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of two correlated Normals.

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Abstract The aim of this paper is to study the distribution of the absolute quotient of two correlated Normals. This study may have many applications because often the researcher expects that a ratio is positive or the sign of the ratio is unimportant. Keeping into account a form of the distribution of the ratio of two correlated Normals proposed by Oksoy and Aroyan in 1994, we find the distribution of the absolute quotient of two correlated Normals random variables. The form here proposed is simple to compute because it is a function of the $T(h, \lambda)$ studied by Owen in 1956. It is also given the distribution of the Absolute ratio as a function of the distribution of the Arctangent r.v. It is also given the distribution of the Absolute ratio as a function of the distribution of the Arctangent r.v..

Keywords: Absolute ratio of two Correlated Normals, Owen's function, Arctangent r.v..

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1 Introduction

Often we are interested in the study of the distribution of a ratio whose value is positive. Many indexes assumes only positive values. In industrial practice measurements are frequently recorded without their algebraic sign in order to add negative values to positive values. The present study may be useful to the researcher interested in the distribution of the ratio of two measures of this kind.

Moreover the ratio of two Normals has no moment. Keeping into account of the results of Cedilnik et al. (2006) it is possible to study the existence of moments for the absolute value of the ratio.

The distribution of the ratio has been studied from many scholars. In particular we remember: Geary (1930), Fieller (1932), Aroian and Oksoy (1986), Marsaglia (2006), Galeone (2007). Kim (2006) considered the ratio of two independent Folded Normals.

Taking into account the result of Aroian and Oksoy (1994) about the CDF of two correlated Normals, we propose a formulation of the CDF for the absolute value of the ratio in a simple way which is very easy to compute.

2 Derivation of the Cumulative Distribution Function

Let's consider a Bivariate Correlated Normal

$$(Y, X) \sim N(\mu_Y, \mu_X, \sigma_Y^2, \sigma_X^2, \rho).$$

The r.v. $w = \frac{Y}{X}$ has CDF given by (Aroian and Oksoy, 1986)

Considerations about the Quotient of two Correlated Normals

$$F_W(w) = L\left(\frac{a - b t_w}{\sqrt{1+t_w^2}}, -b, \frac{t_w}{\sqrt{1+t_w^2}}\right) + L\left(\frac{b t_w - a}{\sqrt{1+t_w^2}}, b, \frac{t_w}{\sqrt{1+t_w^2}}\right),$$

$w \in \mathfrak{R}$, where

$$a = \sqrt{\frac{1}{1-\rho^2}} \left(\frac{\mu_Y}{\sigma_Y} - \rho \frac{\mu_X}{\sigma_X} \right), b = \left(\frac{\mu_X}{\sigma_X} \right),$$

$$t_w = \sqrt{\frac{1}{1-\rho^2}} \left(\frac{\sigma_X}{\sigma_Y} w - \rho \right)$$

and $L(h, k, \rho)$ is the bivariate normal integral according to the indication of Kotz et al. (2000).

An alternative formula for $F_W(w)$ involving the $V(h, q)$ function of Nicholson (1943) or the $T(h, \lambda)$ function of Owen (1956) is

$$F_W(w) = \frac{1}{2} + \frac{1}{\pi} \arctan(t_w) + 2V\left\{\frac{b t_w - a}{\sqrt{1+t_w^2}}, \frac{b + a t_w}{\sqrt{1+t_w^2}}\right\} - 2V(b, a)$$

where

$$V(h, q) = \int_0^h \int_0^y \Phi(x)\Phi(y) dx dy, \quad y = \frac{q}{h} x, \quad \text{and}$$

$$T(h, \lambda) = \frac{1}{2\pi} \arctan \lambda - V(h, \lambda h).$$

It easy to find

$$F_W(w) = \frac{1}{2} + \frac{1}{\pi} \arctan(t_w) + \frac{1}{\pi} \arctan\left(\frac{b + a t_w}{b t_w - a}\right) +$$

$$- 2T\left(\frac{b t_w - a}{\sqrt{1+t_w^2}}, \frac{b + a t_w}{b t_w - a}\right) - \frac{1}{\pi} \arctan\left(\frac{a}{b}\right) + 2T\left(b, \frac{a}{b}\right)$$

Now we want to consider the CDF of the rv $V = \left| \frac{Y}{X} \right|$.

$$\begin{aligned}
F_V(v) &= P\left\{ \left| \frac{Y}{X} \right| < v \right\} = P\left\{ -v < \frac{Y}{X} < v \right\} = P\left\{ \frac{Y}{X} < v \right\} - P\left\{ \frac{Y}{X} < -v \right\} = \\
&= F_W(v) - F_W(-v) = \\
&= \frac{1}{2} + \frac{1}{\pi} \arctan(t_v) + 2V \left\{ \frac{b t_v - a}{\sqrt{1+t_v^2}}, \frac{b + a t_v}{\sqrt{1+t_v^2}} \right\} + \\
&\quad - 2V(b, a) - \frac{1}{2} - \frac{1}{\pi} \arctan(t_{-v}) + \\
&\quad - 2V \left\{ \frac{b t_{-v} - a}{\sqrt{1+t_{-v}^2}}, \frac{b + a t_{-v}}{\sqrt{1+t_{-v}^2}} \right\} + 2V(b, a) = \\
&= \frac{1}{\pi} [\arctan(t_v) - \arctan(t_{-v})] + \\
&\quad + 2V \left\{ \frac{b t_v - a}{\sqrt{1+t_v^2}}, \frac{b + a t_v}{\sqrt{1+t_v^2}} \right\} - 2V \left\{ \frac{b t_{-v} - a}{\sqrt{1+t_{-v}^2}}, \frac{b + a t_{-v}}{\sqrt{1+t_{-v}^2}} \right\} = \\
&= \frac{1}{\pi} [\arctan(t_v) - \arctan(t_{-v})] + \\
&\quad + \frac{1}{\pi} \left[\arctan\left(\frac{b + a t_v}{b t_v - a} \right) - \arctan\left(\frac{b + a t_{-v}}{b t_{-v} - a} \right) \right] + \\
&\quad - 2T \left(\frac{b t_v - a}{\sqrt{1+t_v^2}}, \frac{b + a t_v}{b t_v - a} \right) + 2T \left(\frac{b t_{-v} - a}{\sqrt{1+t_{-v}^2}}, \frac{b + a t_{-v}}{b t_{-v} - a} \right)
\end{aligned}$$

Remembering the formula:

$\arctan(a) - \arctan(b) = \arctan\left(\frac{a-b}{1+ab}\right)$, we obtain

$$F_v(v) = \frac{1}{\pi} \left[\arctan\left(\frac{t_v - t_{-v}}{1 + t_v t_{-v}}\right) \right] + \frac{1}{\pi} \left[\arctan\left(\frac{t_{-v} - t_v}{1 + t_v t_{-v}}\right) \right] +$$

$$- 2T\left(\frac{b t_v - a}{\sqrt{1+t_v^2}}, \frac{b + a t_v}{b t_v - a}\right) + 2T\left(\frac{b t_{-v} - a}{\sqrt{1+t_{-v}^2}}, \frac{b + a t_{-v}}{b t_{-v} - a}\right)$$

$$F_v(v) = 2T\left(\frac{b t_{-v} - a}{\sqrt{1+t_{-v}^2}}, \frac{b + a t_{-v}}{b t_{-v} - a}\right) - 2T\left(\frac{b t_v - a}{\sqrt{1+t_v^2}}, \frac{b + a t_v}{b t_v - a}\right).$$

The above formula is very easy to compute using R program or using the series expression reported in Owen (1956).

3 Relation between the CDF of V and the CDF of the Arctangent random variable

The Arctangent distribution was proposed by Zenga (1979). In 2004 Pollastri and Tornaghi studied the characteristics of the distribution. This distribution has been utilised to obtain the simultaneous confidence regions and for testing hypothesis about the probabilities of a multinomial distribution (Zenga, Fedrizzi, 1981; Brunazzo, 1979; Brunazzo, Fedrizzi, 1980; Pollastri, 1979, 1980, 1982).

Remembering the Cumulative Distribution Function of the Arctangent r.v.

$$F^*(h; a) = 1 - 2\pi \frac{T(h, a)}{\arctan(a)}$$

we obtain

$$\begin{aligned}
 F_v(v) &= \frac{1}{\pi} \left[\arctan\left(\frac{b + a t_{-v}}{b t_{-v} - a}\right) - \arctan\left(\frac{b + a t_v}{b t_v - a}\right) \right] + \\
 &+ \frac{1}{\pi} \left[\arctan\left(\frac{b + a t_v}{b t_v - a}\right) F^* \left(\frac{b t_v - a}{\sqrt{1+t_v^2}}; \frac{b + a t_v}{b t_v - a} \right) + \right. \\
 &\left. - \arctan\left(\frac{b + a t_{-v}}{b t_{-v} - a}\right) F^* \left(\frac{b t_{-v} - a}{\sqrt{1+t_{-v}^2}}; \frac{b + a t_{-v}}{b t_{-v} - a} \right) \right]
 \end{aligned}$$

i.e.

$$\begin{aligned}
 F_v(v) &= \frac{1}{\pi} \left[\arctan\left(\frac{t_v - t_{-v}}{1 + t_v t_{-v}}\right) \right] + \\
 &+ \frac{1}{\pi} \left[\arctan\left(\frac{b + a t_v}{b t_v - a}\right) F^* \left(\frac{b t_v - a}{\sqrt{1+t_v^2}}; \frac{b + a t_v}{b t_v - a} \right) + \right. \\
 &\left. - \arctan\left(\frac{b + a t_{-v}}{b t_{-v} - a}\right) F^* \left(\frac{b t_{-v} - a}{\sqrt{1+t_{-v}^2}}; \frac{b + a t_{-v}}{b t_{-v} - a} \right) \right]
 \end{aligned}$$

So we can assert that the CDF of the absolute ratio of two Correlated Normals is a function of the CDF of the Arctangent r.v..

The result may help in finding the characteristics of the Distribution of the r.v. V .

4 Conclusions

The absolute ratio is important in the study of indexes of many practical situations. Here we propose a simple way of computing the CDF of the absolute value of the ratio of two correlated Normals. We propose also a relation with the CDF of the Arctangent rv. Moreover, if we consider the absolute variable, on the contrary of the ratio, we can study the moments.

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