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Contents

Introduction

One of the greatest advances in theoretical physics over the last years has been the conjectured AdS/CFT correspondence [1]. It states that the strongly coupled regime of a conformal field theory (CFT) in d dimensions can be described by a dual theory of gravity in a weakly curved $d+1$ dimensional anti de–Sitter AdS spacetime.
This offers a tool for investigating field theories at strong coupling, which is notoriously a hard task.
The original realization of the correspondence involves the maximally supersymmetric extension of Yang–Mills theory in four dimensions ($\mathcal{N}=4$ SYM).
At strong coupling its dual is provided by the supergravity limit of type IIB string theory compactified on $AdS_5 \times S^5$.
For almost ten years very little had been known on the AdS/CFT correspondence away from four dimensional field theories, until an explicit realization of the AdS_4/CFT_3 case was first carried out by Aharony, Bergman, Jafferis and Maldacena (ABJM) in 2008 [2].
More precisely, the authors proved that an $\mathcal{N}=6$ supersymmetric Chern–Simons quiver gauge theory with bifundamental matter enjoying $SO(4)$ flavor symmetry is dual to M–theory compactified on $AdS_4 \times S^7/\mathbb{Z}_k$, and describes the low energy dynamics of a stack of $M2$ branes probing an orbifold singularity.
This discovery has triggered a renewed interest in three dimensional Chern–Simons–matter (CSM) theories.

My PhD thesis is mainly devoted to the weak coupling study of such models, restricting to $\mathcal{N}=2$ theories, and particularly focusing on the famous ABJM CFT.
In the weak coupling regime, such theories can be analyzed perturbatively. A manifestly supersymmetric approach is available to perform computations, greatly simplifying their complexity. This is provided by three dimensional $\mathcal{N}=2$ superspace techniques.
In the introduction of my thesis I review how Chern–Simons matter theories are implemented in such a formalism. Then I perform a manifestly supersymmetric quantization of the theory, which is the starting point for all the subsequent perturbative calculations.
These are addressed to two main problems: the determination of exactly marginal deformations and the computation of scattering amplitudes in three dimensional theories.

When dealing with a conformal field theory a natural question is whether it is possible to deform it, by adding operators to the Lagrangian, in such a way that conformal invariance is preserved. If this occurs the operators are dubbed *exactly marginal* and the space of parameters they span is called the *conformal manifold*.
Its determination goes through deforming the CFT by the most general set of operators and analyzing the flow under the renormalization group (RG) [3]. This requires the knowledge of anomalous dimensions and beta–functions, which have to be computed perturbatively. However in four dimensional supersymmetric theories, some properties of the conformal manifold, like its dimension, can be more efficiently inferred from the global symmetries (and their breaking) of the CFT, as recent papers by Green, Komargodski, Seiberg, Tachikawa and Wecht [4], and by Kol [5] pointed out.
The first section of my thesis is aimed at determining the conformal manifold for the ABJM theory. By the explicit two–loop computation of the RG functions, I find the complete spectrum of exactly marginal deformations of the ABJM theory. The main result is that its conformal manifold has complex dimension 3 and is a compact surface, isomorphic to $\mathbb{CP}^3$ at two–loop order.
Pursuing the alternative method based on global symmetries, I reproduce the correct dimension of the conformal manifold, giving evidence that this procedure also applies to $\mathcal{N} = 2$ supersymmetric theories in three dimensions.
More generally I also analyze the renormalization group flow for a wider class of (flavored) $\mathcal{N} = 2$ supersymmetric Chern–Simons matter (CSM) theories, covering some examples already described in literature and discovering a plethora of new conformal field theories, suitable for having an AdS/CFT gravity dual.
In all cases I ascertain that the CFT's are infrared stable fixed points of the RG, confirming a prediction coming from the AdS/CFT correspondence.

The second part of my thesis addresses the problem of computing scattering amplitudes in supersymmetric gauge theories.
The traditional way of evaluating amplitudes in field theory, by means of Feynman diagrams is well-established, but may prove impractical for high loop order and large number of external particles. This motivated the search for new methods for calculating amplitudes efficiently. This novel techniques include unitarity based methods, twistorial formulation, recursive relations and so on. These are of phenomenological interest for their application to QCD, but have also allowed great progress for maximally supersymmetric $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity in four dimensions.
In particular, for $\mathcal{N} = 4$ SYM, where a dual method for computing amplitudes at strong coupling has been provided by the AdS/CFT correspondence, an impressive amount of indications hints at believing that scattering amplitudes may display a hidden simplicity. In particular, by explicit computation, they revealed an underlying dual superconformal and Yangian symmetry, which points towards an integrable structure. Moreover evidence suggests that a net of dualities links amplitudes, Wilson loops (WL) and correlation functions of protected operators. This hidden simplicity also shows up in the conjecture by

Bern, Dixon and Smirnov (BDS) that amplitudes may exponentiate, uncovering an iterative pattern. The origin of such a beautiful structure is far from well–understood, and exploring its existence in theories other than $\mathcal{N} = 4$ SYM is certainly fruitful.

In my thesis I investigate the existence of a Wilson loop / scattering amplitude / correlator duality in $\mathcal{N} = 2$ CSM theories. In particular I explicitly spell out the computation of

- the lightlike Wilson loop to one–loop order for any number of cusps, for any CSM theory;
- the lightlike limit of the one–loop correlator for an arbitrary (even) number of half–BPS operators, for any $\mathcal{N} = 2$ CSM theory;
- the complete set of one–loop four–point amplitudes, for any $\mathcal{N} = 2$ CSM theory;
- the two–loop four–point amplitude, for the ABJM model and a simple generalization thereof;

The bosonic Wilson loop is evaluated in components, whereas correlators and amplitudes are calculated by a direct, manifestly supersymmetric, Feynman diagram approach.

The outcome of the one–loop computation is that the expressions for correlation functions in the light–like limit and Wilson loops match at the level of the Feynman integrals. In particular I prove analytically that both eventually vanish for any $\mathcal{N} = 2$ CSM theory. This leaves open the possibility for a correlator/WL duality to hold for a wide set of theories.

Four point amplitudes at one–loop order actually vanish only for the ABJ(M) case. This restricts the possibility of a WL/amplitude duality to the latter models only.

The first non–trivial correction to the four–point amplitude for the ABJ(M) then comes at two loops. The remarkable result of such a computation is that the amplitude exhibits dual conformal invariance and matches the expression of the four–cusped Wilson loop, hinting at a possible extension of the WL/amplitude duality in three dimensional models. Furthermore the expression for the two–loop amplitude notably resembles its four dimensional one–loop analogue in $\mathcal{N} = 4$ SYM, and I give evidence supporting that it can be thought of as the first order expansion of a BDS–like exponentiation ansatz.
This leads to the intriguing possibility that some of the marvelous properties of amplitudes in $\mathcal{N} = 4$ SYM might be shared by its three dimensional cousin, a perspective which deserves further investigation.

The thesis is arranged over five chapters. The first is devoted to introducing the basics of the ABJM theory, to explaining its $\mathcal{N} = 2$ superspace quantization and to presenting generalizations of the model. In the second chapter I deal with deformations of the ABJM theory. After a brief introductory review on the generalities of exactly marginal deformations in supersymmetric theories, I focus on the three dimensional case provided by the ABJM model and detail the perturbative computation of its exactly marginal deformations, leading to the conformal manifold of the theory. In the third chapter I outline a survey of scattering amplitudes in $\mathcal{N} = 4$ SYM, highlighting their striking features uncovered in recent literature. Finally in the last two chapters I investigate some aspects of scattering amplitudes in the ABJM. Starting with an overview on known results on tree level scattering amplitudes, I then spell out some new advances at one and two loops and comment on their relevance for getting hints on the symmetries and the structure of amplitudes in ABJM.

Chapter 1

ABJM

1.1 The field theory.

The ABJM theory is a conformal field theory in three dimensional space–time. Its basic features are as follows

- a gauge sector consisting of two Chern–Simons actions for A and $\hat{A}$ gauge vectors in the adjoint representations of two unitary gauge groups $U(N)_K \times U(N)_{-K}$, having equal ranks and opposite Chern–Simons levels K and $-K$;
- a matter field content given by four complex scalars A^i and B_i ($i = 1, 2$) transforming in the bifundamental representations of the gauge groups and coupled to the gauge fields;
- the fermionic superpartners of the aforementioned bosonic fields, composing $\mathcal{N} = 2$ supermultiplets;
- a superpotential interaction for matter fields, preserving a $SU(2) \times SU(2)$ global symmetry, which enhances supersymmetry to $\mathcal{N} = 6$.

We briefly examine such points in more detail.

Gauge sector. The gauge sector of the theory is given by a Chern–Simons CS action, which is power counting renormalizable in three dimensions, in contrast to YM which would give rise to a superrenormalizable theory. This fact is obviously essential in getting a conformal field theory. The action of pure Chern–Simons is topological. In the language of forms it reads

$$\frac{iK}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{1.1.1}$$

whereas in components, and conveniently rescaling the gauge fields so that they are real, one gets

$$-\frac{iK}{4\pi}\int_M \mathrm{Tr}\,\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho+\frac{2}{3}\,i\,A_\mu A_\nu A_\rho\right) \tag{1.1.2}$$

The equations of motion simply imply that the gauge connection is flat $F = dA + A\wedge A = 0$, therefore the gauge field is pure gauge and does not possess any physical propagating degrees of freedom, in that there is no quadratic term in derivatives.
In the non–abelian case invariance of the generating functional under large gauge transformations requires the CS level K to be integer–valued. Indeed, the CS action is invariant under gauge transformations

$$A \longrightarrow A' = U^{-1}\,(A+d)\,U \tag{1.1.3}$$

only up to a term proportional to a topological number classifying the (third) homotopy class of the gauge transformation. This is an integral multiple of 2π, so that whenever $K\in\mathbb{Z}$, $\exp(S'_{CS}) = \exp(S_{CS})$ gauge invariance of the functional integral is guaranteed.

The CS term can be made $\mathcal{N}=2$ supersymmetric easily, by adding auxiliary fields to the CS action

$$S_{CS}^{\mathcal{N}=2} = \frac{K}{4\pi}\int \mathrm{Tr}\left(A\wedge dA+\frac{2}{3}\,A\wedge A\wedge A-\chi\,\bar{\chi}-2\,\sigma\,D\right) \tag{1.1.4}$$

where χ are the gaugini, D is the auxiliary scalar and σ is an additional scalar field which may be thought of as the fourth component of the A_μ vector when reducing from $4d$ to $3d$. None of the fields above display any kinetic term in the action, therefore they are all auxiliary and they might be integrated out using their algebraic equations of motion.

The ABJM theory features two such supersymmetric CS actions with $U(N)$ gauge groups and opposite CS levels K and $-K$. Hereafter we will denote the fields in the vector multiplet transforming in the adjoint of the second $U(N)$ gauge group by hats, e.g. $\hat{A}$, $\hat{\chi}$, $\hat{D}$ and so on.

$$\begin{aligned} S \;=\;& S_{CS}^{\mathcal{N}=2}-\hat{S}_{CS}^{\mathcal{N}=2} \\ & \frac{K}{4\pi}\,\mathrm{Tr}\int d^3x\,d^4\theta\,\Big[-i\,\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho+\frac{2}{3}\,i\,A_\mu A_\nu A_\rho-\hat{A}_\mu\partial_\nu\hat{A}_\rho-\frac{2}{3}\,i\,\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho\right)+ \\ & \qquad -\chi\,\bar{\chi}-2\,\sigma\,D+\hat{\chi}\,\bar{\hat{\chi}}+2\,\hat{\sigma}\,\hat{D}\Big] \end{aligned} \tag{1.1.5}$$

Matter. The physical degrees of freedom are introduced by adding matter fields to the content of the model, and coupling them to the gauge sector. These are four complex scalar fields which are dubbed A_1, A_2, B_1 and B_2. The A fields are taken to transform in the bifundamental representation of the gauge groups $(\mathbf{N},\bar{\mathbf{N}})$, whereas B fields transform in the complex conjugate representation $(\bar{\mathbf{N}},\mathbf{N})$. In order to add matter in a supersymmetric fashion we also introduce the fermionic partners of each field, namely two component complex fermions ψ_{A^i/B_i} and auxiliary complex scalars F's.

The kinetic terms for scalars and fermions include the coupling to the gauge vectors

$$\mathcal{L}_{kinetic} = -\left(D^\mu A\right)^*_i D_\mu A^i - \left(D^\mu B\right)^{*\,i} D_\mu B_i + i\,\bar{\psi}_{A^i}\,\gamma^\mu D_\mu \psi_{A^i} + i\,\bar{\psi}_{B_i}\,\gamma^\mu D_\mu \psi_{B_i} \quad (1.1.6)$$

encoded in the gauge covariant derivatives acting on A and B scalars as $D_\mu A^i = \partial_\mu A^i + i\,A_\mu A^i - i\,A^i \hat{A}_\mu$ and $D_\mu B_i = \partial_\mu B_i + i\,B_i A_\mu - i\,\hat{A}_\mu B_i$. Here the flavor index i runs over 1, 2 and the star denotes complex conjugation.
Moreover Yukawa couplings are there

$$\begin{aligned} \mathcal{L}_{Yukawa} \;=\; & -\bar{\psi}_{A^i}\,\bar{\chi}\,A^i - A^*_i\,\chi\,\psi_{A^i} + \bar{\psi}_{A^i}\,A^i\,\bar{\hat{\chi}} + A^*_i\,\psi_{A^i}\,\hat{\chi} \\ & -\bar{\psi}_{B_i}\,\bar{\hat{\chi}}\,B_i - B^{*\,i}\,\hat{\chi}\,\psi_{B_i} + \bar{\psi}_{B_i}\,B_i\,\bar{\chi} + B^{*\,i}\,\psi_{B_i}\,\bar{\chi} \end{aligned} \quad (1.1.7)$$

as well as interactions between scalars and fermions with the σ gauge scalar, together with the usual D–terms

$$\begin{aligned} \mathcal{L}_{aux} \;=\; & A^*_i\,D\,A^i - A^i\,\bar{D}\,A^*_i - B_i\,D\,B^{*\,i} + B^{*\,i}\,\bar{D}\,B_i \\ & -A^*_i\,\sigma^2\,A^i - A^i\,\bar{\sigma}^2\,A^*_i - B_i\,\sigma^2\,B^{*\,i} - B^{*\,i}\,\bar{\sigma}^2\,B_i + 2\,A^*_i\,\sigma\,A^i\,\hat{\sigma} + 2\,B^{*\,i}\,\sigma\,B_i\,\hat{\sigma} \\ & +\bar{\psi}_{A^i}\,\sigma\,\psi_{A^i} - \bar{\psi}_{A^i}\,\psi_{A^i}\,\hat{\sigma} + \bar{\psi}_{B_i}\,\hat{\sigma}\,\psi_{B_i} - \bar{\psi}_{B_i}\,\psi_{B_i}\,\sigma \end{aligned} \quad (1.1.8)$$

Finally the ordinary F–terms $F\bar{F}$ appear in the action, which we do not write down here explicitly. Matter fields are conveniently collected into chiral superfields A and B, whose precise definition in terms of $\mathcal{N}=2$ superspace is reviewed in Appendix A.1. Hereafter A and B will always refer to superfields rather than their lowest components, unless otherwise stated.

$\mathcal{N}=3$ supersymmetric CSM. So far the model enjoys $\mathcal{N}=2$ supersymmetry. One may increase supersymmetry to $\mathcal{N}=3$ by adding chiral multiplets in the adjoint representation of each gauge group ϕ_1 and ϕ_2. The scalar and fermionic components of the latter combine with the scalar σ and the two gaugini, to make up a triplet and a singlet plus a triplet of the $SU(2)_R$ symmetry of $\mathcal{N}=3$. Then one also couples pairs $(A^i,\, B_i)$ (which transform in complex conjugates representations) into hypermultiplets. Their scalar components form doublets of the $SU(2)_R$ R–symmetry, for each index $i=1\,,2$ separately: $(A^1,\, B^{*\,1})$ and $(A^2,\, B^{*\,2})$. Moreover the two pairs of superfields (A^i, B_i) are related by an additional $SU(2)$ symmetry for the two flavors $i=1$ and $i=2$. This does not commute with the $SU(2)_R$ and enhances R–symmetry to $SO(4)_R$, showing that the matter sector alone is $\mathcal{N}=4$ supersymmetric. Hypermultiplets interact with the adjoint chiral superfields ϕ through a marginal superpotential

$$W_1 = \mathrm{Tr}\left(B_i\,\phi_1\,A^i\right) + \mathrm{Tr}\left(B_i\,\phi_2\,A^i\right) + \; h.c. \quad (1.1.9)$$

in which gauge indices are suppressed but gauge invariance is clear recalling the representations in which the fields transform. A relevant mass term for the adjoint fields may be added as well, reading

$$W_2 = \frac{K}{8\pi}\,\mathrm{Tr}\left(\phi_2^2 - \phi_1^2\right) + h.c. \tag{1.1.10}$$

where the mass is fixed by supersymmetry. This relevant coupling induces a flow in the infrared to an effective theory obtained by integrating out ϕ fields, which do not possess any kinetic term and are therefore auxiliary.

$\mathcal{N} = 6$ **supersymmetric CSM.** This procedure yields the superpotential

$$W = \frac{4\pi}{K}\,\mathrm{Tr}\left(A^1 B_1 A^2 B_2 - A^1 B_2 A^2 B_1\right) + h.c. \tag{1.1.11}$$

which may also be rewritten as

$$W = \frac{2\pi}{K}\,\epsilon_{ij}\,\epsilon^{kl}\,\mathrm{Tr}\left(A^i B_k A^j B_l\right) + h.c. \tag{1.1.12}$$

This last form exposes manifestly a $SU(2) \times SU(2)$ global symmetry. Again this symmetry does not commute with the $SU(2)_R$ R–symmetry of the $\mathcal{N} = 3$ theory. Together they generate a $SU(4)_R \simeq SO(6)_R$ R–symmetry, which indicates enhancement of supersymmetry to $\mathcal{N} = 6$. This can be verified in components looking into the scalar potential, derived by integrating out the auxiliary gauge fields D and σ in (1.1.8) and the F–terms from the superpotential. The former contribution gives

$$\begin{aligned} V_D \;=\; \frac{4\pi^2}{K^2}\,\mathrm{Tr}\,\Big[&\left(A^i\,A_i^* + B^{*\,i}\,B_i\right)\left(A^j\,A_j^* - B^{*\,j}\,B_j\right)\left(A^k\,A_k^* - B^{*\,k}\,B_k\right) \\ &+ \left(B_i\,B^{*\,i} + A_i^*\,A^i\right)\left(B_j\,B^{*\,j} - A_j^*\,A^j\right)\left(B_k\,B^{*\,k} - A_k^*\,A^k\right) \\ &+2\,A^i\,\left(B_j\,B^{*\,j} - A_j^*\,A^j\right)\,A_i^*\,\left(B_k\,B^{*\,k} - A_k^*\,A^k\right) \\ &+2\,B_i\,\left(A^j\,A_j^* - B^{*\,j}\,B_j\right)\,B^{*\,i}\,\left(A^k\,A_k^* - B^{*\,k}\,B_k\right)\Big] \end{aligned} \tag{1.1.13}$$

whereas the latter

$$\begin{aligned} V_F \;=\; \frac{16\pi^2}{K^2}\,\mathrm{Tr}\,\Big(&A_i^*\,B^{*\,j}\,A_k^*\,A^k\,B_j\,A^i - A_i^*\,B_{*\,j}\,A_k^*\,A^i\,B_j\,A^k \\ &+B^{*\,i}\,A_j^*\,B^{*\,k}\,B_k\,A^j\,B_i - B^{*\,i}\,A_j^*\,B^{*\,k}\,B_i\,A^j\,B_k\Big) \end{aligned} \tag{1.1.14}$$

It is then easy to rewrite this in a manifestly $SU(4)_R$ invariant fashion, by embedding the four complex scalars into multiplets $C_I = (A^1, A^2, B_1^*, B_2^*)$ and $\bar{C}^I = (A_1^*, A_2^*, B_1, B_2)$ (where $I = 1, \ldots, 4$) in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of the R–symmetry group, leading to the scalar potential

$$V_{D+F} = \frac{4\pi^2}{K^2}\,\mathrm{Tr}\left[C^I\,\bar{C}_{[I}\,C^J\,\bar{C}_{K]}\,C^K\,\bar{C}_J - \frac{1}{3}\,C^I\,\bar{C}_{[I}\,C^J\,\bar{C}_J\,C^K\,\bar{C}_{K]}\right] \tag{1.1.15}$$

To summarize, in $\mathcal{N} = 2$ the global symmetry of the theory is $SU(2) \times SU(2)$, along with a $U(1)_b$ baryonic symmetry under which the A fields have charge 1 and B fields charge −1. This symmetry is gauged into the $U(N) \times U(N)$ gauge group. Before adding the superpotential, an axial $U(1)_a$ is there, under which A and B fields all have charge 1. This is explicitly broken by the superpotential (1.1.12). We remark that these global symmetries will be widely used in Section 2.3, to extract information on the exactly marginal deformations of the ABJM.

In the ABJM model supersymmetry allows for one coupling only, namely $\frac{1}{K}$. This measures the strength of both gauge and matter interactions. A weak coupling regime is therefore available whenever $K \gg 1$. Then, although the CS level is quantized, its inverse may be considered a continuous variable and treated as a usual coupling constant.

The ABJM theory arises in the context of the AdS/CFT correspondence. Therefore we will be usually interested in its large N limit. The relevant effective coupling will then be the 't Hooft parameter

$$\lambda \equiv \frac{N}{K} \tag{1.1.16}$$

and the corresponding weak coupling regime, where the field theoretical description is more suitable, holds for $N \ll K$. Whenever $N \gg K$ the dual description in the AdS/CFT sense, which is reviewed in the next Section, is more adequate.

1.2 The gravity dual.

The strongly coupled regime of the ABJM field theory is described by a dual gravity theory, through the AdS/CFT correspondence, which was explicitly determined in [2]. In close connection to this, the ABJM model arises as the low energy field theory describing the dynamics of a stack of $M2$ branes in M–theory, probing the conical singularity of the orbifold $\mathbb{C}^4/\mathbb{Z}_K$.
Indeed the main features of ABJM outlined in the previous Section are designed to fulfill the basic requirements for engineering such a theory:

- conformality (implying the emergence of the CS action in three dimensions)*;
- four complex scalars, representing the transverse directions in which the branes fluctuate;
- $SO(6)$ R–symmetry matching the $SU(4)$ isometry symmetry of the transverse space $\mathbb{C}^4/\mathbb{Z}_K$†;
- $\left(\mathbb{C}^4/\mathbb{Z}_K\right)^N/S_N$‡ moduli space, precisely matching that of N BPS objects free to move in the $\mathbb{C}^4/\mathbb{Z}_K$ conical background.

*In particular the conformal group $SO(3,2)$ matches the isometry of the AdS_4 spacetime resulting as the near horizon geometry of the brane, corresponding to the low energy limit.

†The $\mathbb{Z}_K$ quotient acts on the four complex coordinates of $\mathbb{C}^4$ as $x_i \to e^{\frac{2\pi i}{K}} x_i$

‡S_N stems for the permutation group of N objects.

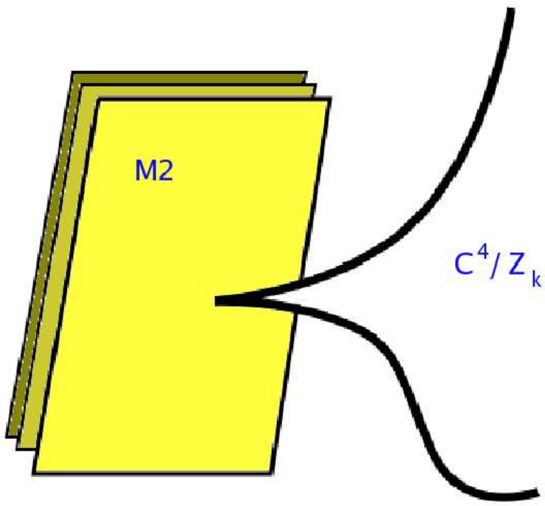


Figure 1.1: A cartoon of $M2$ branes in $\mathbb{C}^4/\mathbb{Z}_k$ geometry.

This M–theory setting may be viewed as the up–lift from a type IIB brane configuration. Indeed the ABJM theory describes the low energy limit field theory on N $D3$ branes wound around a compact direction and breaking on a transverse $NS5$ brane and a $(1,k)$–fivebrane, properly extended so as to preserve $\mathcal{N}=3$ supersymmetry.

The near horizon limit of the M–theoretical configuration sketched above produces the background metric for the AdS/CFT dual of the ABJM. We will not be concerned too much with the dual description of ABJM, however we report here some basic aspects for completeness. Moreover a few elementary facts about the stringy dual description will be employed in Section 5.3.5, where we sketch the computation of a four particle scattering amplitude in ABJM at strong coupling.

Since $\mathbb{C}/\mathbb{Z}_K$ can be viewed as a cone over $S^7/\mathbb{Z}_K$, the dual description is in terms of M–theory on $AdS_4\times S^7/\mathbb{Z}_K$. More precisely one considers a solution of eleven dimensional supergravity on such a background, with N units of four–form F_4 flux sourced by the $M2$'s turned on.

In order to determine such a solution one may consider the flat space case first, originating from N $M2$ branes in $\mathbb{C}^4$. We use the following general solution for the metric [6], applying to p brane solutions in $D=11$ supergravity

$$ds^2 = H^{-2\frac{D-p-3}{\Delta}}\left(-dt^2+\sum_i dy_i^2\right)+H^{2\frac{p+1}{\Delta}}\left(dr^2+r^2\,d\Omega_{D-p-2}\right) \tag{1.2.1}$$

where the harmonic (in the transverse space) function $H(r)$ is

$$H = 1 + \frac{1}{D-p-3}\sqrt{\frac{\Delta}{2(D-2)}}\,\frac{Q}{r^{D-p-3}}, \qquad \Delta = (p+1)(D-p-3) \tag{1.2.2}$$

The factor Q, measures the charge of the solution and its explicit form is

$$Q = \frac{N\,(2\,\pi)^{D-p-3}\,l_p^{D-p-3}}{\Omega_{D-p-2}} \tag{1.2.3}$$

where N is the number of branes and $\Omega_{D-p-2} = \frac{2\,\pi^{\frac{D-p-1}{2}}}{\Gamma\left(\frac{D-p-1}{2}\right)}$ is the volume of a $(D-p-2)$–sphere. In particular, for an $M2$ brane we have

$$ds^2_{M2} = H^{-\frac{2}{3}}\left(-dt^2 + dy_1^2 + dy_2^2\right) + H^{\frac{1}{3}}\left(dr^2 + r^2\,d\Omega_{D-p-2}\right) \tag{1.2.4}$$

where

$$H = 1 + \frac{1}{6}N\,(2\,\pi)^6\,l_p^6\,\Gamma\,(4)\frac{2\,\pi^4}{r^6} = 1 + \frac{N\,2^5\,\pi^2\,l_p^6}{r^6} \tag{1.2.5}$$

The near horizon limit of such a solution gives the desired $AdS_4 \times S^7$ $11d$ supergravity background. In such a limit

$$H \sim \frac{N\,2^5\,\pi^2\,l_p^6}{r^6} \tag{1.2.6}$$

Plugging this into the metric (in Planck units, i.e. setting $l_p = 1$) yields

$$ds^2 = \frac{r^4}{(N\,2^5\,\pi^2\,)^{\frac{2}{3}}}\left(-dt^2 + \sum_i dy_i^2\right) + \frac{(N\,2^5\,\pi^2)^{\frac{1}{3}}}{r^2}\left(dr^2 + r^2\,d\Omega_{D-p-2}\right)$$

Performing the change of variables $r \to r^{\frac{1}{2}}$ casts the metric into the AdS_4 form, after rescaling t and y coordinates

$$ds^2 = \frac{r^2}{(N\,2^5\,\pi^2\,)^{\frac{1}{3}}}\left(-dt^2 + \sum_i dy_i^2\right) + \frac{(N\,2^5\,\pi^2)^{\frac{1}{3}}}{r^2}\left(\frac{1}{4}\,dr^2 + r^2\,d\Omega_{D-p-2}\right)$$

from this we read that the AdS_4 radius in Planck units $R^2_{AdS_4} = \frac{1}{4}\,R^2 = \frac{1}{4}\,(N\,2^5\,\pi^2)^{\frac{1}{3}}$ is half that of the sevensphere.

The desired solution looks synthetically

$$\begin{aligned} ds^2 &= \frac{R^2}{4}\,ds^2_{AdS_4} + R^2\,ds^2_{S^7} \\ F_4 &\sim N\,\epsilon_4 \end{aligned} \tag{1.2.7}$$

where ϵ_4 stems for the volume form of AdS_4.

Now the $\mathbb{Z}_k$ quotient on this solution has to be carried out. To do this it proves convenient to write S^7 as a Hopf fibration over $\mathbb{CP}^3$

$$ds^2_{S^7} = (d\alpha + \omega)^2 + ds^2_{\mathbb{CP}^3} \tag{1.2.8}$$

where $\alpha \in [0, 2\pi]$ and the Fubini–Study metric in homogeneous coordinates for $\mathbb{CP}^3$ reads

$$ds^2_{\mathbb{CP}^3} = \frac{dz_i\, d\bar{z}_i}{\rho^2} - \frac{|z_i\, d\bar{z}_i|}{\rho^4} \tag{1.2.9}$$

where $\rho^2 = \sum |z_i|^2$ and

$$d\alpha + \omega = \frac{i}{2\,\rho^2}\,(z_i\, d\bar{z}_i - \bar{z}_i\, dz_i) \tag{1.2.10}$$

From (1.2.10) it follows that $d\omega$ is the Kaehler form of $\mathbb{CP}^3$

$$d\omega = J = i\, d\left(\frac{z_i}{\rho}\right) \wedge \left(\frac{\bar{z}_i}{\rho}\right) \tag{1.2.11}$$

Then the $\mathbb{Z}_K$ quotient is performed sending $\alpha \to \alpha/K$ (keeping $\alpha \in [0, 2\pi]$) giving the metric

$$ds^2_{S^7/\mathbb{Z}_K} = \frac{1}{K^2}\,(d\alpha + K\,\omega)^2 + ds^2_{\mathbb{CP}^3} \tag{1.2.12}$$

making the volume of the internal space smaller by a factor K, which also means that N has to be sent to KN, so that the F_4 flux is quantized in N units in the orbifolded space. Therefore also the radii of AdS_4 and $\mathbb{CP}^3$ change to $R^2 = (2^5\,\pi^2\,K\,N)^{1/3}$.

This supergravity description is reliable only when the radius of curvature of the background is large. The radius of $\mathbb{CP}^3$, $R^2 = (2^5\,\pi^2\,K\,N)^{1/3}$, is large for $NK \gg 1$. However the radius of the circle S^1 is smaller by a factor of K: $R^2_{S^1} = R^2/K^2 \sim (N/K^5)^{1/3}$, meaning that the M–theoretical description is actually suitable for $N \gg K^5$. As this circle shrinks the theory is pushed into the realm of type IIA string theory. The reduction is performed as in [7]

$$ds^2_{11} = e^{\frac{4}{3}\,\phi}\,(dx_{11} + A)^2 + e^{-\frac{2}{3}\,\phi}\,ds^2_{10} \tag{1.2.13}$$

in terms of the type IIA dilaton ϕ and the RR field A. Comparing to the $11d$ equations one identifies $A = K\,\omega$ and $e^{\frac{4}{3}\,\phi} = \frac{R^2}{K^2}$. This gives the string frame metric

$$ds^2 = \frac{R^3}{K}\left(\frac{1}{4}\,ds^2_{AdS_4} + ds^2_{\mathbb{CP}^3}\right) \tag{1.2.14}$$

with dilaton

$$e^{2\phi} = \frac{R^3}{K^3} \tag{1.2.15}$$

and fluxes

$$\begin{aligned} F_4 &= \frac{3}{8}\,R^3\,\epsilon_4 \\ F_2 &= K J \end{aligned} \tag{1.2.16}$$

In particular the AdS_4 radius is related to the solution above by

$$R^2_{AdS_4} = \frac{R^3}{4\,K} = \frac{\sqrt{2^5\,\pi^2\,K\,N}}{4\,K} = \sqrt{2}\,\pi\,\sqrt{\frac{N}{K}} \tag{1.2.17}$$

This formulation is valid until the radius of curvature is large compared to the string length. One therefore requires

$$R^2_{AdS_4} = 2^{\frac{5}{2}}\,\pi\,\sqrt{\lambda} \gg 1 \tag{1.2.18}$$

where $\lambda = \frac{N}{K}$ is the 't Hooft coupling, which entails $N \gg K$.

Summarizing, three regimes of the parameters N and K are available:

- for $\lambda \ll 1$, i.e. $N \ll K$, the field theoretical description is weakly coupled;
- for $K \ll N \ll K^5$ the type IIA supergravity approximation is valid;
- for $N \gg K^5$ the eleven dimensional supergravity description is reliable.

1.2.1 Moduli space.

To check that the conjectured duality is reasonable, one may look at the moduli space of vacua of the ABJM field theory and check if it matches the transverse space in its dual description.

It is easier to consider the Abelian case first, and then generalize to the non–Abelian theories.
For $U(1)\times U(1)$ gauge groups there is no cubic term in the CS action and no superpotential in the matter sector. Minimizing the scalar potential implies setting $\sigma = \hat{\sigma} = \frac{2\pi}{K}(|A^i|^2 - |B_i|^2)$. One then has to mod by the gauge group, in order not to overcount gauge equivalent vacua. One may use gauge transformations to set the gauge fields A, $\hat{A}$ to zero and the moduli space is then naively $\mathbb{C}^4$, since it is parameterized by the four complex free fields A^i and B_i. This gauge fixing does not fix residual gauge transformations where Λ and $\hat{\Lambda}$ are constant. In this special case the CS action is not invariant since under

$$A \;\rightarrow\; A + d\Lambda \qquad \hat{A} \;\rightarrow\; \hat{A} + d\hat{\Lambda} \tag{1.2.19}$$

the abelian CS term transforms as

$$\delta S_{CS} = i\,\frac{K}{2\pi}\int d\Lambda\wedge dA - d\hat{\Lambda}\wedge d\hat{A} \tag{1.2.20}$$

This is a total derivative

$$\delta S_{CS} = i\,\frac{K}{2\pi}\int d\big(\Lambda\wedge dA\big) - d\big(\hat{\Lambda}\wedge d\hat{A}\big) \tag{1.2.21}$$

that is usually dropped, but not here since the Λ are constant at the boundary. By Stokes' theorem the integral becomes

$$\delta S_{CS} = i\,\frac{K}{2\pi}\int_{B_2} \Lambda \wedge F - \hat{\Lambda} \wedge \hat{F} \tag{1.2.22}$$

where B_2 stems for some closed manifold of dimension 2 and $F = dA$ and $\hat{F} = d\hat{A}$ are the 2–form Abelian field strengths. By flux quantization the boundary integral evaluates in general

$$\delta S_{CS} = i\,\frac{K}{2\pi}\left(\Lambda\, 2\pi n - \hat{\Lambda}\, 2\pi m\right) = i\, K\,\left(\Lambda\, n - \hat{\Lambda}\, m\right) \tag{1.2.23}$$

where n and m are integer numbers. Therefore gauge invariance of the functional integral may be rescued for Λ and $\hat{\Lambda}$ integral multiples of $2\pi/K$. This $\mathbb{Z}_K$ discrete group is the only true residual gauge transformation by which modding out. Since the scalars A^i and $B^{*\,i}$ all transform by $e^{\frac{2\pi i}{K}}$ under the action of $\mathbb{Z}_K$, the resulting moduli space is $\mathbb{C}^4/\mathbb{Z}_K$.

The generalization to the non–Abelian case is done by taking the symmetric product of N abelian moduli spaces by diagonalizing $N \times N$ matrices. The resulting moduli space is therefore $\left(\mathbb{C}^4/\mathbb{Z}_K\right)^N/S_N$. This moduli space matches the one for N $M2$ branes probing the geometry $\mathbb{C}^4/\mathbb{Z}_K$, providing support for the conjecture.

1.2.2 Chiral primaries.

Another check on the conjectured duality concerns the spectrum of protected operators. In this Section we will briefly sketch the spectrum of scalar chiral primary operators. We anticipate that the dimension two operators made up of four fields will be relevant in Section 2.2, when inspecting the exactly marginal deformations of ABJM. Furthermore, chiral primary operators of generic dimension will be also encountered in Section 4.6, where we shall compute the one–loop correction to their n–point correlation functions in the light–like limit.

In principle a chiral primary in ABJM may be easily obtained by traced strings of A and B chiral superfields, producing chiral objects by construction. In order to achieve gauge invariance, the bilinear $(A^i\, B_j)$ has to be considered as the fundamental building block, actually: $\mathcal{O} = \mathrm{Tr}\left((A^i\, B_j)^L\right)$, for $L = 1, 2, \ldots$. The corresponding canonical dimension of the operator is $\Delta = L$. Not all operators built up this way are primaries, a requirement that may be jeopardized by the equations of motions. Indeed, if any combination of antisymmetric indices appears, for instance $(A\,B)^{L_1}\epsilon_{ij}\,\epsilon^{kl}\,A^i\,B_k\,A^j\,B_l\,(A\,B)^{L_2}$, the F–term condition $\bar{D}^2\bar{A}_i = -\frac{4\pi}{K}\,\epsilon_{ij}\,\epsilon^{kl}\,B_k\,A^j\,B_l$ implies that on–shell the operator is equivalent to $(A\,B)^{L_1}\,A^i\,\bar{D}^2\bar{A}_i\,(A\,B)^{L_2}$, which is a descendant. Therefore one concludes that chiral primary operators are obtained by products of (AB) bilinears, symmetrized on their $SU(2)\times SU(2)$ indices. In components, where the $SU(4)_R$ R–symmetry is manifest, one may think of the scalar part of such operators as products $\mathrm{Tr}\left((C^I\,\bar{C}^J)^L\right)$, where the

C fields transform in the **4** and the $\bar{C}$ fields in the $\bar{\mathbf{4}}$ of $SU(4)$. Symmetrizing separately on the tensor product of L **4**'s and L $\bar{\mathbf{4}}$'s gives the representations identified by Dynkin labels $[L,0,0]$ and $[0,0,L]$, respectively. One takes then the tensor product of these two representations, keeping only pieces with no contractions between **4**'s and $\bar{\mathbf{4}}$, which gives

$$[0,0,L] \otimes [L,0,0] = [L,0,L] \oplus \dots \tag{1.2.24}$$

where ... indicate other terms, neglected for the reason above. Hence such chiral primary operators belong to the $[L,0,L]$ representation of $SU(4)$. These operators are all neutral under the $U(1)_b$ baryonic symmetry of the ABJM.

As reviewed in Section 2.1.3, superconformal invariance and unitarity imply that these operators do not receive quantum corrections to their dimension, which is then fixed to its naive value $\Delta = L$. Therefore their spectrum is suitable for comparison to the spectrum of supergravity fields in the dual description on the background $AdS_4 \times S^7/\mathbb{Z}_K$. The spectrum of $11d$ supergravity KK modes on $AdS_4 \times S^7$ was computed in [7], expanding excitations on the harmonics of the internal space. These states are classified by the representations of the $SO(8)$ isometry group of S^7 they belong. To get the spectrum on the orbifolded background a projection on $\mathbb{Z}_K$ invariant states has to be performed. This may be achieved decomposing the $SO(8)$ representations under its $SU(4) \times U(1)$ subgroup. This corresponds to reducing to KK states on $\mathbb{CP}^3$ and was also performed in [7]. Considering $U(1)$ invariant states, it is actually found that the lowest components of supergravity multiplets lie in the same $[L,0,L]$ representations, as the chiral primaries in field theory. In particular the $\mathcal{N}=6$ $SU(4)$ massless supermultiplet contains scalars in the **15** representation, which has Dynkin labels $[1,0,1]$.

1.3 Superspace formulation of the ABJM model.

In this Section we give a survey of the $\mathcal{N}=2$ superspace formulation of the ABJM theory.

In three dimensional euclidean space–time, we consider an $\mathcal{N}=2$ supersymmetric $U(N)\times U(N)$ Chern–Simons theory for vector multiplets $(V,\hat{V})$ coupled to chiral superfields A^i and B_i, $i=1,2$, in the $(\mathbf{N},\bar{\mathbf{N}})$ and $(\bar{\mathbf{N}},\mathbf{N})$ bifundamental representations of the gauge groups, respectively. This setting is described pictorially by the quiver diagram in Fig. 1.2. The vector multiplets $V,\hat{V}$ transform in the adjoint representation of the gauge groups $U(N)$ and $U(N)$ respectively. Therefore they are valued in the Lie algebra of $U(N)$ groups and we can write $V^b{}_a \equiv V^A (T_A)^b{}_a$ and $\hat{V}^{\hat{b}}{}_{\hat{a}} \equiv \hat{V}^A (\hat{T}_A)^{\hat{b}}{}_{\hat{a}}$, where T_A and $\hat{T}_A$ ($A = 0, 1 \dots N^2-1$) are the $U(N)$ generators, whose conventions are in Appendix A.1. Bifundamental matter carries global $SU(2)_A \times SU(2)_B$ indices $A^i, \bar{A}_i, B_i, \bar{B}^i$ and local $U(N)\times U(N)$ indices $A^a{}_{\hat{a}}, \bar{A}^{\hat{a}}{}_a, B^{\hat{a}}{}_a, \bar{B}^a{}_{\hat{a}}$

In $\mathcal{N}=2$ superspace the action reads [8, 9] (for superspace conventions see Appendix

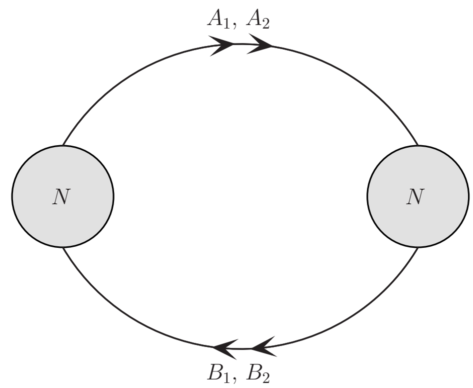


Figure 1.2: Quiver diagram representing the field content of the ABJM theory.

A.1)

$$\mathcal{S} = \mathcal{S}_{\rm CS} + \mathcal{S}_{\rm mat} + \mathcal{S}_{\rm pot} \tag{1.3.1}$$

Gauge sector. We start with the gauge sector. The expression for the CS actions (for the two gauge groups) in $\mathcal{N}=2$ superspace [10] is

$$\mathcal{S}_{\rm CS} = \int d^3x\, d^4\theta \int_0^1 dt\, \left\{ \frac{K}{4\pi} {\rm Tr}\Big[V \bar{D}^\alpha \left(e^{-tV} D_\alpha e^{tV} \right) \Big] - \frac{K}{4\pi} {\rm Tr}\Big[\hat{V} \bar{D}^\alpha \left(e^{-t\hat{V}} D_\alpha e^{t\hat{V}} \right) \Big] \right\} \tag{1.3.2}$$

By integrating fermionic coordinates we can verify that the above action reproduces the bosonic CS action, along with its $\mathcal{N}=2$ partners, as in (1.1.5). As recalled in the introductory section, none of these fields has physical, propagating degrees of freedom and they are all auxiliary.

The superspace action is invariant under superspace gauge transformations

$$e^V \;\rightarrow\; e^{i\bar{\Lambda}_1}\, e^V\, e^{-i\Lambda_1} \qquad\qquad e^{\hat{V}} \;\rightarrow\; e^{i\bar{\Lambda}_2}\, e^{\hat{V}}\, e^{-i\Lambda_2} \tag{1.3.3}$$

where Λ_1 and Λ_2 are chiral superfields which play the role of the parameters of gauge transformations for the $U(M)$ and $U(N)$ gauge groups, respectively. The invariance is ascertained explicitly in Appendix A.2.

Matter sector. Now we turn to the matter sector. Here we consider the kinetic terms for A and B fields, reading

$$\mathcal{S}_{\rm mat} = \int d^3x\, d^4\theta\, {\rm Tr}\left(\bar{A}_i e^V A^i e^{-\hat{V}} + \bar{B}^i e^{\hat{V}} B_i e^{-V} \right) \tag{1.3.4}$$

Finally we include the interaction superpotential (1.1.12)

$$\mathcal{S}_{\rm pot} = \frac{1}{K} \int d^3x\, d^2\theta\, \frac{1}{2}\, \epsilon_{ij}\, \epsilon^{kl}\, {\rm Tr}\left(A^i\, B_k\, A^j\, B_l \right) + \; h.c. \tag{1.3.5}$$

where the coupling of the superpotential is exactly tuned to the value $1/K$, which guarantees $\mathcal{N}=6$ supersymmetry enhancement. The appearance of ϵ tensors makes the global $SU(2)\times SU(2)$ symmetry manifest.

Gauge invariance of the action above under the gauge transformations

$$A^i \;\rightarrow\; e^{i\Lambda_1}\, A^i\, e^{-i\Lambda_2} \qquad\qquad B_i \;\rightarrow\; e^{i\Lambda_2}\, B_i\, e^{-i\Lambda_1} \tag{1.3.6}$$

which are the proper ones for bifundamental fields, follows immediately from the cyclicity of the trace over gauge indices.

The component expansion of this action reproduces the usual CS action and the matter content and interaction of the ABJM theory (1.1.5–1.1.12).

1.4 Superspace quantization.

After we reviewed the superspace formulation of ABJM, we now proceed to the quantization of the theory in a manifestly $\mathcal{N}=2$ supersymmetric setting. Following the conventions of [11], we work in the Euclidean and consider the the path integral $Z=\int \mathcal{D}\phi\, \exp\left(S[\phi]\right)$.
We begin our analysis with the gauge sector. In each $\mathcal{N}=2$ supersymmetric Chern–Simons action we choose gauge-fixing functions $\bar{F}=D^2\,V$, $F=\bar{D}^2\,V$ and insert into the functional integral $Z=\int \mathcal{D}V\, \exp\left(S_{CS}[V]\right)$ the factor

$$\int \mathcal{D}f\,\mathcal{D}\bar{f}\,\Delta(V)\,\Delta^{-1}(V)\,\exp\Big\{-\frac{K}{2\alpha}\int d^3x\,d^2\theta\,\mathrm{Tr}\,(f\,f)-\frac{K}{2\alpha}\int d^3x\,d^2\bar{\theta}\,\mathrm{Tr}\,(\bar{f}\,\bar{f})\Big\} \tag{1.4.1}$$

where $\Delta(V)=\int d\Lambda\,d\bar{\Lambda}\,\delta(F(V,\Lambda,\bar{\Lambda})-f)\,\delta(\bar{F}(V,\Lambda,\bar{\Lambda})-\bar{f})$ and the weighting function has been chosen in order to have a dimensionless gauge parameter α. We note that the choice of the weighting function is slightly different from the four dimensional case [11] where one usually employs $\int \mathcal{D}f\,\mathcal{D}\bar{f}\,\exp\Big\{-\frac{1}{g^2\alpha}\int d^4x\,d^4\theta\,\mathrm{Tr}\,(f\,\bar{f})\Big\}$. Averaging over the weighting functions gives the gauge fixing action

$$\begin{aligned} Z=\int \mathcal{D}V\,\mathcal{D}c\,\mathcal{D}\bar{c}\,\mathcal{D}c'\,\mathcal{D}\bar{c}'\,\exp\;\; &\bigg(S_{CS}[V]-\frac{K}{2\alpha}\int d^3x\,d^2\theta\,\mathrm{Tr}\,(D^2\,V D^2\,V) \\ &-\frac{K}{2\alpha}\int d^3x\,d^2\bar{\theta}\,\mathrm{Tr}\,(\bar{D}^2\,V\bar{D}^2\,V)+S_{FP}[c,c',\bar{c},\bar{c}']\bigg) \end{aligned} \tag{1.4.2}$$

where we have adopted the usual Faddev–Popov trick to write Δ^{-1} as an integral over anticommuting ghosts $c,c',\bar{c},\bar{c}'$.

After using D^2 and $\bar{D}^2$ derivatives to complete the superspace measures in the gauge fixing term, the quadratic part of the gauge–fixed action for the two gauge sectors reads

$$\begin{aligned} S_{CS} + S_{gf} \quad \to \quad & \frac{1}{2} K \int d^3x d^4\theta \ \mathrm{Tr}\, V \left(\bar{D}^\alpha D_\alpha + \frac{1}{\alpha} D^2 + \frac{1}{\alpha} \bar{D}^2 \right) V \\ & - \frac{1}{2} K \int d^3x d^4\theta \ \mathrm{Tr}\, \hat{V} \left(\bar{D}^\alpha D_\alpha + \frac{1}{\alpha} D^2 + \frac{1}{\alpha} \bar{D}^2 \right) \hat{V} \end{aligned} \tag{1.4.3}$$

The operators acting on V and $\hat{V}$ may be inverted, leading to the gauge propagators

$$\langle V^A(1)\, V^B(2) \rangle = -\frac{1}{K} \frac{1}{\Box} \left(\bar{D}^\alpha D_\alpha + \alpha D^2 + \alpha \bar{D}^2 \right) \delta^4(\theta_1 - \theta_2)\, \delta^{AB} \tag{1.4.4}$$

$$\langle \hat{V}^A(1)\, \hat{V}^B(2) \rangle = +\frac{1}{K} \frac{1}{\Box} \left(\bar{D}^\alpha D_\alpha + \alpha D^2 + \alpha \bar{D}^2 \right) \delta^4(\theta_1 - \theta_2)\, \delta^{AB} \tag{1.4.5}$$

In our calculations we will use the analogue of the Landau gauge, $\alpha = 0$.

Expanding $S_{CS} + S_{GF}$ at higher orders in $V, \hat{V}$ we obtain the interaction vertices. For our two–loop calculations we only need the cubic one

$$\begin{aligned} S_{CS} + S_{GF} \quad \to \quad & \frac{i}{6} K f^{ABC} \int d^3x d^4\theta \ \left(\bar{D}^\alpha V^A\, V^B\, D_\alpha V^C \right) \\ & - \frac{1}{24} K f^{ABE} f^{ECD} \int d^3x d^4\theta \ \left(\bar{D}^\alpha V^A\, V^B\, D_\alpha V^C\, V^D \right) \\ & - \frac{i}{6} K f^{ABC} \int d^3x d^4\theta \ \left(\bar{D}^\alpha \hat{V}^A\, \hat{V}^B\, D_\alpha \hat{V}^C \right) \\ & + \frac{1}{24} K f^{ABE} f^{ECD} \int d^3x d^4\theta \ \left(\bar{D}^\alpha \hat{V}^A\, \hat{V}^B\, D_\alpha \hat{V}^C\, \hat{V}^D \right) \end{aligned} \tag{1.4.6}$$

We now turn to the ghost action, which is the same as in the four dimensional $\mathcal{N} = 1$ case [11]. Focusing on one gauge field V, it arises when evaluating the gauge variation of the gauge fixing functions $\Lambda' \left(\frac{\delta F}{\delta \Lambda} \Lambda + \frac{\delta F}{\delta \bar{\Lambda}} \bar{\Lambda} \right)$ and $\bar{\Lambda}' \left(\frac{\delta \bar{F}}{\delta \Lambda} \Lambda + \frac{\delta \bar{F}}{\delta \bar{\Lambda}} \bar{\Lambda} \right)$, after identifying Λ and Λ' with c and c'

$$S_{FP} = i\, \mathrm{Tr} \int d^4x\, d^2\theta\, c'\, \bar{D}^2\, (\delta V) + i\, \mathrm{Tr} \int d^4x\, d^2\bar{\theta}\, \bar{c}'\, \Delta^2\, (\delta V) \tag{1.4.7}$$

Using the non–linear gauge variation for the vector superfield [11] (already in terms of ghosts)

$$\delta V = -\frac{1}{2}\, i\, L_V \left[(c + \bar{c}) + \coth L_{\frac{1}{2} V}\ (c - \bar{c}) \right] \tag{1.4.8}$$

where $L_V X = [V, X]$ and using D^2 and $\bar{D}^2$ in (1.4.7) to complete the fermionic measure

$$S_{FP} = i\, \mathrm{Tr} \int d^4x\, d^4\theta\, (c' + \bar{c}')\ L_{\frac{1}{2} V} \left[(c + \bar{c}) + \coth L_{\frac{1}{2} V}\ (c - \bar{c}) \right] \tag{1.4.9}$$

Expanding up to linear order in V yields

$$S_{gh} = \text{Tr} \int d^4x\, d^4\theta \left\{ \overline{c}' c - c' \overline{c} + \frac{1}{2}(c' + \overline{c}')[V, (c + \overline{c})] \right\} + \mathcal{O}(V^2) \qquad (1.4.10)$$

and gives ghost propagators

$$\langle\, \overline{c}'(1)\, c(2)\, \rangle = \langle\, c'(1)\, \overline{c}(2)\, \rangle = -\frac{1}{\Box}\, \delta^4(\theta_1 - \theta_2) \qquad (1.4.11)$$

and cubic interaction vertices

$$\frac{i}{2}\, f^{ABC} \int d^4x\, d^4\theta \left(c'^A\, V^B\, c^C + \overline{c}'^A\, V^B\, c^C + c'^A\, V^B\, \overline{c}^C + \overline{c}'^A\, V^B\, \overline{c}^C \right) \qquad (1.4.12)$$

When considering the second gauge field $\hat{V}$ an equal contribution arises.

We now quantize the matter sector. From the quadratic part of the action (1.3.4) we read the propagators

$$\langle\, \bar{A}^{\hat{a}}{}_a(1)\, A^b{}_{\hat{b}}(2)\, \rangle = -\frac{1}{\Box}\, \delta^4(\theta_1 - \theta_2)\, \delta^{\hat{a}}{}_{\hat{b}}\, \delta_a{}^b \qquad (1.4.13)$$

$$\langle\, \bar{B}^a{}_{\hat{a}}(1)\, B^{\hat{b}}{}_b(2)\, \rangle = -\frac{1}{\Box}\, \delta^4(\theta_1 - \theta_2)\, \delta^a{}_b\, \delta_{\hat{a}}{}^{\hat{b}} \qquad (1.4.14)$$

From the expansion of (1.3.4) mixed gauge/matter vertices entering two–loop calculations are

$$\begin{aligned} \mathcal{S}_{mat} \;\to\; & \int d^3x d^4\theta\ \text{Tr} \left(\bar{A} V A - \bar{A} A \hat{V} + \bar{B} \hat{V} B - \bar{B} B V \right) \qquad (1.4.15) \\ & + \int d^3x d^4\theta\ \text{Tr} \left(\frac{1}{2} \bar{A} V V A + \frac{1}{2} \bar{A} A \hat{V} \hat{V} - \bar{A} V A \hat{V} + \frac{1}{2} \bar{B} \hat{V} \hat{V} B + \frac{1}{2} \bar{B} B V V - \bar{B} \hat{V} B V \right) \end{aligned}$$

Pure matter vertices can be read from the superpotential (1.3.5).

In the computation of correlation functions we will use propagators in configuration space. We list here these superspace propagators for the gauge sector (in Landau gauge)

$$\begin{aligned} \langle\, V^A(1)\, V^B(2)\, \rangle &= \frac{1}{K}\, \frac{\Gamma(1/2 - \epsilon)}{\pi^{1/2-\epsilon}}\, \bar{D}^\alpha D_\alpha\, \frac{\delta^4(\theta_1 - \theta_2)}{|x_1 - x_2|^{1-2\epsilon}}\, \delta^{AB} \\ \langle\, \hat{V}^A(1)\, \hat{V}^B(2)\, \rangle &= -\frac{1}{K}\, \frac{\Gamma(1/2 - \epsilon)}{\pi^{1/2-\epsilon}}\, \bar{D}^\alpha D_\alpha\, \frac{\delta^4(\theta_1 - \theta_2)}{|x_1 - x_2|^{1-2\epsilon}}\, \delta^{AB} \end{aligned} \qquad (1.4.16)$$

and for matter fields

$$\begin{aligned} \langle\, \bar{A}^{\hat{a}}{}_a(1)\, A^b{}_{\hat{b}}(2)\, \rangle &= \frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}}\, \frac{\delta^4(\theta_1 - \theta_2)}{|x_1 - x_2|^{1-2\epsilon}}\, \delta^{\hat{a}}{}_{\hat{b}}\, \delta_a{}^b \\ \langle\, \bar{B}^a{}_{\hat{a}}(1)\, B^{\hat{b}}{}_b(2)\, \rangle &= \frac{\Gamma(1/2 - \epsilon)}{4\pi^{3/2-\epsilon}}\, \frac{\delta^4(\theta_1 - \theta_2)}{|x_1 - x_2|^{1-2\epsilon}}\, \delta^a{}_b\, \delta_{\hat{a}}{}^{\hat{b}} \end{aligned} \qquad (1.4.17)$$

which can be obtained Fourier transforming (1.4.5) and (1.4.13), with the help of (A.5.8).

1.5 Generalizations.

Since the formulation of the ABJM theory, a number of generalizations was developed for which a dual description is (or might be) available. Here we want to focus on $\mathcal{N}=2$ supersymmetric models only, in such a way that their Lagrangian could be cooked up by enlarging the field content, allowing for more interaction terms and/or by taking more generic values for the data of the theory, namely the ranks of the gauge groups and the CS levels.

In this Section we sketch these extensions, pointing out their Lagrangian description and outlining the AdS/CFT dual, when available.

1.5.1 ABJ.

The simplest generalization we analyze accounts for allowing different ranks of the gauge groups which are taken to be generically $U(M)\times U(N)$. The resulting model is then a Chern–Simons theory for vector multiplets $(V,\hat{V})$ are in the adjoint representations of the gauge groups $U(M)$ and $U(N)$ respectively, coupled to chiral multiplets A^i and B_i, $i=1,2$, in the $(M,\bar{N})$ and $(\bar{M},N)$ bifundamental representations of the gauge groups, respectively.

This deformation preserves $\mathcal{N}=6$ supersymmetry and can be simply implemented at weak coupling, using the same Lagrangian as for the ABJM. However we notice that it explicitly breaks the $\mathbb{Z}_2$ parity invariance under $K\to -K$ of the ABJM model, exchanging the gauge groups. This apparently innocuous fact may on the contrary entail relevant consequences on the properties of the theory. For instance it poses doubts on the possibility that the theory is integrable. A few more remarks on that are given in Section 5.3.4.

The perturbative series in such models is properly organized in terms of two 't Hooft couplings λ and $\hat{\lambda}$

$$\lambda\equiv\frac{M}{K}\qquad \hat{\lambda}\equiv\frac{N}{K} \tag{1.5.1}$$

along with the parity breaking parameter σ

$$\sigma\equiv\frac{\lambda-\hat{\lambda}}{\bar{\lambda}}\qquad \bar{\lambda}\equiv\sqrt{\lambda\hat{\lambda}} \tag{1.5.2}$$

These variables prove very convenient when dealing with the ABJ theory and will be extensively used in Chapters 4 and 5.

At strong coupling the gravity dual was pointed out in [12]. It is basically a perturbation of the ABJM setting, where $l=|M-N|$ units of a torsion flux are introduced. This is deduced by considering the near horizon geometry of a brane configuration where l $D3$

branes are suspended (but not wrapped) between the $NS5$ and the $(1, k)$ fivebranes of the ABJM type IIB brane construction. This changes the rank of one of the gauge groups only, therefore implementing the desired shift. The M–theory lift of such a configuration is in terms of the usual N $M2$ branes probing the $\mathbb{C}^4/\mathbb{Z}_K$ singularity, along with l fractional $M2$ branes, which are $M5$, wrapping the vanishing three cycle at the apex of the orbifold, and are therefore stuck to this point, hence not modifying the moduli space of vacua. They source l units of flux, taking values in $\mathbb{Z}_K$, (i.e. $l + K$ branes are equivalent to l) for the three form potential C_3 of M–theory, on the cycle mentioned above.

The near horizon geometry arising from this configuration gives the AdS/CFT dual of the ABJ theory. The M–theoretical dual, which is the proper description whenever $N \gg k^5$ is the same $AdS_4 \times S^7/\mathbb{Z}_K$ background with N units of F_4 flux as for the ABJM, with additional l units for C_3 on the three cycle $S^3/\mathbb{Z}_K$ inside S^7.

Reducing to type IIA, which is the most suitable description in the regime $K \ll N \ll K^5$, one is led to the background metric of $AdS_4 \times \mathbb{CP}^3$ with F_4 and F_2 flux turned on. The three form potential of M–theory reduces the B–field, which then acquires a non–trivial holonomy on a $\mathbb{CP}^1$ cycle inside $\mathbb{CP}^3$.

In [12] it is argued that the model is meaningful only for parameters satisfying the bound $l < K$. Therefore at strong coupling, where K is supposed to be very small compared to M, N, the difference between λ and $\bar{\lambda}$ is less than

$$\frac{|\lambda - \bar{\lambda}|}{\bar{\lambda}} \leq \frac{1}{\bar{\lambda}} \ll 1 \tag{1.5.3}$$

Therefore the large $\bar{\lambda}$ limit ensures that the 't Hooft coupling can be approximated by N/K and the ABJ deformation can be considered as a probe approximation where one neglects the backreaction of the background.

1.5.2 Romans mass deformation.

The ABJ(M) models feature opposite Chern–Simons levels. One could deform the theory by considering non–opposite values [13]. At the level of the Lagrangian description this is easily performed by considering the CS action

$$\mathcal{S}_{\text{CS}} = \int d^3x\, d^4\theta \int_0^1 dt \left\{ \frac{K_1}{4\pi} \operatorname{Tr}\left[V\, \bar{D}^\alpha \left(e^{-tV} D_\alpha\, e^{tV} \right) \right] + \frac{K_2}{4\pi} \operatorname{Tr}\left[\hat{V}\, \bar{D}^\alpha \left(e^{-t\hat{V}} D_\alpha\, e^{t\hat{V}} \right) \right] \right\} \tag{1.5.4}$$

which straightforwardly determines the following gauge propagators in x and momentum space

$$\begin{aligned}
\langle\, V^A(1)\, V^B(2)\,\rangle \quad &= \quad \frac{1}{K_1}\,\frac{\Gamma(1/2-\epsilon)}{\pi^{1/2-\epsilon}}\,\bar{D}^\alpha D_\alpha\,\frac{\delta^4(\theta_1-\theta_2)}{|x_1-x_2|^{1-2\epsilon}}\,\delta^{AB} \\
&\longrightarrow \quad \frac{4\pi}{K_1}\,\frac{1}{p^2}\,\bar{D}^\alpha D_\alpha\,\delta^4(\theta_1-\theta_2)\,\delta^{AB} \\
\langle\, \hat{V}^A(1)\, \hat{V}^B(2)\,\rangle \quad &= \quad \frac{1}{K_2}\,\frac{\Gamma(1/2-\epsilon)}{\pi^{1/2-\epsilon}}\,\bar{D}^\alpha D_\alpha\,\frac{\delta^4(\theta_1-\theta_2)}{|x_1-x_2|^{1-2\epsilon}}\,\delta^{AB} \\
&\longrightarrow \quad \frac{4\pi}{K_2}\,\frac{1}{p^2}\,\bar{D}^\alpha D_\alpha\,\delta^4(\theta_1-\theta_2)\,\delta^{AB} \qquad (1.5.5)
\end{aligned}$$

This generalization entails deep consequences on the symmetries of the theory. In particular it breaks supersymmetry to $\mathcal{N}=3$ at most, covering a range of theories up to non–supersymmetric models. In [13] it was proven that CFT's always exist in any of these differently supersymmetric deformations, corresponding to some (unknown) dual background in AdS_4. Since we are willing to adopt an $\mathcal{N}=2$ description, here we will only deal with at least $\mathcal{N}=2$ models and disregard the lower supersymmetric ones.

As mentioned above there exist conformal theories with $\mathcal{N}=3$ and $\mathcal{N}=2$.

The $\mathcal{N}=3$ model is pretty unique, given a set of K_1 and K_2 levels.

$$W=\frac{1}{K_1}\,\mathrm{Tr}\,(A^i\,B_i)^2+\frac{1}{K_2}\,\mathrm{Tr}\,(B_i\,A^i)^2 \qquad (1.5.6)$$

Here all couplings are quantized, preventing them from running, and therefore scale invariance follows naturally. This theory possesses manifest $SO(3)_R$ R–symmetry (naturally built–in in our $\mathcal{N}=2$ description) as well as an $SU(2)$ global symmetry simultaneously rotating A and B fields. This is a surviving remnant of the original $SU(2)_A\times SU(2)_B$ of the ABJM theory, which is broken explicitly by the introduction of the operators $(A^1B_1)^2$ and $(A^2B_2)^2$, contained in (1.5.6).

As concerns $\mathcal{N}=2$ models, the existence of a CFT with $SU(2)\times SU(2)$ global symmetry was argued in [13]. Again this symmetry is inherited from the ABJM theory, which displays the same superpotential

$$W=\lambda\,\epsilon_{ij}\,\epsilon^{kl}\,\mathrm{Tr}\,\left(A^i\,B_k\,A^j\,B_l\right) \qquad (1.5.7)$$

The new coupling λ, which is now a mixture of K_1 and K_2, has to be properly tuned, so as to achieve the vanishing of the corresponding β–function. Moreover this model was conjectured to be connected to the $\mathcal{N}=3$ one by a line of fixed points. Indeed, starting from the superpotential

$$W=\mathrm{Tr}\,\left[c_1\,(A^i\,B_i)^2+c_2\,(B_i\,A^i)^2\right] \qquad (1.5.8)$$

one can recover $SU(2)\times SU(2)$ global symmetry whenever $c_1 = -c_2$ and $\mathcal{N} = 3$ supersymmetry fixing $c_1 = \frac{1}{K_1}$ and $c_2 = \frac{1}{K_2}$. Then, null β–functions for c_1 and c_2 can be assured by imposing the anomalous dimensions of A and B fields to be opposite: $\gamma_A + \gamma_B = 0$. This is one constraint on two variables, hence the conjectured line of fixed points. However this one parameter family of CFT's was not worked out explicitly.

The precise determination of the coupling λ^* ensuring scale invariance in the superpotential (1.5.7), and of the exactly marginal operator encompassing the latter and the $\mathcal{N} = 3$ (1.5.6), constitute the main subjects of Section 2.7.

On the dual side of the correspondence, the deformation of the Chern–Simons levels has a physical meaning. It corresponds to the introduction of a Romans mass [14], namely the appearance of non–zero F_0 flux. Therefore whenever the Chern–Simons levels of the given conformal field theory do not sum up to zero, one generally expects $\sum_i K_i$ quanta of F_0 flux to be present in its dual background. Nevertheless the solution of type IIA supergravity deformed with F_0 flux, which is dual to the $\mathcal{N} = 2$ and $\mathcal{N} = 3$ conformal field theories sketched above was derived in [15] just as a first order perturbation of the $\mathcal{N} = 6$ $AdS_4 \times \mathbb{CP}^3$ background.

1.5.3 Generic $\mathcal{N} = 2$ deformation with bifundamental fields.

In the case $M \neq N$ and $K_1 + K_2 \neq 0$ one obtains a generic class of Chern–Simons matter theories with bifundamental superfields. Giving up the global symmetries of the models mentioned above, one may add to the action the most general $\mathcal{N} = 2$ superpotential. The study of such models is the main topic of Section 2.2. Actually there exist ten gauge invariant quartic supersymmetric operators which may serve as superpotential

$$\begin{aligned} W \;=\; \mathrm{Tr}\Big[\, & h_1\,(A^1B_1A^2B_2 - A^1B_2A^2B_1) + h_2\,(A^1B_1A^2B_2 + A^1B_2A^2B_1) \\ & + h_3\,A^1B_1A^1B_1 + h_4\,A^2B_2A^2B_2 + h_5\,A^1B_2A^1B_2 + h_6\,A^2B_1A^2B_1 \\ & + \,h_7\,A^1B_1A^1B_2 + h_8\,A^1B_1A^2B_1 + h_9\,A^2B_2A^2B_1 + h_{10}\,A^2B_2A^1B_2\Big] \;+\; h.c. \end{aligned} \tag{1.5.9}$$

This class of theories contains many conformal fixed points of the RG flow, describing a surface in the space of couplings. This includes all special models reviewed above as subcases, obtained properly tuning the parameters. The basic properties of such a manifold of conformal field theories are analyzed in Section 2.2, whereas some specific examples are given in Section 2.6. We will carry out this program by means of perturbation theory. This implies imposing $h_i \ll 1$, which can be safely assumed. As concerns the Chern–Simons levels, we recall that they are quantized, however, in the weak coupling regime we consider the gauge couplings $\frac{1}{K_1}$ and $\frac{1}{K_1}$ to be very small. In such a $K_1,\ K_2 \gg 1$ limit, the Chern–Simons levels are large enough to pretend that their inverse behave as continuously varying parameters and to allow one to apply ordinary perturbative techniques.

On the strong coupling side, the low amount of symmetries and the large number of superpotential operators and parameters significantly hamper the determination of a gravity dual for these generic set of $\mathcal{N} = 2$ CFT's, which is not known.

1.5.4 Adding flavors.

We introduce flavor fields Q_1 ($\tilde{Q}_1$), Q_2 ($\tilde{Q}_2$), transforming in the (anti)fundamental of each gauge group. Flavor matter carries (anti)fundamental gauge and global $U(M_f) \times U(N_f) \times U(\tilde{M}_f) \times U(\tilde{N}_f)$ indices, $(Q_1^r)^a$, $(\tilde{Q}_{1,\tilde{r}})_a$, $(Q_2^s)^{\hat{a}}$, $(\tilde{Q}_{2,\tilde{s}})_{\hat{a}}$, with $r(\tilde{r}) = 1, \cdots M_f(\tilde{M}_f)$, $s(\tilde{s}) = 1, \cdots N_f(\tilde{N}_f)$. This is represented in Table 1.1.

	gauge		flavor			
	$U(M)$	$U(N)$	$U(M_f)$	$U(\tilde{M}_f)$	$U(N_f)$	$U(\tilde{N}_f)$
Q_1	M	1	M_f	1	1	1
$\tilde{Q}_1$	$\bar{M}$	1	1	$\tilde{M}_f$	1	1
Q_2	1	N	1	1	N_f	1
$\tilde{Q}_2$	1	$\bar{N}$	1	1	1	$\tilde{N}_f$

Table 1.1: The representations of the gauge and global symmetries under which flavor fields transform.

At weak coupling this addition is accounted for by introducing kinetic terms in the Lagrangian

$$\int d^3x\, d^4\theta\, \left(\bar{Q}^1_r\, e^V\, Q_1^r + \tilde{Q}_{1,\tilde{r}}\, e^{-V}\, \bar{\tilde{Q}}^{1,\tilde{r}} + \bar{Q}^2_s\, e^{\hat{V}}\, Q_2^s + \tilde{Q}_{2,\tilde{s}}\, e^{-\hat{V}}\, \bar{\tilde{Q}}^{2,\tilde{s}} \right) \tag{1.5.10}$$

which is invariant under ordinary super gauge transformations for (anti)fundamental fields

$$\begin{aligned} Q_1 &\to e^{i\Lambda_1}\, Q_1 & \qquad \tilde{Q}_1 &\to \tilde{Q}_1\, e^{-i\Lambda_1} \\ Q_2 &\to e^{i\Lambda_2}\, Q_2 & \qquad \tilde{Q}_2 &\to \tilde{Q}_2\, e^{-i\Lambda_2} \end{aligned} \tag{1.5.11}$$

Quantization with flavors. In order to perform computations we need the propagators and the interaction vertices for fundamental matter fields, which can be derived in a completely analogous manner as the bifundamental degrees of freedom. From the general action (2.6.1,2.6.2) we read the propagators

$$\begin{aligned} &\langle\, (\bar{Q}^1_r)_a(1)\, (Q_1^q)^b(2)\, \rangle = -\frac{1}{\Box}\, \delta^4(\theta_1 - \theta_2)\, \delta_a{}^b \delta_r{}^q \\ &\langle\, (\tilde{Q}_{1,r})_a(1)\, (\bar{\tilde{Q}}^{1,q})^b(2)\, \rangle = -\frac{1}{\Box}\, \delta^4(\theta_1 - \theta_2)\, \delta_a{}^b \delta_r{}^q \qquad\qquad r, q = 1, \cdots, N_f \\ &\langle\, (\bar{Q}^2_s)_{\hat{a}}(1)\, (Q_2^{q'})^{\hat{b}}(2)\, \rangle = -\frac{1}{\Box}\, \delta^4(\theta_1 - \theta_2)\, \delta_{\hat{a}}{}^{\hat{b}} \delta_s{}^{q'} \\ &\langle\, (\tilde{Q}_{2,s})_{\hat{a}}(1)\, (\bar{\tilde{Q}}^{2,q'})^{\hat{b}}(2)\, \rangle = -\frac{1}{\Box}\, \delta^4(\theta_1 - \theta_2)\, \delta_{\hat{a}}{}^{\hat{b}} \delta_s{}^{q'} \qquad\qquad s, q' = 1, \cdots, N_f' \end{aligned}$$

and mixed gauge/matter vertices entering two–loop calculations are (besides those in (1.4.15))

$$
\begin{aligned}
&+\int d^3x d^4\theta \ \mathrm{Tr}\, \left(\bar{Q}^1_r \, V \, Q^r_1 - \bar{\tilde{Q}}^{1,r} \, \tilde{Q}_{1,r} \, V + \bar{Q}^2_s \, \hat{V} \, Q^s_2 - \bar{\tilde{Q}}^{2,s} \, \tilde{Q}_{2,s} \, \hat{V} \right) \\
&+\int d^3x d^4\theta \ \mathrm{Tr}\, \left(\frac{1}{2} \, \bar{Q}^1_r \, V \, V \, Q^r_1 + \frac{1}{2} \, \bar{\tilde{Q}}^{1,r} \, \tilde{Q}_{1,r} \, V \, V + \frac{1}{2} \, \bar{Q}^2_s \, \hat{V} \, \hat{V} \, Q^s_2 + \frac{1}{2} \, \bar{\tilde{Q}}^{2,s} \, \tilde{Q}_{2,s} \, \hat{V} \, \hat{V} \right)
\end{aligned}
$$

Superpotential. Moreover one could prescribe superpotential interactions for these new degrees of freedom, which lie into two categories: interactions among fundamental fields alone, or mixed interactions with bifundamental matter

$$(F_{A,B})^{ik}_{\ \ jl} \, \tilde{Q}_{A,\,i} \, Q^j_A \, \tilde{Q}_{B,\,k} \, Q^l_B \qquad , \qquad A, B = 1, 2 \text{ (not summed)} \tag{1.5.12}$$

$$M^{b,i}_{a,j} \, \tilde{Q}_{1\,i} \, A^a \, B_b \, Q^j_1 + \tilde{M}^{b,i}_{a,j} \, \tilde{Q}_{2\,i} \, B_b \, A^a \, Q^j_2 \tag{1.5.13}$$

The presence of these new fields and interactions enlarges the number of conformal field theories which can be realized at special values of the parameters. The dimension of the manifold spanned by these theories is computed in Section 2.5 on general grounds. Some explicit cases are analyzed also in Section 2.6, where a restriction of the general superpotential (1.5.12) is considered, preserving enough symmetries so as to make direct computations feasible.

In particular we stress that there exists a very peculiar choice of flavors, allowing for an enhancement of supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 3$. Such a situation is described by the following superpotential,

$$W = \frac{1}{K_1} \, \mathrm{Tr} \, (A^i \, B_i + Q_1 \, \tilde{Q}_1)^2 + \frac{1}{K_2} \, \mathrm{Tr} \, (B_i \, A^i + Q_2 \, \tilde{Q}_2)^2$$

in which M_f and $\tilde{M}_f$ have been set equal and a global $SU(2) \times SU(M_f) \times SU(N_f)$ combination of the naive $U(2)^2 \times U(M_f)^2 \times U(N_f)^2$ symmetry groups is manifestly preserved. Even if one chooses $K_1 + K_2 = 0$, supersymmetry cannot be enhanced further and this situation corresponds to introducing flavors in the ABJM model.

At strong coupling a dual description is available for addition of flavors in such a $\mathcal{N} = 3$ preserving manner. This is done in [16] where it is argued that fundamental degrees of freedom are accounted for by $D5$ branes in the type IIB brane construction of the ABJM model, which become $D6$ branes wrapping three–cycles of $\mathbb{CP}^3$ in the type IIA dual description of the ABJM.

Chapter 2

Deformations.

When dealing with a conformal field theory, a natural question is whether there exist any deformations of the latter preserving conformal invariance. Once the theory is specified by fixing the gauge symmetry and the field content, one can deform it by adding interactions and tuning their strength. In practice this is done by introducing new operators $\mathcal{O}$ in the Lagrangian, parameterized by some coupling constants. Deformations can be roughly classified according to the naive dimension of the operators they are triggered by. Thus we can distinguish:

- *relevant deformations*, when $\dim(\mathcal{O}) < D$;
- *irrelevant deformations*, when $\dim(\mathcal{O}) > D$
- and *marginal deformations*, when $\dim(\mathcal{O}) = D$

Accordingly, the corresponding coefficients must have positive, negative and vanishing mass dimension. These kind of operators have different influence on the dynamics depending on the energy scale at which the theory is probed. In the ultraviolet the theory is sensitive to irrelevant deformations, whereas the relevant are suppressed by negative powers of the scale, by dimensional analysis. The converse is true in the infrared. The usual way of thinking is to consider a CFT as the low energy effective theory of some more fundamental microscopic theory defined in the ultraviolet. In other words the CFT is the infrared fixed point of a renormalization group flow. From this point of view it is quite meaningless to perturb the UV Lagrangian by irrelevant operators, which would require changing the definition of the microscopic theory, since they strongly affect it in such a regime. Rather, if we deform the model by a relevant operator, the theory would probably flow to some other conformal fixed point, different from the original one. If one adds to the Lagrangian of a given CFT either relevant or irrelevant operators, conformal invariance gets explicitly broken by the appearance of dimensionful parameters. Hence, in order to preserve conformality, one has to restrict deformations to the marginal ones.

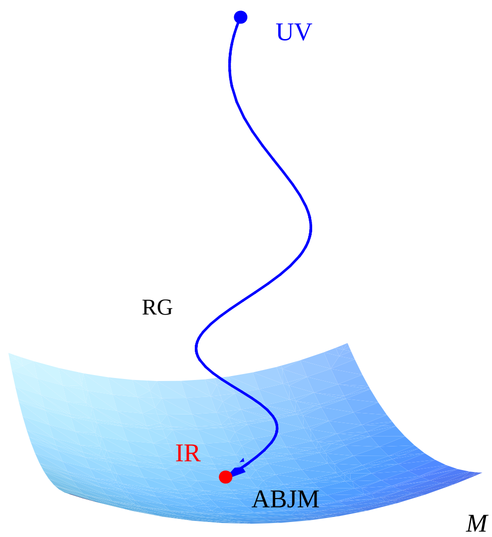


Figure 2.1: A cartoon of a CFT (the ABJM model) arising as a low energy effective theory. It is reached as a fixed point of the RG flow, after the introduction of a relevant operator in the Lagrangian of the UV theory.

Even though the corresponding operators have dimension D classically, they may well acquire an anomalous dimension quantum mechanically, which would spoil scale invariance as well. Depending on the sign of the anomalous dimension marginal deformations are named marginally relevant or marginally irrelevant. The latter are attractive in the infrared, the former are repulsive, causing an instability of the RG flow driving the theory away from the original fixed point when perturbed. Finally, a marginal deformation may remain so even quantum mechanically. These perturbations are suggestively called *exactly marginal*. They manifestly preserve conformal invariance and thus they determine a family of CFT's. A set of n exactly marginal operators, weighted in the Lagrangian by n real parameters spans a manifold of CFT's, which is locally isomorphic to R^n. This manifold is dubbed the *conformal manifold* and its existence is rather non–generic. If a conformal manifold exists one should be interested in determining its properties such as its dimension, its local geometry as well as global issues, for instance (non)compactness or singularities*.

In what follows, we will mainly deal with supersymmetric CFT's. Furthermore, if we just focus on superpotential deformations, the corresponding coupling constants are holomorphic coordinates on an n complex dimensional manifold, by construction.

*It is worth mentioning that the subject is also interesting from the point of view of the AdS/CFT correspondence, the conformal manifold mapping to a space of AdS vacua.

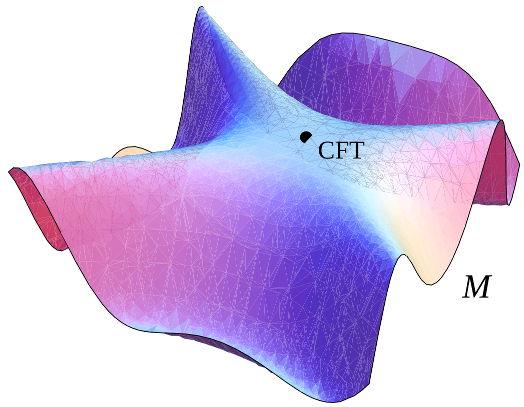


Figure 2.2: A cartoon of the conformal manifold of a given CFT.

2.1 Conformal manifolds in supersymmetric gauge theories.

For a generic CFT the existence of exactly marginal deformations is a highly non–trivial occurrence. However, in the presence of supersymmetry, conformal manifolds appear quite commonly. This phenomenon was first observed and explained in a paper [3] by Leigh and Strassler, for $\mathcal{N} = 1$ theories in four dimensions. We now briefly review their argument and then account for further developments.

2.1.1 Leigh–Strassler analysis

The first issue concerning the conformal manifold of a CFT, namely its existence, was addressed by Leigh and Strassler, who showed that in 4d $\mathcal{N} = 1$ supersymmetric theories exactly marginal deformations arise quite commonly.

Their analysis lies on the direct inspection of the β–functions of the theory. The requirement that the theory is conformal amounts to asking for the β–functions to vanish. If this constraint can be satisfied imposing fewer conditions than the number of couplings, one has a non–trivial manifold of conformal field theories.

The full power of this argument reveals when coping with superconformal theories. In this situation one is usually interested in deformations which preserve supersymmetry. Therefore one should analyze the β–functions of the gauge coupling and of the superpotential interactions. The striking feature of $\mathcal{N} = 1$ theories in four dimensions is that both are exactly determined in terms of the anomalous dimensions of the fields.

This is true thanks to the NSVZ formula for the gauge couplings and because of the non–renormalization theorem for the superpotential. For definiteness let us consider a theory with a gauge group being generically the product $L = \Pi_j L_j$ (L stands for "local"), whose interactions are measured by couplings g_j. The matter content is given by superfields $\phi_{A,a}$, transforming in representations $\mathcal{R}_{A_j}$ of the group G_j ($A_j = 1, \ldots n_{A_j}$), with an index a labeling their multiplicity and running over $a = 1, \ldots N_A$. They are supposed to interact through a cubic (in four dimensions) superpotential $W(\phi_{A,a})$, which we suppose to be polynomial $W = \sum_i h_i W^i(\phi_{A,a})$ and parameterized by couplings h_i.

Here we will only deal with chiral deformations, meaning those triggered by holomorphic operators in the Lagrangian. These include both the gauge deformations (by operators $\int d^2\theta\, W^2$) and superpotential deformations, but not deformations of the Kaehler potential (whose role will be discussed below in Section 2.1.3). The operators considered are furthermore gauge invariant and classically marginal (in order not to break scale invariance explicitly), and will be dubbed *"supermarginals"* throughout this Section.

Now we analyze the RG properties of the deformed $\mathcal{N} = 1$ theory. It is possible to define a regularization scheme where the gauge β–function is exactly given by its one–loop expression

$$\begin{aligned} \beta_{g_j} = f(g) A_{g_j} &= -f(g) \left[3\, C_2(L_j) - \sum_{A_j,a} T(\mathcal{R}_{A_j}) \left(1 - \gamma_{A_j,a}\right) \right] \\ &= -f(g) \left(b_0 + \sum_{A_j,a} T(\mathcal{R}_{A,j})\, \gamma_{A,a} \right) \end{aligned} \qquad (2.1.1)$$

where $f(g)$ is a scheme–dependent unimportant function of the couplings g_j, which is smooth and positive and may eventually hit a pole at strong coupling; $C_2(L_j)$ is the quadratic Casimir of the adjoint representation for the gauge group L_j, $T(\mathcal{R}_{A_j})$ that of the representation $\mathcal{R}_{A_j}$ of the fields $\phi_{A_j,a}$ and finally $\gamma_{A_j,a}$ stems for the anomalous dimension of the fields $\phi_{A_j,a}$. Therefore the last sum runs over all representations of the given gauge group L_j and all fields transforming in them. The anomalous dimensions of such fields are weighted by the quadratic indices $\mathrm{Tr}_{R_A}\left(T^a T^b\right) = T(\mathcal{R}_A)\, \delta^{ab}$ of the representations R_A. In case of mixing (which is allowed only among fields transforming in the same representation under the gauge group) one should diagonalize the anomalous dimensions γ_A, which are then $N_A \times N_A$ matrices. The non–trivial dependence on the field content of the theory, inside the parentheses in (2.1.1) is connected to the anomaly of the $U(1)_R$ R–symmetry of the $\mathcal{N} = 1$ supersymmetric theory. Indeed the factor $1 - \gamma_{A,a}$ is precisely the quantum corrected dimension of the field $\phi_{A,a}$, which for a superconformal theory is linked to its R–charge by the relation $\Delta_{A,a} = \frac{3}{2} R_{A,a}$. Moreover supersymmetry relates the conformal anomaly for the stress–energy tensor $T_{\mu\nu}$ conservation (which produces a non–vanishing β–function and signals departure from a CFT) to that of the other components in the multiplet of currents, in particular the $U(1)$ charge of R–symmetry. Being an anomaly, this quantity is one–loop exact and scheme independent.

Holomorphicity prevents the superpotential from getting perturbatively renormalized [17]. This is because the superpotential coupling may be regarded as a chiral superfield, ensuring holomorphicity of the whole superpotential. The non–renormalization theorem is then derived by inspecting the properties of the effective superpotential at weak coupling, which forbid any coupling renormalization.
Nevertheless the physical superpotential coupling h_i gets renormalized because of the anomalous dimensions of the superfields

$$\beta_{h_i} = h_i\, A_{h_i} = h_i \left[-\dim(W^i) + \sum_{A,a} \left(\dim(\phi_{A,a}) + \frac{1}{2}\,\gamma_{A,a} \right) \right] \tag{2.1.2}$$

where the sum is understood to extend over all fields appearing in the given superpotential W^i.
In many cases one starts from a CFT where the fields have canonical dimension 1 and the superpotential has dimension 3. In this situation the equation above simplifies

$$\beta_{h_i} = h_i\, \frac{1}{2} \sum_{A,a} \gamma_{A,a} \tag{2.1.3}$$

This always occurs whenever the set of deformations is restricted to classically marginal operators, which is the situation we are interested in.

The requirement that both A_{g_j} and A_{h_i} in (2.1.1) and (2.1.2) vanish can be derived [3] from imposing that the anomaly for the supercurrent multiplet $\mathcal{J}_{\alpha\dot{\alpha}}$ is zero, which we want to be satisfied at a superconformal point

$$\bar{D}^{\dot{\alpha}}\, \mathcal{J}_{\alpha\dot{\alpha}} \propto \frac{W_\beta\, W^\beta}{32\pi^2}\, A_g + \sum_i h_i\, A_{h_i} = 0 \tag{2.1.4}$$

The bottom–line of the discussion is that the only conditions to be applied for conformality are on the anomalous dimensions of the fields. Then the problem of finding exactly marginal deformations boils down to counting anomalous dimensions and couplings.

In particular there will be generically n_γ independent conditions to be imposed on anomalous dimensions.

Usually, due to some symmetries and in the presence of a large number of couplings, the set of independent anomalous dimensions is less plentiful than that of couplings. At worst all constraints coming from setting $A_{g_j} = A_{h_i} = 0$ can be satisfied by vanishing anomalous dimensions, as in any homogeneous system of linear equations. In the situation the theory is even finite. However this is just a sufficient but not a necessary condition for conformality.

In any case, whenever n_γ exceeds the number of couplings, or equivalently of β–functions n_β, there will exist exactly marginal deformations. In particular one expects them to be $n_\beta - n_\gamma$. However one has to keep in mind two remarks. The first is that couplings are complex, but constraints are usually real. The second is that this apparent mismatch can be compensated by the possibility of removing phases in the couplings, by a redefinition of the fields. Through a case by case analysis it turns out that the number of independent phase removals is always n_γ, as the number of constraints.

Altogether this implies that the conformal manifold will have complex dimension given by the number of β's minus the number of γ's

$$\dim_{\mathbb{C}}(\mathcal{M}_c) = n_\beta - n_\gamma \tag{2.1.5}$$

Therefore in general the dimension has the form of an index given by the number of classically marginal gauge invariant and supersymmetric operators and a number of obstructions, here represented by constraints on anomalous dimensions. In [3] the reasoning described above was applied and successfully tested to a plethora of examples, for which we refer to the original paper.

We finally comment on the efficiency of the procedure outlined so far. Although well defined in principle, this method can become increasingly impractical if the number of couplings grows large. In these situations one may at best restrict to a subset of marginal deformation and check whether some combinations thereof are exactly marginal. However a thorough description of the whole conformal manifold may be out of reach. Moreover the reasoning above applies only to field theories allowing for a direct evaluation of the β–functions, which is not possible at all if the theory is inherently strongly coupled or even if it lacks a Lagrangian description.

2.1.2 Exactly marginal deformations and symmetries.

A second approach to the study of conformal manifolds of CFT's is based on the study of the global symmetries of the theory and their breaking.

This has been recently derived for $\mathcal{N} = 1$ theories in four dimensions [4, 5]. The main result, which we anticipate, is that the conformal manifold can be determined as the quotient $\mathcal{M} = \{\lambda\}/G_{\mathbb{C}}$ of the set of supermarginal operators of the theory by its complexified (continuous, non R–symmetry) global symmetry group. This method is very powerful since it does not require the evaluation of β–functions and allows for the determination of local properties of the conformal manifold even for supersymmetric theories without a Lagrangian description. For several four dimensional examples it has been successfully checked that the two methods agree [4, 5].

The conformal manifold and global symmetries.

Let us suppose that the theory under exam is invariant under a group of global transformations G. The classical global group at zero couplings, for the theory described above is $G = \Pi_{A=1}^{n_A} U(N_A)$, where N_A is the multiplicity of fields in the representation $\mathcal{R}_A$ of the gauge group.

The classically supermarginal operators are of course invariant under local symmetries, but may not be so under the global ones. In other words the marginal operators may be charged under the global group G.

The matrices of anomalous dimensions γ_A for fields ϕ_A transforming in the representation $\mathcal{R}_A$ are $N_A \times N_A$ hermitian matrices, thus in the adjoint of $U(N_A)$. Therefore the number of fields $\phi_{A,a}$ in the superpotential W^i may be regarded as the charge under the Abelian $U(1)_A$ factor of $U(N_A)$ of the coupling h_i, which we shall name $q_{A,i}$. By analogy, one assigns $U(1)_A$ charges to gauge couplings, given by $-T(\mathcal{R}_A)$. Finally for the non–abelian part of the global symmetry group, the charge matrices can be simply identified as the non–abelian generators of the global group G, i.e. of $SU(N_1), \dots SU(N_{n_A})$.

With these definitions the condition for conformality derived from equations (2.1.3) and (2.1.1) (having set $b_0 = 0$, to start with a conformal field theory at zero coupling) may be restated

$$\beta_{\lambda_i} \propto \sum_A q_{A,i}\, \gamma_A \tag{2.1.6}$$

where λ collects gauge and superpotential couplings $\lambda_i = (g_j, h_i)$, the sum runs over the different representations of the gauge groups, and it is understood that the equation is valued in the adjoint representation of $U(N_A)$, so that there is one different equation for each generator.

Having done this one can recast the conditions for conformal invariance into a suggestive fashion [5]

$$D^{l_A} = \sum_i \bar{\lambda}^i\, T^{l_A}\, \lambda_i = 0 \qquad\qquad \forall A = 1, \dots n_A \tag{2.1.7}$$

where $l_A = 0 \dots N_A^2 - 1$ is an index in the adjoint of $U(N_A)$ and T^{l_A} is its l_A–th generator. This equation has the structure of a D–term for the "fields" $\lambda = (g, h)$. This can be solved thanks to the minus sign in the charges of the gauge couplings under $U(1)$'s, making (2.1.7) not positive definite.

Actually only generators of the global symmetry group, which are not unbroken, produce independent D–term equations. Unbroken symmetries do not contribute, having null charges. The dimension of the conformal manifold will be then indexed by

$$\begin{aligned} \dim_{\mathbb{C}}(\mathcal{M}_c) = & \quad \#\ \text{supermarginals at } W = 0 \\ & - \#\ \text{broken generators of global symmetries} \end{aligned} \tag{2.1.8}$$

This matches the counting in the Leigh–Strassler prescription, where the independent constraints on γ–functions are just reinterpreted as the breaking of global symmetries. To be more precise, starting from the global symmetry group at zero coupling, the breaking of some generators thereof can be certainly caused classically by turning on superpotential operators, however may be also produced quantum mechanically by anomalies in the gauge sector. No matter the origin of the breaking, (2.1.8) provides the correct counting of exactly marginal operators. Moreover it can be shown that (2.1.7) reproduces the Leigh–Strassler constraints on anomalous dimensions at first order in perturbation theory.

The dimension of the conformal manifold (2.1.8) suggests that the latter may be locally described by the holomorphic quotient

$$\mathcal{M}_c = \frac{\{\lambda, D = 0\}}{G} \tag{2.1.9}$$

where $\{\lambda\}$ indicates the vector space of supermarginal operators, and $\{\lambda, D = 0\}$ that after the imposition of the D–term constraints. This construction is also known to be equivalent to modding out by the complexified global symmetry group [18]

$$\mathcal{M}_c = \frac{\{\lambda\}}{G_{\mathbb{C}}} \tag{2.1.10}$$

This claim may be justified as follows. In $\mathcal{N} = 1$ supersymmetric theories any parameter may be regarded as the expectation value of some background chiral superfield. Hence the space of parameters is naturally a manifold endowed with a complex structure. This must be true both for the set of supermarginals and for the conformal manifold of exactly marginal deformations.

In going from the former to the latter one has to mod out by deformations which are equivalent under a global transformation. This operation has to be performed in a holomorphic manner, by the imposition of the D–terms. Stated another way, the supersymmetric theory is invariant under the full complexified group of global transformations, so that the quotient in (2.1.10) is meaningful and the imposition of D–terms just corresponds to modding out by the real part of the complexified transformations.

More precisely, the action of the Lie algebra G over the space of couplings $\mathcal{M}$ preserves supersymmetry and hence holomorphicity. Therefore it leaves the Kaehler (or symplectic) form defined over it invariant. In particular, every element ζ of the Lie algebra induces a vector field X_ζ on $\mathcal{M}$, corresponding to its infinitesimal action, in such a way that its contraction with the Kaehler form J form is closed $d(\iota_{X_\zeta} J) = 0$. The action of G on $\mathcal{M}$ is also Hamiltonian, i.e., $\iota_{X_\zeta} J$ is exact and equal to the differential of some differentiable Hamiltonian function $\iota_{X_\zeta} J = dH_\zeta$, $H : \mathcal{M} \to \mathbb{R}$. This Hamiltonian function may be thought of as the pairing between the moment map $\mu : \mathcal{M} \to G^*$ from the manifold to the dual algebra and the element ζ of the Lie algebra $H_\zeta = \langle \mu , \zeta \rangle$. The D–term (2.1.7) is precisely the moment map for the global symmetry group.

The kernel of the moment map $\mu^{-1}(0)$ (i.e. the locus where the D–terms vanish) is invariant under the action of G. Then if one quotients this kernel by the group G (supposing it is compact), a theorem in symplectic geometry ensures that one obtains a well defined symplectic manifold, which inherits the symplectic structure of $\mathcal{M}$. Then the conformal manifold is expected to be Kaehler, which was indeed proven in [19].

2.1.3 Superconformal algebra and constraints imposed.

Using purely field theoretical arguments, based on the superconformal algebra, the result of the previous Section was proven last year by the authors of [4]. We briefly review their argument here. Starting with a supersymmetric conformal field theory we can use superconformal algebra to classify operators. In particular we divide deformations into two sets: the chiral or *superpotential deformations*, which are implemented through operators integrated over half superspace, and operators integrated over the whole superspace, which can be dubbed *Kaehler deformations*. In order to retain conformality at the classical level, superpotential deformations are required to be chiral operators of dimension 3 and Kaehler deformations to be implemented by real primary operators of weight 2.

We review some basic aspects of the highest weight unitary representations of the superconformal group $SU(2,2|1)$ of $\mathcal{N}=1$ superCFT's in four dimensions. These are given by a direct sum of unitary highest weight representations of the bosonic subalgebra $SU(2,2)\times U(1)$. Its maximal compact subalgebra is $SU(2)\times SU(2)\times U(1)\times U(1)$, where the $SU(2)$ factors accounts for the $SO(4)$ part inside $SO(2,4)\simeq SU(2,2)$. Therefore highest weight representations may be classified by the four quantum numbers of these factors: two spins j_1 and j_2 for each $SU(2)$ factor and two real numbers corresponding to the two $U(1)$ factors, which may be regarded as the dimension Δ and the R–charge R. Unitarity (positivity of the norm of all descendants, which are the operators obtained by acting with supercharges Q and $\bar{Q}$) imposes severe constraints on these quantum numbers: in particular the following conditions have to be met

- $\Delta \geq \frac{3}{2}R+2$ for $j_1=0$
- $\Delta \geq -\frac{3}{2}R+2$ for $j_2=0$
- $\Delta \geq \frac{3}{2}R+2+j_1$ for $j_1>0$
- $\Delta \geq -\frac{3}{2}R+2+j_2$ for $j_2>0$

The first two bounds are dictated by the positivity of the norm of the first two descendants, obtained by (anti)commutators with $\bar{Q}$ (or Q) supercharges: e.g. $[\bar{Q},\mathcal{O}\}$ and $[\bar{Q},[\bar{Q},\mathcal{O}\}\}$ (further application of $\bar{Q}$ identically vanishes). In the most general case these constraints define all long unitary highest weight representations of the superconformal group. However there are special cases when shortening may occur. This happens when some descendants vanish. In these cases the corresponding bounds get saturated. Moreover, in the cases where either j_1 or j_2 are zero, the bounds themselves may become milder.

In particular the constraint $\Delta \geq \frac{3}{2}R+2$ comes from requiring that the norm of the first two descendants $[\bar{Q},\mathcal{O}\}$ and $[\bar{Q},[\bar{Q},\mathcal{O}\}\}$ are positive. If the former vanishes the unitarity bound implies $\Delta = \frac{3}{2}R$. This is called a *chiral primary* operator. If the similar mechanism shows up for $[Q, \mathcal{O}\}$, the corresponding operator is dubbed *antichiral*. Similarly, non–chiral short multiplets may be obtained when $j_1 = j_2 = j$, $\Delta = 2j + 1$ and $R = 0$. These are called conserved.

Having classified highest weight representations of the superconformal algebra, we can consider all possible operators with which deforming the Lagrangian, which are subject to further constraints. By Lorentz invariance these must be obtained by primary operators, applying Q and $\bar{Q}$ operators and contracting indices until the total spin is 0. Since we want to keep only chiral operators of dimension 3 or non–chiral ones of dimension 2, it is easy to see that the total spin $j_1 + j_2$ of the starting primary operator must not exceed 2, and, separately, $j_1 \leq 1$, $j_2 \leq 1$.

These are

1. the spin $j_1 = j_2 = 0$ operator $\mathcal{O}_0$;
2. the spin $j_1 = \frac{1}{2}$ contracted with Q: $\{Q, \mathcal{O}_{\frac{1}{2}}\}$;
3. the spin $j_1 = 1$ contracted with Q^2: $\{Q, [Q, \mathcal{O}_1]\}$;
4. the spin $j_1 = j_2 = \frac{1}{2}$ contracted with $Q\bar{Q}$: $\{Q, [\bar{Q}, \mathcal{O}_{(\frac{1}{2},\frac{1}{2})}]\}$;
5. the spin $j_1 = j_2 = 1$ contracted with $Q^2\bar{Q}^2$: $\{Q, [\bar{Q}, \{Q, [\bar{Q}, \mathcal{O}_2]\}]\}$;

and their complex conjugates. Then one imposes that such operators be supersymmetric, namely that they are annihilated by both Q and $\bar{Q}$. These constraints impose that

- $\mathcal{O}_0$ is a constant operator, which we can neglect;
- $\mathcal{O}_1$ must be a chiral operator ;
- $\mathcal{O}_2$ is not constrained by supersymmetry and corresponds to a Kaehler deformation, however for the operator added to the Lagrangian not to vanish, this operator must not have null descendants, therefore it holds strictly $d > \frac{3}{2}R + 2$, meaning that the operator is irrelevant;
- all other operators listed above must vanish to be supersymmetric;

To summarize, classifying all possible perturbations, only two kinds of operators are compatible with Lorentz invariance, supersymmetry and unitarity: deformations of the superpotential and of the Kaehler potential. The letter are either irrelevant, and are therefore discarded, or conserved currents (having a null descendant), which do not deform the Lagrangian because of the equations of motion. This explains why in the previous analysis of Sections 2.1.2 and 2.1.3 it is possible to neglect Kaehler deformations and why we are left with the space of deformations being that of chiral marginal operators.

Now we report a parallel proof of the prescription outlined above for computing the conformal manifold locally, which highlights some new aspects. The space of supermarginals is naturally endowed with the Zamolodchikov metric, determined by two point functions

$$\langle\, W^i(x)\, \bar{W}^{\bar{j}}(0)\,\rangle = \frac{g^{i\,\bar{j}}}{|x|^6} \tag{2.1.11}$$

It is also possible to define a metric in the space of conserved currents, choosing a suitable basis for two–point functions

$$\langle\, J_a(x)\, J_b(0)\,\rangle = \frac{\gamma_{a\,b}}{|x|^4} \tag{2.1.12}$$

This last metric will be useful in what follows to raise, lower and contract indices relating currents.

We deform a given CFT by a supermarginal operator $\mathcal{L} \to \mathcal{L} + h_i\, W^i$, and see what happens locally, in the vicinity of the CFT. It is possible to choose a renormalization scheme in which, thanks to the non–renormalization theorem for the superpotential, these deformations may only affect the Kaehler potential, inducing a correction proportional to a conserved current operator,

$$\mathcal{L} \rightsquigarrow \mathcal{L} + \int d^4\theta\, Z^a\, J_a \tag{2.1.13}$$

This may potentially entail a renormalization scale dependence, through some logarithmic divergence arising in the correction $Z^a = Z^a(\mu)$. However if no conserved current J_a is available in the theory to couple with, this mechanism is not possible and all deformations automatically preserve conformality. In any case, the general condition for conformal invariance to be preserved is that the deformations in the Kaehler potential be scale invariant, or

$$D^a \equiv \frac{d}{d\,\log\mu}\, Z_a = 0 \tag{2.1.14}$$

Therefore, close to the CFT, the conformal manifold will be given by the space of couplings $\{\lambda\}$ where the vanishing of the D–terms in (2.1.14) has been imposed and where equivalent deformations are identified, modding out by the global symmetry group. In formulae this means

$$\mathcal{M}_c \simeq \frac{\{\lambda,\, D=0\}}{G} \tag{2.1.15}$$

and we therefore recover (2.1.9). However we have not yet determined in this formalism the precise relation between the D–terms in (2.1.14) and the global symmetries of the theory. To do this, we look into the constraints (2.1.14), which can be determined explicitly by conformal perturbation theory from the OPE's of the operators. In particular the lowest order in the OPE of two supermarginals is

$$W^i(x)\, \overline{W}^{\bar{j}}(0) = \frac{g^{i\bar{j}}}{|x|^6} + \frac{(T^a)^{i\bar{j}}}{|x|^4}\, J_a(0) + \dots \tag{2.1.16}$$

The first non–trivial term determines a logarithmic divergence. Indeed, using (A.5.8) for Fourier transforming $\frac{1}{|x|^4}$

$$\frac{1}{|x|^4} \to \pi^2 \Gamma(-\epsilon) + \dots \tag{2.1.17}$$

signaling the typical logarithmic divergence in dimensional regularization. This means that after renormalization these two operators induce in the Lagrangian the scale dependent piece

$$\int d^4x\, d^2\theta\, \lambda_i W^i(x) \int d^2\bar\theta\, \bar\lambda_{\bar j} \bar W^{\bar j}(0) \sim \int d^4x\, d^4\theta\, Z(\mu,\lambda,\bar\lambda)^a\, J_a(0) \tag{2.1.18}$$

and comparing this to the OPE, one concludes that to first order in perturbation theory†

$$D^a = 2\pi^2 \frac{d}{d\,\log\mu} Z(\mu,\lambda,\bar\lambda)^a = \lambda_i\, (T^a)^{i\bar j}\, \bar\lambda_{\bar j} \tag{2.1.19}$$

This has the typical form of a D–term, or equivalently of the moment map for the action of a global symmetry acting on the space of couplings by the representation $(T^a)^{i\bar j}$, showing that indeed the obstruction to conformality is proportional to the D–term for the global symmetry current to which the divergent contribution in the effective action couples: precisely the same as we had found in the previous Section.

Moreover, this formalism offers an explicit first order expression for the non–conservation of the currents:

$$\bar D^2 J^a = \lambda^j\, (T^a)^i_j\, W^j + \dots \tag{2.1.20}$$

This allows to fix the structure of the associated β–function as follows. The deformation by the operator $h_i W^i$ produces a change in the Kaehler potential which may be absorbed into a redefinition of the coupling

$$\lambda_i \longrightarrow \lambda_i - \frac{1}{2}\lambda_j\, T_i^{a\ j}\, Z_a \tag{2.1.21}$$

to do this we have to give up the renormalization scheme where the superpotential does not renormalize, since the change (2.1.21) is not holomorphic. Then, deriving the coupling with respect to the renormalization scale, one gets the corresponding β–function, which reads

$$\begin{aligned} \beta_{\lambda_i} = \frac{\partial\,\lambda_i}{\partial\,\log\mu} &= -\frac{1}{2}\lambda_j\, T_i^{a\ j}\, D_a \\ &= -\pi^2 \left(\lambda_j\, T_i^{a\ j}\right)\left(\lambda_k\, T_a^{k\bar l}\,\bar\lambda_{\bar l}\right) + \dots \\ &\propto -g^{i\bar j}\frac{\partial}{\partial\overline{\lambda}^{\bar j}}\left(\gamma_{ab}\, D^a\, D^b\right) + \dots \end{aligned} \tag{2.1.22}$$

†The numerical factor $2\pi^2$ in front of (2.1.19) can be explained as follows: in dimensional regularization the coefficient of the $\frac{1}{\epsilon}$ pole is $-\pi^2$, from (2.1.17). As explained more thoroughly in Section 2.2.2, the procedure for computing the scale dependence in dimensional regularization accounts for a sign reversal and a factor of 2, naturally coming from the divergent contribution being quadratic in $|\lambda|$. This finally produces the coefficient $2\pi^2$.

This last expression is enlightening because the positivity of the metric $g^{i\bar{j}}$ implies that $\gamma_{ab}\, D^a\, D^b$ is non–negative around the fixed point. In the case where the bound is saturated we get another CFT, so that we are moving in a tangent direction with respect to the conformal manifold.
When the β–function is strictly positive, slightly off the original CFT, the situation can be described pictorially by Fig. 2.3

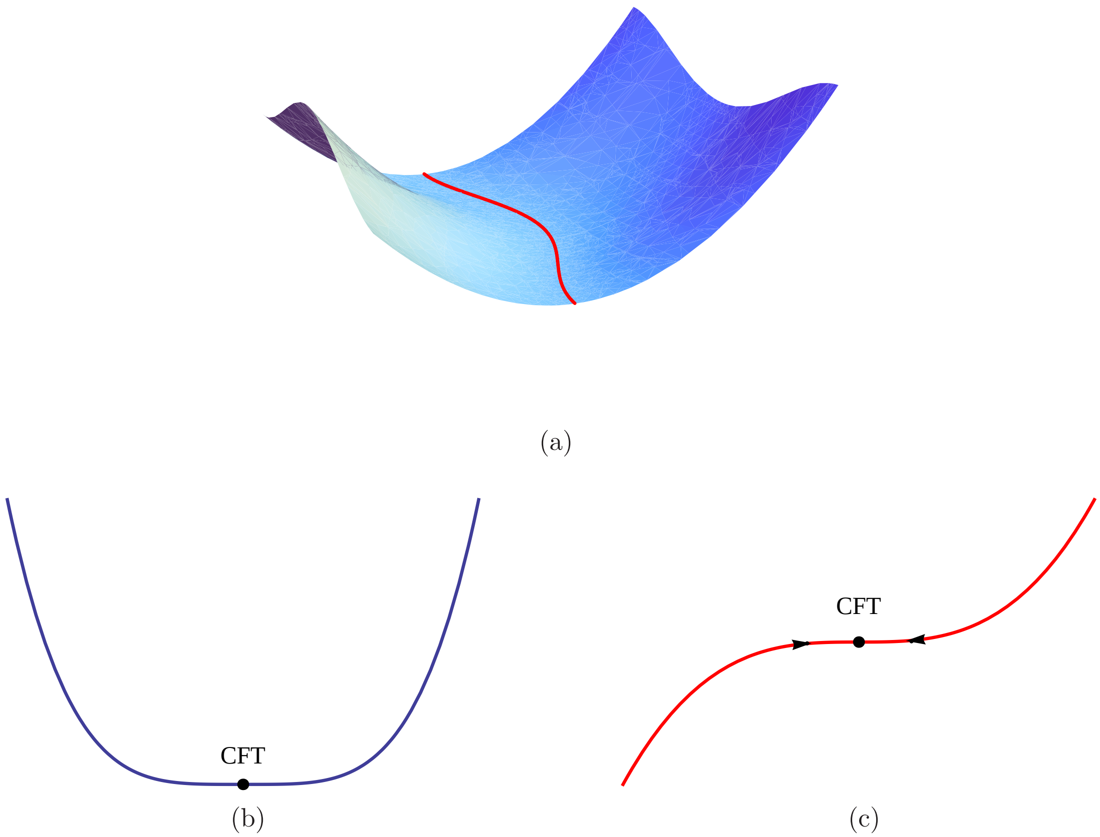


Figure 2.3: A line of fixed points is depicted in the space of D^a's (a). The fixed line lies on the locus where $D^a D_a$ vanishes. In (b) the pictorial representation of $D^a D_a$ in the vicinity of the CFT, in the transverse direction to the conformal line. In (c) the corresponding β–function as the derivative with respect to $\bar{\lambda}$. Arrows indicate the IR flow, which points back towards the CFT, meaning that the point is stable.

Here we realize that the behavior of the β–function is such that the flow towards the IR drives the theory back to the fixed point. This entails that all transverse directions to the conformal manifold are marginally irrelevant and hence attractive, namely that this surface is stable in the IR.

Summarizing, we have a simple algorithm to compute the dimension and local geometry of the conformal manifold, which is based on

1. finding the space of all supermarginals,
2. identifying the group of continuous symmetries,
3. selecting the subgroup under which no supermarginal is charged.

The conclusion is that the conformal manifold of some CFT is locally given by the quotient of its classically marginal deformations by the complexified global symmetry group. Moreover all classically marginal deformations must be either marginal or irrelevant, but never relevant.
The power of this argument is its simplicity and its validity beyond perturbation theory. Indeed the prescription holds also nonperturbatively and if the theory is strongly coupled (since it does not make use of the NSVZ formula) or even if it lacks a UV Lagrangian description.

A $4d$ example: the conformal manifold of $\mathcal{N} = 4$ SYM.

We give here just one simple example for the sake of clarity, namely we derive the dimension of the conformal manifold for $\mathcal{N} = 4$ SYM. In $\mathcal{N} = 1$ language this is a gauge theory with three chiral superfields in the adjoint representation of the gauge group. The global symmetry of the theory is $U(1)_R \times SU(3)$. We consider the theory at zero couplings. The number of superpotential couplings can be enumerated as follows: let us choose the gauge group $SU(N)$, then there exists the following cubic invariants: the antisymmetric $\mathrm{Tr}\left([T^A, T^B]\, T^C\right)$, which is the superpotential of $\mathcal{N} = 4$ SYM, and the symmetric combinations $\mathrm{Tr}\left([\{T^A, T^B\}\, T^C\right)$. The former is unique, the latter are $10 = \frac{(3+2)(3+1)3}{3!}$, as a symmetric tensor with three indices running from 1 to 3. In addition there exists the gauge term $\frac{1}{g^2}\int d^2\theta\, W^2$, which is supermarginal as well. At zero couplings the classical global symmetry group is $U(3) = U(1) \times SU(3)$. Any coupling is charged under the $U(1)$ factor, in particular so is the gauge coupling g, due to instantonic effects. Therefore (as often occurs with $U(1)$ factors) we can discard the $U(1)$ global symmetry, canceled against the gauge coupling, in the computation of supermarginals and global symmetries. Then we are left with the $\mathcal{N} = 4$ superpotential, which is a singlet under the global $SU(3)$ and the ten symmetric operators that are charged. Altogether, the $U(3)$ global symmetry is broken completely by the supermarginal. According to the prescription above, the dimension of the conformal manifold evaluates

$$\dim_{\mathbb{C}}(\mathcal{M}_c) = (1 + 1 + 10) - (1 + 8) = 3 \qquad (2.1.23)$$

and locally it can be described by

$$\mathcal{M}_c \simeq \mathbf{1}_{\mathbb{C}} + \frac{\mathbf{10}}{SU(3)_{\mathbb{C}}} = \mathbf{1}_{\mathbb{C}} + \frac{\mathbf{10}}{Sl(3, \mathbb{C})} \qquad (2.1.24)$$

2.2 The conformal manifold for the ABJM theory.

The only known examples of exactly marginal deformations were carried out for four dimensional supersymmetric theories. As discussed in [4], nothing prevents the new approach from being applied to $\mathcal{N} = 2$ supersymmetric theories in three dimensions, which have a superspace description very similar to $\mathcal{N} = 1$ models in four dimensions. Therefore we want to extend this analysis to the three dimensional case, where of course our preferred example will concern the ABJM theory. Thanks to the power of the superspace approach in computing anomalous dimensions perturbatively, we will be able to provide a complete analysis of the exactly marginal deformations of the ABJM theory, to two-loop order. We will then check that the study of the conformal manifolds addressed from symmetry arguments leads to the same conclusions as in the perturbative computation, thus providing an explicit test. Then, restricting to certain classes of deformations, we will reproduce some extension of the ABJM theory which appeared in literature before, such as beta deformed, Romans mass perturbed and flavored models. Stability of the fixed points on the conformal manifold is also studied, leading to the conclusion that the conformal surface is always stable, hence confirming the expectations from general arguments in [4]. The following discussion in mainly taken from [20].

2.2.1 Exactly marginal deformations of Chern–Simons matter theories from perturbation theory.

In this Section we determine the structure of the conformal manifold of the ABJM theory by a direct a perturbative calculation. As a byproduct, we find the explicit expression of the exactly marginal superpotential (formula (2.3.7)).

To do this we have to start by considering the most general action (1.3.1) with superpotential

$$\begin{aligned} W \;=\; \mathrm{Tr}\Big[& h_1\,(A^1B_1A^2B_2 - A^1B_2A^2B_1) + h_2\,(A^1B_1A^2B_2 + A^1B_2A^2B_1) \\ & + h_3\,A^1B_1A^1B_1 + h_4\,A^2B_2A^2B_2 + h_5\,A^1B_2A^1B_2 + h_6\,A^2B_1A^2B_1 \\ & + h_7\,A^1B_1A^1B_2 + h_8\,A^1B_1A^2B_1 + h_9\,A^2B_2A^2B_1 + h_{10}\,A^2B_2A^1B_2\Big] + h.c. \end{aligned} \tag{2.2.1}$$

which includes all classically supermarginal deformations, which are those of canonical dimension 2, i.e. quartic in the chiral superfields. This can be thought of as the most general marginal perturbation of any of the $\mathcal{N} = 2$ CSM CFT's under exam.

In order to determine perturbatively the conformal manifold of the theory we need to evaluate its renormalization group functions.

2.2.2 Two–loop renormalization and β–functions

It is well known that even in the presence of matter chiral superfields the CS actions cannot receive loop divergent corrections [21, 22, 23]. In fact, gauge invariance requires K_1, K_2 to be integers, so preventing any renormalization except for a finite shift. In particular, for the $\mathcal{N} = 2$ case it has been proved [23] that even finite renormalization is absent.

Divergent contributions are then expected only in the matter sector. Since a non–renormalization theorem still holds for the superpotential (in $\mathcal{N} = 2$ superspace perturbative calculations one can never produce local and chiral divergent contributions) divergences arise only in the Kaehler sector and lead to field functions renormalization.

In odd space–time dimensions there are no UV divergences at odd loops. Therefore, the first non trivial tests for the perturbative quantum properties of the theory arise at two loops.

We therefore perform the calculation at the first non–trivial order, that is two loops [21, 22, 23], using the $\mathcal{N} = 2$ superspace quantization [10] described in [24] and reviewed in Section 1.4.

We now carry out this program explicitly, starting from a power counting proof of the perturbative non–renormalization theorem for the superpotential in superspace Feynman diagrams, then computing the one–loop effective action and finally the two–loop divergent contributions at two loops.

Non–renormalization theorem via power counting.

We consider a superdiagram with E chiral external lines, V vertices, I internal propagators and L loops. The following relation relating these features of the graph is always true:

$$L = I - V + 1 \tag{2.2.2}$$

In the CSM theory under exam there appear several interaction vertices which we can divide into three categories:

- quartic superpotential (anti)chiral vertices V_c ($\hat{V}_c$);
- mixed gauge/matter vertices $V_{c/g} = \sum_n V_{c/g,n}$, where $V_{c/g,n}$ is the number of mixed vertices between a pair of matter fields and n gluons;
- pure gauge vertices $V_g = \sum_n V_{g,n}$, where $V_{g,n}$ is the number of interaction vertices with n gluons.

so that $V = V_c + \hat{V}_c + V_{c/g} + V_g$.
If we want a loop diagram to have chiral external lines, as in the correction to the superpotential, it should at least employ one chiral vertex. Since vertices with gauge fields do not change the chirality of a line in the diagram, the number of external chiral legs will be always given by the number of chiral lines leaving chiral vertices, which do not end onto an antichiral vertex, i.e.

$$E = 4\,V_c - 4\,\bar{V}_c \tag{2.2.3}$$

The same holds true for an antichiral vertex, with reversed roles for V_c and $\bar{V}_c$.
This sets E to be an integral multiple of 4.
The number of internal lines for such diagrams can be obtained as follows.
A diagram without gauge vertices has a number of chiral propagators I_α given by the total amount of (anti)chiral vertices multiplied by 4, minus the number of external legs

$$I_c\Big|_{\text{no gauge}} = \frac{1}{2}\,(4\,V_c + 4\,\bar{V}_c - E)$$

Now we can insert interactions with gauge fields. Since there are no gauge external fields, the number of gauge propagators I_g is given by half the total number of gauge lines leaving all vertices

$$2\,I_g = \sum_n n\,V_{g,n} \tag{2.2.4}$$

Moreover, each mixed vertex increases the number of chiral lines by 1, since it splits them into two propagators. Therefore, the number of chiral lines ought to be

$$I_c = \frac{1}{2}\,(4\,V_c + 4\,\hat{V}_c - E) + \sum V_{c/g}$$

Now the degree of divergence of a diagram can be computed as

$$\omega(G) = dL - 2\,I + M \tag{2.2.5}$$

where M is the number of loop momenta arising in the graph and d the dimension of space–time.
In supergraphs such momenta may be generated by anticommutation of covariant derivatives, so that the power of momenta can be overestimated by counting the number of spinorial derivatives. In order to compute such a quantity we recall that,

- each gauge propagator carries a factor $\bar{D}D$;
- each chiral propagator introduces $\bar{D}^2 D^2$;
- each pure gauge vertex a factor $\bar{D}D$ (independently of n);
- each (anti)chiral vertex absorbs a term $\bar{D}^2$ (D^2);
- each loop in θ–space requires one further $\bar{D}^2 D^2$ to perform D-algebra.

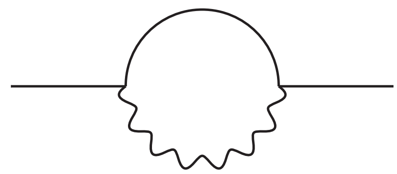


Figure 2.4: One–loop diagrams for scalar propagators.

Thus the number of internal momenta can be at most

$$M = 2I_c + I_g - 2L - 2\,\mathrm{Max}(V_c, \hat{V}_c) + V_g$$

Hereafter we will assume that the number of chiral vertices is greater than that of antichiral. Plugging this into (2.2.5) leads to

$$\omega(G) = (d-2)\,L - I_g - 2V_c + V_g \tag{2.2.6}$$

By use of (2.2.2) to substitute for L

$$\omega(G) = (d-2)(I - V + 1) - I_g - 2V_c + V_g \tag{2.2.7}$$

which in $d = 3$ and expanding $I = I_g + I_c$ and $V = V_c + \hat{V}_c + V_g + V_{c/g}$

$$\omega(G) = I_c - V_{c/g} - 2\,V_c - \hat{V}_c + 1 \tag{2.2.8}$$

Recalling the expression for the number of internal chiral lines (2.2.5) and inserting it in the latter equation the dependence on mixed vertices drops and we are left with

$$\omega(G) = 1 - \frac{1}{2}\,E - \hat{V}_c \tag{2.2.9}$$

In the case of the superpotential we may set $E = 4$, yielding $\omega(G) \leq -1$. All other holomorphic diagrams have even better UV behavior. This concludes the power counting proof that the holomorphic operators do not suffer from renormalization.

One loop results

We first compute the finite quantum corrections to the scalar and gauge propagators which then enter two–loop computations.

The only diagrams contributing to the matter field propagators are those given in Fig. 2.4. It is easy to verify that they vanish because the one–loop integral they give rise to are trivially zero for symmetry reasons.

We then move to the gauge propagator. Gauge one-loop self–energy contributions come from diagrams in Fig. 2.5 where chiral, gauge and ghost loops are present.

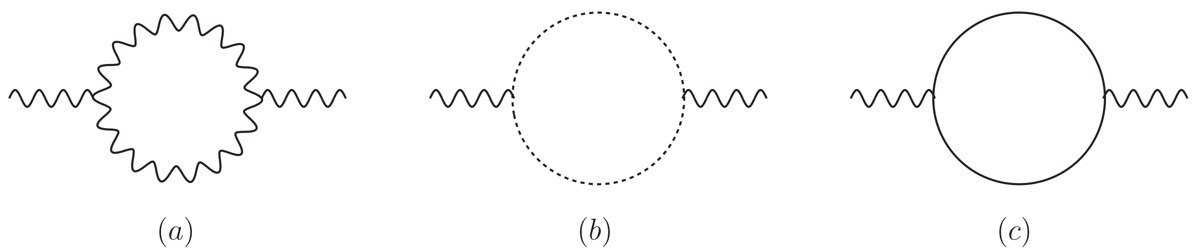


Figure 2.5: One–loop diagrams for gauge propagators.

Performing the calculation in $d = 3 - 2\epsilon$ momentum space and using the superspace projectors [11]

$$\Pi_0 \equiv -\frac{1}{k^2}\{D^2, \bar{D}^2\}(k) \quad , \quad \Pi_{1/2} \equiv \frac{1}{k^2}\bar{D}^\alpha D^2 \bar{D}_\alpha(k) \qquad\qquad \Pi_0 + \Pi_{1/2} = 1 \tag{2.2.10}$$

we find the following finite contributions to the quadratic action for the gauge fields, separating contributions from diagrams in Fig. 2.5

$$\begin{aligned}
\Pi^{(1)}_{gauge}(a) &= N\,\delta^{AA'} \int \frac{d^3k}{(2\pi)^3} d^4\theta\ B_0(k)\, k^2\, V^A(k)\, \Pi_{1/2}\, V^{A'}(-k) \\
\Pi^{(1)}_{gauge}(b) &= -\frac{1}{8}\, f^{ABC} f^{A'BC} \int \frac{d^3k}{(2\pi)^3} d^4\theta\ B_0(k)\, k^2\, V^A(k)\, \left(\Pi_0 + \Pi_{1/2}\right) V^{A'}(-k) \\
\Pi^{(1)}_{gauge}(c) &= \frac{1}{8}\, f^{ABC} f^{A'BC} \int \frac{d^3k}{(2\pi)^3} d^4\theta\ B_0(k)\, k^2\, V^A(k)\, \Pi_0\, V^{A'}(-k)
\end{aligned} \tag{2.2.11}$$

Summing them up one obtains the one–loop V gluon self–energy

$$\Pi^{(1)} = \left[-\frac{1}{8}\, f^{ABC} f^{A'BC} + M\delta^{AA'}\right] \int \frac{d^3p}{(2\pi)^3} d^4\theta\ B_0(p)\, V^A(p)\, \bar{D}^\alpha D^2 \bar{D}_\alpha\, V^{A'}(-p) \tag{2.2.12}$$

Analogously, for the $\hat{V}$ gauge boson

$$\hat{\Pi}^{(1)} = \left[-\frac{1}{8}\, \hat{f}^{ABC} \hat{f}^{A'BC} + N\delta^{AA'}\right] \int \frac{d^3p}{(2\pi)^3} d^4\theta\ B_0(p)\, \hat{V}^A(p)\, \bar{D}^\alpha D^2 \bar{D}_\alpha\, \hat{V}^{A'}(-p) \tag{2.2.13}$$

Finally a mixed V–$\hat{V}$ two–point function arises at one loop from diagram (c) only, when the Abelian $U(1)$ photons couple to bifundamental matter fields running in the loop

$$\tilde{\Pi}^{(1)} = -2\sqrt{NM}\, \delta^{A0}\, \delta^{A'0} \int \frac{d^3p}{(2\pi)^3} d^4\theta\ B_0(p)\, V^A(p)\, \bar{D}^\alpha D^2 \bar{D}_\alpha\, \hat{V}^{A'}(-p) \tag{2.2.14}$$

where

$$B_0(p) = \frac{1}{8\,(p^2)^{\frac{1}{2}+\epsilon}}$$

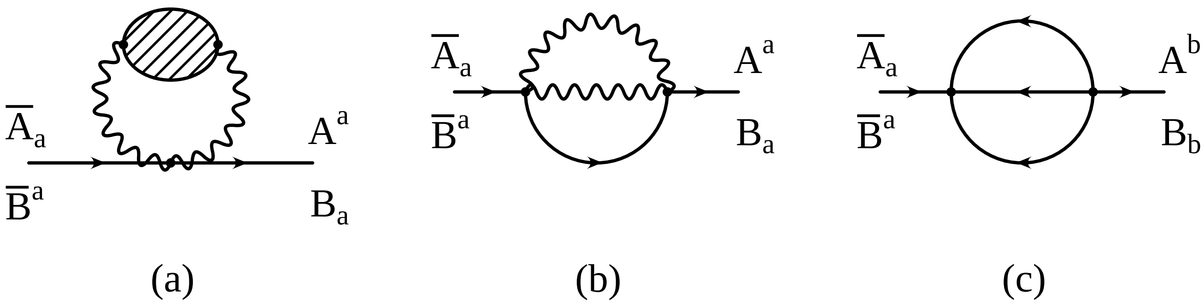


Figure 2.6: Two–loop divergent diagrams contributing to the matter propagators.

is the three dimensional bubble scalar integral (see (A.5.9)).

Summing all the contributions we see that the gauge loop cancels against part of the ghost loop as in the 4D $\mathcal{N} = 1$ case [25] and we find the known results [25, 26]

$$\begin{aligned} \Gamma^{(1)}_{gauge} &= G[1,1]\left(N - \frac{M}{4}\right)\int \frac{d^3p}{(2\pi)^3}\, d^4\theta \ \mathrm{Tr}\Big(V(p)\frac{\bar{D}^\alpha D^2 \bar{D}_\alpha}{|p|^{1+2\epsilon}}V(-p)\Big) \\ \hat{\Gamma}^{(1)}_{gauge} &= G[1,1]\left(M - \frac{N}{4}\right)\int \frac{d^3p}{(2\pi)^3}\, d^4\theta \ \mathrm{Tr}\Big(\hat{V}(p)\frac{\bar{D}^\alpha D^2 \bar{D}_\alpha}{|p|^{1+2\epsilon}}\hat{V}(-p)\Big) \end{aligned} \quad (2.2.15)$$

and $\tilde{\Pi}^{(1)}_{gauge}$ (2.2.14) which mixes the two $U(1)$ gauge vectors.

Two-loop results for the effective action.

We are now ready to evaluate the matter self–energy contributions

The only divergent graphs are self-energy diagrams for the bifundamental fields at two loops, which then contribute to their anomalous dimensions. They are given in Fig. 2.6 where the blob in the first diagram indicates the finite one–loop correction to the gauge vectors propagators 2.2.15. The computation of each diagram proceeds in the standard way by first performing D–algebra in order to reduce supergraphs to ordinary Feynman graphs and evaluate them in momentum space and dimensional regularization ($d = 3-2\epsilon$). The first diagram in Fig. 2.6 gives

$$\begin{aligned} \Pi^{(2)}(a) = -\Bigg[&\frac{(4\pi)^2}{K_1^2}\left(2MN - \frac{1}{2}\left(M^2-1\right)\right) + \\ &+\frac{(4\pi)^2}{K_2^2}\left(2MN - \frac{1}{2}\left(N^2-1\right)\right) + \frac{4(4\pi)^2}{K_1\,K_2}\Bigg]\ \mathcal{F}(0)\ \mathrm{Tr}\left(\bar{A}_i A^i + \bar{B}^i B_i\right) \end{aligned} \quad (2.2.16)$$

in terms of the two-loop $\mathcal{F}(0)$ tadpole like diagram, which we shall compute in a few lines.

The second contribution non–trivially depends on the nature of the external fields and the interaction vertices involved. In contrast to the previous and the following diagram, it allows for mixing between A and B fields separately.

In particular, we may distinguish for A fields

$$\begin{aligned}\Pi^{(2)}(b)^A_{11} =& \ \mathcal{F}(p)\ \mathrm{Tr}\left(\bar{A}_1 A^1\right)\left\{2\,MN\left(|h_1|^2+|h_2|^2\right)+2\,\left(|h_2|^2-|h_1|^2\right)+\right.\\ & \left.(MN+1)\left[4\left(|h_3|^2+|h_5|^2\right)+2|h_7|^2+|h_8|^2+|h_{10}|^2\right]\right\}\\ \Pi^{(2)}(b)^A_{11} =& \ \mathcal{F}(p)\ \mathrm{Tr}\left(\bar{A}_2 A^2\right)\left\{2\,MN\left(|h_1|^2+|h_2|^2\right)+2\,\left(|h_2|^2-|h_1|^2\right)+\right.\\ & \left.(MN+1)\left[4\left(|h_4|^2+|h_6|^2\right)+2|h_9|^2+|h_8|^2+|h_{10}|^2\right]\right\}\\ \Pi^{(2)}(b)^A_{21} =& \ \overline{\Pi^{(2)}(b)^A_{12}}\\ =& \ \mathcal{F}(p)\ \mathrm{Tr}\left(\bar{A}_2 A^1\right)2\left\{h_2\,\overline{h_9}+h_7\,\overline{h_2}+h_3\,\overline{h_8}+h_5\,\overline{h_{10}}+h_8\,\overline{h_6}+h_{10}\,\overline{h_4}\right\}\end{aligned} \tag{2.2.17}$$

where the divergent part comes from the double bubble integral $\mathcal{F}(p)$, which is written down explicitly below.
Finally diagram (c), correcting the diagonal part of the kinetic term, evaluates

$$\Pi^{(2)}(c) = -8\pi^2\left[\frac{M^2+1}{K_1^2}+\frac{N^2+1}{K_2^2}+\frac{4MN}{K_1\,K_2}\right]F(p)\ \mathrm{Tr}\left(\bar{A}_i A^i+\bar{B}^i B_i\right) \tag{2.2.18}$$

where again $\mathcal{F}(p)$ is the two–loop self–energy integral, which we now compute.

The integral $\mathcal{F}(p)$ involved for $p \neq 0$ is ultraviolet divergent and needs to be regularized by analytically continue its dimension to $d = 3-2\epsilon$, yielding

$$\begin{aligned}\mathcal{F}(p) &\equiv \int\frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}}\frac{d^{3-2\epsilon}q}{(2\pi)^{3-2\epsilon}}\ \frac{1}{k^2\,q^2\,(p-k-q)^2}\\ &= G[1,1]\int\frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}}\ \frac{1}{k^2\ \left[(p-k)^2\right]^{\frac{1}{2}+\epsilon}}\\ &= G[1,1]\,G[1,1/2+\epsilon]\,p^{-2\epsilon}\\ &= \frac{\Gamma\left(\frac{1}{2}-\epsilon\right)^3\Gamma\left(2\epsilon\right)}{(4\pi)^{3-2\epsilon}\,\Gamma\left(\frac{3}{2}-3\epsilon\right)}\ \xrightarrow{\epsilon\to 0}\ \frac{1}{64\pi^2}\,\frac{1}{\epsilon}\end{aligned} \tag{2.2.19}$$

where G is the bubble function defined in (A.5.1). In the case where $p = 0$, after performing the first integration, we get a one–loop tadpole integral which is both UV and IR divergent and as a whole evaluates to 0 in dimensional regularization. However, since we are interested here in ultraviolet divergences, we separate the former from the IR one and take its contribution to be again $\frac{1}{64\pi^2}\frac{1}{\epsilon}$ in (2.2.16), therefore concurring to the balance of UV divergences.

Two–loop divergent contributions for B fields are completely analogous to those for A and may be derived from (2.2.17), by the simple exchanges $h_5 \leftrightarrow h_6,\ h_7 \leftrightarrow h_8,\ h_9 \leftrightarrow h_{10}$.

The entire divergent contributions to the effective action can be collected as follows

$$\begin{aligned}\Gamma_{2loops} \quad \rightarrow \quad & \left(1+\frac{1}{\epsilon}\,d^A_{11}\right)\overline{A}_1\,A^1+\left(1+\frac{1}{\epsilon}\,d^A_{22}\right)\overline{A}_2\,A^2 \\ & +\left(1+\frac{1}{\epsilon}\,d^B_{11}\right)\overline{B}^1\,B_1+\left(1+\frac{1}{\epsilon}\,d^B_{22}\right)\overline{B}^2 B_2 \\ & +\frac{1}{\epsilon}\,d^A_{12}\,\overline{A}_1\,A^2+\frac{1}{\epsilon}\,d^A_{21}\,\overline{A}_2\,A^1+\frac{1}{\epsilon}\,d^B_{12}\,\overline{B}^1\,B_2+\frac{1}{\epsilon}\,d^B_{21}\,\overline{B}^2\,B_1\end{aligned} \tag{2.2.20}$$

where, calling

$$\begin{aligned}C \;\equiv\;& 2\,MN\left(|h_1|^2+|h_2|^2\right)+2\,\left(|h_2|^2-|h_1|^2\right) \\ & -(4\pi)^2\,\left(2MN+1\right)\left(\frac{1}{K_1^2}+\frac{1}{K_2^2}\right)-2\,(4\pi)^2\,\frac{MN+2}{K_1K_2}\end{aligned}$$

the coefficients of the ϵ–poles in the A sector are given by

$$\begin{aligned}d^A_{11} &= \frac{1}{64\,\pi^2}\left\{C+(MN+1)\left[4\left(|h_3|^2+|h_5|^2\right)+2|h_7|^2+|h_8|^2+|h_{10}|^2\right]\right\} \\ d^A_{21} &= \overline{d^A_{12}} = \frac{1}{32\,\pi^2}\left\{h_2\,\overline{h_9}+h_7\,\overline{h_2}+h_3\,\overline{h_8}+h_5\,\overline{h_{10}}+h_8\,\overline{h_6}+h_{10}\,\overline{h_4}\right\} \\ d^A_{22} &= \frac{1}{64\,\pi^2}\left\{C+(MN+1)\left[4\left(|h_4|^2+|h_6|^2\right)+2|h_9|^2+|h_8|^2+|h_{10}|^2\right]\right\}\end{aligned} \tag{2.2.21}$$

while the ones in the B sector are obtained again by replacing $h_5 \leftrightarrow h_6,\ h_7 \leftrightarrow h_8,\ h_9 \leftrightarrow h_{10}$. We stress that the interactions $h_7 - h_{10}$ produce mixing both in the A and B sectors due to non–vanishing d^A_{21} and d^B_{21} contributions.

Renormalization.

In this Paragraph we spell out the procedure of renormalization in dimensional regularization we will employ. We develop the subject in full generality and apply this procedure to the case under exam in the next Section.

In dimensional regularization $D = d - 2\epsilon$, we consider a theory with a set of fields Φ_a and couplings λ_i. We begin with the simplest situation where no mixing occurs at quantum level for the fields Φ_a.

In case of UV divergences we renormalize the effective action by adding suitable counterterms. These may be absorbed into a redefinition of couplings and fields, as follows. We define a renormalized coupling, multiplying by a renormalization function Z_{λ_i} for the coupling, and allowing for a renormalization scale μ, which takes care of leaving the action dimensionless

$$\lambda_i^B = \mu^{2\epsilon}\left(\lambda_i+\sum_{r=1}^{\infty}\frac{a_i^r(\lambda)}{\epsilon^r}\right) = \mu^{2\epsilon}\,\lambda_i\,Z_{\lambda_i} \tag{2.2.22}$$

We also renormalize the field wavefunctions by defining

$$\Phi_a^B\,\overline{\Phi}_a^B = \Phi_a\,\overline{\Phi}_a\,\left(1+2\sum_{r=1}^{\infty}\frac{b_a^r(\lambda)}{\epsilon^r}\right) = \Phi_a\,\overline{\Phi}_a\,Z_a \tag{2.2.23}$$

which at first order is the same as

$$\Phi_a^B = \Phi_a\,\left(1+\sum_{r=1}^{\infty}\frac{b_a^r(\lambda)}{\epsilon^r}\right) = \Phi_a\,Z_a^{\frac{1}{2}} \tag{2.2.24}$$

We define the renormalization group functions β and γ as

$$\beta_{\lambda_i} = \mu\,\frac{\partial\,\lambda_i}{\partial\,\mu} \tag{2.2.25}$$

and

$$\gamma_{\Phi_a} = \frac{1}{2}\,\mu\,\frac{\partial\,\log Z_a}{\partial\,\mu} \tag{2.2.26}$$

In order to evaluate them in dimensional regularization we start from the identity

$$\frac{1}{\mu^{2\epsilon-1}}\,\frac{\partial\,\lambda_i^B}{\partial\,\mu} = 0 \tag{2.2.27}$$

and from (2.2.22) we obtain

$$\beta_{\lambda_i} + 2\,\epsilon\,\lambda_i + 2\,a_i^1 + \sum_{r=1}^{\infty}\frac{1}{\epsilon^r}\,\left[\mu\,\frac{\partial\,\lambda_j}{\partial\,\mu}\frac{\partial\,a_i^r}{\partial\,\lambda_j} + 2\,a_i^{r+1}\right] = 0 \tag{2.2.28}$$

The function β_{λ_i} may be series expanded in ϵ

$$\beta_{\lambda_i} = d_i^0 + \epsilon\,d_i^1 + \ldots \tag{2.2.29}$$

which yields

$$\epsilon\,\left(2\,\lambda_i + d_i^1\right) + \left(d_i^0 + 2\,a_i^1 + d_j^1\frac{\partial\,a_i^1}{\partial\,\lambda_j}\right) + \sum_{r=1}^{\infty}\frac{1}{\epsilon^r}\,\left[d_j^0\frac{\partial\,a_i^r}{\partial\,\lambda_j} + d_j^1\,\frac{\partial\,a_i^{r+1}}{\partial\,\lambda_j} + 2\,a_i^{r+1}\right] = 0 \tag{2.2.30}$$

The solution of the equation above, order by order in ϵ entails

$$d_i^1 = -2\,\lambda_i \qquad d_i^0 = -2\,a_i^1 + 2\,\lambda_j\,\frac{\partial\,a_i^1}{\partial\,\lambda_j} \tag{2.2.31}$$

from which it is straightforward to read an explicit expression for the β–function

$$\beta_{\lambda_i} = -2\,\lambda_i\epsilon - 2\,a_i^1 + 2\,\lambda_j\,\frac{\partial\,a_i^1}{\partial\,\lambda_j} \tag{2.2.32}$$

Similarly for the anomalous dimensions γ one finds

$$\begin{aligned}
\gamma_{\Phi_a} &= \frac{1}{2}\,\mu\,\frac{\partial\,\log Z_a}{\partial\,\mu} \\
&= \frac{1}{2}\,\mu\,\frac{\partial}{\partial\,\mu}\,\log\left(1+\sum_{r=1}^{\infty}2\,\frac{b_a^r(\lambda)}{\epsilon^r}\right) \\
&\sim \mu\,\frac{\partial}{\partial\,\mu}\,\frac{b_a^1(\lambda)}{\epsilon} \\
&= \mu\,\frac{\partial\,\lambda_j}{\partial\,\mu}\,\frac{\partial\,b_a^1(\lambda)}{\partial\,\lambda_j}\,\frac{1}{\epsilon} \\
&= -2\,\lambda_j\,\frac{\partial\,b_a^1(\lambda)}{\partial\,\lambda_j}
\end{aligned} \tag{2.2.33}$$

where in the last step (2.2.32) was used to replace $\mu\,\frac{\partial\,\lambda_j}{\partial\,\mu}$, up to subleading in ϵ remainders.

We can easily generalize the diagonal case to a situation were mixing occurs at quantum level between the kinetic terms of the fields Π_a. The only difference from the previous case is that counterterms and the wavefunction renormalization are now matrices

$$\lambda_i^B = \mu^{2\epsilon}\left(\lambda_i+\sum_{r=1}^{\infty}\frac{a_i^r(\lambda)}{\epsilon^r}\right) = \mu^{2\epsilon}\,\lambda_i\,Z_{\lambda_i} \tag{2.2.34}$$

$$\overline{\Phi}_a^B\,\Phi_b^B = \overline{\Phi}_a\left(1_{ab}+2\sum_{r=1}^{\infty}\frac{b_{ab}^r(\lambda)}{\epsilon^r}\right)\Phi_b = Z_{ab}\,\overline{\Phi}_a\,\Phi_b \tag{2.2.35}$$

$$\beta_{\lambda_i} = \mu\,\frac{\partial\,\lambda_i}{\partial\,\mu} \tag{2.2.36}$$

Here

$$Z_{ab} \sim \delta_{ab} + \frac{2\,b_{ab}^1(\lambda)}{\epsilon} \tag{2.2.37}$$

The β–function computation is precisely as above, as for anomalous dimensions their definition is generalized to

$$\gamma_{ab} = \frac{1}{2}\,\mu\,\frac{\partial\,\log Z_{ab}}{\partial\,\mu} \tag{2.2.38}$$

where the logarithm of the matrix Z is handled by expanding it to first order $\log(1+\frac{2}{\epsilon}b)\sim\frac{2}{\epsilon}b$, which gives

$$\gamma_{ab} = \mu\,\frac{\partial\,b_{ab}^1(\lambda)}{\partial\,\mu} = -2\,\lambda_j\,\frac{\partial\,b_{ab}^1(\lambda)}{\partial\,\lambda_j} \tag{2.2.39}$$

In order to make contact with our calculation, b^1's here are related to the divergent contribution to the effective action d's in (2.2.20) by $b^1 = -\frac{1}{2}d$. We also note that the eigenvalues of the operator $\lambda_j \frac{\partial}{\partial \lambda_j}$ just count the degrees of the powers of the couplings appearing in the counterterms. This basically means that the anomalous dimensions are proportional to the coefficients of the divergences in the effective action.

For instance, since in our situation divergences are quadratic in the couplings, it follows that anomalous dimensions are just the coefficients of divergences multiplied by 2.

Supersymmetry severely constrains β–functions. Indeed it ensures that superpotential couplings are not renormalized by divergent contributions to the superpotential itself: its bare and renormalized expressions coincide $\lambda_i^B \prod_a \Phi_a^B = \lambda_i \prod_a \Phi_a$. Since the RHS has to be equal to $\lambda_i Z_{\lambda_i} \prod_a \Phi_a Z_a^{1/2}$, it holds that the couplings renormalization functions Z_{λ_i} only get contributions form the wavefunction renormalization of the fields

$$\begin{aligned} \lambda_i \, Z_{\lambda_i} &= \lambda_i \prod_a Z_a^{-\frac{1}{2}} \\ \lambda_i + \sum_{r=1}^{\infty} \frac{a_i^r(\lambda)}{\epsilon^r} &= \lambda_i \prod_a \left(1 - \sum_{r=1}^{\infty} \frac{b_a^r(\lambda)}{\epsilon^r} \right) \\ a_i^1 &= -\lambda_i \sum_a b_a^1 \end{aligned} \tag{2.2.40}$$

Hence the expression for the β–function turns out to be simply proportional to the sum over the anomalous dimensions of the fields appearing in the superpotential

$$\beta_{\lambda_i} = -2\, a_i^1 + 2\, \lambda_j \frac{\partial\, a_i^1}{\partial\, \lambda_j} = -2\, \lambda_i\, \lambda_j \frac{\partial \sum_a b_a^1}{\partial\, \lambda_j} = \lambda_i \sum_a \gamma_{\Phi_a} \tag{2.2.41}$$

where in the last equation we used (2.2.40) and (2.2.39).

2.2.3 Renormalization for $\mathcal{N} = 2$ CSM theories.

We now apply the general procedure detailed above to our case. We renormalize the effective action (2.2.20) by adding suitable counterterms and defining (the label B stands for bare fields)

$$A^a = \left(Z_{ab}^A\right)^{-\frac{1}{2}} A_B^b \qquad , \qquad B_a = \left(Z_{ab}^B\right)^{-\frac{1}{2}} B_{b\,B} \tag{2.2.42}$$

in such a way that the bare quadratic action goes back being diagonal

$$\begin{aligned} \mathcal{L}^{(2)} + \mathcal{L}^{ct} \quad = \quad & \left(1 - \frac{1}{\epsilon} d_{11}^A\right) \overline{A}_1 A^1 + \left(1 - \frac{1}{\epsilon} d_{22}^A\right) \overline{A}_2 A^2 - \frac{1}{\epsilon} d_{12}^A\, \overline{A}_1 A^2 - \frac{1}{\epsilon} d_{21}^A\, \overline{A}_2 A^1 \\ & + \left(1 - \frac{1}{\epsilon} d_{11}^B\right) \overline{B}^1 B_1 + \left(1 - \frac{1}{\epsilon} d_{22}^B\right) \overline{B}^2 B_2 - \frac{1}{\epsilon} d_{12}^B\, \overline{B}^1 B_2 - \frac{1}{\epsilon} d_{21}^B\, \overline{B}^2 B_1 \\ = \quad & \overline{A}_{1\,B}\, A_B^1 + \overline{A}_{2\,B}\, A_B^2 + \overline{B}_B^1\, B_{1\,B} + \overline{B}_B^2\, B_{2\,B} \end{aligned} \tag{2.2.43}$$

This requires the matrices Z to be

$$Z^A = 1_{2\times 2} - \frac{1}{\epsilon}\begin{pmatrix} d^A_{11} & d^A_{12} \\ d^A_{12} & d^A_{22}\end{pmatrix} \qquad Z^B = 1_{2\times 2} - \frac{1}{\epsilon}\begin{pmatrix} d^B_{11} & d^B_{12} \\ d^B_{12} & d^B_{22}\end{pmatrix} \tag{2.2.44}$$

The matrices of anomalous dimensions are defined as

$$\gamma^A_{ab} \equiv \frac{1}{2}\,\frac{d\,\log Z^A_{ab}}{d\,\log\mu} \qquad \gamma^B_{ab} \equiv \frac{1}{2}\,\frac{d\,\log Z^B_{ab}}{d\,\log\mu} \tag{2.2.45}$$

and can be evaluated explicitly by

$$\gamma^A_{ab} = \sum_k h_k\,\frac{\partial d^A_{ab}}{\partial h_k} \qquad \gamma^B_{ab} = \sum_k h_k\,\frac{\partial d^B_{ab}}{\partial h_k} \tag{2.2.46}$$

Since the divergences are all quadratic in the couplings, this yields

$$\begin{aligned}
\gamma^A_{11} &= \frac{1}{32\,\pi^2}\left\{C + (MN+1)\left[4\left(|h_3|^2+|h_5|^2\right)+2|h_7|^2+|h_8|^2+|h_{10}|^2\right]\right\}\\
\gamma^A_{21} &= \overline{\gamma^A_{12}} = \frac{1}{16\,\pi^2}\left\{h_2\,\overline{h_9}+h_7\,\overline{h_2}+h_3\,\overline{h_8}+h_5\,\overline{h_{10}}+h_8\,\overline{h_6}+h_{10}\,\overline{h_4}\right\}\\
\gamma^A_{22} &= \frac{1}{32\,\pi^2}\left\{C + (MN+1)\left[4\left(|h_4|^2+|h_6|^2\right)+2|h_9|^2+|h_8|^2+|h_{10}|^2\right]\right\}\\
\gamma^B_{11} &= \frac{1}{32\,\pi^2}\left\{C + (MN+1)\left[4\left(|h_3|^2+|h_6|^2\right)+2|h_8|^2+|h_7|^2+|h_9|^2\right]\right\}\\
\gamma^B_{21} &= \overline{\gamma^B_{12}} = \frac{1}{16\,\pi^2}\left\{h_2\,\overline{h_{10}}+h_8\,\overline{h_2}+h_3\,\overline{h_7}+h_6\,\overline{h_9}+h_7\,\overline{h_5}+h_9\,\overline{h_4}\right\}\\
\gamma^B_{22} &= \frac{1}{32\,\pi^2}\left\{C + (MN+1)\left[4\left(|h_4|^2+|h_5|^2\right)+2|h_{10}|^2+|h_7|^2+|h_9|^2\right]\right\}
\end{aligned} \tag{2.2.47}$$

The non–renormalization theorem for $\mathcal{N}=2$ supersymmetric CS theories in three dimensions [27] prevents the appearance of divergent corrections to the superpotential which then gets modified only by field function renormalization. Therefore, we renormalize the couplings as

$$h_i = \mu^{-2\epsilon}\,Z^{-1}_{h_i}\,h_{i\,B} \qquad \text{with} \qquad Z_{h_i} = \prod_{\Phi_i} Z_{\Phi_i}^{-\frac{1}{2}} \tag{2.2.48}$$

where $\Phi_i \equiv (A,\,B)$. As a consequence, the β–functions $\beta_i = \mu\,\frac{d\,h_i}{d\,\mu}$ turn out to be expressed directly in terms of the coefficients of the anomalous dimensions matrices. The calculation is straightforward and we find

$$
\begin{aligned}
\beta_{h_1} &= h_1\,\left(\gamma^A_{11}+\gamma^B_{11}+\gamma^A_{22}+\gamma^B_{22}\right)\\
\beta_{h_2} &= h_2\,\left(\gamma^A_{11}+\gamma^B_{11}+\gamma^A_{22}+\gamma^B_{22}\right)+\Big(h_7\,\overline{\gamma^A_{12}}+h_8\,\overline{\gamma^B_{12}}+h_9\,\gamma^A_{12}+h_{10}\,\gamma^B_{12}\Big)\\
\beta_{h_3} &= 2\,h_3\,\left(\gamma^A_{11}+\gamma^B_{11}\right)+h_7\,\gamma^B_{12}+h_8\,\gamma^A_{12}\\
\beta_{h_4} &= 2\,h_4\,\left(\gamma^A_{22}+\gamma^B_{22}\right)+h_9\,\overline{\gamma^B_{12}}+h_{10}\,\overline{\gamma^A_{12}}\\
\beta_{h_5} &= 2\,h_5\,\left(\gamma^A_{11}+\gamma^B_{22}\right)+h_7\,\overline{\gamma^B_{12}}+h_{10}\,\gamma^A_{12}\\
\beta_{h_6} &= 2\,h_6\,\left(\gamma^A_{22}+\gamma^B_{11}\right)+h_8\,\overline{\gamma^A_{12}}+h_9\,\gamma^B_{12}\\
\beta_{h_7} &= h_7\,\left(2\gamma^A_{11}+\gamma^B_{11}+\gamma^B_{22}\right)+2\,\Big(h_3\,\overline{\gamma^B_{12}}+h_2\,\gamma^A_{12}+h_5\,\gamma^B_{12}\Big)\\
\beta_{h_8} &= h_8\,\left(\gamma^A_{11}+\gamma^A_{22}+2\gamma^B_{11}\right)+2\,\Big(h_3\,\overline{\gamma^A_{12}}+h_2\,\gamma^B_{12}+h_6\,\gamma^A_{12}\Big)\\
\beta_{h_9} &= h_9\,\left(2\gamma^A_{22}+\gamma^B_{11}+\gamma^B_{22}\right)+2\,\Big(h_4\,\gamma^B_{12}+h_2\,\overline{\gamma^A_{12}}+h_6\,\overline{\gamma^B_{12}}\Big)\\
\beta_{h_{10}} &= h_{10}\,\left(\gamma^A_{11}+\gamma^A_{22}+2\gamma^B_{22}\right)+2\,\Big(h_4\,\gamma^A_{12}+h_2\,\overline{\gamma^B_{12}}+h_5\,\overline{\gamma^A_{12}}\Big)
\end{aligned}
\qquad (2.2.49)
$$

We see that due to the mixing of the anomalous dimensions, the β–functions are not diagonal in the coupling constants.

The set of fixed points of the theory, that is its conformal manifold, is determined by setting $\beta_{h_i}=0$. In order to work out these constraints, it is convenient to diagonalize the β–functions (2.2.49). In terms of the eigenvectors h'_i the diagonal β–functions read $\beta'_{h_i}=\lambda_i\,h'_i$, where the eigenvalues λ_i are

$$
\begin{aligned}
\lambda_1 &= \lambda_2\equiv\lambda\\
\lambda_{3,4} &= \lambda\pm\sqrt{\mu_A}\\
\lambda_{5,6} &= \lambda\pm\sqrt{\mu_B}\\
\lambda_{7,8,9,10} &= \lambda\pm\sqrt{\mu_A+\mu_B\pm2\sqrt{\mu_A\mu_B}}
\end{aligned}
$$

where

$$
\lambda=\gamma^A_{11}+\gamma^A_{22}+\gamma^B_{11}+\gamma^B_{22} \qquad (2.2.50)
$$

$$
\mu_A = (\gamma^A_{11}-\gamma^A_{22})^2+4\,\gamma^A_{12}\,\overline{\gamma^A_{12}} \quad , \quad \mu_B=(\gamma^B_{11}-\gamma^B_{22})^2+4\,\gamma^B_{12}\,\overline{\gamma^B_{12}}
$$

Therefore, the set of conditions $\beta_{h_i}=0$ is equivalent to the set $\lambda_i=0$ which implies

$$
\gamma^A_{11}=\gamma^A_{22}\quad,\quad\gamma^B_{11}=\gamma^B_{22}\quad,\quad\gamma^A_{12}=\gamma^B_{12}=0\quad,\quad\gamma^A_{11}+\gamma^B_{11}=0 \qquad (2.2.51)
$$

Since the diagonal coefficients of the anomalous dimensions matrices are real, while the off-diagonal ones are complex (see eq. (2.2.47)), we have obtained seven real conditions. In addition, it is easy to see that by a field redefinition we can remove the phases of seven couplings, as will be detailed more thoroughly below in an explicit example. All together

there are seven complex constraints to be imposed on ten complex coupling constants. According to [3], this means that the conformal manifold of $\mathcal{N}=2$ Chern–Simons matter theories has complex dimension three. In particular, for $K_1 = -K_2$ in (2.6.1) this is the manifold of conformal fixed points connected to the ABJ(M) model by exactly marginal deformations.
In Section 2.3 we shall check this prediction against the dimension of the quotient $\{\lambda\}/G_{\mathbb{C}}$, computed from symmetry arguments.

We look for explicit solutions of the constraints (2.2.51) in the space of the coupling constants in order to determine the expression of the exactly marginal perturbations.

The simplest case is obtained by setting $h_i = 0\ \forall i > 6$. This corresponds to the class of models investigated in [24]. At this locus the off–diagonal elements of the anomalous dimensions matrices vanish. The explicit solution to (2.2.51) in terms of the remaining coupling constants reads

$$\begin{aligned} & |h_3| = |h_4|, \quad |h_5| = |h_6|, \\ & (MN+1)\,\left[|h_2|^2 + 2\,|h_3|^2 + 2\,|h_5|^2\right] + (MN-1)\,|h_1|^2 = X \end{aligned} \tag{2.2.52}$$

where X is coupling independent and given by

$$X = (4\pi)^2\,\left[\frac{MN+2}{K_1K_2} + \frac{2MN+1}{2}\left(\frac{1}{K_1^2}+\frac{1}{K_2^2}\right)\right]$$

By a field redefinition we can fix three phases: Setting $h_i = |h_i|e^{i\phi_i}$, we can for instance choose $\phi_1 = 0$, $\phi_3 = \phi_4$ and $\phi_5 = \phi_6$. This may be achieved, for example, by rotating the phases of the fields A_1 and B_1, by

$$\begin{aligned} \phi_{A_1} &\to \frac{1}{4}\left(-\phi_{h_3} + \phi_{h_4} - \phi_{h_5} + \phi_{h_6} + 4\phi_{A_2}\right) \\ \phi_{B_1} &\to \frac{1}{4}\left(-\phi_{h_3} + \phi_{h_4} + \phi_{h_5} - \phi_{h_6} + 4\phi_{B_2}\right) \end{aligned} \tag{2.2.53}$$

This shift may be reabsorbed into a redefinition of the couplings' phases

$$\begin{aligned} \phi'_{h_1} &= \phi_{h_1} - \frac{1}{2}\phi_{h_3} + \frac{1}{2}\phi_{h_4} + 2\phi_{A_2} + 2\phi_{B_2} \\ \phi'_{h_2} &= \phi_{h_2} - \frac{1}{2}\phi_{h_3} + \frac{1}{2}\phi_{h_4} + 2\phi_{A_2} + 2\phi_{B_2} \\ \phi'_{h_3} &= \phi_{h_4} + 2\phi_{A_2} + 2\phi_{B_2} \\ \phi'_{h_4} &= \phi_{h_4} + 2\phi_{A_2} + 2\phi_{B_2} \\ \phi'_{h_5} &= -\frac{1}{2}\left(\phi_{h_5} + \phi_{h_6} - \phi_{h_3} + \phi_{h_4}\right) + 2\phi_{A_2} + 2\phi_{B_2} \\ \phi'_{h_6} &= \frac{1}{2}\left(\phi_{h_5} + \phi_{h_6} - \phi_{h_3} + \phi_{h_4}\right) + 2\phi_{A_2} + 2\phi_{B_2} \end{aligned} \tag{2.2.54}$$

This shift sets the phases of h_3 and h_4 equal to each other, as well as those of h_5 and h_6, as desired. The remaining freedom in redefining phases can be used e.g. to shift either that of h_1 or that of h_2 to zero, by poperly tuning the phase of A_2 or B_2. Note that since the combination $2\phi_{A_2} + 2\phi_{B_2}$ is common to all the new phases in (2.2.54) this last shift amounts to modding out all of them by a common angle.

It follows that the conformal manifold is parameterized by three complex parameters, for instance h_2, h_3 and h_5. Its parametric equations may be written as

$$\begin{aligned} & h_2 = t, \quad h_3 = h_4 = u, \quad h_5 = h_6 = v, \\ & |h_1|^2 = \frac{1}{MN-1} \left\{ X - (MN+1) \left[|t|^2 + 2\,|u|^2 + 2\,|v|^2 \right] \right\} \\ & h_7 = h_8 = h_9 = h_{10} = 0 \end{aligned} \tag{2.2.55}$$

The precise geometry of the conformal manifold (up to second order in perturbation theory) is as follows: the algebraic equation in (2.2.55) produces a seven–dimensional ellipsoid embedded in $\mathbb{C}^4 = \{h_1, h_2, h_3, h_5\}$. This is isomorphic to an S^7. When we fix the last free phase setting h_1 to zero, we are actually taking a convenient representative in the orbit of a $U(1)$ symmetry which rotates all phases of h_1, h_2, h_3 and h_5 by the same amount. This is equivalent to taking the quotient

$$\mathcal{M}_c = \frac{S^7}{U(1)} = \mathbb{CP}^3 \tag{2.2.56}$$

Therefore we conclude that at two–loop order, the conformal manifold is *globally* isomorphic to $\mathbb{CP}^3$. Some comments on this result are in order

1. The reality of the solution for h_1 requires the $(t,\, u,\, v)$ moduli not to be arbitrarily large, being subject to the condition

$$\left[|t|^2 + 2\,|u|^2 + 2\,|v|^2 \right] \leq (4\pi)^2 \left[\frac{MN+2}{MN+1} \frac{1}{K_1 K_2} + \frac{2MN+1}{2(MN+1)} \left(\frac{1}{K_1^2} + \frac{1}{K_2^2} \right) \right]$$

   Since the limiting value is of order $1/K_{1,2}$, this condition also guarantees that in the large $K_{1,2}$ limit the results are within the range of the perturbative analysis. This is a self–consistency check on the validity of our approach.

2. We have found that the conformal manifold is a compact surface for any $\mathcal{N} = 2$ CS matter theory in three dimensions. In particular, it is true for the ABJ(M) theory and for $SU(2)$ invariant models, including the $\mathcal{N} = 3$ theory.
   This renders the three dimensional models strikingly different from analogous models in four dimensions, where the exactly marginal couplings are not usually constrained [3].

3. The conformal manifold is Kaehler, confirming expectations on general grounds, sketched at the end of Section 2.1.2.

4. The solution (2.2.55) leads to an exactly marginal superpotential explicitly determined at two loops as

$$\begin{aligned} W_{\text{ex mar}} &= \mathrm{Tr}\Big[t\,\left(A^1\,B_1\,A^2\,B_2 + A^1\,B_2\,A^2\,B_1\right) + u\,\left(A^1\,B_1\,A^1\,B_1 + A^2\,B_2\,A^2\,B_2\right) \\ &\quad + \frac{\sqrt{X-(MN+1)\,(|t|^2+2\,|v|^2+2\,|u|^2)}}{\sqrt{MN-1}}\,\left(A^1\,B_1\,A^2\,B_2 - A^1\,B_2\,A^2\,B_1\right) \\ &\quad + v\,\left(A^1\,B_2\,A^1\,B_2 + A^2\,B_1\,A^2\,B_1\right)\Big] \,+\, h.c. \end{aligned} \tag{2.2.57}$$

Exactly marginal operators for the ABJ(M) from perturbation theory. We focus on the ABJ(M) model and look for its exactly marginal deformations.

We start from the parametric equations (2.2.55). Locally, at the ABJ(M) fixed point the exactly marginal directions are given by a basis of operators on the tangent space with respect to the conformal surface $(\partial_{t,u,v}h_i)$

$$\left(1,1,0,-\frac{\left(2\sqrt{2}K_1K_2(1+MN)|h_2|\right)/\sqrt{(-1+MN)}}{\sqrt{2K_1K_2(2+MN)+(1+2MN)(K_1^2+K_2^2)-2K_1^2K_2^2(MN+1)\,(2|h_2|^2+2|h_6|^2+|y_1|^2)}},0,0\right)$$

$$\left(0,0,1,-\frac{\left(\sqrt{2}K_1K_2(1+MN)|y_1|\right)/\sqrt{(-1+MN)}}{\sqrt{2K_1K_2(2+MN)+(1+2MN)(K_1^2+K_2^2)-2K_1^2K_2^2(MN+1)\,(2|h_2|^2+2|h_6|^2+|y_1|^2)}},0,0\right)$$

$$\left(0,0,0,-\frac{\left(2\sqrt{2}K_1K_2(1+MN)|h_6|\right)/\sqrt{(-1+MN)}}{\sqrt{2K_1K_2(2+MN)+(1+2MN)(K_1^2+K_2^2)-2K_1^2K_2^2(MN+1)\,(2|h_2|^2+2|h_6|^2+|y_1|^2)}},1,1\right)$$

Plugging in the data of the ABJ(M) fixed point we find the following basis of vectors

$$(1,1,0,0,0,0), \quad (0,0,1,0,0,0), \quad (0,0,0,0,1,1)$$

which corresponds to the operators

$$\begin{aligned} &\mathrm{Tr}\left(A^1\,B_1\,A^1\,B_1 + A^2\,B_2\,A^2\,B_2\right) \\ &\mathrm{Tr}\left(A^1\,B_1\,A^2\,B_2 + A^1\,B_2\,A^2\,B_1\right) \\ &\mathrm{Tr}\left(A^1\,B_2\,A^1\,B_2 + A^2\,B_1\,A^2\,B_1\right) \end{aligned} \tag{2.2.58}$$

whereas the ABJM operator, which is transverse to the surface, is an irrelevant perturbation, matching the statement [4] that marginal deformations of such superCFT's can be either exactly marginal or marginally irrelevant, but never relevant. The same holds true for all points belonging to the conformal manifold, which are therefore IR attractors.

Remarks. As mentioned before, different solutions to the equations (2.2.51) may be found, corresponding to different conformal manifolds. For instance, we can find another set of solutions by setting $h_3 = h_4 = h_5 = h_6 = 0$. It is easy to see that, again, we obtain a complex three dimensional manifold. It is related to the previous one by a $SO(4)$ rotation of the superpotential. Since this is in general the case, the exactly marginal deformations (2.3.6) are to be regarded as representatives of the equivalence classes of the quotient.

Finally, we prove that all the theories belonging to the conformal manifold are two–loop finite, having vanishing anomalous dimensions. Indeed, from (2.2.51) we read that on the conformal manifold $\gamma_{11}^A = \gamma_{22}^A$ and $\gamma_{11}^B = \gamma_{22}^B$. Using the explicit expressions for the anomalous dimensions given in (2.2.47), these constraints give

$$\begin{aligned} 4\left(|h_3|^2 - |h_4|^2\right) &= |h_9|^2 + |h_{10}|^2 - |h_7|^2 - |h_8|^2 \\ 4\left(|h_5|^2 - |h_6|^2\right) &= |h_8|^2 + |h_9|^2 - |h_7|^2 - |h_{10}|^2 \end{aligned}$$

Plugging these constraints into the two–loop anomalous dimensions (2.2.47) we obtain that not only the matrices of anomalous dimensions are proportional to the identity, but also $\gamma_{11}^A = \gamma_{22}^A = \gamma_{11}^B = \gamma_{22}^B = \gamma$. On the other hand, the last condition in (2.2.51) states that their sum $\gamma_{11}^A + \gamma_{22}^A + \gamma_{11}^B + \gamma_{22}^B = 4\gamma$ should be zero, meaning that all anomalous dimensions are vanishing. We conclude that all theories belonging to the conformal manifold are two–loop finite. A special case is the theory obtained by turning off the superpotential. This model is trivially scale invariant since there are no running couplings. However the anomalous dimensions are non–vanishing, because the bifundamental fields have to be renormalized due to divergent contributions to the effective action coming from gauge interactions. This theory is the UV fixed point of the RG flow. In particular the anomalous dimensions are strictly negative at this fixed point, as can be inferred from (2.2.47). This means that adding a small superpotential deformation $h\mathcal{O}$ the operator $\mathcal{O}$ is relevant and will drive the theory to an IR fixed point, which is precisely what happens for the ABJ(M) model and the other $\mathcal{N} = 2$ CFT's in general.

2.3 Exactly marginal deformations of Chern–Simons matter theories from global symmetry breaking.

In this Section we aim at re–deriving the results from our perturbative analysis, by the general procedure based on global symmetries, explained above. We stress that this method was developed for $\mathcal{N} = 1$ theories in four dimensions. Its extension to three dimensional $\mathcal{N} = 2$ models looks straightforward, but no explicit example has been offered to validate this belief. Here we successfully test it for the ABJM and its $\mathcal{N} = 2$ generalizations.

Sticking to the algorithm outlined in Section 2.1.3 we want to detect the spectrum of supermarginal operators and identify the broken generators of the global symmetry group they break.

For $\lambda = 0$ the classical global symmetry of the theory is $U(2)_A \times U(2)_B$ where the two groups act independently on the A and B doublets. The diagonal $U(1)$ is the baryonic symmetry under which the bifundamentals (A^1, A^2, B_1, B_2) have charges (1, 1, −1, −1). It is gauged into the $U(M) \times U(N)$ gauge group, and therefore it is conserved by any gauge invariant superpotential. The axial $U(1)$ acts on the bifundamental matter with charges (1, 1, 1, 1).

	$SU(2)$	$SU(2)$	$U(1)_b$	$U(1)_a$
A^1	**2**	1	1	1
A^2	**2**	1	1	1
B_1	1	**2**	-1	1
B_2	1	**2**	-1	1

Table 2.1: The behavior of bifundamental fields under the naive global symmetry group.

The addition of the marginal perturbation $\epsilon_{ab}\,\epsilon^{cd}\,\mathrm{Tr}\left(A^a B_c A^b B_d\right)$ breaks the axial $U(1)$ and leads the theory to the infrared fixed point given by [24]

$$\lambda = \sqrt{\frac{2MN+1}{2(MN-1)}\left(\frac{1}{K_1^2}+\frac{1}{K_2^2}\right)+\frac{MN+2}{MN-1}\frac{1}{K_1K_2}} \qquad (2.3.1)$$

where only the $SU(2)_A \times SU(2)_B$ global symmetry survives. For $K_1 = -K_2 \equiv K$ it reduces to the ABJ(M) fixed point $\lambda = 1/K$ [2, 12] where supersymmetry gets enhanced to $\mathcal{N} = 6$.

In order to determine the structure of the exactly marginal deformations we have to find the class of gauge invariant supermarginal operators. For $\mathcal{N} = 2$ supersymmetric theories in three dimensions classical marginal perturbations are quartic in the bifundamental fields and constrained by gauge symmetry to have the form $\lambda_{ab}^{cd}\,\mathrm{Tr}(A^a B_c A^b B_d)$, where λ_{ab}^{cd} is a tensor of $SU(2) \times SU(2)$. For $\lambda_{ab}^{cd} = 1/2\,\epsilon_{ab}\,\epsilon^{cd}$ we find the ABJ(M) superpotential which preserves the $SU(2) \times SU(2)$ global symmetry. In addition, there are 9 operators transforming in the $(\mathbf{3},\mathbf{3})$ representation of $SU(2) \times SU(2)$ which have symmetric upper and lower indices, separately. Explicitly, they are

$$\begin{aligned}
&\mathrm{Tr}\left(A^1 B_1 A^2 B_2 + A^1 B_2 A^2 B_1\right), \\
&\mathrm{Tr}\left(A^1 B_1 A^1 B_1\right), \quad \mathrm{Tr}\left(A^2 B_2 A^2 B_2\right), \\
&\mathrm{Tr}\left(A^1 B_2 A^1 B_2\right), \quad \mathrm{Tr}\left(A^2 B_1 A^2 B_1\right), \\
&\mathrm{Tr}\left(A^1 B_1 A^1 B_2\right), \quad \mathrm{Tr}\left(A^1 B_1 A^2 B_1\right), \quad \mathrm{Tr}\left(A^1 B_2 A^2 B_2\right), \quad \mathrm{Tr}\left(A^2 B_1 A^2 B_2\right)
\end{aligned} \qquad (2.3.2)$$

All together they break $SU(2) \times SU(2) \times U(1)_{axial}$ completely.

The gauge sector has no marginal operators since the Chern–Simons term is not gauge invariant. On the other hand, Yang–Mills contributions cannot be added, being them dimensionful.

In order to investigate the structure of the conformal manifold for this kind of theories, we perturb the action (2.6.1) at its conformal ABJ(M) point by adding the marginal chiral operators (2.3.2). In the class of marginal perturbations we do not include the ABJM superpotential $\epsilon_{ab}\,\epsilon^{cd}\,\mathrm{Tr}(A^a B_c A^b B_d)$ since at the IR fixed point it is irrelevant. In fact, since this operator leads the theory from the UV fixed point (free theory) to the non–trivial IR point, it is marginal but not exactly marginal. As discussed in [4], it becomes irrelevant by pairing with the short current multiplet associated to the axial $U(1)$ global symmetry broken by the operator itself. This will be checked perturbatively in Section 2.4.

According to the general prescription of [4, 5], the conformal manifold is given locally by the symplectic quotient $\{\lambda\}/G_{\mathbb{C}}$, where $\{\lambda\}$ is the set of all marginal perturbations of the CFT and $G_{\mathbb{C}}$ is the complexified global (continuous) symmetry group.

In our case $\{\lambda\}$ is given by the set (2.3.2), whereas the global symmetry group at the IR fixed point is $SU(2) \times SU(2)$. We can thus write

$$\mathcal{M}_c = \frac{(\mathbf{3},\mathbf{3})}{(SU(2)\times SU(2))_{\mathbb{C}}} \tag{2.3.3}$$

and the complex dimension of the conformal manifold is $9-6=3$. This means that there are 3 exactly marginal operators which we now identify.

First of all, the most general linear combination of marginal chirals (2.3.2) can be written as the matrix product [28]

$$W = M_{ij}\, w_i\, w_j \tag{2.3.4}$$

where the vectors w are defined as

$$\begin{aligned} w_1 &= \frac{1}{2}(A^1\,B_1 + A^2\,B_2) \\ w_2 &= \frac{i}{2}(A^1\,B_1 - A^2\,B_2) \\ w_3 &= \frac{i}{2}(A^1\,B_2 + A^2\,B_1) \\ w_4 &= \frac{1}{2}(A^1\,B_2 - A^2\,B_1) \end{aligned} \tag{2.3.5}$$

and M is a symmetric traceless 4×4 matrix. The combinations which survive the quotient (2.3.3) are obviously the diagonal ones. Out of them we can find three independent chiral primary operators

$$\begin{aligned} &\mathrm{Tr}\left(A^1\,B_1\,A^1\,B_1 + A^2\,B_2\,A^2\,B_2\right) \\ &\mathrm{Tr}\left(A^1\,B_1\,A^2\,B_2 + A^1\,B_2\,A^2\,B_1\right) \\ &\mathrm{Tr}\left(A^1\,B_2\,A^1\,B_2 + A^2\,B_1\,A^2\,B_1\right) \end{aligned} \tag{2.3.6}$$

Therefore, including the original ABJM superpotential term, we conclude that the exactly marginal superpotential should have the form

$$
\begin{aligned}
&u\,\mathrm{Tr}\left(A^1\,B_1\,A^1\,B_1+A^2\,B_2\,A^2\,B_2\right)+f(t,u,v)\,\mathrm{Tr}\left(A^1\,B_1\,A^2\,B_2-A^1\,B_2\,A^2\,B_1\right)\\
&t\,\mathrm{Tr}\left(A^1\,B_1\,A^2\,B_2+A^1\,B_2\,A^2\,B_1\right)+v\,\mathrm{Tr}\left(A^1\,B_2\,A^1\,B_2+A^2\,B_1\,A^2\,B_1\right)
\end{aligned}
\qquad (2.3.7)
$$

where f is a solution of the D–term conditions [4]. The explicit form of the function $f(t,u,v)$ which describes the conformal manifold has been determined at two–loops in the Section 2.2.1, equation (2.2.57).

The same analysis can be applied to other remarkable fixed points introduced in [13] and perturbatively determined in [29, 24].

We consider a $\mathcal{N}=2$ theory described by the action (2.6.1) with superpotential

$$
W=\int d^3x\,d^2\theta\;\;\mathrm{Tr}\left[c_1\,(A^a\,B_a)^2+c_2\,(B_a\,A^a)^2\right]\;+\;h.c. \qquad (2.3.8)
$$

For real couplings, the equation

$$
c_1^2+c_2^2+2\,\frac{MN+2}{2MN+1}\,c_1\,c_2=(4\pi)^2\,\left[\frac{1}{4K_1^2}+\frac{1}{4K_2^2}+\frac{MN+2}{2\,K_1\,K_2\,(2MN+1)}\right] \qquad (2.3.9)
$$

describes a line of fixed points [24]. For particular values $c_1=\frac{2\pi}{K_1}$ and $c_2=\frac{2\pi}{K_2}$ supersymmetry gets enhanced to $\mathcal{N}=3$ [13].

We focus on a given point on this line and study the conformal manifold perturbing around it with all classical marginal operators. This time the superpotential preserves a diagonal global $SU(2)$ subgroup, out of the original $SU(2)\times SU(2)\times U(1)_{axial}$ global symmetry of the free theory. Therefore, four classically marginal operators have become irrelevant at the IR fixed point by coupling to the broken currents of the four broken generators. The set of supermarginals is thus reduced to six operators. The dimension of the conformal manifold is then given by

$$
\dim\left(\frac{\{6\text{ operators}\}}{SU(2)_{\mathbb{C}}}\right)=6-3=3 \qquad (2.3.10)
$$

2.4 Currents and OPE

In this Section we study the OPE algebra of marginal chiral operators and global currents at the first non–trivial perturbative order. We show that the constraints for vanishing β–functions previously found are in one to one correspondence with those for the vanishing of the D–terms for global symmetries [4, 5].

Given the set of marginal chiral primary operators $\{\mathcal{O}_i\}$ as appearing in the superpotential (2.2.1) ($\mathcal{O}_i$ is the operator associated to h_i) the OPE algebra has the general structure

$$\mathcal{O}_i(x)\,\overline{\mathcal{O}_j}(0) = \frac{g_{i\bar{j}}}{|x|^4} + \frac{T^a_{i\bar{j}}}{|x|^3}\,J_a + \ldots \tag{2.4.1}$$

where $g_{i\bar{j}}$ is the Zamolodchikov metric and J_a the global symmetry currents. Using the chiral propagators in coordinate space

$$\langle\, A^a(x)\,\bar{A}_b(0)\,\rangle = \frac{1}{4\,\pi}\,\frac{1}{|x|}\,\delta^a_b \qquad , \qquad \langle\, B_a(x)\,\bar{B}^b(0)\,\rangle = \frac{1}{4\,\pi}\,\frac{1}{|x|}\,\delta^b_a \tag{2.4.2}$$

it is easy to determine the metric on the couplings space at the lowest perturbative order

$$g_{i\bar{j}} = \frac{MN(MN+1)}{(4\pi)^4}\,\mathrm{diag}\left(2\,\frac{1-MN}{MN+1},\, 2,\, 2,\, 2,\, 2,\, 2,\, 1,\, 1,\, 1,\, 1\right) \tag{2.4.3}$$

We note that it turns out to be diagonal, thanks to the particular choice of $\mathcal{O}_i$ in (2.2.1).

Performing three contractions we obtain the contributions of order $1/|x|^3$. They turn out to be expressed in terms of the global symmetries currents

$$\begin{aligned}
&J_{U(1)} = \overline{A}_1\,A^1 + \overline{A}_2\,A^2 + \overline{B}^1\,B_1 + \overline{B}^2\,B_2\\
J^A_+ &= \overline{A}_1\,A^2 \quad , \quad J^A_- = \overline{A}_2\,A^1 \quad , \quad J^A_3 = \overline{A}_1\,A^1 - \overline{A}_2\,A^2\\
J^B_+ &= \overline{B}^1\,B_2 \quad , \quad J^B_- = \overline{B}^2\,B_1 \quad , \quad J^B_3 = \overline{B}^1\,B_1 - \overline{B}^2\,B_2
\end{aligned} \tag{2.4.4}$$

where $J_{U(1)}$ is the current of the global axial $U(1)$ under which all chiral superfields have unit charge, whereas J^A_i and J^B_i are the currents associated to the generators of $SU(2)_A$ and $SU(2)_B$, where in particular $J_\pm = J_1 \pm i\,J_2$.

By direct inspection it is easy to obtain the $T^a_{i\bar{j}}$ coefficients in (2.4.1). Up to an overall $1/(4\pi)^3$, they are given by

$$\begin{aligned}
&T^{U(1)}_{11} = (1-MN) \qquad T^{U(1)}_{22} = 2\,(MN+1)\\
&T^{U(1)}_{33} = T^{J^A_3}_{33} = T^{J^B_3}_{33} = 2\,(MN+1) \qquad T^{U(1)}_{44} = -T^{J^A_3}_{44} = -T^{J^B_3}_{44} = 2\,(MN+1)\\
&T^{U(1)}_{55} = T^{J^A_3}_{55} = -T^{J^B_3}_{55} = 2\,(MN+1) \qquad T^{U(1)}_{66} = -T^{J^A_3}_{66} = T^{J^B_3}_{66} = 2\,(MN+1)\\
&T^{U(1)}_{77} = T^{J^A_3}_{77} = (MN+1) \qquad T^{U(1)}_{88} = T^{J^B_3}_{88} = (MN+1)\\
&T^{U(1)}_{99} = -T^{J^A_3}_{99} = (MN+1) \qquad T^{U(1)}_{10\,10} = -T^{J^B_3}_{10\,10} = (MN+1)\\
&T^{J^A_+}_{27} = T^{J^A_+}_{4\,10} = T^{J^A_+}_{68} = T^{J^A_+}_{83} = T^{J^A_+}_{92} = T^{J^A_+}_{10\,5} = 2\,(MN+1)\\
&T^{J^B_+}_{28} = T^{J^B_+}_{49} = T^{J^B_+}_{57} = T^{J^B_+}_{73} = T^{J^B_+}_{96} = T^{J^B_+}_{10\,2} = 2\,(MN+1)\\
&T^{J^A_-}_{ij} = T^{J^A_+}_{ji} \qquad T^{J^B_-}_{ij} = T^{J^B_+}_{ji}
\end{aligned} \tag{2.4.5}$$

Analogously, we can compute the OPE between two gauge currents. It has the general structure

$$J_M(x)\,J_N(0) \,\sim\, \frac{1}{|x|^3}\,\Gamma_{MN}\,k^a\,J_a + \cdots \tag{2.4.6}$$

where J_a are global currents.

In the present case, taking the two gauge currents $J_A = \bar{A}_a e^V A^a e^{-\hat{V}}$ and $J_B = \bar{B}^a e^V B_a e^{-\hat{V}}$, the only non–trivial contribution at order $1/|x|^3$ is proportional to $J_{U(1)}$ with coefficient

$$k^{U(1)} = -(4\pi)^2 (2MN+1) \left(\frac{1}{K_1^2} + \frac{1}{K_2^2} \right) - 2 \frac{MN+2}{K_1 K_2} \tag{2.4.7}$$

This comes from two different contributions: The first one corresponds to expanding J_A and J_B at second order in the gauge prepotentials and contracting one chiral and two gauge superfields. The second contribution comes from normal–ordering the currents and corresponds to contracting two gauge superfields inside a single current with a one–loop corrected propagator.

Since the contraction of two gauge currents as well as of two chiral supermarginals give a logarithmically singular contribution to the Kaehler potential of the form $\int d^4\theta Z^a(\mu) J_a$, the D–terms for the global symmetries have the general form [4]

$$D^a \equiv \mu \frac{\partial}{\partial \mu} Z^a(\mu) \sim h^i T^a_{i\bar{j}} \overline{h^j} + k^a \tag{2.4.8}$$

Vanishing D-term conditions insure independence of the effective action of the energy scale and then its superconformal invariance.

In the present case, using the explicit results (2.4.5, 2.4.7) we find

$$\begin{aligned}
D_{U(1)} &\sim (MN+1) \big(2|h_2|^2 + 2|h_3|^2 + 2|h_4|^2 + 2|h_5|^2 + 2|h_6|^2 + |h_7|^2 + \\
&\quad |h_8|^2 + |h_9|^2 + |h_{10}|^2\big) - 2(MN-1)|h_1|^2 + k^{U(1)} \\
D_+^A &\sim 2(MN+1) \big(h_2 \overline{h_7} + h_4 \overline{h_{10}} + h_9 \overline{h_2} + h_8 \overline{h_3} + h_6 \overline{h_8} + h_{10} \overline{h_5}\big) \\
D_-^A &\sim 2(MN+1) \big(h_2 \overline{h_9} + h_3 \overline{h_8} + h_5 \overline{h_{10}} + h_7 \overline{h_2} + h_8 \overline{h_6} + h_{10} \overline{h_4}\big) \\
D_3^A &\sim (MN+1) \big(2|h_3|^2 - 2|h_4|^2 + 2|h_5|^2 - 2|h_6|^2 + |h_7|^2 - |h_9|^2\big) \\
D_+^B &\sim 2(MN+1) \big(h_2 \overline{h_8} + h_4 \overline{h_9} + h_5 \overline{h_7} + h_7 \overline{h_3} + h_9 \overline{h_6} + h_{10} \overline{h_2}\big) \\
D_-^B &\sim 2(MN+1) \big(h_2 \overline{h_{10}} + h_3 \overline{h_7} + h_6 \overline{h_9} + h_7 \overline{h_5} + h_8 \overline{h_2} + h_9 \overline{h_4}\big) \\
D_3^B &\sim (MN+1) \big(2|h_3|^2 - 2|h_4|^2 - 2|h_5|^2 + 2|h_6|^2 + |h_8|^2 - |h_{10}|^2\big)
\end{aligned} \tag{2.4.9}$$

We recognize the $D_\pm = 0$ conditions to be the same as (2.2.51) for the vanishing of the off–diagonal part of the anomalous dimensions (2.2.47), whereas the combinations $D_{U(1)} \pm D_3^A = 0$ and $D_{U(1)} \pm D_3^B = 0$ precisely match the conditions for the vanishing of the diagonal part of the anomalous dimensions, as derived above. This proves the complete equivalence of the two methods for finding the conformal manifold of the theory.

By using the nonconservation equation for the currents,

$$\bar{D}^2 J_a = h^i T^a{}_i{}^j \mathcal{O}_j + \dots \tag{2.4.10}$$

one can also check which operators are responsible for breaking the various symmetries of the free theory.
For example at the ABJ(M) fixed point, we see that the ABJM superpotential operator couples to the global axial $U(1)$ and thus becomes irrelevant, as we argued above.
At the $\mathcal{N}=3$ fixed point, lying on the $SU(2)$ invariant subset of the conformal manifold, we see that indeed the conservation equations for $J_+^A - J_-^B$, $J_-^A - J_+^B$ and $J_3^A - J_3^B$ are satisfied, corresponding to the preserved diagonal $SU(2)_D$ subgroup of $SU(2)\times SU(2)$. Instead, from the nonconservation of the currents of $(SU(2)\times SU(2))/SU(2)_D$, we see that the operators parameterized by $h_3 - h_4$, h_7+h_{10} and h_8+h_9 have become irrelevant by coupling to the broken currents.

2.5 Conformal manifold for flavored theories

Finally, we study the conformal manifold for $\mathcal{N}=2$ CS-matter theories when additional flavor degrees of freedom are included.

Flavor chiral superfields in the (anti)fundamental representation of the gauge groups can be safely added to the theories described above, without affecting superconformal invariance [30, 16, 31, 24]. For instance, given the model (2.6.1, 1.3.5) we can add M_f fundamental and $\tilde{M}_f$ antifundamental chiral superfields $(Q_1^r, \tilde{Q}_{1,\tilde{r}})$, charged under the first gauge group and N_f fundamentals and $\tilde{N}_f$ antifundamentals $(Q_2^s, \tilde{Q}_{2,\tilde{s}})$, charged under the second gauge group. In the literature only the cases $N_f = \tilde{N}_f$ and $M_f = \tilde{M}_f$ are discussed. However, as long as we are not concerned with surviving non–abelian global symmetries, nothing prevents us from considering a more general case with $N_f \neq \tilde{N}_f$, $M_f \neq \tilde{M}_f$.

For $W=0$, the classical global symmetry of the flavor sector is $U(N_f)\times U(\tilde{N}_f)\times U(M_f)\times U(\tilde{M}_f)$ in addition to the $U(2)\times U(2)$ group of the bifundamental sector already discussed.

The set of classical supermarginals is given by the operators belonging to the bifundamental sector (eqs. (1.3.5, 2.3.2)) plus pure flavor operators (1.5.12)

$$(F_{A,B})^{ik}{}_{jl}\, \tilde{Q}_{A,i}\, Q_A^j\, \tilde{Q}_{B,k}\, Q_B^l \qquad , \qquad A,B = 1,2 \text{ (not summed)} \tag{2.5.1}$$

with $(F_{A,B})^{ik}{}_{jl} = (F_{B,A})^{ki}{}_{lj}$, and mixed marginals (1.5.13) whose form is constrained by gauge invariance to be

$$M^{b,i}_{a,j}\, \tilde{Q}_{1\,i} A^a B_b\, Q_1^j \qquad , \qquad \tilde{M}^{b,i}_{a,j}\, \tilde{Q}_{2\,i} B_b A^a\, Q_2^j \tag{2.5.2}$$

Counting the number of independent M and $\tilde{M}$ couplings we have $4(N_f\tilde{N}_f + M_f\tilde{M}_f)$ operators of the form (2.5.2).

For operators of the form (2.5.1), we have three different countings according to the values of the A,B labels. For $A,B=1$, taking into account the symmetry of the trace, we have multiplicity $\frac{1}{2}\,N_f\tilde{N}_f(N_f\tilde{N}_f+1)$. Analogously, for $A,B=2$ we have $\frac{1}{2}\,M_f\tilde{M}_f(M_f\tilde{M}_f+1)$ operators, whereas for $A=1$ and $B=2$ we find $N_f\tilde{N}_fM_f\tilde{M}_f$ supermarginals.

All together, there are $\frac{1}{2}(N_f\tilde{N}_f+M_f\tilde{M}_f)(N_f\tilde{N}_f+M_f\tilde{M}_f+9)$ marginal operators in addition to the ten supermarginals of the unflavored case. These operators break all the global symmetries except two diagonal $U(1)$'s among $U(N_f)\times U(\tilde{N}_f)$ and $U(M_f)\times U(\tilde{M}_f)$ which are part of the gauge symmetry. The number of broken global generators is then $(N_f^2+\tilde{N}_f^{\,2}+M_f^2+\tilde{M}_f^{\,2}-2)$.

Using the arguments of [4, 5], we find that in the presence of flavor degrees of freedom the dimension of the conformal manifold is

$$\dim\mathcal{M}_c = 5+\frac{1}{2}(N_f\tilde{N}_f+M_f\tilde{M}_f)(N_f\tilde{N}_f+M_f\tilde{M}_f+5)-(N_f-\tilde{N}_f)^2-(M_f-\tilde{M}_f)^2 \quad (2.5.3)$$

Setting $N_f=\tilde{N}_f$ and $M_f=\tilde{M}_f$ we obtain the dimension of the conformal manifold for the $\mathcal{N}=3$ theory introduced in [16] and studied in [24].

2.6 Explicit examples of conformal fixed points and exactly marginal deformations.

The discussion in the previous Section is rather general and abstract. It is mainly focused on figuring out the overall properties of the conformal manifold of the ABJ(M) theory.

In this Section we focus our attention on less symmetric theories and extend the RG flow analysis to a more general set of models. In doing this we get a better insight on the spectrum of $\mathcal{N}=2$ CFT's, however we have to sacrifice generality and consider particular cases of deformations, to make explicit computation feasible. The results which follow are mainly taken from [29] and [24].

In particular we are interested in recovering some of the models proposed in literature to extend the ABJM theory, ascertaining whether and how these may be connected by exactly marginal deformations.

Indeed, since the original formulation of the AdS_4/CFT_3 correspondence, many efforts have been devoted to the its generalization. These were reviewed in Section 1.5 and we just list some of them here for convenience:

- different gauge groups [12, 32];
- less (super)symmetric backgrounds [33]-[34];
- inclusion of flavor degrees of freedom [30, 16, 31, 35, 36];
- non–opposite CS levels (K_1, K_2) [37, 13].

CS matter theories involved in the AdS_4/CFT_3 correspondence are of course at their superconformal fixed point ‡. Compactification of type IIA supergravity on $AdS_4 \times \mathbb{CP}^3$ does not contain scalar tachyons [7]. Since these states are dual to relevant operators in the corresponding field theory, AdS_4/CFT_3 correspondence leads to the prediction that in the far IR fixed points should be stable.

As a non–trivial check of the correspondence, it is then interesting to investigate the properties of these fixed points in the quantum field theory in order to establish whether they are isolated fixed points or they belong to a continuum surface of fixed points, whether they are IR stable and which are the RG trajectories which intersect them. Since for $K \gg N$ the CS theory is weakly coupled, a perturbative approach is available.

With these motivations in mind, we consider a $\mathcal{N}=2$ supersymmetric two–level CS theory for gauge group $U(M) \times U(N)$ with matter in the bifundamental representation

‡A classification of a huge landscape of superconformal Chern–Simons matter theories in terms of matter representations of global symmetries has been given in [38].

and flavor degrees of freedom in the fundamental, perturbed by a rather general matter superpotential compatible with $\mathcal{N}=2$ supersymmetry and capable of encompassing all models cited above. The action we refer to reads

$$\mathcal{S} = \mathcal{S}_{\rm CS} + \mathcal{S}_{\rm mat} + \mathcal{S}_{\rm pot} \tag{2.6.1}$$

with $\mathcal{S}_{\rm CS}$, the standard $\mathcal{N}=2$ CS action already given in (1.3.1), $\mathcal{S}_{\rm mat}$ the complete matter action (including flavors), and the superpotential $\mathcal{S}_{\rm pot}$, featuring pure bifundamental, pure fundamental and mixed interactions

$$\begin{aligned}
\mathcal{S}_{\rm mat} &= \int d^3x\, d^4\theta \;\; {\rm Tr}\left(\bar{A}_i\, e^V\, A^i\, e^{-\hat{V}} + \bar{B}^i\, e^{\hat{V}}\, B_i\, e^{-V}\right) \\
&+ \int d^3x d^4\theta \;\left(\bar{Q}^1_r\, e^V\, Q^r_1 + \tilde{Q}_{1,r}\, e^{-V}\, \bar{\tilde{Q}}^{1,r} + \bar{Q}^2_s\, e^{\hat{V}}\, Q^s_2 + \tilde{Q}_{2,s}\, e^{-\hat{V}}\, \bar{\tilde{Q}}^{2,s}\right) \\
\mathcal{S}_{\rm pot} &= \int d^3x\, d^2\theta \;\Big\{{\rm Tr}\left[(h_1+h_2)\,(A^1\,B_1\,A^2\,B_2) + (h_2-h_1)\,(A^2\,B_1\,A^1\,B_2)\right. \\
&\quad \left. + h_3\,(A^1\,B_1)^2 + h_4\,(A^2\,B_2)^2\right] + \\
&\quad + \lambda_1\,(\tilde{Q}_1\,Q_1)^2 + \lambda_2\,(\tilde{Q}_2\,Q_2)^2 + \lambda_3\,\tilde{Q}_1\,Q_1\,\tilde{Q}_2\,Q_2 \\
&\quad + \alpha_1\,\tilde{Q}_1\,A^1\,B_1\,Q_1 + \alpha_2\,\tilde{Q}_1\,A^2\,B_2\,Q_1 + \alpha_3\,\tilde{Q}_2\,B_1\,A^1\,Q_2 + \alpha_4\,\tilde{Q}_2\,B_2\,A^2\,Q_2\Big\} \;+\; h.c.
\end{aligned} \tag{2.6.2}$$

As recalled several times, $4\pi K_1$, $4\pi K_2$ are two independent integers, as required by gauge invariance of the effective action. In the perturbative regime we take $K_1, K_2 \gg N, M$. The superpotential (2.6.2) is the most general classically marginal perturbation which respects $\mathcal{N}=2$ supersymmetry but allows only for a diagonal $U(M_f)\times U(N_f)$ global symmetry (under which Q and $\tilde{Q}$ transform in conjugates representations, Table 2.2) in addition to a global $U(1)$ under which the bifundamentals have for example charges $(1,0,-1,0)$.

	gauge		flavor	
	$U(M)$	$U(N)$	$U(M_f)$	$U(N_f)$
Q_1	M	1	M_f	1
$\tilde{Q}_1$	$\bar{M}$	1	$\bar{M}_f$	1
Q_2	1	N	1	N_f
$\tilde{Q}_2$	1	$\bar{N}$	1	$\bar{N}_f$

Table 2.2: The representations of the gauge and global symmetries under which flavor fields transform.

As anticipated this is a restriction, in the bifundamental sector, with respect to the most general interaction in (2.2.1), since only four couplings out of the ten possible are retained. This entails great simplification in that as we will see this model does not cause mixing in the anomalous dimensions of the fields. However we in some sense generalize (2.2.1) in that we also introduce interacting flavor degrees of freedom explicitly. Still we select a very small subset of supermarginal operators restricting to a global symmetry preserving superpotential.

For generic values of the couplings, the action (1.3.1) is invariant under the following gauge transformations

$$e^V \to e^{i\bar{\Lambda}_1}\, e^V\, e^{-i\Lambda_1} \qquad\qquad e^{\hat{V}} \to e^{i\bar{\Lambda}_2}\, e^{\hat{V}}\, e^{-i\Lambda_2} \tag{2.6.3}$$

$$\begin{aligned} A^i &\to e^{i\Lambda_1}\, A^i\, e^{-i\Lambda_2} & \qquad B_i &\to e^{i\Lambda_2}\, B_i\, e^{-i\Lambda_1} \\ Q_1 &\to e^{i\Lambda_1}\, Q_1 & \tilde{Q}_1 &\to \tilde{Q}_1\, e^{-i\Lambda_1} \\ Q_2 &\to e^{i\Lambda_2}\, Q_2 & \tilde{Q}_2 &\to \tilde{Q}_2\, e^{-i\Lambda_2} \end{aligned} \tag{2.6.4}$$

where Λ_1, Λ_2 are two chiral superfields parameterizing $U(M)$ and $U(N)$ gauge transformations, respectively. Antichiral superfields transform according to the conjugate of (2.6.4).

For special values of the couplings we can have enhancement of global symmetries and/or R–symmetry with consequent enhancement of supersymmetry. In particular choosing the couplings properly the model reduces to the $\mathcal{N} = 6$ ABJ/ABJM superconformal theories [2, 12] ($\mathcal{N} = 8$ BLG theory [39, 40] for $N = M = 2$) or to the superconformal $\mathcal{N} = 2, 3$ theories with different CS levels studied in [13], in all cases with and without flavors. More generally, it describes marginal (but not necessarily exactly marginal) perturbations which can drive the theory away from the superconformal points. We list the most important cases we will be interested in, dividing theories with flavors from flavored ones.

Theories without flavors. Turning off flavor matter ($M_f = N_f = 0$, $\alpha_j = \lambda_j = 0$) and setting

$$K_1 = -K_2 \equiv K \qquad , \qquad h_3 = h_4 = 0 \tag{2.6.5}$$

we have $\mathcal{N} = 2$ ABJM/ABJ–like theories already studied in [29]. In this case the theory is invariant under two global $U(1)$'s in addition to $U(1)_R$. The transformations are

$$\begin{aligned} U(1)_A: \quad A^1 &\to e^{i\alpha}\, A^1 \qquad , \qquad & U(1)_B: \quad B_1 &\to e^{i\beta}\, B_1 \\ A^2 &\to e^{-i\alpha}\, A^2 \qquad , & B_2 &\to e^{-i\beta}\, B_2 \end{aligned} \tag{2.6.6}$$

When $h_3 = -h_4 \equiv h$, the global symmetry becomes $U(1)_R \times SU(2)_A \times SU(2)_B$ and gets enhanced to $SU(4)_R$ for $h = 4\pi/K$ [2, 33]. For this particular values of the couplings we recover the $\mathcal{N} = 6$ superconformal ABJ theory [12] and for $N = M$ the ABJM theory [2].

More generally, we can select theories corresponding to complex couplings

$$h_1 + h_2 = h\, e^{i\,\pi\,\beta} \quad , \quad h_2 - h_1 = -h\, e^{-i\,\pi\,\beta} \tag{2.6.7}$$

These are $\mathcal{N} = 2$ β–deformations of the ABJ–like theories. For particular values of h and β we find a superconformal invariant theory.

Going back to real couplings, we now consider the more general case $K_1 \neq -K_2$. Setting

$$h_3 = h_4 = h_2 \tag{2.6.8}$$

the corresponding superpotential reads

$$\mathcal{S}_{\rm pot} = \frac{1}{2} \int d^3x \, d^2\theta \;\; {\rm Tr} \left[\, (h_1 + h_2) \, (A^i B_i)^2 + (h_2 - h_1) \, (B_i A^i)^2 \, \right] \, + \, h.c. \tag{2.6.9}$$

This is the class of $\mathcal{N} = 2$ theories studied in [13] with $SU(2)$ invariant superpotential, where $SU(2)$ rotates simultaneously A^i and B_i.

When $h_2 = 0$, that is $h_3 = h_4 = 0$, we have the particular set of $\mathcal{N} = 2$ theories with global $SU(2)_A \times SU(2)_B$ symmetry [13]. This is the generalization of ABJ/ABJM–like theories to $K_1 \neq -K_2$. According to AdS/CFT, for particular values of $h_2 = 0$ we should find a superconformal invariant theory.

Another interesting fixed point should correspond to $h_1 + h_2 = \frac{4\pi}{K_1}$ and $h_2 - h_1 = \frac{4\pi}{K_2}$. The $U(1)_R$ R–symmetry is enhanced to $SU(2)_R$ and the theory is $\mathcal{N} = 3$ superconformal [13].

Theories with flavors. Setting the following identifications for the couplings

$$\begin{aligned}
&K_1 = -K_2 \equiv K \quad , \quad h_3 = h_4 = 0 \quad , \quad h_2 = 0, \quad h_1 = \frac{1}{K} \\
&\lambda_1 = 4\pi \, \frac{a_1^2}{2K} \quad , \quad \lambda_2 = -4\pi \, \frac{a_2^2}{2K} \quad , \quad \lambda_3 = 0 \\
&\alpha_1 = \alpha_2 = 4\pi \, \frac{a_1}{K} \quad , \quad \alpha_3 = \alpha_4 = 4\pi \, \frac{a_2}{K}
\end{aligned} \tag{2.6.10}$$

with arbitrary parameters a_1, a_2, our model reduces to the class of $\mathcal{N} = 2$ theories studied in [30]. Choosing in particular the values $a_1 = -a_2 = 1$ there is an enhancement of R–symmetry and the theory exhibits $\mathcal{N} = 3$ supersymmetry. This set of couplings should correspond to a superconformal fixed point [30, 16, 31].

In the more general case of $K_1 \neq -K_2$, in analogy with the unflavored case we consider the class of theories with

$$h_3 = h_4 = h_2 \quad ; \quad \alpha_1 = \alpha_2 \quad , \quad \alpha_3 = \alpha_4 \tag{2.6.11}$$

For generic couplings these are $\mathcal{N} = 2$ theories with a $SU(2)$ symmetry in the bifundamental sector which rotates simultaneously A^i and B_i. When $h_2 = 0$ this symmetry is enhanced to $SU(2)_A \times SU(2)_B$. The flavor sector has only $U(M_f) \times U(N_f)$ flavor symmetry.

Within this class of theories we can select the one corresponding to

$$\begin{array}{ccccc} \lambda_1 = \dfrac{h_1 + h_2}{2} & , & \lambda_2 = \dfrac{h_2 - h_1}{2} & , & \lambda_3 = 0 \\ \alpha_1 = \alpha_2 = h_1 + h_2 & , & \alpha_3 = \alpha_4 = h_2 - h_1 & & \end{array} \tag{2.6.12}$$

The values $h_1 + h_2 = \frac{4\pi}{K_1}, h_2 - h_1 = \frac{4\pi}{K_2}$ give the $\mathcal{N} = 3$ superconformal theory with flavors mentioned in [16]. It corresponds to flavoring the $\mathcal{N} = 3$ theory of [13].

2.6.1 RG properties.

Having introduced the main ingredients, we now turn to the explicit determination of the RG flow of the theories described by (2.6.2).

We compute the anomalous dimensions and β–functions with this superpotential. Its great advantage with respect to the general case described above, is that the reduced set of interactions prevents matter fields from undergoing mixing, making all the computation more handable.

One–loop results.

We list the results for the RG functions of this class of theories and then use them for analyzing the details of the RG flows.

We can recycle the perturbative computations from the previous Section, just adding new contributions for and from flavor fields. For instance the gauge one-loop self–energy contributions coming from diagrams in Fig. 2.5 where also flavor loops are present determine the following finite contributions to the quadratic action for the V gauge fields

$$\begin{aligned} \Pi^{(1)}_{gauge}(a) &= \left(N + \frac{M_f}{2}\right) \delta^{AA'} \int \frac{d^3k}{(2\pi)^3} d^4\theta \; B_0(k)\, k^2\, V^A(k)\, \Pi_{1/2}\, V^{A'}(-k) \\ \Pi^{(1)}_{gauge}(b) &= -\frac{1}{8} f^{ABC} f^{A'BC} \int \frac{d^3k}{(2\pi)^3} d^4\theta \; B_0(k)\, k^2\, V^A(k) \left(\Pi_0 + \Pi_{1/2}\right) V^{A'}(-k) \\ \Pi^{(1)}_{gauge}(2c) &= \frac{1}{8} f^{ABC} f^{A'BC} \int \frac{d^3k}{(2\pi)^3} d^4\theta \; B_0(k)\, k^2\, V^A(k)\, \Pi_0\, V^{A'}(-k) \end{aligned} \tag{2.6.13}$$

and for $\hat{V}$ gauge fields

$$\begin{aligned}
\hat{\Pi}^{(1)}_{gauge}(a) &= \left(N+\frac{N_f}{2}\right)\delta^{AA'}\int\frac{d^3p}{(2\pi)^3}d^4\theta\; B_0(p)\,p^2\,\hat{V}^A(p)\,\Pi_{1/2}\,\hat{V}^{A'}(-p)\\
\hat{\Pi}^{(1)}_{gauge}(b) &= -\frac{1}{8}\hat{f}^{ABC}\hat{f}^{A'BC}\int\frac{d^3p}{(2\pi)^3}d^4\theta\; B_0(p)\,p^2\,\hat{V}^A(p)\,\left(\Pi_0+\Pi_{1/2}\right)\hat{V}^{A'}(-p)\\
\hat{\Pi}^{(1)}_{gauge}(2c) &= \frac{1}{8}\hat{f}^{ABC}\hat{f}^{A'BC}\int\frac{d^3p}{(2\pi)^3}d^4\theta\; B_0(p)\,p^2\,\hat{V}^A(p)\,\Pi_0\,\hat{V}^{A'}(-p)
\end{aligned} \tag{2.6.14}$$

In addition diagram (a) also allows for mixing of the abelian propagators of V and $\hat{V}$ kind, but flavor fields loop do not contribute to such a term, since they couple with one type of gauge vector boson only

$$\tilde{\Pi}^{(1)}_{gauge}(a) = -2\sqrt{MN}\delta^{A0}\delta^{A'0}\int\frac{d^3p}{(2\pi)^3}d^4\theta\; B_0(p)\,p^2\,V^A(p)\,\Pi_{1/2}\,\hat{V}^{A'}(-p) \tag{2.6.15}$$

Summing all the contributions we find the one–loop corrected gauge propagators

$$\begin{aligned}
\Pi^{(1)}_{gauge} &= \left[-\frac{1}{8}f^{ABC}f^{A'BC}+\left(N+\frac{M_f}{2}\right)\delta^{AA'}\right]\int\frac{d^3p}{(2\pi)^3}d^4\theta\; B_0(p)\,p^2\,V^A(p)\,\Pi_{1/2}\,V^{A'}(-p)\\
\hat{\Pi}^{(1)}_{gauge} &= \left[-\frac{1}{8}f^{ABC}f^{A'BC}+\left(M+\frac{N_f}{2}\right)\delta^{AA'}\right]\int\frac{d^3p}{(2\pi)^3}d^4\theta\; B_0(p)\,p^2\,\hat{V}^A(p)\,\Pi_{1/2}\,\hat{V}^{A'}(-p)
\end{aligned} \tag{2.6.16}$$

together with $\tilde{\Pi}^{(1)}_{gauge}$ in (2.6.15) which mixes the two $U(1)$ gauge sectors.

Two-loop results

We are now ready to evaluate the matter self–energy contributions at two loops, both for the bifundamental and the flavor matter fields. The divergent diagrams are again given in Fig. 2.6, taking into account the contribution to the one–loop effective propagator from flavor degrees of freedom, and considering similar self–energy diagrams for fundamental fields.

Separating the contributions of each diagram, the results for bifundamental matter read (see also 2.2.17)

$$\begin{aligned}
\Pi^{(2)}_{bif}(3a) = -&\left[\frac{(4\pi)^2}{K_1^2}\left(2MN+MM_f-\frac{1}{2}\left(M^2-1\right)\right)+\right.\\
&\left.+\frac{(4\pi)^2}{K_2^2}\left(2MN+NN_f-\frac{1}{2}\left(N^2-1\right)\right)+\frac{4(4\pi)^2}{K_1\,K_2}\right]\mathcal{F}(0)\;\mathrm{Tr}\left(\bar{A}_iA^i+\bar{B}^iB_i\right)
\end{aligned}$$

$$\begin{aligned}
\Pi^{(2)}_{bif}(3b) = &\Big[4\,|h_3|^2\,(MN+1) + 2\,(|h_1|^2+|h_2|^2)\,MN + 2\,(|h_2|^2-|h_1|^2) \\
&+ (|\alpha_1|^2\,MM_f + |\alpha_3|^2\,NN_f)\Big]\,\mathcal{F}(p)\ \mathrm{Tr}\left(\bar{A}_1 A^1 + \bar{B}^1 B_1\right) \\
&+ \Big[4\,|h_4|^2\,(MN+1) + 2\,(|h_1|^2+|h_2|^2)\,MN + 2\,(|h_2|^2-|h_1|^2) \\
&+ (|\alpha_2|^2\,MM_f + |\alpha_4|^2\,NN_f)\Big]\,\mathcal{F}(p)\ \mathrm{Tr}\left(\bar{A}_2 A^2 + \bar{B}^2 B_2\right) \\
\Pi^{(2)}_{bif}(3c) = &-8\pi^2\left[\frac{M^2+1}{K_1^2} + \frac{N^2+1}{K_2^2} + \frac{4MN}{K_1\,K_2}\right]\,\mathcal{F}(p)\ \mathrm{Tr}\left(\bar{A}_i A^i + \bar{B}^i B_i\right)
\end{aligned} \tag{2.6.17}$$

where $\mathcal{F}(p)$ is again the two–loop self–energy integral given in (2.2.19).

Analogously, for fundamental matter we find

$$\begin{aligned}
\Pi^{(2)}_{fund1}(3a) &= -\frac{(4\pi)^2}{K_1^2}\left(2MN + MM_f - \frac{1}{2}\left(M^2-1\right)\right)\,F(0)\,\mathrm{Tr}\left(\bar{Q}^1 Q_1 + \bar{\tilde{Q}}^1 \tilde{Q}_1\right) \\
\Pi^{(2)}_{fund2}(3a) &= -\frac{(4\pi)^2}{K_2^2}\left(2MN + NN_f - \frac{1}{2}\left(N^2-1\right)\right)\,F(0)\,\mathrm{Tr}\left(\bar{Q}^2 Q_2 + \bar{\tilde{Q}}^2 \tilde{Q}_2\right) \\
\Pi^{(2)}_{fund1}(3b) &= \Big[4\,|\lambda_1|^2\,(MM_f+1) + |\lambda_3|^2\,NN_f \\
&\qquad + \left(|\alpha_1|^2 + |\alpha_2|^2\right)\,MN\Big]\,F(p)\,\mathrm{Tr}\left(\bar{Q}^1 Q_1 + \bar{\tilde{Q}}^1 \tilde{Q}_1\right) \\
\Pi^{(2)}_{fund2}(3b) &= \Big[4\,|\lambda_2|^2\,(NN_f+1) + |\lambda_3|^2\,MM_f \\
&\qquad + \left(|\alpha_3|^2 + |\alpha_4|^2\right)\,MN\Big]\,F(p)\,\mathrm{Tr}\left(\bar{Q}^2 Q_2 + \bar{\tilde{Q}}^2 \tilde{Q}_2\right) \\
\Pi^{(2)}_{fund1}(3c) &= -8\pi^2\,\frac{M^2+1}{K_1^2}\,F(p)\,\mathrm{Tr}\left(\bar{Q}^1 Q_1 + \bar{\tilde{Q}}^1 \tilde{Q}_1\right) \\
\Pi^{(2)}_{fund2}(3c) &= -8\pi^2\,\frac{N^2+1}{K_2^2}\,F(p)\,\mathrm{Tr}\left(\bar{Q}^2 Q_2 + \bar{\tilde{Q}}^2 \tilde{Q}_2\right)
\end{aligned} \tag{2.6.18}$$

where $\mathcal{F}(p)$ is still given in (2.2.19).

The renormalization of the theory proceeds as in the previous Section. In order to cancel the divergences in (2.6.17) and (2.6.18) we choose

$$\begin{aligned}
&Z_{A^1} = Z_{\bar{A}_1} = Z_{B_1} = Z_{\bar{B}^1} = 1 - \\
&\frac{1}{64\pi^2}\Big[-(4\pi)^2\left(\frac{2MN + MM_f + 1}{K_1^2} + \frac{2NM + NN_f + 1}{K_2^2} + \frac{2MN+4}{K_1 K_2}\right) \\
&\qquad + 4\,|h_3|^2\,(MN+1) + 2\,(|h_1|^2+|h_2|^2)\,MN + 2(|h_2|^2-|h_1|^2) \\
&\qquad + (|\alpha_1|^2\,MM_f + |\alpha_3|^2\,NN_f)\Big]\frac{1}{\epsilon}
\end{aligned}$$

$$
\begin{aligned}
&Z_{A^2} = Z_{\bar{A}_2} = Z_{B_2} = Z_{\bar{B}^2} = 1 - \\
&\frac{1}{64\pi^2}\Bigg[-(4\pi)^2\left(\frac{2MN+MM_f+1}{K_1^2}+\frac{2MN+NN_f+1}{K_2^2}+\frac{2MN+4}{K_1K_2}\right) \\
&\qquad + 4\,|h_4|^2\,(MN+1) + 2\left(|h_1|^2+|h_2|^2\right)MN + 2\left(|h_2|^2-|h_1|^2\right) \\
&\qquad + \left(|\alpha_2|^2\,MM_f + |\alpha_4|^2\,NN_f\right)\Bigg]\frac{1}{\epsilon} \\
&Z_{Q_1} = Z_{\bar{Q}^1} = Z_{\tilde{Q}_1} = Z_{\bar{\tilde{Q}}^1} = 1 - \\
&\frac{1}{64\pi^2}\Bigg[-(4\pi)^2\,\frac{2MN+MM_f+1}{K_1^2} + 4|\lambda_1|^2\,(MM_f+1) + \\
&\qquad |\lambda_3|^2\,NN_f + \left(|\alpha_1|^2+|\alpha_2|^2\right)MN\Bigg]\frac{1}{\epsilon} \\
&Z_{Q_2} = Z_{\bar{Q}^2} = Z_{\tilde{Q}_2} = Z_{\bar{\tilde{Q}}^2} = 1 - \\
&\frac{1}{64\pi^2}\Bigg[-(4\pi)^2\,\frac{2MN+NN_f+1}{K_2^2} + 4\,|\lambda_2|^2\,(NN_f+1) + \\
&\qquad |\lambda_3|^2\,MM_f + \left(|\alpha_3|^2+|\alpha_4|^2\right)MN\Bigg]\frac{1}{\epsilon}
\end{aligned}
\tag{2.6.19}
$$

Reading the single pole coefficients $Z^{(1)}_{\Phi_j}$ in eqs. (2.6.19) we obtain the anomalous dimensions of all fields

$$
\begin{aligned}
\gamma_{A^1} = \gamma_{B_1} = \frac{1}{32\pi^2}\Bigg[&-(4\pi)^2\left(\frac{2MN+MM_f+1}{K_1^2}-\frac{2MN+NN_f+1}{K_2^2}-\frac{2MN+4}{K_1K_2}\right) \\
&+ 4\,|h_3|^2\,(MN+1) + 2\left(|h_1|^2+|h_2|^2\right)MN + 2\left(|h_2|^2-|h_1|^2\right) \\
&+ \left(|\alpha_1|^2\,MM_f + |\alpha_3|^2\,NN_f\right)\Bigg] \\
\gamma_{A^2} = \gamma_{B_2} = \frac{1}{32\pi^2}\Bigg[&-(4\pi)^2\left(\frac{2MN+MM_f+1}{K_1^2}-\frac{2MN+NN_f+1}{K_2^2}-\frac{2MN+4}{K_1K_2}\right) \\
&+ 4\,|h_4|^2\,(MN+1) + 2\left(|h_1|^2+|h_2|^2\right)MN + 2\left(|h_2|^2-|h_1|^2\right) \\
&+ \left(|\alpha_2|^2\,MM_f + |\alpha_4|^2\,NN_f\right)\Bigg] \\
\gamma_{Q_1} = \gamma_{\tilde{Q}_1} = \frac{1}{32\pi^2}\Bigg[&-(4\pi)^2\,\frac{2MN+MM_f+1}{K_1^2} \\
&+ 4\,|\lambda_1|^2\,(MM_f+1) + |\lambda_3|^2\,NN_f + \left(|\alpha_1|^2+|\alpha_2|^2\right)MN\Bigg] \\
\gamma_{Q_2} = \gamma_{\tilde{Q}_2} = \frac{1}{32\pi^2}\Bigg[&-(4\pi)^2\,\frac{2MN+NN_f+1}{K_2^2} \\
&+ 4\,|\lambda_2|^2\,(NN_f+1) + |\lambda_3|^2\,MM_f + \left(|\alpha_3|^2+|\alpha_4|^2\right)MN\Bigg]
\end{aligned}
\tag{2.6.20}
$$

and the corresponding β–functions are finally obtained by

$$\begin{aligned}
&\beta_{h_3} = 4\,h_3\,\gamma_{A^1} && \beta_{h_4} = 4\,h_4\,\gamma_{A^2} \\
&\beta_{h_1} = 2\,h_1\,(\gamma_{A^1} + \gamma_{A^2}) && \beta_{h_2} = 2\,h_2\,(\gamma_{A^1} + \gamma_{A^2}) \\
&\beta_{\lambda_1} = 4\,\lambda_1\,\gamma_{Q_1} && \beta_{\lambda_2} = 4\,\lambda_2\,\gamma_{Q_2} \\
&\qquad \beta_{\lambda_3} = 2\,\lambda_3\,(\gamma_{Q_1} + \gamma_{Q_2}) && \\
&\beta_{\alpha_1} = 2\,\alpha_1\,(\gamma_{A_1} + \gamma_{Q_1}) && \beta_{\alpha_2} = 2\,\alpha_2\,(\gamma_{A_2} + \gamma_{Q_1}) \\
&\beta_{\alpha_3} = 2\,\alpha_3\,(\gamma_{A_1} + \gamma_{Q_2}) && \beta_{\alpha_4} = 2\,\alpha_4\,(\gamma_{A_2} + \gamma_{Q_2})
\end{aligned} \qquad (2.6.21)$$

2.7 The spectrum of fixed points

In this Section we study solutions to the equations $\beta_{\nu_j} = 0$ where the β–functions are given in (2.6.21). We consider separately the cases with and without flavor matter.

2.7.1 Theories without flavors

We begin by considering the class of theories without flavors. In eqs. (2.6.20) we set $M_f = N_f = 0$, $\lambda_j = \alpha_j = 0$ and solve the equations

$$\begin{aligned}
&\beta_{h_3} = 4\,h_3\,\gamma_{A^1} = 0 && \beta_{h_4} = 4\,h_4\,\gamma_{A^2} = 0 \\
&\beta_{h_1} = 2\,h_1\,(\gamma_{A^1} + \gamma_{A^2}) = 0 && \beta_{h_2} = 2\,h_2\,(\gamma_{A^1} + \gamma_{A^2}) = 0
\end{aligned} \qquad (2.7.1)$$

When $h_j \neq 0$ for any j the conditions (2.7.1) are equivalent to $\gamma_{A^1} = \gamma_{A^2} = 0$, that is no UV divergences appear at two–loops.

On the other hand, if we restrict to the surface $h_3 = h_4 = 0$, the β–functions are zero when $\gamma_{A^1} + \gamma_{A^2} = 0$, which in principle would not require the anomalous dimensions to vanish. However, it is easy to see from (2.6.20) that for $h_3 = h_4 = 0$ we have $\gamma_{A^1} = \gamma_{A^2}$ and again $\beta_{h_1} = \beta_{h_2} = 0$ imply the vanishing of all the anomalous dimensions. Therefore, at two loops the request for vanishing β–functions is equivalent to the request of finiteness.

We first study the class of theories with $h_3 = h_4 = 0$. In this case it is easy to see that h_2 is associated to a $SU(2)_A \times SU(2)_B$ breaking perturbation, whereas h_1 is symmetry preserving.

For real couplings, the anomalous dimensions vanish when

$$h_2^2\,(MN+1) + h_1^2\,(MN-1) = 4\pi^2\,\left[2\,(2\,MN+1)\left(\frac{1}{K_1^2} + \frac{1}{K_2^2}\right) + 2\,\frac{2\,MN+4}{K_1\,K_2}\right] \qquad (2.7.2)$$

This describes an ellipse in the parameter space $(h_1,\, h_2)$, which is drawn in Fig. 2.7. For $K_{1,2}$ sufficiently large it is very close to the origin and solutions fall in the perturbative regime. The ellipse degenerates to a circle in the large M, N limit. Fixed points corresponding to $h_2 \neq 0$ describe $\mathcal{N} = 2$ superconformal theories with $U(1)_A \times U(1)_B$ global symmetry (2.6.6).

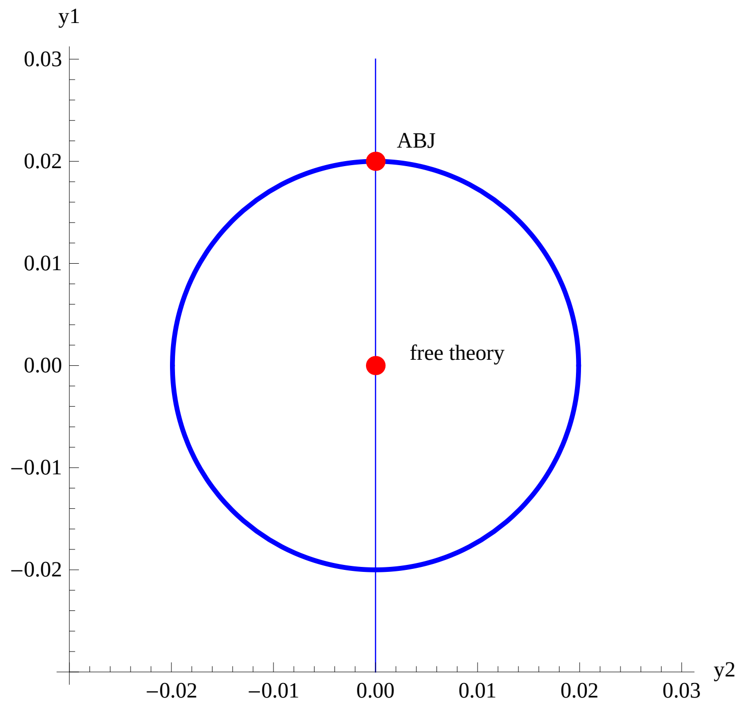


Figure 2.7: The ellipse of fixed points in the $y_1 = h_1/2$, $y_2 = h_2/2$ space. For large M, N its eccentricity is very small.

A more symmetric conformal point is obtained by solving (2.7.2) under the condition $h_2 = 0$. The solution

$$h_2 = 0, \quad h_1 = 4\pi \sqrt{\frac{2MN+1}{2(MN-1)} \left(\frac{1}{K_1^2} + \frac{1}{K_2^2} \right) + \frac{MN+2}{MN-1} \frac{1}{K_1 K_2}} \tag{2.7.3}$$

corresponds to a superconformal theory with $SU(2)_A \times SU(2)_B$ global symmetry. This is the theory conjectured in [13]. When $K_1 = -K_2 \equiv K$ it reduces to $h_2 = 0$, $h_1 = 4\pi/K$ and we recover the $\mathcal{N} = 6$ ABJ model [12] and, for $N = M$, the ABJM one [2].

More generally, we study fixed points with $h_j \neq 0$ for any j. In this case we have two equations, $\gamma_{A^1} = \gamma_{A^2} = 0$, for four unknowns. The spectrum of fixed points then spans a two dimensional surface which for real couplings is given by

$$h_3^2 = h_4^2 = \frac{8\pi^2}{MN+1} \Big[\left(MN + \frac{1}{2} \right) \left(\frac{1}{K_1^2} + \frac{1}{K_2^2} \right) + \frac{MN+2}{K_1 K_2} - MN(h_1^2 + h_2^2) - (h_2^2 - h_1^2) \Big] \tag{2.7.4}$$

This equation describes an ellipsoid in the four dimensional h–space as depicted in Fig. 2.8, localized in the subspace $h_3 = h_4$ (or equivalently $h_3 = -h_4$). A particular point on this surface corresponds to $h_1 + h_2 = 4\pi/K_1$ and $h_2 - h_1 = 4\pi/K_2$ with, consequently, $h_3 = h_4 = 2\pi(\frac{1}{K_1} + \frac{1}{K_2})$. This is the $\mathcal{N} = 3$ superconformal theory discussed in [13] [§].

The locus $h_3 = h_4 = 0$, $h_2 = 0$ of this surface is the $\mathcal{N} = 2$, $SU(2)_A \times SU(2)_B$ invariant superconformal theory (2.7.3). Therefore, the $\mathcal{N} = 3$ and the $\mathcal{N} = 2$, $SU(2)_A \times SU(2)_B$ superconformal points are continuously connected by the surface (2.7.4).

We can select a particular line of fixed points interpolating between the two theories, by setting

$$h_3 = h_4 = h_2 \tag{2.7.5}$$

and, consequently

$$h_1^2 + h_2^2 + \frac{MN+2}{2\,MN+1}\,(h_2^2 - h_1^2) = 8\,\pi^2\left[\frac{1}{K_1^2} + \frac{1}{K_2^2} + 2\,\frac{MN+2}{K_1\,K_2\,(2\,MN+1)}\right] \tag{2.7.6}$$

These are $SU(2)$ invariant, $\mathcal{N} = 2$ superconformal theories with superpotential (2.6.9). The existence of a line of $SU(2)$ invariant fixed points interpolating between the two theories was already conjectured in [13]. So far we have considered real solutions to the equations $\beta_{\nu_j} = 0$. We now discuss the case of complex couplings focusing in particular on the so–called β–deformations.

In the class of theories with $h_3 = h_4 = 0$ we look for solutions of the form

$$h_1 + h_2 = he^{i\,\pi\,\beta} \qquad , \qquad h_2 - h_1 = -he^{-i\,\pi\,\beta} \tag{2.7.7}$$

which implies $h_2 = -i\,2\,h\,\sin\pi\,\beta$, $h_1 = 2\,h\,\cos\pi\,\beta$. The condition for vanishing β–functions then reads

$$h^2\,MN - h^2\,\cos 2\pi\,\beta = 2\,\pi^2\,(2\,MN+1)\left(\frac{1}{K_1^2} + \frac{1}{K_2^2}\right) + \frac{MN+2}{K_1\,K_2} \tag{2.7.8}$$

This describes a line of fixed points which correspond to superconformal β–deformations of the $SU(2)_A \times SU(2)_B$ invariant theory (2.7.3). For $\beta \neq 0$ the global symmetry is broken to $U(1)_A \times U(1)_B$ in (2.6.6) and the deformed theory is only $\mathcal{N} = 2$ supersymmetric. In particular, setting $K_1 = -K_2$ we obtain the β–deformed ABJM/ABJ theories studied in [41].

In the large M, N limit the β–dependence of equation (2.7.8) disappears, consistently with the fact that in planar Feynman diagrams the effects of the deformation are invisible [42]. In this limit the condition for superconformal invariance reads

$$h^2 = (4\pi)^2\left(\frac{1}{K_1^2} + \frac{1}{K_2^2} + \frac{1}{K_1\,K_2}\right) \tag{2.7.9}$$

[§] Finiteness properties of $\mathcal{N} = 3$ CS–matter theories have also been discussed in [34] within the $\mathcal{N} = 3$ harmonic superspace setup.

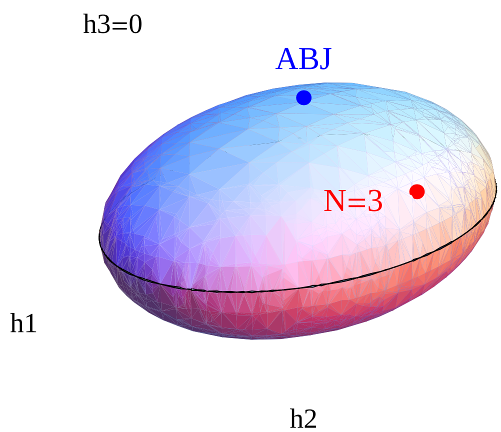


Figure 2.8: The exactly marginal surface of fixed points in the space of h_i couplings, restricted to the subspace $h_3 = h_4$, to make a pictorial representation possible. The parameters have been chosen so that the CS levels are opposite and $M = 43$, $N = 30$. The dots denote the $\mathcal{N} = 3$ (red) and the $\mathcal{N} = 6$ ABJ (blue) fixed points belonging to the ellipsoid. The plane $h_3 = 0$ represents the class of theories (2.6.9) with $SU(2)_A \times SU(2)_B$ global symmetry and its intersection with the ellipsoid is the line described by (2.7.6). The ABJ theory belongs to such a line as well. The $\mathcal{N} = 3$ model does not, since a nonvanishing value for h_3 is required to cook it up. The same picture presents also in the case of non–opposite CS levels, where however the would be ABJ point is no longer $\mathcal{N} = 6$ supersymmetric, but just $\mathcal{N} = 2$. It still belongs to a line of CFT's with $SU(2)_A \times SU(2)_B$ global symmetry, which must be thought of as arising when slicing the ellipsoid with the $h_3 = 0$ plane. An $\mathcal{N} = 3$ model will be present as well, though with some new values $\frac{2\pi}{K_1} + \frac{2\pi}{K_2}$ for h_3.

which reduces to $h = 4\pi/K$ for opposite CS levels.

The analysis of β–deformations can be extended to theories with $h_3, h_4 \neq 0$. Since they enter the anomalous dimensions only through $|h_3|^2$ and $|h_4|^2$ we can take them to be arbitrarily complex and still make the ansatz (2.7.7) for h_1, h_2. The surface of fixed points is then given by

$$|h_3|^2 = |h_4|^2 = \frac{1}{4\,(MN+1)}\Big[(4\pi)^2\,(2\,MN+1)\left(\frac{1}{K_1^2}+\frac{1}{K_2^2}\right) + 2\,(4\pi)^2\frac{MN+2}{K_1K_2} - 2\,h^2\,MN + 2\,h^2\,\cos 2\,\pi\,\beta\Big] \quad (2.7.10)$$

and describes superconformal β–deformations of $\mathcal{N}=2$ invariant theories.

The results of this Section agree with the ones in [26] obtained by using the three–algebra formalism.

2.7.2 Theories with flavors

As in the previous case, when all the couplings are non-vanishing, the request for zero β–functions implies the finiteness conditions $\gamma_{\Phi_i} = 0$. These provide four constraints on a set of eleven unknowns (see eqs. (2.6.20)). Therefore, in the space of the coupling constants the spectrum of fixed points spans a seven dimensional hypersurface given by the equations

$$\begin{aligned}
|\alpha_2|^2 = \frac{1}{MM_fK_1^2K_2^2}\Big\{ & K_2^2\,(2MN+MM_f+1) + K_1^2\,(2MN+NN_f+1) \\
& +2K_1K_2\,(MN+2) - 4|h_4|^2\,K_1^2K_2^2\,(MN+1) \\
& - K_1^2K_2^2\left[2(|h_1|^2+|h_2|^2)MN + 2(|h_2|^2-|h_1|^2) + |\alpha_4|^2NN_f\right]\Big\}
\end{aligned}$$

$$\begin{aligned}
|\alpha_3|^2 = \frac{1}{NN_fK_1^2K_2^2}\Big\{ & K_2^2\,(2MN+MM_f+1) + K_1^2\,(2MN+NN_f+1) \\
& +2K_1K_2\,(MN+2) - 4|h_3|^2\,K_1^2K_2^2\,(MN+1) \\
& - K_1^2K_2^2\left[2(|h_1|^2+|h_2|^2)MN + 2(|h_2|^2-|h_1|^2) + |\alpha_1|^2MM_f\right]\Big\} \\
|\lambda_1|^2 = \frac{1}{4(MM_f+1)K_1^2}\Big\{ & 2MN+MM_f+1 - K_1^2\left[|\lambda_3|^2NN_f + (|\alpha_1|^2+|\alpha_2|^2)MN\right]\Big\} \\
|\lambda_2|^2 = \frac{1}{4(NN_f+1)K_2^2}\Big\{ & 2MN+NN_f+1 - K_2^2\left[|\lambda_3|^2MM_f + (|\alpha_3|^2+|\alpha_4|^2)MN\right]\Big\}
\end{aligned} \quad (2.7.11)$$

When $K_1 = -K_2 \equiv K$ a particular point on this surface corresponds to

$$\begin{aligned} & h_3 = h_4 = 0 \quad , \quad h_2 = 0, \quad h_1 = \frac{4\pi}{K} \\ & \lambda_1 = -\lambda_2 = \frac{2\pi}{K} \quad , \quad \lambda_3 = 0 \\ & \alpha_1 = \alpha_2 = \frac{4\pi}{K} \quad , \quad \alpha_3 = \alpha_4 = -\frac{4\pi}{K} \end{aligned} \tag{2.7.12}$$

and describes the $\mathcal{N} = 3$ ABJ/ABJM models with flavor matter [30, 16, 31].

More generally, allowing $K_2 \neq -K_1$ we find the fixed point

$$\begin{aligned} & h_3 = h_4 = 2\pi\left(\frac{1}{K_1} + \frac{1}{K_2}\right) \quad , \quad h_1 + h_2 = \frac{4\pi}{K_1} \quad , \quad h_2 - h_1 = \frac{4\pi}{K_2} \\ & \lambda_1 = \frac{2\pi}{K_1} \quad , \quad \lambda_2 = \frac{2\pi}{K_2} \quad , \quad \lambda_3 = 0 \\ & \alpha_1 = \alpha_2 = \frac{4\pi}{K_1} \quad , \quad \alpha_3 = \alpha_4 = \frac{4\pi}{K_2} \end{aligned} \tag{2.7.13}$$

which corresponds to a superconformal theory obtained from the $\mathcal{N} = 3$ theory of [13] by the addition of flavor matter [16]. The superpotential

$$\begin{aligned} \mathcal{S}_{\text{pot}} &= \int d^3x\, d^2\theta\, \text{Tr}\left\{2\pi\left(\frac{1}{K_1} + \frac{1}{K_2}\right)\left[(A^1B_1)^2 + (A^2B_2)^2\right]\right. \\ &\quad + \frac{4\pi}{K_1}(A^1B_1A^2B_2) + \frac{4\pi}{K_2}(A^2B_1A^1B_2) + \frac{2\pi}{K_1}(Q_1\tilde{Q}_1)^2 + \frac{2\pi}{K_2}(Q_2\tilde{Q}_2)^2 \\ &\quad \left. + \frac{4\pi}{K_1}\left[\tilde{Q}_1A^iB_iQ_1\right] + \frac{4\pi}{K_2}\left[\tilde{Q}_2B_iA^iQ_2\right]\right\} + h.c. \\ &= \int d^3x\, d^2\theta\, \text{Tr}\left\{\frac{2\pi}{K_1}\left(A^i\, B_i + Q_1\, \tilde{Q}_1\right)^2 + \frac{2\pi}{K_2}\left(B_i\, A^i + Q_2\, \tilde{Q}_2\right)^2\right\} + h.c. \end{aligned} \tag{2.7.14}$$

can be thought of as arising from the action

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_{\text{CS}} + \mathcal{S}_{\text{mat}} \\ &+ \int d^3x\, d^2\theta\, \left[-\frac{K_1}{8\pi}\text{Tr}(\Phi_1^2) + \text{Tr}(B_i\Phi_1A^i) + \text{Tr}(\tilde{Q}_1\Phi_1Q_1)\right] \\ &+ \int d^3x\, d^2\theta\, \left[-\frac{K_2}{8\pi}\text{Tr}(\Phi_2^2) + \text{Tr}(A^i\Phi_2B_i) + \text{Tr}(\tilde{Q}_2\Phi_2Q_2)\right] + h.c. \end{aligned} \tag{2.7.15}$$

after integration on the Φ_1, Φ_2 chiral superfields belonging to the adjoint representations of the two gauge groups and giving the $\mathcal{N} = 4$ completion of the vector multiplet. Therefore, as in the unflavored case, the theory exhibits $\mathcal{N} = 3$ supersymmetry with the couples $(A, B^\dagger)_i$, $(Q, \tilde{Q}^\dagger)_1^r$ and $(Q, \tilde{Q}^\dagger)_2^s$ realizing $(2 + M_f + N_f)$ $\mathcal{N} = 4$ hypermultiplets (The CS terms break $\mathcal{N} = 4$ to $\mathcal{N} = 3$).

As already discussed, in the absence of flavors the $\mathcal{N}=3$ superconformal theory is connected by the line of fixed points (2.7.6) to a $\mathcal{N}=2$, $SU(2)_A\times SU(2)_B$ invariant theory. We now investigate whether a similar pattern arises even when flavors are present.

To this end, we first choose

$$h_3 = h_4 = h_2 \qquad , \qquad \alpha_1 = \alpha_2 \quad , \quad \alpha_3 = \alpha_4 \tag{2.7.16}$$

with λ_j arbitrary. This describes a set of $\mathcal{N}=2$ theories with global $SU(2)$ invariance in the bifundamental sector.

Solving the equations $\beta_{\nu_j}=0$ for real couplings we find a whole line of $SU(2)_A\times SU(2)_B$ invariant fixed points parameterized e.g. by the unconstrained coupling λ_3

$$\begin{aligned}
\alpha_1 = \alpha_3 &= 0, \quad h_2 = 0, \\
h_1 &= \sqrt{\frac{(2MM+NN_f+1)K_1^2 + 2(MN+2)K_1K_2 + (2MN+MM_f+1)K_2^2}{2(MN-1)K_1^2K_2^2}} \\
\lambda_1^2 &= \frac{2MN+MM_f+1-K_1^2NN_f\lambda_3^2}{4K_1^2(MM_f+1)} \\
\lambda_2^2 &= \frac{2MN+NN_f+1-K_2^2MM_f\lambda_3^2}{4K_2^2(NN_f+1)}
\end{aligned} \tag{2.7.17}$$

A four dimensional hypersurface of $\mathcal{N}=2$ fixed points given by

$$\begin{aligned}
\alpha_1^2 &= \frac{1}{2MNK_1^2}\Big[-4K_1^2(MM_f+1)\lambda_1^2 + MM_f + 2MN + 1 - NN_f\lambda_3^2K_1^2\Big] \\
\alpha_3^2 &= \frac{1}{2MNK_2^2}\Big[-4K_2^2(NN_f+1)\lambda_2^2 + NN_f + 2MN + 1 - MM_f\lambda_3^2K_2^2\Big] \\
h_1 + h_2 &= -\frac{1}{2MN+1}\Bigg\{(MN+2)(h_2-h_1) \pm \Bigg[(2MN+1)\Bigg(NN_f\left(-\alpha_3^2+4\lambda_2^2+\lambda_3^2\right) \\
&\quad + 2MN(\alpha_1^2+\alpha_3^2+1) + MM_f\left(-\alpha_1^2+4\lambda_1^2+\lambda_3^2\right) \\
&\quad + 4\left(\lambda_1^2+\lambda_2^2\right) + \frac{4(4\pi)^2}{K_1K_2}\Bigg) - 3(h_2-h_1)^2\left(N^2M^2-1\right)\Bigg]^{1/2}\Bigg\}
\end{aligned} \tag{2.7.18}$$

connects the line of $\mathcal{N}=2$, $SU(2)_A\times SU(2)_B$ invariant theories (2.7.17) to the $\mathcal{N}=3$ theory (2.7.13). This surface is the analogue of the fixed line (2.7.6) found in the unflavored theories.

Before closing this Section we address the question of superconformal invariance versus finiteness for theories with flavor matter. In the bifundamental sector, the only possibility to have vanishing β–functions without vanishing anomalous dimensions is by setting $h_3 = h_4 = 0$. When flavor matter is present, this does not necessarily imply $\gamma_{A^1} = \gamma_{A^2}$, so we can solve for $\beta_{h_1,h_2} = \gamma_{A^1} + \gamma_{A^2} = 0$ without requiring the two γ's to vanish separately. Once these equations have been solved in the bifundamental sector, in the flavor sector we choose $\lambda_1 = \lambda_2 = 0$ and $\alpha_1 = \alpha_4 = 0$ (or equivalently, $\alpha_2 = \alpha_3 = 0$) in order to avoid $\gamma_{Q_1} = \gamma_{Q_2} = 0$. We are then left with five couplings subject to the three equations $\gamma_{A^1} + \gamma_{A^2} = 0$, $\gamma_{A^1} + \gamma_{Q_2} = 0$ and $\gamma_{A^2} + \gamma_{Q_1} = 0$. Solutions correspond to superconformal but *not finite* theories. We note that this is true as long as we work with M, N finite. In the large M, N limit with $M_f, N_f \ll M, N$ we are back to $\gamma_{A^1} = \gamma_{A^2}$, as flavor contributions are subleading. In this case superconformal invariance requires finiteness.

2.8 Infrared stability

We now study the RG flows around the fixed points of main interest in order to establish whether they are IR attractors or repulsors. In particular, we concentrate on the ABJ/ABJM theories, $\mathcal{N} = 3$ and $SU(2)_A \times SU(2)_B$ $\mathcal{N} = 2$ superconformal points, in all cases with and without flavors.

The behavior of the system around a given fixed point ν_0 is determined by studying the stability matrix

$$\mathcal{M}_{ij} \equiv \frac{d\beta_i}{d\nu_j}(\nu_0) \tag{2.8.1}$$

Diagonalizing $\mathcal{M}$, positive eigenvalues correspond to directions of increasing β–functions, whereas negative eigenvalues give decreasing βs. It follows that the fixed point is IR stable if $\mathcal{M}$ has all positive eigenvalues, whereas negative eigenvalues represent directions where a classically marginal operator becomes relevant.

Whenever null eigenvalues are present we need in principle to compute derivatives of the stability matrix along the directions individuated by the corresponding eigenvectors. A null eigenvalue whose eigenvector is tangent to the conformal manifold then signals an exactly marginal deformation. Then a fixed point is locally unstable under such a perturbation, but the theory flows back to the conformal surface, even though on another point. Therefore such operators precisely trigger motion around the conformal manifold.

We apply these criteria to the two–loop β–functions (2.6.21).

2.8.1 Theories without flavors

We begin with the $\mathcal{N} = 2$ theories without flavor discussed in Section 4.1. As shown, the non–trivial fixed points lie on a two dimensional ellipsoid and particular points on it are the $\mathcal{N} = 3$ and the $\mathcal{N} = 2$ $SU(2)_A \times SU(2)_B$ invariant theories. Since the ellipsoid is localized in the subspaces $h_3 = \pm h_4$ we restrict our discussion to the $h_3 = h_4$ case.

When $K_1 = -K_2 \equiv K$, the set of theories with $h_3 = h_4 = 0$ has been already studied in [29]. In this case the ellipsoid reduces to an ellipse in the (h_1, h_2) plane. It has been shown that at the order we are working the RG trajectories are straight lines passing through the origin and intersecting the ellipse. Infrared flows point towards the ellipse so proving that the whole line of fixed points is IR stable. Every single point has only one direction of stability which corresponds to the RG trajectory passing through it. When perturbed along any other direction the system flows to a different fixed point on the curve. In the ABJ/ABJM case the direction of stability is described by $SU(2)_A \times SU(2)_B$ preserving perturbations.

This can be understood by computing the stability matrix at $h_3 = h_4 = 0$, $h_2 = 0$, $h_1 = 4\pi/K$ and diagonalizing it. We find that mutual orthogonal directions are $(h_3 = h_4, h_1, h_2)$ and the corresponding eigenvalues are

$$\mathcal{M} = diag\Big\{0, 16\,\frac{MN-1}{K^2}, 0\Big\} \tag{2.8.2}$$

For $M, N > 1$ the third eigenvalue is positive, so the ABJ/ABJM theory is an attractor along the h_1–direction.

Solving the degeneracy of null eigenvalues requires computing the matrix of second derivatives. In particular, looking at the h_2–direction we find

$$\frac{\partial^2 \beta_{h_1}}{\partial h_1^2} = 16\,\frac{1-MN}{K^2} \tag{2.8.3}$$

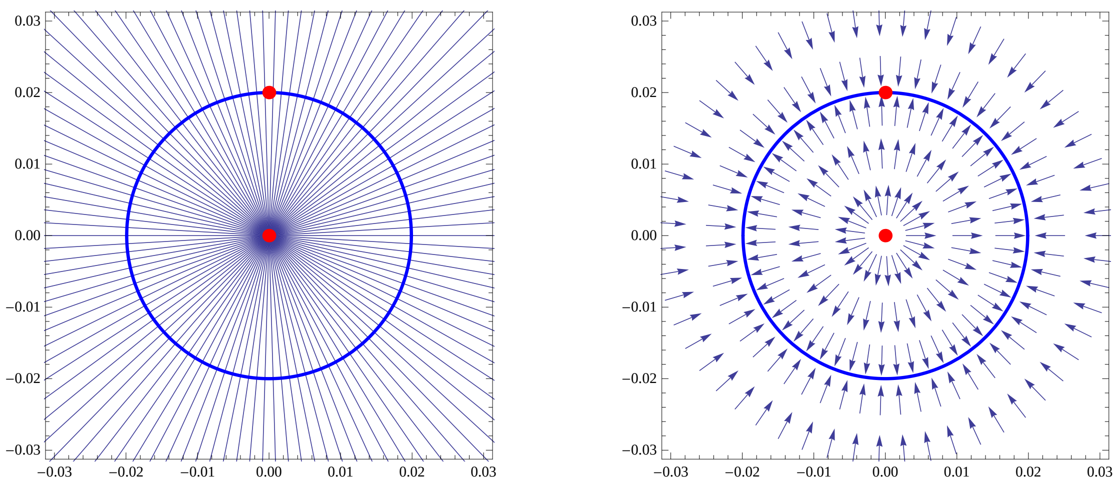
Since it is non–vanishing, the h_1 coordinate is a line of instability. Therefore, when perturbed by a $SU(2)_A \times SU(2)_B$ violating operator the system leaves the ABJ CFT and flows to a less symmetric fixed point along a RG trajectory. The coupling h_2 has a null eigenvalue and represents an exactly marginal direction.

This result states that in the class of scalar, dimension–two composite operators of the form

$$\mathcal{O} = (h_1 + h_2)\,\mathrm{Tr}(A^1 B_1 A^2 B_2) + (h_2 - h_1)\,\mathrm{Tr}(A^2 B_1 A^1 B_2) \tag{2.8.4}$$

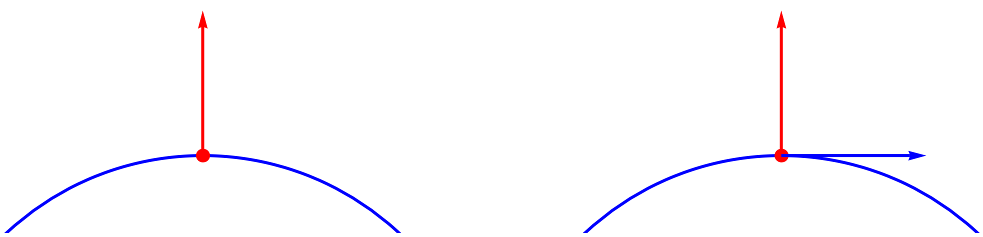
there is only one exactly marginal operator. This is the operator which allows the system to move away from the ABJ point along the fixed line. At the ABJ(M) fixed point this is the deformation $\mathrm{Tr}(A^1 B_1 A^2 B_2) + \mathrm{Tr}(A^2 B_1 A^1 B_2)$ parameterized by the coupling h_2, whereas that labeled by h_1 is irrelevant. The former is therefore tangent to the ellipse, while the second is transverse, as depicted in Fig. 2.10

We now generalize the analysis to the inclusion of the interactions parameterized by h_3 and h_4 couplings. In this case we refer to the surface of fixed points in Fig. 2.11 where for clearness only half of the ellipsoid has been drawn and opposite CS levels were assumed. Moreover $h_3 = h_4$ is understood in the picture, in order to draw the surface in a three dimensional space. The blue spot correspondes to the ABJ mnodel, at $h_2 = h_3 = h_4 = 0$.



(a) (b)

Figure 2.9: In figure (a) the trajectories of the RG flow are depicted. In (b) arrows indicate the IR flow, showing that the fixed line of CFT's is IR stable.



(a) (b)

Figure 2.10: The picture shows a detail of the eigenvectors of the stability matrix at the ABJ point: one is the normal vector to the ellipse, which represents an irrelevant perturbation, the other is tangent to the curve, signaling an exactly marginal deformation.

The red point is instead the $\mathcal{N} = 3$ superconformal theory. A completely analogous pattern arises when considering non–opposite CS levels, where, however, the ABJ point is replaced by an $\mathcal{N} = 2$ CFT with $SU(2)_A \times SU(2)_B$ global symmetry.

From eq. (2.6.20) we see that in the $h_3 = h_4$ subsector we have $\gamma_{A^1} = \gamma_{A^2}$. As a consequence, all the β–functions are equal and the RG flow equations simplify to

$$\frac{dh_i}{dh_j} = \frac{h_i}{h_j} \tag{2.8.5}$$

In the three dimensional parameter space $(h_3 = h_4, h_1 + h_2, h_2 - h_1)$, solutions are all the straight lines passing through the origin and intersecting the ellipsoid.

Infrared flows can be easily studied by plotting the vector $(-\beta_{h_3}, -\beta_{h_1+h_2}, -\beta_{h_2-h_1})$ in each point. The result is given in Fig. 2.11 where it is clear that the entire surface is globally IR stable.

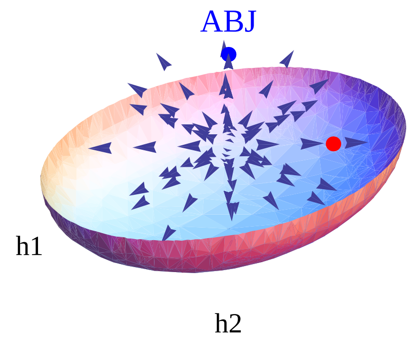


Figure 2.11: The ellipsoid of fixed points and the RG flows for $\mathcal{N} = 2$ theories in the space of couplings $(h_3 = h_4, h_1, h_2)$. Arrows point towards the direction of the IR flow. Here the figure is obtained with opposite CS levels.

In order to study the local behavior of the system in proximity of a given fixed point, we compute the stability matrix at the point (2.7.4) and diagonalize it. Surprisingly, the eigenvalues turn out to be independent of the particular point on the surface

$$\mathcal{M} = diag\left\{0, 0, \frac{K_1^2 + 4K_1K_2 + K_2^2 + 2\left(K_1^2 + K_1K_2 + K_2^2\right)MN}{4K_1^2K_2^2\pi^2}\right\} \tag{2.8.6}$$

The two null eigenvalues characterize exactly marginal directions. In fact, the corresponding eigenvectors span the tangent space with respect to the surface, at a given point.

For example, at the $\mathcal{N} = 2$, $SU(2)_A \times SU(2)_B$ invariant fixed point at $h_3 = 0$, on the line in Fig. 2.8, these eigenvectors are $\{1, 1, 0\}$ and $\{0, 0, 1\}$, which are precisely the directions $h_1 = 0$ and h_3, tangent to the surface at that point. It is clear from Fig. 5 that if we perturb the system along these directions it will intercept a RG trajectory which leads it to another fixed point.

The stability properties of the β–deformed theories are easily inferred from the previous discussion. In fact, performing the following rotation of the couplings

$$h\cos(\pi\beta) = \frac{x}{2}, \qquad h\sin(\pi\beta) = \frac{y}{2} \tag{2.8.7}$$

the condition (2.7.8) for vanishing β–functions becomes

$$\frac{1}{4}(MN-1)x^2 + \frac{1}{4}(MN+1)y^2 = 2\pi^2(2MN+1)\left(\frac{1}{K_1^2}+\frac{1}{K_2^2}\right) + (4\pi)^2\frac{MN+2}{K_1K_2} \tag{2.8.8}$$

This is exactly the ellipse (2.7.2) of the undeformed case. Therefore, the infrared stability properties of this curve are precisely the ones discussed before.

2.8.2 Theories with flavors

We now turn to flavored theories introduced in Section 2.7.2. The form of the stability matrix is quite cumbersome, but we can analyze the effects of the interactions with flavor multiplets by studying particular examples.

As the simplest case we consider the class of theories described by the superpotential (2.6.2) where only λ_i couplings have been turned on. The β-functions of the theory split into two completely decoupled sectors: The former is the four dimensional space of couplings h_3, h_4, h_1, h_2, whose stability was addressed in the previous Section; the latter is the three dimensional space of λ_i couplings.

Looking at the λ_i sector, non–trivial solutions to $\beta_i = 0$ describe a curve of fixed points given by expressing λ_1 and λ_2 as functions of λ_3 (see eqs. (2.7.11)). It is the two–branch curve of Fig. 6. The most general solution includes also isolated points where either λ_1 or λ_2 vanish.

Drawing the vector $(-\beta_{\lambda_1}, -\beta_{\lambda_2}, -\beta_{\lambda_3})$ in each point of the parameter space we obtain the RG flow configurations as given in Fig. 2.12. It is then easy to see that the isolated fixed points are always unstable since the RG flows drive the theory to one of the two branches in the IR.

This behavior can be also inferred from the structure of the stability matrix. In fact, one can check that when evaluated on the curve the matrix has two positive eigenvalues, whereas when evaluated at the isolated solutions it has negative eigenvalues. As before, theories living on the curve have directions of local instability signaled by the presence of a null eigenvalue which can be solved at second order in the derivatives. This direction is tangent to the curve and leads the system to a different fixed point corresponding to the addition of an exactly marginal perturbation.

Finally, we consider the more complicated case of theories with superpotential (2.6.2) where only the α_i couplings are non–vanishing. This time the β–functions for the h_i sector do not decouple from the β–functions of the α_i sector and the analysis of fixed points becomes quite complicated.

In order to perform the calculation we restrict to the class of $U(N) \times U(N)$ theories

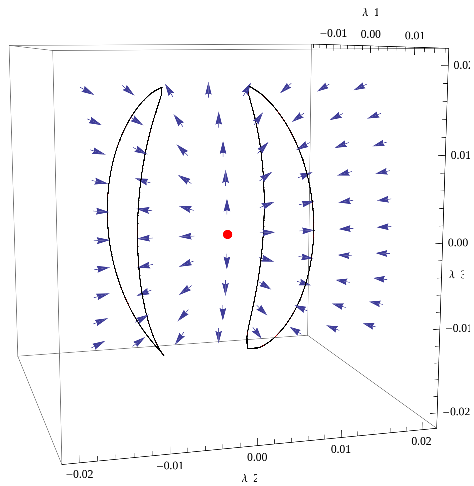


Figure 2.12: A sketch of the RG flow for the λ_i couplings only. A curve of fixed points with two branches is shown, which is IR stable. The red dot represents an isolated IR unstable fixed point, located at $\lambda_i = 0$. Here, the parameters are $K_1 = K_2 = 20$, $M = N = 10$, $M_f = N_f = 1$.

(therefore $M_f = N_f$) with $|K_1| = |K_2|$. This allows to choose α_i all equal to α. Moreover, we set $h_3 = h_4$ and $h_2 = 0$. The spectrum of fixed points and the RG trajectories are then studied in the three-dimensional space of parameters (α, h_1, h_3).

The β–functions are zero for vanishing couplings (free theory) and for $\gamma_{A^1} = \gamma_{Q_1} = 0$. non–trivial solutions for α are obtained from $\gamma_{Q_1} = 0$. Using eqs. (2.6.20), for real couplings we find

$$\alpha = \pm 4\pi \sqrt{\frac{2\,N^2 + N N_f + 1}{2\,N^2 K_1^2}} \tag{2.8.9}$$

Fixing α to be one of the three critical values (zero or one of these two values) we can solve $\gamma_{A^1} = 0$. As in the previous cases this describes an ellipse on the (h_1, h_3) plane localized at $\alpha = \text{const}$. For theories with $K_1 = K_2$ the configuration of fixed points lying on the three ellipses is given in Fig. 2.13.

Renormalization group flows are obtained by plotting the vector $(-\beta_\alpha, -\beta_{h_1}, -\beta_{h_3})$. The stability of fixed points is better understood by projecting RG trajectories on orthogonal planes. Looking for instance at the $h_3 = 0$ plane we obtain the configurations in Fig. 2.14 where the red dots indicate the origin and the intersections of the three ellipses with the plane.

From this picture we immediately infer that the free theory is an IR unstable fixed point since the system is always driven towards non–trivial fixed points. Among them, the ones corresponding to $\alpha \neq 0$ are attractors, whereas $\alpha = 0$ does not seem to be a preferable point for the theory. In fact, it is reached flowing along the $\alpha = 0$ trajectory,

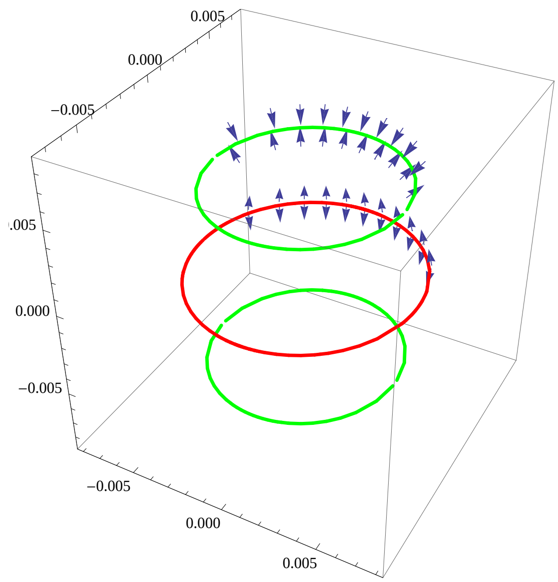


Figure 2.13: The three ellipses of fixed points on (h_1, h_3) planes and the RG flows for $\mathcal{N} = 2$ theories with α couplings (on the vertical axis) turned on. Arrows point towards the IR flow, showing that the green ellipses at $\alpha \neq 0$ are attractive, whereas the red one at $\alpha = 0$ is repulsive. The parameters are $K_1 = K_2 = 20$, $M = N = 10$, $M_f = N_f = 1$.

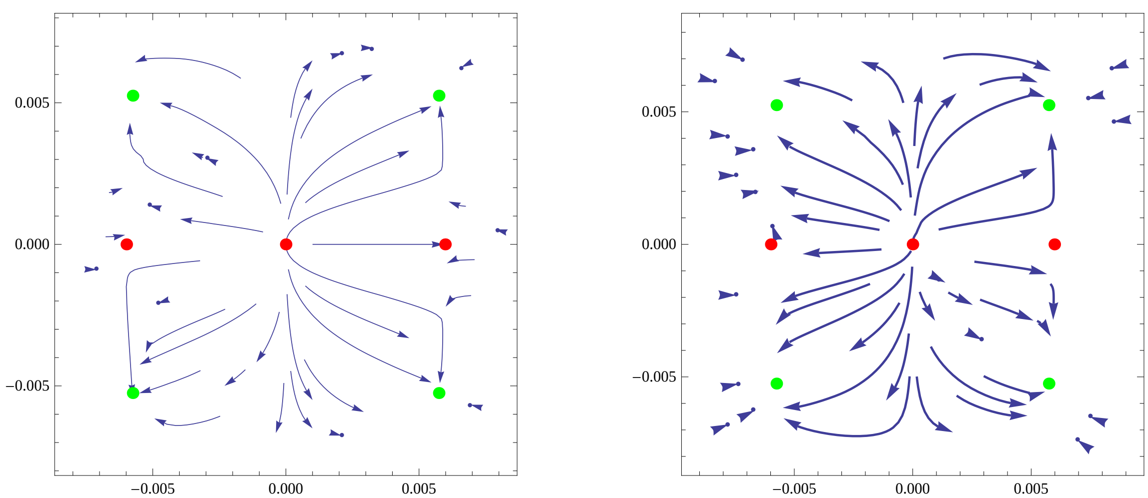


Figure 2.14: RG trajectories on the $h_3 = 0$ plane for $\mathcal{N} = 2$ theories with α couplings turned on. The α coupling spans the vertical axis, whereas h_1 is on the horizontal one. Arrows point towards IR directions. Red dots indicate unstable fixed points at $\alpha = 0$, whereas green ones are at stable CFT's with $\alpha \neq 0$. This picture is obtained from Fig. 2.13, by slicing it on the $h_2 = 0$ plane. Dots are points where the three ellipses intersect such a plane. The parameters are $K_1 = K_2 = 20$, $M = N = 10$, $M_f = N_f = 1$.

but as soon as we perturb the system with a marginal operator corresponding to $\alpha \neq 0$ it will flow to one of the two non–trivial points. We conclude that if we add flavor degrees of freedom the system requires a non–trivial interaction with bifundamental matter in order to reach a stable superconformal configuration in the infrared region.

In the absence of flavors the condition of vanishing β–functions necessarily implies the vanishing of anomalous dimensions for all the elementary fields of the theory. Therefore, the set of superconformal fixed points coincides with the set of superconformal finite theories. When flavors are present this is no longer true and in the space of the couplings we determine a surface of fixed points where the theory is superconformal but not two–loop finite.

Concluding remarks. We conclude this section on explicit examples of $\mathcal{N} = 2$ (flavored) CSM conformal fixed points of the RG flow and on the exactly marginal deformations connecting them by a summary of the results highlighted.

When flavors are turned off we have determined a continuum surface of fixed points which contains as non–isolated fixed points the BLG, the ABJ and ABJM theories. The case of theories with equal CS levels and $U(1)_A \times U(1)_B$ symmetry preserving perturbations was investigated in [29] and is a simple subcase of the more general treatment offered here, covering a larger set of models. Beyond extending the analysis to non–opposite CS levels, we have provided details for a class of theories generalizing the results to the case of less symmetric perturbations breaking $U(1)_A \times U(1)_B$. When the CS levels are different the surface contains as somewhat special points an $\mathcal{N} = 2$, $SU(2)_A \times SU(2)_B$ invariant and an $\mathcal{N} = 3$ superconformal theories. This result confirms the existence of such superconformal points, which was conjectured in [13]. Moreover, we have proved that these two theories are connected by a line of $\mathcal{N} = 2$ fixed points, as proposed in the same paper.

We have extended our analysis to the case of complex couplings, so including fixed points corresponding to β–deformed theories [41].

In the presence of flavor matter the spectrum of fixed points spans a seven dimensional hypersurface in the space of the couplings which contains the CFT's corresponding to the ABJM/ABJ models with flavors studied in [30, 16, 31]. More generally, we find a fixed point which describes a $\mathcal{N} = 3$ theory with *non–opposite* CS levels with the addition of flavor degrees of freedom [16]. As a generalization of the pattern arising in the unflavored case, we find that it is connected by a four real dimensional hypersurface of $\mathcal{N} = 2$ fixed points to a line of $\mathcal{N} = 2$ fixed points with $SU(2)_A \times SU(2)_B$ invariance in the bifundamental sector. This is a submanifold of the three complex dimensional one derived in Section 2.2.

We have then studied the RG trajectories around these fixed points in order to investigate their IR stability. The pattern which arises is common to all these theories, flavors included or not, and can be summarized as follows.

- Infrared stable fixed points always exist and we have determined the RG trajectories which connect them to the UV stable fixed point (free theory).
- In general these fixed points belong to a continuum surface. The surface is globally stable since RG flows always point towards it.
  Locally, each single fixed point has only one direction of stability which corresponds to perturbations along the RG trajectory which intersect the surface at that point and are normal to it. In the ABJ/ABJM case this direction coincides with the maximal global symmetry preserving perturbation [29]. Along transverse directions, perturbations drive the system away from the original point towards a different one on the surface. This corresponds to deforming the original system with an exactly marginal operator.
- When flavors are added, stability is guaranteed by the presence of non–trivial interactions between flavors and bifundamental matter. The fixed point corresponding to setting these couplings to zero is in fact unstable.

Chapter 3

Scattering amplitudes

The main power of quantum field theory consists in being able to predict expectation values of observables, which could be eventually measured in experiments. Among observables cross sections of scattering events and decay rates are the most widely measured. These quantities depend on the quantitative features of the experiment, but the non–trivial contribution arises from the scattering amplitude, which computes the quantum mechanical probability of a transition between an initial and a final state. Formally these are matrix elements of the so–called S–matrix. In the regime where the interaction between particles is weak, perturbation theory may be employed to compute these matrix elements to the desired precision. This perturbative expansion may be represented diagrammatically in terms of the Feynman graphs. Although well established and defined in principle, this approach turns out to be increasingly cumbersome as the loop order and the number of external particles grow large. In particular, gluon amplitudes in QCD are the hardest. This hurdle motivated the quest for novel and more efficient methods for computing scattering amplitudes in gauge theories, and huge progress has been achieved in this subject over the last decade.
Stringy inspired techniques and the advent of the AdS/CFT correspondence have uncovered fascinating properties of amplitudes in supersymmetric theories, hinting at their hidden simplicity. The most astonishing results apply to $\mathcal{N} = 4$ SYM in the planar sector, for which a huge amount of amplitudes has been computed, allowing to discover dualities between apparently unrelated objects such as amplitudes and Wilson loops [43]-[44], and to realize the emergence of an underlying superconformal symmetry [45, 46] which highly constrains their form. Furthermore a prescription has been found to compute amplitudes at strong coupling [47], which has offered a different and fruitful perspective from which gaining deeper understanding of such field theoretical phenomena. Interestingly, the emergent dual superconformal symmetry combined with the original superconformal invariance of the theory generates a Yangian symmetry [48], which has been recently exploited to propose a generating function for all scattering amplitudes in $\mathcal{N} = 4$ SYM [49].

In order to get a deeper insight into the subject of scattering amplitudes it would be certainly desirable to extend the realm of these recent achievements to theories other than $\mathcal{N} = 4$ SYM. One opportunity is offered by Chern–Simons matter theories in three dimensions, which display a large spectrum of interesting models where techniques borrowed from the four dimensional case can be tested. Among these the ABJM model, being conformal and enjoying a high amount of supersymmetry, is the best candidate to look for analogies with maximally supersymmetric Yang-Mills in four dimensions.
In this chapter we want to give an overview of the remarkable properties of $\mathcal{N} = 4$ SYM scattering amplitudes.

3.1 Amplitudes in N=4 SYM

This Section is aimed at providing some basic features of amplitudes in $\mathcal{N} = 4$ SYM, making it very special with respect to less symmetric theories. This would be far from exhaustive and I will put more emphasis on some aspects which I will address in what follows, concerning amplitudes in ABJM. In particular we will introduce and briefly sketch the following topics:

- color ordering and spinor helicity formalism;
- tree level: from Parke–Taylor to superamplitudes;
- BCFW recursion relations;
- unitarity methods;
- amplitudes at strong coupling;
- BDS ansatz;
- dual superconformal and Yangian invariance;
- MHV amplitudes / WL / correlators triality.

3.2 Color ordering.

Color decomposition carries both the advantages of reducing the number of graphs to be computed, and to strip the amplitudes off their group structure, separating the color part from the kinematics. We will here briefly introduce its principle for a gauge theory with unitary $SU(N)$ or $U(N)$ gauge group.

The generators in the fundamental representation of $SU(N)$ are $N \times N$ hermitian matrices, normalized according to $\mathrm{Tr}\,(T^A T^B) = \delta^{AB}$, obeying the group algebra, that is their commutators are fixed by the algebra structure constants. The latter are defined through

$$[T^A, T^B] = i\, f^{ABC}\, T_C \tag{3.2.1}$$

Therefore

$$f^{ABC} = -i\, \mathrm{Tr}\left([T^A, T^B]\, T_C\right) \tag{3.2.2}$$

Moreover the following group theory identity holds for $SU(N)$

$$(T^A)_{i_1\, j_j}\, (T^A)_{i_2\, j_2} = \delta_{i_1\, j_2}\, \delta_{j_1\, i_2} - \frac{1}{N}\, \delta_{i_1\, j_1}\, \delta_{i_2\, j_2} \tag{3.2.3}$$

While evaluating a diagram containing particles in the adjoint representation of the gauge group (such as gluons in QCD or all particles in $\mathcal{N} = 4$ SYM) with ordinary Feynman rules, one encounters interaction vertices whose color data are usually encoded in the structure constants.

If every structure constant in such vertices is replaced with (3.2.1), one is left with products of traces of generators, with some color indices contracted by propagators, which contain δ's of group indices. These contractions can be easily solved using (3.2.3), which sews multiplied traces into one. This can be made very simple if one refers to the group $U(N)$, adding a $U(1)$ factor to $SU(N)$. This is a generator, which commutes with every other generator of $SU(N)$, thus it is proportional to the identity: $T^a_{U(1)} = \frac{1}{\sqrt{N}}\, \delta_{ij}$. This factor is also called the photon. Considering the $U(N)$ gauge group, the structure constants are as above and vanish whenever a 0 index is present, whereas (3.2.3) simplifies in that the last term cancels out. Hence $U(N)$ can be also regarded as the large N, leading color approximation of $SU(N)$.

If external particles in other representations than the adjoint are present (such as fundamental fields) strings of generators $(T^{A_1} \ldots T^{A_n})_{i_1\, j_n}$ for external particles are produced.

It turns out that for a tree diagram containing only gluons the corresponding amplitude can be decomposed into the sum over permutations of the product of a single trace of color operators, which encodes color information, times a gauge invariant coefficient accounting for the kinematics, called the *partial amplitude*

$$A_n^{tree}(1 \ldots n) = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}\left(T^{A_{\sigma(1)}} \ldots T^{A_{\sigma(n)}}\right)\, \mathcal{A}_n^{tree}(\sigma(1) \ldots \sigma(n)) \tag{3.2.4}$$

The sum is over inequivalent permutations, that is the $(n-1)!$ which cannot be realized as a cyclic permutation of indices, under which the trace is invariant. This is the group of permutations of n indices S_n, where cyclic permutations $\mathbb{Z}_n$ are identified.

The importance of this decomposition lies in the fact that partial amplitudes are color ordered, that is they receive contributions only by diagrams sharing the same color structure of the trace. This greatly reduces the amount of diagrams to be taken into account, especially as the number of external particles grows.

By virtue of color decomposition the fundamental constituents, from which constructing an amplitude, are the partial, color ordered, amplitudes, that are of simpler evaluation. Henceforth color decomposition will be always applied in what follows.

As concerns loop amplitudes, color decomposition is derived in the same manner as for tree level. At one loop this leads to up to two traces over color generators, and a sum must be performed over the different spins J of the particles circulating in the loop. If all the particles are in the adjoint representation of the gauge group, color decomposition reads

$$A_n^{1-loop}(1\dots n) = \sum_{\text{spin J}} n_J \sum_{c=1}^{\lfloor \frac{n}{2}+1 \rfloor} \sum_{\sigma \in S_n/S_{n;c}} \mathrm{Tr}\left(T^{A_{\sigma(1)}} \dots T^{A_{\sigma(c-1)}}\right) \mathrm{Tr}\left(T^{A_{\sigma(c)}} \dots T^{A_{\sigma(n)}}\right) \mathcal{A}^J_{n;c}(\sigma(1)\dots\sigma(n)) \tag{3.2.5}$$

where n_J means the number of spin J particles, and $S_{n;c}$ denotes the group of permutations leaving the product of traces invariant. The first coefficient ($c = 1$) in the sum is made of the trace over the n generators, multiplied by a factor N, coming from the trace of the identity; the corresponding partial amplitude is thus called the *leading color amplitude* $\mathcal{A}^J_{n;1}(\sigma(1)\dots\sigma(n))$. For higher loops there appear more and more complicated trace structures, so that performing the planar limit $N \to \infty$ leads to great simplifications, restricting the whole amplitude to the sum over single trace contributions. Namely in this limit only leading color partial amplitudes appear. This is the setting in which we will mainly work, so that the amplitudes we will analyze are always understood to be color ordered.

3.3 Spinor helicity formalism.

Helicity formalism consists in a compact way of representing spinors of different chirality. It was developed to deal with external massless fermions and helicity vectors, required in the presence of external gauge bosons (gluons). This machinery provides a very compact way of expressing spinor products and momentum invariants, which the amplitude is supposed to depend on. Moreover it makes manifest a lot of identities which prove very useful in carrying out the evaluation of diagrams.

The key point is to consider a convenient basis for polarization vectors, which appear in the evaluation of amplitudes with external gluons. It is worth expressing these polarization vectors, of positive or negative helicity in terms of massless spinors. Let us introduce first a compact notation (customary in QCD literature) for the massless solutions to the Dirac equation (being massless, those for fermions and antifermions are equal) of definite chirality

$$\begin{aligned} u_\pm(k_i) &\equiv \frac{1}{2}(1 \pm \gamma_5)u(k_i) = v_\mp(k_i) \equiv 1(1 \pm \gamma_5)v(k_i) = |i^\pm\rangle \\ \overline{u_\pm(k_i)} &\equiv \overline{u(k_i)}\frac{1}{2}(1 \mp \gamma_5) = \overline{v_\mp(k_i)} \equiv \overline{u(k_i)}\frac{1}{2}(1 \mp \gamma_5) = \langle i^\pm| \end{aligned} \tag{3.3.1}$$

Amplitudes are Lorentz symmetric, so that they are expected to consist of kinematic invariants constructed from these spinors. Therefore a convenient way of representing Lorentz vectors and invariants thereof should be of great help. Expressing such quantities in terms of spinors provides such a shorthand notation.

$$\begin{aligned} \langle i\, j\rangle &\equiv \langle i^-\, j^+\rangle = \overline{u_-(k_i)}\, u_+(k_j) = \overline{v_+(k_i)}\, v_-(k_i) \\ [i\, j] &\equiv \langle i^+\, j^-\rangle = \overline{u_+(k_i)}\, u_-(k_j) = \overline{v_-(k_i)}\, v_+(k_i) \end{aligned} \tag{3.3.2}$$

These spinor products are naturally antisymmetric and vanish whenever the same spinor is self contracted

$$\langle i\, j\rangle = -\langle j\, i\rangle \qquad [i\, j] = -[j\, i] \qquad \langle i\, i\rangle = [i\, i] = 0 \tag{3.3.3}$$

It is worth pointing out that spinor invariants behave simply under parity which acts like $\langle i\, j\rangle \to [j\, i]$. This turns convenient when deriving an amplitude from one with the opposite choice of external helicities. Then these amplitudes are clearly related by a parity reversal, and the simple substitution mentioned above implements this operation on the amplitude. There are many identities which can be employed to simplify amplitudes, here we list a few of them:

- Fierz rearrangement

$$\langle i^+|\gamma_\mu|j^+\rangle\langle k^+|\gamma_\mu|l^+\rangle = 2[i\, k]\langle l\, j\rangle \tag{3.3.4}$$

- Charge conjugation of current

$$\langle i^+|\gamma_\mu|j^+\rangle = \langle j^-|\gamma_\mu|i^-\rangle \tag{3.3.5}$$

- Schouten identity

$$\langle i\, j\rangle\langle k\, l\rangle = \langle i\, k\rangle\langle j\, l\rangle + \langle i\, l\rangle\langle k\, j\rangle \tag{3.3.6}$$

- Trace

$$\langle i\, j\rangle[j\, i] = \mathrm{Tr}\left(\frac{1}{2}(1-\gamma_5)\not{k}_i\,\not{k}_j\right) = 2k_i \cdot k_j = (k_i + k_j)2 \equiv s_{ij} \tag{3.3.7}$$

  where the last step follows from the on-shell condition for massless momenta. This generalizes to:

$$\langle i_1\, i_2\rangle[i_2\, i_3]\ldots\langle i_{n-1}\, i_n\rangle[i_n\, i_1] = \mathrm{Tr}\left(\frac{1}{2}(1-\gamma_5)\not{k}_{i_1}\ldots\not{k}_{i_n}\right) \tag{3.3.8}$$

- If momentum is conserved: $\sum_{i=1}^{n} k_i = 0$, then another identity holds

$$\sum_{i=1}^{n} \langle k\, i\rangle [i\, j] = 0 \tag{3.3.9}$$

This formalism allows to represent polarization vectors for gluons in a very convenient fashion, i.e. in terms of spinor products. We won't give details of that here, which could be found for instance in [50].

In many applications to scattering amplitude these tools derived in QCD for polarization vectors are used more generally to represent momenta. A direct application of this will be outlined in Section 3.4.2. This procedure is based on the decomposition of the (euclideanized) Lorentz group $SO(4)$ into $SU(2) \times SU(2)$, such that vectors are expressed as bispinors. More precisely, taking the complexified Lorentz space $SO(3,1;\mathbb{C})$: it is locally isomorphic to $Sl(2;\mathbb{C}) \times Sl(2;\mathbb{C})$, hence its representations are labeled by a pair of half–integer or integer numbers (p,q). The simplest representations are spinors: $\left(\frac{1}{2},0\right)$ is the representation of a negative chirality spinor λ_a, $\left(0,\frac{1}{2}\right)$ that of a positive chirality spinor, denoted by $\tilde{\lambda}_{\dot{a}}$, where indices a and $\dot{a}$ run in the range 1, 2.

Indices a and $\dot{a}$ are raised and lowered by the antisymmetric tensors ϵ_{ab} and $\epsilon^{\dot{a}\dot{b}}$ and their inverse, such that contractions between spinors of the same chirality are easily performed (the minus sign in the definition of the square spinor product is introduced somewhat ad hoc, in order to keep the notation uniform with the previously reviewed, which is the standard in QCD literature)

$$\epsilon_{ab}\, \lambda^a\, \lambda^b \equiv \langle \lambda_1,\, \lambda_2\rangle \qquad , \qquad -\epsilon_{\dot{a}\dot{b}}\, \tilde{\lambda}^{\dot{a}}\, \tilde{\lambda}^{\dot{b}} \equiv [\lambda_1,\, \lambda_2] \tag{3.3.10}$$

The vector representation of $SO(3,1;\mathbb{C})$ is $(\frac{1}{2},\frac{1}{2})$. Hence a four dimensional momentum vector p_μ ($\mu = 0,\dots 3$) can be naturally rewritten as a bispinor $p_{a\dot{a}}$, with indices belonging to different chirality. This can be done by contracting it with the four dimensional vector of Pauli matrices

$$\sigma^\mu_{a\dot{a}} = (1,\sigma)_{a\dot{a}} \qquad , \qquad p_{a\dot{a}} = p_\mu\, \sigma^\mu_{a\dot{a}} \tag{3.3.11}$$

It follows that the squared momentum equals the determinant of the bispinor over spinor indices, due to the ϵ symbols:

$$p^\mu\, p_\mu = p^{a\dot{a}}\, p_{a\dot{a}} = p_{a\dot{a}}\, \epsilon^{ab}\, \epsilon^{\dot{a}\dot{b}}\, p_{b\dot{b}} \equiv 2 \det p_{a\dot{a}} \tag{3.3.12}$$

The light–cone condition $p^2 = 0$ corresponds to the vanishing of this determinant, then if p_μ is light–like, the bispinor can be factorized into the product of two spinors of opposite chirality

$$p_{a\dot{a}} = \lambda_a\, \tilde{\lambda}_{\dot{a}} \tag{3.3.13}$$

This ensures the vanishing of the determinant, because spinor products are antisymmetric and the determinant would equal $\det p_{a\dot{a}} = \langle \lambda\, \lambda\rangle [\lambda\, \lambda]$. The spinors λ and $\tilde{\lambda}$ are determined by p up to a rescaling $(t\lambda, t^{-1}\tilde{\lambda})$ with $t \in \mathbb{C}^*$, so that their product remains constant. The inner product between two light–like vectors is given by

$$2\, p \cdot q = p_{a\dot{a}}\, \epsilon^{ab}\, \epsilon^{\dot{a}\dot{b}}\, q_{b\dot{b}} = +\langle p\, q\rangle [q\, p] \tag{3.3.14}$$

in agreement with (3.3.7). If momenta are supposed to be real (as in a real Minkowski space), then the reality condition reads $\tilde{\lambda} = \pm\bar{\lambda}$, but it is often useful to extend the analysis and work with complexified momenta, thus keeping λ and $\tilde{\lambda}$ independent. We refer also to the former (negative chirality) as the holomorphic spinor and to the latter (positive chirality) as the antiholomorphic. All this applies when working with the Lorentzian signature $+---$; considering the signature $++--$, which may be useful, spinor representations are real, so that λ and $\tilde{\lambda}$ are real and independent.
Finally, the following strings of spinor products can be synthesized in compact form by:

$$\begin{aligned} &\sum_a \langle i\, a\rangle [aj] \equiv \langle i| \sum k_a |j] \\ &\sum_a \sum_b \sum_a \langle i\, a\rangle [a\, b]\langle b\, j\rangle \equiv \langle i| \sum_a \sum_b k_a\, k_b |j] \end{aligned} \tag{3.3.15}$$

and so on. This is a generalization of (3.3.8), where slashes are dropped to avoid clutter.

3.4 Tree level amplitudes.

Poincaré invariance. All amplitudes in field theory must satisfy invariance under the Poincaré group. Lorentz symmetry is automatically built–in, when expressing the amplitude in terms of spinor products, which are manifestly Lorentz invariant. It is well–known that translations determine conservation of total momentum $\sum_{i=1}^n p_i^\mu = 0$. Stated another way, the generator of translations $p_{\alpha\dot{\alpha}}$ has to annihilate the amplitude

$$p^{\alpha\dot{\alpha}}\, \mathcal{A}_n(p) = 0 \qquad , \qquad p^{\alpha\dot{\alpha}} = \sum_{i=1}^n p_i^{\alpha\dot{\alpha}} \tag{3.4.1}$$

a condition which is fulfilled by writing the amplitude as a distribution and exposing a δ–function explicitly enforcing the constraint

$$\mathcal{A}_n(p) \propto \delta^{(4)}\big(\sum_{i=1}^n p_i\big) \tag{3.4.2}$$

Equivalently, using spinor notation

$$p^{\alpha\dot{\alpha}}\, \mathcal{A}_n(p) = 0 \qquad \Longrightarrow \qquad \mathcal{A}_n\left(\lambda, \tilde{\lambda}\right) \propto \delta^{(4)}\big(\sum_{i=1}^n \tilde{\lambda}_i\, \lambda_i\big) \qquad , \qquad p^{\alpha\dot{\alpha}} = \sum_{i=1}^n \tilde{\lambda}_i^\alpha\, \lambda_i^{\dot{\alpha}} \tag{3.4.3}$$

At tree level gluonic amplitudes are the same in QCD and in $\mathcal{N} = 4$. Form now on we will consider only the latter.

Conformal invariance. Given that $\mathcal{N} = 4$ SYM is invariant under the conformal group $SO(2,4)$, amplitudes should be constrained by this symmetry as well. At tree level, where infrared divergences are absent, the amplitudes must be annihilated by the generators of the conformal algebra. At loop level dilatations and special conformal transformations are anomalously broken by the emergence of divergences. In such a situation the $SO(2,4)$ may still play a role in constraining amplitudes, in the form of anomalous Ward identities, in principle. In practice, however, the conformal group acts linearly on configuration space, whereas its action on particles momenta is complicated and it is not efficient to exploit this symmetry*.
For instance the conformal boost generator $k_{\alpha\dot{\alpha}}$ acts on the amplitude in terms of a second order differential operator

$$k_i^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial\lambda_i^{\alpha}\partial\tilde{\lambda}_i^{\dot{\alpha}}} \qquad , \qquad k^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} k_i^{\alpha\dot{\alpha}} \tag{3.4.4}$$

The dilatation operator reads in this formalism

$$d_i = \frac{1}{2}\lambda_i^{\alpha}\frac{\partial}{\partial\lambda_i^{\alpha}} + \frac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\alpha}}} + 1 \qquad , \qquad d = \sum_{i=1}^{n} d_i \tag{3.4.5}$$

The coefficients 1/2 are natural since $\dim[p] = \dim[\lambda\tilde{\lambda}] = 1$, whereas the constant piece is fixed by closure of the $SO(2,4)$ algebra, reviewed in Appendix A.4.

Supersymmetry. Supersymmetry allows to connect amplitudes with different kinds of external particles (scalars, fermions) to gluonic ones. These relations are called supersymmetric Ward identities SWI. They are derived from the fact that any supercharge must annihilate the vacuum $Q|0\rangle = 0$ and from the supersymmetric transformation rules

$$\begin{aligned}
&[Q^a(\eta),\, g^{\pm}(k)] = \mp\Gamma^{\pm}(k,\eta)\,\lambda^{a\pm}(k)\\
&[Q^a(\eta),\, \lambda^{b\pm}(k)] = \mp\Gamma^{\pm}(k,\eta)\,g^{\pm}(k)\,\delta^{ab} \mp i\,\Gamma^{\pm}(k,\eta)\,\phi_{\pm}^{ab}\\
&\Gamma(k,\eta)^{+} = \theta\,[\eta,\, k] \qquad , \qquad \Gamma(k,\eta)^{-} = \theta\,\langle\eta,\, k\rangle
\end{aligned} \tag{3.4.6}$$

where g labels gluons, Λ denote gluinos and ϕ scalars. The action of Q turns gluons into gluinos and viceversa, up to a function Γ which depends on the helicity of the particle and can be chosen to be the product of a spinor contraction times an anticommuting Grassmann parameter θ. The reference spinor η is arbitrary and may be chosen conveniently so as to simplify computations. Besides their utility in relating amplitudes, these SWI may also be employed to show the vanishing of some gluonic amplitudes, as follows. For instance one may start from the amplitude $\langle 0|[\Lambda^+ g^+ g^+ \dots g^+]|0\rangle$, which vanishes identically because fermions can be only created from the vacuum in pairs with opposite helicities

*The situation may be improved by transforming to twistor space, where the conformal group acts linearly on twistor variables and its constraints are manifest [51].

(differently from gluons, fermion interactions are always helicity preserving), and act with Q

$$\begin{aligned} 0 &= \langle 0|[Q(\eta(q)), \Lambda^+ g^+ g^+ \dots g^+]|0\rangle \\ &= -\Gamma^+(q,k_1)A(g^+g^+\dots g^+) - \sum_i \Gamma^-(q,k_i)\mathcal{A}(\Lambda^+ g^+ \dots \Lambda_i^+ g^+) \end{aligned} \tag{3.4.7}$$

which entails that gluonic amplitudes with all the same helicities vanish, since all terms in the sum do so. Similarly, acting on $\langle 0|[Q(\eta(q), \Lambda^+ g^- g^+ \dots g^+]|0\rangle$ gives

$$\begin{aligned} 0 &= \langle 0|[Q(\eta(q)), \Lambda^+ g^- g^+ \dots g^+]|0\rangle \\ &= -\Gamma^+(q,k_1)A(g^+g^-\dots g^+) + \Gamma^-(q,k_2)A(\Lambda^+\Lambda^- g^+ \dots g^+) \\ &\quad - \sum_i \Gamma^+(q,k_i)\mathcal{A}(\Lambda^+ g^- g^+ \dots \Lambda_i^+ g^+) \end{aligned} \tag{3.4.8}$$

Then a clever choice for the reference spinor $q = k_2$, implies the vanishing of the amplitude with one negative helicity gluon as well. Therefore one concludes that

$$A^{\text{tree}}(g^+ \dots g^+) = 0 \qquad , \qquad A^{\text{tree}}(g^- g^+ \dots g^+) = 0 \tag{3.4.9}$$

Classifying gluonic amplitudes by the number of minus helicity gluons, the first non–vanishing, with an arbitrary number of external particles, has the property of showing only two negative helicity gluons and the other positive (mostly plus). We refer to these amplitudes as maximally helicity violating, shortened MHV, because if less negative legs are present the amplitude vanishes identically. These amplitudes play an important role in that they seem to possess the highest degree of simplicity, at tree as well as at loop level, and exhibit many remarkable properties.

Increasing the number of negative helicity gluons the corresponding amplitudes are usually dubbed as $\mathrm{N^kMHV}$, until the mostly minus amplitude is reached which is sometimes called $\overline{\mathrm{MHV}}$. For n external gluons there are $n-4$ non–vanishing helicity configurations, ranging from the MHV at $k=0$ to the $\overline{\mathrm{MHV}}$ at $k=n-4$.

3.4.1 MHV amplitudes.

The first indication that MHV amplitudes are somewhat special comes from their tree level structure, for which an all multiplicity formula exists, which is called the Parke–Taylor amplitude. This looks very simple in the spinor helicity formalism

$$\mathcal{A}_{MHV}^{\text{tree}}(1^+ \dots i^- \dots j^- \dots n) = \delta^4\left(\sum_{i=1}^n k_i\right) \frac{\langle ij\rangle^4}{\prod_{k=1}^n \langle k, k+1\rangle} \tag{3.4.10}$$

where k is understood to be cyclic and i and j label the negative helicity gluons, and we have made explicit the momentum δ–function in front. The case of a mostly minus amplitude with two positive legs is related to the conventional mostly minus MHV by a parity transformation and are therefore obtained from the former by replacing angular spinor products $\langle\ \rangle$ by square ones $[\ ]$.

It proves convenient to organize the whole perturbative series of MHV amplitudes factoring out the tree level result, which naturally pops out in loop calculations, by writing

$$\mathcal{A}_n^{MHV} = \mathcal{A}_n^{tree} \left(1 + \sum_{l=1}^{\infty} a^l \, \mathcal{M}_n^{(l)} \right) \tag{3.4.11}$$

where a denotes the effective coupling $\frac{g^2 N}{8\pi}$, which is customary for loop computations in $\mathcal{N}=4$ SYM.

MHV amplitudes are also taken as a benchmark for $\mathrm{N^k}$MHV ones, which may conveniently be organized as

$$\sum_{k=0}^{\lfloor (n-4)/2 \rfloor} \mathcal{A}_n^{N^k MHV} = \mathcal{A}_n^{MHV} \left(1 + \sum_{k=1}^{\lfloor (n-4)/2 \rfloor} P_n^{(k)} \right) \tag{3.4.12}$$

where only the half of mostly plus amplitudes has been displayed.

Remarkably, MHV amplitudes exhibit rather special properties under the conformal group $SO(2,4)$ of $\mathcal{N}=4$ SYM. Indeed formula (3.4.10) shows that they are holomorphic objects, in that only spinors of definite chirality λ_α appear. In particular this means that they are explicitly annihilated by the conformal boost generator (whose form can be read from A.4.5)

$$k_{\alpha\dot{\alpha}} \, A_{MHV}^{\rm tree}(1^+ \dots i^- \dots j^- \dots n) = 0 \tag{3.4.13}$$

By explicit computation [51] it is found that the momentum δ–function is invariant under the action of k as well, so that the whole amplitude is so. As concerns invariance under dilatation d (3.4.5), the numerator of the MHV amplitude has weight 4 under the action of $\sum_{i=1}^{n} \frac{1}{2} \lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} + \frac{1}{2} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}}$. The δ–function has weight -4, canceling that of the numerator. The denominator in (3.4.10) has uniform degree -1 for each i and cancels out with the $+1$ in the definition (3.4.5) of the generator.

3.4.2 Recursion relations.

At tree level and for pure glue amplitudes, a recurrence relation was introduced by Britto, Cachazo and Feng [52] and later demonstrated by the same authors and Witten (BCFW) [53]. The relation is very simple and allows to derive tree amplitudes from lower points ones, which are supposed to be already known, and the result turns out to be expressed in a very compact form. The starting point is to extend the amplitude $\mathcal{A}_n$ to a complex function $\mathcal{A}_n(z)$, where z is a complex variable. This can be achieved picking up two external momenta i and j and shifting them by a complex term, preserving their on-shell condition and momentum conservation:

$$\tilde{\lambda}_i \to \tilde{\lambda}_i - z\,\tilde{\lambda}_j \qquad , \qquad \lambda_j \to \lambda_j + z\,\lambda_i \tag{3.4.14}$$

while keeping λ_i and λ_j untouched. Momentum conservation is preserved because

$$k_i + k_j = \lambda_i\,\tilde{\lambda}_i + \lambda_j\,\tilde{\lambda}_j \to \lambda_i\,\tilde{\lambda}_i - z\,\lambda_i\,\tilde{\lambda}_j + \lambda_j\,\tilde{\lambda}_j + z\,\lambda_i\,\tilde{\lambda}_j = \lambda_i\,\tilde{\lambda}_i + \lambda_j\,\tilde{\lambda}_j \tag{3.4.15}$$

On–shellness follows from

$$k_i^2 = \langle\lambda_i\,\lambda_i\rangle\,[\tilde{\lambda}_i\,\tilde{\lambda}_i] = 0 \tag{3.4.16}$$

because the product $\langle\lambda_i\,\lambda_i\rangle$ vanishes, being unaffected by the shift†.

It is worth noting that $\mathcal{A}_n(z)$ is thus a physical on–shell amplitude for any z. It follows that $\mathcal{A}_n(z)$ is a rational function of z, because the original tree amplitude was so. Indeed every tree amplitude possesses only single poles arising from internal propagators, whose momenta are always a sum over external adjacent momenta, which we can denote by $K_{rs} = k_r + \cdots + k_s$. If such a sum contains both or neither k_i and k_j it is not z–dependent and no poles arise in z. If instead it contains only one of the two, say j, the propagator momentum K_{rs} does depend on z and is therefore shifted by $K_{rs}(z) = K_{rs} + z\lambda_i\tilde{\lambda}_j$. A single pole in z does arise because the shifted inverse propagator $K_{rs}^2(z) = K_{rs}^2 + z\langle i|K_{rs}|j]$ vanishes at $z_{rs} = -K_{rs}^2/\langle i|K_{rs}|j]$. These are the only poles for $\mathcal{A}_n(z)$. The physical quantity we are interested in is $\mathcal{A}_n(0)$. This can be achieved considering Cauchy's theorem for the contour integral

$$\frac{1}{2\pi i}\oint \frac{dz}{z}\,\mathcal{A}_n(z) \tag{3.4.17}$$

In general this integral is given by two contributions: the residues at poles, one of which is just $\mathcal{A}_n(0)$, because a pole in z was created by $1/z$, and the contribution from the circle at infinity C_∞. These must give zero:

$$\mathcal{A}_n(0) = -C_\infty - \sum_{r,s} \mathrm{Res}_{z=z_{rs}}\,\frac{\mathcal{A}_n(z)}{z} \tag{3.4.18}$$

At tree level the odd contribution from C_∞ drops out, because it can always be made vanishing with a proper choice of the helicities of the reference momenta i and j, as demonstrated in [53]. This means

$$\mathcal{A}_n(0) = -\sum_{r,s} \mathrm{Res}_{z=z_{rs}}\,\frac{\mathcal{A}_n(z)}{z} \tag{3.4.19}$$

†As pointed out in [53] this deformation by a complex amount does not actually make sense in real Minkowski space, because it violates the reality condition for momenta: $\tilde{\lambda} = \pm\bar{\lambda}$. It is valid only in complex Minkowski space or in a real, after Wick rotating to the signature $++--$. Once the amplitude has been computed, it is expressed by spinor products, which are invariant, and thus one can restore the Lorentz signature.

In order to evaluate the residues it is useful to remember that such single poles occur in tree amplitudes multiparticle factorization, that is splitting them into two amplitudes A_L and A_R joined by a scalar propagator $i/K^2_{rs}(0)$, which determines the singularity going on-shell. This observation leads to the final form of the BCFW recurrence formula:

$$\mathcal{A}_n(0) = \sum_{r,s} \sum_{h=\pm} A^h_L(z = z_{rs}) \frac{1}{K^2_{rs}} A^{-h}_R(z = z_{rs}) \tag{3.4.20}$$

where the sum runs over poles and over the two possible choices for the helicity of the internal propagator.

The recurrence relation actually trades the evaluation of a tree amplitude with the sum over products of simpler lower points amplitudes, shifted by a certain amount. A clever choice of the reference momenta can reduce the contributions to be summed. It turns out, that a good choice is often to take them adjacent. This way these momenta are held fixed and the sum is over all possible distributions of the external legs between the left and right amplitudes, as depicted in Figure 3.1.

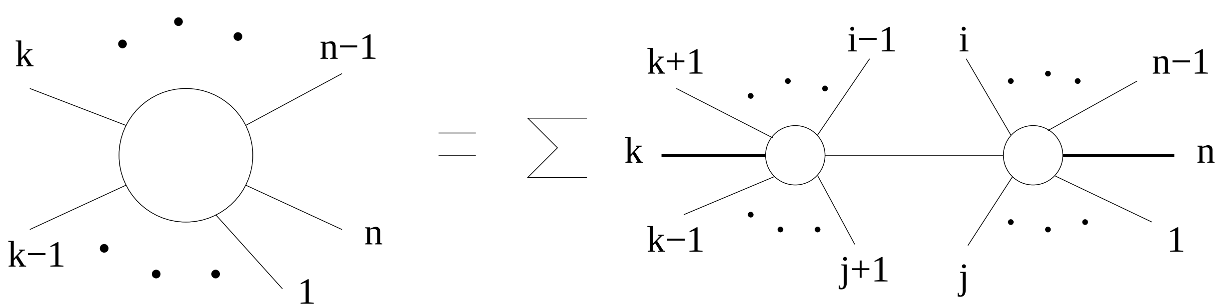


Figure 3.1: Pictorial representation of the BCFW recursive relation.

Some of the combinations may vanish, e.g. those with all the same helicities or one different. The best choice for the helicities of the reference momenta may vary and depends on the particular amplitude one is facing.

The demonstration of formula (3.4.20) lies also on the assumption that the shifted amplitude vanishes at infinity, in order to discard the contribution of the circle C_∞ in (3.4.18). It was pointed out that this is always possible. However if one takes the shift (3.4.14), that is shifting the positive chirality spinor of the first $\tilde{\lambda}_i$ and the negative chirality spinor of the second λ_j, it turns out [53] that the only helicity configurations of the reference momenta (h_i, h_j) for the shifted amplitude to vanish at infinity are $(-,-)$, $(+,+)$ and $(-,+)$. Otherwise one has to exchange the roles of i and j to use the helicities $(+,-)$. It is important to point out that at intermediate stages one works with complexified momenta and this means that three point amplitudes do not vanish and must be evaluated like mostly minus $(++-)$ or mostly plus $(--+)$ MHV amplitudes. To each diagram in the sum is associated a different shift given by:

$$\begin{aligned} \lambda_{\hat{i}} &= \lambda_i \quad , \quad \tilde{\lambda}_{\hat{i}} = \tilde{\lambda}_i + \frac{K_{rs}^2}{\langle i|K_{rs}|j]}\,\tilde{\lambda}_j \\ \tilde{\lambda}_{\hat{j}} &= \tilde{\lambda}_j \quad , \quad \lambda_{\hat{j}} = \lambda_j - \frac{K_{rs}^2}{\langle i|K_{rs}|j]}\,\tilde{\lambda}_i \end{aligned} \tag{3.4.21}$$

Amplitudes are to be evaluated taking into account this shift. No difficulty occurs when the shift is due to a spinor product containing a reference momentum. Nevertheless the shifts in the amplitudes are given also by the presence of a leg corresponding to the internal propagator, which we will call $\hat{P}$. This is also shifted. Two cases can arise: $\langle a\hat{P}\rangle$ and $[\hat{P}b]$, where a and b are generic external momenta. These can be handled as follows:

$$\langle a\,\hat{P}\rangle = \frac{\langle a\,\hat{P}\rangle[\hat{P}\,j]}{[\hat{P}\,j]} = \frac{\langle a|\hat{P}|j]}{[\hat{P}\,j]} \tag{3.4.22}$$

where the last equality descends from (3.3.15). Analogously

$$[\hat{P}\,b] = \frac{\langle i\,\hat{P}\rangle[\hat{P}\,b]}{\langle i\,\hat{P}\rangle} = \frac{\langle i|\hat{P}|b]}{\langle i\,\hat{P}\rangle} \tag{3.4.23}$$

Since the momentum P has opposite helicities on the two side amplitudes, the factors in the denominators of (3.4.22) and (3.4.23) are always paired and can be easily evaluated thanks to (3.3.15):

$$\langle i\,\hat{P}\rangle[\hat{P}\,j] = \langle i|\hat{P}|j] \tag{3.4.24}$$

It is worth mentioning [52] that applying the recursive relation many times on the same amplitude, one can in principle reduce it to the combination of only trivalent vertices $(++-)$ or $(+--)$, suggesting that the theory might be encoded by an effective Lagrangian describing a scalar field with cubic interactions.

3.4.3 On–shell supersymmetry.

$\mathcal{N}=4$ SYM is superconformal invariant. As mentioned in Section 3.4, supersymmetry is a powerful tool for scattering amplitudes. Indeed, instead of using SWI's it proves very convenient to introduce the notion of on–shell superspace, in terms of which amplitudes in $\mathcal{N}=4$ SYM may be compactly rearranged in terms of superamplitudes.

The on–shell particle content of the theory is in terms of gluons $G^\pm$ of plus and minus helicity, eight fermionic gluinos Γ_A and $\bar{\Gamma}^A$, and six complex scalars S_{AB}. The indices $A, B = 1\dots4$ are in the fundamental **4** representation of the R–symmetry group $SU(4)$: fermions transform in the **4** and $\bar{\mathbf{4}}$, respectively, whereas scalars are in the vector of $SO(6)$, which is the **6** and they are antisymmetric in their indices $S_{AB} = -S_{BA}$.

These states are accounted for by the introduction of on–shell superspace. The supersymmetry algebra is

$$\left\{q_\alpha^A\,,\,\bar{q}_{B\,\dot{a}}\right\} = \delta_B^A\,p_{\alpha\dot{a}} \tag{3.4.25}$$

where the supercharges q_α^A and $q_{\dot{\alpha}\,A}$ are in the (anti)fundamental representation of $SU(4)$. On shell, where $p^2 = 0$, one may project the supercharge onto light–cone ones, in terms of which the algebra turns out to be simply of a Clifford type. However this procedure breaks Lorentz symmetry explicitly. A Lorentz covariant manner of performing the projection may be derived, which splits the supercharges into parallel $\lambda_\alpha\,q^A$ ($\tilde{\lambda}_{\dot{\alpha}}\,\bar{q}_A$) and orthogonal $\lambda^\alpha q_\alpha^A$ ($\tilde{\lambda}^{\dot{\alpha}}\,\bar{q}_{\dot{\alpha}\,A}$) components with respect to the spinors λ and $\tilde{\lambda}$, respectively. Then, the orthogonal part decouples, anticommuting with everything, and the algebra becomes simply

$$\{q^A, \bar{q}_B\} = \delta_B^A \tag{3.4.26}$$

These Clifford anticommutation relations can be solved by introducing Grassmann variables η

$$q^A = \eta^A \qquad , \qquad \bar{q}_A = \frac{\partial}{\partial\eta^A} \qquad , \qquad \{\eta^A, \eta^B\} = 0 \tag{3.4.27}$$

and identifying the supergharges as

$$q_\alpha^A = \lambda_\alpha\,\eta^A \qquad , \qquad \bar{q}_{A\,\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}}\,\frac{\partial}{\partial\eta^A} \tag{3.4.28}$$

Then the content of the $\mathcal{N} = 4$ SYM supermultiplet is encoded in a super wavefunction

$$\begin{aligned}\Phi(p,\eta) \;&=\; G^+(p) + \eta^A\,\Gamma_A(p) + \frac{1}{2}\,\eta^A\,\eta^B\,S_{AB}(p) + \frac{1}{3!}\,\eta^A\,\eta^B\,\eta^C\,\epsilon_{ABCD}\,\bar{\Gamma}^D(p)\\ &\quad + \frac{1}{4!}\,\eta^A\,\eta^B\,\eta^C\,\eta^D\,\epsilon_{ABCD}\,G^-(p)\end{aligned} \tag{3.4.29}$$

Thanks to maximal supersymmetry and, consequently, the self–CPT conjugacy property of the supermultiple, a single wave–function is sufficient to span all the content and moreover allows for a completely holomorphic (i.e. $\bar{\eta}$ independent) description, making on–shell $\mathcal{N} = 4$ superspace chiral.

Given this powerful formalism of super wavefunctions, it is natural to consider amplitudes for such objects

$$\mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = \mathcal{A}(\Phi(1)\,\Phi(2)\,\ldots\Phi(n)) \tag{3.4.30}$$

which are referred to as the on–shell superamplitudes. Invariance of the latter under supersymmetry generators $q_\alpha^A = \sum_{i=1}^n q_{i\,\alpha}^A = \sum_{i=1}^n \eta_i^A\,\lambda_{i\,\alpha}$ implies that the superamplitude, as a distribution, contains a further Grassmann δ–function

$$p_{\alpha\dot{\alpha}}\;\mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = q_\alpha^A\;\mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = 0 \;\Rightarrow\; \mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)}(p_{\alpha\dot{\alpha}})\;\delta^{(8)}(q_\alpha^A)\;\mathcal{R}_n(\lambda, \tilde{\lambda}, \eta) \tag{3.4.31}$$

where $\mathcal{R}_n$ is some polynomial in the Grassmann variables η_i^A. Actually the superamplitude has to be a singlet under the R–symmetry $SU(4)$, so that the η's have to appear in quartic invariants $\epsilon_{ABCD}\,\eta_i^A\,\eta_j^B\,\eta_k^C\,\eta_l^D$ (independently of the particle to which they refer), meaning that $\mathcal{R}$ is a polynomial of degree 4. This allows to expand (3.4.31) in pieces of the same Grassmann degree $4k$, corresponding to the k–th degree on non–MHVness of the given part of the superamplitude

$$\mathcal{A}_n(\lambda,\tilde{\lambda},\eta) = i(2\pi)^4\;\delta^{(4)}(\sum_{i=1}^{n}\;\lambda_i^{\alpha}\,\tilde{\lambda}_i^{\dot{\alpha}})\;\delta^{(8)}(\sum_{i=1}^{n}\;\lambda_i^{\alpha}\,\eta_i^{A})\;\left[\mathcal{R}_n^{(0)}+\mathcal{R}_n^{(4)}+\mathcal{R}_n^{(8)}+\ldots+\mathcal{R}_n^{(4n-16)}\right] \tag{3.4.32}$$

where the first component yields the MHV amplitude and the last the $\overline{MHV}$ one. It is also customary to rewrite the result above, factoring out the MHV amplitude as

$$\mathcal{A}(\Phi_1,\ldots,\Phi_n) = \mathcal{A}_n = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\ldots\langle n1\rangle}\mathcal{P}_n(\lambda_i,\tilde{\lambda}_i,\eta_i) = \mathcal{A}_n^{\mathrm{MHV}}\mathcal{P}_n \tag{3.4.33}$$

where $\mathcal{P}$ is again a polynomial in Grassmann variables (just as $\mathcal{R}$, but more conveniently normalized)

$$\mathcal{P}_n = 1+\mathcal{P}_n^{\mathrm{NMHV}}+\mathcal{P}_n^{\mathrm{NNMHV}}+\;\ldots\;+\mathcal{P}_n^{\overline{\mathrm{MHV}}}. \tag{3.4.34}$$

This is a generating function for amplitudes: a given choice of the external particle corresponds to a certain polynomial in the η_i variables, with suitable degree for each point i. Expanding the Grassmann δ–function and the $\mathcal{P}$ polynomial and selecting the desired combination of η's is an algorithmic procedure to find out the amplitude. The Grassmann δ–function making the degree starting from 8, automatically and consistently enforces the vanishing of all–plus and all–but–one–plus amplitudes.

The remaining generators $\bar{q}_{A,\dot{\alpha}} = \sum_{i=1}^{n}\tilde{\lambda}_{i\,\dot{\alpha}}\frac{\partial}{\partial\eta_i^A}$ do annihilate the MHV part, since when acting on the fermionic δ–function they turn it into the momentum conservation one. For other component in the expansion (3.4.33) invariance poses further constraints. In particular the entire superamplitude (at tree level) must be annihilated by all of the superconformal generators, whose form is reviewed in Appendix A.4.

The formalism just introduces proves very useful also when dealing with sums over $\mathcal{N}=4$ SYM states such as those encountered when adopting the BCFW recursion relations. Indeed they may be completely re–expressed in terms of superamplitudes and the sum over helicities of the internal line joining the factorized amplitudes replaced by an integral over the Grassmann η variables associated to it.

3.5 Unitarity based methods.

3.5.1 Unitarity cuts.

Unitarity of the S–matrix provides an efficient tool for constructing loop amplitudes at order L, from amplitudes (with more external legs) at previous orders, if known. The ultimate goal is to use only on–shell information to recover the loop amplitude. Such a procedure was developed by Bern, Dixon, Dunbar and Kosower, and is usually referred to as the unitarity cuts [54, 55]. This method is based on the Cutkoski rules, which relate the imaginary part of a Feynman diagram to its branch cuts. When considering the S–matrix one usually separates its interacting part T as

$$S = 1 + i\,T \tag{3.5.1}$$

Unitarity of the S–matrix then implies

$$i\,(T^{\dagger} - T) = 2\,\mathrm{Im}\,(T) = T\,T^{\dagger} \tag{3.5.2}$$

The naive physical interpretation of such an equation is that the imaginary part of T, at some order in perturbation theory, may be detected from the product of two (on–shell!) lower order amplitudes. The cutting rules provide a systematic way of evaluating this quantity. They consist in considering all possible cuts of a diagram in which the cut propagators can be put simultaneously on–shell. Putting them on–shell means replacing $1/p^2$ by $\delta(p^2)$–functions (i.e. removing the principal value part of the propagator) and evaluating the resulting loop integral. More precisely, one replaces

$$\frac{1}{k^2 + i\epsilon} \longrightarrow -2\pi i\,\delta(k^2) \tag{3.5.3}$$

In the end a sum is performed over all possible cuts. This offers a simple algorithm to get the imaginary part of the amplitude.

From the LHS of (3.5.2) $T^{\dagger} - T$, the imaginary part of the amplitude is understood to be connected to the branch cut discontinuities of the amplitude. More precisely the discontinuity is for the amplitude as a complex function of the Lorentz invariant associated to the squared sum of the cut momenta $(p_{i_1} + \cdots + p_{i_k})^2$, which is also referred to as the channel of the cut. The bottom–line is that such a discontinuity may be easily determined from products of lower order amplitudes.

Applying the method outlined above, the loop integral transforms into a Lorentz invariant phase space integral of the product of two lower order amplitudes, corresponding to the two sides of the cut.

We consider as an example the one–loop four–gluon color–ordered amplitude, and look at the s–channel cut in Fig. 3.2,

$$\begin{aligned} -i\,\mathrm{Disc}\,\mathcal{A}_4^{1-loop}(1,2,3,4) &= 2\,\mathrm{Im}\left(\mathcal{A}_4^{1-loop}\right) \\ &= \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}(2\pi)^2\delta(l_1^2)\mathcal{A}_4^{tree}(1,l_2,-l_1,2)\,\delta(l_2^2)\,\mathcal{A}_4^{tree}(-l_2,3,4,l_1) \end{aligned} \tag{3.5.4}$$

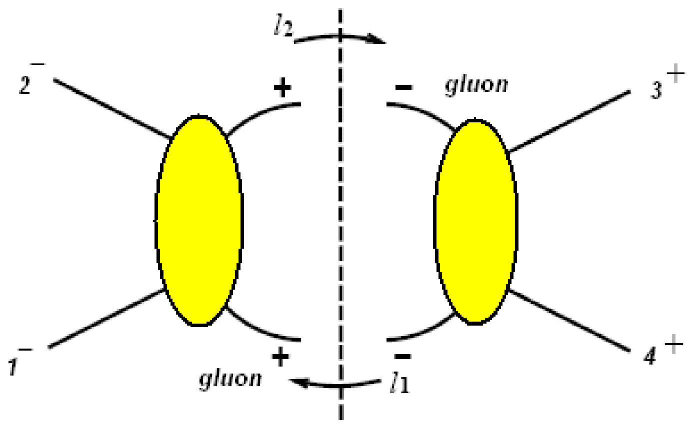


Figure 3.2: Cuts for the one–loop four–point amplitude, s–channel, singlet cut.

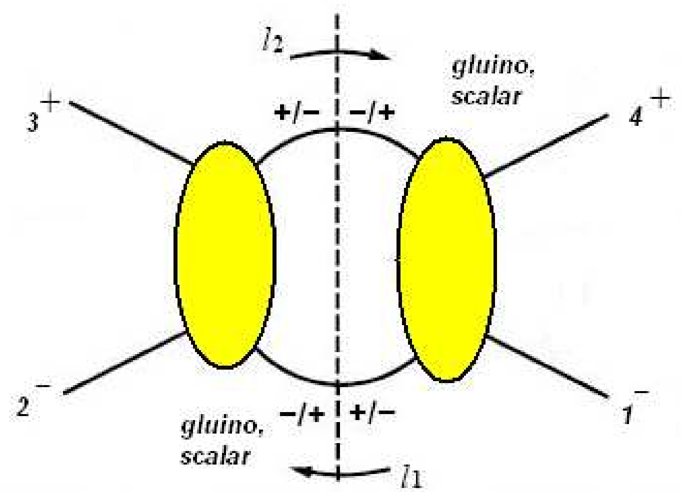


Figure 3.3: Cuts for the one–loop four–point amplitude, t–channel, non–singlet cut.

where l_2 and l_1 are the cut momenta, and Disc stems for the discontinuity across the branch cut. Momenta p, l_2 and l_1 are connected by momentum conservation. For example, choosing $p = l_2$, momentum conservation yields $l_2 = l_1 + k_3 + k_4$ and $l_2 = l_1 - k_1 - k_2$. Some remarks are in order.

- First note that the presence of the δ-functions turns the loop integral into a Lorentz invariant phase space one, whose measure is

$$\frac{d^{4-2\epsilon}p}{(2\pi)^{2-2\epsilon}}\,\delta(l_1^2)\,\delta(l_2^2) \tag{3.5.5}$$

- These amplitudes may contain both ultraviolet and infrared divergences, which have been regulated through dimensional regularization, from four dimensions to $d = 4 - 2\epsilon$. This point is tricky since one should perform all computations (including cuts) in non–integer dimension, which brings much more effort. We will come back to this later.

- For an L–loop amplitude each channel entails cutting $L + 1$ internal propagators, separating the amplitude into two disconnected pieces $\mathcal{A}^{l-loop}$ and $\mathcal{A}^{(L-l)-loop}$ such that the sum of their loop orders gives L.

- Each cut could be singlet Fig. 3.2 or non–singlet Fig. 3.3, meaning that many particles can in principle run in the loop of the cut propagators. One has to take this into account by summing over all particle species which can contribute. Alternatively, working conveniently with superamplitudes the sum may be substituted by an integral over the Grassmann variables η_i corresponding to the cut lines. Therefore the unitarity method can be very efficiently performed directly on the superamplitude.

The advantages of this procedure arise since the lower order amplitudes in (3.5.4) can be simplified before evaluating the cut integral and are usually easier to compute: in the case of a one–loop calculation they are just tree level amplitudes. Moreover the on–shell condition for the cut momenta can be employed to simplify expressions in the numerators of the resulting integrals.

The drawback is that in trying to derive the whole amplitude, i.e. the real part, one has to exploit dispersion relations. For any complex function which is analytic in the upper half plane it holds

$$\mathrm{Re}\,(f(s)) = \mathcal{P} \int_{-\infty}^{+\infty} \frac{dz}{\pi} \frac{\mathrm{Im}\,(f(z))}{z-s} - C_\infty \tag{3.5.6}$$

that in principle allows to get the real part of the amplitude by a Hilbert transform of the cut. However this is true only up to a rational (cut–free) additive piece, which is related to the circle at infinity and cannot be detected by the cut. If this term does not vanish, an ambiguity is there, spoiling the completeness of the result. Instead of working out the imaginary part and then using the Kramers–Kronig relation (3.5.6), one can follow a different route. We said that cutting a propagator means replacing it by a δ–function, which amounts to discarding its principal value part, according to the Feynman prescription on its regularization:

$$\frac{1}{k^2 + i\epsilon} = \mathcal{P}\left(\frac{1}{k^2}\right) - 2\pi i\, \delta(k^2) \tag{3.5.7}$$

The whole singular part of the amplitude, sensible to a given cut, must come from the entire cut loop integral (not just that over the (3.5.5) measure). This is obtained by reinstating the principal value part as well, thus replacing back propagators $1/k^2$ instead of δ–functions, however keeping the amplitudes in the integrand still on–shell. This allows to evaluate the cut part of the amplitude, both real and imaginary, in a given channel as an unrestricted (not phase space) integral over loop momentum substituting the δ–functions by propagators, but still keeping using on–shell conditions in the numerator [55, 54]:

$$\mathcal{A}_4^{1-loop}(1,2,3,4)\Big|_{s-cut} = \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}\, \frac{i}{l_2^2}\, \mathcal{A}_4^{tree}(1,l_2,-l_1,2)\, \frac{i}{l_1^2}\, \mathcal{A}_4^{tree}(-l_2,3,4,l_1) \qquad (3.5.8)$$

Then, by analyzing every channel, one can reconstruct the piece of the amplitude containing logarithms and polylogarithms, which have branch cuts in some regions of the kinematic invariants. On the other hand, the procedure is still unable to determine the rational parts of the amplitude, which are cut free: there is a *rational ambiguity.*

Integral basis.

The basic idea underlying the unitarity method is to reconstruct a loop amplitude from its branch cuts in some particular kinematic region. The singular functions containing such branch cuts are expected to come from loop integrations. Indeed, one can always imagine a n–point loop amplitude as given by the sum over some loop integrals $\mathcal{I}$ weighted by rational coefficients c_m, plus some extra rational remainders $\mathcal{R}$

$$\mathcal{A}_n^{loop} = \sum_{m\leq n} c_m\, \mathcal{I}_m[P(p_\mu)] + \mathcal{R} \qquad (3.5.9)$$

where the sum is over all possible loop integrals of the general form

$$\mathcal{I}_m[P(p_\mu)] = (-1)^{n+1}\, i \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}\, \frac{P(p_\mu)}{p^2(p-K_1)2(p-K_1-K_2)^2\ldots(p+K_n)^2} \qquad (3.5.10)$$

where $P(p_\mu)$ stands for some polynomial of loop momenta.

Given this expansion in terms of integrals one may try to directly read their coefficients, by taking a cut and comparing the integrand of the cut amplitude on the LHS to those of the momentum integrals contributing in the given channel.

Since the coefficients and the remainders $\mathcal{R}$ are rational functions, they do not have branch cuts, hence one expects the singularity to come entirely from loop integrals, giving functions like logarithms or polylogarithms of the kinematic variables, that do possess branch cuts. The computation of a particular cut, as described above, i.e. replacing two propagators with δ–functions, is equivalent to selecting the singular part of the amplitude in some kinematic region. The complete discontinuity must be given by the sum of the contributions from all the loop integrals that have a branch cut singularity in this same channel. Therefore the evaluation of the cut actually gives a combination of overlapping loop integrals that share the same singularity. In other words, the computation of the cut yields an expression containing pieces from entangled loop integrals. In order to pick up all the singular contributions one has to compute all possible cuts. At the end one should combine the various results. However, this combination cannot just be a simple sum, because of integrals sharing the same singularity as mentioned above, so care is needed to avoid overcounting.

Of course if the amplitude contains a cut–free part, this cannot be detected by this procedure, unless it comes entirely from loop integrals. In such a situation it is conceivable that all possible integrals occurring when computing an amplitude possess some peculiar, unique, singularities distinguishing them. If this is the case, each cut selects a single integral and it cannot happen that a weighted sum of more integrals concurs to yield a rational part

$$\sum c_i \mathcal{I}_i \neq \text{rational} \tag{3.5.11}$$

In such a situation the given amplitude is said to be *cut–constructible* and the method of unitarity cuts exhausts it entirely, including rational terms. One–loop amplitudes of $\mathcal{N} = 4$ have been proved to be so, for instance.

If this does not occur the rational remainder has to be found by other means.

Cuts in d dimensions.

The rational remainders of an amplitude cannot be worked out by unitarity cuts because this amounts to computing the singular part of the amplitude, but these pieces do not possess discontinuities, so that they are discarded. However the statement that they are cut–free is only partially true. It holds only through $\mathcal{O}(\epsilon^0)$ order, to which amplitudes are usually evaluated. This is equivalent to considering them in four dimensions, setting the regularization parameter to zero.

The key observation is that, by dimensional analysis of the loop integral, it turns out that every amplitude must be proportional to terms of momentum dimensions $(k^2)^{-\epsilon}$, that must have the form $(-t)^{-\epsilon}$, t being a general momentum invariant. Thus each amplitude contains explicitly branch cuts at order $\mathcal{O}(\epsilon)$, due to the expansion $(-t)^{-\epsilon} \sim (1-\epsilon\log(-t))$, which exhibits a logarithmic singularity, though suppressed by an ϵ factor, and previously neglected. Therefore, by working at $\mathcal{O}(\epsilon)$ order one can in principle detect the rational parts from the cuts too (in [27] this is carried out, using generalized unitarity, though). It is worth mentioning that the evaluation at order $\mathcal{O}(\epsilon)$ can be performed just modifying the on–shell condition for the cut momenta, in order to account for the 2ϵ–dimensional part μ of the loop momentum: $q^2 = \mu^2$, namely loop particles acquire a fictitious or effective mass μ^2. Summarizing, dimensional regularization can in principle be actually employed as a means of constructing amplitudes from d–dimensional cuts. However this requires the knowledge of tree amplitudes with two $(4-2\epsilon)$–dimensional legs, which is not so easy. Indeed there are not as many amplitudes performed this way, as with four–dimensional cuts.

3.5.2 Generalized unitarity.

The original unitarity cuts method described above is well–defined, but may still become cumbersome at high loop level. A more efficient evolution of it is the so–called *generalized unitarity*. It is based on the following observation: the RHS of (3.5.2) includes by definition the part of the L–loop amplitude containing the propagators which have been cut in all possible channels. The point is that these pieces may be reorganized as a sum on the number l of cut loop propagators running from 2 to $L-1$. Each of these may be considered as a product of l lower order amplitudes. Indeed when a cut of l propagators is performed, in a such away that this removal tears it in l disconnected pieces, since the whole amplitude is a sum over Feynman diagrams with the original external particles along with loop propagators, this sum can be partitioned according to which internal propagators are present. Suppose for instance that one of the pieces in which the amplitude is divided has external momenta p_i and loop momenta k_j. Then the sum over all possible Feynman diagrams with the same p_i and k_j is by definition the amplitude $\mathcal{A}(p_i, k_j)$, by virtue of the on–shell conditions for cut momenta.

This observation allows to cut more than $L+1$ propagators as in the regular cuts. The underlying logic is quite different: instead of reconstructing the amplitude from its discontinuity, employing dispersion relation, rather one tries to isolate the contributions to the amplitudes featuring given sets of propagators. By properly combining a sufficiently large sample of such contributions one is legitimated to believe that the full Feynman diagrammatic expansion has been encompassed. However one has to pay care not to overcount contributions, by checking cross consistency between several cuts. There is no general rule on how to perform the generalized cuts, however reiteration of two–particle cuts seems to be a very efficient strategy.

Again, the discussion above refers to d–dimensional cuts. Four dimensional ones are not generally sufficient, especially at higher order. Indeed, when considering generalized cuts for higher than one–loop amplitudes, $\mathcal{O}(\epsilon)$ terms may mix with poles from other loops to give subleading poles or finite pieces, which have to be taken into account. Moreover, in the presence of numerators, such contributions may also come from d–dimensional algebra of on–shell momenta. These integrals contribute with the (-2ϵ)–dimensional part of the vectors and are dubbed μ*–integrals*.

As in the regular cuts, generalized unitarity can be applied to determine the amplitude constructively, namely computing the cut and assembling tree level amplitude so as to make momentum integrals pop out and directly identify their rational coefficients. Rather, knowing in advance a convenient basis of integrals one may perform multiple cuts on the amplitude on the LHS and on the integral expansion on the RHS of (3.5.9). This way one may derive a system of independent equations so as to fix the unknown coefficients. Progress in this direction has been triggered by the discovery of dual conformal invariance (reviewed in Section 3.9) which poses stringent constraints on the structure of integrals which may appear in the basis for amplitudes in $\mathcal{N}=4$ SYM.

3.6 Amplitudes at strong coupling.

Most of the beautiful properties of scattering amplitudes in $\mathcal{N} = 4$ SYM manifest themselves in the planar limit. This is suggestive of a deep connection to the AdS/CFT correspondence, which may shed more light on their origin.

Indeed, the AdS/CFT correspondence has allowed important advances in the field of computing scattering amplitudes. For the best studied case of maximally supersymmetric Yang–Mills theory, scattering amplitudes have been computed explicitly at strong coupling, enabling interesting comparison to the weakly coupled results from field theory.

As pointed out in [47], the problem of evaluating scattering amplitudes at strong coupling, amounts to minimizing the area of the string worldsheet surface, ending on a light–like contour at the boundary of AdS, parametrized by the scattered particles momenta. In the $\mathcal{N} = 4$ SYM case one starts whith the metric af AdS_5 written in T–dual coordinates

$$d\tilde{s}^2 = R^2 \left[\frac{dy_\mu dy^\mu + dr^2}{r^2} \right] , \qquad r = \frac{R^2}{z} \tag{3.6.1}$$

where R is the AdS radius, containing the dependence on the 't Hooft parameter $R^2_{AdS_5} = \sqrt{g^2 N} \equiv \sqrt{\lambda_{SYM}}$). At leading order, one then considers the Nambu–Goto action, which equals the string worldsheet area

$$A = \int d\sigma \, d\tau \, \sqrt{- \det \partial_\alpha \, x^\mu \, \partial_\beta \, x^\nu \, g_{\mu\nu}(x)} \tag{3.6.2}$$

where σ and τ are the worldsheet coordinates and the area is given by the integral of the induced metric. One then solves the corresponding Euler–Lagrange equations, which amounts to minimizing the worldsheet area A with the constraint that the surface should end at the boundary of AdS_5 an a sequence of light–like segments. In the end the leading behaviour of the amplitude at strong coupling is given by

$$\mathcal{A} = e^{-\frac{R^2}{2\pi} A} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}} \right) \tag{3.6.3}$$

having set $l_s = 1$. In the case of a four cusped light–like contour, corresponding to four particle scattering, a general solutions has been found in [47]. Choosing conformal gauge coordinates u_1, u_2 for the worldsheet, this reads

$$\begin{aligned} r &= \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \qquad & y_0 &= \frac{a\sqrt{1+b^2} \sinh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \\ y_1 &= \frac{a \sinh u_1 \cosh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \qquad & y_2 &= \frac{a \cosh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \end{aligned} \tag{3.6.4}$$

The fifth coordinate y_3 does not enter the solution and may be set to 0. Then the amplitude is given by e^{iS} evaluated on this solution.

Actually the amplitude is affected by IR divergences, which in the gravity context map to the worldsheet area being infinite. This divergence may be regularized for instance by considering an IR cutoff, represented by a brane set at a small radius r_c, which imposes new boundary conditions for the solution and prevents the integral on the worldsheet from diverging. Rather, it can be handled by introducing a dual analogue of dimensional regularization in field theory. If we consider a CFT in $p+1$ dimensions which describes the dynamics on a p-brane in the low energy limit, then in the AdS/CFT correspondence the AdS metric is given by the near horizon geometry of the black p-brane solution of supergravity, corresponding to the brane where the CFT lives on. Therefore the suitable regularization amounts to extending the solution from a p-brane to a $p-2\epsilon$.

Taking into account this subtlety and rescaling the 't Hooft coupling constant by an infrared regulator to keep the coupling dimensionless in the $p+1-2\epsilon$ dimensional Lagrangian, it has been shown that the prescription amounts to evaluating the solution in a modified action

$$S = \frac{\sqrt{\lambda_D\, c_D}}{2\pi} \int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon} \tag{3.6.5}$$

where c_D is an ϵ dependent constant which can be read from the brane solution for the $D(3-2\epsilon)$–brane. The final result for the ϵ expanded action evaluated on the solution of the modified equation of motion reads

$$i\, S = -B_\epsilon \left(\frac{\pi \Gamma[-\frac{\epsilon}{2}]^2}{\Gamma[\frac{1-\epsilon}{2}]^2}\, {}_2F_1\left(\frac{1}{2}, -\frac{\epsilon}{2}, \frac{1-\epsilon}{2}; b^2\right) + 1/2 \right) \tag{3.6.6}$$

in terms of the dimensionally regulated AdS radius $B_\epsilon = \frac{\sqrt{\lambda_D\, c_D}}{2\pi}$.

It can be noticed that the structure of such an amplitude may be interpreted in the following manner

$$\mathcal{A} = \exp\left(i\, S\right) = \exp\left(\text{Div} + \frac{\sqrt{\lambda}}{8\pi} \log^2\left(\frac{s}{t}\right) + C \right) \tag{3.6.7}$$

where C is a constant, Div corresponds to the divergence developed in dimensional regularization because of factors $\Gamma\left(-\frac{\epsilon}{2}\right)$, which has the explicit form

$$\text{Div} = \sqrt{\lambda} \left(-\frac{1}{2\pi\epsilon^2} - \frac{1}{4\pi\epsilon}(1-\log 2) \right) \left(\frac{\mu^2}{s}\right)^\epsilon + (s \leftrightarrow t) \tag{3.6.8}$$

and the non–trivial kinematic dependent piece $\log\left(\frac{s}{t}\right)$ is the same as in the one–loop four–point amplitude of $\mathcal{N}=4$ SYM at weak coupling.

This form is very suggestive because it implies that the amplitude in some sense exponentiates, an idea which had been already pointed out at weak coupling [56] and goes under the name of the *BDS ansatz*. We will review this conjecture in a few lines, in Section 3.7.

The computation of the amplitude at strong coupling has also suggested a connection between it and light–like Wilson loops. Indeed the evaluation of the amplitude as a minimal area surface in AdS strikingly parallels that of a Wilson loop [57]. In this case, since the surface ends on a light–like polygon defined by the on–shell gluon momenta, in dual space, the corresponding Wilson loop is a polygonal light–like one. This topic is to be sketched in a forthcoming Section 3.8.

3.7 The BDS ansatz.

In paper [56] the interesting observation that amplitudes in the planar limit of $\mathcal{N} = 4$ SYM could present an iterative structure was pointed out. In particular, by direct insight into the structure of four–point loop amplitudes evidence supports that their whole perturbative series may be thought of as the expansion of an exponential. The exponentiation of infrared divergences is a phenomenon already observed before, the novel non–trivial statement is that the finite part of the amplitude may exponentiate as well.

In practice [56] suggests that an all–loop n–point amplitude in $\mathcal{N} = 4$ SYM, expanded in a series of powers of the perturbative coupling λ as

$$\mathcal{A}_n = \mathcal{A}_n^{tree} \left(1 + \sum_{l=1}^{\infty} \mathcal{M}_n^{(l)} \right) = \mathcal{A}_n^{tree} \mathcal{M}_n \tag{3.7.1}$$

can be expressed as

$$\mathcal{M}_n = \exp \left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon)\, \mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right) \right] \tag{3.7.2}$$

where $\mathcal{M}_n^{(1)}$ is the n–point one–loop amplitude to all orders in ϵ, $C^{(l)}$ are constants which do not depend on the number of external gluons and $E_n^{(l)}(\epsilon)$ are order $\mathcal{O}(\epsilon)$ terms. These do not iterate, but being of order ϵ do not invalidate the exponentiation property of the amplitue (to order ϵ). However they do contribute when the whole amplitude, i.e. the expansion of the exponential is condsidered, because non–trivial contributions will arise when they multiply ϵ poles. Here the perturbative expansion parameter is $a \equiv \frac{\lambda}{8\pi^2}$, where λ is the $\mathcal{N} = 4$ SYM 't Hooft coupling $\lambda = g^2 N$. The functions $f^{(l)}(\epsilon)$ have an expansion in ϵ,

$$f^{(l)}(\epsilon) = f_0^{(l)} + f_1^{(l)}\epsilon + f_2^{(l)}\epsilon^2 \tag{3.7.3}$$

whose zero order in ϵ term coincides with one quarter of the coefficients in the λ expansion of the cusp anomalous dimension $f_{YM}(a)$ of $\mathcal{N} = 4$ SYM

$$f_0^{(l)} = \frac{1}{4} f_{YM}^{(l)} \qquad f_{YM}(a) = \sum_{l=1}^{\infty} a^l\, f_{YM}^{(l)} \tag{3.7.4}$$

We recall the leading behaviour of f_{YM}, as a function of the 't Hooft coupling λ, rather than of the effective a, at weak and strong coupling[‡]

$$f_{YM}(\lambda) = \begin{cases} \frac{\lambda}{2\pi^2}\left(1 - \frac{\lambda}{48} + \dots\right) & \lambda \ll 1 \\ \frac{\sqrt{\lambda}}{\pi} + \dots & \lambda \gg 1 \end{cases} \tag{3.7.5}$$

This quantity controls the leading IR divergencies of n–point amplitudes (and UV of the light–like Wilson loops) which in dimensional regularization exponentiate as

$$\begin{aligned} &\exp \mathrm{Div}_n \\ &\mathrm{Div}_n = -\sum_{i=1}^{n}\left[\frac{1}{4\epsilon^2} f^{(-2)}\left(\frac{\lambda\,\mu_{IR}^{2\epsilon}}{(-s_{i,i+1})^\epsilon}\right) + \frac{1}{2\epsilon} g^{(-1)}\left(\frac{\lambda\,\mu_{IR}^{2\epsilon}}{(-s_{i,i+1})^\epsilon}\right)\right] \end{aligned} \tag{3.7.6}$$

where $s_{i,\,i+1} = (p_i + p_{i+1})^2$ and $f^{(-2)}$ and $g^{(-1)}$ are functions satisfying the relations

$$\left(\lambda \frac{d}{d\lambda}\right)^2 f_{YM}^{(-2)}(\lambda) = f_{YM}(\lambda)\,, \qquad \lambda \frac{d}{d\lambda}\, g_{YM}^{(-1)}(\lambda) = g_{YM}(\lambda) \tag{3.7.7}$$

The subleading ϵ^{-1} pole is thus governed by the function g_{YM}, which is scheme dependent, in contrast to the cusp anomalous dimension.

The ansatz is also often re–expressed more compactly as,

$$\log \mathcal{M}_n = \mathrm{Div}_n + \frac{f_{YM}(\lambda)}{4} F_n^{(1)}(0) + nk(\lambda) + C(\lambda) + \mathcal{O}(\epsilon) \tag{3.7.8}$$

by collecting the divergent part into e^{Div} and up to $\mathcal{O}(\epsilon)$ terms and exposing the one–loop finite remainder $F_n^{(1)}$.

In particular for the four point amplitude the finite remainder in (3.7.8) is simply

$$F_4^{(1)}(0) = \frac{1}{2}\log^2\left(\frac{s}{t}\right) + 4\zeta_2 \tag{3.7.9}$$

giving the full one–loop four point amplitude to order $\mathcal{O}(\epsilon)$

$$\mathcal{M}_4^{(1)} = -\frac{1}{\epsilon^2}\left[\left(\frac{\mu^2}{s}\right)^\epsilon + \left(\frac{\mu^2}{t}\right)^\epsilon\right] + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + 4\zeta_2 + \mathcal{O}(\epsilon) \tag{3.7.10}$$

All the ϵ dependence in the ansatz is captured by $\mathcal{M}_4(l\epsilon)$, i.e. there is no need for a non–trivial $E_4(\epsilon)$: $E_4(\epsilon) = 0$. At list this is what occurs up to three–loop level. Such terms may well appear at higher n order, though.

Expressions for the iterating finite remainders of higher n multiplicity amplitudes $F_n^{(1)}(0)$, though more complicated, are available from direct loop computations and are reported in Appendix A.3.

[‡]The coefficients $f_0^{(l)}$ are readed from those of (3.7.5), multiplying by $\frac{1}{4}(8\pi^2)^l$.

The BDS formula was checked up to the three–loop four–point amplitude in [56] and up to two loops for the five–point. Interestingly enough the computation of amplitudes at strong coupling, which was performed two years later by Alday and Maldacena, shows that precisely the same structure occurs in the strongly coupled regime, where the amplitude looks like an exponential of the first order result, and the leading part of divergencies is controlled by the scaling function $f_{YM}(\lambda)$ at strong coupling as well. However an argument from strong coupling poses some bounds on the validity of the ansatz [58]: indeed it was shown that in the limit where the number of external gluons is large, the strong coupling amplitude is not compatible with the BDS result. Taking $n \to \infty$, both the strong coupling prediction and the BDS guess may be approximated by a rectangular space–like Wilson loop, where the edges are actually a thick light–like zigzag trajectory. Then the two predictions disagree, which is the first indication that the ansatz may fail at large n.

Indeed it has been proposed that a finite remainder should correct the BDS ansatz, which is a function of conformal cross–ratios, so that the result satisfies conformal invariance in dual space, which is suggested by the strong coupling computation (more details on that will follow in the next Section). Since at $n = 4$ and $n = 5$ such conformal cross–ratios are not availbale, the BDS ansatz should be correct as it has originally been formulated. Nevertheless one may expect such non–trivial remainders to show up beginning from six points, as was veified by explicit two–loop computations of both the Wilson loop and the amplitude [59].

3.8 WL/amplitude duality.

The recipe [47] for computing scattering amplitudes at strong coupling provides a very important side observation. This consists in noting that the prescription for evaluating amplitudes closely resembles that for Wilson loops using the AdS/CFT. In particular the computation of an amplitude coincides with evaluating a light–like Wilson loop with a polygonal contour. More precisely the edges of the polygon are displacement vectors equal to the momenta of the scattered particles, written in dual space variables $p_i^\mu = x_{i+1}^\mu - x_i^\mu$.

This points towards a duality relating MHV amplitudes and light–like Wilson loops. Compelling evidence for this to hold was adduced by direct comparisons at weak coupling [60, 44, 61, 62], leading to the proposal

$$\log\left(1+\sum_{l=1}^{\infty} a^l\,\mathcal{M}_n^{(l)}\right) = \log\left(1+\sum_{l=1}^{\infty} a^l\,W_n^{(l)}\right) + \mathcal{O}(\epsilon) \tag{3.8.1}$$

which identifies Wilson loops and MHV amplitudes. More precisely here $\mathcal{M}_n^{(l)}$ represents the l–loop contribution to the amplitude divided by its tree level expression (as in (3.7.2)) and $W_n^{(l)}$ is the l–loop correction to the expectation value of the ligh–like Wilson loop.

C_n is the Wilson loop contour of integration, which is defined in terms of the momenta of the particles scattered in the amplitude, by representing the latter in dual space notation $p_i = x_{i+1} - x_i$: $C_n = \bigcup_{i=1}^{n} [x_{i,i+1}]$, for n–cusped Wilson loop, where i is cyclic. Therefore the Wilson loop under consideration is a light–like polygonal one (closure of the polygon sides is determined by momentum conservation in dual variables). In some sense the Wilson loop considered here is to be intended as defined in dual momentum space rather than usual configuration space. The relation (3.8.1) holds when carefully identifying the kinematic variables and regularization scales. Indeed the two objects are divergent and are dealt with dimensional regularization. The light–like Wilson loops possesses UV divergences associated to the cusps of its contour and strenghtened by light–like edges, whereas the amplitude exhibits infrared singularities. Actually, pursuing the idea of considering the Wilson loop in momentum space, its UV divergences associated to the short distance behavior $x^2 = 0$ in original space are to be interpreted as low energy effects, pretending $x_{i,i+1}^2 = p^2$, and hence could be identified with the IR divergences of the amplitude. This re–definition, however, entails transforming the mass scales of dimensional regularization, which differ between that for the amplitude μ_{IR} and that of the Wilson loop μ_{UV}, as well as the sign in the regularization parameters ϵ. The correct prescription to relate the two objects is to trade

$$x_{ij}^2\, \mu_{UV}^2 \longrightarrow \frac{s_{ij}}{\mu_{IR}^2} \tag{3.8.2}$$

and $\epsilon_{UV} = -\epsilon_{IR}$. After these caveats are taken into consideration, relation (3.8.1) is sensible and well–defined.

We give an explicit example of such a relation for the four cusp Wilson loop at one–loop.

In the weak coupling regime the expectation value of the Wilson loop operator

$$\langle W_{C_n} \rangle = \frac{1}{N} \langle\, 0 \,|\, \mathrm{Tr}\, P \,\exp\left(i\, g \int_{C_n} d\tau A_\mu(x(\tau))\, \dot{x}^\mu(\tau) \right) |\, 0 \,\rangle \tag{3.8.3}$$

may be calculated perturbatively.

To perform the computation, the two point function of gluons in configuration space is needed, which in Feynman gauge reads

$$G_{\mu\nu}(x) = -\eta_{\mu\nu} \frac{\Gamma(1-\epsilon_{UV})}{4\pi^2} \frac{(\pi\mu^2)^{\epsilon_{UV}}}{(-x^2 + i0)^{1-\epsilon_{UV}}} \tag{3.8.4}$$

which has been already evaluated in dimensional regularization $d = 4 - 2\epsilon_{UV}$ with $\epsilon_{UV} > 0$ (this makes the regulator appear in the scaling of the propagator, because the Fourier transform from momentum space has been carried out in non–integer dimension) in view of ultraviolet divergences.

Employing this two-point function (3.8.4), the lowest order correction to the expectation value of the Wilson loop is given by

$$\langle W_{C_n} \rangle = 1 + \frac{1}{2}(ig)^2 C_F \int_{C_n} d\tau \int_{C_n} d\tilde{\tau}\; \dot{x}^\mu(\tau)\dot{x}^\nu(\tilde{\tau}) G_{\mu\nu}(x(\tau) - x(\tilde{\tau})) + \mathcal{O}(g^4) \qquad (3.8.5)$$

where C_F denotes the quadratic Casimir of the adjoint of the gauge group, which we will assume to be $SU(N)$, meaning $C_F = \frac{N^2-1}{2N}$, which we will soon approximate to $\frac{N}{2}$ in the planar limit.

Skipping the details of the computation, which may be found in [43], the complete one–loop contribution to the four–cusped Wilson loop evaluates to

$$\log W_4 = a \left(\mathrm{Div}_4 + \frac{1}{2} \log^2 \frac{s}{t} + 2\zeta_2 + \mathcal{O}(\epsilon_{UV}) \right) + \mathcal{O}(g^4) \qquad (3.8.6)$$

where the effective coupling $a = \frac{Ng^2}{8\pi^2}$ is the same as in (3.7.2). The explicit expression for the divergent part Div

$$\mathrm{Div}_4 = -\frac{1}{\epsilon_{UV}^2} \left(\left(\mu_{UV}^2(-x_{13}^2)\right)^{\epsilon_{UV}} + \left(\mu_{UV}^2(-x_{24}^2)\right)^{\epsilon_{UV}} \right) \qquad (3.8.7)$$

where this compact form is achieved by a change of scheme re–defining $\mu_{UV}^2 = \mu^2 \pi e^\gamma$ Then identifying

$$\epsilon_{UV} = -\epsilon_{IR} \equiv -\epsilon \qquad , \qquad x_{13}^3\, \mu_{UV}^2 = \frac{s}{\mu_{IR}^2} \qquad , \qquad x_{24}^3\, \mu_{UV}^2 = \frac{t}{\mu_{IR}^2} \qquad (3.8.8)$$

one appreciates the matching with the one loop amplitude (3.7.10) up to a constant. In particular the UV divergences of the Wilson loop are mapped to the infrared singularities of the MHV amplitude and the finite remainders are exactly the same.

The correspondence has been significantly sharpened by direct evaluation of two–loop contributions for four [44] and five cusps [61], and by computation of higher n polygons. Remarkably, the conjectured duality passed all tests. When in [63] departures were observed in the two–loop six–cusps Wilson loop with respect to the predicted six–gluon amplitude from the original BDS ansatz, it was soon proved that the correct six–point MHV amplitude indeed is only partially determined by the BDS ansatz and has to be complemented by a remainder function (of conformal cross–ratios). Notably, these pieces were exactly the same as in the Wilson loop calculation [62, 59].

3.9 Dual conformal invariance.

Dual conformal invariance is an emergent symmetry exhibited by amplitudes, even though it is not a symmetry of the Lagrangian of the theory.

Its formulation lays the basis in an empirical observation on the structure of momentum integrals appearing in loop calculations of ampliudes. We recall that in general every correction to the amplitude is given by a sum over integrals weighted by rational coefficients: $\mathcal{M}_n^{(l)} = \sum_i c_i \mathcal{I}_i$. By direct inspection into the integrals appearing in up to three–loop order four–point amplitudes, it turns out that they are unexpectedly invariant under many transformations, which are not symmetries of the Lagrangian. To make these symmetries manifest it is convenient to switch from momentum to dual space and express integrals in terms of dual variables $x_{i,i+1} \equiv p_i$. Moreover the new symmetries to be described are broken by IR divergences of integrals. These require dimensional regularization, spoiling forthcoming arguments which hold for integer d only. Therefore integrals will be considered evaluated off–shell, where (usually, but not always[§]) IR divergences disappear.

Momentum integrals are translation and Lorentz invariant by construction. Then one might easily make them scale invariant by properly adjusting their dimension with momentum invariants of external momenta (which could be eventually absorbed into the coefficients c_i). For instance, starting from the usual massless box in four dimensions depicted in Fig. 3.4

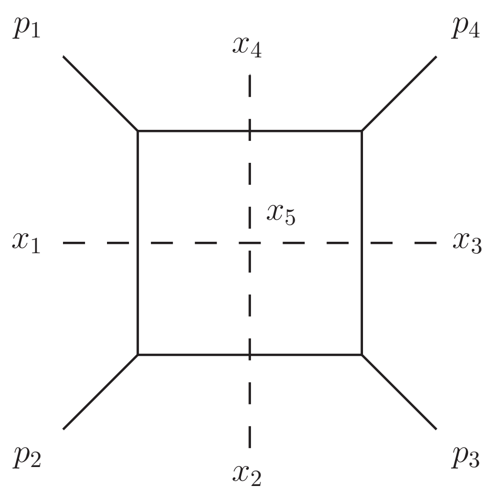


Figure 3.4: Box integral in dual space.

$$\mathcal{I} = \int d^4x_5 \, \frac{1}{x_{51}^2 \, x_{52}^2 \, x_{53}^2 \, x_{54}^2} \tag{3.9.1}$$

can be made to be annihilated by the dilatation operator by inserting a (for the moment

[§]In case an integral persists diverging even off–shell it is removed from the basis. A different situation appears for instance in three dimensions where it is possible to include off–shell divergent integral in the basis, provided that they always combine in such a way to cancel this singularity.

unspecified) numerator of weight 4

$$\mathcal{I}_4 = \int d^4x_5 \, \frac{x^4}{x_{51}^2 \, x_{52}^2 \, x_{53}^2 \, x_{54}^2} \tag{3.9.2}$$

A more stringent requirement is that the integral be invariant under inversion $I = \sum_{i=1}^{5} I_i$, acting on dual variables as

$$I_i : \qquad x_i^\mu \to \frac{x_i^\mu}{x_i^2} \qquad , \qquad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2} \qquad , \qquad d^d x_i \to \frac{d^d x_i}{(x_i^2)^d} \tag{3.9.3}$$

Neglecting momentarily the numerator, the box integral transforms as

$$\mathcal{I} \to \int \frac{d^4x_5}{(x_5^2)^4} \frac{(x_5^2\, x_1^2)(x_5^2\, x_2^2)(x_5^2\, x_3^2)(x_5^2\, x_4^2)}{x_{51}^2 \, x_{52}^2 \, x_{53}^2 \, x_{54}^2} \tag{3.9.4}$$

The important thing to notice is that the variation of the loop variable x_5 cancels, so that at least the integral transforms covariantly, that is up to invariants of external momenta of weight 4. Then it may be easily made invariant suitably chosing the numerator $x_{13}^2 \, x_{24}^2$, which adjusts factors of x_i^2 $(i = 1 \ldots 4)$. This is the only possible choice since $x_{12}^2 = x_{34}^2 = 0$, by on–shell conditions, and even though the whole discussion was understood to hold when considering the integral off–shell, one will eventually put invariants on the ligh–cone to recover the amplitude.

Invariance under inversions I and translations P guarantees that the integral is also symmetric under special conformal transformations

$$K_i : \quad x_i^\mu \ \to \ \frac{x_i^\mu + a^\mu \, x_i^2}{1 + a_\nu \, x_i^\nu + a^2 \, x_i^2} \tag{3.9.5}$$

because conformal boosts are generated by a IPI combination.

Along with Lorentz and dilatation invariance this closes a conformal algebra in four dimensions $SO(2,4)$. This is not the ordinary conformal invariance of the $\mathcal{N} = 4$ action, which would naturally show up in amplitudes, if IR divercences were not be there, breaking it anomalously. Rather, this is a symmetry exhibited by integrals (and consequently amplitudes) only, in particular when embedding them into dual space. Therefore such an emergent symmetry goes under the name of *dual conformal invariance.*

In practice, assuming this symmetry to hold true, highly constrains the form of an amplitude. In particular it may be used as a very powerful criterion for choosing a suitable basis of integrals on which expanding an amplitude at loop level. The requirement that all integrals appearing in such a basis are invariant under conformal transformations in dual space, dramatically reduces the possibilities. Roughly, it excludes triangles and bubbles, not only from the one–loop basis, but also as possible subintegrals. This exclusion descends from the relative scaling of the integration measure and the propagators under inversion: in a d–dimensional integral over x_L containing n propagators, the transformation under I_{x_L} would give $(x_L^2)^{-d+n}$. In four dimensions it means that the first structure which can arise to compensate the variation of the measure is four propagators, which is a box. Lower order integrals such as triangles and bubbles are excluded, since they will never be able to make the integral invariant. Higher n integrals, such as pentagons and hexagons are allowed, provided that aditional numerators compensate an excess of x_L^2 from propagators. These do indeed participate into higher point amplitudes.

Assuming invariance under dual conformal symmetry leads to a decisive simplification of computations and represents a valuable tool for carrying out calculations by (generalized) unitarity methods, which allow to determine the coefficient of the linear combination of integrals reproducing the amplitude.

3.9.1 Dual conformal invariance and Wilson loops.

The WL/amplitude duality is perfectly compatible with the notion of dual conformal invariance. Indeed the Wilson loop is invariant under conformal transformations, in some sense specified below. Were it not divergent, one would be able to write down a Ward identity for the Wilson loop where the generators of dilatations and conformal transformations would annihilate the Wilson loop. However ultraviolet singularities occur, which require regularization. Employing for instance dimensional regularization, explicitly breaks conformal invariance. Therefore the best one can do is to derive an anomalous Ward identity for the Wilson loop. Remarkably, this identity constrains the result in such a way that the finite part can only depend on a fixed set of functions and its only freedom is in the possible addition of functions of conformal cross ratios, moreover its coefficient is connected to the cusp anomalous dimension.

Some basic properties of Wilson loops are as follows:

- Wilson loops are not invariant under genral coordinate transformations, since they are defined on contours which change under such operations. Whenever a theory is described by a Lagrangian enjoying invariance under a general coordinate transformation the expectation value of the Wilson loop inherits this symmetry, provided the contour be suitably redefined

$$\langle W(\tilde{C})\rangle = \langle W(C)\rangle \tag{3.9.6}$$

  In particular let us consider the four dimensional conformal group $SO(2,4)$, under which the $\mathcal{N}=4$ Lagrangian is invariant. Wilson loops are dimensionless operators, however the variation of their contours under the action of the group prevents them from being manifestly symmetric. Light–like contours have the special property of being closed under conformal transformations and in particular under inversion. The expectation value ensures invariance under translations, Lorentz and special conformal transformations, so that since the Lagrangian is conformally invariant too, one concludes that the Wilson loop is so, provided a redefinition of the contour

$$\langle W(\tilde{C}_n)\rangle = \langle W(C_n)\rangle \tag{3.9.7}$$

and Ward identities for dilatations and special conformal transformations follow

$$\langle \mathbb{D}\, W_n\rangle = 0 \qquad , \qquad \langle \mathbb{K}^\mu\, W_n\rangle = 0 \tag{3.9.8}$$

- This conclusion is however too naive, because the presence of ultraviolet divergences spoils it. In particular these are associated to cusps in the contour and worsened by the light–likeness of the edges of the contour. A remarkable property of the Wilson loop divergences is that they exhibit exponentiation. In particular, if one splits the contributions to the Wilson loop into divergent and finite

$$W_n = Z_n\, F_n \tag{3.9.9}$$

the singular piece has a well–defined all–loop structure

$$\log Z_n = -\frac{1}{8}\sum_{l\geq 1} a^l \sum_{i=1}^{n} (-x^2_{i-1,i+1}\mu^2)^{l\epsilon}\left(\frac{f^{(l)}_{YM}}{l^2\epsilon^2} + \frac{2\Gamma^{(l)}}{l\epsilon}\right) \tag{3.9.10}$$

where $f^{(l)}_{YM}$ are again the coefficients in the perturbative expansion of the cusp anomalous dimension of $\mathcal{N} = 4$ SYM and the subleading divergence is controlled by the function $\Gamma^{(l)}$, which we won't be much interested in.

The expectation value of the Wilson loop operator may be expressed as usual as a path integral

$$\langle W_{C_n}\rangle = \frac{1}{N}\int \mathcal{D}A\,\mathcal{D}\lambda\,\mathcal{D}\phi\; e^{iS_\epsilon}\;\mathrm{Tr}\,P\,\exp\left(i\,g\oint_{C_n} A(x)\right) \tag{3.9.11}$$

in terms of the $\mathcal{N} = 4$ SYM action S. In order to deal with divergences dimensional regularization is employed, making the spacetime measure of integration $d = 4 - 2\epsilon$ dimensional and the explicit appearance of the ultraviolet scale μ.

$$S_\epsilon = \frac{1}{g^2\mu^{2\epsilon}}\int d^Dx\,\mathcal{L}(x), \tag{3.9.12}$$

In particular we absorbed all 2ϵ–dimensional dependence in the coupling g and rescaled every field so that their dimension is canonical. Therefore the Lagrangian has the same integral weight four under dilatations as before (the action was scale invariant in $d = 4$), whereas the measure has scaling $4 - 2\epsilon$. This mismatch is detected by applying the dilatation operator, under which the action tranforms non–trivially as

$$\delta_{\mathbb{D}} S_\epsilon = \frac{2\epsilon}{g^2\mu^{2\epsilon}}\int d^Dx\,\mathcal{L}(x) \tag{3.9.13}$$

This induces an anomalous term in the Ward identity for the expectation value of the dilatation generator $\mathbb{D}$ acting on the Wilson loop

$$\mathbb{D}\langle W_n\rangle = \sum_{i=1}^{n} (x_i \cdot \partial_i)\langle\, W(C_n)\rangle = \langle \delta_{\mathbb{D}} S_\epsilon W_n\rangle = \frac{2i\epsilon}{g^2\mu^{2\epsilon}} \int d^D x\, \langle \mathcal{L}(x)\, W_n\rangle \tag{3.9.14}$$

Analogously, application of conformal boosts determines a second anomalous Ward identity

$$\mathbb{K}^\mu \langle W_n\rangle = (2\, x^\mu\, x^\nu\, \partial_\nu - x^2\, \partial^\mu)\, \langle W_n\rangle = \frac{4i\epsilon}{g^2\mu^{2\epsilon}} \int d^D x\; x^\mu \langle \mathcal{L}(x) W_n\rangle \tag{3.9.15}$$

Differently from chiral anomalies, these are not one–loop exhausted and have in principle to be computed order by order in perturbation theory, by considering the expectation value of the Wilson loop operator with the insertion of a Lagrangian. More conveniently on may consider the application of conformal transformations on the logarithm of the Wilson loop

$$\mathbb{D} \log\langle W_n\rangle = \frac{2i\epsilon}{g^2\mu^{2\epsilon}} \int d^D x\, \frac{\langle \mathcal{L}(x)\, W_n\rangle}{\langle W_n\rangle} \tag{3.9.16}$$

$$\mathbb{K}^\mu \log\langle W_n\rangle = \frac{4i\epsilon}{g^2\mu^{2\epsilon}} \int d^D x\, x^\mu\, \frac{\langle \mathcal{L}(x)\, W_n\rangle}{\langle W_n\rangle} \tag{3.9.17}$$

This has been evaluated at one–loop in [61] and generalized at all–loop (by exploiting the known all–order behavior of the divergent part). Indeed, due to the explicit ϵ factor in (3.9.14,3.9.15), the anomaly is determined by the divergent part of $\int d^D x\; x^\mu \langle \mathcal{L}(x) W_n\rangle$ only. This behaves similarly to the original Wilson loop divergences, allowing to conclude that to all–loop orders

$$\mathbb{D} \log\langle W_n\rangle \;=\; -\frac{1}{4} \sum_{l=1} \alpha^l \sum_{i=1}^{n} (-x^2_{i-1,i+1}\mu^2)^{l\epsilon} \left(\frac{f^{(l)}_{YM}}{l\epsilon} + 2\Gamma^{(l)} \right) \tag{3.9.18}$$

$$\mathbb{K}^\mu \log\langle W_n\rangle \;=\; -\frac{1}{2} \sum_{l=1} \alpha^l \sum_{i=1}^{n}\; x_i^\mu\, (-x^2_{i-1,i+1}\mu^2)^{l\epsilon} \left(\frac{f^{(l)}_{YM}}{l\epsilon} + 2\Gamma^{(l)} \right) \tag{3.9.19}$$

Splitting the Wilson loop into the divergent and the finite part, it is also possible to derive conformal Ward identities for its finite remainder. Applying $\mathbb{D}$ on the divergent piece $\log Z_n$ one gets

$$\mathbb{D}_i \log Z_n = -\frac{1}{8} \sum_{l\geq 1} a^l \sum_{i=1}^{n} \left(\frac{f^{(l)}_{YM}}{(l\epsilon)^2} + \frac{2\Gamma^{(l)}}{l\epsilon} \right) l\epsilon(\mu^2)^{l\epsilon}(-2)\, x_i \cdot x_{i,i+2}\, (-x^2_{i,i+2})^{l\epsilon-1} \tag{3.9.20}$$

Summing over $\mathbb{D}_i$'s and pairing adjacent contributions $x_i \cdot x_{i,i+2} - x_{i+2} \cdot x_{i,i+2} = x^2_{i,i+2}$ yields

$$\mathbb{D} \log Z_n = -\frac{1}{4} \sum_{l\geq 1} a^l \sum_{i=1}^{n} \left(\frac{f^{(l)}_{YM}}{l\epsilon} + 2\Gamma^{(l)} \right) (\mu^2)^{l\epsilon}(-x^2_{i,i+2})^{l\epsilon} \tag{3.9.21}$$

which exactly matches the anomaly (3.9.18). One concludes that no anomalous terms are generated in the dilatation Ward identity for the finite remainder. Turning to the special conformal transformations one applies the same manouver. In particulr the action of $\mathbb{K}_i^\mu$ on the divergent part is found by looking at

$$\sum_{i=1}^{n}(2x_i^\mu x_i\cdot\partial_i - x_i^2\partial_i^\mu)\sum_{j=1}^{n}(-x_{j-1,j+1}^2\mu^2)^{l\epsilon} = \sum_{i=1}^{n} 2l\epsilon\,(x_i+x_{i+1})^\mu\,(-x_{i,i+1}^2)^{l\epsilon} \tag{3.9.22}$$

When subtracting the result obtained by substituting this into (3.9.10) to the RHS of (3.9.19), in order to extract the Ward identity for the finite remainder, one encounters terms of the form

$$\sum_{i=1}^{n} x_i^\mu\,(-x_{i-1,i+1}^2)^{l\epsilon} - \sum_{i=1}^{n}(x_i+x_{i+1})^\mu\,(-x_{i,i+1}^2)^{l\epsilon} \tag{3.9.23}$$

Reshuffling terms in the sum one gets a contribution

$$\sum_{i=1}^{n}(x_i-x_{i+1})^\mu\,(-x_{i-1,i+1}^2)^{l\epsilon} - (x_i-x_{i+1})^\mu\,(-x_{i,i+2}^2)^{l\epsilon} \tag{3.9.24}$$

which expanding in ϵ gives only $\mathcal{O}(\epsilon)$ terms, as expected, since the poles have to cancel out in the finite part

$$-\sum_{i=1}^{n}(x_i-x_{i+1})^\mu\,\log\left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\right) \tag{3.9.25}$$

Reinstating this into the whole expression gives an anomalous contribution to the Ward identity for the special conformal transformations

$$\mathbb{D}\log F_n = \sum_{i=1}^{n}(x_i\cdot\partial_i)\,F_n = 0 \tag{3.9.26}$$

$$\mathbb{K}^\mu\,\log F_n = \sum_{i=1}^{n}(2\,x_i^\mu\,x_i\cdot\partial_i - x_i^2\,\partial_i^\mu)\,\log F_n = \frac{1}{4}\,f_{YM}(a)\sum_{i=1}^{n}x_{i,i+1}^\mu\,\log\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} \tag{3.9.27}$$

The consequences of such constraints on the structure of the Wilson loop are

- from (3.9.26) it follows that the finite remainder is a function of dimensionless, scale invariant ratios $f\left(\frac{x_{ij}^2}{x_{kl}^2}\right)$;
- (3.9.27) implies that the coefficient of the finite part is the same as that of the leading singularity, namely the cusp anomaolus dimension;

- the solution to equation (3.9.27) is the sum of a solution to its inhomogeneous version and one to the homogeneous. The former can be detected by a one–loop calculation, whereas the latter is a function which by definition is annhiliated by the special conformal transformations. Recalling that in any Lorentz invaviant of the problem x_{ij}^2 invariance under translations is already granted, the desired function should just be invariant under inversion. As in the previous section this determines that it depends on conformal cross–ratios

$$u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2} \tag{3.9.28}$$

  For four and five cusps these are not available and the conformal Ward identity exhausts the Wilson loop completely. For $n \geq 6$ non–trivial functions of the conformal cross–ratios may appear and do appear.

It is worth mentioning that in [61] it has been veified that the BDS ansatz is indeed a solution of such identities. Therefore, reasoning the other way around and assuming WL/amplitude duality, offers an explanation of the emergence of dual conformal symmetry in the perturbative expansion of amplitudes.

Dual conformal invariance at strong coupling. As for the WL/amplitude duality, dual conformal invariance has a strong coupling explanation.

This was spelled out in [64], where it is explained as the self–duality of the string σ–model in the $AdS_5 \times S^5$ background, which is dual to $\mathcal{N} = 4$ SYM, under a proper combination of bosonic and fermionic T–transformations. This transformation maps the standard conformal invariance of Wilson loops into dual conformal invariance of scattering amplitudes and viceversa, and therefore provides a strong coupling explanation of the duality between scattering amplitudes and Wilson loops observed at weak coupling. Here we are mainly interested in the perturbative, weak coupling regime, and do not attempt to review this remarkable findings.

3.9.2 Dual superconformal and Yangian invariance.

By considering superamplitudes one can extend the dual conformal group acting on dual variables x_i to a dual superconformal group $SU(2,2|4)$ acting on dual supervariables (x_i, θ_i). The fermionic dual variables are obtained in a similar fashion with respect to the bosonic. In particular the latter were derived by solving the momentum conservation constraint

$$\sum_{i=1}^{n} \lambda_i^\alpha \, \tilde{\lambda}_i^{\dot{\alpha}} = 0 \implies x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = \tilde{\lambda}_i^{\dot{\alpha}} \, \lambda_i^\alpha \tag{3.9.29}$$

Then by analogy, supercharge conservation may be solved introducing a dual variable θ by the defining relation

$$\sum_{i=1}^{n} \lambda_i^{\alpha}\, \eta_i^{A} = 0 \Longrightarrow \theta_i^{A\,\alpha} - \theta_{i+1}^{A\,a} = \lambda_i^{\alpha}\, \eta_i^{A} \tag{3.9.30}$$

Since these relations define x_i and θ_i up to the choice of the reference "initial" values x_1, θ_1, it follows that the amplitude is naturally invariant under dual translations and dual supersymmetry transformations

$$P_{\alpha\dot{\alpha}}\, \mathcal{A}_n = 0 \qquad , \qquad Q_{A\,\alpha}\, \mathcal{A}_n = 0 \tag{3.9.31}$$

defined by

$$P_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}} \qquad , \qquad Q_{A\,\alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta_i^{A\,\alpha}} \tag{3.9.32}$$

The defining equations may be used to eliminate $\tilde{\lambda}$ and η in terms of x and θ and have therefore a description of amplitudes in terms of a chiral dual superspace. Rather, one may keep the whole superspace $(\lambda_i, \tilde{\lambda}_i, x_i, \eta_i, \theta_i)$ as the space of all varibles of the problem and consider the amplitude as living on a surface in it, parameterized by the equations (3.9.29,3.9.30). The form of the dual superconformal generators is constructed on this space, by starting from their action on dual (x, θ) variables, which is canonical and extending them to the remaining $(\lambda, \tilde{\lambda}, \eta)$ triplet, in such a way that they preserve the constraints (3.9.29,3.9.30). For instance let us make this explicit with the generator of dual conformal boosts $K_{\alpha\dot{\alpha}}$. In the dual (x, θ) space its canonical infinitesimal action is

$$K^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \left[x_i^{\alpha\dot{\beta}} x_i^{\alpha\dot{\beta}} \frac{\partial}{\partial x_i^{\beta\dot{\beta}}} + x_i^{\dot{\alpha}\beta} \theta_i^{\alpha B} \frac{\partial}{\partial \theta_i^{\beta B}} \right] . \tag{3.9.33}$$

Then if one acts on both sides of (3.9.29) as

$$\left[K_{\alpha\dot{\alpha}}\, , x_{i,i+1}^{\beta\dot{\beta}} \right] = \left[K_{\alpha\dot{\alpha}}\, , \tilde{\lambda}_i^{\dot{\beta}}\, \lambda_i^{\beta} \right] \tag{3.9.34}$$

some dependence of $K_{\alpha\dot{a}}$ on the new variables is needed to preserve the constraint. It turns out that the correct additional generator is

$$x_i^{\alpha\dot{\beta}}\, \lambda_i^{\alpha}\, \frac{\partial}{\partial \lambda_i^{\beta}} + x_{i+1}^{\alpha\dot{\beta}}\, \tilde{\lambda}_i^{\dot{\alpha}}\, \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\beta}}} \tag{3.9.35}$$

Indeed from (3.9.34)

$$\begin{aligned} x_i^{\alpha\dot{b}}\, x_i^{\alpha\dot{\beta}} - x_{i+1}^{\alpha\dot{b}}\, x_{i+1}^{\alpha\dot{\beta}} &= x_i^{\alpha\dot{b}}\, \tilde{\lambda}_i^{\dot{\beta}} \lambda_i^{\alpha} + x_{i+1}^{\alpha\dot{b}}\, \tilde{\lambda}_{i+1}^{\dot{\beta}} \lambda_{i+1}^{\alpha} \\ &= x_i^{\alpha\dot{b}}\, x_{i+1}^{\alpha\dot{\beta}} + x_{i+1}^{\alpha\dot{b}}\, x_i^{\alpha\dot{\beta}} \\ &= x_i^{\alpha\dot{b}}\, x_i^{\alpha\dot{\beta}} - x_{i+1}^{\alpha\dot{b}}\, x_{i+1}^{\alpha\dot{\beta}} \end{aligned} \tag{3.9.36}$$

where the constraint (3.9.29) has been employed. In a very similar manner, acting with $K_{\alpha\dot\alpha}$ on both sides of (3.9.30)

$$\left[K^{\alpha\dot\alpha}\,,\theta^{A\beta}_{i,i+1}\right]=\left[K^{\alpha\dot\alpha}\,,\lambda^{\beta}_i\,\eta^A_i\right] \tag{3.9.37}$$

requires complementing the operator with

$$\tilde\lambda^{\dot\alpha}_i\,\theta^{\alpha B}_{i+1}\,\frac{\partial}{\partial\,\eta^B_i} \tag{3.9.38}$$

so that

$$\begin{aligned} x^{\alpha\dot\gamma}_i\,\theta^{A\,\alpha}_i - x^{\alpha\dot\gamma}_{i+1}\,\theta^{A\,\alpha}_{i+1} &= \tilde\lambda^{\dot\alpha}_i\lambda^{\gamma}_i\,\theta^{A\,\alpha}_{i+1}+x^{\alpha\dot\gamma}_i\,\lambda^{\alpha}_i\,\eta_i \\ &= x^{\alpha\dot\gamma}_{i,i+1}\,\theta^{A\,\alpha}_{i+1}+x^{\alpha\dot\gamma}_i\,\theta^{A\,\alpha}_{i,i+1} \\ &= x^{\alpha\dot\gamma}_i\,\theta^{A\,\alpha}_i - x^{\alpha\dot\gamma}_{i+1}\,\theta^{A\,\alpha}_{i+1} \end{aligned} \tag{3.9.39}$$

again by use of the constraint (3.9.30). Then finally its full form reads

$$K^{\alpha\dot\alpha}=\sum_{i=1}^{n}\left[x^{\alpha\dot\beta}_i x^{\alpha\dot\beta}_i\frac{\partial}{\partial x^{\beta\dot\beta}_i}+x^{\alpha\dot\beta}_i\theta^{\alpha B}_i\frac{\partial}{\partial\theta^{\beta B}_i}+x^{\alpha\dot\beta}_i\lambda^{\alpha}_i\frac{\partial}{\partial\lambda^{\beta}_i}+x^{\alpha\dot\beta}_{i+1}\tilde\lambda^{\dot\alpha}_i\frac{\partial}{\partial\tilde\lambda^{\dot\beta}_i}+\tilde\lambda^{\dot\alpha}_i\theta^{\alpha B}_{i+1}\frac{\partial}{\partial\eta^B_i}\right] \tag{3.9.40}$$

The same reasoning may be applied on the superconformal charge S, leading to

$$S^A_\alpha=\sum_{i=1}^{n}\left[-\theta^B_{i\alpha}\theta^{\beta A}_i\frac{\partial}{\partial\theta^{\beta B}_i}+x_{i\alpha}{}^{\dot\beta}\theta^{\beta A}_i\frac{\partial}{\partial x^{\beta\dot\beta}_i}+\lambda_{i\alpha}\theta^{\gamma A}_i\frac{\partial}{\partial\lambda^{\gamma}_i}+x_{i+1\,\alpha}{}^{\dot\beta}\eta^A_i\frac{\partial}{\partial\tilde\lambda^{\dot\beta}_i}-\theta^B_{i+1\,\alpha}\eta^A_i\frac{\partial}{\partial\eta^B_i}\right] \tag{3.9.41}$$

The bottom–line of all this is that at tree level the whole superamplitude is annihilated by the original superconformal generators, whereas transforms covariantly under the generators of dual superconformal invariance. In particular the polynomial $\mathcal{P}$ in (3.4.33) is invariant, so is the combination of δ–functions, while the denominator from the MHV factor in front makes the complete superamplitude transform covariantly as

$$\begin{aligned} D\,\mathcal{A}_n &= n\,\mathcal{A}_n \quad , \quad C\,\mathcal{A}_n = n\,\mathcal{A}_n \\ K^{\alpha\dot\alpha}\,\mathcal{A}_n &= -\sum_{i=1}^{n}x^{\alpha\dot\alpha}_i\,\mathcal{A}_n \quad , \quad S^A_\alpha\,\mathcal{A}_n=-\sum_{i=1}^{n}\theta^A_{i\alpha}\,\mathcal{A}_n \end{aligned} \tag{3.9.42}$$

where the action under dilatations D and the helicity central charge C has also been explicited. Then, the generators which actually annihilate the superamplitude are

$$\begin{aligned} \tilde D &= D-n \quad , \quad \tilde C = 0 \\ \tilde K^{\alpha\dot\alpha} &= K^{\alpha\dot\alpha}+\sum_{i=1}^{n}x^{\alpha\dot\alpha}_i \quad , \quad \tilde S^A_\alpha = S^A_\alpha+\sum_{i=1}^{n}\theta^A_{i\alpha} \end{aligned} \tag{3.9.43}$$

These, together with the other dual generators

$$j_a^{(1)} \in \left\{ P_{\alpha\dot\alpha},\, Q_{\alpha A},\, \bar{Q}^A_{\dot\alpha},\, M_{\alpha\beta},\, \overline{M}_{\dot\alpha\dot b},\, R^A{}_B,\, \tilde{D},\, \tilde{S}^A_\alpha,\, \overline{S}^{\dot\alpha}_A,\, \tilde{K}^{\alpha\dot\alpha} \right\} \tag{3.9.44}$$

produce a dual copy of the j_a $PSU(2,2|4)$ superconformal algebra annihilating tree level amplitudes

$$j_a\, \mathcal{A}_n^{\text{tree}} = 0 \tag{3.9.45}$$

It is interesting to combine the dual generators, together with the original superconformal ones

$$j_a \in \left\{ p^{\alpha\dot\alpha},\, q^{\alpha A},\, \bar{q}^{\dot\alpha}_A,\, m_{\alpha\beta},\, \bar{m}_{\dot\alpha\dot b},\, r^A{}_B,\, d,\, s^\alpha_A,\, \bar{s}^A_{\dot\alpha},\, k_{\alpha\dot\alpha} \right\} \tag{3.9.46}$$

satisfying

$$[j_a, j_b] = f_{ab}{}^c\, j_c \tag{3.9.47}$$

where $f_{ab}{}^c$ are the $PSU(2,2|4)$ structure constants. This forms a set of charges conserved for tree level $\mathcal{N}=4$ SYM amplitudes.

To perform such a combination the dual superconformal generators have to be put in such a form where they act only on the original variables $(\lambda, \tilde\lambda, \eta)$. Doing this the generators of dual translations and supersymmetry are trivial, whereas all other dual generators but $\tilde{K}$ and $\tilde{S}$ may be paired to the generators of the original superconformal algebra. Identifying the latter as level zero generators j_a, one defines level one operators $j_b^{(1)}$ by the relation

$$j_a^{(1)} = f_a{}^{cb} \sum_{k<l} j_{kb}\, j_{l\,c} \tag{3.9.48}$$

In particular, while level zero generators may be determined by summing over the single particle ones

$$j_a = \sum_{k=1}^{n} j_{k\,a} \tag{3.9.49}$$

level one generators are bilocal in particles. Moreover they transform in the adjoint with respect to the level zero

$$\left[j_a^{(1)}\,, j_b \right\} = f_{ab}{}^c\, j_c^{(1)} \tag{3.9.50}$$

where $f_{ab}{}^c$ are the $PSU(2,2|4)$ structure constants. The point is that level one generators may be identified as the dual superconformal ones and their higher commutators with j, generating an infinite set of charges, are constrained by the Serre relation [48], which is a sufficient condition for defining the Yangian of the superconformal algebra. Therefore tree level amplitudes of $\mathcal{N}=4$ SYM are invariant under the action of a Yangian symmetry

$$y\, \mathcal{A}_n^{\text{tree}} = 0 \tag{3.9.51}$$

for any $y \in Y(PSU(2,2|4))$.

3.10 Dualities involving correlators.

The conjectured duality between Wilson loops and amplitudes is an important achievement in the study of scattering. If true, it would allow to use a notoriously simpler calculation, the Wilson loop one, to determine or at least constrain, MHV amplitudes, which are harder to compute.

Dualities in the subject of scattering amplitudes were extended last year to a a new set of field theoretical objects: correlation functions [65, 66]. More precisely, a correspondence was suggested between Wilson loops and correlation functions of half BPS operators in $\mathcal{N} = 4$ SYM. The statement is that an n–point correlation function $\mathcal{C}_n$ in the limit where adjacent points become light–like separated should be equal to a light–like n–polygonal Wilson loop in the adjoint representation of the gauge group. The precise identification reads

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\mathcal{C}_n}{\mathcal{C}_n^{\,tree}} = \langle \mathrm{Tr}_{adj}\, P \,\exp\left(ig \int_{C_n} dz^\mu A_\mu(z) \right) \rangle \tag{3.10.1}$$

As for amplitudes, this novel duality can either be used to compute Wilson loops from the light–like limit of correlation functions, if known, or the other way around, to place constraints on correlation functions, since their behaviour in the light–like limit should match that of a Wilson loop.

In field theory a physical explanation for this identification to occur can be given in terms of an infinitely fast moving scalar particle interacting with a low energy gluon.

In the light–like limit, in fact, the scalar particle flowing around the loop becomes infinitely energetic compared to the gluon it interacts with.

As a consequence, its propagator becomes an almost free propagator, except for an eikonal phase $P\,\exp\left(ig \int_{C_n} dz^\mu A_\mu(z) \right)$ which arises as the result of a path integral saddle point approximation.

We give a more detailed explanation of this, focusing on the simplest case of scalar bilinear operators $\phi\,\phi$ in a generic theory, transforming in the adjoint representation of a $SU(N)$ gauge group. At tree level, a correlator of n such operators inserted at points $\{x_i\}$ $(i = 1, \ldots n)$ is given by the product of n scalar propagators

$$\mathcal{C}_n^{\mathrm{tree}} = N^2 \sum_{\sigma \in S_n/\mathbb{Z}_n} S(x_{\sigma(i_1),\sigma(i_2)})\, S(x_{\sigma(i_2),\sigma(i_3)}) \,\ldots\, S(x_{\sigma(i_n),\sigma(i_1)}) \tag{3.10.2}$$

where a sum is performed over all non–cyclic permutations of $\{x_i\}$. These include all contributions, both connected and disconnected. Taking the light–like limit $x_{i,i+1}^2 \to 0$ obvioulsy selects as the most singular part the unique connected term with cyclic order in joining operators

$$\mathcal{C}_n^{\rm tree} \xrightarrow{x_{i,i+1}^2 \to 0} N^2\, S(x_{12})\, S(x_{23}) \, \dots \, S(x_{n1}) = \frac{(2\pi)^{-2n}\, N^2}{x_{12}^2\, x_{23}^2 \, \dots \, x_{n1}^2} \tag{3.10.3}$$

All other terms are subleading in this limit.

In an interacting theory, such as $\mathcal{N} = 4$ SYM, the charged scalars exchange gluons and the correlator develops loop corrections. These corrections are finite, however the light–like limit jeopardizes such finiteness and new singularities are expected, beyond the trivial $1/x^2$. These divergent corrections have to be regularized, with dimensional regularization in $d = 4 - 2\epsilon$, for instance. This introduces a UV mass scale μ, to define a dimensionless action. We then consider propagation of a scalar between two operators at x_i and x_{i+1} positions. Then a second scale is there in the problem, which is the squared separation between adjacent operator insertions $x_{i,i+1}^2$. In the light–like limit, we have to choose a hierarchy between these two scales. In particular we consider the limit where $x_{i,i+1}^2\, \mu^2 \to 0$, namely we are probing the dynamics in the deep UV (at length scales much smaller than μ^{-1}). Since the $x_{i,i+1}^2$ scale refers to scalars, whereas μ to gluon dynamics, it follows that these two sectors are somewhat separated: the corresponding picture is that of very fast particles interacting in a slowly varying gauge background. In such a setting it is reasonable to approximate the correlator to the product of scalar propagators in a background field

$$S(x_i, x_{i+1}; A) = S^{tree}(x_{i,i+1})\, P\, \exp\left(i\, g \int_{x_i}^{x_{i+1}} dz \cdot A(z) \right)\, G(x_i, x_{i+1}; A) \tag{3.10.4}$$

which are a solution of the Green function equation

$$i\, D^\mu D_\mu\, S(x_i, x_{i+1}; A) = \delta^{(4-2\epsilon)}(x_{i,i+1}) \tag{3.10.5}$$

Such a solution can be determined from the ansatz (3.10.4), evaluating the function G in the light–like limit by an operator product expansion, whose local operators are weighted by powers of $x_{i,i+1}^2$: $G(x, 0; A) = \sum_{N,\Delta} (x^2)^\Delta\, C_{\Delta,N}(x^2\mu^2)\, x_{\mu_1} \dots x_{\mu_N}\, \mathcal{O}_\Delta^{\mu_1 \dots \mu_N}(0)$. Such powers are determined by the twists τ of the operators by $n = \tau/2$, so that the only operator surviving the light–cone limit is that of twist 0, namely the identity. Therefore one concludes that the proper propagator to be employed in the computation of the correlator in the light–like limit is

$$S(x_i, x_{i+1}; A) \to S^{tree}(x_{i,i+1})\, P\, \exp\left(i\, g \int_{x_i}^{x_{i+1}} dz \cdot A(z) \right) \tag{3.10.6}$$

Then assembling the propagator as in (3.10.3), one naturally factorizes the tree level contribution from the free part S_0, times the product of n consecutive Wilson lines, that traced make naturally a Wilson loop in the adjoint representation of the gauge group

$$\mathcal{C}_n \longrightarrow \mathcal{C}_n^{\rm tree}\, W^{\rm adj}[C_n] \tag{3.10.7}$$

where the contour of integration is $C_n = \bigcup_{i=1}^{n}[x_i, x_{i+1}]$. In the planar limit the Wilson loop in the adjoint representation may be regarded as the square of that in the fundamental one, so that finally one obtains the relation

$$W^{\text{adj}}[C_n] = \big(W[C_n]\big)^2 + \mathcal{O}\left(\frac{1}{N^2}\right) \tag{3.10.8}$$

In particular the UV divergences of the Wilson loop, which are known to be there because of the cusps in the contour of integration, arise from the spurious singularities of the correlation functions, which are an artifact of the light–like limit.

In [65] this considerations have been applied to projections of bilinear half–BPS operators $\phi\,\phi$, $\bar\phi\,\bar\phi$ in $\mathcal{N}=4$ SYM, in such a configuration that the correlator is non–vanishing. However the relation above is expected to hold generally in any theory. The only requirement is that the limit $x_{i,i+1}^2 \to 0$ still probes a regime where the perturbative analysis on which the argument is based is reliable and therefore the particles move with $1/x^2$ propagators, without corrections. This is the theory has to exhibit UV asymptotic freedom. In particular any CFT in any dimension should display this WL/correlator duality.

Since Wilson loops are dual to planar scattering amplitudes, a direct duality between n–point correlation functions in the light–like limit and n–point scattering amplitudes must exist. This relation was defined in [66] by stating

$$\lim_{x_{i,i+1}^2 \to 0} \log \frac{\mathcal{C}_n}{\mathcal{C}_n^0} = 2\,\log \frac{\mathcal{A}_n}{\mathcal{A}_n^0} \tag{3.10.9}$$

While this new dualities are still lacking a proof in the string theory regime, they have been checked perturbatively up to two loops in a number of cases [65]. In particular, the above conjecture was successfully tested at one–loop for generic n and proved at two loops for $n = 4, 5, 6$, by using the Lagrangian insertion technique in a harmonic $\mathcal{N}=2$ setting.

This result closes a new triality WL/amplitudes/correlators in the light–like limit.

We stress again that, according to the explanation above, the connection between correlators and polygonal Wilson loops should be true not only for $\mathcal{N}=4$ SYM but also for general conformal gauge theories in any dimensions [65].

Chapter 4

Amplitudes in ABJM: one–loop computations

4.1 Introduction

Recently, new interest has been devoted to the study of the S–matrix for the non–trivial sector of three dimensional Chern–Simons–matter theories which allow for a string theory dual description. These are the well–known $\mathcal{N} = 6$ superconformal ABJM model [2] for $U(N)_K \times U(N)_{-K}$ gauge group and the more general ABJ model [12] for $U(M)_K \times U(N)_{-K}$, where K is the Chern–Simons level. In the large M, N limit their strongly coupled dual description is given in terms of M–theory on $AdS_4 \times S^7/\mathbb{Z}_K$ background and, for $K \ll N \ll K^5$, by type IIA string theory on $AdS_4 \times \mathbb{CP}^3$.

The main motivation is to understand whether these theories, even if distinguished in nature, being them non–maximally supersymmetric, share fundamental properties of the four dimensional $\mathcal{N} = 4$ SYM theory, like integrability [67], Yangian symmetry [48] of the planar physical sector and scattering amplitudes / WL / correlation functions dualities [61]–[68], [65]–[69]. While going deep into the nature of three dimensional theories, the investigation of these properties should help to understand their actual origin and the role of the AdS/CFT in their determination.

For the ABJM model, preliminary results can be already found in literature, concerning integrability [70]–[71], the related Yangian symmetry [72, 73] and dualities at tree level.

At classical level, scattering amplitudes have been shown to be invariant under dual superconformal symmetry [74, 75] whose generators are the level–one generators of a Yangian symmetry [72]. At strong coupling this symmetry should rely on self–duality properties of type IIA string on $AdS_4 \times \mathbb{CP}^3$ under a suitable combination of bosonic and fermionic T–dualities [47, 64], even if the situation is complicated by the emergence of singularities in the fermionic T–transformations [76]–[77]. Recent progress in this direction has been done in [78].

The results we report here are mainly focused on the perturbative sector of the theory. In this regime the state of the art on loop amplitudes was quite poor until recently. The aim of this chapter is to review our results at quantum level, trying to begin filling in this gap.

In particular we produce evidence supporting the existence of similar dualities with respect to $\mathcal{N} = 4$ SYM and the persistence of dual conformal invariance in three dimensions, by describing recent findings on scattering amplitudes, light–like Wilson loops and correlators of half–BPS operators at one and two loops.

In this introduction we present a survey of the main topics, we are going to elucidate more thoroughly in this chapter:

- Using $\mathcal{N} = 2$ superspace description and a direct Feynman diagrammatic approach, at one loop we compute the whole spectrum of four–point superamplitudes for $\mathcal{N} = 2, 3, 6, 8$ Chern–Simons matter conformal field theories with $U(M) \times U(N)$ gauge group. The result is generically different from zero, except for the $\mathcal{N} = 6$ ABJ(M) and $\mathcal{N} = 8$ BLG* cases where they all vanish. Suitably generalizing the definition of the Wilson loop to the ABJ theory, we easily argue that it also vanishes at one loop. Therefore, we conclude that a scattering amplitude / WL duality may work only for the $\mathcal{N} \geq 6$ case, independently of the fact that conformal symmetry is present in all the theories we analyze.

- For $\mathcal{N} = 2, 3, 6, 8$ Chern–Simons–matter theories, correlation functions of $2n$ BPS operators are computed [79].
  It is proved that the one–loop result for the correlator, divided by the corresponding tree level expression, coincides with the one–loop light–like $2n$–polygon Wilson loop [80]. The identification is at the level of the integrands, independently of the fact that both of them eventually vanish. This provides the first preliminary hints of remarkable properties of $\mathcal{N} = 4$ translating to three dimensions. The one loop computation of scattering amplitudes in Section 4.5.1, that of the Wilson loop is carried out in Section 4.5.2 and the calculation of correlation functions of half–BPS operators and their light–like limit is done in Section 4.6.

- Less trivial evidence for a scattering amplitude / WL duality arises at two loops where these quantities are not supposed to vanish. Focusing on ABJ models, still using a direct Feynman diagram approach, we evaluate the two–loop planar scattering superamplitude of four chiral superfields, two of them in the bifundamental

*The mentioned results hold also for the $\mathcal{N} = 8$ BLG theory [39, 40] described by $SU(2)_K \times SU(2)_{-K}$ Chern–Simons–matter theory in the large K limit. Enhancement to maximal supersymmetry could also be obtained for Chern–Simons levels $K = 1, 2$. However, these values are out of the perturbative regime and will not be considered.

and two in the antibifundamental representation of the $U(M) \times U(N)$ gauge group. The result for the ratio $\mathcal{M}_4^{(2)} \equiv \mathcal{A}_4^{(2\,loops)}/\mathcal{A}_4^{tree}$ is

$$\mathcal{M}_4^{(2)} = \lambda\hat{\lambda} \left[-\frac{(s/\mu'^2)^{-2\epsilon}}{(2\,\epsilon)^2} - \frac{(t/\mu'^2)^{-2\epsilon}}{(2\,\epsilon)^2} + \frac{1}{2} \ln^2\left(\frac{s}{t}\right) + C_{\mathcal{A}}(M,N) + \mathcal{O}(\epsilon) \right] \quad (4.1.1)$$

where $\lambda = M/K$, $\hat{\lambda} = N/K$, μ' is the mass scale and $C_{\mathcal{A}}(M,N)$ is a constant depending on the ranks of the groups. For $M = N$ we are back to the result for the ABJM theory [81, 82]. This is spelled out in Section 5.2.

- In Section 5.3.1 it is proved that for the ABJM theory, at this order the four–point scattering amplitude [81, 82] divided by its tree–level counterpart coincides with the second order expansion of a light–like four–polygon Wilson loop [80] (whose calculation is reviewed in Subsection 5.1).
- The issue of dual conformal invariance in three dimensions is analyzed in Section 5.3.2. This hinges mainly on the results of [81], where the two–loop four–point amplitude has been computed by generalized unitarity, using a basis of dual conformally invariant integrals. The matching occurring between this and our Feynman diagram computation proves that the amplitude does satisfy dual conformal symmetry.
- Finally in Section 5.3.4 we give evidence supporting a BDS–like ansatz in three dimensions.

We start with a brief introduction on amplitudes in Chern–Simons–matter theories, pointing out the differences with respect to $\mathcal{N} = 4$ SYM in four dimensions.

4.2 Scattering amplitudes in CSM

4.2.1 On–shell momenta.

In three space–time dimensions the Lorentz group $SO(1, 2)$ is isomorphic to $Sl(2, \mathbb{R})$.

As in the four dimensional case, it is convenient to adopt spinor notation to represent on–shell momenta. Here there are not different spinor representations of the Lorentz group, so that only λ spinors (and not $\tilde{\lambda}$) exist. Therefore momenta of massless particles may be represented as

$$p^{\alpha\beta} = \lambda^\alpha \, \lambda^\beta \quad (4.2.1)$$

This identification holds up to a sign ambiguity, connected to the discrete little group $\mathbb{Z}_2$ for massless representations. Invariants may be translated into spinor products†

$$2\, p_1 \cdot p_2 = \langle 1\, 2 \rangle^2 = (\lambda_1^\alpha \, \lambda_2^\beta \, C_{\beta\alpha})^2 \quad (4.2.2)$$

†The sign of the invariant is different from other papers in literature ([72]), due to different conventions in the metric and contraction with the antisymmetric symbol. Our notations are listed in Appendix A.1.

Since momentum is real it follows that λ must be either purely real or purely imaginary. The former choice yields $p_0 > 0$, stemming for positive energy momenta, whereas the latter produces negative energy $p_0 < 0$ ones.

4.2.2 External particles.

In Chern–Simons matter theories the only non–trivial scattering amplitudes are matter amplitudes of scalars and fermions, as the vector fields are not propagating. Even though the theory might be interacting, yielding non–trivial equations of motion for the gauge field, considering an amplitude with external gauge vectors is meaningless, since as asymptotic states they obey free equations of motions. These are of the form $dA = 0$, implying that the gauge field is just a flat connection. Therefore one restricts the computation of amplitudes to the physical matter sector.

The on–shell matter content of ABJM is in terms of a multiplet of four scalars, and one of their $\mathcal{N} = 2$ fermionic superpartners, transforming in the fundamental representation of the R–symmetry group $SU(4)$ and in the bifundamental $(\mathbf{N}, \bar{\mathbf{N}})$ of the gauge group. In addition their complex conjugate fields, transforming in the conjugates representations are there. The former are referred to as the particles, whereas the latter as the antiparticles

$$\phi^A(\lambda)\,, \quad \bar{\phi}_A(\lambda)\,, \quad \psi_A(\lambda)\,, \quad \bar{\psi}^A(\lambda)\,, \quad A \in \{1,2,3,4\} \tag{4.2.3}$$

Although we shall not adopt it here, we review how the on–shell superspace formalism of $\mathcal{N} = 4$ SYM translates to the ABJM case. In contrast to the maximally supersymmetric four dimensional case, all physical particles transform in the same representation of the R–symmetry group $SU(4)$. Therefore they cannot be multiplied by different amounts of $SU(4)$ Grassmann variables. In order to embed the fields (4.2.3) in such multiplets it is necessary to break $SU(4)$ into a $U(3)$ symmetry, manifestly realized on the spectrum as well as some remainder. By doing this, fields can be recovered from the expansion of a scalar Φ and a fermionic Ψ multiplet in powers of η^A Grassmann spinors in the fundamental of $SU(3)$

$$\begin{aligned}
\Phi(\Lambda) &= \phi^4(\lambda) + \eta^A\,\psi_A(\lambda) + \frac{1}{2}\,\epsilon_{ABC}\,\eta^A\,\eta^B\,\phi^C(\lambda) + \frac{1}{3!}\,\epsilon_{ABC}\,\eta^A\,\eta^B\,\eta^C\,\psi_4(\lambda) \\
\Psi(\Lambda) &= \bar{\psi}^4(\lambda) + \eta^A\,\bar{\phi}_A(\lambda) + \frac{1}{2}\,\epsilon_{ABC}\,\eta^A\,\eta^B\,\bar{\psi}^C(\lambda) + \frac{1}{3!}\,\epsilon_{ABC}\,\eta^A\,\eta^B\,\eta^C\,\bar{\phi}_4(\lambda)
\end{aligned} \tag{4.2.4}$$

The former contains the particles, whilst the second the antiparticles of the spectrum. Using these superfields, superamplitudes may be computed

$$\mathcal{A}_n = \mathcal{A}_n\left(\Phi_1,\, \bar{\Phi}_2,\, \Phi_3, \ldots, \bar{\Phi}_n\right) \tag{4.2.5}$$

where the components are reproduced expanding in η and retaining the proper pieces.

As mentioned above we will not use the on–shell superspace formalism, rather we want to address the problem of computing amplitudes in CSM theories (and especially in the ABJ(M) models) in manifestly $\mathcal{N}=2$ supersymmetric (off–shell) superspace language. This entails considering A, B superfields and their complex conjugates as external particles. The corresponding superamplitude contains all components in its θ expansion.

Given the structure of the vertices and the fact that we discard amplitudes with external gauge fields, it is straightforward to realize that only those involving an even number of external legs are non–vanishing. This is consistent with the requirement for the amplitudes to be invariant under gauge and Lorentz transformations and under the $SU(3)$ linearly realized R–symmetry in on–shell superspace [74, 72].

In our language each external scalar particle A, B carries an on–shell momentum $p_{\alpha\beta}$ ($p^2=0$), an $SU(2)$ flavor index and color indices corresponding to the two gauge groups. In the more general situation of $U(M)\times U(N)$ gauge groups we classify as particles the ones carrying $(M,\bar{N})$ indices and antiparticles the ones carrying $(\bar{M},N)$ indices. Therefore, $(A^i,\bar{B}^j)$ superfields fall into the former set, whereas $(B_i,\bar{A}_j)$ into the latter.

4.2.3 Color ordering.

The representation of the gauge groups in which (both on and off–shell) superfields transform is the (anti)bifundamental. This means that gauge invariance is ensured in amplitudes whenever each particle has its two indices contracted with two antiparticles and viceversa, so on up to building up a trace of fields. Of course multiple trace configurations are allowed. We will mostly neglect them (only in Section 4.5.1 all trace structures of one–loop four–point amplitudes are taken into account) by restricting to the planar limit, which conveniently simplifies computations and is also the main framework where amplitudes are worked out in $\mathcal{N}=4$ SYM as well.

Therefore amplitudes may be expanded into sums of color ordered pieces, as in (3.2.4)

$$A_n\left(\Phi_1{}^{A_1}_{\bar{A}_1},\bar{\Phi}_2{}^{\bar{B}_2}_{B_2},\Phi_3{}^{A_3}_{\bar{A}_3},\dots,\bar{\Phi}_n{}^{\bar{B}_n}_{B_n}\right)=\sum_{\sigma\,\in\,(S_{n/2}\times S_{n/2}/C_{n/2})}\mathcal{A}_n\left(\Lambda_{\sigma_1},\dots,\Lambda_{\sigma_n}\right)\delta^{A_{\sigma_1}}_{B_{\sigma_2}}\dots\delta^{\bar{B}_{\sigma_n}}_{\bar{A}_{\sigma_1}} \tag{4.2.6}$$

where the sum runs over permutations of even and odd sites separately (in order not to spoil gauge invariance) and cyclic permutations are neglected, being symmetries of the trace. Hereafter color ordering will be always understood and the objects studied will be partial amplitudes.

4.2.4 Tree level results.

Before heading the loop computation of amplitudes in superspace we mention some of the properties highlighted in literature on scattering in ABJM. The language used in this context is mostly in component or in on–shell superspace and the main results concern tree level amplitudes.

- In [72, 83] the tree level superamplitudes for four and six external particles have been computed in ABJM.
- In [72] the constraints from the ordinary $OSp(6|4)$ superconformal group on tree level amplitudes has been analyzed. Moreover it has been shown that four and six–point amplitudes possess Yangian invariance. To do this it has not actually been necessary to determine dual conformal generators, in that the Yangian can be constructed from the superconformal level zero generators alone. This is done by composing the latter into bilocal level one operators (as in Section 3.9.2) and checking that they obey the desired commutation relations with level zero ones (3.9.50) and the Serre relations, ensuring the Yangian structure $Y(OSp(6|4))$.
- In [74] the dual superconformal generators under which tree–level amplitudes are invariant have been identified. Dual space is constructed as in [48] by defining

$$\begin{aligned} x_{i,i+1}^{\alpha\beta} &= x_i^{\alpha\beta} - x_{i+1}^{\alpha\beta} = p_i^{\alpha\beta} = \lambda_i^\alpha \lambda_i^\beta \\ \theta_{i,i+1}^{A\alpha} &= \theta_i^{A\alpha} - \theta_{i+1}^{A\alpha} = q_i^{\alpha A} = \lambda_i^\alpha \eta_i^A \end{aligned} \tag{4.2.7}$$

  Dual $SO(2,2)$ conformal invariance of tree level amplitudes follows quite straightforwardly, constructing the dual conformal boost generator from the canonical one and adding terms so that it commutes with the defining relations 4.2.7.

  When trying to extend to the whole superconformal $OSp(6|4)$, a problem arises since only the $U(3)$ part of the R–symmetry, linearly realized on the on–shell spectrum (4.2.4) commutes with the constraints (4.2.7), whereas the other 6 generators do not. This problem can be remedied by introducing additional dual coordinates y, in terms of which the R–generators act canonically

$$y_{i,i+1}^{AB} = y_i^{AB} - y_{i+1}^{AB} = \eta_i^A \eta_i^B \tag{4.2.8}$$

  Then dual generators are deformed in such a way to commute with the latter constraint as well. After doing this it turns out that level one generators of the $Y(OSp(6|4))$ Yangian symmetry can be found to be equivalent to the dual superconformal generators, restricted to on–shell space using the defining equations (4.2.7) and (4.2.8).
- In [75] recursion relations have been determined for ABJM tree–level amplitudes. The main hurdle in deriving them in a parallel manner with respect to the BCFW relations is that no complex shift of momenta is allowed which preserves momentum conservation and the on–shell conditions. This may be circumvented by considering a non–linear shift, which allows to deform momenta consistently and write down a recursion relation as arising from a Cauchy residue theorem. The recursion relation

derived in [75] preserves dual conformal invariance, meaning that all tree level amplitudes which may be constructed from it are invariant under dual superconformal symmetry.

4.3 Superspace computation of amplitudes.

Now we come back to loop amplitudes in superspace, for which we set up the computation.

We are interested in the simplest non–trivial amplitudes, that is four–point amplitudes. These are chiral superamplitudes $(A^i\, B_j\, A^k\, B_l)$ and non–chiral superamplitudes $(A^i\, \bar{A}_j\, A^k\, \bar{A}_l)$, $(B_i\, \bar{B}^j\, B_k\, \bar{B}^l)$, $(A^i\, \bar{A}_j\, \bar{B}^k\, B_l)$ plus possible permutations. While for the ABJ(M) theories they can all be obtained from $(A\, B\, A\, B)$ by $SU(4)$ R–symmetry transformations [87], for more general $\mathcal{N}=2$ models they are independent objects and need to be computed separately.

As outlined above, the color indices can be stripped out, as we can write

$$\mathcal{A}_4\left(X^{a_1}_{\bar{a}_1}\, Y^{\bar{b}_2}_{b_2}\, Z^{a_3}_{\bar{a}_3} W^{\bar{b}_4}_{b_4}\right) = \sum_\sigma \mathcal{A}_4(\sigma(1),\cdots,\sigma(4))\; \delta^{a_{\sigma(1)}}_{b_{\sigma(2)}}\, \delta^{\bar{b}_{\sigma(2)}}_{\bar{a}_{\sigma(3)}}\, \delta^{a_{\sigma(3)}}_{b_{\sigma(4)}} \delta^{\bar{b}_{\sigma(4)}}_{\bar{a}_{\sigma(1)}} \tag{4.3.1}$$

where (X,Z) stay generically for A or $\bar{B}$, (Y,W) for B or $\bar{A}$ superfields and the sum is over exchanges of even and odd sites between themselves. We can then restrict to color–ordered amplitudes $\mathcal{A}_4(\sigma(1),\cdots,\sigma(4))$ with a fixed order of the external momenta.

Our strategy is to compute amplitudes perturbatively, by a direct superspace Feynman diagrammatic approach. Precisely, for four–point amplitudes, we evaluate the effective action quartic in the scalar matter superfields. Since in a scattering process the external fields are on–shell, it is sufficient to evaluate the *on–shell* effective action. This amounts to requiring the external superfields to satisfy the equations of motion (EOM) [‡]

$$D^2\, A^i = D^2\, B_i = 0 \qquad , \qquad \bar{D}^2\, \bar{A}_i = \bar{D}^2\, \bar{B}^i = 0 \tag{4.3.2}$$

from which further useful equations follow

$$i\, \partial^{\alpha\beta}\, D_\beta\, A^i = i\, \partial^{\alpha\beta}\, D_\beta\, B_i = 0 \qquad , \qquad i\, \partial^{\alpha\beta}\, \bar{D}_\beta\, \bar{A}_i = i\, \partial^{\alpha\beta}\, \bar{D}_\beta\, \bar{B}^i = 0 \tag{4.3.3}$$

In principle, setting the external superfields on–shell might cause problems when IR divergences appear in loop integrals. We dimensionally regularize these divergences working in $D = 3-2\epsilon$ dimensions, while keeping spinors and ϵ_{ijk} tensors strictly in three dimensions. We then use the prescription to set the external superfields on–shell at finite ϵ.

[‡]The actual EOM as derived from the action (1.3.1) would be $D^2\, A^1 = -\bar{h}_1\, \bar{B}^2\, \bar{A}_2\, \bar{B}^1 - \bar{h}_2\, \bar{B}^1\, \bar{A}_2\, \bar{B}^2$, $D^2\, A^2 = -\bar{h}_1\, \bar{B}^1\, \bar{A}_1\, \bar{B}^2 - \bar{h}_2\, \bar{B}^2\, \bar{A}_1\, \bar{B}^1$ plus their hermitian conjugates, and similarly for the B fields. However, being us interested in the quartic terms of the effective action, we can safely approximate the EOM as in (4.3.2).

To summarize, the general strategy is the following: For a given process and at a given order in loops we draw all super–Feynman diagrams with four external scalar superfields. The corresponding contribution will be the product of a color/combinatorial factor times a function of the kinematic variables. We work in the large M, N limit and perturbatively in $\lambda = M/K_1$ and $\hat{\lambda} = N/K_2$. To determine the kinematic function, we perform D–algebra to reduce superdiagrams to a linear combination of ordinary momentum integrals. This is achieved by integrating by parts spinorial derivatives coming from vertices and propagators and using the algebra (A.1.16) up to the stage where only one factor $D^2\bar{D}^2$ for each loop is left. This procedure is highly simplified by the on–shell conditions (4.3.2, 4.3.3) on the external superfields. We then evaluate momentum integrals in dimensional regularization by using standard techniques (Feynman parameterization and Mellin–Barnes representation).

In momentum space the external superfields carry outgoing momenta (p_1, p_2, p_3, p_4), with $p_i^2 = 0$ and $\sum_i p_i = 0$. At the level of the effective action we are allowed to conveniently rename the external momenta, since the p_i's are integrated. When evaluating the amplitude, the total contribution from every single graph will be given by the sum over all possible permutations of the external legs accounting for the different scattering channels.

Mandelstam variables are defined as $s = (p_1 + p_2)^2, t = (p_1 + p_4)^2, u = (p_1 + p_3)^2$.

4.4 Light–like Wilson loops for CSM theories.

The definition of the Wilson loop for CSM theories was proposed in [84, 80] for the $U(N) \times U(N)$ gauge group, namely for the ABJM case

$$\langle W_4 \rangle_{\rm ABJM} = \frac{1}{2N} \left\{ {\rm Tr}\, P\, e^{i \int_{C_n} A_\mu \, dz^\mu} + {\rm Tr}\, P\, e^{i \int_{C_n} \hat{A}_\mu \, dz^\mu} \right\} \tag{4.4.1}$$

In the situation we are interested in C_n is a light–like polygonal closed path. It is given by n points x_i $(i = 1 \dots n)$ in three dimensional spacetime. The edges $x_i^\mu - x_{i+1}^\mu$ are parameterized by the parameters $\alpha_i \in [0, 1]$ in the following manner

$$z_i^\mu = x_i^\mu + (x_{i+1} - x_i)^\mu \, \alpha_i \tag{4.4.2}$$

For later convenience we refer to the displacement vectors $x_i^\mu - x_j^\mu$ as x_{ij}^μ and to the edges $x_{i+1,i}^\mu$ as p_i^μ. In the light–like Wilson loop $p_i^2 = 0$ for all i, where the invariants are taken by contraction with the Minkowskian metric ($g = {\rm diag}\,(+1, -1, -1)$)[§].

[§]All computations with Wilson loops are performed here in Lorentzian signature, so as to stick to the mainstream in literature. In order to match computations for amplitudes and correlators which are carried out in the Euclidean, it is then straightforward to Wick rotate the final results to this signature.

We want to restrict to real Wilson loops, for which imposing all non–vanishing displacements $-x_{ij}^2 > 0$ is required. This is possible only for an even number n of sides, which will be understood hereafter.

Up to two loops the precise CSM theory where we set the computation of the Wilson loop does not really matter, since there are no contribution involving superpotential interactions. The only information entering the computation is the field content of the theory, which is the same for all the models with bifundamental matter we deal with.

We can sensibly deform the Wilson loop away from the ABJM, by changing the ranks of the gauge groups (ABJ), or taking non–opposite CS levels, in addition. In what follows we will be mainly interested in the extension to ABJ, henceforth we will generalize the definition of the Wilson loop, taking into account different ranks for the gauge groups, which we hereafter take to be $U(M) \times U(N)$, but leaving opposite CS levels. In doing this, we change the normalizations of the two contributions from A and $\hat{A}$ gauge fields. We suggest the following definition for the Wilson loop

$$\langle W_4 \rangle_{\rm ABJ}^{(1)} = \frac{1}{M+N} \left\{ {\rm Tr}_{U(M)}\, P\, e^{i\int A_\mu\, dz^\mu} + {\rm Tr}_{U(N)}\, P\, e^{i\int \hat{A}_\mu\, dz^\mu} \right\} \tag{4.4.3}$$

This should be the natural generalization of the Wilson loop for a unitary gauge group, just considering $U(M) \times U(N)$ as the whole gauge group and taking its gauge field $\mathcal{A} = {\rm diag}(A, \hat{A})$ to be a block diagonal $(N+M) \times (N+M)$ square matrix)

$$\begin{aligned} \langle W_4 \rangle_{\rm ABJ}^{(1)} &= \frac{1}{M+N}\, {\rm Tr}\, P\, e^{i\int \mathcal{A}_\mu\, dz^\mu} \\ &= \frac{1}{M+N}\, {\rm Tr}\, P\, e^{i\int \begin{pmatrix} A & 0 \\ 0 & \hat{A} \end{pmatrix} dz^\mu} \\ &= \frac{1}{M+N} \left\{ {\rm Tr}_{U(M)}\, P\, e^{i\int A_\mu\, dz^\mu} + {\rm Tr}_{U(N)}\, P\, e^{i\int \hat{A}_\mu\, dz^\mu} \right\} \end{aligned} \tag{4.4.4}$$

We can arbitrarily propose a slightly modified candidate

$$\langle W_4 \rangle_{\rm ABJ}^{(2)} = \frac{1}{2M}\, {\rm Tr}_{U(M)}\, P\, e^{i\int A_\mu\, dz^\mu} + \frac{1}{2N}\, {\rm Tr}_{U(N)}\, P\, e^{i\int \hat{A}_\mu\, dz^\mu} \tag{4.4.5}$$

which differs from the previous one by the different choice of relative normalization of the two contributions.

Both (4.4.3) and (4.4.5) reproduce the ABJM Wilson loop if the parity violating parameter σ is set to zero, or, said another way, when we set $M = N$.

In the computations we will perform, up to two loops we will see that the two definitions are pretty indistinguishable and we do not have a favorite one in that both will be able to match the four point amplitude, up to an unobservable change of regularization scale. We include (4.4.5) in the discussion because we will show that its dependence on M and N in loop computations seems to be slightly more similar to that for amplitudes than (4.4.3).

4.5 One loop computations.

In this section we address the computation of scattering amplitudes in general $\mathcal{N}=2$ CSM theories to one–loop order in perturbation theory.

The aim of this calculation is to study the properties of loop amplitudes in three dimensions and ascertain whether they share some of the marvelous features of the $\mathcal{N}=4$ SYM ones.

For instance we are interested in investigating whether the WL/amplitude/correlator triality may hold in CSM theories. In particular here we want to test if this conjecture passes the tests at one–loop and eventually selects subclasses of theories for which the triality or some corner thereof is valid. To do this we also review how to compute light–like Wilson loops, and correlation functions of half–BPS operators, detailing the steps and quoting the main results.

We start by spelling out the evaluation of the scattering amplitude of four scalar fields. We embed the problem into superspace language by suitably rearranging the scalar into supermultiplets and then considering the corresponding superamplitude. As explained in more detailed in Section 4.2, in the most symmetric theories supersymmetry constrains amplitudes in such a way that only one component is independent and all others can be obtained by multiplication by a simple kinematic factor. In less symmetric theories any $\mathcal{N}=2$ superamplitude is pretty independent. Working in a generic $\mathcal{N}=2$ entails therefore the calculation of many possible configurations of external superfields. We are able to accomplish such a task and we list the final expression for all amplitudes. The main outcome is that four–point amplitudes vanish at one loop for the ABJ(M) case, whereas are generically non–zero for all other theories. These results are contained in [85].

In Section 4.5.2 we deal with the computation of a light–like polygonal bosonic Wilson loop, whose definition in CSM theories was outlined in the last Section. Their perturbative calculation is performed directly in components, by expanding the Wilson loop exponential and employing the gauge vectors propagators. We show how to generalize the result from the simplest four cusped Wilson loop to an arbitrary even number n of cusps and we derive a closed expression for it, in terms of an integral over the affine parameters of the Wilson loop, which we work out later on. The bulk of the computation is taken from [80], whereas the extension to any n can be found in [79].

The last part of this Chapter is devoted to the one–loop computation of the light–like limit of correlation functions of half–BPS operators. Setting this calculation in $\mathcal{N}=2$ superspace it amounts to considering the prototype of a one–loop correction, given by the exchange of a vector superfield and sum over all channels. The main hurdle is due to a quite involved Feynman integral which we handle in a very non–trivial way. At the end of the day we are able to show that the final expression for the correlator of an arbitrary even number of such operators reproduces the same expression as the Wilson loop, in the light–like limit. This is again given as a sum over different contributions and in terms of a still unresolved integral.

Finally we face the problem of evaluating this integral. After a lengthy calculation we manage to prove that summing over all possible exchange channels yields a vanishing outcome for any even number of insertions. Since the expression is formally the same as for a polygonal light–like Wilson loop, we conclude that both evaluate to zero at one loop. These section is inspired by [79].

4.5.1 Scattering at one–loop

For a generic $\mathcal{N}=2$ model described by the action (2.6.1), we first concentrate on the chiral amplitudes $(A^i\, B_j\, A^k\, B_l)$.

At tree level and one loop the corresponding contributions are depicted in Fig. 4.1 where the four–point interaction comes from the superpotential term in (1.3.5).

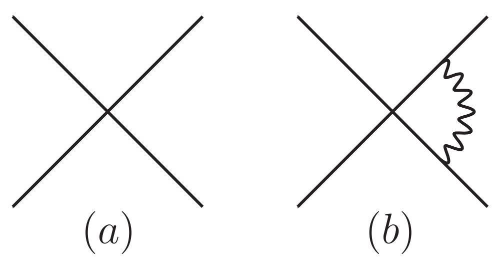


Figure 4.1: Diagrams contributing to the tree level and one–loop four–point chiral scattering amplitude.

The tree–level amplitudes as coming from Fig. 4.1(a) are simply given by

$$\begin{aligned}\mathcal{A}_4^{tree}\left(A^1(p_1),\, B_1(p_2),\, A^2(p_3),\, B_2(p_4)\right) &= h_1\\ \mathcal{A}_4^{tree}\left(A^1(p_1),\, B_2(p_2),\, A^2(p_3),\, B_1(p_4)\right) &= h_2\end{aligned} \tag{4.5.1}$$

At one loop, we need evaluate diagram 4.1(b). Performing on–shell D–algebra and going to momentum space, the corresponding term in the effective action turns out to be proportional to

$$\begin{aligned}&\int d^4\theta\, \mathrm{Tr}\left(A^i(p_1)\, B_j(p_2)\, D^\alpha\, A^k(p_3)\, D^\beta\, B_l(p_4)\right) \times \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}}\, \frac{(k+p_4)_{\alpha\beta}}{k^2(k-p_3)^2(k+p_4)^2}\\ &\xrightarrow{\epsilon\to 0} \frac{1}{8}\int d^4\theta\, \mathrm{Tr}\left(A^i(p_1)\, B_j(p_2)\, D^\alpha\, A^k(p_3)\, D^\beta\, B_l(p_4)\right)\, \frac{(p_4-p_3)_{\alpha\beta}}{|p_3+p_4|^3}\end{aligned} \tag{4.5.2}$$

where in the second line we have used the results (A.5.11, A.5.12) for the scalar and vector–like triangles in dimensional regularization.

Now, using the on–shellness conditions (4.3.3), which in the case under exam read $p_3^{\alpha\beta} D_\alpha A^k(p_3) = 0$ and $p_4^{\alpha\beta} D_\beta B_l(p_4) = 0$, it is easy to see that the final result is zero. Since the same pattern occurs for all the permutations of the external momenta, we conclude that the chiral four–point amplitude is one–loop vanishing. This occurs not only in the planar limit, but also for any finite value of M, N.

Notably, the one–loop vanishing of the effective action, quartic in the chiral superfields, can be proved to be true even off–shell [86]. Indeed diagram 4.1(b) (where momenta are labeled starting from $p1$ upperleft following counterclockwise order) yields the following contribution

$$\int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{k_{\alpha\beta}}{k^2(k-p_3)^2(k-p_3-p_4)^2} \int d^4\theta \; A(p_1)\, B(p_2)\, D^\alpha A(p_3)\, D^\beta B(p_4) \qquad (4.5.3)$$

where k stems for loop momentum. Performing the spinorial integration over $\bar{\theta}$ to recover the chiral measure gives

$$\begin{aligned} &\int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{k_{\alpha\beta}(p_4)^{\beta\gamma}(p_3)_g{}^{\alpha}}{k^2(k-p_3)^2(k-p_3-p_4)^2} \int d^2\theta \; A(p_1)\, B(p_2)\, A(p_3)\, B(p_4) \\ &= \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{\epsilon(k\, p_4\, p_3)}{k^2(k-p_3)^2(k-p_3-p_4)^2} \int d^2\theta \; A(p_1)\, B(p_2)\, A(p_3)\, B(p_4) \qquad (4.5.4) \end{aligned}$$

where the notation $\epsilon(k\, p_4\, p_3) = \epsilon_{\mu\nu\rho}\, k^\mu\, p_4^\nu\, p_3^\rho$ is understood. By a Passarino–Veltman decomposition of the vector integral, we can realize that this contribution is proportional to $\epsilon(p_4\, p_4\, p_3)$ and $\epsilon(p_3\, p_4\, p_3)$, which are identically zero due to the antisymmetry of the ϵ tensor.

We now go through non–chiral amplitudes of the form $(A^i\, \bar{A}_j\, A^k\, \bar{A}_l)$. For theories with low supersymmetry, such as the generic $\mathcal{N} = 2$ CSM, these are not connected to the chiral one by super Ward identities. Thus we do not expect them to vanish in general.

These amplitudes get contributions from several supergraphs, which we depict in Fig. 4.2.

For each graph we compute the corresponding color/combinatorial factor and perform on–shell D–algebra. We list the results valid for M, N finite (for the time being, no large M, N limit is taken). Since we work at the level of the effective action, an overall integral over p_i momenta is understood. For convenience we also neglect an overall $(4\pi)^2$ coming from the gauge propagators (1.4.5).

Diagram 4.2(a) : This is the only diagram involving the chiral interaction vertices proportional to h_1, h_2. In this case D–algebra is trivial and the resulting color structure gives only double traces. Exploiting the possibility to relabel the integrated momenta,

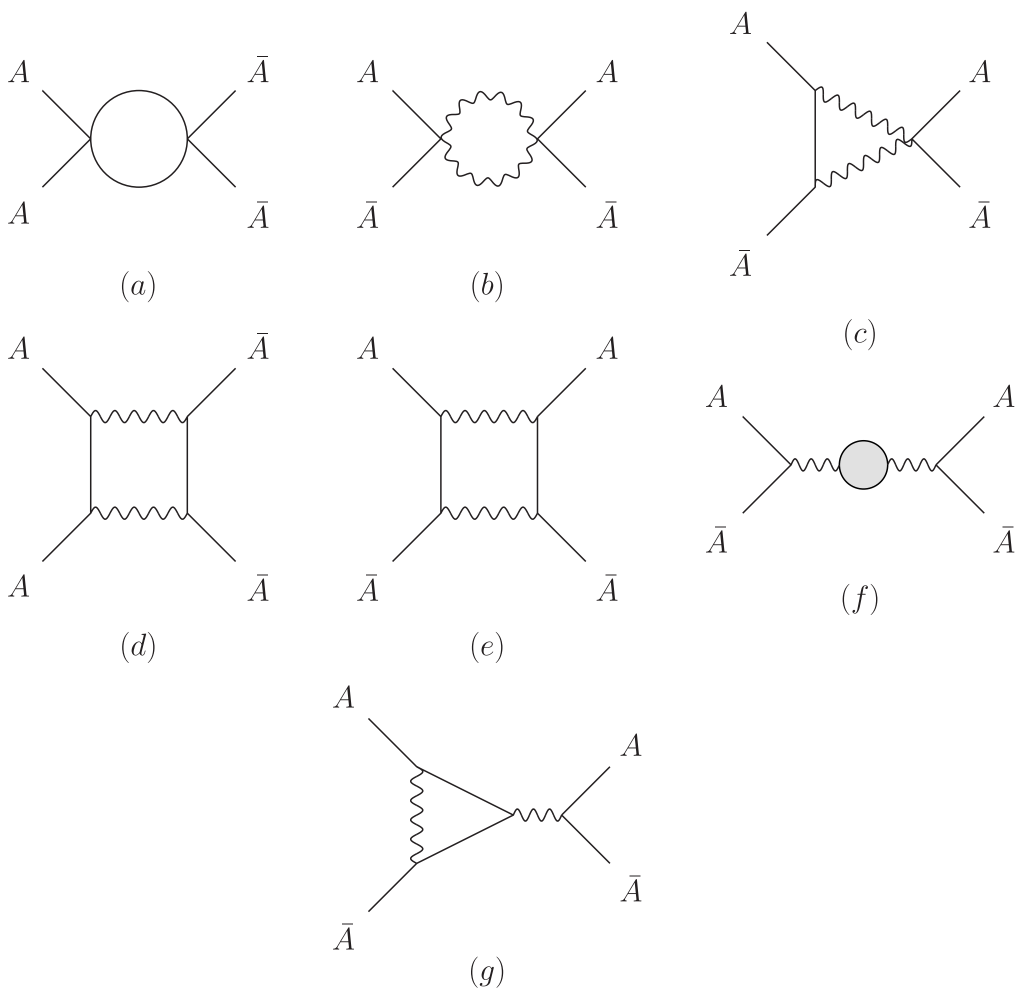


Figure 4.2: One–loop diagrams contributing to non–chiral amplitudes $(A^i\,\bar{A}_j\,A^k\,\bar{A}_l)$.

the result can be written in a quite compact form

$$2(a) = \frac{1}{32\pi^2} \int d^4\theta \quad \Big\{ (|h_1|^2 + |h_2|^2)\, \mathrm{Tr}\,(A^i(p_1)\bar{A}_i(p_2))\, \mathrm{Tr}\,(A^j(p_4)\bar{A}_j(p_3)) \qquad (4.5.5)$$
$$+(h_1\bar{h}_2 + h_2\bar{h}_1)\, \mathrm{Tr}\,(A^i(p_1)\bar{A}_j(p_3))\, \mathrm{Tr}\,(A^j(p_4)\bar{A}_i(p_2))$$
$$-|h_1 + h_2|^2\, \mathrm{Tr}\,(A^i(p_1)\bar{A}_i(p_2))\, \mathrm{Tr}\,(A^i(p_4)\bar{A}_i(p_3)) \Big\}\, \mathcal{B}(p_1 + p_4)$$

where $\mathcal{B}(p_1 + p_4)$ is the bubble integral defined in (A.5.9). Repeated flavor indices are understood to be summed.

Diagram 4.2(b) : In this case the result is a linear combination of single and double traces. Single traces are associated with planar graphs and are leading in the large M, N limit. D–algebra is easily performed and leads to

$$2(b) = \int d^4\theta \Big\{ - \frac{M}{4K_1^2}\, \mathrm{Tr}\,(A^i(p_1)\bar{A}_i(p_2)A^j(p_4)\bar{A}_j(p_3)) \qquad (4.5.6)$$
$$- \frac{N}{4K_2^2}\, \mathrm{Tr}\,(A^i(p_1)\bar{A}_j(p_3)A^j(p_4)\bar{A}_i(p_2))$$
$$- \left(\frac{1}{4K_1^2} + \frac{1}{4K_2^2}\right)\, \mathrm{Tr}\,(A^i(p_1)\bar{A}_i(p_2))\, \mathrm{Tr}\,(A^j(p_4)\bar{A}_j(p_3))$$
$$- \frac{1}{K_1K_2}\, \mathrm{Tr}\,(A^i(p_1)\bar{A}_j(p_3))\, \mathrm{Tr}\,(A^j(p_4)\bar{A}_i(p_2)) \Big\}\, \mathcal{B}(p_1 + p_2)$$

Diagram 4.2(c) : With a convenient choice for the internal momentum, this diagram gives rise to

$$2(c) = \int d^4\theta \quad \Big\{ \frac{M}{2K_1^2}\, \mathrm{Tr}\,(D^\alpha A^i(p_1)\bar{D}^\beta \bar{A}_i(p_2)A^j(p_4)\bar{A}_j(p_3))$$
$$+\frac{N}{2K_2^2}\, \mathrm{Tr}\,(D^\alpha A^i(p_1)\bar{A}_j(p_3)A^j(p_4)\bar{D}^\beta \bar{A}_i(p_2))$$
$$+\left(\frac{1}{2K_1^2} + \frac{1}{2K_2^2}\right)\, \mathrm{Tr}\,(D^\alpha A^i(p_1)\bar{D}^\beta \bar{A}_i(p_2))\, \mathrm{Tr}\,(A^j(p_4)\bar{A}_j(p_3))$$
$$+\frac{2}{K_1K_2}\, \mathrm{Tr}\,(D^\alpha A^i(p_1)\bar{A}_j(p_3))\, \mathrm{Tr}\,(A^j(p_4)\bar{D}^\beta \bar{A}_i(p_2)) \Big\}$$
$$\times \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}}\, \frac{k_{\alpha\beta}}{k^2(k-p_1)^2(k+p_2)^2} \qquad (4.5.7)$$

where D–algebra requires integrating two spinorial derivatives on the external fields.

Using the result (A.5.12) for the vector–like triangle in dimensional regularization, this contribution vanishes due to the equations of motion (4.3.3) $p_1^{\alpha\beta} D_\alpha A(p_1) = 0$ and $p_2^{\alpha\beta} \bar{D}_\beta \bar{A}(p_2) = 0$.

Diagram 4.2(d) : This is the first case where on–shell D–algebra and repeated use of the equations of motion allow for a drastic simplification of the final result. We spell out the calculation in full detail.

Computing the color factors we obtain only double trace structures. After performing D–algebra we are led to

$$\begin{aligned}\int d^4\theta \quad & \Big\{ \left(\frac{1}{2K_1^2} + \frac{1}{2K_2^2} \right) \mathrm{Tr}\left(D^\alpha A^i(p_1) \bar{D}^\beta \bar{A}_i(p_2) \right) \mathrm{Tr}\left(D^\gamma A^j(p_4) \bar{D}^\delta \bar{A}_j(p_3) \right) \\ & - \frac{1}{K_1 K_2} \mathrm{Tr}\left(D^\alpha A^i(p_1) \bar{D}^\delta \bar{A}_j(p_3) \right) \mathrm{Tr}\left(D^\gamma A^j(p_4) \bar{D}^\beta \bar{A}_i(p_2) \right) \Big\} \\ & \times \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{k_{\alpha\gamma} k_{\beta\delta}}{k^2 (k+p_1)^2 (k+p_1+p_4)^2 (k-p_2)^2} \end{aligned} \tag{4.5.8}$$

We first integrate by parts the $\bar{D}^\beta$ derivative.
Using the equations of motions (4.3.2,4.3.3) we obtain

$$\begin{aligned}\int d^4\theta \quad & \Big\{ \left(\frac{1}{2K_1^2} + \frac{1}{2K_2^2} \right) \mathrm{Tr}\left(A^i(p_1) \bar{A}_i(p_2) \right) \mathrm{Tr}\left(D^\gamma A^j(p_4) \bar{D}^\delta \bar{A}_j(p_3) \right) \\ & - \frac{1}{K_1 K_2} \mathrm{Tr}\left(A^i(p_1) \bar{D}^\delta \bar{A}_j(p_3) \right) \mathrm{Tr}\left(D^\gamma A^j(p_4) \bar{A}_i(p_2) \right) \Big\} \\ & \times \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{p_1^{\alpha\beta} k_{\alpha\gamma} k_{\beta\delta}}{k^2 (k+p_1)^2 (k+p_1+p_4)^2 (k-p_2)^2} \\ & - \Big\{ \left(\frac{1}{2K_1^2} + \frac{1}{2K_2^2} \right) \mathrm{Tr}\left(D^\alpha A^i(p_1) \bar{A}_i(p_2) \right) \mathrm{Tr}\left(A^j(p_4) \bar{D}^\delta \bar{A}_j(p_3) \right) \\ & + \frac{1}{K_1 K_2} \mathrm{Tr}\left(D^\alpha A^i(p_1) \bar{D}^\delta \bar{A}_j(p_3) \right) \mathrm{Tr}\left(A^j(p_4) \bar{A}_i(p_2) \right) \Big\} \\ & \times \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{p_4^{\beta\gamma} k_{\alpha\gamma} k_{\beta\delta}}{k^2 (k+p_1)^2 (k+p_1+p_4)^2 (k-p_2)^2} \end{aligned} \tag{4.5.9}$$

We concentrate on the first integral. The numerator can be rewritten as

$$p_1^{\alpha\beta} k_{\alpha\gamma} k_{\beta\delta} = p_1^{\alpha\beta} \left[k_{\delta\gamma} k_{\beta\alpha} - k^2 C_{\delta\alpha} C_{\beta\gamma} \right] = (k+p_1)^2 k_{\gamma\delta} - k^2 (k+p_1)_{\gamma\delta} \tag{4.5.10}$$

Now, simplifying the squares at numerator against the ones at denominators we are left with a linear combination of scalar and vector–like triangle integrals

$$\begin{aligned}\int \frac{d^3k}{(2\pi)^3} & \left\{ \frac{k_{\alpha\beta}}{k^2 (k+p_1+p_4)^2 (k+p_1+p_4+p_3)^2} + \right. \\ & \left. \frac{-k_{\alpha\beta}}{k^2 (k+p_4)^2 (k+p_4+p_3)^2} + \frac{-p^2\, k_{\alpha\beta}}{k^2 (k+p_1)^2 (k+p_1+p_4)^2 (k-p_2)^2} \right\} \end{aligned} \tag{4.5.11}$$

Exploiting the fact that in dimensional regularization the scalar triangle is zero, while the vector–like one is proportional to a bubble integral (see Appendix B), the first term in

(4.5.9) reduces to

$$\int d^4\theta \quad \left\{ \left(\frac{1}{2K_1^2} + \frac{1}{2K_2^2} \right) \mathrm{Tr}\,\left(A^i(p_1)\bar{A}_i(p_2)\right)\,\mathrm{Tr}\,\left(D^\gamma A^j(p_4)\bar{D}^\delta \bar{A}_j(p_3)\right) \right. \tag{4.5.12}$$
$$\left. -\frac{1}{K_1K_2}\,\mathrm{Tr}\,\left(A^i(p_1)\bar{D}^\delta \bar{A}_j(p_3)\right)\,\mathrm{Tr}\,\left(D^\gamma A^j(p_4)\bar{A}_i(p_2)\right) \right\} \times \frac{(p_2)_{\gamma\delta}}{(p_1+p_4)^2}\,\mathcal{B}(p_1+p_4)$$

where equations of motion and momentum conservation have been used. Now, integrating by parts the $\bar{D}^\delta$ derivative and using on–shell conditions, it can be further simplified to

$$\int d^4\theta \quad \left\{ \left(\frac{1}{2K_1^2} + \frac{1}{2K_2^2} \right) \mathrm{Tr}\,\left(A^i(p_1)\bar{A}_i(p_2)\right)\,\mathrm{Tr}\,\left(A^j(p_4)\bar{A}_j(p_3)\right) \right. \tag{4.5.13}$$
$$\left. +\frac{1}{K_1K_2}\,\mathrm{Tr}\,\left(A^i(p_1)\bar{A}_j(p_3)\right)\,\mathrm{Tr}\,\left(A^j(p_4)\bar{A}_i(p_2)\right) \right\} \times \frac{(p_2+p_4)^2}{(p_1+p_4)^2}\,\mathcal{B}(p_1+p_4)$$

We can apply the same tricks to the second integral in eq. (4.5.9). After a bit of algebra, we obtain a similar expression which, summed to the rest, leads to the final expression for the box diagram $2(d)$ in terms of a linear combination of bubbles

$$2(d) = \int d^4\theta \left\{ \left(\frac{1}{2K_1^2} + \frac{1}{2K_2^2} \right) \mathrm{Tr}\,\left(A^i(p_1)\bar{A}_i(p_2)\right)\,\mathrm{Tr}\,\left(A^j(p_4)\bar{A}_j(p_3)\right) \right. \tag{4.5.14}$$
$$\left. +\frac{1}{K_1K_2}\,\mathrm{Tr}\,\left(A^i(p_1)\bar{A}_j(p_3)\right)\,\mathrm{Tr}\,\left(A^j(p_4)\bar{A}_i(p_2)\right) \right\} \times \left[\mathcal{B}(p_1+p_2) - \mathcal{B}(p_1+p_4)\right]$$

<u>Diagram 4.2(e)</u> : The result for this diagram reads

$$2(e) = -\int d^4\theta \quad \left\{ \frac{M}{2K_1^2}\,\mathrm{Tr}\,\left(A^i(p_1)\bar{A}_i(p_2)D^\alpha A^j(p_4)\bar{D}^\beta \bar{A}_j(p_3)\right) \right. \tag{4.5.15}$$
$$-\frac{N}{2K_2^2}\,\mathrm{Tr}\,\left(A^i(p_1)\bar{D}^\beta \bar{A}_j(p_3)D^\alpha A^j(p_4)\bar{A}_i(p_2)\right)$$
$$\left. -\frac{1}{K_1K_2}\,\mathrm{Tr}\,\left(A^i(p_1)\bar{D}^\beta \bar{A}_j(p_3)\right)\,\mathrm{Tr}\,\left(\bar{A}_i(p_2)D^\alpha A^j(p_4)\right) \right\}$$
$$\times\,(p_2)_\alpha^{\ \gamma} \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}}\,\frac{k_\gamma^{\ \delta}\,(k+p_1+p_3)_{\delta\beta}}{k^2(k+p_1)^2(k+p_1+p_3)^2(k-p_2)^2}$$

Elaborating its numerator, the integral can be rewritten as

$$\int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}}\,\frac{k^2 C_{\beta\gamma} + k_\gamma^{\ \delta}(p_1+p_3)_{\delta\beta}}{k^2(k+p_1)^2(k+p_1+p_3)^2(k-p_2)^2} = C_{\beta\gamma}\,\mathcal{T}(p_3,p_4) + (p_1+p_3)_{\delta\beta}\,\mathcal{Q}_{\mathcal{V}_\gamma}^{\ \ \delta} \tag{4.5.16}$$

As proved in Appendix B, the triangle and vector–like box integrals are $\mathcal{O}(\epsilon)$ in dimensional regularization (see eqs. (A.5.11, A.5.27)). Therefore, this diagram can be discarded when $\epsilon \to 0$.

Diagram 4.2(f) : We now consider 1P–reducible diagrams. For graph 4.2(f) using the one–loop correction to the gauge propagator given in eq. (2.2.15) we obtain

$$\begin{aligned}\int d^4\theta \quad & \Bigg\{ \frac{1}{K_1^2}\left(N-\frac{M}{4}\right) \mathrm{Tr}\,\left(D^\alpha A^i(p_1)\bar{A}_i(p_2)A^j(p_4)\bar{D}^\beta\bar{A}_j(p_3)\right) \\ & + \frac{1}{K_2^2}\left(M-\frac{N}{4}\right) \mathrm{Tr}\,\left(\bar{A}_i(p_2)D^\alpha A^i(p_1)\bar{D}^\beta\bar{A}_j(p_3)A^j(p_4)\right) \\ & + \left(\frac{1}{4\,K_1^2}+\frac{1}{4\,K_2^2}+\frac{2}{K_1K_2}\right) \mathrm{Tr}\,\left(D^\alpha A^i(p_1)\bar{A}_i(p_2)\right) \mathrm{Tr}\,\left(A^j(p_4)\bar{D}^\beta\bar{A}_j(p_3)\right) \Bigg\} \\ & \times \frac{(p_4)_{\alpha\beta}}{(p_1+p_2)^2}\mathcal{B}(p_1+p_2)\end{aligned} \tag{4.5.17}$$

We can integrate by parts the D^α derivative. Exploiting the equations of motion, the only non–vanishing term is the one where the derivative hits $\bar{D}^\beta\bar{A}_j(p_3)$, giving a factor $-p_3^{\alpha\beta}(p_4)_{\alpha\beta} = -(p_3+p_4)^2$. By momentum conservation, this cancels against the denominator in (4.5.17) and we finally obtain

$$\begin{aligned}2(f) \;=\; & -\int d^4\theta \Bigg\{ \frac{1}{K_1^2}\left(N-\frac{M}{4}\right) \mathrm{Tr}\,\left(A^i(p_1)\bar{A}_i(p_2)A^j(p_4)\bar{A}_j(p_3)\right) \\ & + \frac{1}{K_2^2}\left(M-\frac{N}{4}\right) \mathrm{Tr}\,\left(\bar{A}_i(p_2)A^i(p_1)\bar{A}_j(p_3)A^j(p_4)\right) \\ & + \left(\frac{1}{4\,K_1^2}+\frac{1}{4\,K_2^2}+\frac{2}{K_1K_2}\right) \mathrm{Tr}\,\left(A^i(p_1)\bar{A}_i(p_2)\right) \mathrm{Tr}\,\left(A^j(p_4)\bar{A}_j(p_3)\right) \Bigg\} \times \mathcal{B}(p_1+p_2)\end{aligned} \tag{4.5.18}$$

Diagram 4.2(g) : Finally, we consider the reducible triangle diagram. Performing on–shell D–algebra we produce terms with four spinorial derivatives acting on the external fields. After integrating by parts one of these derivatives, using on–shell conditions and momentum conservation and relabeling internal and external momenta, we can write the result as

$$\begin{aligned}-\int d^4\theta \quad & \Bigg\{ \frac{N}{K_1K_2} \mathrm{Tr}\,\left(A^i(p_1)\bar{D}^\beta\bar{A}_i(p_2)D^\alpha A^j(p_4)\bar{A}_j(p_3)\right) \\ & - \frac{M}{K_1K_2} \mathrm{Tr}\,\left(A^i(p_1)\bar{A}_j(p_3)D^\alpha A^j(p_4)\bar{D}^\beta\bar{A}_i(p_2)\right) \\ & + \left(\frac{1}{K_1^2}+\frac{1}{k_2^2}\right) \mathrm{Tr}\,\left(A^i(p_1)\bar{D}^\beta\bar{A}_i(p_2)\right) \mathrm{Tr}\,\left(D^\alpha A^j(p_4)\bar{A}_j(p_3)\right) \Bigg\} \\ & \times \frac{(p_1+p_2)_{\gamma\alpha}}{(p_1+p_2)^2} \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{(k+p_1)^{\gamma\delta}\,k_{\delta\beta}}{k^2(k+p_1)^2(k-p_2)^2}\end{aligned} \tag{4.5.19}$$

We can elaborate the numerator of the integrand to obtain

$$\begin{aligned}\int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \, \frac{k^2\delta_\beta^\gamma + p_1^{\gamma\delta}\, k_{\delta\beta}}{k^2(k+p_1)^2(k-p_2)^2} &= \delta_\beta^\gamma \, \mathcal{B}(p_1+p_2) + p_1^{\gamma\delta}\, \mathcal{T}_{\mathcal{V}\delta\beta} \\ &= \mathcal{B}(p_1+p_2) \left[\delta_\beta^\gamma - \frac{p_1^{\gamma\delta}(p_2)_{\delta\beta}}{(p_1+p_2)^2} \right] \qquad (4.5.20)\end{aligned}$$

where eq. (A.5.12) has been used together with on–shell conditions. Inserting back into eq. (4.5.19), observing that on–shell $(p_1+p_2)_{\gamma\alpha} p_1^{\gamma\delta}(p_2)_{\delta\beta} = (p_1+p_2)^2(p_2)_{\alpha\beta}$ where $(p_2)_{\alpha\beta}$ vanishes when contracted with $\bar{D}^\beta \bar{A}_i(p_2)$, we obtain

$$\begin{aligned}- \int d^4\theta \quad & \left\{ \frac{N}{K_1K_2} \, \mathrm{Tr}\, \left(A^i(p_1) \bar{D}^\beta \bar{A}_i(p_2) D^\alpha A^j(p_4) \bar{A}_j(p_3) \right) \right. \\ & - \frac{M}{K_1K_2} \, \mathrm{Tr}\, \left(A^i(p_1) \bar{A}_j(p_3) D^\alpha A^j(p_4) \bar{D}^\beta \bar{A}_i(p_2) \right) \\ & \left. + \left(\frac{1}{K_1^2} + \frac{1}{K_2^2} \right) \, \mathrm{Tr}\, \left(A^i(p_1) \bar{D}^\beta \bar{A}_i(p_2) \right) \, \mathrm{Tr}\, \left(D^\alpha A^j(p_4) \bar{A}_j(p_3) \right) \right\} \\ & \times \frac{(p_1)_{\alpha\beta}}{(p_1+p_2)^2} \, \mathcal{B}(p_1+p_2) \qquad (4.5.21)\end{aligned}$$

On–shell conditions are once again helpful for reducing the structure of spinorial derivatives acting on the external fields. In fact, integrating by parts the D^α derivative we produce a term $p_2^{\alpha\beta}$ that, contracted with $(p_1)_{\alpha\beta}$, cancels $(p_1+p_2)^2$ at the denominator. We finally obtain

$$\begin{aligned}2(g) = - \int d^4\theta \quad & \left\{ \frac{N}{K_1K_2} \, \mathrm{Tr}\, \left(A^i(p_1) \bar{A}_i(p_2) A^j(p_4) \bar{A}_j(p_3) \right) \right. \qquad (4.5.22) \\ & + \frac{M}{K_1K_2} \, \mathrm{Tr}\, \left(A^i(p_1) \bar{A}_j(p_3) A^j(p_4) \bar{A}_i(p_2) \right) \\ & \left. + \left(\frac{1}{K_1^2} + \frac{1}{K_2^2} \right) \, \mathrm{Tr}\, \left(A^i(p_1) \bar{A}_i(p_2) \right) \, \mathrm{Tr}\, \left(A^j(p_4) \bar{A}_j(p_3) \right) \right\} \times \mathcal{B}(p_1+p_2)\end{aligned}$$

We are now ready to sum all the results and obtain the one–loop effective action needed for the evaluation of $(A^i\, \bar{A}_j\, A^k\, \bar{A}_l)$ amplitudes.

Having reduced all the expressions to strings of external superfields with no derivatives acting on them, multiplied by bubble integrals, we can group them according to their trace structure. We have single trace contributions from diagrams $2(b)$, (f), (g) and double trace contributions from $2(a), (b), (d), (f), (g)$. Collecting them all, we obtain

$$\begin{aligned}\int d^4\theta \quad \Big\{ &-\mathrm{Tr}\,(A^i(p_1)\bar{A}_i(p_2)A^j(p_4)\bar{A}_j(p_3))\;\hat{\lambda}\left(\frac{1}{K_1}+\frac{1}{K_2}\right)\times\mathcal{B}(p_1+p_2)\\ &-\mathrm{Tr}\,(A^i(p_1)\bar{A}_j(p_3)A^j(p_4)\bar{A}_i(p_2))\;\lambda\left(\frac{1}{K_1}+\frac{1}{K_2}\right)\times\mathcal{B}(p_1+p_2)\\ &+\mathrm{Tr}\,(A^i(p_1)\bar{A}_i(p_2))\,\mathrm{Tr}\,(A^j(p_4)\bar{A}_j(p_3))\times\\ &\left[\frac{1}{2}\left(\frac{|h_1|^2+|h_2|^2}{16\pi^2}-\frac{1}{K_1^2}-\frac{1}{K_2^2}\right)\times\mathcal{B}(p_1+p_4)-\left(\frac{1}{K_1}+\frac{1}{K_2}\right)^2\times\mathcal{B}(p_1+p_2)\right]\\ &+\mathrm{Tr}\,(A^i(p_1)\bar{A}_j(p_3))\,\mathrm{Tr}\,(A^j(p_4)\bar{A}_i(p_2))\left(\frac{h_1\bar{h}_2+h_2\bar{h}_1}{32\pi^2}-\frac{1}{K_1K_2}\right)\times\mathcal{B}(p_1+p_4)\\ &-\frac{1}{32\pi^2}\,\mathrm{Tr}\,(A^i(p_1)\bar{A}_i(p_2))\,\mathrm{Tr}\,(A^i(p_4)\bar{A}_i(p_3))\;|h_1+h_2|^2\times\mathcal{B}(p_1+p_4)\Big\}\end{aligned} \tag{4.5.23}$$

First of all, we observe that for M, N finite the quartic effective action, and consequently the amplitude, vanishes when $K_2 = -K_1$ and $h_2 = -h_1 = 4\pi/K$. This is exactly the $\mathcal{N} = 6$ superconformal fixed point corresponding to the ABJ theory. This result was expected and provides a non–trivial check of our calculation. In fact, in the ABJ model the non–chiral amplitude is related to the chiral one by $SU(4)$ symmetry and we have already checked that the chiral amplitude is one–loop vanishing.

Taking M, N large and assuming the h_i couplings of order of $1/K_i$, only single trace contributions survive in (4.5.23). In this case, the amplitude will vanish for $K_2 = -K_1$, independently of the values of the chiral couplings. In particular, we have a vanishing non–chiral amplitude for the whole set of $\mathcal{N} = 2$ superCFT's given by the condition (see eq. (2.7.2) when $M = N$ and $K_1 = K_2$)

$$|h_1|^2 + |h_2|^2 = \frac{32\pi^2}{K^2} \tag{4.5.24}$$

We observe that the amplitude never vanishes for theories with $K_2 \neq -K_1$, in particular for superCFT's which correspond to turning on a Romans mass in the dual supergravity background. The same pattern occurs for the $(B_i\,\bar{B}^j\,B_k\,\bar{B}^l)$ amplitudes. In fact, repeating the previous calculation we obtain exactly the same expression (4.5.23) as a consequence of the $\mathbb{Z}_2$ symmetry of the action under exchange $V \leftrightarrow \hat{V}$, $A \leftrightarrow B$, $M \leftrightarrow N$, $K_1 \leftrightarrow K_2$ and $h_1 \leftrightarrow h_2$.

Finally, we need consider mixed amplitudes of the type $(A^i\,\bar{A}_j\,\bar{B}^k\,B_l)$. These give rise to rather different trace structures. The contributing diagrams are still the ones drawn in Fig. (4.2) with obvious substitution of one $(A, \bar{A})$ couple with a $(B, \bar{B})$ couple. Applying the same procedure as before, we get

$$\begin{aligned}
\int d^4\theta \quad \Big\{ & 2\,\mathrm{Tr}\,(A^i(p_1)\bar{A}_i(p_2)\bar{B}^j(p_3)B_j(p_4))\;\hat{\lambda}\left(\frac{1}{K_1}+\frac{1}{K_2}\right)\times\mathcal{B}(p_1+p_2) \\
& + 2\,\mathrm{Tr}\,(\bar{A}_i(p_2)A^i(p_1)B_j(p_4)\bar{B}^j(p_3))\;\lambda\left(\frac{1}{K_1}+\frac{1}{K_2}\right)\times\mathcal{B}(p_1+p_2) \\
& + \mathrm{Tr}\,(A^i(p_1)\bar{A}_i(p_2)\bar{B}^j(p_3)B_j(p_4))_{i\neq j}\;M\left(\frac{|h_1|^2}{16\pi^2}-\frac{1}{K_1^2}\right)\times\mathcal{B}(p_1+p_4) \\
& + \mathrm{Tr}\,(A^i(p_1)\bar{A}_i(p_2)\bar{B}^i(p_3)B_i(p_4))\;M\left(\frac{|h_2|^2}{16\pi^2}-\frac{1}{K_1^2}\right)\times\mathcal{B}(p_1+p_4) \\
& + \mathrm{Tr}\,(\bar{A}_i(p_2)A^i(p_1)B_j(p_4)\bar{B}^j(p_3))_{i\neq j}\;N\left(\frac{|h_2|^2}{16\pi^2}-\frac{1}{K_2^2}\right)\times\mathcal{B}(p_1+p_4) \\
& + \mathrm{Tr}\,(\bar{A}_i(p_2)A^i(p_1)B_i(p_4)\bar{B}^i(p_3))\;N\left(\frac{|h_1|^2}{16\pi^2}-\frac{1}{K_2^2}\right)\times\mathcal{B}(p_1+p_4) \\
& + 2\,\mathrm{Tr}\,(A^i(p_1)\bar{A}_i(p_2))\,\mathrm{Tr}\,(B_j(p_4)\bar{B}^j(p_3))\left(\frac{1}{K_1}+\frac{1}{K_2}\right)^2\times\mathcal{B}(p_1+p_2) \\
& + 2\,\mathrm{Tr}\,(A^i(p_1)B_j(p_4))\,\mathrm{Tr}\,(\bar{A}_i(p_2)\bar{B}^j(p_3))\left(\frac{h_1\bar{h}_2+h_2\bar{h}_1}{32\pi^2}-\frac{1}{K_1K_2}\right)\times\mathcal{B}(p_1+p_4)\Big\}
\end{aligned} \tag{4.5.25}$$

Some comments are in order. For M, N finite, these amplitudes vanish only at the ABJ(M) fixed point, as expected. However, in contrast with the previous case, in the large M, N limit a non–trivial dependence on the chiral couplings survives, which restricts the set of superCFT's with vanishing one–loop amplitudes to be only the ABJ(M) models.

In the ABJM case, this result is consistent with what has been found in Ref. [87] in components and massive regularization, and in [81] by means of the generalized unitarity cuts method.

4.5.2 The one–loop light–like Wilson loop.

The perturbative evaluation of the bosonic Wilson loop is simply obtained by expanding the exponential to a given order in the gauge connections and then Wick contracting the latter in the expectation value. This produces gauge propagators and integrations over the affine variables α_i (4.4.2), parameterizing the edges of the polygonal contour. Up to one loop there is no need to specify the exact CSM theory we are considering, since the contributions only come from the CS sector as we will show briefly. Moreover, pieces from the two gauge groups are separated, hence we can perform the calculation for a generic CS theory and eventually massage the result to reproduce the ABJ(M) cases.

The whole computation is intrinsically not supersymmetric and will be performed in components, namely employing the propagator and the interaction vertices for the gauge vectors A and $\hat{A}$, which can be read off from the component action in (1.1.5). We now review the essential ingredients for the calculation.

The propagator for the vector component of the gauge superfield is given by taking

$$A_{\alpha\beta} = \frac{1}{\sqrt{2}}\,\bar{D}_\alpha\,D_\beta\,V\,\Big| \tag{4.5.26}$$

Using the algebra in Appendix A.1, we can turn from bispinor to vector notation

$$A_\mu = \frac{1}{\sqrt{2}}\,\gamma_\mu^{\alpha\beta}\,A_{\alpha\beta} \tag{4.5.27}$$

Then the CS sector of the action in components reads

$$\mathcal{S}_{CS} = -i\,\frac{K}{4\pi}\int d^3x\,\mathrm{Tr}\,\left(A_\mu\,\partial_\nu\,A_\rho + \frac{2}{3}\,i\,A_\mu\,A_\nu\,A_\rho\right)\,\epsilon^{\mu\nu\rho} \tag{4.5.28}$$

The latter is invariant under the bosonic gauge transformation

$$A_\mu \;\to\; g(x)\;(A_\mu - i\,\partial_\mu)\;g(x)^{-1} \tag{4.5.29}$$

which can be derived from gauge invariance in superspace

$$e^V \;\to\; e^{i\bar{\Lambda}}\,e^V\,e^{-i\Lambda} \tag{4.5.30}$$

by applying $\bar{D}_\alpha\,D_\beta$.

In order to compare to existing literature and for the sake of clarity we can also transform the action from Euclidean to $(+,-,-)$ signature, by Wick rotating $x^0 \to -i\,t$ and then changing sign to the invariants

$$\mathcal{S}_{CS}^{(-,+,+)} = -\frac{K}{4\pi}\int d^3x\,\mathrm{Tr}\,\left(A_\mu\,\partial_\nu\,A_\rho - \frac{2}{3}\,i\,A_\mu\,A_\nu\,A_\rho\right)\,\epsilon^{\mu\nu\rho} \tag{4.5.31}$$

where the minus sign arises from $\epsilon^{0\mu\nu}\,A_0\ldots$,

$$\mathcal{S}_{CS}^{(+,-,-)} = \frac{K}{4\pi}\int d^3x\,\mathrm{Tr}\,\left(A_\mu\,\partial_\nu\,A_\rho - \frac{2}{3}\,i\,A_\mu\,A_\nu\,A_\rho\right)\,\epsilon^{\mu\nu\rho} \tag{4.5.32}$$

which is the same as in [80]. We will use these conventions in this Section.

For CS theories the following convention relating the cubic interaction, gauge invariance and the Wilson loop holds: given the action

$$\mathcal{S} = \frac{K}{4\pi}\int d^3x\,\mathrm{Tr}\,\left(A_\mu\,\partial_\nu\,A_\rho - \frac{2}{3}\,s\,A_\mu\,A_\nu\,A_\rho\right)\,\epsilon^{\mu\nu\rho} \tag{4.5.33}$$

invariant under the gauge transformation

$$A_\mu \to g(x) \left(A_\mu - \frac{1}{s} \partial_\mu \right) g(x)^{-1} \tag{4.5.34}$$

the Wilson loop is defined as

$$W = \frac{1}{N} \mathrm{Tr}\, P\, e^{s \int A_\mu \, dz} \tag{4.5.35}$$

Hereafter we choose $s = i$ and Minkowski signature.

After defining notations we come to the first order computation.

The one–loop contribution to the Wilson loop is obtained by expanding the path–ordered exponential at second order in the gauge fields. Concentrating on one of the gauge fields, let us say A_μ, it is given by

$$\langle W(A) \rangle^{1-loop} = \frac{i^2}{N} \sum_{i \geq j} \int d\alpha_i \, d\alpha_j \, \dot{z}_i^\mu \, \dot{z}_j^\nu \, \langle \mathrm{Tr} \left(A_\mu(z_i) \, A_\nu(z_j) \right) \rangle \tag{4.5.36}$$

where $\alpha_i, \alpha_j, i \neq j$ run independently between 0 and 1, whereas for $i = j$ the integration domain is meant to be $0 \leq \alpha_i \leq 1$ and $0 \leq \alpha_j \leq \alpha_i$. Dots indicate derivatives with respect to the affine parameters.

Taking the expectation value yields gauge propagators. Plugging their explicit expression, which in our gauge reads

$$\langle (A_\mu)^a_b(z_i) \, (A_\nu)^c_d(z_j) \rangle = -\frac{1}{8 \pi K} \, \epsilon_{\mu\nu\rho} \, \frac{(z_i - z_j)^\rho}{|z_i - z_j|^3} \, \delta^a_d \, \delta^c_b \tag{4.5.37}$$

into (4.5.36) determines the one–loop contributions.

According to the relative positions of the x_i and x_j points of the contour where the gauge vectors get radiated, we obtain three different configurations, which are depicted in Fig. 4.3 (taken from [80]).

Thanks to the antisymmetry of the ϵ tensor in (4.5.37), we may discard diagrams (b) and (c), where the gauge vector is emitted and reabsorbed within the same edge, or between two adjacent, respectively.

Only the contribution from diagram (c), where the gauge vector connects the (x_i, x_{i+1}) and (x_j, x_{j+1}) edges, survives and is proportional to $\epsilon_{\mu\nu\rho} \, x^\mu_{i,i+1} \, x^\nu_{i+1,j} \, x^\rho_{j,j+1} \, \mathcal{K}(i,j)$, where

$$\mathcal{K}(i,j) = \frac{\pi^4}{2} \int_0^1 d\alpha_i d\alpha_j \; \times \tag{4.5.38}$$

$$\frac{1}{\left[(1-\alpha_i)\,(1-\alpha_j)\, x^2_{i,j} + \alpha_i \, \alpha_j \, x^2_{i+1,j+1} + \alpha_j \,(1-\alpha_i)\, x^2_{i,j+1} + \alpha_i \,(1-\alpha_j)\, x^2_{i+1,j} \right]^{\frac{3}{2}}}$$

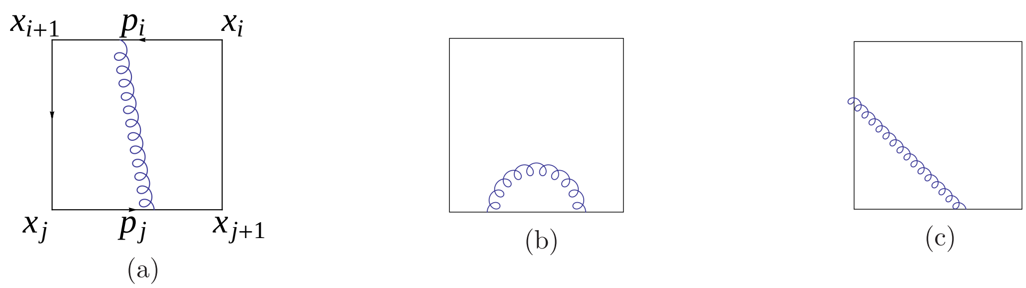


Figure 4.3: One-loop contributions to the Wilson loop. Diagrams (b) and (c) are actually vanishing.

Now, including all the coefficients and summing the analogous contribution coming from $\hat{A}$, the one–loop Wilson loop can be written as

$$\langle W(A,\hat{A})\rangle^{1-loop} = -\frac{1}{4\pi^5}\left(\frac{M}{K_1}+\frac{N}{K_2}\right)\sum_{i=1}^{n-2}\sum_{j=i+2}^{n-\delta_{i,1}} \epsilon_{\mu\nu\rho}\, x^{\mu}_{i,i+1}\, x^{\nu}_{i+1,j}\, x^{\rho}_{j,j+1}\, \mathcal{K}(i,j) \quad (4.5.39)$$

where the sum runs over all possible ways to connect two non–adjacent lines. In a polygon with n edges these are $\frac{n(n-3)}{2}$ different contributions. We note that at this order matter fields do not enter the calculation, and what we need is just the CS free action. Therefore, this result is valid also for pure Chern–Simons theories, and we have left generic ranks and CS levels to stress this generality.

One–loop Wilson loop in the ABJM model. We can derive the one–loop result for the Wilson loop in ABJM without even performing a single calculation. This is due to the definition of the Wilson loop (4.4.1). According to this the final result is given by the sum over the two separate pure CS Wilson loops for the two gauge groups. Since their coupling constants (the CS levels) are opposite, the sum exactly cancels, leaving a trivially vanishing outcome. This cancelation will occur at any odd perturbative order due to this $\mathbb{Z}_2$ symmetry, but is no longer true at even loops.

One–loop Wilson loop for generic CSM theories. The final result for the sum (4.5.39) depends non–trivially on the explicit form of the integral $\mathcal{K}_{ij}$ (4.5.38), which is involved (although it is straightforward to get a close answer for the $n = 4$ case, in which the Wilson loop trivially vanish by symmetry arguments). The evaluation of the integral and the sum takes considerable effort and is postponed to the final part of this Chapter.

We next turn to the calculation of correlation functions of half–BPS operators. Their light–like separation limit will turn out to be strikingly similar to the result (4.5.39) for the Wilson loop, hence solving the sum over integrals (4.5.38) involved in it, will produce the final result for both Wilson loops and correlators, at one loop.

4.6 Correlation functions of half–BPS operators.

As recalled in the introductory Section 3.10, novel dualities were highlighted last year, relating scattering amplitudes and Wilson loops to correlation functions of protected operators in $\mathcal{N}=4$ SYM. In particular, the ratio between the correlation function of n such operators at L loops, divided by its tree level counterpart is considered. When the limit is taken where operators are inserted at light–like separated points x_i $(i=1\dots n)$, this reproduces the L loop computation of a light–like Wilson loop whose contour is the polygon defined by the set $\{x_i\}$, in the adjoint representation

$$\lim_{x^2_{i,i+1}\to 0} \frac{\mathcal{C}^L_n}{\mathcal{C}^0_n}(x_1,\dots,x_n) = \langle\, W^{adj}_{C_n(\{x_i\})}\,\rangle^{(L)} \tag{4.6.1}$$

Since in the planar limit the Wilson loop in the adjoint representation coincides with the square of that evaluated in the fundamental one, the relation above can be recast into

$$\lim_{x^2_{i,i+1}\to 0} \frac{\mathcal{C}^L_n}{\mathcal{C}^0_n}(x_1,\dots,x_n) = \left[\langle\, W^{fund}_{C_n(\{x_i\})}\,\rangle^{(L)}\right]^2 \tag{4.6.2}$$

On the other hand (either independently or owing to the well–known amplitude/WL duality), correlation functions are also connected to scattering amplitudes. More precisely the conjectured relation reads

$$\lim_{x^2_{i,i+1}\to 0} \log \frac{\mathcal{C}^L_n}{\mathcal{C}^0_n}(x_1,\dots,x_n) = \log \mathcal{M}^2_n\,(p_i = x_{i+1,i}) \tag{4.6.3}$$

Some examples were provided in favor of this triality for $\mathcal{N}=4$ SYM.

It is interesting to investigate whether the amplitudes/WL/correlators dualities and the existence of underlying hidden symmetries extend to classes of theories in different dimensions for which a string dual description is known.
Here we will give preliminary evidence that this is the case.

In particular we will of course consider the general set of three dimensional $\mathcal{N}=2$, $U(M)_{K_1}\times U(N)_{K_2}$ Chern–Simons matter theories with arbitrary (K_1,K_2) CS levels and generic superpotential, which is the main subject of this thesis.

We will compute the first order corrections to correlation functions of gauge invariant half–BPS scalar operators at weak coupling in perturbation theory.
Then the light–like limit of such a quantity, divided by the tree level contribution, is taken. Finally we compare this to Wilson loops and amplitudes at the same perturbative order to check whether and how the triality holds.

The main outcome of our analysis is that the light–like limit of one–loop correlation functions, divided by the corresponding tree level expression, coincides with the one–loop light–like n–polygonal Wilson loop, once the Feynman combining parameters of the correlator integral are identified with the affine parameters which parameterize the light–like edges of the Wilson loop polygon.

While in the ABJM case, and whenever $K_2 = -K_1$ and $M = N$, the identification gets trivialized by the fact that both the correlator and the Wilson loop are proportional to a vanishing color factor, in the more general cases the color factor in front is not zero and a non–trivial identification emerges.

In particular we find that both the light–cone limit of the correlator and the light–like Wilson loop depend at one loop on a five dimensional two–mass–easy box integral. We manage to compute the five dimensional box integral analytically and prove that a very non–trivial cancelation between different permutations occurs yielding a vanishing result. Nevertheless we stress that our identification between correlators and Wilson loops is valid independently of the fact that they both eventually vanish, since it is already verified at the level of their expressions in terms of integrals. Therefore, our result is a first non–trivial hint at a correlator/WL duality at work in three dimensions.

In formulae, our final statement is then

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\mathcal{C}_n^{\,1-loop}}{\mathcal{C}_n^{\,tree}} = \langle W_n \rangle^{1-loop} \; = \; 0 \qquad , \qquad \text{for any } n \tag{4.6.4}$$

This identity holds for any $\mathcal{N} = 2$ CSM theory, for any value of the CS levels (allowed by the perturbative regime) and for N, M finite (no planar limit is required). Hence it is valid independently of the degree of supersymmetry and of conformality. Actually this is a little bit more than expected: this is likely to be a consequence of the low perturbative order to which we are working. Indeed the superpotential, which distinguishes theories in supersymmetry, does not influence the one loop calculation at all. Moreover at one loop order all $\mathcal{N} = 2$ CSM theories are still superconformal, since as we proved in Section 2.2.2, the beta–functions are trivially zero [29, 24] and get non–vanishing starting from two loops. We therefore speculate that theories with different number of supersymmetries and with or without superconformal invariance will undergo a different destiny starting from two–loops. In particular the conjectured WL/correlator duality is expected to continue to hold for conformal theories, but is not granted for those which are not scale invariant.

As a byproduct, we can prove a quite general result on Wilson loops in Chern–Simons theories. Indeed, having shown that the n–polygonal Wilson loop is zero at first order for any value of the CS levels and independently of the chiral couplings, implies that our reasoning can also apply to pure Chern–Simons theories, just setting matter fields and one of the two gauge fields to zero. Therefore, our findings provide the analytical proof of the conjecture made in [80] according to which one–loop light–like Wilson loops should vanish in pure Chern–Simons theories.

Let us go back and consider the other side of the duality, namely the correlator/MHV amplitude. The question whether this correspondence arises in three dimensional theories can be answered quite partially at one–loop order for the particular case of four point amplitudes, due to our poor knowledge on higher point results beyond tree level. Indeed we demonstrated in Section 4.2 that one–loop four point amplitudes might have quite different behavior depending on the degree of symmetry of the corresponding theory. In particular we confirmed the result that for $\mathcal{N} > 4$, four–point scattering amplitudes vanish at one loop [87], including the most interesting ABJ case. This suggests that the duality might work for this theory. One should ask how far can this reasoning apply, either considering higher point amplitudes or taking into account more general theories. As concerns the first question we stress that the six point is already trickier since the corresponding ABJ amplitude is not MHV from its Grassmann degree, hence it is not clear which its dual objects should be, possibly supersymmetric extensions of Wilson loops and correlation functions. The same obstruction holds for all higher point amplitudes as well. As for theories with lower supersymmetry the concept of MHV amplitudes may be by itself ill–defined and it is difficult to predict which duality, if any, may occur. Therefore any tentative extension of the latter correlator/amplitude duality should not be straightforward.

In what follows we will go into the details of the computation of correlation functions of half BPS operators in Chern–Simons matter theories. In the first Section we lay the basis of the calculation, then we account for a rather technical treatment of the integrals emerging at one loop, and in Section 4.6.6 we write a closed expression for the light–like limit of a generic n–point correlator. Then we demonstrate that the latter expression actually evaluates to zero and finally make a comparison with Wilson loops and amplitudes at one loop, establishing the aforementioned dualities at one loop order.

4.6.1 The n–point correlation functions

We begin with some generalities on correlators of BPS operators in CS matter theories and their light–like limit.

Here we focus on weight one operators, which are the bilinear $A\,B$ ones, leaving the generalization to operators of higher dimension as an aside for Section 4.6.5. Hereafter we will refer to these operators as $\mathcal{O}$, together with their conjugates $\bar{\mathcal{O}}$

$$\mathcal{O}^i_j\,(x) = \mathrm{Tr}\,(A^i(Z)\,B_j(Z)) \qquad , \qquad \bar{\mathcal{O}}^j_i(x) = \mathrm{Tr}\,(\bar{A}_i(Z)\,\bar{B}^j(Z)) \tag{4.6.5}$$

where $Z = (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ and i, j are flavor indices that we omit in what follows.

Furthermore we want to restrict to the lowest component piece of the correlation functions, belonging to the scalar sector of the theory. Keeping the computation manifestly supersymmetric, this task is accomplished by selecting the surviving terms after setting the fermionic superspace coordinates to 0 in the correlation function

$$\mathcal{C}_n = \left\langle \mathcal{O}(Z_1)\,\bar{\mathcal{O}}(Z_2)\cdots\mathcal{O}(Z_{n-1})\,\bar{\mathcal{O}}(Z_n)\right\rangle\Big|_{\theta_i=\bar{\theta}_i=0} \tag{4.6.6}$$

In terms of superdiagrams, this is operatively performed by moving the spinorial derivatives in such a way that a $D^2\bar{D}^2$ acts on each fermionic delta function connecting external fermionic coordinates $\delta(\theta_i - \theta_{i+1})$. This yields the scalar part of the correlators since $D^2\bar{D}^2\,\delta(\theta_i - \theta_{i+1})\Big|_{\theta=0} = 1$.

In particular, at tree level, the correlation function simply gives the product of free chiral propagators (1.4.17) which, evaluated at $\theta = \bar{\theta} = 0$, are simply $\frac{1}{4\pi}\,\frac{1}{|x_i - x_j|}$ (in $d = 3$, since no divergences are there at one loop).

In computing a generic correlation function one should also take into account all the possibilities of contracting the fields. This entails that the correlator (4.6.6) will be in fact a linear combination of connected and disconnected diagrams.

We want to concentrate on the connected part only. With this constraint on Wick contractions, the correlator reads

$$\mathcal{C}_n^{\,tree} = \frac{MN}{(4\pi)^n} \sum_{\{i_1,\cdots,i_n\}} \frac{1}{x_{i_1,\,i_2}}\,\frac{1}{x_{i_2,\,i_3}} \cdots \frac{1}{x_{i_n,\,i_1}} \tag{4.6.7}$$

where $x_{i,\,j}$ obviously stems for $|x_i - x_j|$, and the permutations are restricted to all the non–cyclic ones which are allowed by chirality and flavor. Moreover a connected correlation function of n bilinear operators $\mathcal{O}$ and $\bar{\mathcal{O}}$ may be constructed only with an even number of points. We will thus concentrate on the case $n = 2m$.

One further constraint comes when pursuing the light–like limit we are eventually interested in. This picks up the most singular piece as $x^2_{i,\,i+1} \to 0$ in the sum over connected contributions (4.6.7). Since the light–like limit is taken between adjacent points where operators are inserted, the cyclic order $\{1, 2, \cdots, n\}$ corresponds to the most divergent term

$$\mathcal{C}_n^{\,tree} \;\to\; \frac{MN}{(4\pi)^n} \prod_{i=1}^{n} \frac{1}{x_{i,\,i+1}} \tag{4.6.8}$$

where i indices are all understood in $\mathbb{Z}_n$, $x_{n+1} = x_1$.

Depicting the n insertion points on a plane, the Wick contractions associated to this contribution draw a planar n–polygon with the operators at the vertices (See Fig. 4.4). Of course, having chosen a different light–like limit would have selected another planar ordering. In view of a correlation function/MHV amplitude duality, this procedure parallels color ordering, yielding a particular sequence of scattered particles.

4.6.2 One–loop corrections.

One loop corrections of any correlation function are trivially vanishing for the ABJM theory. This can be ascertained diagrammatically by noticing that every one-loop correction to any object built from chiral propagators is obtained by joining two of them through

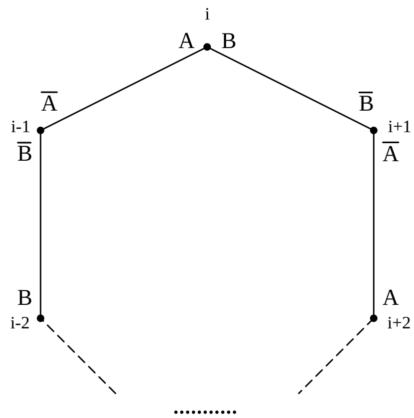


Figure 4.4: The correlation function of dimension–one operators in the light–like limit.

a vector line. Then, due to the quiver structure of ABJM and its opposite couplings for the two gauge groups, every V correction between two legs is exactly canceled by the $\hat{V}$ contribution. This will be still true in any quiver theory where $\sum k_i = 0$, however the above argument fails for generic theories where the CS levels are unrelated, such as many of those analyzed in Chapter 1, which will generally acquire first order corrections in the $\frac{M}{K_1}$, $\frac{N}{K_2}$ couplings, from gluons of the two gauge groups connecting edges of the n–polygon. At this order, chiral interaction vertices from the superpotential do not contribute, so the results are valid for any $\mathcal{N} = 2$ theory. As we have shown in Chapter 1, a plethora of such theories exists which preserves conformal invariance. We therefore expect the correlation functions to vanish (in the light–like limit) for these theories at least, in order for the WL/correlator duality to hold. Actually we will argue that the one loop vanishing occurs non–trivially for all Chern–Simons matter theories with $\mathcal{N} = 2$ supersymmetry.

We now study one loop corrections systematically.

There are basically two possible sources of such contributions: those arising when a gluon is radiated and reabsorbed within the same chiral propagator and those where two propagators exchange a gauge vector. The former are ruled out by D–algebra, as reviewed in section 2.2.2 ¶, therefore one is left with a single kind of diagram.

Since the topology of supergraphs producing first order corrections is pretty much unique, it is useful to analyze it in full generality. The prototype of one–loop interactions is depicted in Fig. 4.5, representing a building block, which we will refer to as $\mathcal{T}_{ij}$, to be inserted in the correlation function polygon. From this we will be able to reconstruct, by suitable permutations, the final answer for a generic correlator. In $\mathcal{T}_{ij}$ the edges $x_{i,\,i+1}$ and $x_{j,\,j+1}$ are connected by a wavy line representing either a V or a $\hat{V}$ propagator.

Actually, there are two different configurations for the one–loop building block, de-

¶It is possible to perform D–algebra in such a way that no enough spinorial derivatives survive inside the loop.

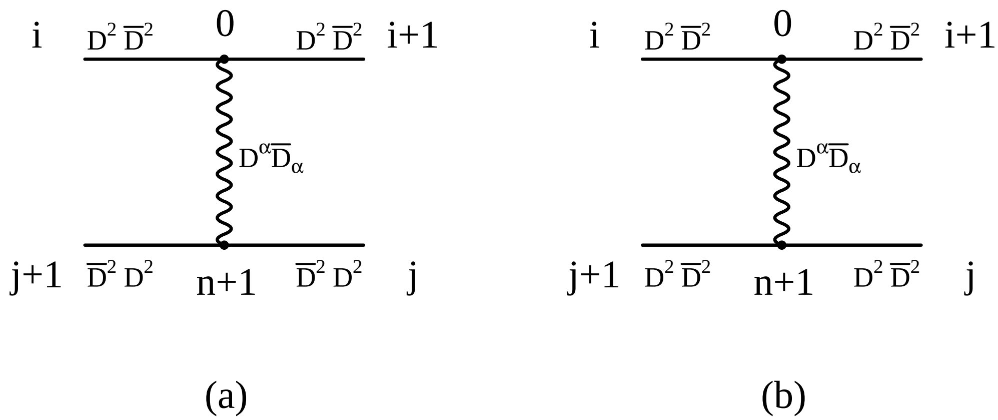


Figure 4.5: Building blocks for one–loop corrections.

pending on the chirality of the external vertices, as shown in figure 4.5. Diagram 4.5(a) displays antichiral fields inserted at x_i and x_j and chiral fields at x_{i+1} and x_{j+1}, whereas in diagram 4.5(b) the converse occurs.

In order to evaluate these graphs, we keep in mind the prescription outlined above to get the scalar component of the correlator, carrying out D–algebra in order to end up with a non–vanishing result when evaluated at $\theta_k = \bar{\theta}_k = 0$, $k = i, i+1, j, j+1$. Starting with the configurations of Fig. 4.5 for the spinorial derivatives, we first free the gauge line from spinorial derivatives. This allows to use the associated δ_{56} to get rid of, say, the θ_6 integration. Then we go on freeing one of the chiral lines from derivatives by integrating by parts at one of the θ_5 vertex. This further allows to perform integration over θ_5 immediately. Among different terms which have got produced, the only non–trivial contribution in the $\theta_k = \bar{\theta}_k = 0$ limit corresponds to a $D^2\bar{D}^2$ operator surviving on the three chiral propagators left in θ space. These derivatives are sufficient to kill the fermionic delta functions setting $\theta_k = \bar{\theta}_k = 0$, leading to a non–vanishing expression. As a result of the D–algebra, the ordinary Feynman diagram we are left with has three space–time derivatives acting on chiral propagators, contracted by the three dimensional Levi–Civita tensor.

Summing the contributions from the V and $\hat{V}$ insertions, the final answer for the two configurations is

$$\begin{aligned}
\mathcal{T}^{(a)}_{ij} &= -\frac{2}{(4\pi)^4}\left(\frac{1}{K_1}+\frac{1}{K_2}\right)\epsilon_{\mu\nu\rho}\,\partial_i^\mu\,\partial_{i+1}^\nu\,\partial_{j+1}^\rho\,\mathcal{I}(i,j)\\
\mathcal{T}^{(b)}_{ij} &= \frac{2}{(4\pi)^4}\left(\frac{1}{K_1}+\frac{1}{K_2}\right)\epsilon_{\mu\nu\rho}\,\partial_i^\mu\,\partial_{i+1}^\nu\,\partial_{j+1}^\rho\,\mathcal{I}(i,j)
\end{aligned} \tag{4.6.9}$$

in terms of the integral in configuration space

$$\mathcal{I}(i,j) = \int \frac{d^3x_0\, d^3x_{n+1}}{x_{0\,i}\, x_{0\,i+1}\, x_{0\,n+1}\, x_{j,\,n+1}\, x_{j+1,\,n+1}} \tag{4.6.10}$$

In the next subsection we will elaborate on this integral. The bottom–line of our computation is that it may be evaluated as a Feynman integral in five dimensions. In particular, interpreting the x_j variables as the dual coordinates of a set of $5d$ momenta $p_j = x_{j+1} - x_j$, it reduces to a single box integral in five dimensions.

$$\epsilon_{\mu\nu\rho}\, \partial_i^\mu\, \partial_{i+1}^\nu\, \partial_{j+1}^\rho\, \mathcal{I}(i,j) = \frac{8}{\pi^2}\, \frac{\epsilon_{\mu\nu\rho}\, x_{i,\,i+1}^\mu\, x_{i+1,\,j}^\nu\, x_{j,\,j+1}^\rho}{x_{i,\,i+1}\, x_{j,\,j+1}} \times \int d^5x_0\, \frac{1}{x_{0i}^2\, x_{0i+1}^2\, x_{0j}^2\, x_{0j+1}^2} \tag{4.6.11}$$

Besides reducing the number of involved integrations, this result also allows to easily extract the divergent part in the light–like limit. Indeed the factor $\frac{1}{x_{i,\,i+1}\, x_{j,\,j+1}}$, is such to complete, together with the other $n-2$ free propagators, the tree level correlator which can then be reconstructed in front of the loop contribution.

The derivation of this result is rather technical and hinges on Feynman parameterization and Mellin–Barnes representation, which are reviewed in Appendix A.5.1. The antisymmetry of the Levi–Civita tensor is essential to finally get to (4.6.11). The lazy reader may skip to Section 4.6.4, where this non–trivial result is used to push the computation of the correlator on and take its light–like limit.

4.6.3 The emergence of a five dimensional integral

Here we spell out the detailed proof of eq. (4.6.11) which allows to express the double three dimensional integral (4.6.10) as a one–loop five dimensional box integral.

We will adopt the shorthand notation, $\epsilon_{\mu\nu\rho}\, \partial_i^\mu\, \partial_{i+1}^\nu\, \partial_{j+1}^\rho\, \mathcal{I}(i,j)$ and choose $i = 1, j = 3$.

Although derivatives might have kept external, it proves convenient to let them act on the propagators in the integrand

$$\begin{aligned} &\epsilon^{\mu\nu\rho}\, \partial_{1\mu}\, \partial_{2\nu}\, \partial_{4\rho} \int d^3x_0\, d^3x_5\, \frac{1}{x_{10}\, x_{20}\, x_{05}\, x_{35}\, x_{45}} \\ &= -\epsilon_{\mu\nu\rho} \int d^3x_0\, d^3x_5\, \frac{x_{10}^\mu\, x_{20}^\nu\, x_{45}^\rho}{(x_{10}^2)^{3/2}\, (x_{20}^2)^{3/2}\, (x_{05}^2)^{1/2}\, (x_{35}^2)^{1/2}\, (x_{45}^2)^{3/2}} \equiv \mathcal{I} \end{aligned} \tag{4.6.12}$$

We employ Feynman parameterization to the x_0–integral, yielding

$$\begin{aligned} &\epsilon_{\mu\nu\rho} \int d^3x_0\, \frac{x_{10}^\mu\, x_{20}^\nu}{(x_{10}^2)^{3/2}\, (x_{20}^2)^{3/2}\, (x_{05}^2)^{1/2}} = \\ &\frac{4}{\pi^{3/2}}\, \Gamma\left(\frac{7}{2}\right) \int \prod_{i=1}^{3} dy_i\, \delta(\sum y_i - 1) y_1^{1/2} y_2^{1/2} y_3^{-1/2} \int d^3x_0 \frac{\epsilon_{\mu\nu\rho}\, x_{10}^\mu\, x_{20}^\nu}{[(x_0-\rho_1)^2 + \Omega_1]^{7/2}} \end{aligned} \tag{4.6.13}$$

where $\rho_1^\mu = y_1\, x_1^\mu + y_2\, x_2^\mu + y_3\, x_5^\mu$ and $\Omega_1 = y_1\, y_2\, x_{12}^2 + y_1\, y_3\, x_{15}^2 + y_2\, y_3\, x_{25}^2$.

Performing the shift $x_0^\mu \to x_0^\mu + \rho_1^\mu$ and integrating over x_0 we obtain

$$4\,\epsilon_{\mu\nu\rho}\,x_{15}^\mu\,x_{25}^\nu \int \prod_{i=1}^{3} dy_i\,\delta(\sum_i y_i - 1)\frac{(y_1\,y_2\,y_3)^{1/2}}{(y_1\,y_2\,x_{12}^2 + y_1\,y_3\,x_{15}^2 + y_2\,y_3\,x_{25}^2)^2} \tag{4.6.14}$$

At this stage we need to separate contributions in the denominator of (4.6.14), so as to make the remaining x_5 integration in (4.6.12) feasible. This is done thanks to the Mellin–Barnes integral representation. According to the general identity

$$\frac{1}{(k^2+A^2+B^2)^a} = \frac{1}{(k^2)^a\Gamma(a)}\int_{-i\infty}^{+i\infty}\frac{dsdt}{(2\pi i)^2}\Gamma(-s)\Gamma(-t)\Gamma(a+s+t)\left(\frac{A^2}{k^2}\right)^s\left(\frac{B^2}{k^2}\right)^t \tag{4.6.15}$$

we can reexpress (4.6.14) as

$$\frac{4\,\epsilon_{\mu\nu\rho}\,x_{15}^\mu\,x_{25}^\nu}{\sqrt{\pi}}\int_{-i\infty}^{i\infty}\frac{du\,dv}{(2\pi i)^2}\frac{\Gamma\left(-u|-\frac{1}{2}-u|-v|-\frac{1}{2}-v|2+u+v|\frac{3}{2}+u+v\right)}{(x_{12}^2)^{u+v+2}\,(x_{15}^2)^{-u}\,(x_{25}^2)^{-v}} \tag{4.6.16}$$

where we have introduced the shorthand notation $\Gamma(z_1|...|z_n) \equiv \Gamma(z_1)...\Gamma(z_n)$.

This allows to readily perform the x_5–integration in (4.6.12). Once again, using Feynman combining we can write (we neglect factors which are independent of x_5)

$$-\epsilon_{\mu\nu\rho}\int d^3x_5\,\frac{x_{15}^\mu\,x_{25}^\nu\,x_{45}^\rho}{(x_{15}^2)^{-u}\,(x_{25}^2)^{-v}\,(x_{35}^2)^{1/2}\,(x_{45}^2)^{3/2}} = \tag{4.6.17}$$
$$-\frac{2\Gamma(2-u-v)}{\Gamma(-u|-v)\pi}\int\prod_{i=1}^{4}dy_i\,\delta(\sum y_i - 1)y_1^{-u-1}y_2^{-v-1}y_3^{-1/2}y_4^{1/2}\int\frac{d^3x_5\,\epsilon_{\mu\nu\rho}\,x_{15}^\mu\,x_{25}^\nu\,x_{45}^\rho}{[(x_5-\rho_2)^2+\Omega_2]^{2-u-v}}$$

provided the following definitions hold

$$\rho_2^\mu = y_1\,x_1^\mu + y_2\,x_2^\mu + y_3\,x_3^\mu + y_4\,x_4^\mu \tag{4.6.18}$$
$$\Omega_2 = y_1\,y_2\,x_{12}^2 + y_2\,y_3\,x_{23}^2 + y_3\,y_4\,x_{34}^2 + y_4\,y_1\,x_{41}^2 + y_1\,y_3\,x_{13}^2 + y_2\,y_4\,x_{24}^2$$

We the shift the integration momentum $x_5^\mu \to x_5^\mu + \rho_2^\mu$, and integrate over x_5 to produce

$$\epsilon_{\mu\nu\rho}\,x_{31}^\mu\,x_{32}^\nu\,x_{34}^\rho\,\frac{2\sqrt{\pi}\,\Gamma(\frac{1}{2}-u-v)}{\Gamma(-u|-v)}\int\prod_{i=1}^{4}dy_i\,\frac{\delta(\sum y_i - 1)\,y_1^{-u-1}y_2^{-v-1}y_3^{1/2}y_4^{1/2}}{\Omega_2^{1/2-u-v}} \tag{4.6.19}$$

Quite remarkably this expression exactly coincides with the Feynman parameterization of a five dimensional scalar square integral with indices $(-u, -v, 3/2, 3/2)$. More precisely, we have

$$(4.6.19) \quad = \epsilon_{\mu\nu\rho}\,x_{31}^\mu\,x_{32}^\nu\,x_{34}^\rho\,\frac{1}{2\pi}\int d^5x_5\,\frac{1}{(x_{15}^2)^{-u}\,(x_{25}^2)^{-v}\,(x_{35}^2)^{3/2}\,(x_{45}^2)^{3/2}} \tag{4.6.20}$$

The identification with a higher dimensional integral is just formal, and should be intended at the level of its Feynman–parameterized form, where only Lorentz invariant quantities appear. Therefore one can unambiguously identify five and three dimensional invariants $x_{i,j}^2$ and deduce the observation above. Notice that the antisymmetry of the ϵ tensor plays a central role, selecting a particular numerator, which produces (4.6.20) after Feynman parameterization.

Back to the complete integral, we can plug (4.6.20) into (4.6.16) and we are left with

$$\begin{aligned}\mathcal{I} \;\; &= \;\; \frac{2}{\pi^{\frac{3}{2}}}\,\epsilon_{\mu\nu\rho}\,x_{13}^{\mu}\,x_{23}^{\nu}\,x_{34}^{\rho}\int\frac{d^5x_5}{(x_{35}^2)^{3/2}\,(x_{45}^2)^{3/2}}\,\times \\ &\int_{-i\infty}^{+i\infty}\frac{du\,dv}{(2\pi i)^2}\frac{\Gamma\left(-u|-v|-\frac{1}{2}-u|-\frac{1}{2}-v|2+u+v|\frac{3}{2}+u+v\right)}{(x_{12}^2)^{u+v+2}\,(x_{15}^2)^{-u}\,(x_{25}^2)^{-v}}\end{aligned} \tag{4.6.21}$$

Again, this MB–representation can be identified with the one for a five dimensional scalar triangle integral with exponents $(3/2, 3/2, 3/2)$ for the propagators

$$\mathcal{I} = \frac{1}{4\pi^2}\,\epsilon_{\mu\nu\rho}\,x_{13}^{\mu}\,x_{23}^{\nu}\,x_{34}^{\rho}\int d^5x_0\,d^5x_5\,\frac{1}{(x_{01}^2)^{3/2}\,(x_{02}^2)^{3/2}\,(x_{05}^2)^{3/2}\,(x_{35}^2)^{3/2}\,(x_{45}^2)^{3/2}} \tag{4.6.22}$$

The crucial point at this stage is that the integral above is quite easily computable resorting to unique triangles relations, derived in [88].

This technique, which we review in Appendix A.5.5, can be applied to particular triangle integrals in D dimensions where the sum over propagator exponents matches the dimension of the spacetime integration (some other similar relations are also available in other cases as well [88]).

Under these uniqueness conditions, this relations allow to replace the triangle integral by [88]

$$\begin{aligned}&\int\frac{d^Dx_0}{(x_{01}^2)^{\alpha_1}\,(x_{02}^2)^{\alpha_2}\,(x_{05}^2)^{\alpha_3}}\bigg|_{\alpha_1+\alpha_2+\alpha_3=D} = \\ &\pi^{D/2}\prod_i\frac{\Gamma(D/2-a_i)}{\Gamma(a_i)}\,\frac{1}{(x_{12}^2)^{D/2-\alpha_3}\,(x_{15}^2)^{D/2-\alpha_2}\,(x_{25}^2)^{D/2-\alpha_1}}\end{aligned} \tag{4.6.23}$$

Despite our integral does not look of such kind, we can apply the following identity

$$\begin{aligned}\mathcal{T}[D;\alpha_1,\alpha_2,\alpha_3;x_{03}^2,x_{04}^2,x_{34}^2] \;\; &= \;\; \frac{\Gamma(\sum_i\alpha_i-\frac{D}{2})}{\Gamma(D-\sum_i\alpha_i)}\prod_i\frac{\Gamma(\frac{D}{2}-\alpha_i)}{\Gamma(\alpha_i)}\times\frac{1}{(x_{34}^2)^{\alpha_2+\alpha_3-D/2}}\times \\ &\mathcal{T}\left[D;\sum\alpha_i-\frac{D}{2},\frac{D}{2}-\alpha_3,\frac{D}{2}-\alpha_2;x_{03}^2,x_{04}^2,x_{34}^2\right]\end{aligned} \tag{4.6.24}$$

which is valid for a generic triangle

$$\mathcal{T}[D;\alpha_1,\alpha_2,\alpha_3;x_{03}^2,x_{04}^2,x_{34}^2] = \int \frac{d^D x_5}{(x_{05}^2)^{\alpha_1}\,(x_{35}^2)^{\alpha_2}\,(x_{45}^2)^{\alpha_3}}, \tag{4.6.25}$$

with arbitrary indices [88], to our x_5–triangle in (4.6.22).

Identify $D=5$ and $\alpha_1=\alpha_2=\alpha_3=3/2$, we thus obtain

$$\mathcal{I} = \frac{2}{\pi^4}\,\frac{\epsilon_{\mu\nu\rho}\,x_{13}^{\mu}\,x_{23}^{\nu}\,x_{34}^{\rho}}{x_{34}} \int d^5x_0\, d^5x_5\, \frac{1}{(x_{01}^2)^{3/2}\,(x_{02}^2)^{3/2}\,(x_{05}^2)^2\,x_{35}^2\,x_{45}^2} \tag{4.6.26}$$

By this step we can realize that the exponents of the x_0 triangle are now $(3/2,3/2,2)$ and therefore satisfy the uniqueness condition $\alpha_1+\alpha_2+\alpha_3=D$ in five dimensions. Hence, we can use the general result for *unique* triangles and finally write

$$\mathcal{I} = \frac{8}{\pi^2}\,\frac{\epsilon_{\mu\nu\rho}\,x_{13}^{\mu}\,x_{23}^{\nu}\,x_{34}^{\rho}}{x_{12}\,x_{34}} \int d^5x_5 \frac{1}{x_{5,1}^2\,x_{5,2}^2\,x_{5,3}^2\,x_{5,4}^2} \tag{4.6.27}$$

This is the result quoted above in (4.6.11), which we will employ in the next section to get the final answer for the correlator and its light–like limit.

4.6.4 One–loop results and their light–like limit

The result for the block (4.6.9), and its explicit realization in terms of the five dimensional box integral (4.6.11) allow to reconstruct the whole one–loop contribution to the correlation function.

This is achieved by simply summing over all possible insertions of the block into the tree correlator, or, stated another way, by summing over any exchange of gluons between two edges of the n–polygon.

Whenever such a gauge line extends between two adjacent edges the corresponding contribution identically vanishes. This is due to the antisymmetry of the ϵ tensor. Indeed, corrections to a corner of the polygon are realized at the level of the block $\mathcal{T}_{ij}$, by identifying two insertion points carrying same chirality, say setting $x_i = x_{j+1}$ or $x_j = x_{i+1}$. Then the structure $\epsilon_{\mu\nu\rho}\,x^{\mu}_{i,\,i+1}\,x^{\nu}_{i+1,\,j}\,x^{\rho}_{j,\,j+1}$ trivially gives a vanishing result. Hence the sum must be restricted on gluon exchanges joining non–adjacent propagators. By a simple combinatorial exercise we can ascertain that these are $\binom{n}{2} - n = n(n-3)/2$. A very similar circumstance occurred in the calculation of the Wilson loop above.

Since the correlation functions alternates insertions of chiral and anti–chiral bilinear operators, it follows that two lines where the gauge propagators are attached must have the same chirality when they are separated by an odd number of matter propagators. In this situation the block $\mathcal{T}_{ij}^{(a)}$ is suitable to be employed. In the opposite situation one has to use the block $\mathcal{T}_{ij}^{(b)}$. The difference between the two choices just amounts to a an exchange in the order of two displacement vectors in the contraction with the ϵ tensor. This entails a relative minus sign. Furthermore, due to the choice of bilinear operators of the form AB, whenever two matter propagators are separated by an odd number of lines, they must carry the same flavor A or B. Since the cubic vertices with a gauge vector and two chiral superfields appear with different signs depending on the A or B nature of matter (see eq. (1.4.15)), a further minus sign emerges, compensating that from D–algebra and making all contributions have the same sign.

In conclusion, taking into account color factors, the leading term of the correlation function at one–loop reads

$$\mathcal{C}_n^{\,1-loop} \to \mathcal{C}_n^{tree} \times \frac{-1}{\pi^4} \left[\frac{M}{K_1} + \frac{N}{K_2}\right] \sum_{i=1}^{n-2} \sum_{j=i+2}^{n-\delta_{i,1}} \epsilon_{\mu\nu\rho}\, x^{\mu}_{i,\,i+1}\, x^{\nu}_{i+1,\,j}\, x^{\rho}_{j,\,j+1}\, \mathcal{J}(i,j) \tag{4.6.28}$$

where the sum extends to the $n(n-3)/2$ ways to connect two non–adjacent edges, and $\mathcal{J}(i,j)$ is the five dimensional integral

$$\mathcal{J}(i,j) = \int d^5x_0\, \frac{1}{x^2_{0i}\, x^2_{0i+1}\, x^2_{0j}\, x^2_{0j+1}} \tag{4.6.29}$$

As anticipated above it is straightforward to realize that in the ABJM case and more generally whenever $K_2 = -K_1$ and $M = N$ the color factor in front of the correlator vanishes, so that correlation functions are trivially zero at one loop for all these parity preserving theories. The same happens for the BLG theory, as it can be easily checked by computing the color factor for special unitary gauge groups $SU(M) \times SU(N)$ which turns out to be $(M - 1/M)/K_1 + (N - 1/N)/K_2$.

We want to also address more general theories for which the color factor does not vanish. The ABJ model, with different gauge group ranks, belongs to this class.

Before solving the problem in full generality for any n, we shall content with a simpler case, namely the four–point correlation function, to investigate what happens.

There are only two possible ways of joining non–adjacent edges

$$\mathcal{C}_4^{\,1-loop} \propto \epsilon_{\mu\nu\rho}\, (x^{\mu}_{1\,2}\, x^{\nu}_{2\,3}\, x^{\rho}_{3\,4} + x^{\mu}_{2\,3}\, x^{\nu}_{3\,4}\, x^{\rho}_{4\,1}) \int d^5x_0\, \frac{1}{x^2_{01}\, x^2_{02}\, x^2_{03}\, x^2_{04}} \tag{4.6.30}$$

Therefore, in this situation, one can conclude that the correlation function naively vanishes, due to the contraction with the ϵ tensor, without even computing the integral or taking the light–cone limit. This phenomenon was already noticed in [80].

This simplicity is spoiled when looking into higher point correlators. In this situation one really has to work out the $\mathcal{J}(i,j)$ integral. It is worth mentioning that this task was accomplished numerically in [80], ending up with a vanishing result for the six–point correlator at one–loop. This led the authors to conjecture that this pattern should also hold for higher points. In what follows we will confirm this intuition by an explicit analytic computation.

To do this we want to have an analytical and handable expression for the five dimensional integral. We recall that we eventually want to get the result for the limit where the operators become light–like separated, $x^2_{i,\,i+1} \to 0$, in order to test a correspondence to light–like Wilson loops. Since the prefactor $\mathcal{C}_n^{\,tree}$ in (4.6.28) is divergent in this limit, we consider the ratio of the one-loop correlator to the tree level result. Then we can directly take the light–like limit on the integral, which greatly simplifies the job, lowering the number of possible invariants on which it could depend. These invariants correspond geometrically to the proper length of the diagonals in the polygon. Actually each integral should only depend on four of them at most.

In order to get a real output, we require the $n(n-3)/2$ diagonals of the n–polygon to be space–like ($x^2_{i,j} > 0$, $j \neq i+1$).

We now focus on the integral (4.6.29). Shifting the integration variable $x_0 \to x_0 + x_i$ it becomes

$$\mathcal{J}(i,j) = \int d^5x_0 \, \frac{1}{x_0^2 \, \left(x_0 + x_{i,\,i+1}\right)^2 \, \left(x_0 + x_{i,\,j}\right)^2 \, \left(x_0 + x_{i,\,j+1}\right)^2} \tag{4.6.31}$$

and can be recognized to be a Feynman scalar box integral with external momenta $x_{i,\,i+1}$, $x_{i+1,\,j}$, $x_{j,\,j+1}$ and $x_{j+1,\,i}$, in five dimensions though.

Taking the light–like limit generically only two of the external momenta are space–like, whereas the other two are null. The massive legs being flowing onto opposite corners of the box make it the *two mass easy* box. A special kinematic occurs whenever $j = i+2$, i.e. when the two edges are separated by a single line in the polygon. Then one more external leg becomes massless and the integral simplifies further, becoming a *one mass* box. These integrals are depicted in Fig. 4.6

We can now Feynman parameterize the scalar five dimensional box and perform the x_0 integration to get

$$\mathcal{J}(i,j) = \frac{\pi^3}{2} \int_0^1 [d\alpha]_4 \frac{1}{\left(\alpha_1\,\alpha_3\,x^2_{i,j} + \alpha_2\,\alpha_4\,x^2_{i+1,j+1} + \alpha_1\,\alpha_4\,x^2_{i,j+1} + \alpha_2\,\alpha_3\,x^2_{i+1,j}\right)^{\frac{3}{2}}} \tag{4.6.32}$$

where the conventional measure $[d\alpha]_4 = \delta(1 - \sum_{k=1}^4 \alpha_k) \prod_{k=1}^4 d\alpha_k$ on Feynman parameter space is present.

The following change of variables is quite customary to solve the δ function automatically

$$\alpha_1 = (1-\beta_1)(1-\beta_3) \quad , \quad \alpha_2 = \beta_1(1-\beta_3) \quad , \quad \alpha_3 = (1-\beta_2)\beta_3 \quad , \quad \alpha_4 = \beta_2\beta_3 \tag{4.6.33}$$

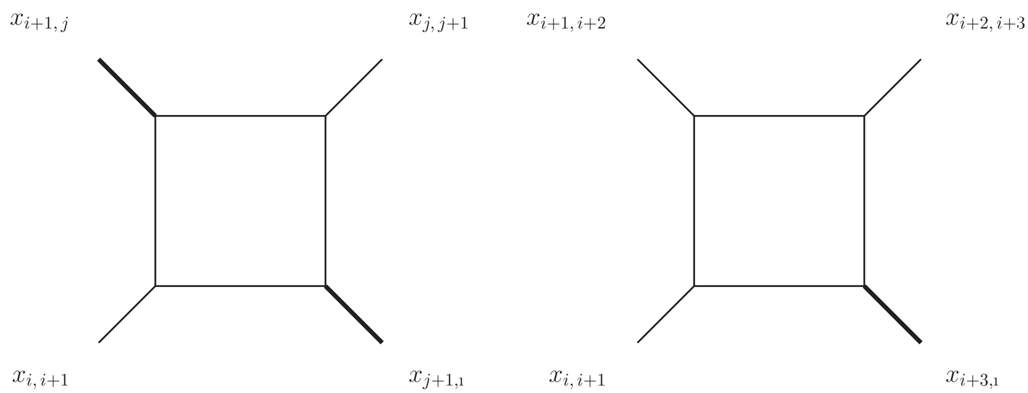


Figure 4.6: The $5d$ integrals: two–mass easy and one–mass.

According to the shift above, the integral reads

$$\mathcal{J}(i,j) = \frac{\pi^3}{2}\int_0^1 \prod_{i=1}^{3} d\beta_i \times \tag{4.6.34}$$
$$\frac{\beta_3^{-\frac{1}{2}}(1-\beta_3)^{-\frac{1}{2}}}{\left[(1-\beta_2)(1-\beta_1)\, x_{i,j}^2 + \beta_1\,\beta_2\, x_{i+1,j+1}^2 + \beta_2\,(1-\beta_1)\, x_{i,j+1}^2 + \beta_1\,(1-\beta_2)\, x_{i+1,j}^2\right]^{\frac{3}{2}}}$$

We notice that the β_3–integration can be trivially performed, leading to

$$\mathcal{J}(i,j) = \frac{\pi^4}{2}\int_0^1 d\beta_1 d\beta_2 \times \tag{4.6.35}$$
$$\frac{1}{\left[(1-\beta_2)(1-\beta_1)\, x_{i,j}^2 + \beta_1\,\beta_2\, x_{i+1,j+1}^2 + \beta_2\,(1-\beta_1)\, x_{i,j+1}^2 + \beta_1\,(1-\beta_2)\, x_{i+1,j}^2\right]^{\frac{3}{2}}}$$

In the last paragraph we comment on this result and on its implications in the conjectured WL/correlator duality.

Connection with light–like Wilson loops As for the Wilson loops, the overall color factor in (4.6.28) vanishes for all the theories with $K_2 = -K_1$ and $M = N$, ABJM case included. For this set of theories the correlation functions/WL duality is then trivial at the first perturbative order.

Interesting non–trivial results can be found, instead, for theories where the color factor does not vanish. In fact, the main observation is that, identifying the affine parameters α_i, α_j with the Feynman parameters β_1, β_2 in (4.6.35), the $\mathcal{K}(i,j)$ integral is precisely the same as the integral $\mathcal{J}(i,j)$ arising in the computation of an n–point correlation function in the light–like limit. Since the integral (4.6.35) is the Feynman parameterization of a $5d$ box integral, we can claim that also the one–loop Wilson loop can be formally expressed in terms of a $5d$ scalar integral.

Therefore we stress that also in the ABJM case the identification is stronger than it may appear, since the expressions for the Wilson loop and the light–like correlator are equal already at the level of the integrands, before pursuing the computation to the end.

We can therefore claim the following relation to be true at first order

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\mathcal{C}_n^{\,1-loop}}{\mathcal{C}_n^{tree}} \;=\; \langle W(A,\hat{A})\rangle^{1-loop} \tag{4.6.36}$$

all in terms of the $5d$ integral (4.6.29).

We note that the two expressions coincide, independently of the values of the couplings K_1, K_2 and for any value of the gauge ranks (N, M), as no planar limit is required.

As pointed out in the introduction to this part, this result is more than expected for the WL/correlator duality, since it ingenuously suggests that this relation should hold for any CSM theory. Nevertheless we know that this conclusion is naive. Indeed no information on the degree of supersymmetry and on conformality is present in both computations. This dependence is expected at two loops, where non–trivial corrections appear involving the matter fields and not only the CS gauge sector. We believe that a higher loop calculation may severely restrict the class of theories for which the duality survives, in particular conformal field theories are expected to exhibit this duality on general grounds, as reviewed in the introduction.

4.6.5 Generalization to higher dimensional operators

Here we outline how to extend the result obtained at one–loop for bilinear operators to higher dimensional ones. In particular we aim at showing that the computation for larger half–BPS operators is simply obtainable from that for bilinears, provided the light–like limit is taken, in order to select Wick contractions between adjacent operators only. The derivation is nevertheless completely independent of the gauge groups involved and does not require any planar limit.

For definiteness, suppose we consider $n = 2m$ half–BPS operators of the general form $\mathcal{O}_i = (A\,B)^{k_i}$ and complex conjugates $\bar{\mathcal{O}}_{\bar{j}} = (\bar{A}\,\bar{B})^{k_{\bar{j}}}$.

The most divergent part of connected correlators of higher dimensional operators in

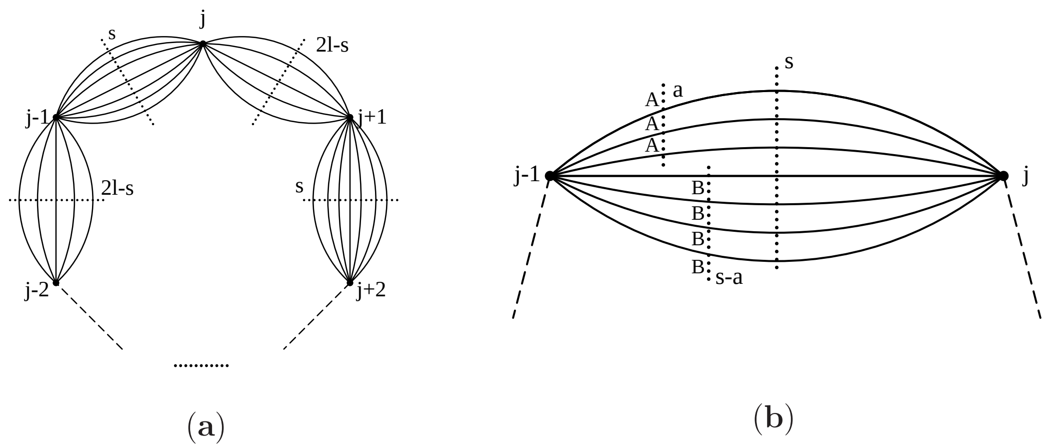


Figure 4.7: General form of the contributions to $\mathcal{C}^{tree}_{n,2l}$. In Fig. (a), structure of the leading divergent terms in the limit $x^2_{i,i+1} \to 0$. In Fig. (b), the parameter a counts the number of $\langle A\bar{A}\rangle$ propagators in a set of s lines.

the light–like limit $x^2_{i,i+1} \to 0$ at tree level reads

$$\begin{aligned} \mathcal{C}^{tree}_{n,2l} &\propto \sum_{s=1}^{2l-1} \mathcal{T}^{tree}_s \\ \mathcal{T}^{tree}_s &= \prod_{j=1}^{n/2} \left(\frac{1}{x_{2j-1,2j}}\right)^s \left(\frac{1}{x_{2j,2j+1}}\right)^{2l-s} \end{aligned} \tag{4.6.37}$$

Eq. (4.6.37) extends (4.6.8) to the $l > 1$ case. The general contribution in the sum (4.6.37) is a polygon with edges alternately made by s and $2l - s$ propagators (see Fig. 4.6.5(a)). Each value of s defines a different topology $\mathcal{T}_s$. In the rest of the discussion, it is useful to divide each topology $\mathcal{T}_s$ into classes $\mathcal{T}_{s,a}$ where the parameter a counts the number of $\langle A\bar{A}\rangle$ propagators inside a block of s lines (see Fig. 4.6.5(b)).

One–loop corrections to $\mathcal{C}_{n,2l}$ are obtained by the insertion of a gauge propagator V or $\hat{V}$ in all possible ways between the edges of the polygon $\mathcal{C}^{tree}_{n,2l}$.

As in the $l = 1$ case, the only non–trivial vector exchanges occur when the gauge propagator connects two non–consecutive edges in the polygon. All other possible insertions vanish due to D–algebra constraints (on the same leg) or to the antisymmetry of the ϵ tensor (on adjacent edges).

The non–trivial corrections have the form (4.6.9). However, since now we have more than one chiral propagator in each edge at disposal, we have more than one possibility to insert a gauge line between the same two edges of the correlator.

The combinatorial factor is in principle different for corrections involving different pairs of edges in each class $\mathcal{T}_{s,a}$. However, a careful computation taking into account the relative signs between A and B vertices (1.4.15) and between the two building blocks (4.6.9) shows that the combinatorics depends only on the (a, s) parameters and it thus evaluates to a common factor for all corrections inside each $\mathcal{T}_{s,a}$ class, which can be then factored out. Precisely, the one–loop correction to the generic $\mathcal{T}_{s,a}$ class reads

$$\mathcal{T}_{s,a}^{1-loop} \propto \mathcal{T}_{s,a}^{tree} \times (s-2a)^2 \sum_{i=1}^{n-2} \sum_{j=i+2}^{n-\delta_{i,1}} \epsilon_{\mu\nu\rho}\, x_{i,\,i+1}^{\mu}\, x_{i+1,\,j}^{\nu}\, x_{j,\,j+1}^{\rho}\, \mathcal{J}(i,j) \tag{4.6.38}$$

where $\mathcal{J}(i,j)$ is the five dimensional box integral (4.6.29).

This formula closely resembles eq. (4.6.28). In particular, the sums in these two expressions coincide. Thus, having computed the one–loop corrections to the n–point function for dimension–one operators, we immediately have the result for any $\mathcal{T}_{s,a}^{1-loop}$. The complete one–loop correction to the correlator $\mathcal{C}_{n,2l}$ can be then recovered through (4.6.37).

4.6.6 One–loop vanishing of correlators and Wilson loops

In the final part of this Section we want to work out a simpler expression for both Wilson loops and correlators in the light–like limit, out of the complicated sum (4.6.28). To accomplish this task we first need to evaluate the five dimensional integral and then combine the different contributions of the sum. The former is done in the following paragraph, in the second we address the latter.

The five dimensional integral. We want to solve the integral over Feynman/affine parameters

This can be done straightforwardly from (4.6.35), just with the help of *Mathematica*. The outcome is that the general one–loop contribution to the n–point correlator corresponding to a Feynman diagram where a vector line connects the $x_{i,i+1}$ and $x_{j,j+1}$ free propagators, in the light–cone limit reads

$$\epsilon_{\mu\nu\rho}\, x_{i,\,i+1}^{\mu}\, x_{i+1,\,j}^{\nu}\, x_{j,\,j+1}^{\rho}\, \mathcal{J}(i,j) \; = \tag{4.6.39}$$

$$\pi^4\, \mathcal{S}_{i,\,j}\, \log\left[\frac{(1+x_{i+1,j}\,\mathcal{L}_{i,\,j})\,(1+x_{i,j+1}\,\mathcal{L}_{i,\,j})}{(1-x_{i+1,j}\,\mathcal{L}_{i,\,j})\,(1-x_{i,j+1}\,\mathcal{L}_{i,\,j})}\frac{(1-x_{i,j}\,\mathcal{L}_{i,\,j})\,(1-x_{i+1,j+1}\,\mathcal{L}_{i,\,j})}{(1+x_{i,j}\,\mathcal{L}_{i,\,j})\,(1+x_{i+1,j+1}\,\mathcal{L}_{i,\,j})}\right]$$

where we have defined

$$\mathcal{S}_{i,\,j} = \frac{2\,\epsilon_{\mu\nu\rho}\, x_{i,\,i+1}^{\mu}\, x_{i+1,\,j}^{\nu}\, x_{j,\,j+1}^{\rho}}{\sqrt{x_{i,j}^2 + x_{i+1,j+1}^2 - x_{i+1,j}^2 - x_{i,j+1}^2}\sqrt{x_{i,j}^2 x_{i+1,j+1}^2 - x_{i+1,j}^2 x_{i,j+1}^2}} \tag{4.6.40}$$

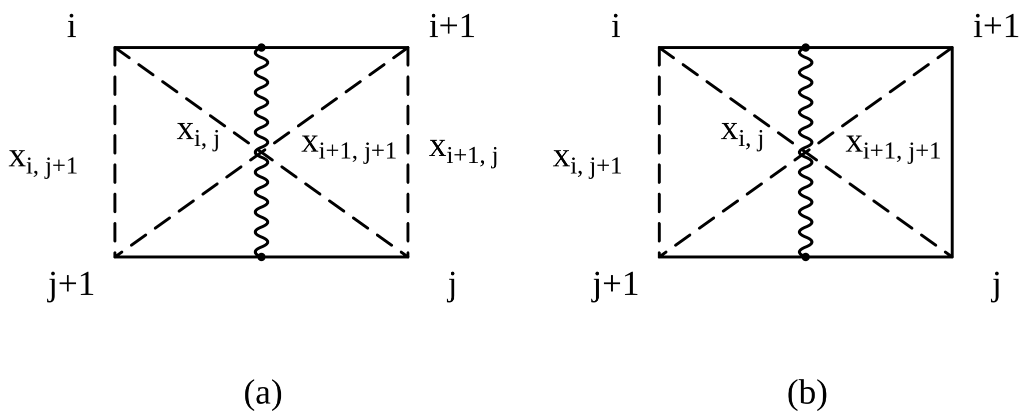


Figure 4.8: The building blocks for the correlation functions only depend on the diagonals of the polygon, which are drawn with dashed lines. Case (a) corresponds to the $n(n-5)/2$ blocks where all the involved diagonals are long. Case (b) depicts one of the n blocks with short diagonals.

and

$$\mathcal{L}_{i,j} = \frac{\sqrt{x_{i,j}^2 + x_{i+1,j+1}^2 - x_{i+1,j}^2 - x_{i,j+1}^2}}{\sqrt{x_{i,j}^2 x_{i+1,j+1}^2 - x_{i+1,j}^2 x_{i,j+1}^2}} \tag{4.6.41}$$

Focusing on the argument of the logarithm in (4.6.39) we note that it depends only on the diagonals connecting the four vertices of the block x_i, x_{i+1}, x_j and x_{j+1}, as depicted in Fig. 4.8(a). This is due to the fact that the correlator, being Poincaré invariant, has to be a function of the only invariants that we can construct. In the light–like limit these are the $n(n-3)/2$ space–like diagonals ‖.

We distinguish two sets of diagonals. We call "short" diagonals those connecting two vertices separated by a pair of light–like edges, whereas we call "long" diagonals the remaining $n(n-5)/2$ ones.

An example of the appearance of short diagonals is depicted in Fig. 4.8(b), where the vertices x_{i+1} and x_j are connected by a null edge, so the space–like segments $x_{i,j}$ and $x_{i+1,j+1}$ are short diagonals. In this case, the corresponding contribution can be obtained from the general expression (4.6.39) by collapsing $x_{i+1,j} \to 0$, and as a result the logarithm contains just three factors instead of four.

Going back to (4.6.39), by straightforward algebra we can rewrite the argument of the

‖Actually not all diagonals are independent and their number could in principle be reduced by the Gram constraints. Since these constraints are difficult to implement we will not pursue this technique.

logarithm as

$$\frac{(1+x_{i+1,j}\,\mathcal{L}_{i,j})\,(1+x_{i,j+1}\,\mathcal{L}_{i,j})}{(1-x_{i+1,j}\,\mathcal{L}_{i,j})\,(1-x_{i,j+1}\,\mathcal{L}_{i,j})}\frac{(1-x_{i,j}\,\mathcal{L}_{i,j})\,(1-x_{i+1,j+1}\,\mathcal{L}_{i,j})}{(1+x_{i,j}\,\mathcal{L}_{i,j})\,(1+x_{i+1,j+1}\,\mathcal{L}_{i,j})} = \frac{(1+x_{i+1,j}\,\mathcal{L}_{i,j})^2\,(1+x_{i,j+1}\,\mathcal{L}_{i,j})^2}{(1+x_{i,j}\,\mathcal{L}_{i,j})^2\,(1+x_{i+1,j+1}\,\mathcal{L}_{i,j})^2} \tag{4.6.42}$$

As proven in Section 4.6.6, $\mathcal{L}_{i,j}$'s are real functions as long as all the diagonals are space–like. Under this assumption, eq. (4.6.42) is the square of a real expression and the logarithm in (4.6.39) is well defined. A similar argument applies also to the case of short diagonals, leading to the same conclusions.

Finally, inserting the result (4.6.39) back into eq. (4.6.28) and summing over all possible contractions, we obtain the complete analytical result for the ratio $\mathcal{C}_n^{1-loop}/\mathcal{C}_n^{tree}$ in the light–like limit. The positiveness of the arguments of all logarithms allows us to safely rewrite the sum as

$$\frac{\mathcal{C}_n^{1-loop}}{\mathcal{C}_n^{tree}} = -\left[\frac{M}{K_1}+\frac{N}{K_2}\right]\log\Bigg\{ \prod_{i=1}^{n-2}\prod_{j=i+2}^{n-\delta_{i,1}}\left[\frac{(1+x_{i+1,j}\,\mathcal{L}_{i,j})\,(1+x_{i,j+1}\,\mathcal{L}_{i,j})}{(1-x_{i+1,j}\,\mathcal{L}_{i,j})\,\left(1-x_{i,j+1}^2\,\mathcal{L}_{i,j}\right)}\frac{(1-x_{i,j}\,\mathcal{L}_{i,j})\,(1-x_{i+1,j+1}\,\mathcal{L}_{i,j})}{(1+x_{i,j}\,\mathcal{L}_{i,j})\,(1+x_{i+1,j+1}\,\mathcal{L}_{i,j})}\right]^{\mathcal{S}_{i,j}}\Bigg\} \tag{4.6.43}$$

In general, this expression is not zero as long as the distances $x_{i,j}$ are arbitrary. However they are not all independent, being the diagonals of a polygon in three spacetime dimensions. In Section 4.6.6 we come back to this result and prove that it is actually zero when implementing an explicit parameterization which constrains the $x_{i,j}$ segments to be the diagonals of a three dimensional polygon.

4.6.7 Evaluation of the sum: the final answer.

In this part we give an analytical proof that the expression (4.6.43) vanishes for any value of n. In other words, the light–like limit of n–point correlation functions of dimension–one BPS operators and the light–like polygonal Wilson loop are zero at one loop.

Given the identification (4.6.36), as a byproduct we also prove that light–like n–polygonal Wilson loops vanish at first order. This result generalizes the one in [80] valid only for $n = 4, 6$ and proves the conjecture made there that Wilson loops should be one–loop vanishing for any n.

As we read in (4.6.43), the one–loop correction to a correlation function is proportional

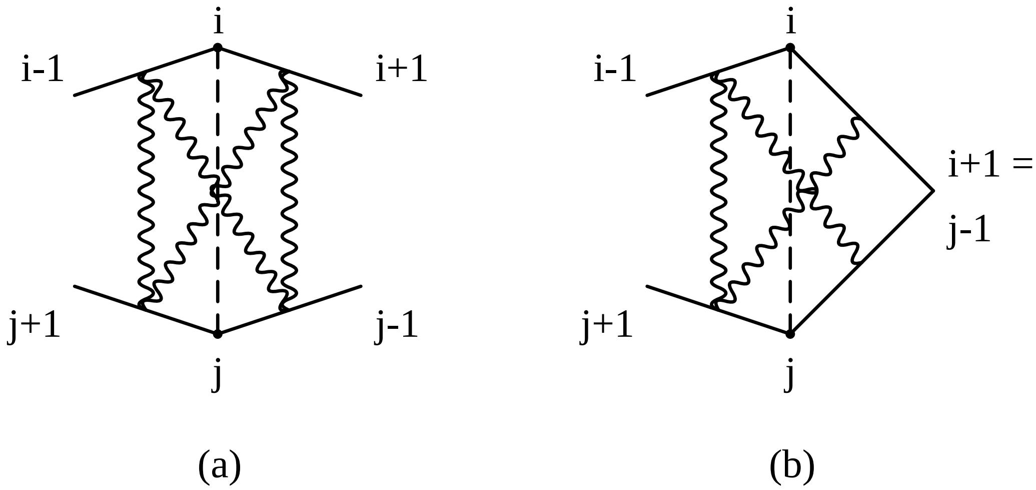


Figure 4.9: In picture (a) the four blocks in which the reference diagonal $x_{i,j}$ is involved are depicted. In picture (b) the case of a short diagonal and its three blocks is shown. Each wiggled line has to be interpreted separately.

to the logarithm of a product of factors with schematic form $\left(\frac{1 \pm x\, \mathcal{L}_{i,\,j}}{1 \mp x\, \mathcal{L}_{i,\,j}}\right)^{\mathcal{S}_{i,\,j}}$. We prove that this product always evaluates to 1.

In (4.6.43) the factors are grouped according to the pair of edges involved in a given gauge vector exchange (see blocks in Fig. 4.5). The basic idea of the proof is to reorganize them by grouping together all the factors which depend on the same diagonal $x_{i,j}$. It is easy to ascertain that each long diagonal is involved in four contributions, coming from the four possible interactions connecting the edges which are adjacent to the diagonal itself (See Fig.4.9 (a)). In the case of a short diagonal, one of these contributions vanishes (it would be a correction to the vertex), thus we are left with just three pieces (See Fig.4.9(b)).

Once this reshuffling of factors has been performed in (4.6.43), we prove that the product of contributions involving the same reference diagonal evaluates to $+1$ for long diagonals and to -1 for short ones. We consider a generic diagonal and parameterize all distances in full generality, so that once we establish this property for one diagonal, we can apply it to all the contributions to the correlator.

Let us focus on one particular diagonal $x_{i,j}$, and suppose it is long. The corresponding block of factors then depends only on the nearest neighbors of the vertices x_i and x_j, which are x_{i-1}, x_{i+1}, x_{j-1}, x_{j+1}. These six points are parametrized by 18 coordinates. However, four of them can be eliminated by light–likeness of the edges $x_{i,\,i+1}$, $x_{i,\,i-1}$, $x_{j,\,j+1}$, $x_{j,\,j-1}$. By using translation invariance, we choose a convenient reference frame where $x_i^\mu = (0,0,0)$, so removing three more coordinates. Using rotational invariance, we eliminate two further parameters by choosing $x_j^\mu = (0,b,0)$ where $b > 0$. In this way, the

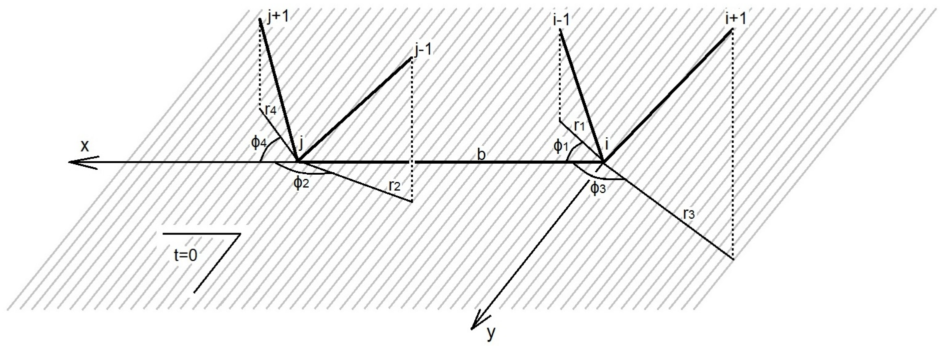


Figure 4.10: Parameterization of the block of contributions involving the same reference diagonal $x_{i,\,j}$.

reference diagonal lies in the $t = 0$ plane. We parameterize the rest of the block in terms of the nine remaining variables as follows

$$\begin{aligned} x^\mu_{i-1} &= r_1\,(1, \cos\phi_1, \sin\phi_1)\,, & x^\mu_{i+1} &= r_3\,(1, \cos\phi_3, \sin\phi_3) \\ x^\mu_{j-1} &= x^\mu_j + r_2\,(1, \cos\phi_2, \sin\phi_2)\,, & x^\mu_{j+1} &= x^\mu_j + r_4\,(1, \cos\phi_4, \sin\phi_4) \end{aligned} \qquad (4.6.44)$$

This parameterization is sketched in Fig. 4.10: the ϕ_i's are the angles held by the projections of the light–like lines on the $t = 0$ plane, while the moduli of the r_i's measure the lengths of these same projections. It is obvious that the edges are light–like and the reference diagonal $x_{i,\,j}$ is space–like by construction. At this stage, the other diagonals are not necessarily space–like. The request for them to be space–like implies that r_1, r_3 and r_2, r_4 should have separately the same sign, in order for adjacent segments to point alternatively to the future and to the past. In the following we will assume that they are all positive, but the final statement can be exhaustively shown to be valid for any choice of these signs.

Let us now evaluate the product of the four contributions for the reference diagonal $x_{i,\,j}$, namely

$$\begin{aligned} &\left(\frac{1 + x_{i,j}\;\mathcal{L}_{i,\,j}}{1 - x_{i,j}\;\mathcal{L}_{i,\,j}}\right)^{\mathcal{S}_{i,\,j}} \left(\frac{1 + x_{i,j}\;\mathcal{L}_{i-1,\,j-1}}{1 - x_{i,j}\;\mathcal{L}_{i-1,\,j-1}}\right)^{\mathcal{S}_{i-1,\,j-1}} \\ &\left(\frac{1 - x_{i,j}\;\mathcal{L}_{i-1,\,j}}{1 + x_{i,j}\;\mathcal{L}_{i-1,\,j}}\right)^{\mathcal{S}_{i-1,\,j}} \left(\frac{1 - x_{i,j}\;\mathcal{L}_{i,\,j-1}}{1 + x_{i,j}\;\mathcal{L}_{i,\,j-1}}\right)^{\mathcal{S}_{i,\,j-1}} \end{aligned} \qquad (4.6.45)$$

By plugging in the parameterization (4.6.44) we obtain a nice symmetric expression

$$\left(\frac{1+\left|\frac{\sin\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}\right|}{1-\left|\frac{\sin\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}\right|}\right)^{\mathrm{Sign}\left[\frac{\sin\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}\right]}\left(\frac{1-\left|\frac{\sin\left(\frac{\phi_3-\phi_2}{2}\right)}{\cos\left(\frac{\phi_3+\phi_2}{2}\right)}\right|}{1+\left|\frac{\sin\left(\frac{\phi_3-\phi_2}{2}\right)}{\cos\left(\frac{\phi_3+\phi_2}{2}\right)}\right|}\right)^{\mathrm{Sign}\left[\frac{\sin\left(\frac{\phi_3-\phi_2}{2}\right)}{\cos\left(\frac{\phi_3+\phi_2}{2}\right)}\right]}$$
$$\left(\frac{1-\left|\frac{\sin\left(\frac{\phi_1-\phi_4}{2}\right)}{\cos\left(\frac{\phi_1+\phi_4}{2}\right)}\right|}{1+\left|\frac{\sin\left(\frac{\phi_1-\phi_4}{2}\right)}{\cos\left(\frac{\phi_1+\phi_4}{2}\right)}\right|}\right)^{\mathrm{Sign}\left[\frac{\sin\left(\frac{\phi_1-\phi_4}{2}\right)}{\cos\left(\frac{\phi_1+\phi_4}{2}\right)}\right]}\left(\frac{1+\left|\frac{\sin\left(\frac{\phi_3-\phi_4}{2}\right)}{\cos\left(\frac{\phi_3+\phi_4}{2}\right)}\right|}{1-\left|\frac{\sin\left(\frac{\phi_3-\phi_4}{2}\right)}{\cos\left(\frac{\phi_3+\phi_4}{2}\right)}\right|}\right)^{\mathrm{Sign}\left[\frac{\sin\left(\frac{\phi_3-\phi_4}{2}\right)}{\cos\left(\frac{\phi_3+\phi_4}{2}\right)}\right]} \qquad (4.6.46)$$

where Sign(x) is the sign function. We notice that the explicit parameterization allows us to fix a loose end from Section 4.6.4, namely we have ascertained that the terms $x_{i,j}\,\mathcal{L}_{\dots}$ are real positive functions. Furthermore we observe that the apparently awkward exponents $\mathcal{S}_{i,j}$ (4.6.40) are surprisingly just $\pm$ signs.
Expression (4.6.46) can be written in a compact fashion (here and in the following $\phi_5=\phi_1$ is understood)

$$\prod_{i=1}^{4}\left(\frac{1+\left|\frac{\sin\left(\frac{\phi_i-\phi_{i+1}}{2}\right)}{\cos\left(\frac{\phi_i+\phi_{i+1}}{2}\right)}\right|}{1-\left|\frac{\sin\left(\frac{\phi_i-\phi_{i+1}}{2}\right)}{\cos\left(\frac{\phi_i+\phi_{i+1}}{2}\right)}\right|}\right)^{\mathrm{Sign}\left[\frac{\sin\left(\frac{\phi_i-\phi_{i+1}}{2}\right)}{\cos\left(\frac{\phi_i+\phi_{i+1}}{2}\right)}\right]} \qquad (4.6.47)$$

We observe that the expression depends exclusively on the four angles of the parameterization but not on any of the five dimensionful parameters. We also note that in each contribution the arguments of absolute values and Sign functions are the same. Because of that and using the fact that $\left(\frac{1\pm x}{1\mp x}\right)^{\pm}=\frac{1+x}{1-x}$ we may simplify expression (4.6.47) to obtain

$$\prod_{i=1}^{4}\frac{\cos\left(\frac{\phi_i+\phi_{i+1}}{2}\right)+\sin\left(\frac{\phi_i-\phi_{i+1}}{2}\right)}{\cos\left(\frac{\phi_i+\phi_{i+1}}{2}\right)-\sin\left(\frac{\phi_i-\phi_{i+1}}{2}\right)} \qquad (4.6.48)$$

This is equivalent to

$$\prod_{i=1}^{4}\cot\left(\frac{\phi_i}{2}-\frac{\pi}{4}\right)\,\tan\left(\frac{\phi_{i+1}}{2}-\frac{\pi}{4}\right)=1 \qquad (4.6.49)$$

Therefore, this completes the proof for long diagonals.

For a short diagonal $x_{i,\,j}$, it suffices to take the result above and set e.g., $x_{i+1} = x_{j-1}$. Then the contribution involving $\mathcal{L}_{i,\,j-1}$ vanishes by construction leaving

$$\frac{\cos\left(\frac{\phi_1+\phi_2}{2}\right)+\sin\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)-\sin\left(\frac{\phi_1-\phi_2}{2}\right)}\,\frac{\cos\left(\frac{\phi_1+\phi_4}{2}\right)-\sin\left(\frac{\phi_1-\phi_4}{2}\right)}{\cos\left(\frac{\phi_1+\phi_4}{2}\right)+\sin\left(\frac{\phi_1-\phi_4}{2}\right)}\,\frac{\cos\left(\frac{\phi_3+\phi_4}{2}\right)+\sin\left(\frac{\phi_3-\phi_4}{2}\right)}{\cos\left(\frac{\phi_3+\phi_4}{2}\right)-\sin\left(\frac{\phi_3-\phi_4}{2}\right)} \tag{4.6.50}$$

When parameterizing as in eq. (4.6.44) the condition $x_{i+1} = x_{j-1}$ is forced by choosing

$$r_2 = r_3, \quad r_3\,\cos(\phi_3) = b + r_2\,\cos(\phi_2), \quad \sin(\phi_3) = \sin(\phi_2) \tag{4.6.51}$$

These equations are solved by $\phi_3 = \pi - \phi_2$ and some function $r_2 = r_2(a, \phi_3)$ which is irrelevant. Plugging it into (4.6.50) finally simplifies the expression to -1, in a completely analogous way as in the long diagonal case. Since there are n contributions of the short type, and since n is even, the overall contribution of short diagonals is equal to $+1$.

Summarizing, we have shown that the combined collection of all short and long diagonal contributions to the argument of the logarithm is equal to $+1$. Therefore, the logarithm is equal to zero, thus proving the vanishing of the n–point correlator and Wilson loops at one loop.

Moreover, the argument developed in Section 4.6.5 to connect the light–like limit of correlation functions of bilinear operators to those of higher dimension, entails that since the former vanish, so the latter do.

Chapter 5

Amplitudes in ABJM: two–loop computations

In this Chapter we want to push the computation of four–point scattering amplitudes for the ABJ(M) model to two–loop order. These are expected to be non–vanishing and therefore able to give more useful insights on the fate of the WL/amplitude duality and on dual conformal symmetry in three dimensional theories at loop level.

We first resum the calculation of the light–like Wilson loop at two loops on a contour given by a four cusps polygon, performed in [80]. This is the best candidate to be the dual object to the four–point amplitude, which is "MHV" on Grassmann counting grounds. Unlike the first perturbative order, the two–loop correction is non–trivial and thus suitable to test the conjectured duality more properly.

Then we set up for the computation of the corresponding amplitude, which we will approach by a direct manifestly $\mathcal{N}=2$ supersymmetric Feynman diagram analysis. Since the difficulty of the calculation increases sensibly from one to two loops, we limit our analysis to the simplest superamplitude, i.e. the chiral one (4.5.1), arising from corrections to the superpotential. For the ABJ(M) models supersymmetry ensures that all other amplitudes are related to the chiral one [87]. Hence the simplest computation allows to completely determine the whole spectrum of scattering amplitudes of four external particles. Therefore we will restrict to $\mathcal{N}=6$ supersymmetric theories only. The evaluation of the chiral amplitude turns out to be feasible using a traditional Feynman diagrammatic approach, since the number of involved graphs turns out to be quite limited. Henceforth we succeed in determining it in an entirely analytical fashion.

The final answer reveals to be quite readable: in the ABJ theory the ratio between the two–loop correction and the tree level result

$$\mathcal{M}_4^{(2)} \equiv \frac{\mathcal{A}_4^{(2-loops)}}{\mathcal{A}_4^{tree}} \tag{5.0.1}$$

gives

$$\mathcal{M}_4^{(2)} = \lambda\hat{\lambda}\left[-\frac{(s/\mu'^2)^{-2\epsilon}}{(2\,\epsilon)^2} - \frac{(t/\mu'^2)^{-2\epsilon}}{(2\,\epsilon)^2} + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + C_{\mathcal{A}}(M,N) + \mathcal{O}(\epsilon)\right] \tag{5.0.2}$$

where $\lambda = M/K$, $\hat{\lambda} = N/K$, μ' is the mass scale and $C_{\mathcal{A}}(M,N)$ is a constant depending on the ranks of the groups.

Setting the same ranks, i.e. in the ABJM model, the result reads

$$\mathcal{M}_4^{(2)} = \lambda^2\left[-\frac{(s/\mu'^2)^{-2\epsilon}}{(2\,\epsilon)^2} - \frac{(t/\mu'^2)^{-2\epsilon}}{(2\,\epsilon)^2} + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + \mathcal{C} + \mathcal{O}(\epsilon)\right] \tag{5.0.3}$$

This amplitude exhibits remarkable properties, which we now briefly list anticipating a more thorough discussion below

- First of all, up to an additive, scheme–dependent constant, this expression exactly matches the second order expansion of the ABJ light–like four–polygonal Wilson loop, once the IR regularization of the former is formally identified with the UV one of the latter, and the particle momenta are expressed in terms of dual coordinates ($s = x_{13}^2$ and $t = x_{24}^2$). Therefore, at least for the four–point amplitude, there is evidence that the following identity embodying the *WL/amplitude duality*

  $$\log\mathcal{M}_4 = \log\langle W_4\rangle + \text{const.} \tag{5.0.4}$$

  should hold true order by order in the perturbative expansion of the two objects.

- The WL/amplitude duality is intimately related to *dual conformal invariance*. Whenever an amplitude possesses this symmetry (eventually broken anomalously by IR divergences) it can be expressed as a linear combination of dual conformally invariant integrals. The fact that the four–point amplitude in ABJ(M) matches the corresponding Wilson loop hints at the suspicion that it might invariant under dual conformal symmetry. This can be expected since tree level amplitudes have been shown to be dual conformally invariant, and through unitarity this property can propagate to loop level, although in principle to the cut constructible piece of the amplitude only. In fact, for the ABJM case, following [81] we can rewrite the result (5.0.2) as a linear combination of scalar momentum integrals which are dual to three dimensional truly conformally invariant integrals, well defined off–shell. As a consequence, the four–point amplitude satisfies anomalous Ward identities associated to dual conformal transformations [61], as dual conformal invariance is broken in the on–shell limit by the appearance of IR divergences which require introducing a mass regulator.

  For the ABJ model the situation is slightly complicated by the appearance of a non–trivial dependence on the parity–violating parameter $\sigma = (M-N)/\sqrt{MN}$ in the mass–scale and in the constant $C_{\mathcal{A}}$. In fact, for $M \neq N$ the two–loop ratio (5.0.2) can be still written as a linear combination of dual invariant integrals only up to an additive constant proportional to σ^2. Therefore, for the ABJ theory the dual conformal invariance principle combined with the unitarity cuts method is not sufficient to uniquely fix the amplitude already in the case of four external particles.

For $M = N$, our result (5.0.2) coincides with the one in [81] obtained by making the ansatz that dual conformal invariance should hold also at loop level. Therefore our calculation supports that ansatz and provides a direct proof of the assumption that dual conformal invariance should be the correct symmetry principle to select the scalar master integrals contributing to the on–shell sector of the theory.

Having computed the two–loop amplitude by a genuine perturbative approach without any *a priori* ansatz on its form, we can investigate whether dual conformal properties can be detected even at the level of Feynman diagrams. We have then studied the two loop diagrams entering our calculation, out of the mass–shell and in three dimensions. Since in three dimensions dual conformal symmetry rules out bubbles, it is immediate to realize that, being some of our diagrams built by bubbles, it cannot work at the level of the integrand on every single diagram. A less stringent scenario could still allow for the possibility to see dual conformal invariance realized at the level of the integrals and after summation of all the contributions. We have made many numerical checks but the output is always negative: dual conformal invariance is definitively broken at the level of Feynman diagrams and hence shadowed by the traditional method. It is only restored in the final on–shell answer. This is not in contrast with what claimed before, since in three dimensions and in dimensional regularization the integrals do not have in general a smooth limit on the mass–shell.

- As outlined in section 3.7, another wonderful feature of scattering amplitudes in $\mathcal{N} = 4$ SYM in four dimensions is their underlying *iterative structure*, uncovered in [56]. The result (5.0.2) strikingly resembles its four dimensional cousin, therefore one might conjecture to identify it with the first order expansion of a BDS–like exponentiation ansatz for the ABJ(M) model

$$\mathcal{M}_4 = e^{Div + \frac{f_{CS}(\lambda,\hat{\lambda})}{8}\left(\log^2\left(\frac{s}{t}\right) + \frac{4\pi^2}{3}\right) + C(\lambda,\hat{\lambda})} \tag{5.0.5}$$

where $C(\lambda, \hat{\lambda})$ is a scheme–dependent constant. The form of the ansatz is exactly the same as in the four dimensional case in which the non–trivial part at first order is completely encoded in the scaling function f, where here the four dimensional scaling function of $\mathcal{N} = 4$ SYM has been replaced by the three dimensional one, $f_{CS}(\lambda, \hat{\lambda})$. One might object that this ansatz is quite meaningless because any function may be regarded as the first term in an expansion series. However a non–trivial indication supporting the ansatz is provided by the fact that the correct value for $f_{CS}(\lambda, \hat{\lambda})$ matching our result, coincides exactly with that obtained in a totally independent setting, from the conjectured asymptotic Bethe equations of ABJM

[89]. Moreover, due to the similarity to the four dimensional result and since the WL/amplitude duality seems to be working in three dimensions as well for four particles (at least at first perturbative order), we expect our result to satisfy the same conformal Ward identities as the four dimensional amplitude, which are a key ingredient in explaining their exponentiation.

- In the $\mathcal{N}=4$ SYM case, the BDS exponentiation of scattering amplitudes occurs at *strong coupling* as well. In this regime, according to the AdS/CFT correspondence and owing to the Alday–Maldacena prescription [47], amplitudes evaluate to the exponential of a minimal–area surface in the AdS_5 dual background ending on a light–like polygon, whose edges are determined by the particle momenta. In particular, since this algorithm is equivalent to that for computing light–like Wilson loops in AdS, it provides evidence in favor of the amplitude/WL duality at strong coupling, in agreement with the findings at weak coupling.

  We investigate whether a similar prescription can be employed to compute scattering amplitudes at strong coupling in the ABJ(M) models. Motivated by our hints supporting the amplitudes/WL duality at weak coupling and BDS exponentiation, we expect it to be the case. Indeed, focusing on the ABJM theory in the intermediate regime $K \ll N \ll K^5$, where a string–theoretical dual description is available, we discuss the generalization of the Alday–Maldacena recipe to $AdS_4 \times \mathbb{CP}^3$. We find that, apart from a rigorous prescription for the regularization procedure in AdS that we have not developed properly, the five dimensional solution can be adapted to the four dimensional case and the output is an expression for the four–point amplitude given by eq. (5.0.5) where the scaling function assumes its leading value at strong coupling, $f_{CS}(\lambda) \sim \sqrt{2\lambda}$.

As announced we start reviewing the computation of the Wilson loop, then we turn to amplitudes and after spelling out their explicit evaluation, we comment on the result analyzing in more detail the items outlined above.

5.1 Wilson loop at two loops.

The two–loop correction to the Wilson loop is derived perturbatively, by expanding the path–ordered exponential up to order four in the gauge field A $(\hat{A})^*$. We first derive its expression for the ABJM theory, setting the ranks of the gauge groups to be equal, then we generalize this result to the ABJ model. We focus on one of the gauge groups and afterwards consider the combination of the two.

The planar limit where the number of colors N is large is considered throughout

*We temporarily switch to Lorentzian signature in order to stick to literature, transforming back to the Euclidean in the end of this section.

the computation. This allows for a considerable simplification, suppressing subleading diagrams by a power of N^2. Therefore non–planar contributions will be neglected.

The different topologies arise as follows:

- The second order expansion yields diagrams in Fig. 5.1$(c - f)$, depending on the relative positions of the insertion points, and contribute to two–loops when considering the one–loop corrected effective propagator. Diagrams where the insertion points lie on the same edge again vanish as in the one–loop computation and have been already neglected. In contrast to one–loop level, when the corrected gluon propagator joins two adjacent edges (Fig. 5.1(f)), the result does not vanish. We have shown the matter loop contribution only, since gluon and ghost corrections exactly cancel in DRED regularization scheme employed here. This implies that diagrams (c) and (d) actually cancel.

- The third order expansion produces diagram Fig. 5.1(b), after contraction with an internal cubic gauge interaction vertex.

- The fourth order expansion leads to the topology drawn in Fig. 5.1(a). This is the only one since corrections between two adjacent edges vanish as already ascertained at one loop, and all other ways of contracting gauge fields are subleading in color.

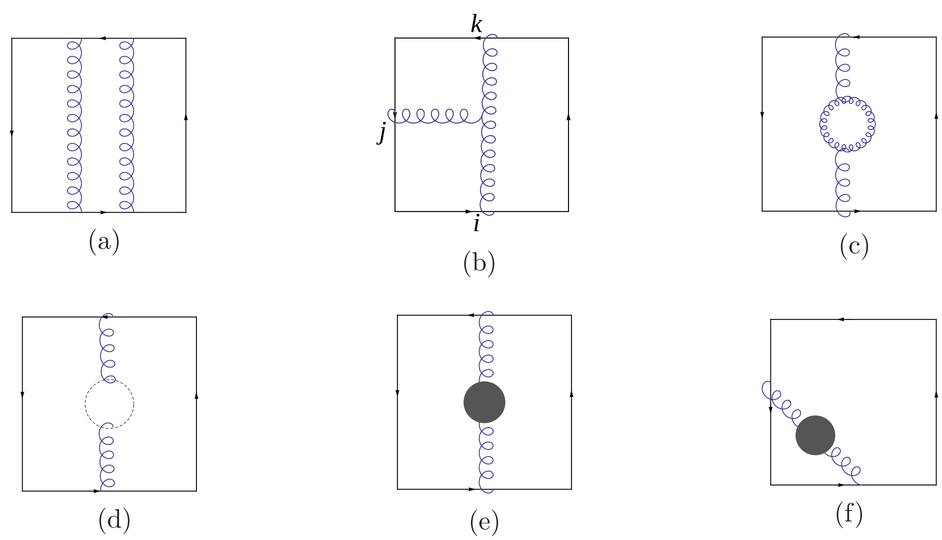


Figure 5.1: Two–loop contributions to the four cusps Wilson loop in CSM theories in the planar limit.

5.1.1 Pure Chern–Simons Wilson loop.

Let us analyze contributions from the pure CS sector first. We start computing the ladder diagram depicted in Fig. 5.1(a)

$$\langle W_4\rangle^{(2)}_{\text{ladder}} = \frac{1}{N}(i)^4 \oint_{z_i>z_j>z_k>z_l} dz^{\mu,\nu,\rho,\sigma}_{i,j,k,l}\,\langle\,\mathrm{Tr}A_\mu(z_i)\,A_\nu(z_j)\,A_\rho(z_k)\,A_\sigma(z_l)\,\rangle$$
$$= 2\left(\frac{N}{k}\right)^2\left(\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}}\right)^2 I_{\text{ladder}}(x_{13}^2, x_{24}^2) \tag{5.1.1}$$

in terms of the integral

$$I_{\text{ladder}}(x_{13}^2, x_{24}^2) = \int ds_{i,j,k,l}\,\frac{\epsilon(\dot z_i, \dot z_l, z_i - z_l)}{[-(z_i - z_l)^2]^{\frac{d}{2}}}\,\frac{\epsilon(\dot z_j, \dot z_k, z_j - z_k)}{[-(z_j - z_k)^2]^{\frac{d}{2}}} \tag{5.1.2}$$

For definiteness we choose e.g. $i = j = 3$, $k = l = 1$ in the numerator we get

$$\epsilon(\dot z_3, \dot z_1, z_3 - z_1)^2 = \epsilon(p_3, p_1, z_3 - z_1)^2 = \epsilon(x_{43}, x_{21}, x_{31})^2 = \epsilon(x_{12}, x_{23}, x_{34})^2 =$$
$$= -\frac{1}{4}\,x_{13}^2\,x_{24}^2\,(x_{13}^2 + x_{24}^2) \tag{5.1.3}$$

Since the integral is finite in $d = 3$ and we have a $(-1)^3$ sign from the invariants in the denominators we obtain

$$\frac{-1}{[x_{13}^2\bar s_i\bar s_l + x_{24}^2 s_i s_l]^{\frac{3}{2}}\,[x_{13}^2\bar s_j\bar s_k + x_{24}^2 s_j s_k]^{\frac{3}{2}}} \tag{5.1.4}$$

and the integral evaluates to

$$I_{\text{ladder}}(x_{13}^2, x_{24}^2) =$$
$$= \frac{1}{4}\int_0^1 ds_i \int_0^{s_i} ds_j \int_0^1 ds_k \int_0^{s_k} ds_l\,\frac{x_{13}^2 x_{24}^2 (x_{13}^2 + x_{24}^2)}{[x_{13}^2\bar s_i\bar s_l + x_{24}^2 s_i s_l]^{\frac{3}{2}}\,[x_{13}^2\bar s_j\bar s_k + x_{24}^2 s_j s_k]^{\frac{3}{2}}} + \mathcal{O}(\epsilon)$$
$$= \frac{1}{2}\left[\log\left(\frac{x_{13}^2}{x_{24}^2}\right) + \pi^2\right] + \mathcal{O}(\epsilon) \tag{5.1.5}$$

We then come to the vertex diagram Fig. 5.1(b), which turns out to be the most involved. It gives

$$\langle W_4\rangle^{(2)}_{\text{vertex}} = \frac{1}{N}\,\langle\,(i)^3 \oint_{z_i>z_j>z_k} dz^{\mu,\nu,\rho}_{i,j,k}\,\mathrm{Tr}\,(A_\mu\,A_\nu\,A_\rho)\left(i\int d^dw\mathcal{L}_{\text{int}}(w)\right)\rangle$$
$$= -\frac{1}{N}\,\frac{k}{4\pi}\,\frac{2}{3}\,(i)^5 \oint dz^{\mu,\nu,\rho}_{i,j,k}\int d^dw\langle\mathrm{Tr}\,(A_\mu\,A_\nu\,A_\rho)\,\mathrm{Tr}\,(A_\alpha\,A_\beta\,A_\gamma(w))\,\epsilon^{\alpha\beta\gamma}\rangle$$

We give a few details on its evaluation in Appendix A.6. Basically, after performing contractions and retaining the leading color pieces only, the diagram gives a sum over integrals I_{ijk}

$$\langle W_4 \rangle^{(2)}_{\text{vertex}} = i \left(\frac{N}{k} \right)^2 \frac{1}{2\pi} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}} \right)^3 \sum_{i>j>k} I_{ijk} \tag{5.1.6}$$

where

$$I_{ijk} = \int dz_i^\mu \, dz_j^\nu \, dz_k^\rho \, \epsilon^{\alpha\beta\gamma} \, \epsilon_{\mu\alpha\sigma} \, \epsilon_{\nu\beta\lambda} \, \epsilon_{\rho\gamma\tau} \int d^3w \, \frac{(w - z_i)^\sigma \, (w - z_j)^\lambda \, (w - z_k)^\tau}{(-(z_i - w)^2)^{\frac{d}{2}} \, (-(z_j - w)^2)^{\frac{d}{2}} \, (-(z_k - w)^2)^{\frac{d}{2}}} \tag{5.1.7}$$

Choosing for definiteness $i = 3$, $j = 2$ and $k = 1$, a lengthy calculation spelled out in the Appendix shows that it finally reduces to the following integral over the affine parameters

$$I_{321} = i \, \frac{8\,\pi^{\frac{d}{2}}}{\Gamma\left(\frac{1}{2}\,d\right)^3} \, x_{13}^2 x_{42}^2 \int_0^1 ds_{1,2,3} \, (\beta_1 \, \beta_2 \, \beta_3)^{\frac{1}{2}\,d-1} \left(\frac{\Gamma\,(d-1)}{\Delta^{d-1}} - 2\,\beta_1\,\bar{s}_1\,\beta_3\,s_3\,(x_{13}^2 + x_{24}^2)\,\Gamma\,(d)\,\frac{1}{\Delta^d} \right) \tag{5.1.8}$$

where $\Delta = -\beta_1\,\beta_2\,z_{12}^2 - \beta_2\,\beta_3\,z_{23}^2 - \beta_1\,\beta_3\,z_{13}^2$. This integral is quite nasty and was solved in [80] by resorting to numerical evaluation, whose result is given in the Appendix.

In the dimensional reduction scheme and Landau gauge where the computation is performed, the contributions in the effective one–loop propagator, coming from a loop of gauge vectors and ghosts (configurations 5.1$(c - d)$) exactly cancel. Then the two diagrams evaluated suffice to determine the two–loop Wilson loop in pure CS

$$\langle W_4 \rangle^{(2)}_{CS} = -\frac{1}{K^2} \left[\frac{1}{2} \log 2 \, \frac{(x_{13}^2\,\pi e^{\gamma_E} \mu^2)^{2\epsilon} + (x_{24}^2\,\pi e^{\gamma_E} \mu^2)^{2\epsilon}}{\epsilon} + \frac{1}{4}(a_6 - 8\log 2 - \pi^2) \right] \tag{5.1.9}$$

a_6 being a constant determined numerically (see Ref. [80])

5.1.2 ABJM Wilson loop.

The remaining contribution is enclosed in diagrams 5.1$(e - f)$ In particular, focusing on the matter contribution, we can insert the one–loop corrected gauge propagator [80] into the second order expansion of the Wilson loop exponential and get

$$G^{(1)}_{\mu\nu}(x) = \left(\frac{2\pi}{K} \right)^2 \frac{N\delta^I_I}{8} \frac{\Gamma(1-\frac{d}{2})\Gamma(\frac{d}{2})^2}{\Gamma(d-1)\,\pi^d} \left(\frac{\Gamma(d-2)}{\Gamma(2-\frac{d}{2})} \frac{\eta_{\mu\nu}}{(-x^2)^{d-2}} - \frac{\Gamma(d-3)}{4\,\Gamma(3-\frac{d}{2})} \partial_\mu \partial_\nu \, \frac{1}{(-x^2)^{d-3}} \right) \tag{5.1.10}$$

neglecting the derivative piece, which is possible in our gauge choice this simplifies

$$G^{(1)}_{\mu\nu}(x) = -\frac{1}{N}\left(\frac{N}{k}\right)^2 \pi^{2-d}\Gamma\left(\frac{d}{2}-1\right)^2 \frac{\eta_{\mu\nu}}{(-x^2)^{d-2}} \tag{5.1.11}$$

The contribution from the matter sector is obtained by summing over permutations of the endpoints of the gauge effective propagator

$$\begin{aligned} \langle W_4\rangle^{(2)}_{\text{matter}} &= \frac{i^2}{N}\text{Tr}\int_{z_i>z_j} dz_i^\mu dz_j^\nu \langle A_\mu A_\nu\rangle^{(1)} \\ &= -N\sum_{i>j}\int ds_i ds_j p_i^\mu p_j^\nu G^{(1)}_{\mu\nu}(z_i-z_j) \\ &= \left(\frac{N}{k}\right)^2\left((4\pi e^{\gamma_E})^{2\epsilon}+\frac{\pi^2}{2}\epsilon^2+\mathcal{O}(\epsilon^3)\right)\sum_{i>j} I_{ij} \end{aligned} \tag{5.1.12}$$

where $I_{ij} = \int ds_i ds_j\, p_i\cdot p_j(-(z_i-z_j)^2)^{2-d}$. Setting e.g. $i=2, j=1$, we have

$$\begin{aligned} I_{21} &= \int_0^1 ds_2 ds_1 \frac{p_2\cdot p_1}{(-(z_2-z_1)^2)^{d-2}} = \\ &= \int_0^1 ds_2 ds_1 \frac{x_{32}\cdot x_{21}}{(-(z_2-z_1)^2)^{d-2}} = \\ &= \int_0^1 ds_2 ds_1 \frac{\frac{1}{2}\, x_{13}^2}{(-(z_2-z_1)^2)^{d-2}} = \\ &= \int_0^1 ds_2 ds_1 \frac{\frac{1}{2}\, x_{13}^2}{(-x_{13}^2\,\bar{s}_1 s_2)^{d-2}} = \\ &- \frac{1}{2}(-x_{13}^2)^{3-d}\int_0^1 ds_2 ds_1 \frac{1}{(\bar{s}_1 s_2)^{1-2\epsilon}} = \frac{1}{8}\frac{(-x_{13}^2)^{2\epsilon}}{\epsilon^2} \end{aligned} \tag{5.1.13}$$

which is UV divergent.
Furthermore, there are two finite diagrams $I_{i+2,i}$. Setting again $d=3$ and taking $i=3, j=1$ the integral yields

$$\begin{aligned} I_{31} &= \int_0^1 ds_3 ds_1 \frac{p_3\cdot p_1}{-(z_3-z_1)^2} = \int_0^1 ds_3 ds_1 \frac{x_{43}\cdot x_{21}}{-(z_3-z_1)^2} \\ &= -\frac{1}{2}\int_0^1 ds_3 ds_1 \frac{x_{13}^2+x_{24}^2}{-(x_{13}^2\,\bar{s}_1\,\bar{s}_3 + x_{24}^2\,s_1\,s_3)} \\ &= \frac{1}{2}\int_0^1 ds_3 ds_1 \frac{x_{13}^2+x_{24}^2}{x_{13}^2\bar{s}_1\bar{s}_3 + x_{24}^2 s_1 s_3} \\ &= \frac{1}{4}\left[\log^2\left(\frac{x_{13}^2}{x_{24}^2}\right)+\pi^2\right]. \end{aligned} \tag{5.1.14}$$

Taking the sum over all contributions we obtain

$$\sum_{i>j} I_{ij} = -\frac{1}{4}\left[\frac{(-x_{13}^2)^{2\epsilon}}{\epsilon^2} + \frac{(-x_{24}^2)^{2\epsilon}}{\epsilon^2} - 2\log^2\left(\frac{x_{13}^2}{x_{24}^2}\right) - 2\pi^2\right], \tag{5.1.15}$$

and thus the full matter part reads

$$\begin{aligned}\langle W_4\rangle^{(2)}_{\text{matter}} &= -\frac{1}{4}\left(\frac{N}{k}\right)^2\left[\frac{(-x_{13}^2 4\pi e^{\gamma_E}\mu^2)^{2\epsilon}}{\epsilon^2} + \frac{(-x_{24}^2 4\pi e^{\gamma_E}\mu^2)^{2\epsilon}}{\epsilon^2}\right.\\ &\qquad\left. -2\log^2\left(\frac{x_{13}^2}{x_{24}^2}\right) - \pi^2 + \mathcal{O}(\epsilon)\right]\end{aligned} \tag{5.1.16}$$

where we have restored the regularization scale μ^2, which was dropped throughout the computation. The complete two–loop correction is obtained by adding the CS part (5.1.9)

$$\langle W_4\rangle^{(2)}_{\text{ABJM}} = N^2\left[\langle W_4\rangle^{(2)}_{CS} + \langle W_4\rangle^{(2)}_{matter}\right] \tag{5.1.17}$$

and the result can be rewritten in a form in which the ϵ^{-1} terms cancel

$$\langle W_4\rangle^{(2)}_{\text{ABJM}} = \lambda^2\left[-\frac{1}{2}\frac{(x_{13}^2\,\mu_{WL}^2)^{2\epsilon}}{(2\epsilon)^2} - \frac{1}{2}\frac{(x_{24}^2\,\mu_{WL}^2)^{2\epsilon}}{(2\epsilon)^2} + \frac{1}{4}\log^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \frac{1}{2}C\right] \tag{5.1.18}$$

where $\lambda \equiv N/K$, $\mu_{WL}^2 = 8\pi e^{\gamma_E}\mu^2$ and

$$C = 3\,\zeta_2 + 2\,\log 2 + 5\,\log^2 2 - \frac{a_6}{4} \tag{5.1.19}$$

Actually the numerical constant may be fitted by the following combination *const.* = $\frac{8}{3}\pi^2 + 12\log^2(2) - 8\log(2)$, where curiously a piece of lower transcendentality appears.

Finally we have to add to this contribution the one coming from the other gauge field $\hat{A}$. Since at two loops the coupling appears quadratically the $U(N)_{-K}$ piece is exactly equal to the previous one, in contrast to what happens at one loop. Hence the final answer can be read from (5.1.18), just doubling that expression

$$\langle W_4\rangle^{(2)}_{\text{ABJM}} = \lambda^2\left[-\frac{(x_{13}^2\,\mu_{WL}^2)^{2\epsilon}}{(2\epsilon)^2} - \frac{(x_{24}^2\,\mu_{WL}^2)^{2\epsilon}}{(2\epsilon)^2} + \frac{1}{2}\log^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + C\right] \tag{5.1.20}$$

Recently a generalization of this result was derived in [90], for light–like polygons with any arbitrary number n of edges, finding a remarkable similarity with Wilson loops at one loop in $\mathcal{N} = 4$ SYM. We will not account for this here, since our main interest is in the computation of the four–cusps Wilson loop, to match with scattering amplitudes.

In order to prepare for this comparison we turn to Euclidean signature, by just changing the signs of the invariants. This does not produce any sensible change, since only the ratio of the two Lorentz invariants in the problem appears.

5.1.3 ABJ Wilson loop.

Then we wish to extend the result (5.1.20) to the larger class of ABJ models, taking into account the possibility for the two gauge groups to have different ranks. At two loops, the contributing diagrams are the same as in the ABJM case, but with different color coefficients. It is convenient to introduce two 't Hooft couplings $\lambda = M/K$ and $\hat{\lambda} = N/K$. It follows that the most convenient perturbative parameter is $\bar{\lambda} = \sqrt{\lambda\hat{\lambda}}$, while

$$\sigma = \frac{\lambda - \hat{\lambda}}{\bar{\lambda}}, \tag{5.1.21}$$

measures the deviation from the ABJM theory.

As stated in Section 4.4, we have two different candidates for the ABJ Wilson loop at disposal, given in (4.4.3) and (4.4.5). We analyze the two–loop correction for both.

1. Following the definition (4.4.3),

$$\langle W_4 \rangle_{\text{ABJ}} = \frac{1}{M+N} \left\{ \text{Tr}_{U(M)} \, P \, e^{i \int_{C_n} A_\mu \, dz^\mu} + \text{Tr}_{U(N)} \, P \, e^{i \int_{C_n} \hat{A}_\mu \, dz^\mu} \right\} \tag{5.1.22}$$

which naturally arises when writing the gauge field as a $(M+N) \times (M+N)$ square matrix, $\mathcal{A} = \text{diag}(A, \hat{A})$, we just have to take the results from the Wilson loop in ABJM and change the relative coefficients of the pure gauge and the matter contributions. The former gets multiplied by a color factor M^3 and N^3, for the A and $\hat{A}$ pieces respectively. The color factors of latter are $M^2\, N$ and $N^2\, M$, respectively.

$$\begin{aligned} \langle W_4 \rangle^{(2)}_{\text{ABJ}} &= \frac{1}{M+N} \left\{ (M^3 + N^3) \, \langle W_4 \rangle^{(2)}_{CS} + (M^2\, N + N^2\, M) \, \langle W_4 \rangle^{(2)}_{matter} \right\} \\ &= (M^2 - MN + N^2) \, \langle W_4 \rangle^{(2)}_{CS} + MN \, \langle W_4 \rangle^{(2)}_{matter} \end{aligned} \tag{5.1.23}$$

where $\langle W_4 \rangle^{(2)}_{CS}$ and $\langle W_4 \rangle^{(2)}_{matter}$ are still given in eqs. (5.1.9) and (5.1.16), respectively. Inserting their explicit expressions and rescaling the regularization parameter as

$$\mu''^2_{WL} = 2^{3+\sigma^2} \, \pi \, e^{\gamma_E} \, \mu^2 \tag{5.1.24}$$

up to terms of order ϵ, the final answer reads

$$\langle W_4 \rangle^{(2)}_{\text{ABJ}} = \bar{\lambda}^2 \Big\{ -\frac{(x_{13}^2 \, \mu''^2_{WL})^{2\epsilon}}{(2\epsilon)^2} - \frac{(x_{24}^2 \, \mu''^2_{WL})^{2\epsilon}}{(2\epsilon)^2} + \frac{1}{2} \log^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + C'' \Big\} \tag{5.1.25}$$

where

$$C'' = \frac{3}{2} \, (\sigma^2 + 2) \, \zeta_2 + 2(\sigma^2 + 1) \log 2 - \frac{\sigma^2 + 1}{4} \, a_6 + (\sigma^2 + 1)(\sigma^2 + 5) \, \log^2 2 \tag{5.1.26}$$

It is straightforward to check that taking $\sigma \to 0$, the above expression reduces to the result (5.1.20) for the Wilson loop in ABJM, rescaling (5.1.29) and constant (5.1.31) included.

2. We take under exam the second possible candidate for the Wilson loop in ABJ theory whose expression we recall (4.4.5)

$$\langle W_4 \rangle^{(2)}_{\text{ABJ}} = \frac{1}{2M}\,\text{Tr}_{U(M)}\, P\, e^{i\int_{C_n} A_\mu\, dz^\mu} + \frac{1}{2N}\,\text{Tr}_{U(N)}\, P\, e^{i\int_{C_n} \hat{A}_\mu\, dz^\mu} \tag{5.1.27}$$

At two loops, it is distinguished from the previous one by a slightly different combination of the expressions (5.1.9, 5.1.16)

$$\langle W_4 \rangle^{(2)}_{\text{ABJ}} = \frac{1}{2}\,(M^2+N^2)\,\langle W_4\rangle^{(2)}_{CS} + MN\,\langle W_4\rangle^{(2)}_{matter} \tag{5.1.28}$$

However, provided that we define

$$\mu'^2_{WL} = 2^{3+\sigma^2/2}\,\pi\, e^{\gamma_E}\,\mu^2 \tag{5.1.29}$$

the calculation leads exactly to the same result as before

$$\langle W_4 \rangle^{(2)}_{\text{ABJ}} = \bar{\lambda}^2 \Big\{ -\frac{\left(x^2_{13}\,\mu'^2_{WL}\right)^{2\epsilon}}{(2\epsilon)^2} - \frac{\left(x^2_{24}\,\mu'^2_{WL}\right)^{2\epsilon}}{(2\epsilon)^2} + \frac{1}{2}\log^2\left(\frac{x^2_{13}}{x^2_{24}}\right) + C' \Big\} \tag{5.1.30}$$

where now

$$C' = \frac{3}{4}\,(\sigma^2+4)\,\zeta_2 + (\sigma^2+2)\log 2 - \frac{\sigma^2+2}{8}\,a_6 + \frac{(\sigma^2+2)(\sigma^2+10)}{4}\,\log^2 2 \tag{5.1.31}$$

Again, taking $\sigma \to 0$ we are back to the ABJM result (5.1.20).

Up to two loops, the two definitions (4.4.3) and (4.4.5) for the Wilson loop in ABJ theory differ only by the choice of the mass scale and the scheme–dependent C constants. At this stage we do not have any tool to discriminate between the two.

The above result suggests that one should identify $\frac{\sqrt{MN}}{k}$ as the effective 't Hooft coupling for ABJM, at least at two loops. At weak coupling it may be reasonable, at strong coupling it would sound strange since the 't Hooft coupling is read from the AdS_4 radius, which only depends on N in the ABJ solution. However in the same paper it is argued that supersymmetric $U(M)_k \times U(N)_{-k}$ CS is well defined only if $|M-N| < k$. At weak coupling, where $k \gg N, M$ all values of M and N are allowed, however at strong coupling, where $k \ll N, M$ it seems that the difference $|M-N|$ should be very small compared to N, so that

$$\frac{\frac{N}{k} - \frac{\sqrt{MN}}{k}}{\frac{N}{k}} \sim \frac{M-N}{2N} < \frac{1}{2}\,\frac{k}{N} \ll 1 \tag{5.1.32}$$

meaning that the 't Hooft couplings $\frac{N}{k}$ and $\frac{\sqrt{MN}}{k}$ are basically equivalent (or, said another way, the ABJ solution is considered as far as the probe limit of $|M-N|$ small holds).

This feature that the dependence on the parity violating parameter is quite trivial is a peculiarity due to the low perturbative order. A significant contribution may come at four loops and higher, and is on general grounds expected to break parity invariance under the exchange $M \leftrightarrow N$. However, as far as we are concerned, all perturbative computation has always displayed a dependence on σ in even powers, preserving parity invariance. We will come back to this point later on.

In the next Section we turn to the subject of scattering and, after deriving a closed result for the four–point amplitude at two loops, we make the desired comparison with the Wilson loop.

5.2 Scattering at two loops

We restrict to the ABJ model for which we have found that all the four–point amplitudes vanish at one loop. This result is consistent with the one–loop vanishing of the Wilson loop and leads to conjecture the existence of a WL/amplitudes duality for this theory. To give evidence to this conjecture, we evaluate four–point scattering amplitudes at two loops.

We study amplitudes of the type $(A^i\, B_j\, A^k\, B_l)$, where the external A, B particles carry outgoing momenta $p_1, \ldots, p_4$ $(p_i^2 = 0)$.

At two loops, in the planar sector, the amplitude can be read from the single trace part of the two–loop effective superpotential

$$\Gamma^{(2)}[A,B] = \int d^2\theta d^3p_1 \ldots d^3p_4\, (2\pi)^3\, \delta^{(3)}(\textstyle\sum_i p_i) \times$$
$$\frac{2\pi}{K}\epsilon_{ik}\epsilon^{jl}\, \mathrm{tr}\left(A^i(p_1)B_j(p_2)A^k(p_3)B_l(p_4)\right)\sum_{X=a}^{g}\mathcal{M}^{(X)}(p_1,\ldots,p_4) \tag{5.2.1}$$

where the sum runs over the six 1PI diagrams in Fig. 5.2, plus the contribution from the 1P–reducible (1PR) graph in Fig. 5.2(g) where the bubble indicates the two–loop correction to the chiral propagator.

In (5.2.1) we have factorized the tree level expression, so that $\mathcal{M}^{(X)}(p_1,\ldots,p_4)$ are contributions to $\mathcal{A}_4^{(2)}/\mathcal{A}_4^{tree}$.

In order to evaluate the diagrams we fix the convention for the upper–left leg to carry outgoing momentum p_1 and name the other legs counterclockwise. The momentum–dependent contributions in (5.2.1) are the product of a combinatorial factor times a sum of ordinary Feynman momentum integrals arising after performing D–algebra on each supergraph (more details can be found in [86]). There are a total of four diagrams of the classes (b), (c), (f) and (g), eight diagrams of the classes (d) and (e) and two diagrams

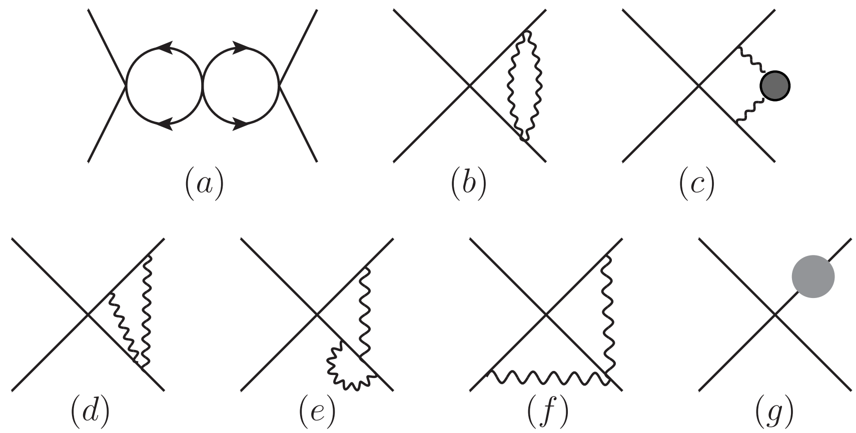


Figure 5.2: Diagrams contributing to the two–loop four–point scattering amplitude. The dark–gray blob represents one–loop corrections and the light–gray blob two–loop ones.

of the class (a). The color/flavor factors $\mathcal{C}_i$ for them are given by

$$\begin{array}{lll} \mathcal{C}_a = (4\pi)^2\,\frac{\lambda^2+\hat\lambda^2}{2} & \mathcal{C}_b = (4\pi)^2\,\frac{\lambda^2+\hat\lambda^2}{8} & \mathcal{C}_c = (4\pi)^2\,\frac{8\lambda\hat\lambda-\lambda^2-\hat\lambda^2}{2} \\ \mathcal{C}_d = (4\pi)^2\,\frac{\lambda^2+\hat\lambda^2}{4} & \mathcal{C}_e = (4\pi)^2\,\lambda\hat\lambda & \mathcal{C}_f = -(4\pi)^2\,\lambda\hat\lambda \end{array} \tag{5.2.2}$$

while diagram (g) contains subdiagrams with different color/flavor factors and these cannot be factorized.

Diagram 5.2(a) : We begin with the simplest graph which, after D–algebra, reduces to the following factorized Feynman integral

$$\mathcal{D}_a^s = \mu^{4\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{-(p_1+p_2)^2}{k^2\,(k+p_1+p_2)^2\,l^2\,(l-p_3-p_4)^2} \tag{5.2.3}$$

where μ is the mass scale of dimensional regularization.

The k and the l bubble integrals can be separately evaluated using the result (A.5.9), so obtaining

$$\mathcal{D}_a^s = -G[1,1]^2 \left(\frac{\mu^2}{s}\right)^{2\epsilon} \tag{5.2.4}$$

In order to determine the corresponding contribution to the amplitude, we need to sum over all the independent configurations of the external momenta. Inserting the corresponding color/flavor factors we obtain

$$\mathcal{M}^{(a)} = -8\pi^2(\lambda^2+\hat\lambda^2)G[1,1]^2 \left(\left(\frac{\mu^2}{s}\right)^{2\epsilon} + \left(\frac{\mu^2}{t}\right)^{2\epsilon} \right) = -\frac{3}{2}\zeta_2(\lambda^2+\hat\lambda^2) + \mathcal{O}(\epsilon) \tag{5.2.5}$$

Diagram 5.2(b) : After D–algebra, it gives

$$\mathcal{D}_b^{s_1} = \mu^{4\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{2(p_3+p_4)^2}{l^2\,(l+k)^2\,(k-p_4)^2\,(k+p_3)^2} \tag{5.2.6}$$

Performing the l integration with the help of Eq. (A.5.9) , we obtain a triangle integral with a modified exponent in one of its propagators

$$\mathcal{D}_b^{s_1} = \mu^{4\epsilon}\, G[1,1] \int \frac{d^dk}{(2\pi)^d} \frac{2(p_3+p_4)^2}{(k^2)^{1/2+\epsilon}\,(k-p_4)^2\,(k+p_3)^2} \tag{5.2.7}$$

We Feynman–parameterize the denominator and integrate over momentum k. Taking into account that we are working on–shell ($p_i^2=0$) we finally get

$$\begin{aligned} \mathcal{D}_b^{s_1} &= \frac{\mu^{4\epsilon}\, 2s\, G[1,1]\,\Gamma(1+2\epsilon)}{(4\pi)^{d/2}\Gamma(1/2+\epsilon)} \int_0^1 \frac{d\beta_1 d\beta_2 d\beta_3\, \delta(\sum_i \beta_i - 1)\beta_1^{-1/2+\epsilon}}{(\beta_1\beta_2 p_4^2 + \beta_1\beta_3 p_3^2 + \beta_2\beta_3 s)^{1+2\epsilon}} \\ &\xrightarrow{p_{3,4}^2\to 0} \frac{2G[1,1]\Gamma(1+2\epsilon)\Gamma^2(-2\epsilon)}{(4\pi)^{d/2}\Gamma(1/2-3\epsilon)} \left(\frac{\mu^2}{s}\right)^{2\epsilon} \end{aligned} \tag{5.2.8}$$

where $\Gamma^2(-2\epsilon)$ signals the presence of an on–shell IR divergence.

Summing over the four inequivalent configurations of the external legs multiplied by the correct vertex factors, the contribution to the amplitude reads

$$\begin{aligned} \mathcal{M}^{(b)} &= 8\pi^2(\lambda^2+\hat\lambda^2) \frac{G[1,1]\Gamma(1+2\epsilon)\Gamma^2(-2\epsilon)}{(4\pi)^{d/2}\Gamma(1/2-3\epsilon)} \left(\left(\frac{\mu^2}{s}\right)^{2\epsilon} + \left(\frac{\mu^2}{t}\right)^{2\epsilon} \right) \\ &= \frac{\lambda^2+\hat\lambda^2}{8} \left[\frac{1}{(2\epsilon)^2} \left(\frac{s}{\pi e^{-\gamma_E}\mu^2}\right)^{-2\epsilon} + \frac{1}{(2\epsilon)^2} \left(\frac{t}{\pi e^{-\gamma_E}\mu^2}\right)^{-2\epsilon} - \frac{5}{2}\,\zeta_2 + \mathcal{O}(\epsilon) \right] \end{aligned} \tag{5.2.9}$$

Diagram 5.2(c) : This diagram may result problematic, being infrared divergent even when evaluated off–shell. In fact, after D–algebra, the particular diagram drawn in Fig. 5.2(c) gives rise to the following integral

$$\mathcal{D}_c^{s_1} = \frac{\mu^{4\epsilon}}{2} \int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{Tr(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)\, p_3^\mu\, (k-p_3)^\nu\, (k+p_4)^\rho\, p_4^\sigma}{l^2\,(l+k)^2\,k^2\,(k-p_4)^2\,(k+p_3)^2} \tag{5.2.10}$$

which, performing the bubble l integral and using the identity

$$Tr(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)\, p_3^\mu\, (k-p_3)^\nu\, (k+p_4)^\rho\, p_4^\sigma = (p_3+p_4)^2k^2 - p_3^2(k+p_4)^2 - p_4^2(k-p_3)^2 \tag{5.2.11}$$

can be separated into three pieces

$$\mathcal{D}_c^{s_1} = \frac{1}{4}\,\mathcal{D}_b^{s_1} - \frac{1}{2}G[1,1]G[1,3/2+\epsilon]\,(p_3^2)^{-2\epsilon} - \frac{1}{2}G[1,1]G[1,3/2+\epsilon]\,(p_4^2)^{-2\epsilon} \tag{5.2.12}$$

While the first term is the off–shell well–behaving Feynman integral in Eq. (5.2.8) that in the on–shell limit produces $1/\epsilon$ poles, the second and third terms are badly divergent even off–shell. However, we can show that these unphysical divergences are cured when we add the 1PR diagrams corresponding to two–loop self–energy corrections to the superpotential, as depicted in Fig. 5.2(g).

In fact, for example the contribution from the diagram with the two–loop correction on the fourth leg as drawn in the picture, yields

$$\mathcal{D}_g^4 = -8\pi^2(8\lambda\hat{\lambda} - \lambda^2 - \hat{\lambda}^2)\, G[1,1]\, G[1,3/2+\epsilon]\,(p_4^2)^{-2\epsilon} + 32\pi^2\lambda\hat{\lambda}\; p_4^2\, \mathcal{B}(p_4)^2 \tag{5.2.13}$$

where color factors have been included.

It is easy to realize that the first term of this expression is off–shell infrared divergent, but precisely cancels the third term in (5.2.12) when all the vertex factors of diagram 5.2(c) are taken into account. On the other hand, the second term in (5.2.13) comes from a double factorized bubble which vanishes on–shell.

Since in a similar way the second term in (5.2.12) gets canceled by a diagram with a two–loop correction on the third leg, summing diagrams 5.2(c),(g) and their permutations we are finally led to the following interesting identity

$$\mathcal{M}^{(c)} + \mathcal{M}^{(g)} = \frac{\lambda^2 + \hat{\lambda}^2 - 8\lambda\hat{\lambda}}{\lambda^2 + \hat{\lambda}^2}\, \mathcal{M}^{(b)} \tag{5.2.14}$$

Diagram 5.2(d) : Diagrams of type (d) may be calculated using Mellin-Barnes techniques. Specifically, after D–algebra the diagram in figure gives rise to the integral

$$\mathcal{D}_d^{s1} = \mu^{4\epsilon} \int \frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\frac{Tr(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)\, p_4^\mu\,(p_3+p_4)^\nu\,(k+p_4)^\rho\,(l-p_4)^\sigma}{(k+p_4)^2\,(k-p_3)^2\,(k+l)^2\,(l-p_4)^2\,l^2} \tag{5.2.15}$$

Using the identity (A.5.6) and the on–shell conditions, it can be rewritten as

$$\begin{aligned}\mathcal{D}_d^{s1} =& \frac{-s\,\Gamma(1/2-\epsilon)}{(4\pi)^{d/2}\Gamma(1-2\epsilon)} \int_{-i\infty}^{i\infty} \frac{d\mathbf{z}}{2\pi i}\Gamma(-\mathbf{z}|1+\mathbf{z}|3/2+\epsilon+\mathbf{z}|-1/2-\epsilon-\mathbf{z}) \\ &\times\ \mu^{4\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^{3/2+\epsilon+\mathbf{z}}\,[(k+p_4)^2]^{-\mathbf{z}}\,(k-p_3)^2}\end{aligned} \tag{5.2.16}$$

The integral over k can be easily performed by Feynman parameterization, leading to

$$\begin{aligned}\mathcal{D}_d^{s1} &= -\mu^{4\epsilon}\,\frac{\Gamma(1/2-\epsilon|1+2\epsilon|-2\epsilon)}{(4\pi)^d\Gamma(1-2\epsilon|1/2-3\epsilon)} \\ &\quad\times \int_{-i\infty}^{i\infty}\frac{d\mathbf{z}}{2\pi i}\,\Gamma(1+\mathbf{z}|3/2+\epsilon+\mathbf{z}|-1/2+\epsilon-\mathbf{z}|-1-2\epsilon-\mathbf{z}) \\ &= -\frac{\Gamma^3(1/2-\epsilon)\Gamma(1+2\epsilon)\Gamma^2(-2\epsilon)}{(4\pi)^d\Gamma^2(1-2\epsilon)\Gamma(1/2-3\epsilon)\,(s/\mu^2)^{2\epsilon}}\end{aligned} \tag{5.2.17}$$

where the remaining integral over the complex variable $\mathbf{z}$ has been performed by using the Barnes first Lemma (A.5.7).

Taking into account the eight permutations with corresponding flavor/color factors we obtain

$$\mathcal{M}^{(d)} = -16\pi^2(\lambda^2+\hat{\lambda}^2)\frac{\Gamma^3(1/2-\epsilon)\Gamma(1+2\epsilon)\Gamma^2(-2\epsilon)}{(4\pi)^d\Gamma^2(1-2\epsilon)\Gamma(1/2-3\epsilon)}\left(\left(\frac{\mu^2}{s}\right)^{2\epsilon}+\left(\frac{\mu^2}{t}\right)^{2\epsilon}\right)$$
$$= \frac{\lambda^2+\hat{\lambda}^2}{4}\left[-\frac{1}{(2\epsilon)^2}\left(\frac{s}{4\pi e^{-\gamma_E}\mu^2}\right)^{-2\epsilon}-\frac{1}{(2\epsilon)^2}\left(\frac{t}{4\pi e^{-\gamma_E}\mu^2}\right)^{-2\epsilon}+\frac{7}{2}\zeta_2+\mathcal{O}(\epsilon)\right] \quad (5.2.18)$$

Diagram 5.2(e) : Using the identities derived in [86] it is possible to write this diagram as a combination of contributions (b) and (d). It holds that

$$(1+\mathcal{S}_{34})\mathcal{D}_e^{s1} = (1+\mathcal{S}_{34})\mathcal{D}_d^{s1}+\mathcal{D}_b^{s1}-p_3^2\mathcal{B}(p_3)^2-p_4^2\mathcal{B}(p_4)^2 \quad (5.2.19)$$

where $\mathcal{D}_e^{s1}$ is the particular diagram of type (e) drawn in the figure and the operator $(1+\mathcal{S}_{34})$ symmetrizes the diagram with respect to the third and fourth leg. Notice the presence of a double factorized bubble which can be dropped on–shell. Accounting for all permutations and flavor/color factors we thus find that

$$\mathcal{M}^{(e)} = 4\frac{\lambda\hat{\lambda}}{\lambda^2+\hat{\lambda}^2}\left(\mathcal{M}^{(d)}+2\,\mathcal{M}^{(b)}\right) \quad (5.2.20)$$

Diagram 5.2(f) : The most complicated contribution comes from this diagram, as it involves a non–trivial function of the s/t ratio. Surprisingly, after some cancelations it turns out to be finite.

The D–algebra for the specific choice of external momenta as in figure results in the Feynman integral

$$\mathcal{D}_f^{234} = \mu^{4\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\frac{-Tr(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)\,p_4^\mu\,p_2^\nu\,k^\rho\,l^\sigma}{k^2\,(k-p_2)^2\,(k+l+p_3)^2\,(l-p_4)^2\,l^2} \quad (5.2.21)$$

Again, using Eq. (A.5.6) for the k integral and working on-shell, we obtain

$$\mathcal{D}_f^{234} = \frac{\Gamma(1/2-\epsilon)}{(4\pi)^d\Gamma(1-2\epsilon)}\int_{-i\infty}^{i\infty}\frac{d\mathbf{z}}{2\pi i}\Gamma(-\mathbf{z})\Gamma(1+\mathbf{z})\Gamma(3/2+\epsilon+\mathbf{z})\Gamma(-1/2-\epsilon-\mathbf{z})$$
$$\times\,\mu^{4\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{Tr(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)\,p_4^\mu\,p_2^\nu\,(l+p_3)^\rho\,l^\sigma}{l^2\,(l-p_4)^2\,[(l+p_3)^2]^{-\mathbf{z}}\,[(l+p_2+p_3)^2]^{3/2+\epsilon+\mathbf{z}}} \quad (5.2.22)$$

It is convenient to separate the l integral in two pieces by using the on–shell identity

$$Tr(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)\, p_4^\mu\, p_2^\nu\, (l+p_3)^\rho\, l^\sigma\Big|_{\text{on-shell}} = -(s+t)\, l^2 + Tr(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)\, p_4^\mu\, p_2^\nu\, p_3^\rho\, l^\sigma \tag{5.2.23}$$

The first piece in (5.2.23) contains an l^2 factor which cancels the l^2 propagator in (5.2.22) leading to a simple triangle which is straightforwardly evaluated as we did for the previous diagram.

The second piece is a vector–box integral which after Feynman parameterization can be written in terms of a second 1–fold Mellin–Barnes integral. Interchanging the order of the two Mellin–Barnes integrals and solving for the original one in (5.2.22) with the first Barnes lemma (A.5.7), the total result is

$$\mathcal{D}_f^{234} = \frac{(s+t)\Gamma^3(1/2-\epsilon)\mu^{4\epsilon}}{(4\pi)^d\Gamma(1/2-3\epsilon)\Gamma^2(1-2\epsilon)}\Bigg[-\frac{\Gamma(1+2\epsilon)\Gamma^2(-2\epsilon)}{s^{1+2\epsilon}} +$$

$$+\frac{1}{t^{1+2\epsilon}}\int_{-i\infty}^{i\infty}\frac{d\mathbf{v}}{2\pi i}\Gamma(-\mathbf{v})\Gamma(-2\epsilon-\mathbf{v})\Gamma(-1-2\epsilon-\mathbf{v})\Gamma^2(1+\mathbf{v})\Gamma(2+2\epsilon+\mathbf{v})\left(\frac{s}{t}\right)^{\mathbf{v}}\Bigg] \tag{5.2.24}$$

The contour of the Mellin–Barnes integral in the second term of the last expression is not well–defined in the limit $\epsilon \to 0$, reflecting the presence of poles in ϵ. The reason is that in this limit the first pole of $\Gamma(-1-2\epsilon-\mathbf{v})$ collapses with the first pole of $\Gamma^2(1+\mathbf{v})$. In order to have a well defined contour in the $\epsilon \to 0$ limit, we can deform the contour so that it passes on the right of the point $\mathbf{v} = -1-2\epsilon$ and include the residue of the integrand in this point. Surprisingly, it turns out that this residue exactly cancels the first term in (5.2.24) so that we obtain a simple 1–fold Mellin-Barnes integral which is finite in the limit $\epsilon \to 0$

$$\mathcal{D}_f^{234} = \frac{(1+s/t)\Gamma^3(1/2-\epsilon)}{(4\pi)^d\Gamma^2(1-2\epsilon)\Gamma(1/2-3\epsilon)(t/\mu^2)^{2\epsilon}} \tag{5.2.25}$$

$$\times\int_{-i\infty}^{+i\infty}\frac{d\mathbf{v}}{2\pi i}\Gamma(-\mathbf{v})\Gamma(-2\epsilon-\mathbf{v})\Gamma^*(-1-2\epsilon-\mathbf{v})\Gamma^2(1+\mathbf{v})\Gamma(2+2\epsilon+\mathbf{v})\left(\frac{s}{t}\right)^{\mathbf{v}} \tag{5.2.26}$$

This integral can be calculated in the $\epsilon \to 0$ limit by closing the contour and performing the infinite sum of all the residues inside it. Taking into account all four permutations of the diagram and flavor/color factors we finally obtain

$$\mathcal{M}^{(f)} = \lambda\hat{\lambda}\left(\tfrac{1}{2}\log^2(s/t) + 3\zeta_2\right) + \mathcal{O}(\epsilon) \tag{5.2.27}$$

We are now ready to collect the partial results (5.2.5, 5.2.9, 5.2.14, 5.2.18, 5.2.20, 5.2.27) and find the four–point chiral superamplitude at two loops. After some algebra, and redefining the mass scale as

$$\mu'^2 = 2^{\sigma^2/2}\,(8\pi e^{-\gamma_E}\,\mu^2) \tag{5.2.28}$$

the result can be cast into the following compact form

$$\boxed{\mathcal{M}^{(2)} \equiv \frac{\mathcal{A}_4^{(2\,loops)}}{\mathcal{A}_4^{tree}} = \bar{\lambda}^2 \left[-\frac{(s/\mu'^2)^{-2\epsilon}}{(2\,\epsilon)^2} - \frac{(t/\mu'^2)^{-2\epsilon}}{(2\,\epsilon)^2} + \frac{1}{2} \log^2\left(\frac{s}{t}\right) + C_{\mathcal{A}} + \mathcal{O}(\epsilon) \right]} \quad (5.2.29)$$

where $\bar{\lambda} = \sqrt{\lambda\hat{\lambda}} = \sqrt{MN}/k$ and $C_{\mathcal{A}}$ is a constant given by

$$C_{\mathcal{A}} = \left(4 - \tfrac{5}{4}\sigma^2\right)\zeta_2 + \left(1 + \tfrac{1}{2}\sigma^2\right)\left(3 + \tfrac{1}{2}\sigma^2\right)\log^2 2 \quad (5.2.30)$$

We note that the mass scale and the constant depend non–trivially on the parity–violating parameter σ defined in eq. (5.1.21). Since it only appears as a square, parity is not violated at this stage. As a check, we observe that for $\sigma = 0$ the result reduces to the ABJM amplitudes computed in [82].

5.3 Discussion

We now discuss the main properties of our result (5.3.9) for the four–point amplitude at two loops.

First of all, in the ABJM case ($\sigma = 0$) the result coincides with the one in [81] obtained by applying generalized unitarity methods. In particular, the effective mass scale is the same and the analytical expression for the constant $C_{\mathcal{A}}|_{\sigma\to 0} = 4\zeta_2 + 3\log^2 2$ matches the numerical result of [81].

In Ref. [81] the result has been found by assuming *a priori* that in the planar limit dual conformal invariance should work also at quantum level. In fact, an ansatz has been made on the general structure of the amplitude which turns out to be a linear combination of integrals that, if extended off–shell, are well defined in three dimensions and exhibit conformal invariance in the dual x–space ($p_i = x_i - x_{i+1}$).

On the other hand, our calculation relies on a standard Feynman diagram approach which does not make use of any assumption. The identification of the two results is then a remarkable proof of the validity of on–shell dual conformal invariance for this kind of theories.

5.3.1 Amplitudes/WL duality

In the general ABJ case, if we write the Mandelstam variables in terms of the dual ones, $s = x_{13}^2$ and $t = x_{24}^2$, up to a (scheme–dependent) constant our result matches those in Eqs. (5.1.30, 5.1.25) for the two–loop expansion of a light–like Wilson loop, once we formally identify the IR and UV rescaled regulators of the scattering amplitude and the Wilson loops as $\mu'^2 = 1/\mu'^2_{WL}$ or $\mu'^2 = 1/\mu''^2_{WL}$.

Since the Wilson loop is conformally invariant in the ordinary configuration space, the identification of the two–loop amplitude with the corresponding term in the WL expansion is a further proof of dual conformal invariance in the on–shell sector of the theory.

We remind that the two results (5.1.30, 5.1.25) in Section 3 correspond to two possible definitions of Wilson loop in ABJ models. At this stage, the result (5.3.9) seems to match both. However, if we compare the rescaled mass regulators, we see that apart from the sign of the Euler constant, in the result for the amplitude μ' looks like μ'_{WL} in eq. (5.1.29), whereas it is quite different from μ''_{WL} in Eq. (5.1.24). Although there is no particular reason for the two mass scales to match exactly, this might be a first indication that the definition (4.4.3) for the light–like Wilson loop dual to scattering amplitudes is preferable.

5.3.2 Dual conformal invariance

As for the $\mathcal{N} = 4$ SYM case, in the ABJ models the two–loop on–shell amplitude divided by its tree–level contribution, when written in terms of dual variables has the same functional structure as the second order expansion of a light–like Wilson loop. Wilson loops are invariant under the transformations of the standard conformal group of the ABJ theory, even though UV divergences break this symmetry anomalously. Hence, the on–shell amplitude should inherit this symmetry, possibly anomalously broken by IR divergences.

In fact, in the $\mathcal{N} = 4$ SYM case where the amplitudes/WL duality also works, the perturbative results for planar MHV scattering amplitudes divided by their tree–level contribution can be expressed as linear combinations of scalar integrals that are off–shell finite and dual conformally invariant [91, 43] in four dimensions. Precisely, once written in terms of dual variables, the integrands times the measure are invariant under translations, rotations, dilatations and special conformal transformations. Dual conformal invariance is broken on–shell by IR divergences that require introducing a mass regulator. Therefore, conformal Ward identities acquire an anomalous contribution [61].

A natural consequence of our findings is that the two–loop result for three dimensional ABJ(M) models should also exhibit dual conformal invariance, and then it should be possible to rewrite the final expression (5.3.9) for the on–shell amplitude as a linear combination of scalar integrals which are off–shell finite in three dimensions and manifestly dual conformally invariant at the level of the integrands. Indeed, for the ABJM case this has been proved in Ref. [81] where the amplitude has been obtained by the generalized unitarity cuts method, based on the ansatz for the amplitude to be dual conformally invariant.

Following [81], we introduce a set of independent scalar integrals $I_{1s}, I_{2s}, I_{3s}, I_{5s} \equiv I_{1s} - I_{4s}$ which correspond to the following off–shell, three dimensional, dual conformally

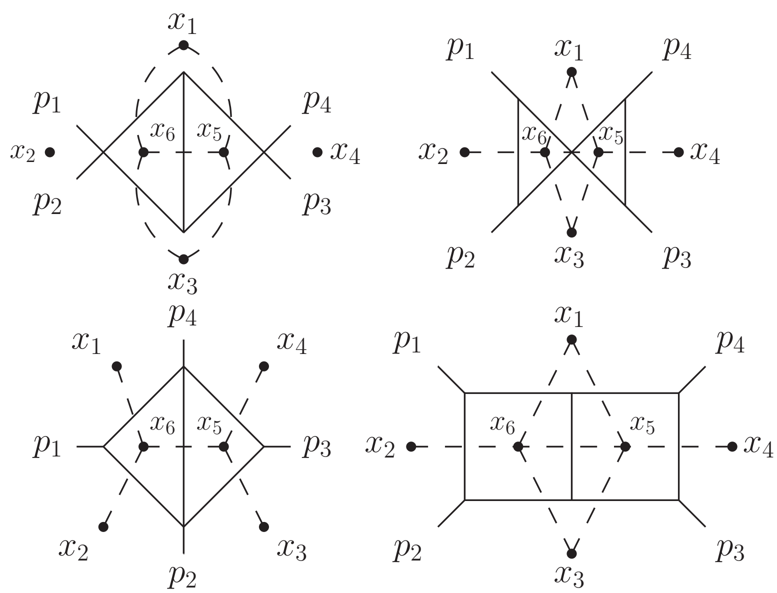


Figure 5.3: Graphical representation of dual conformally invariant integrals.

invariant expressions

$$I_{1s} = \int \frac{d^3x_5\, d^3x_6}{(2\pi)^6} \, \frac{x_{13}^4}{x_{15}^2\, x_{35}^2\, x_{56}^2\, x_{16}^2\, x_{36}^2} \tag{5.3.1}$$

$$I_{2s} = \int \frac{d^3x_5\, d^3x_6}{(2\pi)^6} \, \frac{x_{13}^2\, x_{24}^2}{x_{15}^2\, x_{35}^2\, x_{45}^2\, x_{16}^2\, x_{26}^2\, x_{36}^2} \tag{5.3.2}$$

$$I_{3s} = \int \frac{d^3x_5\, d^3x_6}{(2\pi)^6} \, \frac{x_{13}^2\, x_{24}^2}{x_{35}^2\, x_{45}^2\, x_{56}^2\, x_{26}^2\, x_{16}^2} \tag{5.3.3}$$

$$I_{4s} = \int \frac{d^3x_5\, d^3x_6}{(2\pi)^6} \, \frac{x_{13}^4\, x_{25}^2\, x_{46}^2}{x_{15}^2\, x_{35}^2\, x_{45}^2\, x_{56}^2\, x_{16}^2\, x_{26}^2\, x_{36}^2} \tag{5.3.4}$$

plus their t–counterparts obtained by cyclic permutation of the $(1, 2, 3, 4)$ indices. Their graphical representation is given in Fig. 5.3.

The appearance of the particular combination $I_{1s} - I_{4s}$ is not an accident. In fact, due to the presence of internal cubic vertices, the integrals I_{1s}, I_{4s} are IR divergent also off–shell and then ill–defined in three dimensions. Dual conformal invariance would require to discharge these integrals. However, as we show in Appendix C, taking the linear combination $I_{1s} - I_{4s}$ the off–shell divergences cancel and I_{5s} is well–defined in three dimensions.

The on–shell evaluation of these integrals in $D = 3 - 2\epsilon$ dimensions reveals that

$$I_{2s} \sim O(\epsilon^2) \qquad\qquad I_{3s} + I_{3t} = -I_{1s} - I_{1t} \tag{5.3.5}$$

Thus, the actual basis for the four–point scattering amplitude reduces to $I_{1s}, I_{5s}, I_{1t}, I_{5t}$. Evaluating them and defining $\bar{\mu}^2 = 8\pi e^{-\gamma_E}\mu^2$, one finds [81] †

$$I_{1s} + I_{1t} = -\frac{1}{16\pi^2}\left[\frac{(s/\bar{\mu}^2)^{-2\epsilon}}{2\epsilon} + \frac{(t/\bar{\mu}^2)^{-2\epsilon}}{2\epsilon} + 2 - 2\log 2\right] + O(\epsilon) \tag{5.3.6}$$

$$I_{5s} + I_{5t} = -\frac{1}{8\pi^2}\left[\frac{(s/\bar{\mu}^2)^{-2\epsilon}}{(2\epsilon)^2} + \frac{(t/\bar{\mu}^2)^{-2\epsilon}}{(2\epsilon)^2} - \frac{(s/\bar{\mu}^2)^{-2\epsilon}}{2\epsilon} - \frac{(t/\bar{\mu}^2)^{-2\epsilon}}{2\epsilon} \right.$$
$$\left. -\frac{1}{2}\log^2\left(\frac{s}{t}\right) + 2\log 2 - 2 - 3\log^2 2 - 4\,\zeta_2\right] + O(\epsilon) \tag{5.3.7}$$

Using these results, it is easy to see that for the ABJM theory, the two–loop amplitude (5.3.9) for $\sigma = 0$ can be written as

$$\mathcal{M}_4^{(2)}\Big|_{\text{ABJM}} = (4\pi\lambda)^2\,\left[\frac{1}{2}\,I_{5s} + I_{1s} + (s \leftrightarrow t)\right] \tag{5.3.8}$$

Remarkably, this linear combination not only reproduces correctly the non–trivial part of the amplitude, but also fits the numerical constant $C_{\mathcal{A}}$. In particular, it is such that the non–maximal transcendentality terms in (5.3.6), (5.3.7) cancel.

We can now generalize this analysis to the ABJ models where the amplitude has the same functional structure of the ABJM one, except for a non–trivial dependence on the σ parameter in the mass scale and in the $C_{\mathcal{A}}$ constant. Because of the appearance of σ, we find that in terms of the integrals (5.3.6, 5.3.7) given as functions of the $\bar{\mu}^2$ scale, the $\mathcal{M}_4^{(2)}$ ratio can be written as

$$\mathcal{M}_4^{(2)}\Big|_{\text{ABJ}} = (4\pi\bar{\lambda})^2\,\left[\frac{1}{2}\,I_{5s}(\bar{\mu}^2) + \left(1 + \frac{\sigma^2}{2}\log 2\right)I_{1s}(\bar{\mu}^2) + (s \leftrightarrow t)\right] + C_{\text{res}} \tag{5.3.9}$$

where C_{res} is a residual constant given by

$$C_{\text{res}} = \bar{\lambda}^2\,\sigma^2\left(\log^2 2 - \frac{5}{4}\,\zeta_2 + \log 2\right) \tag{5.3.10}$$

The non–trivial appearance of σ^2 in the coefficients might reflect the fact that in the ABJ case one cannot factorize completely the color dependence outside the combination of integrals. This would suggest that a generalization of the unitarity cuts method should be employed where the trace structures are not stripped out. Nevertheless, it is not difficult to see that the application of such a method would never reproduce the $\log 2$ coefficient in front of I_{1s}, I_{1t}.

One would be tempted to conclude that ABJ amplitudes cannot be computed by unitarity cuts methods. However, a way out is to start from a linear combination of (5.3.6,

†In [81], the constant part of $(I_{5s} + I_{5t})$ has been evaluated numerically. However, *a posteriori* one can check that the numerical factor is well reproduced by the analytical expression in (5.3.7).

5.3.7) integrals where the mass parameter has been rescaled as $\bar{\mu}^2 \to \mu'^2 = \bar{\mu}^2 2^{\sigma^2/2}$. Doing that, we find that the two loop ratio can now be written as

$$\mathcal{M}_4^{(2)}\Big|_{\rm ABJ} = (4\pi\bar{\lambda})^2 \left[\frac{1}{2} I_{5s}(\mu'^2) + I_{1s}(\mu'^2) + (s \leftrightarrow t)\right] + C'_{\rm res} \tag{5.3.11}$$

where

$$C'_{\rm res} = \bar{\lambda}^2 \, \sigma^2 \left[\left(2 + \frac{\sigma^2}{4}\right) \log^2 2 - \frac{5}{4}\zeta_2\right] \tag{5.3.12}$$

The situation has drastically improved, since rational coefficients in front of the integrals indicate that the same result could be obtained by unitarity cuts method. However, in that approach the question of why and how fixing *a priori* a non–standard mass scale in the dual invariant integrals remains an open problem.

Except for the particular case $M = N$, in general the basis of scalar integrals selected by dual conformal symmetry reproduces the four–point amplitude only up to a constant. This is a quite different result compared to what happens in the ABJM and $\mathcal{N} = 4$ SYM cases where dual conformal integrals reproduce exactly the four–point amplitude. However, at the order we are working, the difference is only by a constant and dual conformal invariance is safe, as well as the anomalous Ward identities which follow.

At higher loops, we expect the non–trivial dependence on σ to affect also the terms depending on the kinematic variables. It would be very interesting to check whether this phenomenon may spoil dual conformal invariance or higher order amplitudes could still be expressed as (σ–dependent) combinations of dual conformal invariant integrals. For this reason, it would be extremely important to evaluate the amplitude at four loops.

In the ABJM theories, comparing the result for the amplitude obtained by ordinary perturbative methods with the one obtained by unitarity cuts method, we can write

$$\mathcal{M}_4^{(2)} = \sum_i c_i \, J_i \Big|_{\substack{D=3-2\epsilon \\ on-shell}} = \sum_n \alpha_n \, I_n \Big|_{\substack{D=3-2\epsilon \\ on-shell}} \tag{5.3.13}$$

where J_i are momentum integrals associated to the Feynman diagrams in Fig. 5.2, whereas I_n are the scalar integrals (5.3.1-5.3.4).

Since the linear combination on the right hand side, when written strictly in $D = 3$ with $x_{i,i+1}^2 \neq 0$ is dual conformally invariant, the natural question which arises is whether also the left hand side shares the same property.

In order to answer this question, we investigate the behavior of the Feynman integrals J_i under dual conformal transformations, off–shell and in three dimensions. After rewriting them in terms of dual space variables, we implement the inversion $x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$ and $d^d x_i \to \frac{d^d x_i}{(x_i^2)^d}$, which is the only non–trivial conformal transformation to be checked.

As in four dimensions the invariance under inversion rules out triangle and bubble–like diagrams, similarly in three dimensions it forbids the appearance of bubbles. Therefore, just looking at the integrands, we see that the integrals associated to diagrams 5.2(a)–5.2(b) cannot be separately dual conformal invariant. Moreover, despite the fact that diagrams 5.2(d)–5.2(f) consist of triangles only, non–trivial numerators spoil invariance under inversion, as well.

Nevertheless, these considerations on the integrands may fail in very special cases. As an example, we consider the double bubble diagram 5.2(a). In dual coordinates, the corresponding integral reads

$$\mathcal{B} = x_{13}^2 \int \frac{d^3x_5}{x_{15}^2\, x_{35}^2} \int \frac{d^3x_6}{x_{16}^2\, x_{36}^2} \tag{5.3.14}$$

Performing inversion, this integral gets mapped into a double triangle integral

$$\mathcal{T} = x_{13}^2\, x_1^2\, x_3^2 \int \frac{d^3x_5}{x_5^2\, x_{15}^2\, x_{35}^2} \int \frac{d^3x_6}{x_6^2\, x_{16}^2\, x_{36}^2} \tag{5.3.15}$$

If we evaluate them off–shell, we obtain $\mathcal{B} = \mathcal{T} = 1/64$ (see eqs. (A.5.9, A.5.10)). Therefore, the double bubble diagram is invariant under inversion at the level of the integral, even if it is not invariant at the level of the integrand.

Motivated by this example we may wonder whether dual conformal invariance on the left hand side of eq. (5.3.13) could be restored at the level of the integrals. By numerical evaluating the integrals associated to the independent topologies 5.2(b), 5.2(d) and 5.2(f) and to their duals, obtained by acting with conformal inversion, we find that every single integral is not by itself dual conformally invariant.

However, one may still doubt that the situation could improve when summing over all scattering channels, or summing all the contributions to get the total off–shell amplitude.

We find that, even if for every single diagram the sum over permutations of external legs definitively improves the result, as for a large sample of momentum configurations the integral and its dual look very close to each other, they never appear to be exactly invariant, neither does the total sum.

We thus conclude that the off–shell amplitude computed by Feynman diagrams is not dual conformally invariant. In other words, the identity (5.3.13) is not an algebraic relation between different basis of integrals, as if it were the case it should be valid for any value of the kinematic variables. Instead, it holds only when the integrals are evaluated on the mass–shell and in dimensional regularization. This is not puzzling if we take into account that in three dimensions and in dimensional regularization the on–shell limit is not a smooth limit for the integrals. It would be interesting to investigate what happens when using a different regularization, for example the one suggested in [92, 93].

In any case, our result reinforces the statement that dual conformal invariance is a (anomalous) symmetry *only* of the on–shell sector of the theory.

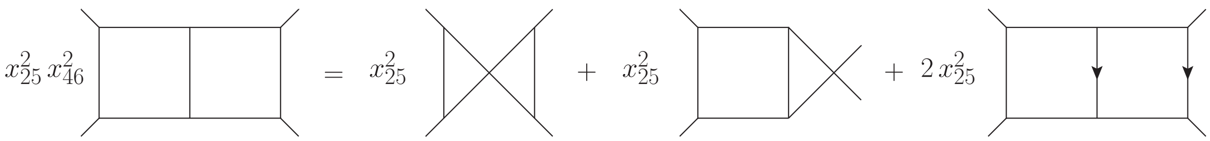
5.3.3 The basis of dual conformally invariant integrals

In this Section we give the proof that the linear combination $I_{5s} \equiv I_{1s} - I_{4s}$ of scalar integrals defined in (5.3.1, 5.3.4), when evaluated in three dimensions, is free from IR divergences. This allows to conclude that the actual basis for two–loop amplitudes is I_{1s}, I_{2s}, I_{3s}, I_{5s} plus their t–counterparts.

We consider I_{4s} in (5.3.4) and apply the following identity

$$x_{46}^2 = x_{56}^2 + x_{45}^2 + 2\, x_{45} \cdot x_{56} \tag{5.3.16}$$

to its numerator, thus decomposing it as (we neglect the factor x_{13}^2)

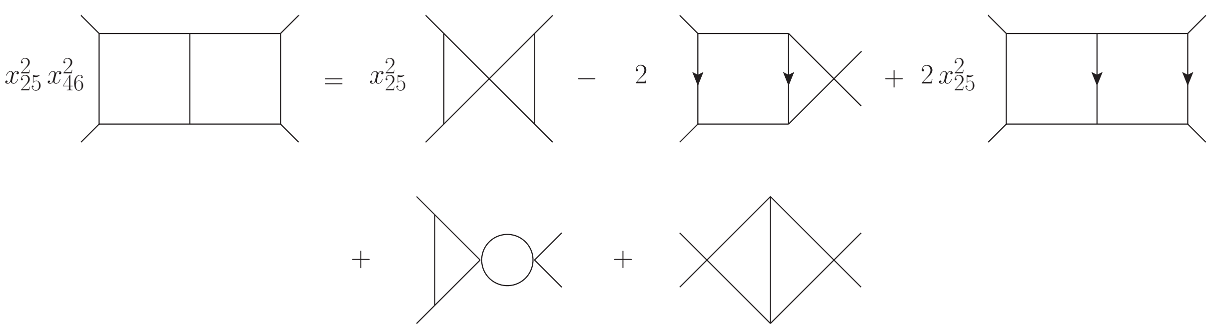


where arrows indicate contractions of the corresponding variables at the numerator (see Fig. 5.2 for the labeling of momenta and dual variables). Here we already recognize the emergence of the infrared divergent integral I_{1s}, a triangle–box which could diverge, having unprotected cubic vertices, and a double–box whose cubic vertices are mitigated by the presence of a non–trivial numerator.

We focus on the triangle–box and handle it by using the identity

$$x_{25}^2 = x_{56}^2 + x_{26}^2 - 2\, x_{56} \cdot x_{26} \tag{5.3.17}$$

in the numerator factor. The final result can be cast into the following graphical relation



where the IR divergence has been completely isolated in the last term. Therefore, taking the linear combination $I_{1s} - I_{4s}$ the divergences cancel at the level of the integrands and we are left with a well–defined dual conformally invariant integral.

5.3.4 BDS–like ansatz

The striking correspondence between the four–point amplitudes of ABJM theory at two loops and the one of 4d $\mathcal{N} = 4$ SYM at one loop led us to conjecture [82] (see also [81]) that a BDS–like ansatz [56] may be formulated also for the three–dimensional case

$$\mathcal{M}_4 = e^{Div + \frac{f_{CS}(\lambda)}{8}\left(\log^2\left(\frac{s}{t}\right) + 8\,\zeta_2 + 6\,\log^2 2\right) + C(\lambda)} \tag{5.3.18}$$

where $f_{CS}(\lambda)$ is the scaling function of ABJM. The analogy between the two theories is due to the fact that they share similar integrable structures, with asymptotic Bethe equations related by an unknown function $h(\lambda)$ [89]-[94]. This leads to a connection between anomalous dimensions of composite operators and, in particular, to the following relation between the scaling functions [89]

$$f_{CS}(\lambda) = \frac{1}{2} f_{\mathcal{N}=4}(\lambda)\bigg|_{\frac{\sqrt{\lambda}}{4\pi} \to h(\lambda)} \tag{5.3.19}$$

in terms of the interpolating function $h(\lambda)$ that needs to be determined in perturbation theory. Since the scaling function governs the coefficients of the kinematic part of the four–point amplitude in $\mathcal{N} = 4$ SYM by means of the BDS exponentiation, one is tempted to conjecture that an analogue resummation may also hold in the ABJM case, giving rise to equation (5.3.18). This formula is confirmed at two loops by the results of [82, 81]. Since at weak coupling $h(\lambda)$ is known up to the forth order [95, 96, 71], we easily find

$$f_{CS}(\lambda) = 4\,\lambda^2 - 24\,\zeta_2\,\lambda^4 + \mathcal{O}(\lambda^6) \tag{5.3.20}$$

and the ansatz (5.3.18) provides a prediction for the four–loop expression of the finite remainder $F_4^{(4)}$ (in the notation of [56]) for the ABJM four–point scattering amplitude [82]

$$F_4^{(4)} = \frac{\lambda^4}{8}\,\log^4\left(\frac{s}{t}\right) + \lambda^4\left(\frac{3}{2}\,\log^2 2 - \zeta_2\right)\log^2\left(\frac{s}{t}\right) + \text{Consts} \tag{5.3.21}$$

Now we discuss how this scenario might be affected by the generalization to the ABJ case, where integrability is not expected to be trivially preserved. We first note that the final expression (5.3.9) is the result of summing many contributions which in general are proportional to the homogeneous couplings λ^2, $\hat{\lambda}^2$ and to the mixed $\lambda\hat{\lambda}$ one. It is interesting to observe that at this order, it is possible to redefine μ^2 in such a way that in all the terms depending on the kinematic variables, the contributions proportional to the homogeneous couplings cancel, leading to an expression which is basically identical to the one for ABJM, except for the substitution $N^2 \to MN$. This also happens for the WL computed in Section 2.

This phenomenon has been also observed in the evaluation of the spin–chain Hamiltonian associated to the two–loop anomalous dimension matrix for single–trace operators

[97] and in the two–loop contribution to the dispersion relation of magnons [70]. This special dependence on the coupling constants is a signal that parity symmetry along with integrability are preserved at least at two–loop order even if ABJ theory is manifestly parity breaking. At four loops only the dispersion relation, i.e. the eigenvalue of the Hamiltonian of spin chains with a single excitation, is known to date. It has been confirmed by explicit computations and the following expansion for the interpolating function has been found [95, 96, 71]

$$h^2(\bar{\lambda}, \sigma) = \bar{\lambda}^2 - \bar{\lambda}^4 \left[4\, \zeta_2 + \zeta_2\, \sigma^2\right] \tag{5.3.22}$$

From the last term of this expression it is clear that, even if the departure from the ABJM case becomes non–trivial, still the function turns out to depend quadratically on σ and thus parity breaking is not visible. This might indicate that integrability is not broken also at the four-loop level.

Therefore it seems plausible that, at least up to four loops, the planar limit of ABJ scattering amplitude could behave in the same way as in the ABJM case, being governed by the scaling function $f(\bar{\lambda}, \sigma)$ obtained through (5.3.19) where we replace $h(\lambda)$ with $h(\bar{\lambda}, \sigma)$.

Since at weak coupling $h(\bar{\lambda}, \sigma)$ is known up to four loops [95, 96, 71], we find

$$f_{CS}(\bar{\lambda}, \sigma) = 4\, \bar{\lambda}^2 - 4\,(6 + \sigma^2)\, \zeta_2\, \bar{\lambda}^4 + \mathcal{O}(\bar{\lambda}^6) \tag{5.3.23}$$

It is then easy to see that the four–point amplitude at order $\bar{\lambda}^2$ can be identified with the first order expansion of an exponential of the type in eq. (5.3.18) with $f_{CS}(\bar{\lambda}, \sigma) = 4\bar{\lambda}^2$.

Moreover, this would lead to a prediction for the four-loop expression for the finite remainder of the ABJ four–point amplitude to be given by

$$F_4^{(4)} = \frac{\bar{\lambda}^4}{8} \log^4\left(\frac{s}{t}\right) + \bar{\lambda}^4 \left[\frac{1}{2}\left(1 + \tfrac{1}{2}\sigma^2\right)\left(3 + \tfrac{1}{2}\sigma^2\right) \log^2 2 - \left(1 + \frac{9}{8}\sigma^2\right) \zeta_2\right] \log^2\left(\frac{s}{t}\right) + \text{Consts} \tag{5.3.24}$$

At four loops the parity–violating parameter is expected to play an active role and the theory could present a very different behavior with respect to the ABJM case. It would be interesting to check this expression by a direct computation.

5.3.5 The amplitude at strong coupling

In $\mathcal{N} = 4$ SYM a recipe for computing scattering amplitudes at strong coupling has been proposed by Alday and Maldacena [47] within the context of the AdS/CFT correspondence. According to their prescription, the amplitude for n gluons is obtained by computing the minimal area of a surface in the AdS_5 dual background, ending on a light–like n–polygon, whose edges are determined by the gluon momenta

$$\mathcal{M} = e^{-\frac{R^2_{AdS_5}}{2\pi} A} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) \tag{5.3.25}$$

Here R_{AdS_5} stems for the AdS radius and determines the dependence on the 't Hooft coupling ($R^2_{AdS_5} = \sqrt{g^2 N} \equiv \sqrt{\lambda_{SYM}}$).

After a suitable regularization of this area, the four–gluon amplitude $\mathcal{M}_4$, to first order in $\sqrt{\lambda_{SYM}}$, matches exactly the BDS ansatz, where the strong coupling scaling function is plugged in. This provides a remarkable check on the BDS ansatz as well as a hint towards the WL/scattering amplitude duality, since the strong coupling computation of the amplitude strikingly parallels that of a light–like Wilson loop.

Motivated by the analogy with the four dimensional case and by evidence in favor of WL/amplitude duality and BDS exponentiation, we investigate the ABJM four–point amplitude at strong coupling, by following the same steps as in [47].

At strong coupling where the 't Hooft parameter $\lambda = N/K$ is large, and in the intermediate regime $K \ll N \ll K^5$ the AdS/CFT correspondence provides a dual description of the ABJM in terms of type IIA supergravity on $AdS_4 \times \mathbb{CP}^3$.

The dual background in the string frame is

$$ds^2 = \frac{R^3}{K}\left(\frac{1}{4}\, ds^2_{AdS_4} + ds^2_{\mathbb{CP}^3}\right) \tag{5.3.26}$$

where in l_s units the AdS_4 radius is given by $R^2_{AdS_4} = \frac{R^3}{4\,K} = \frac{\sqrt{2^5\,\pi^2\,K\,N}}{4\,K} = \sqrt{2}\,\pi\,\sqrt{\lambda}$.

A first indication that the general prescription for computing scattering amplitudes at strong coupling could still be (5.3.25) with the parameters conveniently adapted to the three dimensional model, comes from observing that the ratio of the two AdS radii coincides with the ratio of the scaling functions at leading order in the couplings. In fact, taking into account that for $\mathcal{N} = 4$ SYM at strong coupling

$$f(\lambda_{SYM}) = \frac{\sqrt{\lambda_{SYM}}}{\pi} + \mathcal{O}\left(\lambda^0\right) \tag{5.3.27}$$

whereas for the ABJM theory [89]–[98]

$$f_{CS}(\lambda) = \sqrt{2\,\lambda} + \mathcal{O}\left(\lambda^0\right) \tag{5.3.28}$$

it is easy to see that

$$\frac{R^2_{AdS_5}}{R^2_{AdS_4}} = \frac{f(\lambda_{SYM})}{f_{CS}(\lambda)}\Big|_{\text{leading}} \tag{5.3.29}$$

Moreover, the string solution (3.6.4) in AdS_5 describing a four–point amplitude/light–like WL for $\mathcal{N} = 4$ SYM at strong coupling [47] may be straightforwardly embedded in AdS_4 as well, since the fifth coordinate of AdS_5 was set to 0 there. Therefore the four–point amplitude should be trivially readable from the $\mathcal{N} = 4$ SYM result (5.3.25) by changing the AdS radius. Indeed this supports the extension of the BDS–like ansatz (5.3.18) to strong coupling.

The subtle point in identifying the $\mathcal{N}=4$ solution with the ABJM one comes with regularization of infrared divergences. In [47], the strong coupling analogue of dimensional regularization is spelled out. This amounts to continuing the dimensions of the D_p–branes sourcing the $AdS_5 \times S^5$ background from $p=3$ to $p=(3-2\epsilon)$. Correspondingly, the new solution for the modified metric leads to the expression

$$S_\epsilon = \frac{\sqrt{\lambda_D\, c_\epsilon}}{2\pi} \int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon} \tag{5.3.30}$$

for the regularized world–sheet action that, once minimized, will provide the four–point amplitude (here c_ϵ is an ϵ dependent constant and λ_D is the dimensionless 't Hooft coupling in dimensional regularization).

In the context of ABJM we have not been able to find a similarly well–motivated regularization procedure ‡. However, guided by the analogy between the four–point ABJM and $\mathcal{N}=4$ SYM amplitudes at weak coupling, we are tempted to employ the prescription (5.3.30) to regularize the action also in the AdS_4 context. Following the same steps as for the $\mathcal{N}=4$ SYM case, it leads to a strong coupling version of the ABJM four–point amplitude given by

$$\mathcal{M}_4 = e^{Div + \frac{\sqrt{2\lambda}}{8}\left(\log^2\left(\frac{s}{t}\right) + \frac{4\pi^2}{3}\right) + \text{Consts} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)} \tag{5.3.31}$$

where the leading infrared divergence is

$$Div\big|_{leading} = -\frac{\sqrt{2}}{\epsilon^2}\sqrt{\frac{\lambda\,\mu^{2\epsilon}}{s^\epsilon}} - \frac{\sqrt{2}}{\epsilon^2}\sqrt{\frac{\lambda\,\mu^{2\epsilon}}{t^\epsilon}} \tag{5.3.32}$$

Even though this prescription lacks strong motivations, it definitely captures the essential features of the amplitude, such as the leading singularity and the coefficient of the non–trivial finite piece, which matches the strong coupling value of the ABJM scaling function (5.3.28).

The generalization to the ABJ model is not straightforward. Here the situation is slightly subtler, since concerns have arisen on the integrability of the corresponding σ–model in the dual description [70]. Nevertheless, to first order at strong coupling we still expect the amplitude to be described by (5.3.31). The reason is that unitarity requires $l=|M-N|<K$ [12]. Hence at strong coupling, where $M, N \gg K$, the shift in the ranks is negligible compared to $\sqrt{\lambda}$ of ABJM, and it affects the solution at higher orders only, starting from $\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$ [99].

‡Although cutoff regularization works fine in three dimensions, we prefer to insist on a dimensional–like one in order to compare with our expression (5.3.9).

Appendix A

Appendix

A.1 Notations and conventions

We work in three dimensional euclidean $\mathcal{N} = 2$ superspace described by coordinates $(x^\mu, \theta^\alpha \, \bar{\theta}^\beta)$, $\alpha, \beta = 1, 2$.

Spinorial indices are raised and lowered as (we follow conventions of [11])

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta \qquad \psi_\alpha = \psi^\beta C_{\beta\alpha} \tag{A.1.1}$$

where the C matrix

$$C^{\alpha\beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad C_{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{A.1.2}$$

obeys the relation

$$C^{\alpha\beta} \, C_{\gamma\delta} \ \ = \delta^\alpha{}_\gamma \, \delta^\beta{}_\delta - \delta^\alpha{}_\delta \, \delta^\beta{}_\gamma \tag{A.1.3}$$

Spinors are contracted according to

$$\psi\chi = \psi^\alpha \, \chi_\alpha = \chi^\alpha \, \psi_\alpha = \chi\psi \qquad \psi^2 = \frac{1}{2} \, \psi^\alpha \, \psi_\alpha \tag{A.1.4}$$

Dirac $(\gamma^\mu)^\alpha{}_\beta$ matrices are defined to satisfy the algebra

$$(\gamma^\mu)^\alpha{}_\gamma \, (\gamma^\nu)^\gamma{}_\beta = -g^{\mu\nu}\delta^\alpha{}_\beta - \epsilon^{\mu\nu\rho} \, (\gamma_\rho)^\alpha{}_\beta \tag{A.1.5}$$

An explicit realization of such matrices can be found in terms of Pauli matrices

$$(\gamma^\mu)^{\alpha\beta} = \{-i \, \sigma_0, \, \sigma_1, \, \sigma_3\} = \left\{ \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{A.1.6}$$

$$(\gamma^\mu)_{\alpha\beta} = \{i\,\sigma_0,\, \sigma_1,\, \sigma_3\} = \left\{ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{A.1.7}$$

$$(\gamma^\mu)^\alpha{}_\beta = \{-i\,\sigma_2,\, i\,\sigma_3,\, -i\,\sigma_1\} = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right\} \tag{A.1.8}$$

$$(\gamma^\mu)_\alpha{}^\beta = \{i\,\sigma_2,\, i\,\sigma_3,\, -i\,\sigma_1\} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right\} \tag{A.1.9}$$

Trace identities needed for loop calculations can be easily obtained from the above algebra

$$\begin{aligned}
\mathrm{Tr}(\gamma^\mu\,\gamma^\nu) &= (\gamma^\mu)^\alpha{}_\beta\,(\gamma^\nu)^\beta{}_\alpha = -2\,g^{\mu\nu} && \text{(A.1.10)}\\
\mathrm{Tr}(\gamma^\mu\,\gamma^\nu\,\gamma^\rho) &= -(\gamma^\mu)^\alpha{}_\beta\,(\gamma^\nu)^\beta{}_\gamma\,(\gamma^\rho)^\gamma{}_\alpha = -2\,\epsilon^{\mu\nu\rho} && \text{(A.1.11)}\\
\mathrm{Tr}(\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma) &= (\gamma^\mu)^\alpha{}_\beta\,(\gamma^\nu)^\beta{}_\gamma\,(\gamma^\rho)^\gamma{}_\delta\,(\gamma^\sigma)^\delta{}_\alpha = \\
&= 2\,(g^{\mu\nu}\,g^{\rho\sigma} - g^{\mu\rho}\,g^{\nu\sigma} + g^{\mu\sigma}\,g^{\nu\rho}) && \text{(A.1.12)}
\end{aligned}$$

Using these matrices, vectors and bispinors are exchanged according to

$$\begin{array}{lll}
\text{coordinates :} & x^\mu = (\gamma^\mu)_{\alpha\beta}\,x^{\alpha\beta} & x^{\alpha\beta} = \frac{1}{2}\,(\gamma_\mu)^{\alpha\beta}\,x^\mu \\
\text{derivatives :} & \partial_\mu = \frac{1}{2}\,(\gamma_\mu)^{\alpha\beta}\,\partial_{\alpha\beta} & \partial_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta}\,\partial_\mu \\
\text{fields :} & A_\mu = \frac{1}{\sqrt{2}}\,(\gamma_\mu)^{\alpha\beta}\,A_{\alpha\beta} & A_{\alpha\beta} = \frac{1}{\sqrt{2}}\,(\gamma^\mu)_{\alpha\beta}\,A_\mu
\end{array} \tag{A.1.13}$$

It follows that the scalar product of two vectors can be rewritten as

$$p \cdot k \;=\; \frac{1}{2}\,p^{\alpha\beta}\,k_{\alpha\beta} \tag{A.1.14}$$

Superspace covariant derivatives are defined as

$$D_\alpha = \partial_\alpha + \frac{i}{2}\,\bar\theta^\beta\,\partial_{\alpha\beta} \qquad , \qquad \bar D_\alpha = \bar\partial_\alpha + \frac{i}{2}\,\theta^\beta\,\partial_{\alpha\beta} \tag{A.1.15}$$

and satisfy the anticommutator

$$\{D_\alpha,\, \bar D_\beta\} = i\,\partial_{\alpha\beta} \tag{A.1.16}$$

The components of a chiral and an anti-chiral superfield, $Z(x_L,\theta)$ and $\bar Z(x_R,\bar\theta)$, are a complex boson ϕ, a complex two-component fermion ψ and a complex auxiliary scalar F. Their expansions are given by

$$\begin{aligned}
Z &= \phi(x_L) + \theta^\alpha\psi_\alpha(x_L) - \theta^2\,F(x_L)\\
\bar Z &= \bar\phi(x_R) + \bar\theta^\alpha\bar\psi_\alpha(x_R) - \bar\theta^2\,\bar F(x_R)
\end{aligned} \tag{A.1.17}$$

where $x_L^\mu = x^\mu + i\theta\gamma^\mu\bar{\theta}$, $x_R^\mu = x^\mu - i\theta\gamma^\mu\bar{\theta}$.

The components of the real vector superfield $V(x, \theta, \bar{\theta})$ in the Wess-Zumino gauge ($V| = D_\alpha V| = D^2 V| = 0$) are the gauge field $A_{\alpha\beta}$, a complex two-component fermion λ_α, a real scalar σ and an auxiliary scalar D, such that

$$V = \theta^\alpha \bar{\theta}_\alpha \, \sigma(x) + \theta^\alpha \bar{\theta}^\beta \, \sqrt{2} \, A_{\alpha\beta}(x) - \theta^2 \, \bar{\theta}^\alpha \bar{\lambda}_\alpha(x) - \bar{\theta}^2 \, \theta^\alpha \lambda_\alpha(x) + \theta^2 \, \bar{\theta}^2 \, D(x) \tag{A.1.18}$$

The vector superfields $(V, \hat{V})$ are in the adjoint representation of the two gauge groups $U(M) \times U(N)$, that is $V = V_A T^A$ and $\hat{V} = \hat{V}_A \hat{T}^A$, where T^A are the $U(M)$ generators and $\hat{T}^A$ are the $U(N)$ ones.

The $U(M)$ generators are defined as $T^A = (T^0, T^a)$, where $T^0 = \frac{1}{\sqrt{N}}$ and T^a ($a = 1, \dots, M^2 - 1$) are a set of $M \times M$ hermitian matrices. The generators are normalized as $\mathrm{Tr}(T^A T^B) = \delta^{AB}$. The same conventions hold for the $U(N)$ generators.

For any value of the couplings, the action (1.3.1) is invariant under the following gauge transformations

$$e^V \to e^{i\bar{\Lambda}_1} e^V e^{-i\Lambda_1} \qquad\qquad e^{\hat{V}} \to e^{i\bar{\Lambda}_2} e^{\hat{V}} e^{-i\Lambda_2} \tag{A.1.19}$$

$$A^i \to e^{i\Lambda_1} A^i e^{-i\Lambda_2} \qquad\qquad B_i \to e^{i\Lambda_2} B_i e^{-i\Lambda_1} \tag{A.1.20}$$

where Λ_1, Λ_2 are two chiral superfields parameterizing $U(M)$ and $U(N)$ gauge transformations, respectively. Antichiral superfields transform according to the conjugate of (A.1.20). The action is also invariant under the $U(1)_R$ R–symmetry group under which the A^i and B_i fields have $\frac{1}{2}$–charge.

A.2 Gauge invariance of the superspace CS action.

In this Appendix we explicitly compute the variation of the CS action in $\mathcal{N} = 2$ supersymmetric formalism and show that it yields a total spinorial derivative, namely the action is gauge invariant. We consider just one gauge group

$$\mathcal{S}_{\mathrm{CS}} = \frac{K}{4\pi} \int d^3x \, d^4\theta \int_0^1 dt \, \mathrm{Tr}\Big[V \bar{D}^\alpha \left(e^{-tV} D_\alpha e^{tV} \right) \Big]$$

and gauge transformations

$$e^V \to e^{i\bar{\Lambda}} e^V e^{-i\Lambda} \qquad\qquad e^{-V} \to e^{i\Lambda} e^{-V} e^{-i\bar{\Lambda}} \tag{A.2.1}$$

We conveniently rewrite the CS action as

$$\mathcal{S}_{\rm CS} = \frac{K}{4\pi}\int d^3x\, d^4\theta \int_0^1 dt\, {\rm Tr}\Big[\bar{D}^\alpha \left(e^{-tV} D_\alpha e^{tV}\right) e^{-tV}\partial_t e^{tV}\Big] \tag{A.2.2}$$

We apply gauge transformations

$$\begin{aligned}\mathcal{S}'_{\rm CS} &= \frac{K}{4\pi}\int d^3x\, d^4\theta \int_0^1 dt\, {\rm Tr}\Big[\bar{D}^\alpha \left(e^{it\Lambda}e^{-tV}e^{-it\bar{\Lambda}}D_\alpha e^{it\bar{\Lambda}}e^{tV}e^{-it\Lambda}\right) \\ &\qquad e^{it\Lambda}e^{-tV}e^{-it\bar{\Lambda}}\partial_t e^{it\bar{\Lambda}}e^{tV}e^{-it\Lambda}\Big]\end{aligned} \tag{A.2.3}$$

which gives

$$\mathcal{S}'_{\rm CS} = \int d^3x\, d^4\theta \int_0^1 dt\, {\rm Tr}\Big[\bar{D}^\alpha \left(e^{it\Lambda}e^{-tV}D_\alpha e^{tV}e^{-it\Lambda}\right) e^{it\Lambda}e^{-tV}e^{-it\bar{\Lambda}}\partial_t e^{it\bar{\Lambda}}e^{tV}e^{-it\Lambda}\Big] \tag{A.2.4}$$

Then we apply the following identity for (anti)commuting differential operators A and B

$$A\left(MBM^{-1}\right) = \pm MB\left(M^{-1}AM\right)M^{-1} \tag{A.2.5}$$

where the minus(plus) sign occurs for (anti)commuting operators. Here we identify $A = \bar{D}^\alpha$, $B = D_\alpha$, which anticommute, and $M = e^{it\Lambda}e^{-tV}$

$$\begin{aligned}\mathcal{S}'_{\rm CS} &= \frac{K}{4\pi}\int d^3x\, d^4\theta \int_0^1 dt\, {\rm Tr}\Big[e^{it\Lambda}e^{-tV}D_\alpha \left(e^{tV}e^{-it\Lambda}\bar{D}^\alpha e^{it\Lambda}e^{-tV}\right) \\ &\qquad e^{tV}e^{-it\Lambda}e^{it\Lambda}e^{-tV}e^{-it\bar{\Lambda}}\partial_t e^{it\bar{\Lambda}}e^{tV}e^{-it\Lambda}\Big]\end{aligned} \tag{A.2.6}$$

simplifying to

$$\mathcal{S}'_{\rm CS} = \frac{K}{4\pi}\int d^3x\, d^4\theta \int_0^1 dt\, {\rm Tr}\Big[e^{it\Lambda}e^{-tV}D_\alpha \left(e^{tV}\bar{D}^\alpha e^{-tV}\right) e^{-it\bar{\Lambda}}\partial_t e^{it\bar{\Lambda}}e^{tV}e^{-it\Lambda}\Big] \tag{A.2.7}$$

We can transform back, using (A.2.5) the other way around

$$\mathcal{S}'_{\rm CS} = \frac{K}{4\pi}\int d^3x\, d^4\theta \int_0^1 dt\, {\rm Tr}\Big[e^{it\Lambda}\bar{D}^\alpha \left(e^{-tV}D_\alpha e^{tV}\right) e^{-tV}e^{-it\bar{\Lambda}}\partial_t e^{it\bar{\Lambda}}e^{tV}e^{-it\Lambda}\Big] \tag{A.2.8}$$

Now we perform the t derivative

$$\begin{aligned}\mathcal{S}'_{\rm CS} &= \frac{K}{4\pi}\int d^3x\, d^4\theta \int_0^1 dt\, {\rm Tr}\Big[e^{it\Lambda}\bar{D}^\alpha \left(e^{-tV}D_\alpha e^{tV}\right) e^{-tV}e^{-it\bar{\Lambda}} \\ &\quad \Big[\partial_t\left(e^{it\bar{\Lambda}}\right)e^{tV}e^{-it\Lambda} + e^{it\bar{\Lambda}}\partial_t\left(e^{tV}\right)e^{-it\Lambda} + e^{it\bar{\Lambda}}e^{tV}\partial_t\left(e^{-it\Lambda}\right)\Big]\Big]\end{aligned} \tag{A.2.9}$$

and rearrange the expression as

$$\begin{aligned}\mathcal{S}'_{\rm CS} &= \frac{K}{4\pi}\int d^3x\, d^4\theta \int_0^1 dt\, {\rm Tr}\Big[\bar{D}^\alpha \left(e^{-tV}D_\alpha e^{tV}\right) e^{-tV}e^{-it\bar{\Lambda}}\partial_t\left(e^{it\bar{\Lambda}}\right)e^{tV} \\ &\quad +\bar{D}^\alpha\left(e^{-tV}D_\alpha e^{tV}\right)e^{-tV}\partial_t\left(e^{tV}\right) + e^{it\Lambda}\bar{D}^\alpha\left(e^{-tV}D_\alpha e^{tV}\right)\partial_t\left(e^{-it\Lambda}\right)\Big]\end{aligned}$$

where in the second term we recognize the original CS superspace action (again by use of (A.2.5))

$$\begin{aligned} \mathcal{S}'_{\rm CS} &= \mathcal{S}_{\rm CS} + \frac{K}{4\pi}\int d^3x\, d^4\theta \int_0^1 dt\, {\rm Tr}\Big[\bar{D}^\alpha \left(e^{-tV} D_\alpha e^{tV}\right) e^{-tV} e^{-it\bar{\Lambda}} \partial_t \left(e^{it\bar{\Lambda}}\right) e^{tV} \\ &\qquad + e^{it\Lambda} \bar{D}^\alpha \left(e^{-tV} D_\alpha e^{tV}\right) \partial_t \left(e^{-it\Lambda}\right)\Big] \end{aligned} \tag{A.2.10}$$

Applying (A.2.5) on the first term and using the cyclicity of the trace yields

$$\begin{aligned} \mathcal{S}'_{\rm CS} &= \mathcal{S}_{\rm CS} + \frac{K}{4\pi}\int d^3x\, d^4\theta \int_0^1 dt\, {\rm Tr}\Big[D_\alpha \left(e^{tV} \bar{D}^\alpha e^{-tV}\right) e^{-it\bar{\Lambda}} \partial_t \left(e^{it\bar{\Lambda}}\right) \\ &\qquad + \bar{D}^\alpha \left(e^{-tV} D_\alpha e^{tV}\right) \partial_t \left(e^{-it\Lambda}\right) e^{it\Lambda}\Big] \end{aligned} \tag{A.2.11}$$

and using chirality of Λ and $\bar{\Lambda}$ the variation of the action under gauge transformations can be cast into a total spinorial derivative

$$\begin{aligned} \mathcal{S}'_{\rm CS} &= \mathcal{S}_{\rm CS} + \frac{K}{4\pi}\int d^3x\, d^4\theta \int_0^1 dt\, {\rm Tr}\Big[D_\alpha \left(e^{tV} \bar{D}^\alpha e^{-tV} e^{-it\bar{\Lambda}} \partial_t \left(e^{it\bar{\Lambda}}\right)\right) \\ &\qquad + \bar{D}^\alpha \left(e^{-tV} D_\alpha e^{tV} \partial_t \left(e^{-it\Lambda}\right) e^{it\Lambda}\right)\Big] \end{aligned} \tag{A.2.12}$$

A.3 Finite remainders for higher point amplitudes.

In this Appendix we list the finite remainders for $\mathcal{N}=4$ SYM amplitudes with more than four external gluons.

The finite remainders are given by

$$F_n^{(1)}(0) = \frac{1}{2}\sum_{i=1}^{n} g_{n,i}\,, \tag{A.3.1}$$

where

$$g_{n,i} = -\sum_{r=2}^{\lfloor n/2\rfloor-1} \log\left(\frac{-t_i^{[r]}}{-t_i^{[r+1]}}\right)\log\left(\frac{-t_{i+1}^{[r]}}{-t_i^{[r+1]}}\right) + D_{n,i} + L_{n,i} + \frac{3}{2}\zeta_2\,, \tag{A.3.2}$$

In the above formula $\lfloor x\rfloor$ stems for the greatest integer less than or equal to x and all indices are understood to be cyclic $\mod n$. Momentum invariants are indicated by $t_i^{[r]} = (k_i + \cdots + k_{i+r-1})^2$.

The structure of $D_{n,i}$ and $L_{n,i}$ depends upon whether n is odd or even. In the even case $n=2m$ they read

$$\begin{aligned} D_{2m,i} &= -\sum_{r=2}^{m-2} \mathrm{Li}_2\left(1-\frac{t_i^{[r]}t_{i-1}^{[r+2]}}{t_i^{[r+1]}t_{i-1}^{[r+1]}}\right) - \frac{1}{2}\mathrm{Li}_2\left(1-\frac{t_i^{[m-1]}t_{i-1}^{[m+1]}}{t_i^{[m]}t_{i-1}^{[m]}}\right) \\ L_{2m,i} &= \frac{1}{4}\log^2\left(\frac{-t_i^{[m]}}{-t_{i+1}^{[m]}}\right) \end{aligned} \tag{A.3.3}$$

In the odd case $n=2m+1$ they are

$$\begin{aligned} D_{2m+1,i} &= -\sum_{r=2}^{m-1} \mathrm{Li}_2\left(1-\frac{t_i^{[r]}t_{i-1}^{[r+2]}}{t_i^{[r+1]}t_{i-1}^{[r+1]}}\right) \\ L_{2m+1,i} &= -\frac{1}{2}\log\left(\frac{-t_i^{[m-1]}}{-t_i^{[m+1]}}\right)\log\left(\frac{-t_{i+1}^{[m]}}{-t_{i-1}^{[m+1]}}\right). \end{aligned} \tag{A.3.4}$$

A.4 The $SU(2,2|4)$ superconformal algebra.

Here are listed the (anti)commutator algebra of $su(2,2|4)$. The Lorentz generators $\mathbb{M}_{\alpha\beta}$, $\overline{\mathbb{M}}_{\dot\alpha\dot\beta}$ and the $SU(4)$ R–symmetry ones $\mathbb{R}^A{}_B$ are understood to act canonically. The dilatation generator $\mathbb{D}$ acts on others $\mathbb{J}$

$$[\mathbb{D},\mathbb{J}] = \dim(\mathbb{J})\,\mathbb{J} \tag{A.4.1}$$

and its eigenvalues are the dimensions of the operators, in particular

$$\dim(\mathbb{P}) = 1 \qquad , \qquad \dim(\mathbb{Q}) = \dim(\overline{\mathbb{Q}}) = \frac{1}{2} \qquad , \qquad \dim(\mathbb{S}) = \dim(\overline{\mathbb{S}}) = -\frac{1}{2} \tag{A.4.2}$$

The remaining non-trivial commutation relations are,

$$\begin{aligned} &\{\mathbb{Q}_{\alpha A},\overline{\mathbb{Q}}^B_{\dot\alpha}\} = \delta^B_A\mathbb{P}_{\alpha\dot\alpha}, \qquad \{\mathbb{S}^A_\alpha,\overline{\mathbb{S}}_{\dot\alpha B}\} = \delta^A_B\mathbb{K}_{\alpha\dot\alpha} \\ &[\mathbb{P}_{\alpha\dot\alpha},\mathbb{S}^{\beta A}] = \delta^\beta_\alpha\overline{\mathbb{Q}}^A_{\dot\alpha} \qquad [\mathbb{K}_{\alpha\dot\alpha},\mathbb{Q}^\beta_A] = \delta^\beta_\alpha\overline{\mathbb{S}}_{\dot\alpha A}, \\ &[\mathbb{P}_{\alpha\dot\alpha},\overline{\mathbb{S}}^{\dot\beta}_A] = \delta^{\dot\beta}_{\dot\alpha}\mathbb{Q}_{\alpha A}, \qquad [\mathbb{K}_{\alpha\dot\alpha},\overline{\mathbb{Q}}^{\dot\beta A}] = \delta^{\dot\beta}_{\dot\alpha}\mathbb{S}^A_\alpha \\ &[\mathbb{K}_{\alpha\dot\alpha},\mathbb{P}^{\beta\dot\beta}] = \delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha}\mathbb{D} + \mathbb{M}_\alpha{}^\beta\delta^{\dot\beta}_{\dot\alpha} + \overline{\mathbb{M}}_{\dot\alpha}{}^{\dot\beta}\delta^\beta_\alpha \\ &\{\mathbb{Q}^\alpha_A,\mathbb{S}^B_\beta\} = \mathbb{M}^\alpha{}_\beta\delta^B_A + \delta^\alpha_\beta\mathbb{R}^B{}_A + \tfrac{1}{2}\delta^\alpha_\beta\delta^B_A(\mathbb{D}+\mathbb{C}) \\ &\{\overline{\mathbb{Q}}^{\dot\alpha A},\overline{\mathbb{S}}_{\dot\beta B}\} = \overline{\mathbb{M}}^{\dot\alpha}{}_{\dot\beta}\delta^A_B - \delta^{\dot\alpha}_{\dot\beta}\mathbb{R}^A{}_B + \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\delta^A_B(\mathbb{D}-\mathbb{C}) \end{aligned} \tag{A.4.3}$$

In the following the following shorthand notation will be employed

$$\partial_{i\alpha\dot\alpha} = \frac{\partial}{\partial x_i^{\alpha\dot\alpha}}, \qquad \partial_{i\alpha A} = \frac{\partial}{\partial\theta_i^{\alpha A}}, \qquad \partial_{i\alpha} = \frac{\partial}{\partial\lambda_i^\alpha}, \qquad \partial_{i\dot\alpha} = \frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}, \qquad \partial_{iA} = \frac{\partial}{\partial\eta_i^A} \tag{A.4.4}$$

The generators of the ordinary, configuration space superconformal algebra.

$$
\begin{aligned}
p^{\dot\alpha\alpha} &= \sum_i \tilde\lambda_i^{\dot\alpha}\lambda_i^{\alpha}\,, & k_{\alpha\dot\alpha} &= \sum_i \partial_{i\alpha}\partial_{i\dot\alpha}\,,\\
\overline{m}_{\dot\alpha\dot\beta} &= \sum_i \tilde\lambda_{i(\dot\alpha}\partial_{i\dot\beta)}, & m_{\alpha\beta} &= \sum_i \lambda_{i(\alpha}\partial_{i\beta)}\\
d &= \sum_i [\tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} + \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} + 1], & r^A{}_B &= \sum_i [-\eta_i^A\partial_{iB} + \tfrac{1}{4}\delta^A_B\eta_i^C\partial_{iC}]\\
q^{\alpha A} &= \sum_i \lambda_i^{\alpha}\eta_i^A & \bar q^{\dot\alpha}_A &= \sum_i \tilde\lambda_i^{\dot\alpha}\partial_{iA}\\
s_{\alpha A} &= \sum_i \partial_{i\alpha}\partial_{iA}, & \bar s^A_{\dot\alpha} &= \sum_i \eta_i^A\partial_{i\dot\alpha}\\
c &= \sum_i [1 + \tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} - \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} - \tfrac{1}{2}\eta_i^A\partial_{iA}]
\end{aligned}
\tag{A.4.5}
$$

The generators of the dual superconformal algebra are constructed starting from their canonical action on dual (x, θ) coordinates, and complementing them in order the preserve the constraints

$$
(x_i - x_{i+1})_{\alpha\dot\alpha} - \lambda_{i\,\alpha}\,\tilde\lambda_{i\,\dot\alpha} = 0\,, \qquad (\theta_i - \theta_{i+1})^A_\alpha - \lambda_{i\alpha}\eta_i^A = 0 \tag{A.4.6}
$$

which define dual space itself, modulo constraints. Then

$$
P_{\alpha\dot\alpha} = \sum_i \partial_{i\alpha\dot\alpha}\,, \qquad Q_{\alpha A} = \sum_i \partial_{i\alpha A}\,, \qquad \overline{Q}^A_{\dot\alpha} = \sum_i [\theta_i^{\alpha A}\partial_{i\alpha\dot\alpha} + \eta_i^A\partial_{i\dot\alpha}], \tag{A.4.7}
$$

$$
M_{\alpha\beta} = \sum_i [x_{i(\alpha}{}^{\dot\alpha}\partial_{i\beta)\dot\alpha} + \theta^A_{i(\alpha}\partial_{i\beta)A} + \lambda_{i(\alpha}\partial_{i\beta)}]\,, \qquad \overline{M}_{\dot\alpha\dot\beta} = \sum_i [x_{i(\dot\alpha}{}^{\alpha}\partial_{i\dot\beta)\alpha} + \tilde\lambda_{i(\dot\alpha}\partial_{i\dot\beta)}] \tag{A.4.8}
$$

$$
R^A{}_B = \sum_i [\theta_i^{\alpha A}\partial_{i\alpha B} + \eta_i^A\partial_{iB} - \tfrac{1}{4}\delta^A_B\theta_i^{\alpha C}\partial_{i\alpha C} - \tfrac{1}{4}\delta^A_B\eta_i^C\partial_{iC}] \tag{A.4.9}
$$

$$
D = \sum_i [-x_i^{\dot\alpha\alpha}\partial_{i\alpha\dot\alpha} - \tfrac{1}{2}\theta_i^{\alpha A}\partial_{i\alpha A} - \tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} - \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha}] \tag{A.4.10}
$$

$$
C = \sum_i [-\tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} + \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} + \tfrac{1}{2}\eta_i^A\partial_{iA}] \tag{A.4.11}
$$

$$
S^A_\alpha = \sum_i [-\theta^B_{i\alpha}\theta_i^{\beta A}\partial_{i\beta B} + x_{i\alpha}{}^{\dot\beta}\theta_i^{\beta A}\partial_{i\beta\dot\beta} + \lambda_{i\alpha}\theta_i^{\gamma A}\partial_{i\gamma} + x_{i+1\,\alpha}{}^{\dot\beta}\eta_i^A\partial_{i\dot\beta} - \theta^B_{i+1\,\alpha}\eta_i^A\partial_{iB}]\,, \tag{A.4.12}
$$

$$
\overline{S}_{\dot\alpha A} = \sum_i [x_{i\dot\alpha}{}^{\beta}\partial_{i\beta A} + \tilde\lambda_{i\dot\alpha}\partial_{iA}] \tag{A.4.13}
$$

$$
K_{\alpha\dot\alpha} = \sum_i [x_{i\alpha}{}^{\dot\beta}x_{i\dot\alpha}{}^{\beta}\partial_{i\beta\dot\beta} + x_{i\dot\alpha}{}^{\beta}\theta^B_{i\alpha}\partial_{i\beta B} + x_{i\dot\alpha}{}^{\beta}\lambda_{i\alpha}\partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot\beta}\tilde\lambda_{i\dot\alpha}\partial_{i\dot\beta} + \tilde\lambda_{i\dot\alpha}\theta^B_{i+1\,\alpha}\partial_{iB}] \tag{A.4.14}
$$

A.5 Integrals in dimensional regularization.

In this Appendix we list a number of properties for momentum integrals entering the evaluation of four–point scattering amplitudes.

We work in dimensional regularization, $d = 3-2\epsilon$, with dimensional reduction (spinors and ϵ–tensors are kept strictly in three dimensions).

A.5.1 Tools.

When evaluating one–loop bubbles and two–loop factorized and composite bubbles we used the $G[a,b]$ functions defined by

$$G[a,b] = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(a+b-d/2)\,\Gamma(d/2-a)\,\Gamma(d/2-b)}{\Gamma(a)\,\Gamma(b)\,\Gamma(d-a-b)}. \qquad \text{(A.5.1)}$$

Moreover, we introduce the compact notation $\Gamma(a_1|...|a_n) = \Gamma(a_1)...\Gamma(a_n)$.

In order to write a momentum integral in its Feynman parameterized form, the basic identity is

$$\frac{1}{A_1^{\alpha_1}} \cdots \frac{1}{A_n^{\alpha_n}} = \frac{\Gamma(\alpha_1+\cdots+\alpha_n)}{\Gamma(\alpha_1\,|\,\ldots\,|\,\alpha_n)} \int_0^1 \frac{d\beta_1 \ldots d\beta_n\, \delta(\beta_1+\cdots+\beta_n-1)\, \beta_1^{\alpha_1-1} \ldots \beta_n^{\alpha_n-1}}{(\beta_1\, A_1+\cdots+\beta_n\, A_n)^{\alpha_1+\cdots+\alpha_n}}, \qquad \text{(A.5.2)}$$

where A_j are generic propagators. The integration over Feynman parameters makes often use of the identity

$$\int_0^1 d\beta_1 \ldots d\beta_n\, \delta(\beta_1+\cdots+\beta_n-1)\, \beta_1^{\alpha_1-1} \ldots \beta_n^{\alpha_n-1} = \frac{\Gamma(\alpha_1\,|\,\ldots\,|\,\alpha_n)}{\Gamma(\alpha_1+\cdots+\alpha_n)} \qquad \text{(A.5.3)}$$

The most complicated computations of the one and two–loop on–shell amplitude were performed using Mellin-Barnes representations [100, 101]. These representations are based on the identity

$$\frac{1}{(k^2+M^2)^a} = \frac{1}{(M^2)^a\Gamma(a)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\mathbf{s}\, \Gamma(-\mathbf{s})\Gamma(\mathbf{s}+a) \left(\frac{k^2}{M^2}\right)^{\mathbf{s}}, \qquad \text{(A.5.4)}$$

where the contour is given by a straight line along the imaginary axis such that indentations are used if necessary in order to leave the series of poles $\mathbf{s} = 0, 1, \cdots, n$ to the right of the contour and the series $\mathbf{s} = -a, -a-1, \cdots, -a-n$ to the left of the contour.

After Feynman–parameterizing a triangle integral and using (A.5.4), the following formula holds

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^{2\mu_1}(k-p)^{2\mu_2}(k+q)^{2\mu_3}} = \frac{1}{(4\pi)^{d/2}\prod_i \Gamma(\mu_i)\Gamma(d-\sum_i \mu_i)}\times$$
$$\times \int_{-i\infty}^{i\infty} \frac{d\mathbf{s}\, d\mathbf{t}}{(2\pi i)^2}\, \frac{\Gamma\left(-\mathbf{s}|-\mathbf{t}|\frac{d}{2}-\mu_1-\mu_2-\mathbf{s}|\frac{d}{2}-\mu_1-\mu_3-\mathbf{t}|\mu_1+\mathbf{s}+\mathbf{t}|\sum_i\mu_i-\frac{d}{2}+\mathbf{s}+\mathbf{t}\right)}{(p^2)^{-\mathbf{s}}\,(q^2)^{-\mathbf{t}}\,(p+q)^{2(\mathbf{s}+\mathbf{t}+\sum_i\mu_i-d/2)}} \tag{A.5.5}$$

while for a vector–like triangle we have

$$\int \frac{d^dk}{(2\pi)^d}\frac{k^\nu}{k^{2\mu_1}(k-p)^{2\mu_2}(k+q)^{2\mu_3}} =$$
$$= \frac{(4\pi)^{-d/2}}{\prod_i \Gamma(\mu_i)\Gamma(d-\sum_i \mu_i+1)} \int_{-i\infty}^{i\infty} \frac{d\mathbf{s}\, d\mathbf{t}}{(2\pi i)^2}\, \frac{\Gamma\left(-\mathbf{s}|-\mathbf{t}|\mu_1+\mathbf{s}+\mathbf{t}|\sum_i\mu_i-\frac{d}{2}+\mathbf{s}+\mathbf{t}\right)}{(p^2)^{-\mathbf{s}}\,(q^2)^{-\mathbf{t}}\,(p+q)^{2(\mathbf{s}+\mathbf{t}+\sum_i\mu_i-d/2)}}\times$$
$$\left[\Gamma(\tfrac{d}{2}-\mu_1-\mu_2-\mathbf{s}|\tfrac{d}{2}-\mu_1-\mu_3-\mathbf{t}+1)p^\nu-\Gamma(\tfrac{d}{2}-\mu_1-\mu_2-\mathbf{s}+1|\tfrac{d}{2}-\mu_1-\mu_3-\mathbf{t})q^\nu\right] \tag{A.5.6}$$

where the multiple contours are taken using the convention already mentioned for the relative position of the poles. When the position of a pole is chosen differently compared to the convention, it is customary to use the notation $\Gamma^\star(z)$ for the gamma function involved.

We can proceed along similar lines for writing the Mellin–Barnes representation for box diagrams, vector–like boxes, etc. Using these representations, along with the Barnes first lemma

$$\int_{-i\infty}^{i\infty} \frac{d\mathbf{s}}{2\pi i}\,\Gamma(a+\mathbf{s})\Gamma(b+\mathbf{s})\Gamma(c-\mathbf{s})\Gamma(d-\mathbf{s}) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)} \tag{A.5.7}$$

we have been able to compute one and two–loop amplitudes in a manifestly analytical way, without performing numerical evaluations.

In passing from momentum to configuration space the following formula for multidimensional Fourier transforming (in Euclidean signature) proves useful

$$\int d^dp\,\frac{e^{-ipx}}{(p^2)^\nu} = \pi^{\frac{d}{2}}\,2^{d-2\nu}\,\frac{\Gamma\left(\frac{d}{2}-\nu\right)}{\Gamma(\nu)}\,\frac{1}{(x^2)^{\frac{d}{2}-\nu}} \tag{A.5.8}$$

A.5.2 Bubbles.

At one–loop, the evaluation of simple bubbles is required. Feynman parameterizing the integrand and working off–shell ($p^2 \neq 0$), we easily obtain

$$\mathcal{B}(p) \equiv \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}}\,\frac{1}{k^2(k+p)^2} = G[1,1]\,\frac{1}{|p|^{1+2\epsilon}} \sim \frac{1}{8\,|p|} + \mathcal{O}(\epsilon) \tag{A.5.9}$$

where we have defined $|p| \equiv \sqrt{p^2}$ and $G[a,b]$ is given in (A.5.1). On the other hand, if we are on–shell, $p^2 = 0$, the integral reduces to a tadpole–like integral and in dimensional regularization it vanishes.

A.5.3 Triangles.

We begin by evaluating the scalar triangle diagram of Fig. A.1.

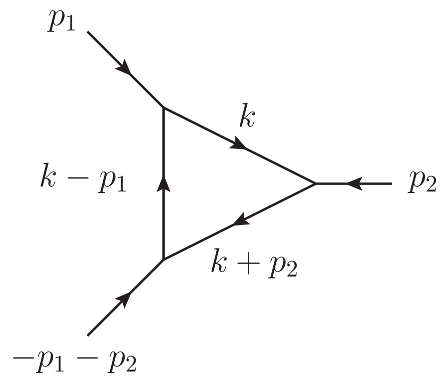


Figure A.1: The triangle diagram.

When the external momenta are off-shell ($p_i^2 \neq 0$), the integral can be computed in three dimensions with no need for regularization. Since for $D = 3$ the triangle with propagator exponents $(1,1,1)$ satisfies the uniqueness condition [88], it evaluates to a rational function

$$\begin{aligned}\mathcal{T}(p_1,p_2) &\equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2\,(k-p_1)^2\,(k+p_2)^2} \\ &= \frac{1}{8|p_1||p_2||p_1+p_2|} \qquad \text{(A.5.10)}\end{aligned}$$

The corresponding integral, when evaluated on–shell ($p_i^2 = 0$) and in dimensional regularization, can be easily treated by Feynman parameterization and is given by

$$\begin{aligned}\mathcal{T}(p_1,p_2) &\equiv \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{1}{k^2\,(k-p_1)^2\,(k+p_2)^2}\bigg|_{p_i^2=0} \\ &= \frac{\Gamma(3/2+\epsilon)\Gamma^2(-1/2-\epsilon)}{(4\pi)^{\frac{3}{2}-\epsilon}\Gamma(-2\epsilon)} \frac{1}{|p_1+p_2|^{3+2\epsilon}} \\ &\sim -\epsilon\,\frac{1}{2|p_1+p_2|^3} + \mathcal{O}(\epsilon^2) \qquad \text{(A.5.11)}\end{aligned}$$

Therefore, in dimensional regularization and on–shell limit, the scalar triangle can be set to zero.

We then consider the on–shell, vector–like triangle. Again, by Feynman parameterization, it is straightforward to show that the integral is given by

$$\begin{aligned} \mathcal{T}_{\mathcal{V}}^{\alpha\beta}(p_1,p_2) &\equiv \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{k^{\alpha\beta}}{k^2\,(k-p_1)^2\,(k+p_2)^2}\bigg|_{p_i^2=0} \\ &= \frac{\Gamma(3/2+\epsilon)\Gamma(1/2-\epsilon)\Gamma(-1/2-\epsilon)}{(4\pi)^{\frac{3}{2}-\epsilon}\Gamma(1-2\epsilon)}\,\frac{(p_1-p_2)^{\alpha\beta}}{|p_1+p_2|^{3+2\epsilon}} \\ &\xrightarrow{\epsilon\to 0} \mathcal{B}(p_1+p_2)\,\frac{(p_2-p_1)^{\alpha\beta}}{(p_1+p_2)^2} \end{aligned} \tag{A.5.12}$$

where $\mathcal{B}(p_1+p_2)$ is the bubble in (A.5.9).

An important observation is that, as a consequence of the on–shell conditions, the vector–like triangle satisfies the following identities

$$\mathcal{T}_{\mathcal{V}}(p_1,p_2)\cdot(p_1+p_2) = -\mathcal{T}_{\mathcal{V}}(p_1,p_2)\cdot(p_3+p_4) = 0 \tag{A.5.13}$$

A.5.4 Boxes.

We now consider scalar and vector–like box diagrams drawn in Fig. A.2.

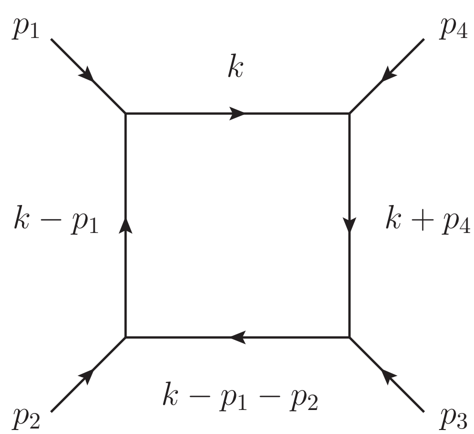


Figure A.2: The box diagram.

In terms of the Mandelstam variables, the scalar integral is written as

$$\mathcal{Q}(s,t) = \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{1}{k^2\,(k-p_1)^2\,(k-p_1-p_2)^2\,(k+p_4)^2}\bigg|_{p_i^2=0} \tag{A.5.14}$$

while the vector–like integral is

$$\mathcal{Q}_{\mathcal{V}}^{\mu} = \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{k^{\mu}}{k^2\,(k-p_1)^2\,(k-p_1-p_2)^2\,(k+p_4)^2}\bigg|_{p_i^2=0} \tag{A.5.15}$$

The scalar integral can be evaluated at leading order in ϵ. Feynman parameterizing the integrand in eq. (A.5.14) we obtain

$$\mathcal{Q}(s,t) = \frac{\Gamma(5/2+\epsilon)}{(4\pi)^{3/2-\epsilon}} \int \frac{dy_1\, dy_2\, dy_3\, dy_4\, \delta(\sum_i y_i - 1)}{(y_1\, y_3\, s + y_2\, y_4\, t)^{5/2+\epsilon}} \tag{A.5.16}$$

Expressing the denominator as a Mellin Barnes integral

$$\frac{1}{(y_1\, y_3\, s + y_2\, y_4\, t)^{5/2+\epsilon}} = \frac{1}{\Gamma(5/2+\epsilon)} \int \frac{du}{2\pi i} \Gamma(-u)\Gamma(u+5/2+\epsilon) \frac{(y_1\, y_3\, s)^u}{(y_2\, y_4\, t)^{5/2+\epsilon+u}} \tag{A.5.17}$$

and integrating on the Feynman parameters we obtain a one–fold representation

$$\frac{2\,\epsilon\,(1+2\epsilon)}{(4\pi)^{D/2}\,\Gamma(1-2\epsilon)\,t^{5/2+\epsilon}} \int \frac{du}{2\pi i} \Gamma(-u)\Gamma^2(-3/2-\epsilon-u)\Gamma(5/2+\epsilon+u)\Gamma^2(1+u) X^u \tag{A.5.18}$$

We note that the MB integral is multiplied by an ϵ factor and the integral itself is well defined when $\epsilon \to 0$. Therefore, to leading order in ϵ we have

$$\begin{aligned} \mathcal{Q}(s,t) &= \frac{\epsilon}{4\pi^{3/2}\, t^{5/2}} \int \frac{du}{2\pi i} \Gamma(-u)\Gamma^2(-3/2-u)\Gamma(5/2+u)\Gamma^2(1+u) X^u + \mathcal{O}(\epsilon^2) \\ &\equiv \frac{\epsilon}{4\pi^{3/2}\, t^{5/2}} (f_1(X) + f_2(X)) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{A.5.19}$$

where $X = s/t$. By closing the contour on the right, $f_1(X)$ is the sum of the residues at the poles of $\Gamma(-u)$, whereas $f_2(X)$ is the contribution from the poles of $\Gamma^2(-3/2-u)$.

The $f_1(X)$ function is easily computed and gives

$$f_1(X) = \pi^2 \sum_{n=0}^{\infty} \frac{(-X)^n\, n!}{\Gamma(5/2+n)} = 4\pi^{3/2} \left(\frac{\sqrt{1+X}}{X^{3/2}} \log(\sqrt{X} + \sqrt{1+X}) - \frac{1}{X} \right) \tag{A.5.20}$$

The $f_2(X)$ function is more complicated since we have to deal with double poles. A first set of manipulations leads to

$$f_2(X) = \frac{\pi}{X^{3/2}} \sum_{n=0}^{\infty} \frac{\Gamma(-1/2+n)(-X)^n}{n!} (\log X + \Psi(-1/2+n) - \Psi(1+n)) \tag{A.5.21}$$

where $\Psi(x)$ is the digamma function.

The first term in (A.5.21) is easily summed to

$$\sum_{n=0}^{\infty} \frac{(-X)^n\, \Gamma(-1/2+n)}{n!} = -2\sqrt{\pi}\,\sqrt{1+X} \tag{A.5.22}$$

The second series in (A.5.21) can be summed by using the trick

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-X)^n\, \Gamma(-1/2+n)\Psi(-1/2+n)}{n!} &= \frac{d}{da} \left(\sum_{n=0}^{\infty} \frac{(-X)^n\, \Gamma(a+n)}{n!} \right)_{a=-1/2} \\ = \frac{d}{da} \left(\frac{\Gamma(a)}{(1+X)^a} \right)_{a=-1/2} &= 2\sqrt{\pi}\sqrt{1+X}\,(\log(1+X) - \Psi(-1/2)) \end{aligned} \tag{A.5.23}$$

For the third term in (A.5.21) we use the following identity

$$\Psi(1+n) = -\gamma_E + \int_0^1 \frac{1-t^n}{1-t}\, dt \tag{A.5.24}$$

to rewrite the digamma function inside the series. By exchanging the order of the sum and the integral and summing the series, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-X)^n\,\Gamma(-1/2+n)\Psi(1+n)}{n!} &= 2\sqrt{\pi}\left(\gamma_E\sqrt{1+X} - \int_0^1 dt\,\frac{\sqrt{1+X}-\sqrt{1+tX}}{1-t}\right)\\ &= -2\sqrt{\pi}\sqrt{1+X}\left(\Psi(-1/2) + 2\log(1+\tfrac{1}{\sqrt{1+X}})\right) + 4\sqrt{\pi}\end{aligned} \tag{A.5.25}$$

where the integral in the first line has been performed using *Mathematica*.

Summing (A.5.22, A.5.23, A.5.25), after many non–trivial cancelations and simplifications we obtain as a final result

$$\mathcal{Q}(s,t) = \frac{\epsilon}{(\sqrt{s\,t})^3}\left[\sqrt{s+t}\,\log\left(\frac{\sqrt{s}+\sqrt{t}+\sqrt{s+t}}{\sqrt{s}+\sqrt{t}-\sqrt{s+t}}\right) - (\sqrt{s}+\sqrt{t})\right] + \mathcal{O}(\epsilon^2) \tag{A.5.26}$$

The vector–like box integral (A.5.15) can be computed by using the same methods as before or, equivalently, by using Passarino–Veltman reduction to write it as a linear combination of scalar integrals. In any case, at leading order in ϵ, we obtain

$$\begin{aligned}\mathcal{Q}^{\mu}_{\mathcal{V}} = \frac{\epsilon}{2(\sqrt{s\,t})^3}\Bigg\{&\frac{1}{\sqrt{s+t}}\,\log\left(\frac{\sqrt{s}+\sqrt{t}+\sqrt{s+t}}{\sqrt{s}+\sqrt{t}-\sqrt{s+t}}\right)\left[s\,(p_1-p_4)^{\mu} + t\,(p_1+p_2)^{\mu}\right]\\ &- \sqrt{t}\,(p_1-p_4)^{\mu} - \sqrt{s}\,(p_1+p_2)^{\mu}\Bigg\} + \mathcal{O}(\epsilon^2)\end{aligned} \tag{A.5.27}$$

It is interesting to note that the projections of $\mathcal{Q}_{\mathcal{V}}$ in the directions of p_1 and p_4 become very simple

$$\mathcal{Q}.p_1 = \frac{\epsilon}{4}\left(\frac{1}{s^{3/2}} - \frac{1}{t^{3/2}}\right) + \mathcal{O}(\epsilon^2) \quad \text{and} \quad \mathcal{Q}.p_4 = \frac{\epsilon}{4}\left(\frac{1}{t^{3/2}} - \frac{1}{s^{3/2}}\right) + \mathcal{O}(\epsilon^2) \tag{A.5.28}$$

since the logarithm term drops.

In fact, these two projections, can be calculated to all orders in ϵ. Writing $k \cdot p_1$ and $k \cdot p_4$ in the numerator of (A.5.15) as the difference of two squares, the integral reduces to the difference of two triangles. Therefore, using the results of Subsection B.2, we obtain

$$\begin{aligned}\mathcal{Q}.p_1 &= \frac{\Gamma(3/2+\epsilon)\Gamma^2(-1/2-\epsilon)}{2\,(4\pi)^{D/2}\Gamma(-2\epsilon)}\left(\frac{1}{t^{3/2+\epsilon}} - \frac{1}{s^{3/2+\epsilon}}\right)\\ \mathcal{Q}.p_4 &= \frac{\Gamma(3/2+\epsilon)\Gamma^2(-1/2-\epsilon)}{2\,(4\pi)^{D/2}\Gamma(-2\epsilon)}\left(\frac{1}{s^{3/2+\epsilon}} - \frac{1}{t^{3/2+\epsilon}}\right).\end{aligned} \tag{A.5.29}$$

Since the leading term for $\epsilon \to 0$ coincides with the expressions (A.5.28), this is a consistency check of our results.

A.5.5 Uniqueness.

In this section we review the technique of *unique triangles*, allowing a straightforward computation of some particular integrals. A triangle integral is said to be unique, whenever the indices (powers) of its propagators (α, β, γ) sum up to the dimension d of the integration measure $\alpha + \beta + \gamma = d$. If this is the case, then [88] the triangle evaluates to a simple rational function. In x–space it is

$$\begin{aligned} \mathcal{T}\left(d;\, \alpha_1, \alpha_2, \alpha_3\right) &\equiv \int d^d x \, \frac{1}{[(x-x_1)^2]^{\alpha_1}\,[(x-x_2)^2]^{\alpha_2}\,[(x-x_3)^2]^{\alpha_3}} \\ &\xrightarrow{\alpha_1+\alpha_2+\alpha_3=d} \frac{\pi^{\frac{d}{2}}\,\Gamma\left(\frac{d}{2}-\alpha_1\right)\Gamma\left(\frac{d}{2}-\alpha_2\right)\Gamma\left(\frac{d}{2}-\alpha_3\right)}{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)\Gamma\left(\alpha_3\right)} \\ &\qquad \frac{1}{[(x_2-x_1)^2]^{\frac{d}{2}-\alpha_3}\,[(x3-x_2)^2]^{\frac{d}{2}-\alpha_1}\,[(x3-x_1)^2]^{\frac{d}{2}-\alpha_2}} \end{aligned} \tag{A.5.30}$$

When a triangle is not unique, some relations between triangles in generic dimension d may prove useful to transform indices of inverse propagators

$$\begin{aligned} \mathcal{T}\left(d;\, \alpha_1, \alpha_2, \alpha_3\right) &= \frac{\Gamma\left(\sum_{i=1}^3 \alpha_i - \frac{d}{2}\right)\prod_{i=1}^3 \Gamma\left(\frac{d}{2}-\alpha_i\right)}{\Gamma\left(d-\sum_{i=1}^3 \alpha_i\right)\prod_{i=1}^3 \Gamma\left(\alpha_i\right)\,(x_{32}^2)^{\alpha_2+\alpha_3-\frac{d}{2}}} \\ &\qquad \mathcal{T}\left(d;\, \sum_{i=1}^3 \alpha_i - \frac{d}{2}, \frac{d}{2}-\alpha_3, \frac{d}{2}-\alpha_2\right) \end{aligned} \tag{A.5.31}$$

$$\mathcal{T}\left(d;\, \alpha_1, \alpha_2, \alpha_3\right) = \frac{1}{(x_{32}^2)^{\alpha_1+\alpha_2-\frac{d}{2}}\,(x_{32}^2)^{\alpha_1+\alpha_2-\frac{d}{2}}}\,\mathcal{T}\left(d;\, \alpha_2, \alpha_1, d-\sum_{i=1}^3 \alpha_i\right) \tag{A.5.32}$$

$$\begin{aligned} \mathcal{T}\left(d;\, \alpha_1, \alpha_2, \alpha_3\right) &= \frac{\Gamma\left(\sum_{i=1}^3 \alpha_i - \frac{d}{2}\right)\prod_{i=1}^3 \Gamma\left(\frac{d}{2}-\alpha_i\right)}{\Gamma\left(d-\sum_{i=1}^3 \alpha_i\right)\prod_{i=1}^3 \Gamma\left(\alpha_i\right)} \\ &\qquad \frac{\mathcal{T}\left(d;\, \frac{d}{2}-\alpha_1, \frac{d}{2}-\alpha_2, \frac{d}{2}-\alpha_3\right)}{(x_{21}^2)^{\alpha_1+\alpha_2-\frac{d}{2}}\,(x_{31}^2)^{\alpha_1+\alpha_3-\frac{d}{2}}\,(x_{32}^2)^{\alpha_2+\alpha_3-\frac{d}{2}}} \end{aligned} \tag{A.5.33}$$

A.6 Vertex diagram of the two–loop Wilson loop correction.

In this Appendix we give full details of the calculations concerning the vertex diagram in the computation of the two–loop corrections to the Wilson loop. The vertex diagram Fig. 5.1(b) is given by

$$\begin{aligned}\langle W_4\rangle^{(2)}_{\rm vertex} &= \frac{1}{N}\,\langle\,(i)^3 \oint_{z_i>z_j>z_k} dz^{\mu,\nu,\rho}_{i,j,k}\,{\rm Tr}\,(A_\mu\,A_\nu\,A_\rho)\left(i\int d^dw\mathcal{L}_{\rm int}(w)\right)\rangle\\ &= -\frac{1}{N}\,\frac{k}{4\pi}\,\frac{2}{3}\,(i)^5 \oint dz^{\mu,\nu,\rho}_{i,j,k}\int d^dw\langle{\rm Tr}\,(A_\mu\,A_\nu\,A_\rho)\,{\rm Tr}\,(A_\alpha\,A_\beta\,A_\gamma(w))\,\epsilon^{\alpha\beta\gamma}\rangle\end{aligned}$$

There are two non–vanishing contraction, plus cyclic permutations per each. However one is subleading in N:

$$(A^A_\mu)_{ab}\,(A^B_\nu)_{bc}\,(A^C_\rho)_{ca}\,(A^D_\alpha)_{de}\,(A^E_\beta)_{ef}\,(A^F_\gamma)_{fd}\,\delta^{AD}\,\delta^{BE}\,\delta^{CF}\rightarrow\delta_{ae}\,\delta_{bd}\,\delta_{bf}\,\delta_{ce}\,\delta_{af}\,\delta_{cd}=N \tag{A.6.1}$$

and the other order N^3

$$(A^A_\mu)_{ab}\,(A^B_\nu)_{bc}\,(A^C_\rho)_{ca}\,(A^D_\alpha)_{de}\,(A^E_\beta)_{ef}\,(A^F_\gamma)_{fd}\,\delta^{AE}\,\delta^{BD}\,\delta^{CF}\rightarrow-\delta_{af}\,\delta_{be}\,\delta_{be}\,\delta_{cd}\,\delta_{af}\,\delta_{cd}=-N^3 \tag{A.6.2}$$

where the minus sign comes from the (suppressed) ϵ to compensate the different contraction of Lorentz indices. Hence this gives

$$\begin{aligned}&= i\left(\frac{N}{k}\right)^2\frac{1}{2\pi}\left(\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}}\right)^3\sum_{i>j>k}\int dz^\mu_i\,dz^\nu_j\,dz^\rho_k\,\epsilon^{\alpha\beta\gamma}\,\epsilon_{\mu\alpha\sigma}\,\epsilon_{\nu\beta\lambda}\,\epsilon_{\rho\gamma\tau}\\ &\int d^3w\,\frac{(z_i-w)^\sigma\,(z_j-w)^\lambda(z_k-w)^\tau}{(-(z_i-w)^2)^{\frac{d}{2}}\,(-(z_j-w)^2)^{\frac{d}{2}}\,(-(z_k-w)^2)^{\frac{d}{2}}}\\ &= -i\left(\frac{N}{k}\right)^2\frac{1}{2\pi}\left(\frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d-2}{2}}}\right)^3\sum_{i>j>k}\int dz^\mu_i\,dz^\nu_j\,dz^\rho_k\,\epsilon^{\alpha\beta\gamma}\,\epsilon_{\mu\alpha\sigma}\,\epsilon_{\nu\beta\lambda}\,\epsilon_{\rho\gamma\tau}\\ &\int d^3w\,\frac{(w-z_i)^\sigma\,(w-z_j)^\lambda\,(w-z_k)^\tau}{(-(z_i-w)^2)^{\frac{d}{2}}\,(-(z_j-w)^2)^{\frac{d}{2}}\,(-(z_k-w)^2)^{\frac{d}{2}}}\end{aligned} \tag{A.6.3}$$

The numerator of the expression above can be simplified, leading to

$$\begin{aligned}&dz^\mu_i\,dz^\nu_j\,dz^\rho_k\,\epsilon^{\alpha\beta\gamma}\,\epsilon_{\mu\alpha\sigma}\,\epsilon_{\nu\beta\lambda}\,\epsilon_{\rho\gamma\tau}\,(w-z_i)^\sigma\,(w-z_j)^\lambda\,(w-z_k)^\tau=\\ &=p^\mu_i\,ds_i\;p^\nu_j\,ds_j\;p^\rho_k\,ds_k\;\epsilon^{\alpha\beta\gamma}\,\epsilon_{\mu\alpha\sigma}\,\epsilon_{\nu\beta\lambda}\,\epsilon_{\rho\gamma\tau}\,(w-x_i)^\sigma\,(w-x_j)^\lambda\,(w-x_k)^\tau=\\ &=p^\mu_i\,ds_i\;p^\nu_j\,ds_j\;p^\rho_k\,ds_k\;\left(\delta^\beta_\sigma\,\delta^\gamma_\mu-\delta^\gamma_\sigma\,\delta^\beta_\mu\right)\,\epsilon_{\nu\beta\lambda}\,\epsilon_{\rho\gamma\tau}\,(w-x_i)^\sigma\,(w-x_j)^\lambda\,(w-x_k)^\tau\end{aligned}$$

where in the first line terms proportional to p inside z have been dropped thanks to antisymmetry of ϵ's.
Selecting for definiteness $i = 3$, $j = 2$ and $k = 1$

$$\begin{aligned}
&ds_3\, ds_2\, ds_1\, \left(p_3^\gamma\, p_2^\nu\, p_1^\rho\, (w-x_3)^\beta (w-x_2)^\lambda (w-x_1)^\tau + \right.\\
&\left. -p_3^\beta\, p_2^\nu\, p_1^\rho\, (w-x_3)^\gamma (w-x_2)^\lambda (w-x_1)^\tau \right)\, \epsilon_{\nu\beta\lambda}\epsilon_{\rho\gamma\tau}\\
&= ds_3\, ds_2\, ds_1\, \left[\epsilon\left(p_2 (w-x_3)(w-x_2)\right)\, \epsilon\left(p_1 p_3 (w-x_1)\right) + \right.\\
&\left. -\epsilon\left(p_2 p_3 (w-x_2)\right)\, \epsilon\left(p_1 (w-x_3)(w-x_1)\right)\right]\\
&= ds_3\, ds_2\, ds_1\, \left[-\left(\epsilon\left(p_2 w x_2\right) + \epsilon\left(p_2 x_3 w\right)\right)\, \epsilon\left(p_1 p_3 (w-x_1)\right) + \right.\\
&\left. +\epsilon\left(p_2 p_3 (w-x_2)\right)\, \left(\epsilon\left(p_1 w x_1\right) + \epsilon\left(p_1 x_3 w\right) - \epsilon\left(p_1 x_3 x_1\right)\right)\right]\\
&= ds_3\, ds_2\, ds_1\, \left[-\left(\epsilon\left(x_3 w x_2\right) - \epsilon\left(x_2 x_3 w\right)\right)\, \epsilon\left(p_1 p_3 (w-x_1)\right) + \right.\\
&\left. +\epsilon\left(p_2 p_3 (w-x_2)\right)\, \left(\epsilon\left(p_1 x_{31} w\right) - \epsilon\left(p_1 x_3 x_1\right)\right)\right]\\
&= ds_3\, ds_2\, ds_1\, \epsilon\left(p_2 p_3 (w-x_2)\right)\, \left(\epsilon\left(p_1 x_{31} w\right) - \epsilon\left(p_1 x_{31} x_1\right)\right)\\
&= ds_3\, ds_2\, ds_1\, \epsilon\left(p_2 p_3 (w-x_2)\right)\, \epsilon\left(p_1 x_{31} (w-x_1)\right)\\
&= ds_3\, ds_2\, ds_1\, \epsilon\left(p_2 p_3 (w-x_2)\right)\, \epsilon\left(p_1 p_2 (w-x_1)\right)
\end{aligned}$$

Finally the integral becomes

$$I_{321} = \int d^d w \int_0^1 ds_{1,2,3} \frac{\epsilon\left(p_2 p_3 (w-x_2)\right)\, \epsilon\left(p_2 p_1 (w-x_1)\right)}{(-(z_3-w)^2)^{\frac{d}{2}} (-(z_2-w)^2)^{\frac{d}{2}} (-(z_1-w)^2)^{\frac{d}{2}}} \tag{A.6.4}$$

Or alternatively, shifting $w \to w + z_2$ and neglecting pieces vanishing due to the antisymmetry of the ϵ symbol

$$I_{321} = \int d^d w \int_0^1 ds_{1,2,3} \frac{\epsilon\left(p_2 p_3 w\right)\, \epsilon\left(p_2 p_1 w\right))}{(-(z_{32}-w)^2)^{\frac{d}{2}} (-w^2)^{\frac{d}{2}} (-(z_{12}-w)^2)^{\frac{d}{2}}} \tag{A.6.5}$$

On this we perform Feynman parameterization

$$\begin{aligned}
&-\int d^d w \int_0^1 ds_{1,2,3} (-1)^{\frac{3}{2}d} \frac{\Gamma\left(\frac{3}{2}d\right)}{\Gamma\left(\frac{1}{2}d\right)^3} (\beta_1\, \beta_2\, \beta_3)^{\frac{1}{2}d-1}\\
&\frac{\epsilon\left(p_2 p_3 (w-x_2)\right)\, \epsilon\left(p_1 p_2 (w-x_1)\right)}{\left[\beta_3\, (z_3-w)^2 + \beta_2\, (z_2-w)^2 + \beta_1\, (z_1-w)^2\right]^{\frac{3}{2}d}}\\
&= -\int d^d w \int_0^1 ds_{1,2,3} (-1)^{-\frac{3}{2}d} \frac{\Gamma\left(\frac{3}{2}d\right)}{\Gamma\left(\frac{1}{2}d\right)^3} (\beta_1\, \beta_2\, \beta_3)^{\frac{1}{2}d-1}\\
&\frac{\epsilon\left(p_2 p_3 (w-x_2)\right)\, \epsilon\left(p_1 p_2 (w-x_1)\right)}{\left[(w-\rho)^2 + \beta_1\, \beta_2\, z_{12}^2 + \beta_2\, \beta_3\, z_{23}^2 + \beta_1\, \beta_3\, z_{13}^2\right]^{\frac{3}{2}d}}
\end{aligned} \tag{A.6.6}$$

We shift $w \to w + \rho$ where $\rho = \beta_1\, z_1 + \beta_2\, z_2 + \beta_3\, z_3$. In the numerator we get

$$
\begin{aligned}
&\epsilon\,(p_2p_3(w - x_2)) \to \epsilon\,(p_2p_3(w + \beta_1\, z_1 + \beta_2\, z_2 + \beta_3\, z_3 - x_2))\\
&= \epsilon\,(p_2p_3(w + \beta_1\, z_1 + \beta_2\, x_2 + \beta_3\, x_3 - x_2))\\
&= \epsilon\,\big(p_2p_3(w + \beta_1\, z_1 - \bar{\beta}_2\, x_2 + \beta_3\, x_3)\big)\\
&= \epsilon\,(p_2p_3w) + \beta_1\,\epsilon\,(p_2p_3z_1) - \bar{\beta}_2\epsilon\,(p_2p_3x_2) + \beta_3\epsilon\,(p_2p_3x_3)\\
&= \epsilon\,(p_2p_3w) + \beta_1\,\epsilon\,(p_2p_3(x_1 + s_1\, p_1)) - \bar{\beta}_2\epsilon\,(x_3x_4x_2) - \beta_3\epsilon\,(x_2x_4x_3)\\
&= \epsilon\,(p_2p_3w) + \beta_1\,\epsilon\,(p_2p_3x_1) + \beta_1\, s_1\,\epsilon\,(p_2p_3p_1)) - \bar{\beta}_2\epsilon\,(x_3x_4x_2) - \beta_3\epsilon\,(x_2x_4x_3)\\
&= \epsilon\,(p_2p_3w) + \beta_1\,\epsilon\,(p_2p_3x_1) - \beta_1\, s_1\,\epsilon\,(p_2p_3x_1)) + \beta_1\, s_1\,\epsilon\,(p_2p_3x_2)) +\\
&- \bar{\beta}_2\epsilon\,(x_3x_4x_2) - \beta_3\epsilon\,(x_2x_4x_3)\\
&= \epsilon\,(p_2p_3w) + \beta_1\,\bar{s}_1\,\epsilon\,(p_2p_3x_1)) + \beta_1\, s_1\,\epsilon\,(x_3x_4x_2)) + \bar{\beta}_2\epsilon\,(x_4x_3x_2) +\\
&- \beta_3\epsilon\,(x_2x_4x_3)\\
&= \epsilon\,(p_2p_3w) + \beta_1\,\bar{s}_1\,\epsilon\,(p_2p_3x_1)) - \beta_1\, s_1\,\epsilon\,(x_4x_3x_2)) + \beta_1\epsilon\,(x_2x_4x_3)\\
&= \epsilon\,(p_2p_3w) + \beta_1\,\bar{s}_1\,\epsilon\,(p_2p_3x_1)) + \beta_1\,\bar{s}_1\,\epsilon\,(x_4x_3x_2))\\
&= \epsilon\,(p_2p_3w) + \beta_1\,\bar{s}_1\,\epsilon\,(p_2p_3x_1)) + \beta_1\,\bar{s}_1\,\epsilon\,(p_3p_2x_2))\\
&= \epsilon\,(p_2p_3w) - \beta_1\,\bar{s}_1\,\epsilon\,(p_2p_3p_1))
\end{aligned}
$$

and

$$
\begin{aligned}
&\epsilon\,(p_1p_2(w - x_1)) \to \epsilon\,(p_1p_2(w + \beta_1\, z_1 + \beta_2\, z_2 + \beta_3\, z_3 - x_1))\\
&= \epsilon\,(p_1p_2(w + \beta_1\, x_1 + \beta_2\, x_2 + \beta_3\, z_3 - x_1))\\
&= \epsilon\,(p_1p_2w) - \bar{\beta}_1\,\epsilon\,(p_1p_2x_1) + \beta_2\,\epsilon\,(p_1p_2x_2) + \beta_3\,\epsilon\,(p_1p_2z_3)\\
&= \epsilon\,(p_1p_2w) - \bar{\beta}_1\,\epsilon\,(x_2x_3x_1) - \beta_2\,\epsilon\,(x_1x_3x_2) + \beta_3\,\epsilon\,(p_1p_2z_3)\\
&= \epsilon\,(p_1p_2w) - \bar{\beta}_1\,\epsilon\,(x_2x_3x_1) - \beta_2\,\epsilon\,(x_1x_3x_2) + \beta_3\,\epsilon\,(p_1p_2x_3) + \beta_3\, s_3\,\epsilon\,(p_1p_2p_3)\\
&= \epsilon\,(p_1p_2w) - \bar{\beta}_1\,\epsilon\,(x_2x_3x_1) - \beta_2\,\epsilon\,(x_1x_3x_2) + \beta_3\,\epsilon\,(x_1x_2x_3) + \beta_3\, s_3\,\epsilon\,(p_1p_2p_3)\\
&= \epsilon\,(p_1p_2w) + \bar{\beta}_1\,\epsilon\,(x_1x_3x_2) - \beta_2\,\epsilon\,(x_1x_3x_2) - \beta_3\,\epsilon\,(x_1x_3x_2) + \beta_3\, s_3\,\epsilon\,(p_1p_2p_3)\\
&= \epsilon\,(p_1p_2w) + \beta_3\, s_3\,\epsilon\,(p_1p_2p_3)
\end{aligned}
$$

Taking the product we retain only the quadratic and constant terms in w, the linear being vanishing when integrated.

$$
\epsilon\,(p_2p_3w)\,\epsilon\,(p_1p_2w) - \beta_1\,\bar{s}_1\,\beta_3\, s_3\,\epsilon\,(p_1p_2p_3)^2 \tag{A.6.7}
$$

The quadratic term is proportional to $\frac{1}{d}\, g_{\alpha\beta}\epsilon^{\mu\nu\alpha}\epsilon^{\rho\sigma\beta}\, p_2^\mu p_3^\nu p_1^\rho p_2^\sigma\, w^2$, giving $\frac{1}{d}\,(p_2\cdot p_1\, p_3\cdot p_2 - p_2^2\, p_3\cdot p_1)\, w^2 = \frac{1}{d}\, p_2\cdot p_1\, p_3\cdot p_2\, w^2 = \frac{1}{4\,d}\, x_{13}^2 x_{42}^2\, w^2$. The Feynman integral over w gives

$$\begin{aligned} &\frac{1}{4\,d}\,(-1)^{-\frac{3}{2}\,d}\int d^d w\, \frac{x_{13}^2 x_{42}^2\, w^2}{[w^2+\beta_1\,\beta_2\, z_{12}^2+\beta_2\,\beta_3\, z_{23}^2+\beta_1\,\beta_3\, z_{13}^2]^{\frac{3}{2}\,d}} \\ &= i\,\frac{1}{d}\,(-1)^{-\frac{3}{2}\,d}\int d^d w_E\, \frac{x_{13}^2 x_{42}^2\, -w_E^2}{[-w_E^2+\beta_1\,\beta_2\, z_{12}^2+\beta_2\,\beta_3\, z_{23}^2+\beta_1\,\beta_3\, z_{13}^2]^{\frac{3}{2}\,d}} \\ &= -i\,\frac{1}{4\,d}\int d^d w_E\, \frac{x_{13}^2 x_{42}^2\, w_E^2}{[w_E^2-\beta_1\,\beta_2\, z_{12}^2-\beta_2\,\beta_3\, z_{23}^2-\beta_1\,\beta_3\, z_{13}^2]^{\frac{3}{2}\,d}} \\ &= -i\,\pi^{\frac{d}{2}}\,\frac{1}{4\,d}\,\frac{d}{2}\,\frac{x_{13}^2 x_{42}^2\,\Gamma\,(d-1)}{\Gamma\left(\frac{3}{2}\,d\right)\,[-\beta_1\,\beta_2\, z_{12}^2-\beta_2\,\beta_3\, z_{23}^2-\beta_1\,\beta_3\, z_{13}^2]^{d-1}} \end{aligned} \tag{A.6.8}$$

The piece without any w in the numerator yields

$$\begin{aligned} &-\beta_1\,\bar{s}_1\,\beta_3\, s_3\,\epsilon\,(p_1 p_2 p_3)^2\;(-1)^{-\frac{3}{2}\,d}\int d^d w\,\frac{1}{[w^2+\beta_1\,\beta_2\, z_{12}^2+\beta_2\,\beta_3\, z_{23}^2+\beta_1\,\beta_3\, z_{13}^2]^{\frac{3}{2}\,d}} \\ &= -i\,\beta_1\,\bar{s}_1\,\beta_3\, s_3\,\epsilon\,(p_1 p_2 p_3)^2\;(-1)^{-\frac{3}{2}\,d}\int d^d w_E\,\frac{1}{[-w_E^2+\beta_1\,\beta_2\, z_{12}^2+\beta_2\,\beta_3\, z_{23}^2+\beta_1\,\beta_3\, z_{13}^2]^{\frac{3}{2}\,d}} \\ &= -i\,\beta_1\,\bar{s}_1\,\beta_3\, s_3\,\epsilon\,(p_1 p_2 p_3)^2\int d^d w_E\,\frac{1}{[w_E^2-\beta_1\,\beta_2\, z_{12}^2-\beta_2\,\beta_3\, z_{23}^2-\beta_1\,\beta_3\, z_{13}^2]^{\frac{3}{2}\,d}} \\ &= -i\,\pi^{\frac{d}{2}}\,\beta_1\,\bar{s}_1\,\beta_3\, s_3\,\epsilon\,(p_1 p_2 p_3)^2\;\frac{\Gamma\,(d)}{\Gamma\left(\frac{3}{2}\,d\right)}\,\frac{1}{[-\beta_1\,\beta_2\, z_{12}^2-\beta_2\,\beta_3\, z_{23}^2-\beta_1\,\beta_3\, z_{13}^2]^{d}} \\ &= i\,\pi^{\frac{d}{2}}\,\beta_1\,\bar{s}_1\,\beta_3\, s_3\,\frac{1}{4}\,x_{13}^2 x_{24}^2 (x_{13}^2+x_{24}^2)\,\frac{\Gamma\,(d)}{\Gamma\left(\frac{3}{2}\,d\right)}\,\frac{1}{[-\beta_1\,\beta_2\, z_{12}^2-\beta_2\,\beta_3\, z_{23}^2-\beta_1\,\beta_3\, z_{13}^2]^{d}} \end{aligned}$$

Collecting everything altogether

$$\begin{aligned} &-\int_0^1 ds_{1,2,3}(-1)^{\frac{3}{2}\,d}\,\frac{\Gamma\left(\frac{3}{2}\,d\right)}{\Gamma\left(\frac{1}{2}\,d\right)^3}\,(\beta_1\,\beta_2\,\beta_3)^{\frac{1}{2}\,d-1} \\ &\left(-i\,\pi^{\frac{1}{8}}\,\frac{d}{2}\,\frac{x_{13}^2 x_{42}^2\,\Gamma\,(d-1)}{\Gamma\left(\frac{3}{2}\,d\right)\,\Delta^{d-1}}+i\,\pi^{\frac{d}{2}}\,\beta_1\,\bar{s}_1\,\beta_3\, s_3\,\frac{1}{4}\,x_{13}^2 x_{24}^2 (x_{13}^2+x_{24}^2)\,\frac{\Gamma\,(d)}{\Gamma\left(\frac{3}{2}\,d\right)}\,\frac{1}{\Delta^d}\right) \\ &= i\,8\,\pi^{\frac{d}{2}}\,\frac{1}{\Gamma\left(\frac{1}{2}\,d\right)^3}\,x_{13}^2 x_{42}^2\int_0^1 ds_{1,2,3}\,(\beta_1\,\beta_2\,\beta_3)^{\frac{1}{2}\,d-1} \\ &\left(\frac{\Gamma\,(d-1)}{\Delta^{d-1}}-2\,\beta_1\,\bar{s}_1\,\beta_3\, s_3\,(x_{13}^2+x_{24}^2)\,\Gamma\,(d)\,\frac{1}{\Delta^d}\right) \\ &= i\,\frac{8\,\pi^{\frac{d}{2}}}{\Gamma\left(\frac{1}{2}\,d\right)^3}\,x_{13}^2 x_{42}^2\int_0^1 ds_{1,2,3}\,(\beta_1\,\beta_2\,\beta_3)^{\frac{1}{2}\,d-1} \\ &\left(\frac{\Gamma\,(d-1)}{\Delta^{d-1}}-2\,\beta_1\,\bar{s}_1\,\beta_3\, s_3\,(x_{13}^2+x_{24}^2)\,\Gamma\,(d)\,\frac{1}{\Delta^d}\right) \end{aligned} \tag{A.6.9}$$

where $\Delta = -\beta_1\,\beta_2\,z_{12}^2 - \beta_2\,\beta_3\,z_{23}^2 - \beta_1\,\beta_3\,z_{13}^2$. This integral is quite nasty and was solved in [80] by resorting to numerical evaluation, which leads to

$$I_{321} = \frac{i\pi^{\frac{d}{2}+1}}{8}\frac{\Gamma(d-1)}{\Gamma\left(\frac{d}{2}\right)^3}\left[2\log(2)\frac{(-x_{13}^2)^{2\epsilon}+(-x_{24}^2)^{2\epsilon}}{\epsilon} + \log^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + a_6 + \mathcal{O}(\epsilon)\right] \quad \text{(A.6.10)}$$

where a_6, is a numerical constant explicitly worked out in [80]. Finally the result for the full diagram yields

$$\langle W_4\rangle^{(2)}_{\text{vertex}} = -\left(\frac{N}{k}\right)^2\left[\frac{\log(2)}{4}\sum_{i=1}^{4}\frac{(-x_{i,i+2}^2\,\mu^2\pi e^{\gamma_E})^{2\epsilon}}{\epsilon} + \frac{1}{4}\log^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \frac{1}{4}a_6 - 2\log(2)\right]. \quad \text{(A.6.11)}$$

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