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and Dynamic Instability**

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Money Targeting, Heterogeneous Agents and Dynamic Instability

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Abstract

Christiano et al. (2005) have shown that a standard medium-sized DSGE model can successfully replicate VAR IRFs to a money supply shock. This important result vanishes under limited asset market participation. Further, even a moderate fraction of constrained consumers is sufficient to dampen the real interest rate reaction to inflation, thereby causing instability. The introduction of a simple fiscal automatic stabilizer restores stability and improves the dynamic performance of the model.

JEL classification: E52.

Keywords: Rule of Thumb Consumers, DSGE, Determinacy, Limited Asset Market Participation

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1 Introduction

New Keynesian business cycle analysis is characterized by optimizing agents (households and firms), and by a number of nominal and real frictions in goods, labor and financial markets. Due to its success in replicating estimated impulse responses of key macroeconomic variables to a money supply shock, the Christiano et al. (2005, CEE henceforth) model is widely regarded as the epitome of this approach.

Following a seminal contribution by Mankiw (2000), who introduced the notion of heterogeneous consumers (savers and spenders), a second strand of New Keynesian literature emphasizes the role of non-optimizing agents, i.e. agents that adopt a rule-of-thumb and fully consume their current income (RT consumers henceforth). Gali et al (2004, 2007), and Bilbiie (2008), showed how RT consumers can substantially affect both stability and aggregate dynamics of New Keynesian business cycle models. De Graeve et al. (2010) introduce RT consumers to model financial risk premia. Empirical research cannot reject the RT consumers hypothesis. Estimated structural equations for consumption growth report a share of RT consumers ranging from 26 to 40% (Jacoviello, 2004; Campbell and Mankiw, 1989) More recent estimates of dynamic stochastic general equilibrium models (Coenen and Straub, 2005; Forni, Monteforte and Sessa, 2009) obtain estimates around 35%. Erceg, Guerrieri and Gust (2006) calibrate the share of RT consumers to 50% in order to replicate the dynamic performance of the Federal Reserve Board Global Model. Critics of the approach might argue that the empirical relevance of RT consumers is bound to gradually decline along with the development of financial markets (Bilbiie, Meier and Müller, 2008). In fact, increasing regulation in the aftermath of the 2008 crisis (OECD 2009) is likely to increase the share of liquidity constrained households.

The paper brings together this two strands of literature. More specifically, we investigate the robustness of the CEE model response to money supply shocks when a fraction of households does not participate to financial markets. Our proposed modification to the CEE model is quite simple, but has profound implications. In fact we find that the model is dynamically unstable unless the share of non-optimizing consumers falls short of 35%. In addition, the dynamic performance of the model is dramatically affected even when the share of non-optimizing agents is restricted to less than 30%, and its celebrated ability to replicate the business cycle response to a monetary shock simply vanishes.

The intuition behind our results is rather simple. Under an exogenous money supply rule, optimizing households' consumption drives money demand and interest rate dynamics. RT consumers generate a "Keynesian multiplier", weakening the link between output and the nominal interest rate. Instability arises when the wedge between output and consumption of optimizing agents is sufficiently large. Two frictions play an important role in determining instability. Nominal wage stickiness dampens the real wage response to shocks and substantially weakens the multiplier effect of RT consumption decisions. The opposite effect is induced by consumption habits, which limit optimizing consumers responses to shocks.

Since atheoretical VAR models suggest that some stabilizing mechanism eventually forces the economy back to steady state when monetary policy is exogenous, we explore whether a fiscal automatic stabilizer can solve the instability problem. In the original CEE model Ricardian equivalence obtains and automatic stabilizers essentially play no role. In our framework they are quite effective in driving RT consumption. In fact we obtain that the model now is stable irrespective of the share of RT consumers, and the dynamic performance of the system closely follows the original CEE model.

The rest of the paper is organized as follows: In the next section we describe in detail the model structure, we then present the results concerning the model stability in section 3. Section 4 proposes alternative ways to regain stability of the model. Section 5 concludes.

2 The Model

We augment the CEE model to account for both Ricardian RT consumers. The behavior of these latter agents is characterized by a simple rule of thumb: they consume their available labor income in each period, and do not save or smooth consumption over time. The key distinction between the two groups concerns intertemporal optimization. Ricardian consumers' choices take into account future utility when choosing consumption and portfolio composition. Rule-of-Thumb consumers spend their whole income every period, thus they do not hold any wealth.

In the paper we maintain the financial structure defined in CEE. This implies that a cash-in-advance constraint is imposed on firms. The latter must hold money in order to finance the wage bill before production is sold. Ricardian consumers' demand for money is derived from their portfolio optimization. Money holdings of Rule-of-Thumb consumers correspond to their (firms-financed) nominal labour income, and are entirely used to finance current consumption.

2.1 Households preferences

We assume a continuum of households indexed by $j \in [0, 1]$. RT consumers are defined over the interval $[0, \theta]$. The rest of the households, interval $(\theta, 1]$ accounts for Ricardian consumers. All households share the same utility function:

$$U_t^i = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln (C_t^i - bC_{t-1}^i) - \frac{\psi_l}{1 + \phi_l} (h_t^i)^{1+\phi_l} + \frac{\psi_q}{1 - \sigma_q} (q_t^i)^{1-\sigma_q} \right\} \quad (1)$$

where $i : o, rt$ stands for household type, $q_t^i = \frac{Q_t}{P_t}$ represents households real money balances, C_t^i represents total individual consumption, b denotes consumption internal habits and h_t^i denotes individual labour supply.

2.1.1 Consumption Bundles

C_t^i is a standard consumption bundle

$$C_t^i = \left[\int_0^1 c(z)_t^{\frac{\eta-1}{\eta}} dj \right]^{\frac{\eta}{\eta-1}} \quad (2)$$

where η represents the price elasticity of demand for the individual goods.

$$P_t = \left(\int_0^1 p(z)_t^{1-\eta} dj \right)^{\frac{1}{1-\eta}}$$

is the aggregate consumption price index.

2.2 Firms

Goods markets are monopolistically competitive, and good z is produced with the following technology:

$$y_t(z) = (k_t(z))^\alpha (h_t(z))^{1-\alpha}$$

where $k_t(z)$ defines the physical capital services obtained from households (see section 2.4 below) and $h_t(z)$ is the composite labor input used by each firm z . The latter is defined as follows

$$h_t(z) = \left(\int_0^1 (h_t^j(z))^{\frac{\alpha_w-1}{\alpha_w}} dj \right)^{\frac{\alpha_w}{\alpha_w-1}} \quad (3)$$

where the parameter $\alpha_w > 1$ is the intratemporal elasticity of substitution between labor inputs. For any given level of its labor demand $h_t(z)$, the optimal allocation across labor inputs implies

$$h_t^j(z) = \left(\frac{W_t^j}{W_t} \right)^{-\alpha_w} h_t^d(z) \quad (4)$$

where $W_t = \left(\int_0^1 (W_t^j)^{1-\alpha_w} dj \right)^{1/(1-\alpha_w)}$ is the standard wage index.

Firms are subject to a cash-in-advance constraint, i.e. they must borrow the wage bill $W_t h_t$ at the beginning of the period t and have to repay it at the end of the period at the gross interest rate R_t .

Firm z 's nominal total production cost is given by

$$TC_t(z) = R_t W_t h_t(z) + (1 + R_t^k) k_t(z) \quad (5)$$

The real marginal costs are:

$$mc_t = \left(\frac{r_t^k}{\alpha} \right)^\alpha \left(\frac{w_t R_t}{(1-\alpha)} \right)^{1-\alpha} \quad (6)$$

where $w_t = \frac{W_t}{P_t}$ and $r_t^k = \frac{R_t^k}{P_t}$.

2.2.1 Sticky Prices

Price stickiness is based on the Calvo mechanism. In each period firm z faces a probability $1 - \lambda_p$ of being able to reoptimize its price. When a firm is not able to reoptimize, it adjusts its price to the previous period inflation, $(1 + \pi_{t-1}) = \frac{P_{t-1}}{P_{t-2}}$. The price-setting condition therefore is:

$$p_t(z) = (1 + \pi_{t-1})^{\gamma_p} p_{t-1}(z) \quad (7)$$

where $\gamma_p \in [0, 1]$ represents the degree of price indexation.

All the $1 - \lambda_p$ firms which reoptimize their price at time t will face symmetrical conditions and set the same price \tilde{P}_t . When choosing \tilde{P}_t the optimizing firm will take into account that in the future it might not be able to reoptimize. In this case, the price at the generic period $t + s$ will read as $\tilde{P}_t \Pi_{t,t+s-1}^{\gamma_p}$ where $\Pi_{t,t+s-1} = (1 + \pi_t) \dots (1 + \pi_{t+s-1}) = \frac{P_{t+s-1}}{P_{t-1}}$.

\tilde{P}_t is chosen so as to maximize a discounted sum of expected future profits:

$$E_t \sum_{s=0}^{\infty} (\beta \lambda_p)^s \lambda_{t+s} \left(\tilde{P}_t \Pi_{t,t+s-1}^{\gamma_p} - P_{t+s} mc_{t+s} \right) y_{t+s}(z)$$

subject to:

$$y_{t+s}(z) = Y_{t+s}^d \left(\frac{\tilde{P}_t \Pi_{t,t+s-1}^{\gamma_p}}{P_{t+s}} \right)^{-\eta} \quad (8)$$

where Y_t^d is aggregate demand and λ_t is the stochastic discount factor.

The F.O.C. for this problem is

$$E_t \sum_{s=0}^{\infty} (\beta \lambda_p)^s \lambda_{t+s} Y_{t+s}^d \left[\begin{aligned} & (1 - \eta) \left(\Pi_{t,t+s-1}^{\gamma_p} \right)^{1-\eta} \tilde{P}_t^{-\eta} (P_{t+s})^{\eta} + \\ & + \eta \tilde{P}_t^{-\eta-1} P_{t+s}^{\eta+1} mc_{t+s} \left(\Pi_{t,t+s-1}^{\gamma_p} \right)^{-\eta} \end{aligned} \right] = 0 \quad (9)$$

2.3 Labor market

There is a continuum of differentiated labor inputs indexed by $j \in [0, 1]$. For each labor input there is a union j which monopolistically supplies the labor input j in the labor market j .

Each union sets the nominal wage, W_t^j , subject to (4). Each household i supplies all labour types at the given wage rate¹ and the total number of hours allocated to the different labor markets must satisfy the time resource constraint

$$h_t^i = \int_0^1 h_t^j dj = \int_0^1 \left(\frac{W_t^j}{W_t} \right)^{-\alpha_w} h_t^d dj \quad (10)$$

¹Under the assumption that wages always remain above all households' marginal rate of substitution, households are willing to meet firms' labour demand.

As in Galì (2007), we assume that the fraction of RT and Ricardian consumers is uniformly distributed across unions, and demand for each labour type is uniformly distributed across households. Ricardian and non-Ricardian households therefore work for the same amount of time, h_t . Individual labor income is

$$h_t^d W_t = \int_0^1 W_t^j \left(\frac{W_t^j}{W_t} \right)^{-\alpha_w} h_t^d dj \quad (11)$$

We posit that the union objective function is a weighted average $(1-\theta, \theta)$ of the utility functions of the two households types. This, in turn, implies that with flexible wages

$$w_t = \frac{W_t}{P_t} = \frac{\alpha_w}{\alpha_w - 1} \frac{\psi_l h_t^{\phi_l}}{[(1-\theta)U'(C_t^o - bC_{t-1}^o) + \theta U'(C_t^{rt} - bC_{t-1}^{rt})]} \quad (12)$$

where $\frac{\alpha_w}{(\alpha_w-1)}$ represents the wage markup over the average marginal rate of substitution.

2.4 Ricardian Households

Ricardian households maximize utility subject to the following period budget constraint.

Budget constraints in nominal terms:

$$M_{t+1} = R_t [M_t - Q_t + (\mu_t - 1)M_t] + A_{j,t} + Q_t + R_t^k u_t \bar{k}_t + \quad (13)$$

$$+ D_t - P_t [i_t + c_t + a(u_t) \bar{k}_t] + h_t^d \int_0^1 W_t^j \left(\frac{W_t^j}{W_t} \right)^{-\alpha_w} dj \quad (14)$$

Where M_t is the total amount of money and Q_t represents the nominal households cash balances. $R_t [M_t - Q_t + (\mu_t - 1)M_t]$ defines interest payments from firms which are subject to a cash-in-advance constraint $A_{j,t}$ and D_t are respectively the net cash flow from participating in state-contingent securities at time t and firm dividends.

Optimizing households own the physical stock of capital k_t , and choose the degree of its utilization, u_t , that rent to firms at the real rental rate r_t^k . The term $a(u_t)$ defines the real cost of using the capital stock with intensity u_t . Finally, i_t denotes time t real purchases of investment goods. The household's stock of physical capital evolves as:

$$\bar{k}_{t+1} = (1 - \delta) \bar{k}_t + i_t \left[1 - S \left(\frac{i_t}{i_{t-1}} \right) \right] \quad (15)$$

$$k_t = u_t \bar{k}_t \quad (16)$$

where δ and S respectively denote the physical rate of depreciation and investment adjustment costs.

The solution for the household problem closely follows CEE. The Euler equation is

$$\lambda_t^o = \beta E_t \lambda_{t+1}^o \frac{R_{t+1}}{\pi_{t+1}} \quad (17)$$

where

$$\frac{1}{C_t^o - bC_{t-1}^o} - \frac{\beta b}{C_{t+1}^o - bC_t^o} = \lambda_t^o \quad (18)$$

Ricardian households money demand depends therefore positively on current consumption and negatively on current interest rate.

$$\psi_q(q_t)^{-\sigma_q} = (R_t - 1) \lambda_t^o \quad (19)$$

The following first order conditions describe demand functions for capital² and investment and the optimal degree of capital utilization.

$$P_{k',t} = \beta E_t \left\{ \lambda_{t+1}^o \frac{r_{t+1}^k u_{t+1} - a(u_{t+1}) + (1 - \delta) P_{k',t+1}}{\lambda_t^o} \right\} \quad (20)$$

The first order condition for investment is

$$\lambda_t^o = E_t \left\{ \begin{aligned} & \lambda_t^o P_{k',t} \left[1 - S\left(\frac{i_t}{i_{t-1}}\right) - S'\left(\frac{i_t}{i_{t-1}}\right) \frac{i_t}{i_{t-1}} \right] + \\ & + \beta \lambda_{t+1}^o P_{k',t+1} \left[S'\left(\frac{i_{t+1}}{i_t}\right) \left(\frac{i_{t+1}}{i_t}\right)^2 \right] \end{aligned} \right\} \quad (21)$$

$$r_t^k = a'(u_t) \quad (22)$$

Following CEE and SGU the investment adjustment cost function and the capital utilization function are given by:³

$$S\left(\frac{i_t}{i_{t-1}}\right) = \frac{\kappa}{2} \left(\frac{i_t}{i_{t-1}} - 1\right)^2$$

$$a(u_t) = \gamma_1 (u_t - 1) + \frac{\gamma_2}{2} (u_t - 1)^2$$

² $P_{k',t}$ is the shadow relative price of one unit of capital with respect to one unit of consumption (Tobin's q).

³ Function $S(\cdot)$ satisfies the following properties. $S(1) = S'(1) = 0$ and $S''(1) > 0$. These restrictions imply the absence of adjustment costs up to a first order approximation around the deterministic steady state. The function $a(\cdot)$, instead, is assumed to satisfy $a(1) = 0$ and $a'(1), a''(1) > 0$. Moreover the parameters γ_1 and γ_2 are fixed given that $a'(u) = r^k$ at steady state.

2.4.1 Loan Market Clearing

The financial sector is characterized by a financial intermediary that, at the beginning of the period, receives a money transfer $(\mu_t - 1)M_t$ from the monetary authority and $M_t - Q_t$ from Ricardian households. Part of this money stock is lent to firms, who need to finance their wage bill. The rest is redistributed to the Ricardian households. Loan market clearing requires that

$$W_t L_t = \mu_t M_t - Q_t \quad (23)$$

2.5 Rule-of-Thumb Households

As pointed out above, RT consumers neither save or borrow. It is worth to recall that RT consumers also receive an amount of money at the beginning of the period in form of wage bill and spend the whole amount of money by the end of the period. Due to the labour market monopolistic structure, these agents are entirely passive. In fact both their consumption and their within-period money holdings are determined by union's (wage) and firms (worked hours) decisions.

$$c_t^{rt} = q_t^{rt} = \frac{h_t^i \int_0^1 \left(\frac{w_t^j}{w_t}\right)^{-\alpha_w} w_t^j dj}{P_t} \quad (24)$$

2.6 Sticky wages

In each period a union faces a constant probability $1 - \lambda_w$ of being able to reoptimize the nominal wage. Unions that cannot reoptimize simply index their wages to lagged inflation:

$$W_t^j = W_{t-1}^j \left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_w} = W_{t-1}^j (\pi_{t-1})^{\gamma_w}$$

where γ_w stands for the degree of wage indexation. Just like firms, when choosing the current wage, \tilde{W}_t , the optimizing union will anticipate that in the future it might not be able to reoptimize. In this case, the real wage at the generic period $t + s$ will read as

$$w_{t+s} = \tilde{w}_t \prod_{k=1}^s \frac{\pi_{t+k-1}^{\gamma_w}}{\pi_{t+k}} \quad (25)$$

Following Colciago(2008), the representative union objective function is defined as

$$L^u = \sum_{s=0}^{\infty} (\beta \lambda_w)^s \{ [(1 - \theta) U^o(C_{t+s}^o) + \theta U^{rt}(C_{t+s}^{rt})] - U(h_{t+s}) \} \quad (26)$$

Where U_s^o, U_s^{rt} are defined as in (1). Thus the wage-setting decision maximizes a weighted average of the two household types conditional to the probability that the wage cannot be reoptimized in the future. The relevant constraints are (10), (13), (24), (25).

The union's first-order condition is:

$$\sum_{s=0}^{\infty} (\beta\lambda_w)^s [(1-\theta)\lambda_{t+s}^o + \theta\lambda_{t+s}^{rt}] h_{t+s}^d (w_{t+s})^{\alpha_w} \left(\prod_{k=1}^s \frac{\pi_{t+k-1}^{\gamma_w}}{\pi_{t+k}} \right)^{-\alpha_w} \cdot \left[\tilde{w}_t \left(\prod_{k=1}^s \frac{\pi_{t+k-1}^{\gamma_w}}{\pi_{t+k}} \right) - \frac{\alpha_w}{(\alpha_w - 1)} \frac{\psi_l h_{t+s}^{\phi_l}}{[(1-\theta)\lambda_{t+s}^o + \theta\lambda_{t+s}^{rt}]} \right] = 0 \quad (27)$$

where $\lambda_{t+s}^{rt} = \frac{1}{C_t^o - bC_{t-1}^o} - \frac{\beta b}{C_{t+1}^o - bC_t^o}$. It is worth noting that the combination of centralized wage setting and wage stickiness introduces an indirect form of consumption smoothing for RT consumers.

2.7 Aggregation

Aggregating budget constraints for each sector, after few manipulations we get the aggregate resource constraint as

$$Y_t = C_t + I_t + a(u_t) K_t$$

where

$$C_t = \int_0^1 C_t^i(j) dj = \int_0^{\theta} C_t^{rt}(j) dj + \int_{\theta}^1 C_t^o(j) dj = \theta C_t^{rt} + (1-\theta)C_t^o \quad (28)$$

$$I_t = (1-\theta) \int_{\theta}^1 I_t^o(j) dj \quad (29)$$

$$K_t = (1-\theta) \int_{\theta}^1 K_t^o(j) dj \quad (30)$$

2.8 Monetary Policy

We assume a passive monetary authority which follows a simple rule on the money growth rate

$$\mu_t = 0.5\mu_{t-1} + \varepsilon_t \quad (31)$$

where $\mu_t = \frac{M_t}{M_{t-1}}$ and ε_t is an i.i.d. exogenous shock with zero mean and standard deviation σ_{ε}

3 Stability Analysis

After standard log-linearization⁴, it is possible to reduce the model to a system of just dynamic equations in the form

$$\widehat{X}_{t+1} = A^{-1}B \widehat{X}_t + \varepsilon_t \quad (32)$$

where the vector \widehat{X} contains the variables of the reduced system: $\widehat{X}_t = \left[\widehat{\pi}_t \quad \widehat{w}_t \quad \widehat{c}_t \quad \widehat{k}_{t-1} \quad \widehat{m}_{t-1} \quad \widehat{R}_t \quad \widehat{u}_t \quad \widehat{P}_{k',t} \quad \widehat{h}_t \quad \widehat{q}_t \quad \widehat{y}_t \right]$, and ε_t is a vector representing an exogenous shock, with zero mean and standard deviation σ_ε , to the money growth rate.

Given the complexity of the system, numerical methods are the only way to study its determinacy properties. In table 1 we present the parameters chosen for our baseline simulations. They follow CEE(2005) and Schmitt-Grohe, Uribe (2004) with the obvious exception of the RT consumers share, which is set at 0.5, as in Galí (2004). The parameter governing the degree of habit persistence, b , is set at 0.7, as in Boldrin et al. (2001). We calibrate the parameters γ_1 and γ_2 in order to have $\frac{a''}{a'} = 2.01$ as in Altig, et al. (2005)

Table 1

Parameter	Value	Description
θ	0.5	share of RT consumers
b	0.7	degree of habit persistence
β	$1.03^{-(0.25)}$	subjective discount factor
α	0.36	share of capital
δ	0.025	depreciation rate
η	6	price-elasticity of demand for a differentiated good
α_w	6	intra-temporal elasticity of substitution between labor inputs
κ	2.48	parameter governing investment adjustment costs
λ_w	0.64	degree of wage stickiness
λ_p	0.6	degree of price stickiness
γ_1	0.0324	parameter governing capacity adjustment costs
γ_2	0.0652	parameter governing capacity adjustment costs
σ_m	10.62	money demand elasticity
ϕ_m	0.5393	preference parameter
γ_p	1	indexation on prices
γ_w	1	indexation on wages
ϕ_l	3	preference parameter
σ_ε	0.15	std. deviation of the exogenous shock

3.1 Results

The baseline version of the model is unstable. Stability is recovered only for $\theta \leq 0.36$. In the following we check the robustness of this result to changes in

⁴See Appendix A.1

the model parameters and, at the same time, investigate the economic factors behind it. Given the size of the model, and the variety of nominal and real dynamic frictions it is very difficult to understand the mechanism through which the presence of rule of thumb consumers generates instability. To facilitate intuition, we begin with a very simple version of the model (model 1), where capital is fixed, wages are flexible, there is no habit in consumption and no cash in advance constraint on firms. We shall use this very simple model to sketch our interpretation of the instability result, which points at the weak response of the interest rate to output and inflation when monetary policy is exogenous and some consumers are non-Ricardian. Then, we introduce frictions in the following sequence: cash-in-advance constraint, wage stickiness, investment adjustment costs and variable capacity utilization, consumption habits. We will show that our interpretation is robust to these additions, and that the effect of each friction on stability depends on how it impacts on the co-movements of nominal interest rate and output.

To simplify presentation, we consider the combinations of price stickiness and share of RT consumers (parameters λ_p, θ) that define the stability frontier for each of the versions of the model considered. Our results are summarized in Figure 1, where we show how the stability frontiers shift when new frictions are introduced.

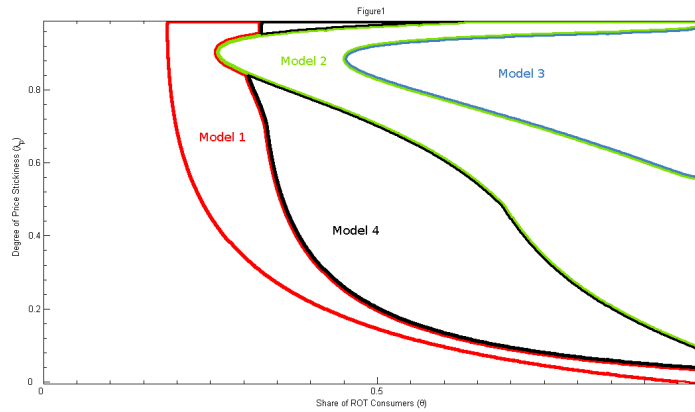


Figure 1: Determinacy Regions

3.1.1 Model 0

Using the relevant baseline parameters of Table 1, the model is unstable for $\theta \geq 0.23$. The threshold combinations λ_p, θ that define the stability frontiers tend to move in opposite directions: an increase in price stickiness requires a fall in the share of RT consumers.

From (18) and (19) it is easy to see that in this simple version of the model, the nominal interest rate is driven down by a money supply shock, but positively

reacts to Ricardian consumption and to an inflation increase. In log-linear form, interest rate dynamics are described by

$$\hat{R}_t = (R - 1) \hat{c}_t^o - \sigma_m (R - 1) \hat{m}_t = \quad (33)$$

$$= (R - 1) \left[\frac{\hat{y}_t}{1 - \theta} - \frac{\theta \hat{c}_t^{rt}}{1 - \theta} \right] + \sigma_m (R - 1) [\hat{\pi}_t - (0.5 \hat{\mu}_{t-1} + \hat{\varepsilon}_t)] + \quad (34)$$

$$- \sigma_m (R - 1) \hat{m}_{t-1} \quad (35)$$

where R denotes the steady-state value of the gross nominal interest rate.

Note that the interest rate response to current inflation is very weak. In our baseline simulations $\sigma_m (R - 1) = 0.0792$.⁵ The weak interest rate response to inflation is a structural feature of a policy regime based on an exogenous money supply rule. In standard models, where all agents are Ricardian, this is offset by the interest rate reaction to consumption. This is shown in figure 2 (solid line), where all consumers are Ricardian and consumption coincides with output. By contrast, as shown in (33), RT consumers generate a "Keynesian multiplier effect" on the initial surge of the Ricardian households and produce a wedge between output and consumption of Ricardian consumers, the variable that drives nominal interest rates in the model. Dashed lines in Figure 2 show that even with a small share of RT consumers ($\theta = 0.2$) the link between output and the nominal interest rate is weakened and substantial differences emerge in the dynamic performance of the model.

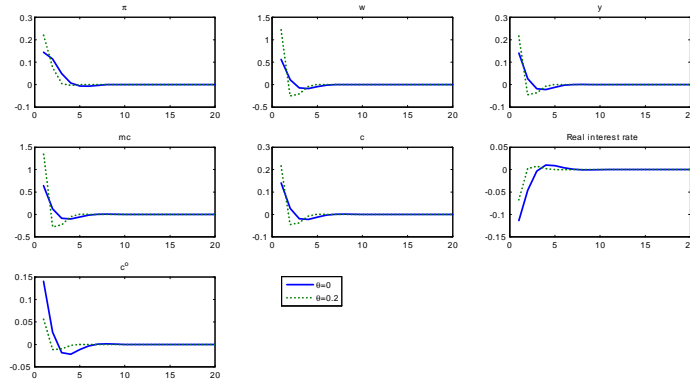


Figure 2: Responses to a Monetary Shock

In Figure 3 we depict the impact responses of y , c^o , π and R as functions of θ . The distance between y and c^o is increasing in θ . Note that inflation also

⁵In fact, by raising σ_m to 1500 from the baseline value of 10.62 it would be possible to obtain stability. Note, however, that σ_m is the inverse of the income elasticity of money demand, and that this would be in sharp contrast with consolidated empirical evidence and theoretical work. Several studies find an income elasticity between .5 and 1 (Choi and Oh, 2003; Knell and Stix, 2005).

grows with θ , whereas the nominal interest rate adjustment remains constant. The growing output "multiplier effect" associated with an increasing share of RT consumers and the apparent inability of the nominal interest rate to react to the stronger output response is the key mechanism driving the instability result.

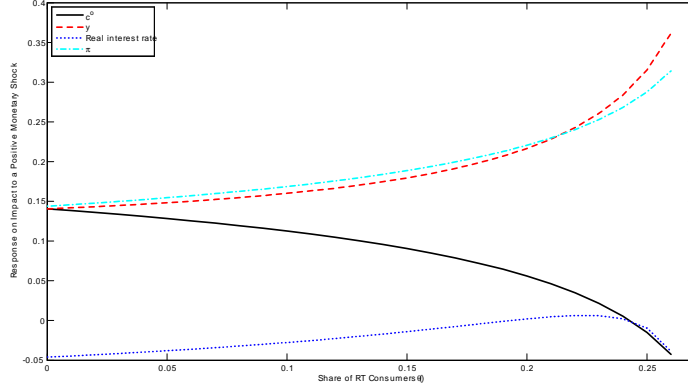


Figure 3: Responses on Impact to a Monetary Shock

3.1.2 Cash-in-advance constraint on firms (Model 1)

The cash-in-advance constraint implies that, in addition to Ricardian consumers money demand, we must now consider firms demand for money, i.e. the wage bill (see eq. 23). Money demand from optimizing consumers now is given by:

$$\hat{q}_t = \frac{1}{\sigma_m} \left(\hat{c}_t^o - \frac{R}{R-1} \hat{R}_t \right) \quad (36)$$

The market-clearing condition in the money market is

$$\hat{h}_t + \hat{w}_t = \frac{\frac{M}{Y}}{\frac{M}{Y} - \frac{Q}{Y}} (\hat{M}_{t-1} + \hat{\mu}_t) - \frac{\frac{Q}{Y}}{\frac{M}{Y} - \frac{Q}{Y}} \hat{q}_t \quad (37)$$

Substituting 24, 36 into 37, and taking into account that $\hat{c}_t^o = \frac{\hat{y}_t}{1-\theta} - \frac{\theta \hat{c}_t^{rt}}{1-\theta}$, we obtain the following expression for the nominal interest rate \hat{R}_t

$$\hat{R}_t = \frac{R-1}{R} \left(\frac{\hat{y}_t}{1-\theta} - \frac{\theta \hat{c}_t^{rt}}{1-\theta} \right) + \frac{R-1}{R} \sigma_m \frac{M-Q}{Q} \hat{c}_t^{rt} - \frac{R-1}{R} \sigma_m \frac{M}{Q} (0.5 \hat{\mu}_{t-1} + \hat{\varepsilon}_t + \hat{m}_{t-1}) \quad (38)$$

It is interesting to note an additional (positive) effect of RT households' consumption decisions on the interest rate, but the latter no longer responds to inflation. Relative to model 0, the stability frontier of the model is substantially unaffected.

3.1.3 Sticky wages (Model 2)

Wage stickiness dampens the real wage bill (only a fraction of wage setters can react to current inflation) and limits the output multiplier effect of RT consumers (Figure 4).⁶ As a result, the stability frontier of the model markedly shifts to the right (Figure 1).

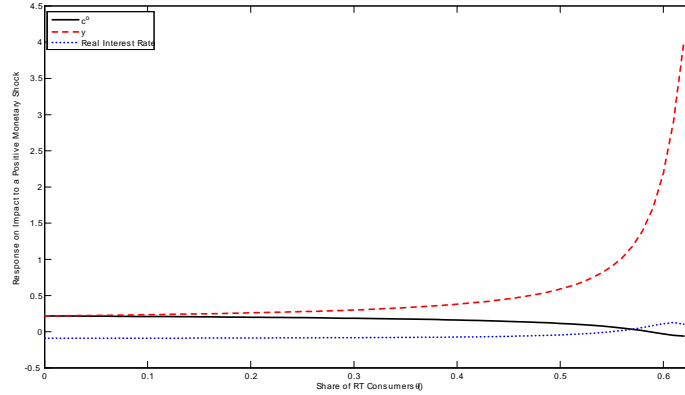


Figure 4: Responses on Impact to a Monetary Shock

3.1.4 Endogenous capital stock (Model 3)

The inclusion of capital enhances the beneficial effects of wage stickiness (Figure I). The key role is played by variable capacity utilization, which increases following the monetary shock. This, in turn, reduces labour demand and the wage bill, dampening RT consumption and its effects on marginal costs. (Figure 5).

⁶See Colciago(2008) for a detailed discussion about the role of wage stickiness in in presence of ROT consumers.

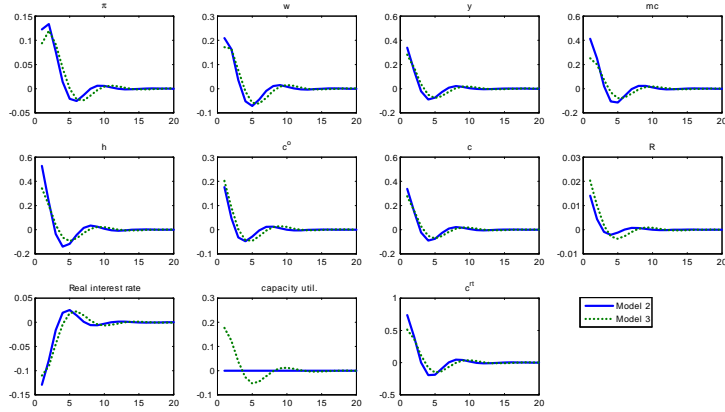


Figure 5: Responses to a Monetary Shock

3.1.5 Consumption habits (Model 4)

We return to the full model by adding habits on consumption in households' utility functions. The stability frontier now markedly shifts to the left (Figure 1).

Habit significantly dampens Ricardian households consumption in response to the monetary shock (Figure 6a). This, in turn, limits the interest rate adjustment to the monetary shock (Figure 6b). In Figure 7 we show that habit increases the wedge between output and Ricardian consumption reaction to the shock, thus confirming our intuition about the cause of model instability.

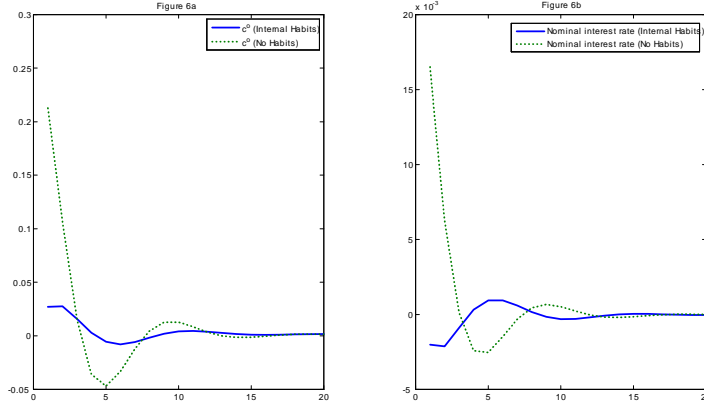


Figure 6: Responses to a Monetary Shock

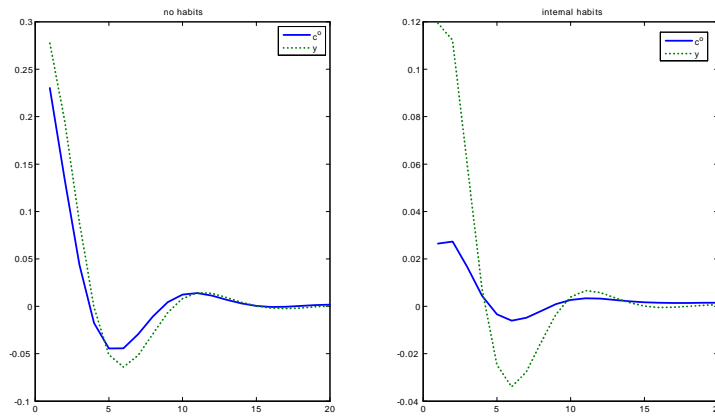


Figure 7: Responses to a Monetary Shock

3.1.6 Sensitivity analysis

As we pointed out in the previous section, habit persistence in consumption strongly affects model stability by dampening the interest rate response to Ricardians' consumption. Coeteris paribus, lowering b to 0.65⁷ enlarges the stability area and model's stability is guaranteed for $\theta < 0.47$. If we shift the degree of habit persistence to 0.8, that is, the value estimated in Fuhrer(2000) and Erceg et al. (2006) we see that the model is stable for $\theta \in [0, 0.19)$. Our results are ro-

⁷ $b = 0.65$ corresponds to the estimates in CEE(2005)

bust to alternative plausible values of $\kappa \in [0.5, 5]$, $\varphi_l \in [0.5, 10]$, $\sigma_m \in [1, 100]$ ⁸. Given the calibration on the other parameters, changing the values for money elasticity, the Frish elasticity and the degree of investment costs does not significantly change the threshold of constrained agents generating instability in this framework.

The intriguing role of indexation The last robustness check concerns wage and price indexation to past inflation. When we impose no indexation, i.e. $\gamma_p = \gamma_w = 0$, the stability area remains almost unaffected whereas outside it the model is stable but undetermined (figure 8).

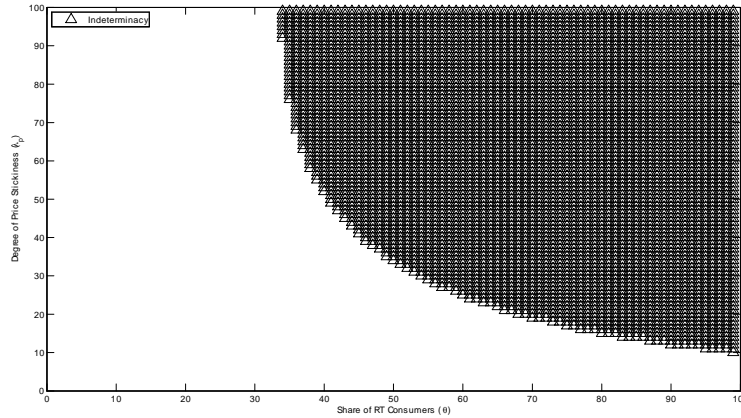


Figure 8: Determinacy Region

Our intuition is the following: as θ increases, the impact response of inflation grows with θ , whereas the real rate response is unaffected (figure 9). The response of the factor driving forward-looking adjustment, i.e. jumps in Ricardian agents' consumption and investment, becomes weaker. As result, with indexation-induced persistence, the initial inflation increase causes further inflation growth, a wage run up and instability. Without indexation, the initial inflation surge is always reversed with an almost monotonic pattern (figure 10). This happens because the inflation increase eventually cuts down the real wage bill and disposable income of RT consumers. This, in turn, implies that inflation reversal obtains irrespective of the real interest rate response, generating indeterminacy.

⁸Results available on request.

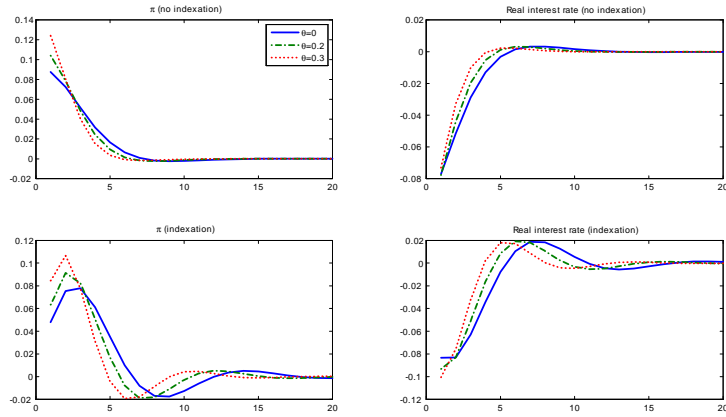


Figure 9: Responses to a Monetary Shock

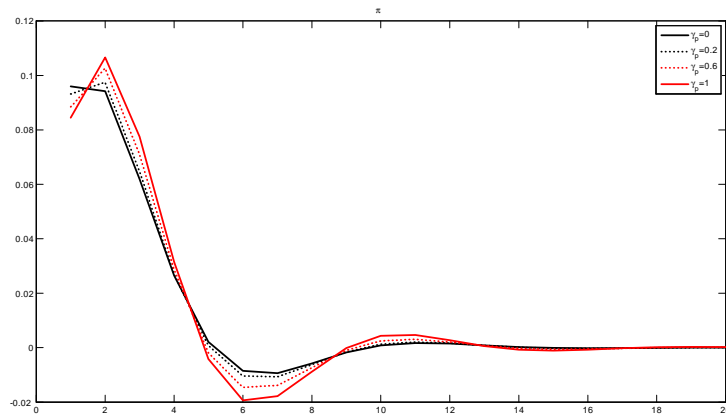


Figure 10: Responses to a Monetary Shock

4 Dynamics

We analyze the model responses to a monetary shock when the share of RT consumers is just below the threshold which would generate instability. As shown in figure 11, the dynamic properties of this model are upset when a relatively small share of RT consumer is considered in the economy. Aggregate consumption strongly rises on impact, and the hump-shaped dynamic response disappears. The multiplier effect of RT consumers reverses the nominal interest rate response, which turns positive on impact. Given the stronger response of

nominal interest rate and wages, profits now fall on impact.

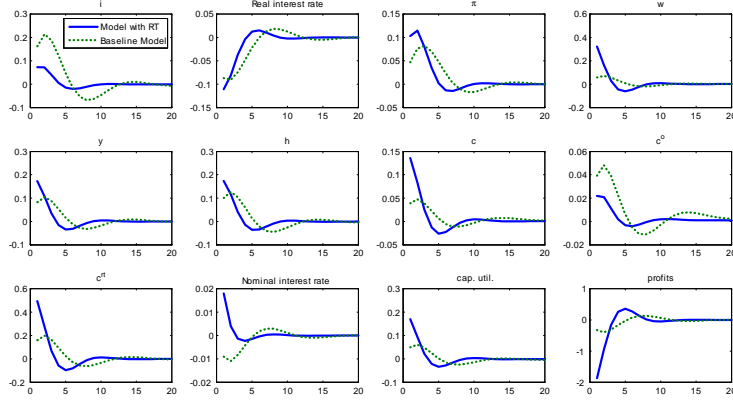


Figure 11: Responses to a Monetary Shock

5 Can we rescue the model?

Atheoretical VAR models suggest that some stabilizing mechanism eventually forces the economy back to steady state when monetary policy is exogenous. We have shown that replicating this result in a microfounded model accounting for even a limited share of RT consumers may be difficult. Since instability is given mainly by the limited interest rate response to output dynamics, we check whether a fiscal automatic stabilizer can solve the problem. To minimize modifications to the original CEE model, we assume that households must pay a lump sum tax whose amount, in turn, depends on the aggregate output gap. By definition, ricardian consumers decisions are not affected by this tax. To the contrary, RT consumption is modified as follows:

$$\hat{c}_t^{rt} = \hat{w}_t + \hat{h}_t - \widehat{tax}_t$$

where

$$\widehat{tax}_t = \gamma_y \hat{y}_t$$

As we see in figure 12, for plausible values of γ_y ($\gamma_y = 0.55$) the instability region shifts on the right. Moreover, this latter version of the model⁹ is characterized by impulse responses which are almost identical to the case of no RT consumers (figure 13). The mechanism of this taxation is similar to the one of the keynesian multiplier on income, the fluctuations are reduced proportionally with the increase in taxes.¹⁰

⁹We simulate the model with $\theta = 0.5$

¹⁰This result is akin to the one in Andres et al. (2008)

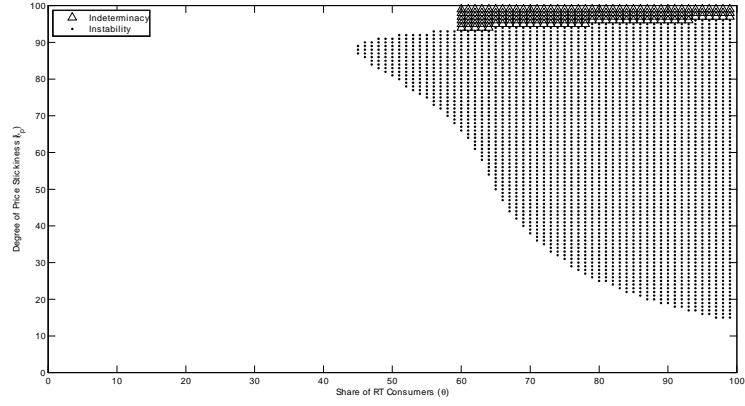


Figure 12: Determinacy Region

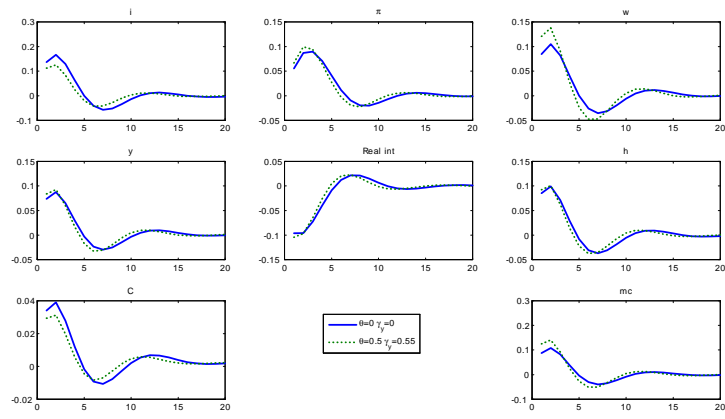


Figure 13: Responses to a Monetary Shock

6 Conclusion

We embodied limited asset market participation in a well known medium scale New Keynesian framework. We showed that when monetary policy is conducted following an exogenous rule on the money growth rate, the model is unstable for a limited share of Rule-of-Thumb consumers. The reason of this instability is that RT consumers behavior multiplies the response of output to a money supply shock, which cannot be restrained by the monetary rule. To restore dynamic stability we need to embed a fiscal stabilizer. This modified model maintains the dynamic performance and the consistency with empirical which characterized the original CEE framework based on a representative agent. A key result therefore is that, under limited asset market participation, plausible macromodels models should explicitly account for fiscal policies, at least in the simple form of automatic stabilizers.

We found that consumption habits play a key role in driving our results. Further research should investigate how different habits specifications could alter the model dynamic properties and how a proper fiscal setup can further improve the model's performance.

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7 Appendix A.1

7.1 log-linearized model

The stability analysis is conducted using a linearized version of the model presented above. Lower case letters from now on denote the log of the corresponding variable or their log deviations from the steady state.

Aggregate consumption is defined by:

$$\hat{C}_t = (1 - \theta) \frac{c^o}{C} \hat{c}_t^o + \theta \frac{c^{rt}}{C} \hat{w}_t + \theta \frac{c^{rt}}{C} \hat{h}_t \quad (39)$$

The next equations describe the market clearing condition and money demand:

$$\hat{h}_t + \hat{w}_t = \frac{\frac{M}{Y}}{\frac{M}{Y} - \frac{Q}{Y}} (\hat{M}_{t-1} + \hat{\mu}_t) - \frac{\frac{Q}{Y}}{\frac{M}{Y} - \frac{Q}{Y}} \hat{Q}_t \quad (40)$$

$$\hat{R}_t + \frac{R-1}{R} \hat{\lambda}_t^o + \frac{R-1}{R} \sigma_m \hat{q}_t = 0 \quad (41)$$

Marginal costs are given by

$$\widehat{mc}_t = (1 - \alpha) (\hat{w}_t + \hat{R}_t) + \alpha \hat{r}_t^k \quad (42)$$

The following equation combines firms' F.o.c. with respect to production factors

$$\hat{h}_t + \hat{w}_t + \hat{R}_t = \hat{k}_{t-1} + \left(1 + \frac{\gamma_1}{\gamma_2}\right) \hat{r}_t^k \quad (43)$$

Production function is given by

$$\hat{y}_t = \alpha \hat{k}_{t-1} + \alpha \frac{\gamma_1}{\gamma_2} \hat{r}_t^k + (1 - \alpha) \hat{h}_t \quad (44)$$

Aggregate resource constraint

$$\hat{y} = \frac{i}{y} \hat{i}_t + \frac{c}{y} \hat{c}_t + \gamma_1 \frac{\gamma_1}{\gamma_2} \frac{k}{y} \hat{r}_t^k \quad (45)$$

RT consumption

$$\hat{c}_t^{rt} = \hat{w}_t + \hat{h}_t \quad (46)$$

Euler equation

$$\hat{\lambda}_t^o = \hat{\lambda}_{t+1}^o + \hat{R}_{t+1} - \hat{\pi}_{t+1} \quad (47)$$

Households marginal utility of consumption

$$\lambda_t^o = \frac{\beta b}{(1 - \beta b)(1 - b)} c_{t+1}^o - \frac{(1 + \beta b^2)}{(1 - \beta b)(1 - b)} c_t^o + \frac{b}{(1 - \beta b)(1 - b)} c_{t-1}^o \quad (48)$$

$$\lambda_t^{rt} = \frac{\beta b}{(1 - \beta b)(1 - b)} c_{t+1}^{rt} - \frac{(1 + \beta b^2)}{(1 - \beta b)(1 - b)} c_t^{rt} + \frac{b}{(1 - \beta b)(1 - b)} c_{t-1}^{rt} \quad (49)$$

Investment decisions

$$\hat{i}_t - \frac{1}{k(1+\beta)} \hat{P}_{k',t} - \frac{1}{(1+\beta)} i_{t-1} - \frac{\beta}{(1+\beta)} \hat{i}_{t+1} = 0 \quad (50)$$

$$\hat{\pi}_{t+1} + \beta(1-\delta) \hat{P}_{k',t+1} - \hat{P}_{k',t} = \hat{R}_{t+1} - \beta r^k \hat{r}_{t+1}^k \quad (51)$$

Capital accumulation

$$\hat{k}_t = (1-\delta) \hat{k}_{t-1} + \delta \hat{i}_t \quad (52)$$

Phillips Curve

$$\frac{\lambda_p}{1-\lambda_p} (\hat{\pi}_t - \gamma_p \hat{\pi}_{t-1}) = (1-\beta\lambda_p) \widehat{mc}_t + \beta\lambda_p (\hat{\pi}_{t+1} - \gamma_p \hat{\pi}_t) + \beta \frac{\lambda_p^2}{1-\lambda_p} (\hat{\pi}_{t+1} - \gamma_p \hat{\pi}_t) \quad (53)$$

money growth rate

$$\mu_t = m_t - m_{t-1} + \pi_t \quad (54)$$

Wage inflation

$$\left[\begin{array}{l} \left(\frac{1}{1-\lambda_w} + \beta \frac{\lambda_w^2}{1-\lambda_w} \right) \hat{w}_t - \beta \frac{\lambda_w}{1-\lambda_w} \hat{w}_{t+1} + \\ - \left(\beta\lambda_w + \beta \frac{\lambda_w^2}{1-\lambda_w} \right) \hat{\pi}_{t+1} + \\ + \left(\beta\lambda_w\gamma_w + \beta \frac{\lambda_w^2}{1-\lambda_w} \gamma_w + \frac{\lambda_w}{1-\lambda_w} \right) \hat{\pi}_t + \\ - \frac{\lambda_w}{1-\lambda_w} \hat{w}_{t-1} - \frac{\lambda_w}{1-\lambda_w} \gamma_w \hat{\pi}_{t-1} \end{array} \right] = (1-\beta\lambda_w) \varphi \hat{h}_t - (1-\beta\lambda_w) \hat{\psi}_t \quad (55)$$

8 Appendix A.2

8.1 Steady State in the benchmark model

Relative to the CEE model, the presence of RT consumers influences the steady state uniquely for what concerns households individual consumption level.

From equation 19 and 20, and assuming zero inflation steady state, it holds true that

$$R = \frac{1}{\beta} \quad (56)$$

$$r^k = \frac{1}{\beta} - 1 + \delta \quad (57)$$

From cost minimization problem come the equations:

$$r^k = mc \alpha \left(\frac{k}{h} \right)^{\alpha-1} \quad (58)$$

$$w = \frac{mc(1-\alpha) \left(\frac{k}{h} \right)^\alpha}{R} \quad (59)$$

Combining the last two equation we get the real wage computed at steady state

$$mc = \frac{r^k}{\alpha} \left(\frac{k}{h} \right)^{1-\alpha} \quad (60)$$

$$w = \frac{r^k (1-\alpha)}{\alpha (R)} \left(\frac{k}{h} \right) \quad (61)$$

Combining (60) and $mc = \frac{\eta-1}{\eta}$ we get the ratio:

$$\frac{K}{h} = \left(\frac{r^k \eta}{\alpha \eta - 1} \right)^{\frac{1}{\alpha-1}} \quad (62)$$

From the production function we get

$$\frac{Y}{h} = \left(\frac{K}{h} \right)^\alpha \quad (63)$$

and as

$$\frac{I}{Y} = \delta \frac{K}{Y} \quad (64)$$

The aggregate resource constraint reads as:

$$Y = C + I \quad (65)$$

$$1 = \frac{C}{Y} + \frac{I}{Y} \quad (66)$$

the aggregate consumption-output ratio is given by

$$\frac{C}{Y} = 1 - \delta \frac{K}{h} \left(\frac{Y}{h} \right)^{-1} \quad (67)$$

The equation for the optimal wage allows us to derive the hours worked at steady state as

$$h = \left[\frac{\alpha_w - 1}{\alpha_w} \left[(1-\theta) \left(\frac{(1-\beta b) c}{(1-b) C^o} \right) + \theta \left(\frac{(1-\beta b) c}{(1-b) C^{rt}} \right) \right] \frac{c^{rt}}{c} \right]^{\frac{1}{(\phi_l+1)}}$$

so that

$$K = \frac{K}{h} h \quad (68)$$

Since RT individual consumption is given at steady state by

$$c^{rt} = w h$$

we can easily derive its relationship with aggregate consumption as

$$\frac{c^{rt}}{C} = \left(\frac{C}{Y} \right)^{-1} w \left(\frac{Y}{h} \right)^{-1} \quad (69)$$

Total consumption is the weighted average of the two groups components:

$$C = (1 - \theta) c^o + \theta c^{rt} \quad (70)$$

From the latter, it comes straightforward

$$\frac{c^o}{C} = \frac{1}{1 - \theta} - \frac{\theta}{1 - \theta} \frac{c^{rt}}{C} \quad (71)$$

Optimizing households consumption at steady state is given by the sum of labour income, firms profits return of capital and returns of money rents to firms:

$$c^o = wh + \frac{1}{1 - \theta} (\Pi + r^k K + (R - 1) wh) \quad (72)$$

where Π are firms profits and are defined as

$$\Pi = (1 - \frac{mc}{P})y = (1 - \frac{1}{\mu})y \quad (73)$$

with μ representing firms markup over prices. Thus optimizing agents are the richer the higher share of RT consumers.

Aggregate consumption can be finally rewritten as

$$C = (1 - \theta) c^o + \theta c^{rt} = wh + \Pi + r^k K + (R - 1) wh \quad (74)$$

8.2 Steady State in the Simplest Version of the Model.

In case of no cash in advance the steady state is described by:

$$R = \frac{1}{\beta}$$

$$Y = C \quad (75)$$

$$Y = h^{1-\alpha} \quad (76)$$

$$c^{rt} = wh \quad (77)$$

$$\frac{w_t}{(1 - \alpha)} h^\alpha = mc = \frac{\vartheta - 1}{\vartheta} = \frac{1}{\mu} \quad (78)$$

$$\frac{(1 - \alpha)}{\mu} h^{-\alpha} = w \quad (79)$$

$$\frac{(1 - \alpha)}{\mu} h^{1-\alpha} = wh \quad (80)$$

$$w = \frac{\frac{\alpha_w}{\alpha_w - 1} h^{\phi_t}}{\left[(1 - \theta) \frac{1}{c^o} + \theta \frac{1}{c^{rt}} \right]}$$

$$C = (1 - \theta) c^o + \theta c^{rt}$$

$$\begin{aligned}
\frac{wh}{y} &= \frac{c^{rt}}{c} = \frac{(1-\alpha)}{\mu} & (81) \\
\frac{c^o}{c} &= \frac{1}{1-\theta} - \frac{\theta}{1-\theta} \frac{(1-\alpha)}{\mu} \\
\left(\frac{(1-\alpha)}{\mu} \frac{\alpha_w - 1}{\alpha_w} \left[(1-\theta) \frac{c}{c^o} + \theta \frac{c}{c^{rt}} \right] \right)^{\frac{1}{\phi_l+1}} &= h
\end{aligned}$$