFINITION (Type of Coxeter graph). Let $\Gamma = (S, \mathfrak{E}, m)$ and $\Gamma' = (S', \mathfrak{E}', m')$ be Coxeter graphs. One defines $\Gamma \sim \Gamma'$ if, and only if, there exists a map $f: S \rightarrow S'$ such that

- f is a bijection;
- if $\mathfrak{e} \in \mathfrak{E}$, then $f(\mathfrak{e}) \in \mathfrak{E}'$;
- the induced map $f^2: \mathfrak{E} \rightarrow \mathfrak{E}'$, $f^2(\{v, u\}) = \{f(v), f(u)\}$ is a bijection;
- $m'(f(\mathfrak{e})) = m(\mathfrak{e})$ for all $\mathfrak{e} \in \mathfrak{E}$.

Then \sim is an equivalence relation such that, if Γ is connected and Γ' is not connected, then $\Gamma \not\sim \Gamma'$. Thus, let \mathcal{X} be the set of the equivalence classes of *connected* Coxeter graphs. Any such equivalence class $[\Gamma]$ is called a *type*.

Define on \mathcal{X} the partial order relation $[\Gamma'] \leq [\Gamma]$ if, and only if, there exist representatives $\bar{\Gamma}' \in [\Gamma']$ and $\bar{\Gamma} \in [\Gamma]$ such that $\bar{\Gamma}' \leq \bar{\Gamma}$.

4.9. NOTATION. By an abuse of notation, the class $[\Gamma]_\sim \in \mathcal{X}$ will be written as $\Gamma \in \mathcal{X}$, and moreover $[\Gamma'] \leq [\Gamma]$ will be written as $\Gamma' \leq \Gamma$.

4.10. DEFINITION (The classes \mathcal{S} , \mathcal{A} , \mathcal{Hc} , \mathcal{Hnc} and \mathcal{M}). Let \mathcal{S} (resp. \mathcal{A}) be the set of all types $[\Gamma]$ of Coxeter graphs such that Γ is spherical and connected (resp. affine).

The class \mathcal{M} is defined to be the set of the elements in \mathcal{X} which are minimal with respect to the property of being neither spherical nor affine:

$$\mathcal{M} = \min(\mathcal{X} \setminus (\mathcal{S} \cup \mathcal{A})).$$

Moreover, let \mathcal{Hc} (resp. \mathcal{Hnc}) be the set of all types $[\Gamma]$ of Coxeter graphs such that Γ is hyperbolic cocompact (resp. hyperbolic non cocompact).

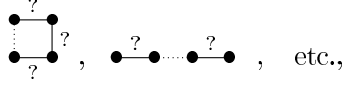
2.1. Some classification lemmas. This section contains four technical lemmas (they are basically a lengthy case-by-case analysis of Coxeter graphs) which will be useful to split (and, hence, to improve the readability of) the proof of Theorem 4.16.

4.11. LEMMA. *If Γ is hyperbolic, then Γ is in \mathcal{M} , i.e.,*

$$\mathcal{Hc} \sqcup \mathcal{Hnc} \subseteq \mathcal{M}.$$

PROOF. A graph Γ is in \mathcal{M} if, and only if, $\Gamma \notin \mathcal{S} \cup \mathcal{A}$, and for all connected component Γ'' of a maximal proper subgraph $\Gamma' \leq \Gamma$, one has $\Gamma'' \in \mathcal{S} \cup \mathcal{A}$. From the very definition of hyperbolic and hyperbolic cocompact one immediately has that every proper parabolic is either of spherical or affine type, hence the claim. ■

4.12. NOTATION. Along the proof of the following lemmas, the symbols



will denote arbitrarily labelled circuits and linear graphs. A *branching point* is a vertex contained in at least 3 edges. An edge ϵ has a *large label* if $m(\epsilon) > 3$. A vertex is called *terminal* if, and only if, it is contained in at most one edge.

The type with diagram

$$(4.3) \quad \begin{array}{ccccccc} & m_1 & m_2 & & m_{k-1} & & \\ \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet \\ 1 & 2 & 3 & \dots & k-1 & k & \end{array}$$

will be identified with the $(k-1)$ -tuple in square brackets $[m_1, m_2, \dots, m_{k-1}]$.

The type with diagram

$$(4.4) \quad \begin{array}{ccccccc} & & 1 & m_1 & 2 & & \\ & & \bullet & \text{---} & \bullet & & \\ & m_k & & & m_2 & & \\ & \bullet & & & \bullet & & \\ & k & & & 3 & & \\ & m_{k-1} & & & m_3 & & \\ & \bullet & & & \bullet & & \\ & k-1 & & & 4 & & \end{array}$$

will be identified with the k -tuple in angled braces $\langle m_1, m_2, \dots, m_k \rangle$.

The above identifications are obviously not unique: $[m_1, \dots, m_{k-1}]$ and $[m_{k-1}, \dots, m_1]$ denote the same type, as well as $\langle m_1, \dots, m_k \rangle$ and $\langle m_{n+\varepsilon 1}, \dots, m_{n+\varepsilon k} \rangle$, for $n \in \mathbb{Z}$ and $\varepsilon \in \{1, -1\}$, where the subscript are taken modulo k .

Moreover, one has

$$[m_1, \dots, m_{k-2}] \leq \langle m_1, \dots, m_{k-1}, m_k \rangle.$$

The first lemma deals with the case of string graphs.

4.13. LEMMA. *Let $\Gamma \in \mathcal{M}$ be a Coxeter graph consisting of a string (4.3). Then Γ is in the following list (cf. Tables B.3, B.4).*

Rank 3: HC1,

Rank 4: HC3, HC4, HC5,

HNC13, HNC14, HNC15, HNC16, HNC17, HNC18, HNC19,

Rank 5: HC12, HC13, HC14,

HNC24

Rank 6: HNC33, HNC34, HNC35.

PROOF. For $k \in \{0, 1, 2\}$, all graphs are finite or affine.

Assume $k = 3$. The type $[a, b]$ is finite or affine if, and only if, $a^{-1} + b^{-1} \geq 1/2$, by direct computation. On the other hand $[a, b]$, with $a^{-1} + b^{-1} < 1/2$ has every proper subgraph of finite or affine type (the label ∞ is allowed and the convention $\frac{1}{\infty} = 0$ is adopted).

Let now $k = 4$. The type $[a, b, c]$ has subdiagrams $[a, b]$ and $[b, c]$ which must lie in $\mathcal{S} \cup \mathcal{A}$, thus

$$[a, b], [b, c] \in \{[3, 3], [3, 4], [4, 3], [3, 5], [5, 3], [4, 4], [3, 6], [6, 3]\},$$

hence the possibilities are

a	3	3	3	3	3	4	4	4	4	3	5	5	5	5	4	4	3	6	6	6	3
b	3	3	3	4	4	3	3	3	3	5	3	3	3	3	4	4	6	3	3	3	3
c	4	5	6	3	4	3	4	5	6	3	3	4	5	6	3	4	3	3	4	5	6
Type	\mathcal{S}	\mathcal{S}	HNC17	\mathcal{S}	HNC13	\mathcal{S}	\mathcal{A}	HC3	HNC15	HC4	\mathcal{S}	HC3	HC5	HNC16	HNC13	HNC14	HNC18	HNC17	HNC15	HNC16	HNC19

The same strategy works for $k = 5$, i.e., the diagram $[a, b, c, d]$ has subdiagrams

$$[a, b, c], [b, c, d] \in \{[3, 3, 3], [4, 3, 3], [3, 3, 4], [3, 4, 3], [3, 3, 5], [5, 3, 3]\},$$

thus the possibilities are the following

a	3	3	4	4	4	3	3	3	3	5	5	5	4
b	3	3	3	3	3	3	4	4	3	3	3	3	3
c	3	3	3	3	3	4	3	3	5	3	3	3	4
d	4	5	3	4	5	3	3	4	no	3	4	5	3
Type	\mathcal{S}	HC13	\mathcal{S}	\mathcal{A}	HC12	\mathcal{A}	\mathcal{A}	HNC24	None	HC13	HC12	HC14	HNC24

For $k = 6$ one checks that the diagram $[a, b, c, d, e]$ must have the subdiagrams $[a, b, c, d], [b, c, d, e] \in \{[3, 3, 3, 3], [4, 3, 3, 3], [3, 3, 3, 4], [4, 3, 3, 4], [3, 3, 4, 3], [3, 4, 3, 3]\}$, hence

a	3	4	3	4	3	3	3
b	3	3	3	3	3	4	4
c	3	3	3	3	4	3	3
d	3	3	4	4	3	3	3
e	4	4	3	3	3	3	4
Type	\mathcal{S}	\mathcal{A}	HNC33	HNC35	HNC34	HNC33	HNC35

Finally, suppose $k \geq 7$, hence Γ identifies with $[m_1, m_2, \dots, m_{k-1}]$. The only linear graphs of rank $k - 1$ which are finite or affine are the ones of type A_{k-1} , B_{k-1} , C_{k-1} or \tilde{C}_{k-2} , thus

$$[m_1, \dots, m_{k-2}], [m_2, \dots, m_{k-1}] \in \{[3, \dots, 3], [4, 3, \dots, 3], [3, \dots, 3, 4], [4, 3, \dots, 3, 4]\}.$$

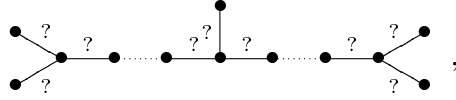
Then $m_i = 3$ for all $i \in \{2, \dots, k-2\}$ and $m_1, m_{k-1} \in \{3, 4\}$. In any case Γ **Errata: Corrige:** is either finite or affine. ■

The second lemma classifies the graphs which have no circuits and at least one branching point.

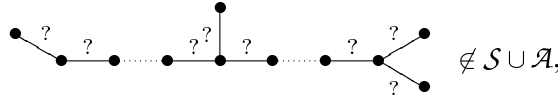
4.14. LEMMA. *Suppose that $\Gamma \in \mathcal{M}$, Γ contains no circuit and Γ has at least one branching point. Then Γ is one of the following types (cf. Tables B.3, B.4).*

- Rank 4: HC6,
HNC20, HNC21, HNC22,
Rank 5: HC15,
HNC25, HNC26, HNC27, HNC30,
Rank 6: HNC36, HNC37, HNC38, HNC39, HNC40, HNC41,
Rank 7: HNC45, HNC46,
Rank 8: HNC48, HNC49, HNC50,
Rank 9: HNC52, HNC53, HNC54,
Rank 10: HNC56, HNC57, HNC58.

PROOF. Let t be the number of branching points, $t \geq 1$. If $t \geq 3$, then Γ has a (non-necessarily proper) subgraph of type

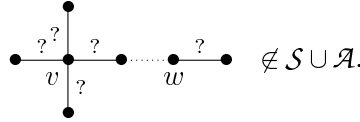


which, in turn, has a proper subgraph of type

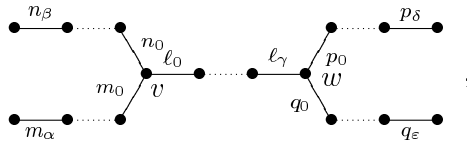


thus $t \in \{1, 2\}$.

Case $t = 2$: in this case Γ has one vertex v contained in $a+1$ edges and another vertex w contained in $b+1$ edges, $a, b \geq 2$. If $a \geq 3$, then removing $a-3$ edges incident to v and $b-1 > 0$ edges incident to w : then one has a proper subgraph



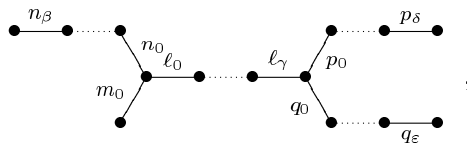
Thus $a = 2$, and a similar argument shows that $b = 2$. Then, since no circuits can exist, the graph has the form



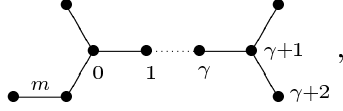
for suitable $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{N}_0$ and labelling functions m, n, ℓ, p, q .

If $\alpha = \beta = \gamma = \varepsilon = 0$, then at least one label must be large. Removing any terminal vertex, one obtains a proper subgraph with a ramification: thus, $\ell_0 = \dots = \ell_\gamma = 3$. Moreover, at most one large label can occur, and it must be 4. Without loss of generality, suppose $m_0 = 4$ and all other labels are equal to 3. Then, removing any terminal vertex containing w , leads to a contradiction.

Thus, assume $\max\{\alpha, \beta, \gamma, \varepsilon\} \geq 1$, and suppose without loss of generality that $\alpha \geq 1$. Then, there is a proper subgraph



which is in $\mathcal{S} \cup \mathcal{A}$ only if $\beta = \delta = \varepsilon = 0$ and $\ell_0 = \dots = \ell_\gamma = m_0 = n_0 = p_0 = q_0 = 3$. Thus if $\alpha \geq 1$, by minimality one may suppose $\alpha = 1$, and the graph is

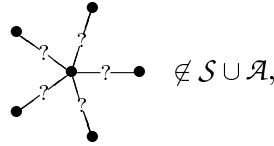


then remove $(\gamma + 2)$ nd vertex and obtain a proper subgraph, say Γ' . Then one has $m = 3$ and Γ' must be one among $E_6, E_7, E_8, \tilde{E}_8$, and hence Γ is one among HNC46, HNC49, HNC54, HNC58.

Case $t = 1$: Suppose the branching point v lies in exactly a edges, $a \geq 3$, i.e., its valency is a .

Since any graph in $\mathcal{S} \cup \mathcal{A}$ has branching point of valency at most 4, then $a \leq 5$.

Subcase $a = 5$: in this case Γ must have a subgraph Γ' of type



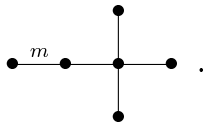
thus $\Gamma = \Gamma'$. Moreover removing any terminal vertex, one has that no large label can occur, and hence one has HNC41.

Subcase $a = 4$: Γ has a subgraph of the type



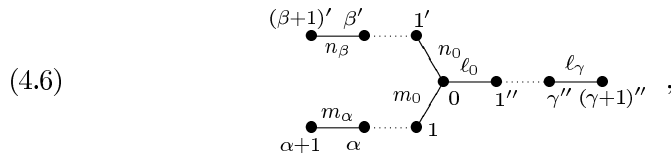
If the graph Γ coincides with (4.5), then at least one label must be large, say $a > 3$. Thus, removing one terminal vertex at a time, one has subgraphs which only can be of type D_4 or \tilde{B}_3 . Thus one has HNC30

If Γ contains (4.5) as a proper subgraph, then by minimality there can be at most 6 vertices. Moreover, the labels of the edges containing v cannot be large, since the only spherical or affine type having a vertex of valency 4 is \tilde{D}_3 . Since there are no circuits, the graph is



Then, removing any terminal vertex (not contained in the edge with the label m) one has $m \in \{3, 4\}$, i.e., Γ is either HNC39 or HNC40.

Subcase $a = 3$: in this case there is a unique branching point with valency 3 and no circuits, then the general shape of Γ is



for suitable α, β, γ and labelling functions m, n, ℓ . Without loss of generality, suppose $0 \leq \alpha \leq \beta \leq \gamma$.

Few simple observations give a bound to the list of cases to be analysed.

- When $\gamma \geq 2$ consider the proper subgraph with vertices $\{0, 1, 1', 1'', 2''\}$, which must be in $\mathcal{S} \cup \mathcal{A}$: thus $m_0 = n_0 = \ell_0 = 3$.

- When $(\alpha, \beta, \gamma) = (1, 1, 3)$, the graph contains a proper subgraph which is neither of spherical nor affine type, thus if $\Gamma \in \mathcal{M}$ is of type (4.6), with $\gamma \geq 3$, then $\alpha = 0$.
- When $\gamma \geq 4$, the graph obtained removing the vertex $5''$ is in $\mathcal{S} \cup \mathcal{A}$ (being one of E_8, \tilde{E}_8) only if $\beta \leq 1$.
- The very same consideration implies that if $\gamma \geq 4$ and $\beta = 1$, then $\gamma \leq 5$.
- Finally, for $\alpha = \beta = 0$ and $\gamma \geq 3$ the only possible graphs are of types $D_{\gamma+4} \in \mathcal{S}$ or $\tilde{B}_{\gamma+3} \in \mathcal{A}$.

Only the following case can thus occur for (α, β, γ) :

- (0, 0, 0): Suppose that there are at least two large labels, without loss of generality let $m_0, n_0 > 3$. Removing $1''$ one has $m_0 = n_0 = 4$, and hence the graphs HNC21, HNC22 arise.
 Suppose, on the other hand, that there is exactly one large label, say $m_0 > 3$. If $m_0 = 4$ one has type $\tilde{B}_3 \in \mathcal{A}$. Thus, removing $1''$, one has $m_0 \in \{5, 6\}$, i.e., Γ is HC6 or HNC20.
- (0, 0, 1): Removing the vertex $2''$ one has that there is at most one large label among m_0, n_0, ℓ_0 , and it is equal to 4. Removing the vertex 1 one has that $\ell_1 \in \{3, 4, 5\}$. Thus one has the types HC15, HNC25, HNC26, HNC27.
- (0, 1, 1): Remove the vertex $2'$ and obtain $m_0 = n_0 = \ell_0 = 3$ and $\ell_1 \in \{3, 4\}$. Remove the vertex $2''$ and further obtain $n_1 \in \{3, 4\}$. Since at least one large label must exist one has the types HNC37 and HNC38.
- (1, 1, 1): Removing the vertex 2, one has that $m_0 = n_0 = \ell_0 = n_1 = \ell_1 = 3$, and removing $2'$ one further has $m_1 = 3$, but this is a graph of affine type.
- (0, 0, 2): The only large labels can be ℓ_1 or ℓ_2 , and removing 1 one has that at most one of the two is actually large, and it is a 4. Thus one has the affine $\tilde{B}_5 \notin \mathcal{M}$ or HNC36.
- (0, 1, 2): Removing $3''$ one has that there is exactly one large label, $\ell_2 \geq 3$. Removing 1 one has that $\ell_2 = 4$, and one has HNC45.
- (1, 1, 2): Removing $3''$ one has that there is at most one large label, ℓ_2 . Removing 1 one has that also $\ell_2 = 3$, and one has HNC50.
- (0, 2, 2): Removing $3''$ (resp. $3'$) one has the unique large label is possibly ℓ_2 (resp. n_2). Thus there are no large labels and Γ is of the affine type $\tilde{E}_7 \notin \mathcal{M}$.
- (1, 2, 2): No graphs in the class \mathcal{M} occur in this case, as one deduces removing, e.g., the vertex $3''$.
- (2, 2, 2): No graphs in the class \mathcal{M} occur in this case, as one deduces removing, e.g., the vertex $3''$.
- (0, 1, 3): Removing $2'$ one has $m_0 = n_0 = \ell_0 = \ell_1 = \ell_2 = 3$, and $\ell_3 \in \{3, 4\}$. Removing $4''$ one has $n_1 = 3$. Thus the only possible cases are $E_8 \notin \mathcal{M}$ or HNC48.
- (0, 2, 3): Removing the vertices $3'$ and $4''$ one-by-one, one deduces that there are no large labels, and one has HNC52.
- (0, 1, 4): Removing the vertex $5''$ one has that there is at most one large label, $\ell_4 \in \{3, 4\}$. The case $\ell_4 = 3$ is affine, the case $\ell_4 = 4$ is HNC53.
- (0, 1, 5): Removing the vertex $6''$ one has that there is at most one large label, $\ell_5 \in \{3, 4\}$. This gives HNC56, HNC57.

Thus, no other cases can exist, and the proof is complete. ■

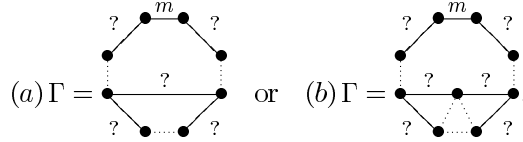
The third lemma classifies the types containing a circuit.

4.15. LEMMA. *Let $\Gamma \in \mathcal{M}$ and suppose that Γ contains a circuit (4.4) of k vertices, $k \geq 3$. Then Γ is one of the following types (cf. Tables B.3, B.4).*

- Rank 3: HC2,
 $\Gamma_\infty(3)$,
 Rank 4: HC7, HC8, HC9, HC10, HC11,
 HNC1, HNC2, HNC3, HNC4, HNC5, HNC6, HNC7,
 HNC8, HNC9, HNC10, HNC11, HNC12, HNC23,
 Rank 5: HC16,
 HNC28, HNC29, HNC31, HNC32,
 Rank 6: HNC42, HNC43, HNC44,
 Rank 7: HNC47
 Rank 8: HNC51,
 Rank 9: HNC55.

PROOF. **Part 1:** Suppose there is an edge \mathfrak{e} of the circuit with $m(\mathfrak{e}) > 3$.

If there is more than one vertex not lying on the circuit, there would be a proper subdiagram consisting of a circuit with a label greater than 3, which is neither finite nor affine. Thus, $|S| \leq k + 1$. Suppose Γ has branching point, then either



These cases can be both excluded since they contain a proper subgraph (the ‘‘top half’’) which is circuit with a label $m > 3$, which is not affine.

Thus, a circuit with a large label $m > 3$ cannot branch. Removing a vertex then removes exactly two edges, and since in $\mathcal{S} \cup \mathcal{A}$ every graph has at most 2 large labels, then Γ has at most 4 large labels, say ℓ their number. Moreover, if $\ell = 4$ then any vertex must be contained in two edges with large labels, while if $\ell = 3$, then any vertex lies in at least one edge with large label.

Then, only the following possibilities can occur (the labels a, \dots, d are large):

- $\ell = 4$ (a) $\langle a, b, c, d \rangle$
 $\ell = 3$ (b) $\langle a, b, c \rangle$, (c) $\langle a, b, c, 3 \rangle$, (d) $\langle a, b, 3, c, 3 \rangle$, (e) $\langle a, 3, b, 3, c, 3 \rangle$,
 $\ell = 2, k = 3$ (f) $\langle a, b, 3 \rangle$, with $a^{-1} + b^{-1} < 2/3$, otherwise in $\mathcal{S} \cup \mathcal{A}$
 $\ell = 2, k = 4$ (g) $\langle a, b, 3, 3 \rangle$, (h) $\langle a, 3, b, 3 \rangle$,
 $\ell = 2, k = 5$ (i) $\langle a, b, 3, 3, 3 \rangle$, (j) $\langle a, 3, b, 3, 3 \rangle$,
 $\ell = 2, k = 6$ (k) $\langle a, b, 3, 3, 3, 3 \rangle$, (l) $\langle a, 3, b, 3, 3, 3 \rangle$, (m) $\langle a, 3, 3, b, 3, 3 \rangle$.
 $\ell = 2, k \geq 7$ If the large labels are adjacent, then there is a subdiagram of type $[a, b, 3]$, which is not in $\mathcal{S} \cup \mathcal{A}$. If the large labels are not adjacent, since $k \geq 7$, then there is a subdiagram of type $[a, 3, \dots, 3, b, 3]$, which is not in $\mathcal{S} \cup \mathcal{A}$. Thus, there are no possibilities in this case.
 $\ell = 1, k = 3$ (n) $\langle a, 3, 3 \rangle$,
 $\ell = 1, k = 4$ (o) $\langle 3, a, 3, 3 \rangle$,
 $\ell = 1, k = 5$ (p) $\langle 3, 3, a, 3, 3 \rangle$,
 $\ell = 1, k = 6$ (q) $\langle 3, 3, a, 3, 3, 3 \rangle$.
 $\ell = 1, k \geq 7$ In this case there is a proper subdiagram of type $[3, a, 3, 3, 3] \notin \mathcal{S} \cup \mathcal{A}$.

Then one has to discuss the values of the large labels for each case (a) to (q).

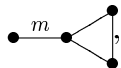
- (a) There are subdiagrams $[a, b]$ and $[c, d]$, thus $a = b = c = d = 4$ and one has HNC3.
 (b) Since labels are large, this is HC2 or $\Gamma_\infty(3)$.
 (c) There are subdiagrams $[a, b]$ and $[b, c]$, thus $a = b = c = 4$ and one has HNC2.
 (d) There is a proper subdiagram $[a, b, 3] \notin \mathcal{S} \cup \mathcal{A}$.
 (e) There is a proper subdiagram $[a, 3, b, 3] \notin \mathcal{S} \cup \mathcal{A}$.


- (f) Since $a, b > 3$, this is HC2.
- (g) There is a subdiagram $[a, b]$, thus $a = b = 4$ and one has HNC1.
- (h) There are subdiagrams $[a, 3], [b, 3]$, thus $a, b \in \{4, 5, 6\}$ and one has types HC8, HC9, HNC5, HC11, HNC6, HNC7.
- (i) There is a proper subdiagram $[a, b, 3] \notin \mathcal{S} \cup \mathcal{A}$.
- (j) There is a proper subdiagram $[a, 3, b]$, thus $a = b = 4$ and one has HNC32.
- (k) There is a proper subdiagram $[a, b, 3, 3] \notin \mathcal{S} \cup \mathcal{A}$.
- (l) There is a proper subdiagram $[a, 3, b, 3] \notin \mathcal{S} \cup \mathcal{A}$.
- (m) There is a proper subdiagram $[a, 3, 3, b]$, then $a = b = 4$ and one has HNC43.
- (n) This is HC2, since $a > 3$.
- (o) There is a proper subdiagram $[a, 3]$, then $a \in \{4, 5, 6\}$ and one has HC7, HC10, HNC4.
- (p) There is a proper subdiagram $[a, 3, 3]$, then $a = 4$ and one has HC16.
- (q) There is a proper subdiagram $[3, a, 3, 3]$, then $a = 4$ and one has HNC42.

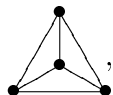
Part 2: Suppose Γ has a circuit with k vertices and, for all edge ϵ contained in the circuit, $m(\epsilon) = 3$.


One immediately knows that Γ has at most $k+1$ vertices, otherwise there would exist a proper subgraph not in $\mathcal{S} \cup \mathcal{A}$. In particular, all the circuits contained in Γ must have the same number of vertices and no edge contained in any circuit can have a large label.


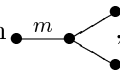
Thus, the possible cases are the following.

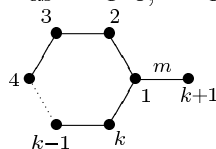
- (a)  , then removing a vertex not contained in the labelled edge one has a subgraph $[3, m]$, hence $m \in \{3, 4, 5, 6\}$ and one has HNC9, HNC10, HNC11, HNC12.

- (b)  , which is HNC8.

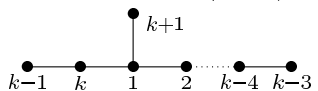
- (c)  , which is HNC23.

- (d)  , which is HNC31.

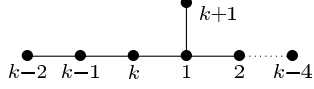
- (e)  which has a subgraph  , thus $m \in \{3, 4\}$, and one has HNC28, HNC29.

- (f)  , for $k \geq 5$. In this case one first considers the subgraph obtained removing the $(k-1)$ st vertex, and thus $m = 3$.

Then, remove the $(k-2)$ nd vertex and obtain a proper subgraph of

- type  , thus $k \in \{5, 6, 7, 8, 9\}$.

Finally, remove the $(k - 3)$ rd vertex and obtain a proper subgraph

of type  , thus $k \neq 9$. This gives the types HNC44, HNC47, HNC51, HNC55.

This completes the proof of the lemma. \blacksquare

2.2. The class \mathcal{M} .

4.16. THEOREM. *The class \mathcal{M} consists of the graphs listed in Table B.3 and B.4.*

PROOF. The following cases exhaust all the possibilities:

- Γ is a tree
 - without branching points,
 - with at least one branching point,
- Γ is not a tree, and hence Γ contains a circuit.

Let $\Gamma \in \mathcal{M}$: the three cases are discussed, respectively in Lemma 4.13, Lemma 4.14, Lemma 4.15.

Thus, every graph in \mathcal{M} appears in either Table B.3 or B.4. Moreover, a direct verification shows that every graph appearing in Table B.3 or B.4 is actually in the class \mathcal{M} . \blacksquare

4.17. COROLLARY (Lannér, [Lan50]). *The class \mathcal{Hc} consists of the graph listed in Table B.3 and the class \mathcal{Huc} consists of the ones listed in Table B.4.*

PROOF. By Lemma 4.11, one only has to check that every graph in \mathcal{M} is actually hyperbolic, and Theorem 4.16 provides the list of the diagrams to inspect. \blacksquare

4.18. COROLLARY. *If (W, S) is an irreducible Coxeter system, with (connected) Coxeter graph Γ , then exactly one of the following conditions hold.*

- (W, S) is either of spherical or affine type.
- (W, S) has a parabolic subsystem of hyperbolic cocompact or hyperbolic type.

3. An application

As an application of Theorem 4.16, one recovers a result by De La Harpe about Coxeter groups.

Before stating it, the following lemmas are needed.

4.19. LEMMA. *Under the conventions stated in Notation B.2, the diagrams of types HC1 and HC2 (cf. Table B.3) can be both considered as diagrams*

$$\Gamma(a, b, c) = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \quad \text{with} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1.$$

Let (W, S) be a Coxeter system with Coxeter diagram Γ , and let $p_{(W,S)}(t)$ be its Poincaré series, with convergence radius $\rho_{(W,S)}$. Then

$$\frac{1}{2} \leq \rho_{(W,S)} < 1.$$

PROOF. Using the recursive formula (4.2), one has

$$p_{\Gamma}(t) = \frac{(1+t)(1-t^a)(1-t^b)(1-t^c)}{1-2t+t^{a+1}+t^{b+1}+t^{c+1}-t^{a+b}-t^{a+c}-t^{b+c}+2t^{a+b+c}-t^{a+b+c+1}}.$$

Thus, by Lemma C.2(2), the convergence radius is the first positive real root of the denominator

$$f(t) = 1 - 2t + t^{a+1} + t^{b+1} + t^{c+1} - t^{a+b} - t^{a+c} - t^{b+c} + 2t^{a+b+c} - t^{a+b+c+1}.$$

One has that $f(0) = 1$ and, denoting $(-)' = \frac{d}{dt}$,

$$f(1) = f'(1) = f''(1) = 0, \quad \text{while} \quad f'''(1) = 6(abc - ab - bc - ac).$$

Then, the condition $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$ forces $f'''(1) > 0$, and hence by Lemma C.4, there is a positive number ε such that $f(y) < 0$ if $1 - \varepsilon < y < 1$. Thus, since f is continuous, it has at least one root in the interval $\{t \in \mathbb{R} \mid 0 < t \leq 1 - \varepsilon\}$. Hence $\rho < 1$, and $\rho \geq 1/2$ follows from Proposition 4.7(2). ■

4.20. LEMMA. *Let (W, S) be a Coxeter system which is hyperbolic cocompact or hyperbolic, and suppose it is neither HC1 nor HC2. Then, if $\rho_W = \rho_{(W, S)}$ is the convergence radius, one has*

$$\rho_W < 1.$$

PROOF. One has to check a finite number of cases. With GAP3 [S⁺97] and the package CHEVIE [GHL⁺96] one may compute the Poincaré series of (W, S) with (4.2). Then, by Lemma C.2(2), one finds the value of ρ .

The convergence radii are listed in Tables B.3 and B.4.

Further details about the computations are collected in Appendix D. ■

4.21. PROPOSITION. *Let (W, S) be a Coxeter system which is neither finite nor affine.*

If $\rho_{(W, S)}$ is the convergence radius of the Poincaré series $p_{(W, S)}(t)$, then

$$\rho_{(W, S)} < 1.$$

PROOF. By Theorem 4.16, (W, S) has a parabolic subsystem (W_I, I) of hyperbolic cocompact or hyperbolic type. Thus, by Proposition 4.2(6),

$$\rho_{(W, S)} \leq \rho_{(W_I, I)} < 1,$$

where the last inequality follows from Lemmas 4.19 and 4.20. ■

Thus, one may give an alternative proof of the following result.

4.22. THEOREM ([dlH87]). *Let (W, S) be an irreducible Coxeter system with connected Coxeter graph Γ and convergence radius $\rho = \rho_{(W, S)}$. Then the following conditions are equivalent:*

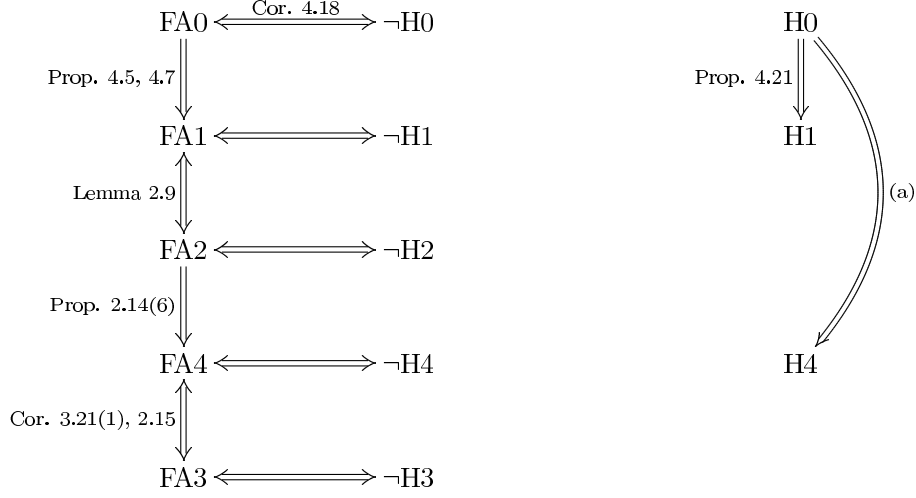
- FA0 (W, S) is of finite or affine type;
- FA1 $\rho \geq 1$;
- FA2 W is a group of polynomial growth;
- FA3 W does not contain a copy of a non-abelian free group;
- FA4 W is an amenable group.

Moreover, the following conditions are equivalent:

- H0 (W, S) has a (non-necessarily proper) parabolic subsystem (W_I, I) of hyperbolic type;
- H1 $\rho < 1$;
- H2 W is a group of exponential growth;
- H3 W contains a copy of a non-abelian free group;
- H4 W is not amenable;

In particular, an irreducible Coxeter system satisfies exactly one of the two blocks of conditions FA or H.

PROOF. The scheme of the proof is the following.



Thus, it suffices to prove (a): assume W contains a parabolic subgroup of hyperbolic cocompact or hyperbolic type, say W_I . Thus, by Theorem 3.20(3), and Definition 3.24, W_I is a discrete subgroup in the orthogonal group $O(n - 1, 1)$ with respect to a non-degenerate bilinear form with 1 negative and $n - 1$ positive eigenvalues, where $n = |S|$.

Moreover, it is well-known **Errata:** (cf. [Bou07c, Ch. 5 §4, Ex. 12]) **Corrige:** (cf. [Bou07c, Ch. V §4, Ex. 12]) that hyperbolic cocompact and hyperbolic Coxeter groups have finite covolume in $O(n - 1, 1)$, thus W is a lattice in the indefinite orthogonal group.

Thus, by Proposition 2.14(7) and 2.14(5), W_I is not amenable.

Finally, one deduces the non-amenability of W from 2.14(3). ■

4.23. COROLLARY. *Let (W, S) be a (possibly non-irreducible) Coxeter system, with Coxeter graph*

$$\Gamma = \Gamma_1 \sqcup \Gamma_2 \sqcup \dots \sqcup \Gamma_k, \quad \Gamma_i \text{ connected.}$$

Then

- *If Γ_i is either spherical or affine for all $i \in \{1, \dots, k\}$, then FA1, FA2, FA3 and FA4 hold true.*
- *Otherwise H0, H1, H2, H3 and H4 hold.*

Theorem 4.22 has another consequence, namely:

4.24. COROLLARY. *Let (W, S) be an irreducible Coxeter system. Then it is possible to determine wheter (W, S) is spherical, affine or it has a parabolic subgroup of hyperbolic or hyperbolic cocompact type, only considering the abstract group W .*

This result might be compared to Lemma 3.22, which tells that it is possible to determine if a group with a presentation à la Coxeter is actually a Coxeter group, only by looking at the abstract group.

CHAPTER 5

Hecke algebras

This chapter introduces Hecke algebras starting with the following datum:

- a Coxeter system (W, S) ,
- a commutative ring R ,
- a set \mathbf{q} of parameters in R .

Hecke algebras appear frequently in diverse contexts of mathematics, hence there exist many (slightly different) definitions of this kind of algebras.

The definition given in §2 comprehends

- the classical case of the double coset algebra of a finite group with a Tits' system (or (B, N) -pair);
- the algebra of averaging operators on a building.

What is *not* covered by the present definition is the Hecke algebra of a complex (pseudo-)reflection group (cf. Chapter 3 §1.3 and references therein), which is outside the scope of this thesis.

1. Some examples

1.1. Finite or algebraic group theory. In the theory of finite groups of Lie type, or in general in the context of groups with a Tits' system (cf. [Bou07c, Ch. IV §2]), Hecke algebras arise as (B, B) -double coset algebras. The most important feature is that the character theory of the Hecke algebra determines most of the character theory of the group (cf. [CR81, §11D]).

Given a finite group of Lie type G with a fixed Tits system (G, B, N, S) and with Weyl group W , let $e = \frac{1}{|B|} \sum_{b \in B} b$ be the idempotent associated to B . The Hecke algebra $\mathcal{H}(G, B)$ is defined to be $eCGe$, and it is well-known that the Bruhat decomposition

$$G = \coprod_{w \in W} BwB$$

implies that a \mathbb{C} -basis of \mathcal{H} is in 1-1 correspondence with W . Moreover, $\mathcal{H}(G, B)$ has a presentation of the form

$$\mathcal{H}(G, B) = \left\langle T_w, w \in W \mid T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ q_s T_{sw} + (q_s - 1)T_w & \text{if } \ell(sw) < \ell(w) \end{cases}, s \in S, w \in W \right\rangle,$$

where $\ell: W \rightarrow \mathbb{N}_0$ is the canonical length function of (W, S) and the parameters q_s are “intersection multiplicities”:

$$q_s = [B : B \cap {}^s B].$$

The Hecke algebras arising in this way are very particular: they are \mathbb{C} -algebras of *finite type* and *crystallographic* i.e., W is a Weyl group.

1.2. Knot theory. One of the most surprising and important application of Hecke algebras appears in [Jon87].

To each (tame) knot one may associate a braid, and a Hecke algebra \mathcal{H} of type A_n is a quotient of the braid algebra.

The fundamental result of Jones is that, using a suitably defined trace on the Hecke algebra, the HOMFLY-PT polynomial can be recovered from a representation of \mathcal{H} .

Further generalizations of the algebraic side of the problem (in particular, the classification of the suitable traces) were made for the type B_n by Geck and Lampropoulou (cf. [GL97]).

1.3. Harmonic analysis. Hecke algebras were studied in the context of harmonic analysis and buildings by Parkinson (cf. [Par06]).

A building Δ is a geometric/combinatorial structure constituted by a cell complex with a distinguished class of subcomplexes isomorphic to a fixed Coxeter complex of type W , and satisfying several axioms. The cells of maximal dimension are called *chambers*, and the set of all chambers is denoted $\mathcal{C}(\Delta)$.

In particular, the building Δ of type W can be given a W -valued metric space structure (cf. [AB08, Ch. 5]) via

$$\delta: \mathcal{C} \times \mathcal{C} \rightarrow W.$$

Under a geometric regularity condition on the size of the δ -spheres, one may define an algebra $\mathcal{B}(\Delta)$ of averaging operators.

For $f \in \mathcal{C} \rightarrow \mathbb{C}$, $w \in W$ and $c \in \mathcal{C}$, one puts

$$B_w(f)(c) = \frac{1}{|\{d \in \mathcal{C} \mid \delta(c, d) = w\}|} \sum_{\{d \in \mathcal{C} \mid \delta(c, d) = w\}} f(d),$$

and \mathcal{B} is the algebra generated by the averaging operators B_w , $w \in W$.

Then, one has the following.

5.1. THEOREM ([Par06, Thm. 3.10]). *Let Δ be a regular building of type W . Then, the algebra \mathcal{B} is isomorphic to the \mathbb{C} -Hecke algebra $\mathcal{H}(W, S, \mathbb{C}, \mathbf{q})$ (cf. Definition 5.2), where*

$$\mathbf{q}(s) = \frac{1}{|\{d \in \mathcal{C} \mid \delta(c, d) = s\}|} \quad \text{for all } s \in S.$$

The Hecke algebras which are defined in this context are very general, e.g., one may start with any Coxeter group. However, one obvious necessary condition to perform Parkinson's construction, is the *existence* of Δ , but it is well-known that the existence of a building of type W with prescribed \mathbf{q} is in general not guaranteed (cf. [FH64] or [KS73]).

2. Definition and first properties

5.2. DEFINITION. Let $\Gamma = (S, \mathfrak{E}, m)$ be a Coxeter graph and let (W, S) be the corresponding Coxeter system (cf. Chapter 3 §2 and Appendix A). Let R be a commutative ring with unit $1_R \in R$.

Let $\mathbf{q}: S \rightarrow R$ be a function constant on the connected components of Γ_{odd} ; it is called the *parameter function*.

Let $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$ be the associative R -algebra consisting of the free R -module $R^{(W)}$ with basis $\{T_w \mid w \in W\}$, and the associative multiplication defined by the relations

$$(5.1) \quad T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ \mathbf{q}(s)T_{sw} + (\mathbf{q}(s) - 1)T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

for $s \in S$ and $w \in W$.

The algebra $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$ is called the R -Hecke algebra of type (W, S) with parameters \mathbf{q} .

5.3. REMARK. (a) The parameter function \mathbf{q} can be identified with a list $(q_s \mid s \in S)$ of elements in R , such that $q_s = q_t$ if s and t are conjugate in W (cf. Proposition 3.13).

(b) If \mathbf{q} is constant $\mathbf{q}(s) = q$ for all $s \in S$, then the corresponding Hecke algebra is called *one-parameter Hecke algebra*.

(c) For a commutative ring R_0 , the R_0 -generic Hecke algebra is defined as follows. If Γ_{odd} has k connected components I_1, \dots, I_k let $R = R_0[q_1, \dots, q_k]$ for indeterminates q_j . Moreover let the parameter \mathbf{q} be defined by $\mathbf{q}(s) = q_j$ for the unique j such that $s \in I_j$. Then let $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$.

(d) Interesting phenomena happen when R is the ring $R_0[[q]]$ of power series (in one indeterminate q) over a commutative ring R_0 and $\mathbf{q}(s) = q$ for all $s \in S$. These phenomena are the subject of Theorem 5.38.

From the above definition, it is not immediate to see that a R -Hecke algebra exists and is unique, which is actually the case.

5.4. THEOREM. For a Coxeter system (W, S) , a commutative ring R and parameters \mathbf{q} satisfying the conditions of Definition 5.2, there exists an associative algebra structure \mathcal{H} on the free R -module $R^{(W)}$ such that (5.1) holds and T_1 is the unit.

Moreover if \mathcal{H} and \mathcal{H}' are such algebras, then there is an isomorphism of associative, unital R -algebras $\mathcal{H} \simeq \mathcal{H}'$.

PROOF. Cf. [Hum90, §§7.1–7.3]. ■

5.5. PROPOSITION. Let $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$ be a Hecke algebra. Then the following hold.

(1) The algebra \mathcal{H} is generated, as associative, unital R -algebra by $\{T_s \mid s \in S\}$ together with the relations

$$(5.1') \quad \begin{cases} \pi_m(T_s, T_t) = \pi_m(T_t, T_s), & \text{where } m = m(s, t) \\ T_s^2 = \mathbf{q}(s)T_1 + (\mathbf{q}(s) - 1)T_s, & \text{if } s \in S. \end{cases}$$

(2) The elements T_w , $w \in W$ are invertible in \mathcal{H} .

(3) If $w, w' \in W$ and $\ell(ww') = \ell(w) + \ell(w')$, then $T_w T_{w'} = T_{ww'}$.

(4) If $\mathbf{1}(s) = 1_R$ is the constant function equal to the identity, then

$$\mathcal{H}(W, S, R, \mathbf{1}) \simeq R[W].$$

PROOF. Cf. [GP00, §4.4]. ■

Since (5.1') is right-left symmetric, one has the following.

5.6. COROLLARY. If $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$ is a Hecke algebra, then a “right-handed” analogue of (5.1), holds:

$$(5.1'') \quad T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w) \\ \mathbf{q}(s)T_{ws} + (\mathbf{q}(s) - 1)T_w & \text{if } \ell(ws) < \ell(w), \end{cases}$$

for $s \in S$ and $w \in W$.

One more fact is needed in the sequel.

5.7. LEMMA. Let (W, S) be a Coxeter group and let R be a commutative ring.

Let $\mathbf{t} = \{t_s \in R \mid s \in S\}$ a set of elements of R such that $t_s = t_r$ if s and r are conjugate in W .

By Matsumoto's Theorem (cf. [GP00, Thm. 1.2.2]) there is a well-defined map induced by the map $T_s \mapsto t_s$: let ε_t be defined by

$$\begin{aligned} \varepsilon_t: \mathcal{B} = \{T_w \mid w \in W\} &\rightarrow R \\ T_w &\mapsto \prod_{k=1}^{\ell(w)} t_{s_k}, \end{aligned}$$

where $w = s_1 s_2 \dots s_{\ell(w)}$ is a reduced word.

2.1. Basis, augmentation, antipode, and canonical trace. The purpose of this paragraph is to give a canonical structure $(\mathcal{H}, _{}^{\natural}, \varepsilon_q, \mathcal{B}, \mu)$ of associative, augmented R -algebra with antipode to each Hecke algebra $\mathcal{H}(W, S, R, \mathbf{q})$.

The free R -basis \mathcal{B} is taken to be

$$\mathcal{B} = \{T_w \mid w \in W\},$$

while the antipodal map is defined as

$$_{}^{\natural}: \mathcal{H} \rightarrow \mathcal{H}^{\text{op}}, \quad (T_w)^{\natural} = T_{w^{-1}}.$$

Since $\ell(sw) = \ell(w^{-1}s^{-1})$ for all $w \in W$, $s \in S$, one easily sees that $_{}^{\natural}$ is a morphism of R -algebras, satisfying $_{}^{\natural\sharp} = \text{id}_{\mathcal{H}}$.

Then, take for augmentation the map $\varepsilon_q: \mathcal{H} \rightarrow R$ defined in Lemma 5.7. It is immediate to check that

$$\varepsilon_q(T_w) = \varepsilon_q(T_w^{\natural}),$$

since R is a commutative ring.

The trace function μ is determined as follows.

5.8. PROPOSITION. *Let \mathcal{H} be a R -Hecke algebra associated to (W, S) , and, by the above discussion, let $(\mathcal{H}, _{}^{\natural}, \varepsilon_q, \mathcal{B})$ be the 4-tuple consisting of the augmented R -Hecke algebra with distinguished basis and antipode.*

Then the R -bilinear map $\langle _, _ \rangle: \mathcal{H} \times \mathcal{H} \rightarrow R$ defined by (1.14) satisfies (1.15). In particular, $\tilde{\mu}_{\mathcal{B}} = \langle T_1, _ \rangle$ is a trace function.

PROOF. By Lemma 1.24, one has to show that

$$(5.2) \quad \langle T_u T_v, T_w \rangle = \langle T_v, T_{u^{-1}} T_w \rangle \quad \text{for all } u, v, w \in W.$$

Using induction one easily concludes that it suffices to show (5.2) in the case that $u = s \in S$. In this case one has:

$$(5.3) \quad \lambda = \langle T_s T_v, T_w \rangle = \begin{cases} \delta_{sv, w} \varepsilon_q(T_{sv}) & \text{if } \ell(sv) > \ell(v), \\ (q_s - 1) \delta_{v, w} \varepsilon_q(T_v) + q_s \delta_{sv, w} \varepsilon_q(T_{sv}) & \text{if } \ell(sv) < \ell(v), \end{cases}$$

and

$$(5.4) \quad \rho = \langle T_v, T_s T_w \rangle = \begin{cases} \delta_{v, sw} \varepsilon_q(T_v) & \text{if } \ell(sw) > \ell(w), \\ (q_s - 1) \delta_{v, w} \varepsilon_q(T_v) + q_s \delta_{v, sw} \varepsilon_q(T_v) & \text{if } \ell(sw) < \ell(w). \end{cases}$$

We proceed by a case-by-case analysis.

Case 1: $\ell(sv) > \ell(v)$ and $\ell(sw) > \ell(w)$. Suppose that $\lambda \neq 0$. Then $sv = w$, but $\ell(w) = \ell(sv) > \ell(v) = \ell(sw)$, a contradiction. Hence $\lambda = 0$. Reversing the rôles of v and w yields $\lambda = \rho = 0$ and thus the claim.

Case 2: $\ell(sv) > \ell(v)$ and $\ell(sw) < \ell(w)$. Then, $v \neq w$. If $\lambda \neq 0$, then $sv = w$. Hence $\ell(w) = \ell(sv) = \ell(v) + 1$, and $\lambda = \varepsilon_q(T_w) = \varepsilon_q(T_s) \varepsilon_q(T_v)$. On the other hand, $\rho = (q - 1) \delta_{v, w} \varepsilon_q(T_v) + q \delta_{v, sw} \varepsilon_q(T_v) = q \varepsilon_q(T_v) = \lambda$. If $\lambda = 0$, then $sv \neq w$. Hence $\rho = (q - 1) \delta_{v, w} \varepsilon_q(T_v) + q \delta_{v, sw} \varepsilon_q(T_v) = 0 = \lambda$.

Case 3: $\ell(sv) < \ell(v)$ and $\ell(sw) > \ell(w)$. Reversing the rôles of v and w one can transfer the proof for Case 2 verbatim.

Case 4: $\ell(sv) < \ell(v)$ and $\ell(sw) < \ell(w)$. Suppose that $sv = w$, or, equivalently, $v = sw$. Then $\ell(sv) < \ell(v) = \ell(sw) < \ell(w)$, a contradiction. Hence $sv \neq w$ and $v \neq sw$. Thus $\lambda = \rho$. This completes the proof. ■

5.9. REMARK. The trace function $\tilde{\mu}: \mathcal{H} \rightarrow R$ can be seen as the *canonical trace function* on \mathcal{H} . It is straight forward to verify that for Hecke algebras of type A_n , B_n or D_n this trace function coincides with the Jones–Ocneanu trace evaluated in 0 (cf. [Gec98]).

3. Parabolic Hecke subalgebras

Let (W, S) be a Coxeter system and let $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$ be the R -Hecke algebra with parameters \mathbf{q} . The functor

$$\mathcal{H}_- : 2^S \rightarrow R\text{-Alg}, \quad \mathcal{H}_I = \mathcal{H}(W_I, I, R, \mathbf{q}|_I)$$

is called the *Hecke algebra parabolic structure*. By Propositions 3.15 and 5.5(1) one has that, for all $I \subseteq S$, there is a canonical isomorphism $\mathcal{H}_I \simeq \langle T_s \mid s \in I \rangle$ of R -algebras. In particular $\mathcal{H}_S \simeq \mathcal{H}$ and $\mathcal{H}_\emptyset \simeq R$.

5.10. DEFINITION (Parabolic Hecke subalgebra). Any subalgebra $\langle T_s \mid s \in I \rangle$ is called *parabolic Hecke subalgebra*.

In the case of Coxeter group algebras, there exists a distinguished set W^I of left W_I -coset representatives; for Hecke algebras the situation is very similar.

5.11. PROPOSITION. Let $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$ be a Hecke algebra and let $I \subseteq S$. Let \mathcal{H}^I be the free R -module over the set W^I . Then the map of right \mathcal{H}_I -modules

$$m_I: \mathcal{H}^I \otimes_R \mathcal{H}_I \rightarrow \mathcal{H},$$

induced by multiplication in \mathcal{H} , is an isomorphism for all $I \subseteq S$. Thus, \mathcal{H} is a free, right \mathcal{H}_I -module.

PROOF. It suffices to say that $\ell(vu) = \ell(v) + \ell(u)$ for $v \in W^I$, $u \in W_I$ (cf. Proposition 3.18(5)), hence $T_v T_u = T_{vu}$ by Proposition 5.5(3). ■

4. Modules

4.1. One-dimensional modules. In the present work a central place is occupied by one-dimensional \mathcal{H} -modules, i.e. module which are isomorphic —as R -modules— to the free R -module R . It turns out that their structure is particularly easy.

Morphisms of R -algebras $\varepsilon \in \text{Hom}_{R\text{-Alg}}(\mathcal{H}, R)$ (i.e. linear characters) and 1-dimensional modules determine each other by the formula $h.r = \varepsilon(h)r$, where $h \in \mathcal{H}$ and $r \in R$.

Let $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$, and let Γ be the Coxeter graph of (W, S) . Any such $\varepsilon \in \text{Hom}_{R\text{-Alg}}(\mathcal{H}, R)$, by (5.1'), must satisfy

$$(5.5) \quad \pi_m(\varepsilon(T_s), \varepsilon(T_t)) = \pi_m(\varepsilon(T_t), \varepsilon(T_s)) \quad \text{for all } s, t \in S$$

and

$$(5.6) \quad \varepsilon(T_s)^2 = q_s + (q_s - 1)\varepsilon(T_s) \quad \text{for all } s \in S.$$

Condition (5.6) forces $\varepsilon(T_s) \in \{-1, q_s\}$. Condition (5.5) is trivial when $2 \mid m$, and it is satisfied for m odd, e.g. when $\varepsilon(T_s) = \varepsilon(T_t)$ if s and t are connected in Γ_{odd} —since \mathbf{q} is locally constant on Γ_{odd} .

The analogue of the trivial module and the sign module for group algebras are given by the following definition.

5.12. DEFINITION (Trivial and sign \mathcal{H} -modules). The one dimensional module R_q determined by $\varepsilon_q(T_s) = q_s$ for all $s \in S$ is the *trivial \mathcal{H} -module*. The one dimensional module R_{-1} determined by $\varepsilon_{-1}(T_s) = -1$ for all $s \in S$ is the *sign \mathcal{H} -module*.

5.13. REMARK. In the case of a Hecke algebra with $\mathbf{q}(s) \neq 1$ for all $s \in S$, then there exist exactly 2^k pairwise non-isomorphic 1-dimensional \mathcal{H} -modules, where k is the number of connected components of Γ_{odd} .

4.2. Generalized Poincaré series. Let $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$ be the R -Hecke algebra associated to (W, S) with parameters \mathbf{q} .

Let ε_q be the map defined in Lemma 5.7.

5.14. DEFINITION (Generalized Poincaré series). The *generalized Poincaré series* of $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$ is defined to be the formal sum

$$p_{\mathcal{H}}(\mathbf{q}) = \sum_{w \in W} \varepsilon_q(T_w) \in \mathbb{Z}[\mathbf{q}].$$

For evident reasons, the generalized Poincaré series $p_{\mathcal{H}}(\mathbf{q})$ is also denoted $p_{(W, S)}(\mathbf{q})$.

4.3. Induced e restricted modules. Let $I \subseteq S$, then applying the machinery of Chapter 1 §1.4 one defines the induction

$$\mathbf{ind}_I^S(-) = \mathbf{ind}_{\mathcal{H}_I}^{\mathcal{H}}(-): \mathcal{H}_I\text{-Mod} \rightarrow \mathcal{H}\text{-Mod}$$

and the restriction

$$\mathbf{res}_I^S(-) = \mathbf{res}_{\mathcal{H}}^{\mathcal{H}_I}(-): \mathcal{H}\text{-Mod} \rightarrow \mathcal{H}_I\text{-Mod}.$$

Induction behaves well on projective modules, indeed the following holds.

5.15. COROLLARY (of Proposition 5.11). *The induction functor $\mathbf{ind}_I^S(-)$ is exact and takes projective left \mathcal{H}_I -modules to projective left \mathcal{H} -modules.*

5.16. NOTATION. Let $\eta_I = T_1 \otimes_{\mathcal{H}_I} 1 \in \mathcal{H} \otimes_{\mathcal{H}_I} R_q$. Then the elements appearing in (5.7) will be rewritten as $T_w \eta_I$.

As already mentioned (cf. §2.1), the trivial module occupies a special place in the present thesis, and modules induced from R_q satisfy the following lemma.

5.17. LEMMA. *Let $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$ be a Hecke algebra, let $I \subseteq S$ and let $R_q(I)$ the trivial \mathcal{H}_I -module. Then the induced module $\mathbf{ind}_I^S R_q(I)$ is a free R -module with basis*

$$(5.7) \quad \{T_w \otimes_{\mathcal{H}_I} 1 \mid w \in W^I\}.$$

The canonical map $c_I: \mathcal{H}^I \rightarrow \mathbf{ind}_I^S R_q$ given by $c_I(T_w) = T_w \eta_I$, where $w \in W^I$, is an isomorphism of R -modules. Moreover, for $w \in W$, one has $T_w \eta_I = \varepsilon_q(T_{w_I}) T_{w^I} \eta_I$.

PROOF. The elements of the module $\mathbf{ind}_I^S R_q \simeq \mathcal{H} \otimes_{\mathcal{H}_I} R_q$ can be written as finite R -linear combinations of monomials of the type $T_w \otimes 1$, for $w \in W$. Moreover, using the decomposition of Proposition 3.18(1), write $w = w^I w_I$: then

$$T_w \otimes 1 = T_{w^I} T_{w_I} \otimes 1 = T_{w^I} \otimes T_{w_I} \cdot 1 = T_{w^I} \otimes \varepsilon_q(T_{w_I}) = \varepsilon_q(T_{w_I}) T_{w^I} \eta_I. \quad \blacksquare$$

Under suitable conditions of finiteness a invertibility of the Poincaré series, the modules $\mathbf{ind}_I^S R_q$ and R_{-1} admit the following characterizations.

5.18. PROPOSITION. *Let I be a subset of S such that W_I is finite. Put $\tau_I = \sum_{w \in W_I} T_w$. Then one has the following:*

$$(1) \quad \tau_I^2 = p_{(W_I, I)}(\mathbf{q}) \tau_I.$$

Moreover if $p_{(W_I, I)}(\mathbf{q}) \in R^\times$ is invertible in R , let $e_I = (p_{(W_I, I)}(\mathbf{q}))^{-1} \tau_I$. Then:

- (2) the element e_I is a central idempotent in \mathcal{H}_I satisfying $e_I^{\natural} = e_I$;
- (3) the left ideal $\mathcal{H}e_I$ is a finitely generated, projective, left \mathcal{H} -module isomorphic to $\mathbf{ind}_I^S R_q$;
- (4) $T_w e_I = \varepsilon_q(T_{wI})T_{wI} e_I$.

PROOF. For $s \in I$ put $X_s = \sum_{w \in \{s\}(W_I)} T_w$. Then $\tau_I = (T_1 + T_s)X_s$ (cf. Prop. 3.18(6)) and therefore

$$T_s \tau_I = T_s(T_1 + T_s)X_s = [T_s + q_s T_1 + (q_s - 1)T_s]X_s = q_s(T_1 + T_s)X_s = \varepsilon_q(T_s)\tau_I.$$

This shows (1). Part (2) is an immediate consequence of (1), and the first part of (3) follows from the decomposition of the regular module ${}^{\text{reg}}\mathcal{H} = \mathcal{H}e_I \oplus \mathcal{H}(T_1 - e_I)$. The canonical map $\pi: \mathcal{H} \rightarrow \mathbf{ind}_I^S R_q$, $\pi(T_w) = T_w \eta_I$, is a surjective morphism of \mathcal{H} -modules with $\ker(\pi) = \mathcal{H}(T_1 - e_I)$. This yields the second part of (3). Part (4) follows from part (2) and Proposition 3.18(1). \blacksquare

5.19. PROPOSITION. *Let W be finite with longest element w_0 . Assume further that $p_{(W,S)}(\mathbf{q}) \in R^\times$ and let*

$$z = (p_{(W,S)}(\mathbf{q}))^{-1} \sum_{w \in W} \varepsilon_{-1}(T_w) \varepsilon_q(T_{w_0 w}) T_w \in \mathcal{H}.$$

Then one has the following.

- (1) For $w \in W$ one has $T_w z = \varepsilon_{-1}(T_w)z$, i.e., $\mathcal{H}z$ is isomorphic to R_{-1} as \mathcal{H} -module.
- (2) The element $z \in \mathcal{H}$ is a central idempotent satisfying $z^{\natural} = z$.
- (3) The left ideal $\mathcal{H}z$ is a finitely generated, projective, left \mathcal{H} -module.

PROOF. Let $\alpha(w) = (p_{(W,S)}(\mathbf{q}))^{-1} \varepsilon_{-1}(T_w) \varepsilon_q(T_{w_0 w})$. Then

$$\begin{aligned} T_s z &= \sum_{w \in W} \alpha(w) T_s T_w = \sum_{w \in \{s\}W} \alpha(w) T_{sw} + \sum_{w \notin \{s\}W} \alpha(w) (q_s T_{sw} + (q_s - 1)T_w) \\ &= \underbrace{\sum_{w \in \{s\}W} \alpha(w) T_{sw}}_A + \underbrace{\sum_{v \in \{s\}W} -\alpha(v) T_v}_B + \underbrace{\sum_{w \notin \{s\}W} \alpha(w) (q_s - 1) T_w}_C. \end{aligned}$$

Here we used the fact that for $w \notin \{s\}W$ it follows that $\ell(sw) < \ell(w)$. Hence for $v = sw$, one has $\ell(v) = \ell(w) - 1$ and therefore $q_s \alpha(w) = -\alpha(v)$.

For $w \in \{s\}W$ and $y = sw \notin \{s\}W$ one has $\ell(y) = \ell(w) + 1$. Hence $\alpha(w) = -q_s \alpha(y)$ and A can be rewritten as $\sum_{v \notin \{s\}W} -q_s \alpha(v) T_v$. Then

$$A + C = \sum_{x \notin \{s\}W} \alpha(x) [-q_s + (q_s - 1)] T_x = - \sum_{x \notin \{s\}W} \alpha(x) T_x.$$

This yields (1).

It is easy to check that $z^{\natural} = z$. Thus, by (1), one has $z \in Z(\mathcal{H})$. Moreover,

$$z^2 = (p_{(W,S)}(\mathbf{q}))^{-1} \sum_{w \in W} \varepsilon_{-1}(T_w) \varepsilon_q(T_{w_0 w}) T_w \cdot z = (p_{(W,S)}(\mathbf{q}))^{-1} \sum_{w \in W} \varepsilon_q(T_{w_0 w}) z = z.$$

This shows (2), and (3) is a direct consequence of (2). \blacksquare

Finally, there are canonical maps between induced modules, that will turn out to be useful for the purposes of this thesis.

For $I \subseteq S$, let $R_q(I)$ be the trivial \mathcal{H}_I -module. It is equal (not just isomorphic) to the restriction $\mathbf{res}_I^S(R_q)$. Then, for $I \subseteq J \subseteq S$ the natural isomorphism, expressing the adjunction $\mathbf{res}_I^J \dashv \mathbf{ind}_I^J$,

$$\varphi_I^J: \text{Hom}_{\mathcal{H}_I}(R_q(I), \mathbf{res}_I^J(R_q(J))) \rightarrow \text{Hom}_{\mathcal{H}_J}(\mathbf{ind}_I^J(R_q(I)), R_q(J))$$

can be simply written as

$$(5.8) \quad \varphi_I^J: \text{Hom}_{\mathcal{H}_I}(R_q, R_q) \rightarrow \text{Hom}_{\mathcal{H}_J}(\mathbf{ind}_I^J(R_q), R_q)$$

and one has

$$\varphi_I^J(\alpha)(h \otimes r) = h.\alpha(r)$$

for $\alpha \in \text{Hom}_{\mathcal{H}_I}(R_q, R_q)$, $h \in \mathbf{ind}_I^J R_q$ and $r \in R_q$. These φ 's allow the definition of canonical maps as follows, which will turn out to be useful in the subsequent §5.

5.20. DEFINITION. For $I \subseteq J \subseteq S$, let d_I^J be the map of left \mathcal{H} -modules

$$d_I^J = \mathbf{ind}_J^S(\varphi_I^J(\text{id}_{R_q})): \mathbf{ind}_I^S R_q \rightarrow \mathbf{ind}_J^S R_q.$$

5.21. LEMMA. The maps d_I^J can be written in coordinates over R as

$$(5.9) \quad d_I^J(T_w \eta_I) = \varepsilon_q(T_{w_J}) T_{w^J \eta_J},$$

for $w \in W^I$.

PROOF. Up to identifying naturally isomorphic functors $\mathbf{ind}_J^S \mathbf{ind}_I^J \simeq \mathbf{ind}_I^S$, one has

$$\begin{aligned} d_I^J(T_w \eta_I) &= \mathbf{ind}_J^S(\varphi_I^J(\text{id}_{R_q}))(T_w \eta_I) = (\mathcal{H} \otimes_{\mathcal{H}_J} \varphi_I^J(\text{id}_{R_q}))(T_w \otimes_{\mathcal{H}_I} 1) \\ &= T_w \otimes_{\mathcal{H}_J} 1 = T_{w^J} T_{w_J} \otimes_{\mathcal{H}_J} 1 = \varepsilon_q(T_{w_J}) T_{w^J \eta_J}. \quad \blacksquare \end{aligned}$$

5. The Hecke–Coxeter chain complex of a Hecke algebra

As described in §2.1, a Hecke algebra $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$ has the properties described in Chapter 1 §5, in particular one may consider the 4-tuple $(\mathcal{H}, -^{\natural}, \varepsilon_q, \mathcal{B})$, i.e., the Hecke algebra as an associative, augmented, R -Hecke algebra with a distinguished basis and an antipode.

Thus, if one had a “nice” complex, quasi-isomorphic to $R_q[[0]]$, it would be possible to define the Euler characteristic $\chi_{\mathcal{H}} = \chi_{(\mathcal{H}, -^{\natural}, \varepsilon_q, \mathcal{B})}$ (cf. Chapter 1 §5.2). The purpose of the present section is exactly to provide such a resolution.

Some preliminary definitions and considerations are needed.

5.22. REMARK. The complex that will be defined is canonical up to a fixed total ordering “ $<$ ” on the finite set S , which determines a sign map appearing in the boundary maps.

5.23. DEFINITION (Degree). For $I \subseteq S$ define $\deg(I) = |S| - |I| - 1$.

5.24. DEFINITION (Sign map). Let the *sign map* be defined by

$$(5.10) \quad \text{sgn}: S \times \mathcal{P}(S) \longrightarrow \{\pm 1\}, \quad \text{sgn}(s, I) = (-1)^{|\{t \in S \setminus I \mid t < s\}|},$$

where $\mathcal{P}(S)$ denotes the set of subsets of S .

The following identities are elementary.

$$(5.11) \quad \text{sgn}(s, I \sqcup \{t\}) = \text{sgn}(s, I) \quad \text{for } t \geq s,$$

$$(5.12) \quad \text{sgn}(s, I \sqcup \{t\}) = -\text{sgn}(s, I) \quad \text{for } t < s,$$

$$(5.13) \quad \text{sgn}(s, I \setminus \{t\}) = \text{sgn}(s, I) \quad \text{for } t \geq s,$$

$$(5.14) \quad \text{sgn}(s, I \setminus \{t\}) = -\text{sgn}(s, I) \quad \text{for } t < s.$$

Moreover, the following holds.

5.25. LEMMA. If $I \subseteq S$ and $s, t \notin I$, $s \neq t$, then

$$(5.15) \quad \text{sgn}(t, I) \text{sgn}(s, I \sqcup \{t\}) + \text{sgn}(s, I) \text{sgn}(t, I \sqcup \{s\}) = 0.$$

PROOF. Note that either $s < t$ or $t < s$. By (5.11) and (5.12), the left-hand side of (5.15) reduces in the first case to

$$\operatorname{sgn}(t, I) \operatorname{sgn}(s, I) + \operatorname{sgn}(s, I) (-\operatorname{sgn}(t, I)) = 0;$$

while in the second case one has

$$\operatorname{sgn}(t, I) (-\operatorname{sgn}(s, I)) + \operatorname{sgn}(s, I) \operatorname{sgn}(t, I) = 0. \quad \blacksquare$$

Finally, the definition of the Hecke–Coxeter chain complex of a Hecke algebra can be given.

5.26. DEFINITION. Let (W, S) be a Coxeter system, and suppose the finite set $(S, <)$ has a fixed total ordering. Let R be a commutative ring and let $\mathbf{q}: S \rightarrow R$ be a parameter function. Let $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$ the Hecke algebra and let $R_{\mathbf{q}}$ be its trivial module.

For $k \in \mathbb{Z}$ let

$$(5.16) \quad \tilde{C}_k = \coprod_{\substack{I \subseteq S \\ \deg(I)=k}} \operatorname{ind}_I^S R_{\mathbf{q}}.$$

Moreover, let $\tilde{\partial}_k: \tilde{C}_k \rightarrow \tilde{C}_{k-1}$ be defined to be the map

$$(5.17) \quad \tilde{\partial}_k = \sum_{\substack{I, J \subseteq S \\ \deg(I)=k, \\ \deg(J)=k-1}} \partial_{I, J},$$

where

$$(5.18) \quad \partial_{I, J} = \begin{cases} \operatorname{sgn}(s, I) d_I^J & \text{if } J = I \sqcup \{s\} \\ 0 & \text{if } J \not\supseteq I, \end{cases}$$

and d_I^J is given as in (5.9) and the $\operatorname{sgn}(-, -)$ is given in Definition 5.24.

Few remarks about the definition are needed

- 5.27. REMARK. (1) The \mathcal{H} -modules \tilde{C}_k are zero for $k \notin \{-1, \dots, |S| - 1\}$, by Definition 5.23.
(2) The maps $\tilde{\partial}_k$ are maps of left \mathcal{H} -modules.
(3) $\tilde{C}_{-1} = \operatorname{ind}_\emptyset^S R_{\mathbf{q}} \simeq R_{\mathbf{q}}$ and $\tilde{C}_{|S|-1} = \operatorname{ind}_\emptyset^S R_{\mathbf{q}} \simeq {}^{\operatorname{reg}}\mathcal{H}$.
(4) The module \tilde{C}_k , considered as R -module, is free with basis

$$\{T_w \eta_I \mid I \subseteq S, \deg(I) = k, w \in W^I\}.$$

5.28. LEMMA. *The sequence of \mathcal{H} -modules $(\tilde{C}_\bullet, \tilde{\partial}_\bullet)$ is a chain complex.*

PROOF. By Remark 5.27(2) and (2), it suffices to prove that if $I \subseteq S$ and $\deg I = k$ one has $\tilde{\partial}_{k-1} \tilde{\partial}_k(\eta_I) = 0$. Indeed,

$$\begin{aligned} \tilde{\partial}_{k-1} \tilde{\partial}_k(\eta_I) &= \tilde{\partial}_{k-1} \left(\sum_{s \in S \setminus I} \operatorname{sgn}(s, I) \eta_{I \sqcup \{s\}} \right) \\ &= \sum_{s \in S \setminus I} \operatorname{sgn}(s, I) \sum_{t \in S \setminus (I \sqcup \{s\})} \operatorname{sgn}(t, I \sqcup \{s\}) \eta_{I \sqcup \{s, t\}} \\ &= \sum_{\substack{s, t \in S \setminus I \\ t \neq s}} \operatorname{sgn}(s, I) \operatorname{sgn}(t, I \sqcup \{s\}) \eta_{I \sqcup \{s, t\}} \\ &= \sum_{\substack{s, t \in S \setminus I \\ t < s}} [\operatorname{sgn}(s, I) \operatorname{sgn}(t, I \sqcup \{s\}) + \operatorname{sgn}(t, I) \operatorname{sgn}(s, I \sqcup \{t\})] \eta_{I \sqcup \{s, t\}}, \end{aligned}$$

which vanishes by Lemma 5.25. \blacksquare

5.29. DEFINITION (Hecke–Coxeter chain complex of a Hecke algebra). Under the conditions of Definition 5.26, one says that the *Hecke–Coxeter chain complex* of \mathcal{H} is

$$C = (C_\bullet, \partial_\bullet) = (\tilde{C}_\bullet, \tilde{\partial}_\bullet)^{\geq 0}.$$

I.e., the Hecke–Coxeter chain complex is

$$0 \longrightarrow C_{|S|-1} \xrightarrow{\partial_{|S|-1}} \cdots \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0.$$

The import of the Hecke–Coxeter chain complex for a Hecke algebra is determined by the following Theorem.

5.30. THEOREM ([**TW**]). *Let (W, S) be a Coxeter group with $|S| \geq 2$. Let C be the Hecke–Coxeter chain complex of the associated R -Hecke algebra $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$.*

- (1) *If (W, S) is spherical then $H_k(C) = 0$ unless $k = 0$ or $k = |S| - 1$. Moreover, $H_0(C) \simeq R_q$ and $H_{|S|-1}(C) \simeq R_{-1}$.*
- (2) *If (W, S) is non-spherical then C is acyclic with $H_0(C) \simeq R_q$.*

The cases of small rank are almost trivial:

5.31. REMARK. (1) If $|S| = 1$ then $H_k(C) = 0$ for all $k \neq 0$ and $H_0(C) \simeq R_q \oplus R_{-1}$.

Indeed, $W = \mathbf{C}_2$ and the Hecke–Coxeter chain complex is concentrated at zero with $C_0 \simeq {}^{\text{reg}}\mathcal{H}$; moreover $p_{(W,S)}(\mathbf{q}) = 1 + q$ and

$$e_{\{s\}} = \frac{T_1 + T_s}{1 + q} \quad \text{and} \quad z = \frac{\varepsilon_q(T_s)T_1 - T_s}{1 + q},$$

(cf. Propositions 5.18 and 5.19) hence $e_{\{s\}} + z = T_1$ and $e_{\{s\}}z = 0$, then ${}^{\text{reg}}\mathcal{H} \simeq R_q \oplus R_{-1}$, which proves the statement.

- (2) If $S = \emptyset$, there is an isomorphism $\mathcal{H} \simeq R$ of R -algebras and an isomorphism of \mathcal{H} -modules $R_q \simeq {}^{\text{reg}}R$. Then $(R[[0]], 0, \text{id}_R)$ is a finite, free resolution* of R_q .

6. Proof of Theorem 5.30

The cases with $|S| \geq 2$ are more interesting, and the following property will be useful to prove acyclicity of the Hecke–Coxeter chain complex C_\bullet .

5.32. PROPOSITION. *Let $h = \sum_{\deg(I)=k} \sum_{w \in W^I} \alpha(w, I) T_w \eta_I \in C_k$, $k \geq 0$, and*

$$\partial_k(h) = \sum_{\deg(J)=k-1} \sum_{v \in W^J} \beta(v, J) T_v \eta_J.$$

Then one has, for $J \subseteq S$, $\deg(J) = k - 1$ and $v \in W^J$,

$$(5.19) \quad \beta(v, J) = \sum_{t \in J} \sum_{x \in W_J^J \setminus \{t\}} \text{sgn}(t, J \setminus \{t\}) \alpha(vx, J \setminus \{t\}) \varepsilon_q(T_x).$$

In particular, if (\bar{w}, \bar{I}) is such that $\bar{w} \in W^{\bar{I}}$, $\alpha(\bar{w}, \bar{I}) \neq 0$ and $\alpha(w, I) = 0$ for all $w \in W$ with $\ell(w) > \ell(\bar{w})$ and $\deg(I) = k$, then

$$(5.20) \quad \beta(\bar{w}, J) = \sum_{t \in J} \text{sgn}(t, J \setminus \{t\}) \alpha(\bar{w}, J \setminus \{t\}).$$

*This complex is *not* a Hecke–Coxeter chain complex, the latter being the zero complex when $S = \emptyset$.

PROOF. For $I \subset J \subseteq S$ one has $W_J = W_J^I W_I$. As $W^J \subset W^I$, one concludes that $W^I = W^J W_J^I$ (cf. Proposition 3.18). Hence

$$\begin{aligned} \partial_k(h) &= \sum_{\deg(I)=k} \sum_{w \in W^I} \alpha(w, I) \sum_{t \in S \setminus I} \operatorname{sgn}(t, I) T_w \eta_{I \sqcup \{t\}} \\ &= \sum_{\deg(J)=k-1} \sum_{t \in J} \sum_{w \in W^{J \setminus \{t\}}} \operatorname{sgn}(t, J \setminus \{t\}) \alpha(w, J \setminus \{t\}) T_w \eta_J \end{aligned}$$

and thus by the previous remark

$$= \sum_{\deg(J)=k-1} \sum_{v \in W^J} \sum_{t \in J} \operatorname{sgn}(t, J \setminus \{t\}) \sum_{x \in W_J^{J \setminus \{t\}}} \alpha(vx, J \setminus \{t\}) T_{vx} \eta_J$$

Thus by Lemma 5.17 one concludes that

$$= \sum_{\deg(J)=k-1} \sum_{v \in W^J} \sum_{t \in J} \operatorname{sgn}(t, J \setminus \{t\}) \sum_{x \in W_J^{J \setminus \{t\}}} \alpha(vx, J \setminus \{t\}) \varepsilon_q(T_x) T_v \eta_J$$

This yields (5.19), and (5.20) is a direct consequence of (5.19). \blacksquare

In order to prove statements (1) and (2), one needs the following fact.

5.33. PROPOSITION. *If (W, S) is a Coxeter system with $|S| \geq 2$. If W is infinite, then $\partial_{|S|-1}$ is injective; while for W finite and $p_{(W,S)}(\mathbf{q}) \in R^\times$ one has $\ker(\partial_{|S|-1}) = \mathcal{H}z \simeq R_{-1}$ (cf. Prop. 5.19).*

PROOF. Put $\partial = \partial_{|S|-1}$. Let $\zeta = \sum_{w \in W} \beta(w) T_w \eta_\emptyset \in \ker \partial \subseteq C_{|S|-1}$. Proposition 3.18(6) yields

$$\begin{aligned} 0 = \partial(\zeta) &= \sum_{w \in W} \beta(w) \sum_{s \in S} \operatorname{sgn}(s, \emptyset) T_w \eta_{\{s\}} \\ &= \sum_{s \in S} \operatorname{sgn}(s, \emptyset) \left(\sum_{w \in W^{\{s\}}} \beta(w) T_w \eta_{\{s\}} + \sum_{v \in W^{\{s\}}} \beta(vs) T_v T_s \eta_{\{s\}} \right) \\ &= \sum_{s \in S} \operatorname{sgn}(s, \emptyset) \sum_{x \in W^{\{s\}}} (\beta(x) + \beta(xs) q_s) T_x \eta_{\{s\}}. \end{aligned}$$

Hence one must have

$$(5.21) \quad \beta(x) + q_s \beta(xs) = 0 \quad \text{for all } s \in S \text{ and } x \in W^{\{s\}}.$$

Suppose W is infinite and that there exists $x_0 \in W$ such that $\beta(x_0) \neq 0$. Then—because W is infinite—there exists a sequence of elements $(x_k)_{k \in \mathbb{N}}$, $x_k \in W$ such that $x_{k+1} = x_k s_k$, $s_k \in W$ and $\ell(x_{k+1}) = \ell(x_k) + 1$. In particular $x_k \in W^{\{s_k\}}$. By induction and (5.21), one concludes that $\beta(x_k) \neq 0$ for all $k \in \mathbb{N}$, a contradiction, and this shows that $\partial_{|S|-1}$ is injective in this case.

Let W be finite with longest element w_0 . Then by (5.21) and induction, one concludes that $\beta(x) = \varepsilon_{-1}(T_{w_0 x}) \varepsilon_q(T_{w_0 x}) \beta(w_0)$ for all $x \in W$. In particular, for $b = \varepsilon_{-1}(T_{w_0}) p_{(W,S)}(\mathbf{q}) \beta(w_0) \in R$ one verifies easily that $\zeta = bz \eta_\emptyset$. This yields the claim. \blacksquare

PROOF OF THEOREM 5.30. By hypothesis, $|S| \geq 2$.

Let “ \preceq ” be the lexicographic order on $\mathbb{N}_0 \times \mathbb{N}_0$, i.e., $(\mathbb{N}_0 \times \mathbb{N}_0, \preceq)$ is a well-ordered set. For $k \in \{-1, \dots, |S| - 2\}$ and $h \in C_k \setminus \{0\}$ put

$$h = \sum_{\deg(I)=k} \alpha_I \eta_I, \quad \alpha_I = \sum_{w \in W^I} \alpha(w, I) T_w \in \mathcal{H}^I,$$

where $\alpha(w, I) \in R$ (cf. Lemma 5.17). Then the following are well-defined:

$$\begin{aligned} \text{supp}(h) &= \{ (w, I) \mid I \subseteq S, \deg(I) = k, w \in W^I, \alpha(w, I) \neq 0 \}, \\ \lambda(h) &= \max\{ \ell(w) \mid (w, I) \in \text{supp}(h) \} \in \mathbb{N}_0, \\ \nu(h) &= |\{ (w, I) \in \text{supp}(h) \mid \ell(w) = \lambda(h) \}| \in \mathbb{N}_0. \end{aligned}$$

Obviously, for $h, h' \in C_k$, $h \neq h'$, and $r \in R$ with $rh \neq 0$, one has

$$(5.22) \quad (\lambda, \nu)(h - h') \preceq \max\{(\lambda, \nu)(h), (\lambda, \nu)(h')\} \quad \text{and} \quad (\lambda, \nu)(rh) \preceq (\lambda, \nu)(h).$$

For short we put $\Omega_k = \ker \partial_k \setminus \text{im} \partial_{k+1}$, and define

$$\Delta_k = (\lambda, \nu)|_{\Omega_k} : \Omega_k \rightarrow \mathbb{N}_0 \times \mathbb{N}_0.$$

Obviously, $\tilde{\partial}_0 : \tilde{C}_0 \rightarrow \tilde{C}_{-1}$ is surjective and, by Lemma 1.4, one has $H_0 C_\bullet = \tilde{C}_{-1} \simeq R_q$.

On the other hand, Proposition 5.33 determines the top-dimensional homology, i.e.,

$$H_{|S|-1}(C_\bullet) = \begin{cases} 0 & \text{if } |W| = \infty \\ R_{-1} & \text{if } |W| < \infty \text{ and } p_{(W, S)}(\mathbf{q}) \in R^\times \end{cases}.$$

Thus, in order to prove (1) and (2), one only has to check that $H_k(C) = 0$ for all $k \in \{1, \dots, |S| - 2\}$.

Assumption: Suppose that $k \in \{1, \dots, |S| - 2\}$ and that the set $\Omega = \Omega_k$ is non-empty, and put $\Delta = \Delta_k$.

As $(\mathbb{N}_0 \times \mathbb{N}_0, \preceq)$ is well-ordered, there exists a unique minimal element $\min \Delta \in \text{im}(\Delta) \subseteq \mathbb{N}_0 \times \mathbb{N}_0$. Let $h \in \Omega$ be such that $\Delta(h) = \min \Delta$. As Ω does not contain zero, $h \neq 0$. Hence there exists a pair $(\bar{w}, \bar{I}) \in \text{supp}(h)$ such that $\ell(\bar{w}) = \lambda(h)$. Let $A = A^p(\bar{w})$ (cf. (3.6)). By Proposition 3.18(8), one has to distinguish two cases.

Case 1: $\bar{I} = A$. By the hypothesis on k , $\bar{I} \neq \emptyset$. Choose any element $\bar{s} \in \bar{I}$, and let $\bar{J} = \bar{I} \setminus \{\bar{s}\}$. Then one has $\deg(\bar{J}) = k + 1$, and by Proposition 3.18(7), $\bar{w} \in W^{\bar{J}}$. Hence $T_{\bar{w}}\eta_{\bar{J}}$ is an element of the standard basis of C_{k+1} , and

$$\partial_{k+1}(T_{\bar{w}}\eta_{\bar{J}}) = \sum_{\substack{\deg(I)=k \\ I=\bar{J} \sqcup \{\bar{s}\}}} \text{sgn}(s, \bar{J}) \varepsilon_q(T_{\bar{w}_I}) T_{\bar{w}^I} \eta_I.$$

Since $\bar{I} = \bar{J} \sqcup \{\bar{s}\}$ and $\bar{w}^{\bar{I}} = \bar{w}$, one has

$$\text{sgn}(\bar{s}, \bar{J}) \partial_{k+1}(T_{\bar{w}}\eta_{\bar{J}}) = T_{\bar{w}}\eta_{\bar{I}} + \overbrace{\sum_{s \in S \setminus \bar{I}} \text{sgn}(\bar{s}, \bar{J}) \text{sgn}(s, \bar{J}) \varepsilon_q(T_{\bar{w}_{\bar{J} \sqcup \{s\}}}) T_{\bar{w}^{\bar{J} \sqcup \{s\}}} \eta_{\bar{J} \sqcup \{s\}}}^X.$$

For $s \in S \setminus \bar{I}$, one has $\bar{J} \sqcup \{s\} \not\subseteq \bar{I} = A$. Thus the elements $\bar{w}^{\bar{J} \sqcup \{s\}}$ are of shorter length than \bar{w} (cf. Proposition 3.18(9)). Hence, if $X \neq 0$, then $\lambda(X) \leq \ell(\bar{w}) = \lambda(h)$. Put

$$(5.23) \quad h' = h - \alpha(\bar{w}, \bar{I}) \text{sgn}(\bar{s}, \bar{J}) \partial_{k+1}(T_{\bar{w}}\eta_{\bar{J}}) \in \ker(\partial_k).$$

As $h \notin \text{im}(\partial_{k+1})$, one has also $h' \notin \text{im}(\partial_{k+1})$. Hence $h' \in \Omega$. Moreover, by (5.22), $\Delta(h') \preceq \Delta(h)$. Thus the minimality of $\Delta(h)$ implies that $\Delta(h') = \Delta(h)$. In particular, $\lambda(h') = \lambda(h)$. However, by construction,

$$\{ (w, I) \in \text{supp}(h') \mid \ell(w) = \lambda(h') \} = \{ (w, I) \in \text{supp}(h) \mid \ell(w) = \lambda(h) \} \setminus \{ (\bar{w}, \bar{I}) \},$$

and thus $\nu(h') < \nu(h)$, a contradiction, showing that Case 1 is impossible.

Case 2: $\bar{I} \subsetneq A$. For the chosen (\bar{w}, \bar{I}) define the disjoint sets

$$\begin{aligned} \mathcal{A} &= \{(\bar{w}, I) \mid I \subseteq A, \deg(I) = k\}, \\ \mathcal{B} &= \{(w, I) \mid \ell(w) = \lambda(h), w \neq \bar{w}, I \subseteq A^\rho(w), \deg(I) = k\}, \\ \mathcal{C} &= \{(w, I) \mid \ell(w) \lesssim \lambda(h), I \subseteq A^\rho(w), \deg(I) = k\}. \end{aligned}$$

Then $\text{supp}(h) \subseteq \mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{C}$. Let $h = h_{\mathcal{A}} + h_{\mathcal{B}} + h_{\mathcal{C}}$ be the corresponding additive decomposition of h (cf. Lemma 5.17). Then $h_{\mathcal{A}} \neq 0$, $\lambda(h_{\mathcal{A}}) = \lambda(h)$, $\lambda(h_{\mathcal{B}}) \leq \lambda(h)$, and $\Delta(h_{\mathcal{C}}) \prec \Delta(h)$.

If $\bar{I} \subsetneq J \subseteq A$ with $\deg(J) = k - 1$, the element $T_{\bar{w}}\eta_J$ is an element of the standard R -basis of C_{k-1} . By hypothesis, the coefficient of $\partial_k(h)$ on $T_{\bar{w}}\eta_J$ equals 0. Thus by the maximality of $\ell(\bar{w})$ and Proposition 5.32, one has

$$(5.24) \quad \sum_{t \in J} \text{sgn}(t, J \setminus \{t\}) \alpha(\bar{w}, J \setminus \{t\}) = 0.$$

Let

$$(5.25) \quad \varphi = \sum_{\substack{I \subseteq A \\ \deg(I) = k}} \alpha(\bar{w}, I) \eta_I,$$

i.e., $T_{\bar{w}}\varphi = h_{\mathcal{A}}$. Define D_k , $k \geq -1$, to be the R -submodule

$$D_k = \text{span}_R(\{\eta_I \mid I \subseteq A, \deg(I) = k\}) \subseteq C_k,$$

and let $d_k: D_k \rightarrow D_{k-1}$, $k \geq 0$, be the R -linear map given by

$$d_k(\eta_I) = \sum_{t \in A \setminus I} \text{sgn}(t, I) \eta_{I \sqcup \{t\}};$$

one easily sees that $d_k d_{k+1} = 0$ for all k (cf. Lemma 5.25). Hence (D_\bullet, d_\bullet) is a chain complex. Moreover, for $I \subseteq A$, $\deg(I) = k$, one has

$$(5.26) \quad (\partial_k - d_k)(\eta_I) = \sum_{t \in S \setminus A} \text{sgn}(t, I) \eta_{I \sqcup \{t\}}.$$

The chain complex of R -modules (D_\bullet, d_\bullet) concentrated in degrees $k \geq -1$ is contractible (as $(D_k, d_k)_{k \geq 0}$ coincides with the singular chain complex of an $(|A| - 1)$ -dimensional simplex with coefficients in R). Thus there exist homomorphisms of R -modules $\sigma_k: D_k \rightarrow D_{k+1}$, $k \geq -1$, satisfying $d_{k+1}\sigma_k + \sigma_{k-1}d_k = \text{id}_{D_k}$. Hence for $\psi \in \ker d_k$, one has $d_{k+1}(\sigma_k(\psi)) = \psi$. Moreover, by (5.24),

$$\begin{aligned} d_k(\varphi) &= \sum_{t \in A \setminus I} \text{sgn}(t, I) \alpha(\bar{w}, I) \eta_{I \sqcup \{t\}} \\ (5.27) \quad &= \sum_{\substack{J \subseteq A \\ \deg(J) = k-1}} \sum_{t \in J} \text{sgn}(t, J \setminus \{t\}) \alpha(\bar{w}, J \setminus \{t\}) \eta_J = 0 \end{aligned}$$

CLAIM. For all $(w, I) \in \text{supp}(h_{\mathcal{A}} - T_{\bar{w}}\partial_{k+1}(\sigma_k(\varphi)))$ one has $\ell(w) < \ell(\bar{w})$.

PROOF OF CLAIM. Note that $h_{\mathcal{A}} = T_{\bar{w}}\varphi$. Since $d_k(\varphi) = 0$ (cf. (5.27)), one has $d_{k+1}(\sigma_k(\varphi)) = \varphi$. Thus, by the previous remark,

$$h_{\mathcal{A}} - T_{\bar{w}}(\partial_{k+1}(\sigma_k(\varphi))) = T_{\bar{w}}(\varphi - \partial_{k+1}(\sigma_k(\varphi))) = T_{\bar{w}}(d_{k+1} - \partial_{k+1})(\sigma_k(\varphi)).$$

By (5.26), $(d_{k+1} - \partial_{k+1})(\sigma_k(\varphi))$ is an R -linear combination of elements η_I with $I \not\subseteq A$, $\deg(I) = k$. As $I \not\subseteq A$, one has $\bar{w}_I \neq 1$ (cf. Prop. 3.18(9)), and therefore, $\ell(\bar{w}^I) < \ell(\bar{w})$. This yields the claim. \blacksquare

Note that $h_0 = h - T_{\bar{w}}\partial_{k+1}(\sigma_k(\varphi)) \in \Omega$. As

$$(5.28) \quad \mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{C} = \{(w, I) \mid \ell(w) \leq \ell(\bar{w}), \deg(I) = k, w \in W^I\},$$

one concludes from the Claim that

$$(5.29) \quad h_1 = h_{\mathcal{A}} - T_{\bar{w}}(\partial_{k+1}(\sigma_k(\varphi))) \in \text{span}_R\{T_w\eta_I \mid (w, I) \in \mathcal{C}\}.$$

In particular, $\lambda(h_0) \leq \lambda(h)$. Thus by the minimality of $\min \Delta$ one must have $\lambda(h_0) = \lambda(h)$. But in this case one has by construction that $\nu(h_0) < \nu(h)$, a contradiction, showing that Case 2 is impossible. From this one concludes that $\Omega = \emptyset$ and hence $\ker \partial_k = \text{im } \partial_{k+1}$ for all $k \in \{1, \dots, |S| - 2\}$. Thus, $H_k(C_\bullet) = 0$ for $k \in \{1, \dots, |S| - 2\}$, and the proof is complete. \blacksquare

7. Projectivity and FP-property

When W is finite, i.e., in the cases (1), of Theorem 5.30 and (1) and (2) of Remark 5.31, if $p_{(W,S)}(\mathbf{q}) \in R^\times$, then the trivial \mathcal{H} -module R_q is finitely generated and projective, by Proposition 5.18(3).

If W is infinite of cocompact type (cf. Definition 3.24), all the proper parabolic subgroups are finite, hence C_k , $k \geq 0$ is a finite sum of modules \mathbf{ind}_I^S for $I \subsetneq S$. Thus, Propositions 5.18(3) and 5.33 imply that the Hecke–Coxeter chain complex is a finite, projective resolution of R_q as soon as the Poincaré series $p_{(W,I)}(\mathbf{q}) \in R^\times$ for all proper parabolic subsystem (W_I, I) .

5.34. PROPOSITION. *Let (W, S) be a finitely generated Coxeter group, which is either affine, or compact hyperbolic and let $\mathbf{q}: S \rightarrow R$ be such that $p_{(W_I, I)}(\mathbf{q}) \in R^\times$ for any proper parabolic subgroup (W_I, I) . Then the Hecke–Coxeter chain complex $(C_\bullet, \partial_\bullet, \varepsilon)$ is a finite projective resolution of R_q .*

In the general case, i.e., for W infinite, but not of cocompact type, one has the following **FP**-property.

5.35. PROPOSITION ([**TW**, Prop. 5.4]). *Let (W, S) be a finitely generated Coxeter group, and let $\mathbf{q}: S \rightarrow R$ be such that $p_{(W_I, I)}(\mathbf{q}) \in R^\times$ for any finite parabolic subgroup (W_I, I) . Then $(C_\bullet, \partial_\bullet)$ is a chain complex of left \mathcal{H} -modules of type **FP**; in particular, R_q is a left \mathcal{H} -module of type **FP**.*

PROOF. By hypothesis and the previously mentioned remark, $\mathbf{ind}_{\mathcal{H}_I}^{\mathcal{H}}(R_q)$ is a finitely generated projective \mathcal{H} -module for any finite parabolic subgroup (W_I, I) . We proceed by induction on $d = |S|$. For $|S| \leq 2$, there is nothing to prove. Assume that the claim holds for all Coxeter groups (W_J, J) with $|J| < d$, and that $|S| = d$. By induction, for $K \subsetneq S$, R_q is a left \mathcal{H}_K -module of type **FP**. Hence $\mathbf{ind}_{\mathcal{H}_K}^{\mathcal{H}}(R_q)$ is a left \mathcal{H} -module of type **FP**. Thus C_k is a left \mathcal{H} -module of type **FP** for $0 \leq k \leq d - 1$. If (W, S) is spherical, then R_q is a finitely generated, projective, left \mathcal{H} -module by the first remark. If (W, S) is non-spherical, $(C_\bullet, \partial_\bullet)$ is acyclic. Hence R_q is a left \mathcal{H} -module of type **FP** by Proposition 1.19. This completes the proof. \blacksquare

The above proposition, together with the discussion in §2.1, imply the generic Euler-ness (cf. Chapter 1 §5) of Hecke algebras.

5.36. THEOREM. *Let $\mathcal{H} = \mathcal{H}(W, S, R, \mathbf{q})$ be a Hecke algebra. If the Poincaré series of all the finite parabolic subgroups are invertible in R , then \mathcal{H} is an Euler algebra.*

8. Euler characteristics

Proposition 5.35 has the following consequence.

5.37. PROPOSITION. *Let (W, S) be a finitely generated, non-spherical Coxeter group, and let \mathbf{q} be a parameter function such that $p_{(W_I, I)}(\mathbf{q}) \in R^\times$ for any finite parabolic subgroup (W_I, I) . Then*

$$(5.30) \quad r_{R_q} = \sum_{I \subsetneq S} (-1)^{|S \setminus I| - 1} r_{\mathbf{ind}_I^S(R_q)}.$$

PROOF. By (1.12) and (1.19), one has

$$(5.31) \quad r_{R_q} = \sum_{0 \leq k < |S|} (-1)^k r_{C_k} = \sum_{I \subsetneq S} (-1)^{|S \setminus I| - 1} r_{\mathbf{ind}_I^S(R_q)}.$$

This yields the claim. ■

5.38. THEOREM ([TW, Thm. C]). *Let (W, S) be any Coxeter system, and let $R = R_0[[q]]$ be the ring of formal power series in one indeterminate q , over the commutative ring R_0 . Let $\mathcal{H} = \mathcal{H}(W, S, R, q)$ be the associated R -Hecke algebra.*

Then $p_{(W, S)}(q)$ is an invertible element of R and

$$p_{(W, S)}(q) \chi_{\mathcal{H}} = 1.$$

PROOF. The invertibility descends from this general fact: an element

$$\sum_{k \geq 0} a_k q^k \in R_0[[q]]$$

is invertible in the ring of power series if, and only if, a_0 is invertible in R_0 .

In the case of Poincaré series, $a_0 = 1$ by Proposition 3.10(1).

If (W, S) is spherical, $R_q \simeq \mathcal{H}e_S$ where e_S is given as in Proposition 5.18. By Lemma 1.18 one has $r_{R_q} = e_S + [\mathcal{H}, \mathcal{H}]$, and hence $\chi_{\mathcal{H}} = \mu(r_{R_q}) = p_{(W, S)}(q)^{-1}$.

If (W, S) is non-spherical, we proceed by induction on $|S|$. Proposition 1.28, Corollary 1.29 and Proposition 5.37 imply that

$$\begin{aligned} \chi_{\mathcal{H}} &= \mu_{\mathcal{H}}(r_{R_q}) = \sum_{I \subsetneq S} (-1)^{|S \setminus I| - 1} \mu_{\mathcal{H}}(r_{\mathbf{ind}_I^S(R_q)}) \\ &= \sum_{I \subsetneq S} (-1)^{|S \setminus I| - 1} \mu_{\mathcal{H}_I}(R_q) \end{aligned}$$

and thus by induction

$$(5.32) \quad = \sum_{I \subsetneq S} (-1)^{|S \setminus I| - 1} p_{(W_I, I)}(q)^{-1},$$

which is equal to $p_{(W, S)}(q)^{-1}$ by Proposition 4.6. ■

5.39. REMARK. The motivation for defining the notion of Euler algebra came essentially from [Sta65], [Hat65] and the discussion of the Hattori–Stallings rank in [Bro82, Ch. IX]. Both group rings $\mathbb{C}G$ and Hecke algebras \mathcal{H} fit it the framework of Euler algebras, but they have somehow “opposite” behaviour.

Consider now finite groups G and Hecke algebras \mathcal{H} of spherical type.

- By Swan’s Theorem (cf. [Swa60]) the Hattori–Stallings rank element of $\mathbb{Z}G$ is only supported over the conjugacy class $[1]$, while for Hecke algebras this is not the case.

- Hattori–Stallings element r can be computed through idempotents, (cf. Lemma 1.18), and a result of A. Zalesskii (cf. [Zal72] and [BV98]) implies that the trace of r can only take rational values (or, in general, in the prime field of the base field).

On the other hand, Theorem 5.38 says that the trace of the Hattori–Stallings rank of a Hecke algebra can assume “almost” any value.

The differences pointed out might be interpreted as phenomena of “rigidity” (resp. “genericity”) of group algebras (resp. Hecke algebras).

APPENDIX A

Graphs

The purpose of this appendix is to establish several notions about graphs (cf. [Bou07c, Annexe au Ch. IV]).

A.1. DEFINITION ((Labelled) combinatorial graph). A graph Γ is a triple $\Gamma = (\mathfrak{V}, \mathfrak{E}, m)$ consisting of

- (1) a (possibly empty) set of vertices \mathfrak{V} ,
- (2) a set \mathfrak{E} consisting of subsets of \mathfrak{V} of cardinality 2; an element $\mathfrak{e} = \{s, r\} \in \mathfrak{E}$ is called an edge (joining s and r),
- (3) a labelling function $m: \mathfrak{E} \rightarrow \Lambda$, where Λ is some set of labels.

A.2. DEFINITION (Ω -restricted combinatorial graphs). Let $\Gamma = (\mathfrak{V}, \mathfrak{E}, m: \mathfrak{E} \rightarrow \Lambda)$ be a labelled combinatorial graph, and let $\Omega \subseteq \Lambda$.

Let

$$\mathfrak{F} = \{\mathfrak{e} \in \mathfrak{E} \mid m(\mathfrak{e}) \in \Omega\},$$

and then define the Ω -restricted graph of Γ as the graph

$$\Gamma_\Omega = (\mathfrak{V}, \mathfrak{F}, m|_{\mathfrak{F}}: \mathfrak{F} \rightarrow \Omega).$$

In particular, if $\Lambda \subseteq \mathbb{Z}$ one denotes $\Gamma_{\text{odd}} = \Gamma_{\Lambda \setminus 2\mathbb{Z}}$.

There exist situations where the labelling is irrelevant: then just think $m: \mathfrak{E} \rightarrow \{\emptyset\}$ or completely forget about m .

A combinatorial graph is *locally finite* if, for all $s \in \mathfrak{V}$, the set $\mathfrak{E}_s = \{\{s, r\} \in \mathfrak{E} \mid r \in \mathfrak{V}\}$ is finite; it is *finite* if \mathfrak{V} is a finite set.

Forgetting about the labelling, a locally finite combinatorial graph is a simplicial set, then it admits a geometric realization (cf. [Mil57]) and hence, a topology.

A *path* in Γ is a finite sequence of edges $(e_i)_{i=1}^n$ such that $e_i \cap e_{i+1} \neq \emptyset$ for all $i \in \{1, \dots, n-1\}$. If $s \neq r \in \mathfrak{V}$ are vertices, one says that r is *joined* to s if there exists, for a suitable n , a path $(e_i)_{i=1}^n$ in Γ such that $s \in e_1$ and $r \in e_n$. A vertex is connected to itself by the empty path.

One can prove that two vertices lie in the same connected component if, and only if, they are joined. By an abuse of language one says that \mathfrak{V}' is a *connected component* of Γ if, and only if, \mathfrak{V}' consists of the vertices lying in a connected component of the geometric realization of Γ .

A.3. DEFINITION (Sum of graphs). Let $\Gamma_1 = (\mathfrak{V}_1, \mathfrak{E}_1, m_1)$ and $\Gamma_2 = (\mathfrak{V}_2, \mathfrak{E}_2, m_2)$ be combinatorial graph. Then define the sum

$$\Gamma_1 \sqcup \Gamma_2 = (\mathfrak{V}_1 \sqcup \mathfrak{V}_2, \mathfrak{E}_1 \sqcup \mathfrak{E}_2, m),$$

where $m: \mathfrak{E}_1 \sqcup \mathfrak{E}_2 \rightarrow \Lambda_1 \cup \Lambda_2$, is given by $m(\mathfrak{e}) = \begin{cases} m_1(\mathfrak{e}) & \text{if } \mathfrak{e} \in \mathfrak{E}_1 \\ m_2(\mathfrak{e}) & \text{if } \mathfrak{e} \in \mathfrak{E}_2 \end{cases}$.

For historical reasons, if Γ is of type \mathcal{T}_1 and Γ_2 is of type \mathcal{T}_2 , one writes that the type of $\Gamma_1 \sqcup \Gamma_2$ is $\mathcal{T}_1 \times \mathcal{T}_2$.

In particular, a graph Γ is the sum of its connected components.

The only notion of subgraph of interest in this thesis is the following*.

A.4. DEFINITION (Subgraph). If $\Gamma = (\mathfrak{V}, \mathfrak{E}, m)$ and $\Gamma' = (\mathfrak{V}', \mathfrak{E}', m')$ are combinatorial graphs and if

- (1) there is an inclusion $\mathfrak{V}' \subseteq \mathfrak{V}$,
- (2) one has $\mathfrak{E}' = \{ \{s, r\} \in \mathfrak{E} \mid s, r \in \mathfrak{V}' \}$,
- (3) $m' = m|_{\mathfrak{E}'}$.

In this case write $\Gamma' \leq \Gamma$.

For a given graph Γ , a subgraph Γ is completely determined by the subset of vertices it contains.

A.5. DEFINITION (Graph category). Let $\Gamma = (\mathfrak{V}, \mathfrak{E}, m)$ be a combinatorial graph, and let $\mathcal{P}(\Gamma)$ be the poset of subgraphs of Γ , ordered with \leq . Then $\mathcal{P}(\Gamma)$ can be viewed as a small category with

$$\text{Obj}(\mathcal{P}(\Gamma)) = \{ \Gamma' \mid \Gamma' \leq \Gamma \}$$

and

$$\text{Hom}_{\mathcal{P}(\Gamma)}(\Gamma', \Gamma'') = \begin{cases} \{ \leq \} & \text{if } \Gamma' \leq \Gamma'' \\ \emptyset & \text{otherwise} \end{cases} .$$

Note that Γ is a terminal object in $\mathcal{P}(\Gamma)$ —actually, the unique one.

Let 2^X denote the small category with objects the subsets of X and with arrows their inclusions. For a fixed graph Γ let

$$(A.1) \quad \pi_\Gamma: 2^{\mathfrak{V}} \rightarrow \mathcal{P}(\Gamma)$$

be the functor associating to a subset \mathfrak{V}' of \mathfrak{V} the subgraph Γ' of Γ having \mathfrak{V}' as vertex set, and to an inclusion $\mathfrak{V}' \subseteq \mathfrak{V}''$ the arrow $\pi_\Gamma(\mathfrak{V}') \leq \pi_\Gamma(\mathfrak{V}'')$.

1. Cayley graphs

A.6. DEFINITION (Cayley graph). Let G be a finitely generated group and let $X \subseteq G$ be a finite generating set such that $1 \notin X$ and $X = X^{-1}$. Then define the Cayley graph $\text{Cay}(G, X)$ as the (non-labelled) combinatorial graph with vertex set G and edges $\{g, gx\}$ for $g \in G$ and $x \in X$.

One immediately verifies the following.

- (1) The edges are non-oriented, since $X = X^{-1}$ and $\{g, gx\} = \{gx, (gx)x^{-1}\}$.
- (2) Since $1 \notin X$, one has $g \neq gx$ for all $g \in G$ and $x \in X$, hence there are no loops.
- (3) $\text{Cay}(G, X)$ is connected.
- (4) There is a discrete metric $\delta: G \times G \rightarrow \mathbb{N}_0$ on G defined by
$$\delta(g_1, g_2) = \min\{n \mid (e_i)_{i=1}^n \text{ is a path joining } g_1 \text{ and } g_2\}.$$
- (5) The length function $\ell: G \rightarrow \mathbb{N}_0$ is defined as $\ell(g) = \delta(1, g)$.
- (6) For all $n \in \mathbb{N}_0$ the set $\mathcal{S}(n) = \ell^{-1}(n)$ is finite.
- (7) The group G is finite if, and only if, the length function is bounded above.

*It is worth remarking that, in general, the Ω -restriction of a graph Γ is not a subgraph of Γ

APPENDIX B

Data about Coxeter groups

1. Classification results

The following statement is a collection of several classification results.

B.1. THEOREM. *Let (W, S) be an irreducible Coxeter system, with Coxeter graph Γ . Then*

$$(W, S) \text{ is } \begin{cases} \text{spherical} \\ \text{affine} \\ \text{hyperbolic cocompact} \\ \text{hyperbolic} \end{cases} \iff \Gamma \text{ appears in } \begin{cases} \text{Table B.1.} \\ \text{Table B.2.} \\ \text{Table B.3.} \\ \text{Table B.4.} \end{cases}$$

2. Lists of Coxeter graphs

This section contains lists of Coxeter graphs referred to along the thesis.

As a general rule for drawing Coxeter graphs, labels equal to 3 are omitted.

For finite and affine Coxeter systems there are established naming conventions (types), listed in the first column of the tables.

B.2. NOTATION. In some cases it is convenient to draw Coxeter graphs using parameters to label some edge \mathfrak{e} . Then the following conventions hold:

- (1) $m(\mathfrak{e}) \in \mathbb{Z}_{\geq 2}$,
- (2) edges labelled (in the picture) by $m(\mathfrak{e}) = 2$ are not edges in the graph, e.g.

$\bullet \overset{k}{\text{---}} \bullet$ has to be interpreted as $\bullet \quad \bullet$ for $k = 2$,
i.e. the type $I_2(2)$, is $A_1 \times A_1$.

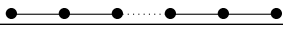
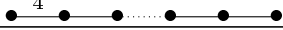
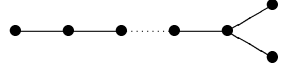
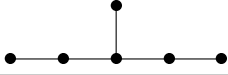
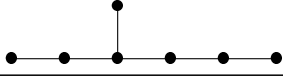
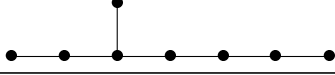
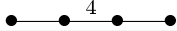




Name	(conditions)	Coxeter Graph	Degrees
A_n	$n \geq 0$		$2, 3, \dots, n+1$
$B_n = C_n$	$n \geq 2$		$2, 4, \dots, 2n$
D_n	$n \geq 4$		$2, 4, \dots, 2(n-1), n$
E_6			$2, 5, 6, 8, 9, 12$
E_7			$2, 6, 8, 10, 12, 14, 18$
E_8			$2, 8, 12, 14, 18, 20, 24, 30$
F_4			$2, 6, 8, 12$
G_2			$2, 6$
H_3			$2, 6, 10$
H_4			$2, 12, 20, 30$
$I_2(m)$	$m \in \mathbb{Z}_{>5} \setminus \{6\}$		$2, m$

Table B.1: The finite, irreducible Coxeter systems (W, S) , with $|S| = n$. Moreover, the types A_n to G_2 are crystallographic, i.e., they determine Weyl groups. The degrees (cf. Proposition 4.4) are printed in the last column.

Name	(conditions)	Coxeter Graph
\tilde{A}_1		
\tilde{A}_n	$n \geq 2$	
$\tilde{B}_2 = \tilde{C}_2$		
\tilde{B}_n	$n \geq 3$	
\tilde{C}_n	$n \geq 3$	
\tilde{D}_n	$n \geq 4$	
\tilde{E}_6		
\tilde{E}_7		
\tilde{E}_8		
\tilde{F}_4		
\tilde{G}_2		

Table B.2: The affine, irreducible Coxeter systems (W, S) , with $|S| = n + 1$.



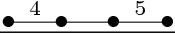
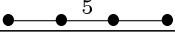
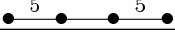
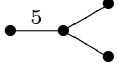








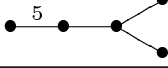
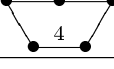









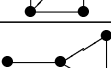
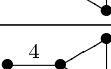
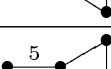
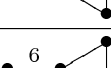
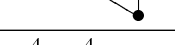
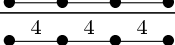
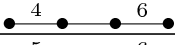
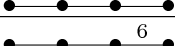
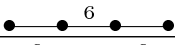
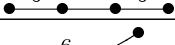

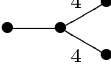
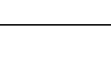
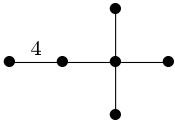
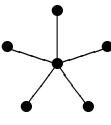
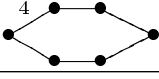
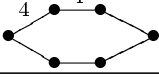
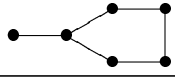
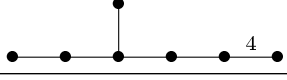
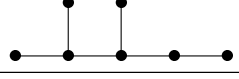
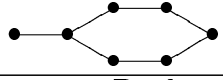
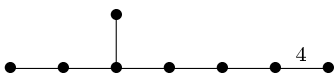
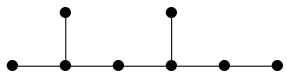
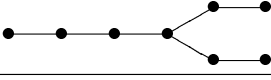
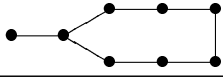
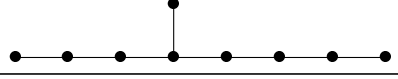
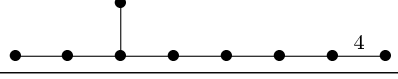
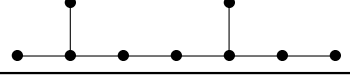
Name	(conditions)	Coxeter Graph	$\rho(W,S)$
Rank 3			
HC1	$a \leq b,$ $\frac{1}{a} + \frac{1}{b} < \frac{1}{2}$		cf. Lemma 4.19
HC2	$3 \leq a \leq b \leq c,$ $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1,$ $a < \infty$		cf. Lemma 4.19
Rank 4			
HC3			0.735294...
HC4			0.740203...
HC5			0.668132...
HC6			0.690406...
HC7			0.642661...
HC8			0.561279...
HC9			0.53101...
HC10			0.583648...
HC11			0.509281...
Rank 5			
HC12			0.720106...
HC13			0.833415...
HC14			0.659358...
HC15			0.689797...
HC16			0.61621...

Table B.3: The irreducible hyperbolic cocompact Coxeter systems (W, S) .

Name	Coxeter Graph	$\rho(W,S)$
Rank 3		
$\Gamma_\infty(3)$		$\frac{1}{2}$, cf. Prop. 4.7
Rank 4		
HNC1		0.551753...
HNC2		0.504138...
HNC3		0.469396...
HNC4		0.561856...
HNC5		0.51879...
HNC6		0.500245...
HNC7		0.492432...
HNC8		0.51879...
HNC9		0.682328...
HNC10		0.588985...
HNC11		0.552531...
HNC12		0.537613...
HNC13		0.639243...
HNC14		0.636883...
HNC15		0.552965...
HNC16		0.708134...
HNC17		0.618034...
HNC18		0.682328...
HNC19		0.639025...
HNC20		0.771327...
HNC21		0.667961...

→

←		
Name	Coxeter Graph	$\rho(w,s)$
HNC22		0.618034...
HNC23		0.434259...
Rank 5		
HNC24		0.698956...
HNC25		0.72899...
HNC26		0.654741...
HNC27		0.627864...
HNC28		0.662566...
HNC29		0.55887...
HNC30		0.579431...
HNC31		0.491695...
HNC32		0.537456...
Rank 6		
HNC33		0.801198...
HNC34		0.741226...
HNC35		0.697036...
HNC36		0.667522...
HNC37		0.744209...
HNC38		0.634641...
HNC39		0.702245...
	→	

←		
Name	Coxeter Graph	$\rho(w,S)$
HNC40		0.59287...
HNC41		0.542596...
HNC42		0.604368...
HNC43		0.533802...
HNC44		0.657119...
Rank 7		
HNC45		0.755431...
HNC46		0.726032...
HNC47		0.656509...
Rank 8		
HNC48		0.763804...
HNC49		0.741982...
HNC50		0.77866...
HNC51		0.657583...
Rank 9		
HNC52		0.826149...
HNC53		0.770075...
HNC54		0.753304...
		→

←		
Name	Coxeter Graph	$\rho_{(W,S)}$
HNC55		0.659124...
Rank 10		
HNC56		0.878674...
HNC57		0.774744...
HNC58		0.761587...

Table B.4: The irreducible hyperbolic non-cocompact Coxeter systems (W, S) .

APPENDIX C

Complex power series

Denote by $\mathbb{C}[[z]]$ the ring of complex power series in one indeterminate z .

Let $a_n \in \mathbb{C}$, for $n \in \mathbb{N}_0$, be the coefficients of the power series

$$(C.1) \quad p(z) = \sum_{n \in \mathbb{N}_0} a_n z^n \in \mathbb{C}[[z]].$$

The convergence radius of $p(z)$ is

$$\rho = \left(\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} \right)^{-1}.$$

Thus, the power series converges to a complex analytic function which is holomorphic in an open ball of radius ρ .

To a series $p(z)$ as in (C.1) one may associate another series (the *cumulative series*) as follows: let $\bar{a}_n = \sum_{k=0}^n a_k$ and denote the associated power series $\bar{p}(z) = \sum \bar{a}_n z^n \in \mathbb{C}[[z]]$ and its convergence radius by $\bar{\rho}$.

C.1. LEMMA. *Let $p(z) = \sum_{n \in \mathbb{N}_0} a_n z^n$ and let $\bar{p}(z)$ the associated cumulative series. Then,*

- (1) *in the ring $\mathbb{C}[[z]]$ one has $(1 - z)\bar{p}(z) = p(z)$,*
- (2) *if $\bar{p}(z)$ converges to a complex analytic function $\bar{\varphi}(z)$ with convergence radius $\bar{\rho}$, then $p(z)$ converges to $\varphi(x) = \frac{\bar{\varphi}(z)}{1-z}$ at least in an open disk of radius $\min\{1, \bar{\rho}\}$,*

PROOF. Then the n th coefficient of $(1 - z)\bar{p}(z)$ is $1 \cdot \bar{a}_n + (-1)\bar{a}_{n-1} = a_n$, proving (1). Then (2) is immediate. ■

If the coefficients of the power series are non-negative integers, then one has the following obvious facts.

C.2. LEMMA. *Let $a_n \in \mathbb{Z}_{\geq 0}$ and let $p(z) = \sum_{n \in \mathbb{N}_0} a_n z^n$.*

- (1) *If $a_n = 0$ for $n > n_0$, then $p(z)$ is a polynomial and $\rho = \infty$. In this case $\bar{\rho} = (\limsup_{n \rightarrow \infty} \sqrt[n]{\bar{a}_n})^{-1} = (\limsup_{n \rightarrow \infty} \sqrt[n]{\bar{a}_{n_0}})^{-1} = 1$.*
- (2) *If, for all $n \in \mathbb{N}$, there exists $m > n$ with $a_m \neq 0$, then $\rho \leq 1$. By Lemma C.1(1), in this case $\bar{\rho} = \rho$.*

Moreover, if the power series converges to a rational function, then $\rho \in \mathbb{C}$ is the minimal non-negative real pole of the limit function.

- (3) *If $a'_n \in \mathbb{Z}_{\geq 0}$ is another sequence of coefficients with $a'_n \leq a_n$ for all $n \in \mathbb{N}_0$, and if $p'(z)$ is the associated power series with convergence radius ρ' , then $\rho \leq \rho'$.*

Moreover, the following result will be useful to compute convergence radii.

C.3. LEMMA. *Let $b: \mathbb{N}_0 \rightarrow \mathbb{N}$ be a sequence such that $b_{n+m} \leq b_n b_m$ for all $n, m \in \mathbb{N}_0$.*

Then the limit $\lim_{n \rightarrow \infty} \sqrt[n]{b_n}$ exists and

$$\limsup_{n \rightarrow \infty} \sqrt[n]{b_n} = \lim_{n \rightarrow \infty} \sqrt[n]{b_n} = \inf_{n \in \mathbb{N}_0} \sqrt[n]{b_n}.$$

PROOF. One only needs to prove that the sequence converges to its (non-negative) infimum, say $\beta \geq 0$. Let $\varepsilon > 0$, there exists then some $\nu \in \mathbb{N}$ such that $\beta \leq \sqrt[\nu]{b_\nu} < \beta + \varepsilon/2$, furthermore let $M = \max\{b_r \mid r \in \{0, \dots, \nu - 1\}\}$.

There exists some $N \in \mathbb{N}$, $N \geq r$ such that

$$\begin{cases} (\beta + \varepsilon/2)M^{1/N} < \beta + \varepsilon, \\ (\beta + \varepsilon/2)^{1-\nu/N}M^{1/N} < \beta + \varepsilon, \end{cases}$$

and hence, for $n > N$, one has $(\beta + \varepsilon/2)^{1-r/n}M^{1/n} < \beta + \varepsilon$ for all $r \in \{0, \dots, \nu - 1\}$.

Now choose any $n > N$ and write $n = q\nu + r$, with $r \in \{0, \dots, \nu - 1\}$.

Using the submultiplicativity hypothesis, on has

$$\begin{aligned} \sqrt[n]{b_n} &\leq \sqrt[q\nu+r]{b_\nu^q b_r} = (b_\nu)^{\frac{q}{q\nu+r}} \sqrt[r]{b_r} = \left(\sqrt[\nu]{b_\nu}\right)^{q\nu/(q\nu+r)} \sqrt[r]{b_r} \\ &= \left(\sqrt[\nu]{b_\nu}\right)^{1-r/n} \sqrt[r]{b_r} \leq (\beta + \varepsilon/2)^{1-r/n} M^{1/n} < \beta + \varepsilon, \end{aligned}$$

yielding the claim. ■

Denote $(-)' = \frac{d}{dt}$ the derivative and let $f^{(k)} = (f^{(k-1)})'$ for $k > 0$ and $f^{(0)} = f$.

C.4. LEMMA. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial function and suppose that*

$$f^{(k)}(x_0) = 0 \quad \forall k < n \quad \text{and} \quad f^{(n)}(x_0) > 0.$$

Then there is a neighborhood U of x_0 such that

$$\operatorname{sgn}(f(y)) = \operatorname{sgn}((y - x_0)^n) \quad \forall y \in U \setminus \{x_0\}.$$

PROOF. One writes the truncated Taylor series

$$f(y) = \frac{f^{(n)}(x_0)}{n!}(y - x_0)^n + q(y),$$

with $\lim_{y \rightarrow x_0} \frac{|q(y)|}{(y - x_0)^n} = 0$. Since $\operatorname{sgn}: \mathbb{R}^\times \rightarrow \{1, -1\}$ is a map of multiplicative groups,

$$\operatorname{sgn}(f(y)) = \operatorname{sgn}\left(\frac{f^{(n)}(x_0)}{n!} - \frac{|q(y)|}{(y - x_0)^n}\right) \operatorname{sgn}((y - x_0)^n)$$

and, up to choosing y close enough to x_0 ,

$$= \operatorname{sgn}\left(\frac{f^{(n)}(x_0)}{n!}\right) \operatorname{sgn}((y - x_0)^n) = \operatorname{sgn}((y - x_0)^n). \quad \blacksquare$$

APPENDIX D

Computations

“ I wish to God these calculations had been executed by steam.

C. BABBAGE

Some explicit computations were done with the aid of a computer and GAP.

1. GAP and CHEVIE

I used GAP, [S⁺97], in the prepackaged version available at J. Michel's web page <http://www.math.jussieu.fr/~jmichel/gap3/>, which in particular auto-loads CHEVIE [GHL⁺96] and Vkcurve.

2. Source code

At URI <http://goo.gl/GZKpG> you may find all the source code I wrote.

- The file `functions.gap` contains the definitions of the GAP-functions used to compute the Poincaré series of any Coxeter systems.
- The file `hyp-list.gap` contains the definition of the Coxeter matrices of the exceptional (i.e., not in the series HC1, HC2) hyperbolic cocompact and hyperbolic types.
- Finally, the file `run.gap` is the main file to be run by GAP to compute the Poincaré series. The computation lasted few days on a 4core, 2.67GHz computer.
- At the end, one obtains a file readable by Mathematica® 8.0, which, in turn, computes the convergence radii of the Poincaré series, and writes a L^AT_EX table.

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