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The geometry of second symmetric products of curves

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A Pietro, Antonietta e Claudio,
senza i quali tutto ciò
non sarebbe stato possibile.

Contents

Introduction

This work concerns the study of some fundamental aspects of the geometry of symmetric products of curves. These varieties play a very important role in the development of the theory of algebraic curves. On one hand, symmetric products of curves are exploited by Brill-Noether theory to study special divisors on curves. On the other hand, they are deeply involved in the classical theory of correspondences on curves.

We deal throughout with several problems on this topic and we mainly focus on the case of the second symmetric product of a curve. In particular, we treat both some classical problems - as the study of the ample cone in the Néron-Severi group - and some attempts at extending the notion of gonality for curves. In order to present the contents of this work we would like to introduce some piece of notation together with our results.

Given a smooth complex projective curve C, we define its *gonality* as the minimum integer d such that the curve admits a covering $f\colon C \longrightarrow \mathbb{P}^1$ of degree d on the projective line and we denote it by $gon(C)$. If C is a singular curve, we denote by $\nu\colon \widetilde{C} \longrightarrow C$ its normalization and we define the gonality of C to be the gonality of the smooth curve $\widetilde{C}$.

A first generalization of gonality is the notion of degree of irrationality introduced by Moh and Heinzer in [41], which has been deeply studied by Yoshihara (see [58], [53] and [59]). If X is an irreducible complex projective variety of dimension n, the *degree of irrationality* of X is defined to be the integer

$$d_r(X) := min \left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a dominant rational map} \\ F\colon X \dashrightarrow \mathbb{P}^n \text{ of degree } d \end{array} \right\}.$$

This number is clearly a birational invariant and having $d_r(X) = 1$ is equivalent to rationality. Moreover, as any dominant rational map $f\colon C \dashrightarrow \mathbb{P}^1$ can be resolved to a morphism, it follows that $d_r(C) = gon(C)$ and hence the notion of degree of irrationality does provide an extension of gonality to n-dimensional varieties.

We would like to recall that if there exists a dominant rational map $C \dashrightarrow C'$ between curves, then $gon(C) \geq gon(C')$. On the other hand, the existence of a dominant rational map $X \dashrightarrow Y$ between varieties of dimension $n \geq 2$, does not lead to an analogous inequality for the degrees of irrationality. Indeed there are counterexamples in the case of surfaces (cf. [59] and [21]) and there are examples of non-rational threefolds that are unirational (see for instance [20] and [31]).

Turning to symmetric products of curves, we deal with the problem of computing the degree of irrationality of the second symmetric product $C^{(2)}$ of a smooth complex projective curve C of geometric genus g. Clearly, there is a strong connection between the existence of a dominant rational map $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$ and the genus g of the curve C. For instance, rational and elliptic curves are such that the degree of irrationality of their second symmetric product is one and two respectively, whereas we shall see that if the genus of C is $g \geq 2$, then $C^{(2)}$ is non-rational and it does not admit a dominant rational map on $\mathbb{P}^2$ of degree 2 (cf. Lemma 4.3.1).

Moreover, the degree of irrationality of the second symmetric product seems to depend on the existence of linear series on the curve as well. In particular, if C admits a degree d covering $f\colon C \longrightarrow \mathbb{P}^1$, it is always possible to define a morphism $C^{(2)} \longrightarrow \mathbb{P}^2$ of degree d^2 by sending a point $p+q \in C^{(2)}$ to the point $f(p)+f(q) \in (\mathbb{P}^1)^{(2)} \cong \mathbb{P}^2$. Hence the degree of irrationality of the second symmetric product of a curve is bounded from above by the square of the gonality. Furthermore, if C admits a birational mapping onto a non-degenerate curve of degree d in $\mathbb{P}^2$ we may construct a dominant rational map $C^{(2)} \dashrightarrow \mathbb{P}^2$ of degree $\binom{d}{2}$, which sends a point $p+q \in C^{(2)}$ to the line $l \in \mathbb{G}(1,2) \cong \mathbb{P}^2$ passing through the images of p and q in $\mathbb{P}^2$. Moreover, it is possible to provide other dominant rational maps by using g^3_d's as well (cf. Example 4.2.1). Thus we have the following upper bound (see Proposition 4.2.2).

Proposition 1. *Let C be a smooth complex projective curve. Let δ_1 be the gonality of C and for $m = 2,3$, let δ_m be the minimum of the integers d such that C admits a birational mapping onto a non-degenerate curve of degree d in $\mathbb{P}^m$. Then*

$$d_r(C^{(2)}) \leq min\left\{\delta_1^2\,,\ \frac{\delta_2(\delta_2-1)}{2}\,,\ \frac{(\delta_3-1)(\delta_3-2)}{2}-g\right\}.$$

In the case of hyperelliptic curves we prove the following (see Theorem 4.2.4).

Theorem 2. *Let C be a smooth complex projective curve of genus $g \geq 2$ and assume that C is hyperelliptic. Then*

(i) $3 \leq d_r(C^{(2)}) \leq 4$ *when either* $g=2$ *or* $g=3$*;*

(ii) $d_r(C^{(2)}) = 4$ *for any* $g \geq 4$.

When the curve is assumed to be non-hyperelliptic, the situation is more subtle and it is no longer true that the degree of irrationality of $C^{(2)}$ equals the square of the gonality of C for high enough genus. The following result summarizes the lower bounds we prove on the degree of irrationality of second symmetric products of non-hyperelliptic curves and we list them by genus (cf. Proposition 4.2.6 and Theorem 4.2.9).

Theorem 3. *Let C be a smooth complex projective curve of genus $g \geq 3$ and assume that C is non-hyperelliptic. Then the following hold:*

(i) *if $g = 3, 4$, then $d_r(C^{(2)}) \geq 3$;*

(ii) *if $g = 5$, then $d_r(C^{(2)}) \geq 4$;*

(iii) *if $g = 6$, then $d_r(C^{(2)}) \geq 5$;*

(iv) *if $g \geq 7$, then*
$$d_r(C^{(2)}) \geq max\,\{\,6,\, gon(C)\,\}\,.$$

Furthermore, if C is assumed to be very general in the moduli space $\mathcal{M}_g$ with $g \geq 4$, then
$$d_r(C^{(2)}) \geq g - 1.$$

We point out that in Chapter 4 we shall present some examples of curves such that $d_r(C^{(2)})$ does not satisfy equality in Proposition 1. It shall be clear that the constructions of those examples do not apply to very general curves. Moreover, under this assumption, the minimum degree we are able to present for a dominant rational map $C^{(2)} \dashrightarrow \mathbb{P}^2$ is one of those in the proposition above. Therefore we conjecture that for a curve C very general in the moduli space $\mathcal{M}_g$ with $g \geq 2$, the bound in Proposition 1 is actually an equality, but Theorem 3 is at the moment our best bound.

Another attempt to extend the notion of gonality to n-dimensional varieties is the following. Given an irreducible complex projective variety X, we define the number

$$d_g(X) := min \left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a family } \mathcal{E} = \{E_t\}_{t \in T} \\ \text{covering } X \text{ whose generic member is} \\ \text{an irreducible } d\text{-gonal curve} \end{array} \right\}$$

and we may call it the *degree of gonality* of X. Notice that the generic member E_t is a possibly singular d-gonal curve, i.e. its normalization $\widetilde{E}_t$ admits a degree d covering $f_t \colon \widetilde{E}_t \longrightarrow \mathbb{P}^1$. The degree of gonality is a birational invariant and $d_g(X) = 1$ if and only if X is an uniruled variety. Moreover, $d_g(C) = gon(C)$ for any complex projective curve C.

Although this second extension of the notion of gonality appears less intuitive and more artificial than the degree of irrationality, the degree of gonality has a nice behavior with respect to dominance. Namely, if there exists a dominant rational map $X \dashrightarrow Y$ between two irreducible complex projective varieties of dimension n, it is easy to see that $d_g(X) \geq d_g(Y)$, as in the one dimensional case.

We note that another proposal to extend the notion of gonality has been recently presented in terms of pencils and fibrations (cf. [33]).

Dealing with the problem of computing the degree of gonality of the second symmetric product $C^{(2)}$ of a smooth complex projective curve C, it is easy to check that $d_g(C^{(2)}) = 1$ when the curve is either rational or elliptic, and $d_g(C^{(2)}) = 2$ for any curve of genus two. Moreover, we prove the following (cf. Theorem 3.2.1).

Theorem 4. *Let C be a smooth complex projective curve of genus $g \geq 4$. For a positive integer d, let $\mathcal{E} = \{E_t\}_{t \in T}$ be a family of curves on $C^{(2)}$ parametrized over a smooth variety T, such that the generic fiber E_t is an irreducible d-gonal curve and for any point $P \in C^{(2)}$ there exists $t \in T$ such that $P \in E_t$. Then $d \geq gon(C)$.*

For any smooth complex projective curve C, its second symmetric product is covered by the family of curves $\mathcal{X} = \{X_p\}_{p \in C}$ parametrized over C, where $X_p := \{p + q \in C^{(2)} \,|\, q \in C\}$ is isomorphic to C. Hence we deduce the following (see Theorem 3.2.2).

Theorem 5. *Let C be a smooth complex projective curve of genus $g \geq 4$. Then $d_g(C^{(2)}) = gon(C)$.*

In order to prove our results, the main technique is to use holomorphic differentials, following Mumford's method of induced differentials (cf. [43, Section 2]). In the spirit of [37], we rephrase our settings in terms of correspondences on the product $Y \times C^{(2)}$, where Y is an appropriate ruled surface. A general 0-cycle of such a correspondence $\Gamma \subset Y \times C^{(2)}$ is a Cayley-Bacharach scheme with respect to the canonical linear series $|K_{C^{(2)}}|$, that is, any holomorphic 2-form vanishing on all but one the points of a 0-cycle vanishes in the remaining point as well. The latter property imposes strong conditions on the correspondence Γ, and the crucial point is to study the restrictions descending to the second symmetric product and then to the curve C.

A further important technique involved in the proofs is monodromy. In particular, we consider the generically finite dominant map $\pi_1 \colon \Gamma \longrightarrow Y$ projecting a correspondence Γ on the first factor, and we study the action of the monodromy group of π_1 on the generic fiber. Finally, an important role is played by Abel's theorem and some basic facts of Brill-Noether theory.

Another problem we treat on symmetric products of curves is the description of the cone $Nef\,(C^{(2)})_{\mathbb{R}}$ of all numerically effective $\mathbb{R}$-divisors classes in the Néron-Severi space $N^1(C^{(2)})_{\mathbb{R}}$. This problem is reduced to estimate the slope $\tau(C)$ of one of the rays bounding the two-dimensional convex cone $Nef\,(C^{(2)})_{\mathbb{R}}$. In [48], Ross uses a degeneration argument to prove a result connecting the real number $\tau(C)$ for a generic curve C of genus g, with Seshadri constants on second symmetric products of curves of genus $g-1$. Then he applies the latter result to improve the bound on $\tau(C)$ when C is a very general curve of genus five. We follow Ross' argument and - as an application of Theorem 4 stated above - we give a further improvement on the bounds on $\tau(C)$ in the cases of genus $5 \leq g \leq 8$. In particular, we prove the following (cf. Theorem 5.1.2).

Theorem 6. *Consider the rational numbers*

$$\tau_5 = \frac{9}{4}, \quad \tau_6 = \frac{32}{13}, \quad \tau_7 = \frac{77}{29} \quad \textit{and} \quad \tau_8 = \frac{17}{6}.$$

Let C be a smooth complex projective curve of genus $5 \leq g \leq 8$ and assume that C has very general moduli. Then

$$\tau(C) \leq \tau_g.$$

Let us now consider a smooth complex projective curve C of genus two. The Abel map $C^{(2)} \longrightarrow J(C)$ is a generically finite morphism from the second symmetric product to the two-dimensional Jacobian variety. It is then possible to shift the problem of computing the degree of irrationality of $C^{(2)}$ to $J(C)$. The idea of constructing the examples we present in Chapter 6 comes both from the approach above, and from a joint work with Gian Pietro Pirola and Lidia Stoppino, that deals with Galois closures of rational coverings and Lagrangian varieties.

Let X be a smooth complex algebraic surface and consider the homomorphism

$$\psi_2 \colon \bigwedge^2 H^0(X, \Omega^1_X) \longrightarrow H^0(X, \Omega^2_X).$$

The non-triviality of the kernel of this map leads to unexpected topological consequences. The main classical result is the Castelnuovo-de Franchis Theorem asserting that if there exist two non-zero forms $\omega_1, \omega_2 \in H^0(X, \Omega^1_X)$ such that $\omega_1 \wedge \omega_2 \neq 0$ and $\omega_1 \wedge \omega_2 \in Ker\,\psi_2$, then X admits a fibration over a curve of genus $g \geq 2$. This result has been generalized by Catanese in [14]. Moreover, if ψ_2 is not injective, the fundamental group of X turns out to be a non-abelian group (see for instance [3] and [42]) and other topological consequences have been studied in [7] in terms of topological index.

In the light of Castelnuovo-de Franchis Theorem it is interesting to study when there exist non-trivial elements of $Ker\,\psi_2$ that do not induce a fibration on X. Some examples of this situation have been presented in [11], [13] and [51].

Following [7], we say that X is a *Lagrangian* surface if there exist a map of degree one $a\colon X \longrightarrow a(X) \subset A$ into an Abelian variety A of dimension 4 and a holomorphic 2-form $\omega \in H^{2,0}(A)$ of rank 4 such that $a^*(\omega) = 0$. In that paper, the authors provide a sufficient condition for Lagrangian surfaces to have non-negative topological index, and they conjecture that the assertion holds for any Lagrangian surface (see [7, Conjecture 2]). We shall present a family of examples of Lagrangian surfaces having negative topological index, hence disproving the conjecture above. In particular, the differential form ω shall be a non-trivial element of $Ker\,\psi_2$ which does not come from a fibration on X.

In order to produce our examples, the main point shall be to take the Galois closure of suitable rational maps between surfaces. In [53], Tokunaga and Yoshihara prove that the degree of irrationality of an Abelian surface S containing a curve D of genus three is $d_r(S) = 3$ (in particular, it could be the case of the Jacobian variety of a genus two curve). Since D induces a polarization of type (1,2) on the Abelian surface S, we shall follow the study of Barth (cf. [8]), to give a detailed description of the linear pencil induced by $|D|$. By opportunely blowing up S, we shall construct a degree three covering $\overline{S} \longrightarrow \mathbb{F}_3$ of the Hirzebruch surface $\mathbb{F}_3$ and we shall define the surface X to be the Galois closure of such covering. Then we shall compute the birational invariants of X that shall turn out to be a Lagrangian surface with negative topological index.

Let us summarize the plan of this work. Chapter 1 has a preliminary character. We shall recall several classical results in the theory of algebraic curves and generalities on symmetric products of curves that will be useful to understand the following.

In Chapter 2 we shall deal with correspondences with null trace on the k-fold symmetric product of a smooth complex projective curve C. In order to develop the main techniques to treat the problems of the following chapters, we shall see how the existence of a correspondence with null trace on $C^{(k)}$ influences the geometry of C (see Theorem 2.2.2 and Corollary 2.2.3). In particular, these results shall descend as consequences from the study of linear subspaces of $\mathbb{P}^n$ enjoying a condition of Cayley-Bacharach type (cf. Section 2.3).

Chapter 3 is devoted to study deformations and gonality of curves lying on second symmetric products of curves. The first result we shall present is an extension of a result in [47] on curves lying on generic three-dimensional Abelian varieties. Namely, we shall prove the following (see Proposition 3.1.1).

Theorem 7. *Let C be a smooth complex projective curve of genus $g \geq 3$ and* *assume that C is very general in the moduli space $\mathcal{M}_g$. Then the Jacobian variety $J(C)$ contains neither rational nor hyperelliptic curves.*

Clearly, under the assumption of the theorem, the second symmetric product $C^{(2)}$ embeds into the Jacobian of C and hence there are neither hyperelliptic nor rational curves lying on $C^{(2)}$ as well.

Then we shall turn to study the gonality of moving curves on second symmetric products of curves. In particular, we shall discuss the degree of gonality of $C^{(2)}$ and we shall prove Theorem 4 and Theorem 5 above.

Chapter 4 concerns the degree of irrationality of second symmetric products of curves. Initially, we shall present an overview of the known results on the degree of irrationality of n-dimensional varieties. Then we shall focus on the case of second symmetric products of curves and we shall prove Theorem 2 and Theorem 3 we stated above.

In Chapter 5 we shall turn to the nef cone $Nef\,(C^{(2)})_{\mathbb{R}}$ on the second symmetric product of a generic curve C and we shall prove Theorem 6. The argument of the proof is based on the main theorem in [48] together with the techniques used by Ross, due to Ein and Lazarsfeld (see [22]).

Moreover, to be able to provide new bounds on the real number $\tau(C)$, we shall discuss the self-intersection of moving curves on surfaces. Let X be a smooth complex projective surface and let $\{(E_t, x_t)\}_{t\in T}$ be a non trivial family of pointed curves covering X such that $mult_{x_t}E_t \geq m$ for any $t \in T$ and for some $m \geq 1$. In [22] the authors prove that the self-intersection of the general member the family is $E_t^2 \geq m(m-1)$. Under the additional hypothesis $m \geq 2$, Xu proved that $E_t^2 \geq m(m-1)+1$ (see [54]). We shall improve the latter bound, and our result will turn out to be sharp. Namely, we shall prove the following (see Theorem 5.2.2). We would like to note the the same result has been independently obtained by Knutsen, Syzdek and Szemberg in a recent paper (see [32]).

Theorem 8. *Let X be a smooth complex projective surface. Let T be a smooth variety and consider a family $\{(E_t, x_t)\}_{t\in T}$ consisting of a curve $E_t \subset X$ through a very general point $x_t \in X$ such that $mult_{x_t}E_t \geq m$ for any $t \in T$ and for some $m \geq 2$.*
If the central fibre E_0 is a reduced irreducible curve and the family is non-trivial, then

$$E_0^2 \geq m(m-1) + gon(E_0).$$

Finally, the main result of this chapter shall follow by combining the latter theorem with Theorem 3.2.1 on the gonality of moving curves on second symmetric products of curves.

In Chapter 6 we shall develop examples of Lagrangian surfaces with negative topological index. Any such a surface X shall turn out to be provided of a holomorphic differential form $\omega \in \bigwedge^2 H^0(X, \Omega^1_X)$ vanishing on $H^0(X, \Omega^2_X)$ which is not induced by a fibration on X.

Let $\mathcal{W}(1,2)$ denote the moduli space of smooth complex Abelian surfaces with a polarization of type (1,2). We shall prove the following (see Theorem 6.1.4).

Theorem 9. *Let S be a smooth complex Abelian surface and let $\mathcal{L}$ be a line bundle on S providing a $(1,2)$-polarization. Suppose further that the pair $(S, \mathcal{L})$ is general in $\mathcal{W}(1,2)$. Then there exists a dominant degree three morphism $\overline{S} \longrightarrow \mathbb{F}_3$ from a suitable blow-up $\overline{S}$ of S to the Hirzebruch surface $\mathbb{F}_3$. The minimal desingularization X of the Galois closure of the covering is a surface of general type with invariants*

$$K_X^2 = 198 \quad c_2(X) = 102 \quad \chi(\mathcal{O}_X) = 25 \quad q = 4 \quad p_g = 28 \quad \tau(X) = -2.$$

Furthermore, X is a Lagrangian surface.

Chapter 1

Generalities on symmetric products of curves

The contents of this chapter are preliminary to the subjects we develop in the rest of this work. In particular, we present several definitions and known results concerning the symmetric products of curves. We omit the proofs and we explicitly refer to the literature. We give a proof of those results of which we were not able to find adequate references.

As there is a strong connection between the geometry of a curve and its k-fold symmetric product, we start in Section 1 with the classical theory of algebraic curves.

In the second Section we deal with the concept of monodromy for a generically finite dominant morphism between varieties of the same dimension and we give some basic definitions and results on this topic.

Section 3 concerns the Néron-Severi group of k-fold symmetric products of curves. At first, we recall some important definitions and results on numerical properties of divisors on algebraic varieties. Then we focus on symmetric products of projective curves by describing the numerical behavior of particular divisors on them.

The fourth Section deals with linear series of special divisors on algebraic curves and some basic notions on Brill-Noether theory. Moreover, we introduce a class of subvarieties of the symmetric product of a projective curve induced by linear series on the curve.

Finally, in Section 5 we point out some connections between symmetric products of curves and Grassmannians. Then, by recovering the varieties presented in the previous section, we give a description of the canonical linear series on the symmetric product of a projective curve.

We shall work throughout over the field $\mathbb{C}$ of complex numbers. As customary, we shall make the identification of invertible sheaves with line bundles and of locally free sheaves with vector bundles. Furthermore, given a sheaf $\mathcal{F}$ of complex vector spaces over a topological space V, we shall set, as usual

$$h^i(V, \mathcal{F}) = dim\, H^i(V, \mathcal{F})$$

and

$$\chi(\mathcal{F}) = \sum(-1)^i h^i(V, \mathcal{F}).$$

Given a variety X, we say that a property holds for a *general* point $x \in X$ if it holds on an open non-empty subset of X. Moreover, we say that $x \in X$ is a *very general* point if there exists a countable collection of proper subvarieties of X such that x is not contained in the union of those subvarieties.

1.1 Preliminaries on curves

In this preliminary section we shall recall some definitions and some important results of the classical theory of curves that will turn out to be very useful in the following chapters. Moreover, we shall define symmetric products of curves and we shall present some basic facts on these varieties. The main reference sources shall be [1] and [26].

Throughout this work, by *curve* we mean a complete reduced algebraic curve over the field of complex numbers. When we speak of *smooth curve*, we always implicitly assume it to be irreducible. Given a curve C, we define its *arithmetic genus* to be

$$p_a(C) := 1 - \chi(\mathcal{O}_C).$$

Moreover, when C is assumed to be irreducible, we define the *geometric genus* of C as the arithmetic genus of its normalization and we denote it by g.

We say that a smooth curve of genus g is *very general* if its corresponding point in the moduli space $\mathcal{M}_g$ is very general, i.e. it is outside of a countable union of proper analytic subvarieties in $\mathcal{M}_g$.

For a smooth curve C of geometric genus g, let K denote its canonical line bundle. We recall that $deg\, K = 2g - 2$ and $h^0(C, K) = g$. The first result we present is the following.

Riemann-Roch Theorem. *For any line bundle L on a smooth curve C of genus g*

$$h^0(C, L) - h^0(C, K\, L^{-1}) = deg\, L - g + 1.$$

Let us consider the canonical map of C

$$\phi_K : C \longrightarrow \mathbb{P}\left(H^0(C, K)\right) \cong \mathbb{P}^{g-1},$$

that is the holomorphic map associated to the base-point-free complete linear series $|K|$ of divisors of holomorphic 1-forms on C. The *canonical image* $\phi_K(C)$ of C is a non-degenerate curve in $\mathbb{P}^{g-1}$ and if C is non-hyperelliptic, then ϕ_K is an embedding.

Looking at the canonical image $\phi_K(C)$ it is possible to restate Riemann-Roch theorem in a geometric form. To this aim, we introduce a notation that we will often use throughout this work. Given an effective divisor $D = p_1 + \ldots + p_d$, we denote by $\overline{\phi_K(D)}$ the intersection of the hyperplanes $H \subset \mathbb{P}^{g-1}$ such that $\phi_K(C) \subset H$ or $\phi_K^*(H) \geq D$. Notice that when ϕ_K is an embedding and the p_i's are distinct points, then $\overline{\phi_K(D)}$ is the ordinary linear span in $\mathbb{P}^{g-1}$ of these points.

With the notation above, Riemann-Roch theorem may be rephrased as follows.

Geometric Version of the Riemann-Roch Theorem. *If C is a smooth curve of genus $g > 1$ and $D \in Div(C)$ is an effective divisor, then*

$$dim\,|D| = deg\,D - 1 - dim\,\overline{\phi_K(D)}.$$

In order to estimate the dimension of a given complete linear series, another important classical result is given by

Clifford's Theorem. *Let C be a smooth curve of genus g. If $D \in Div(C)$ is an effective divisor of degree $deg\,D \leq 2g - 1$, then*

$$dim\,|D| \leq \frac{deg\,D}{2}\,.$$

Furthermore, if equality holds then either D is zero, D is a canonical divisor, or C is hyperelliptic and D is linearly equivalent to a multiple of a hyperelliptic divisor.

Proof. See [1, p. 107]. □

Now, let C be a smooth curve of genus g. For an integer $d \geq 1$, let us consider the *d-fold ordinary product* of C

$$C^d = C \times \ldots \times C$$

and let S_d be the d-th symmetric group. The *d-fold symmetric product* of C is the quotient $C^{(d)}$ of the ordinary d-fold product C^d by the natural action of S_d, that is

$$C^{(d)} := \frac{C^d}{S_d}\,.$$

Therefore $C^{(d)}$ is the projective variety of dimension d parametrizing the effective divisors of degree d on C or, equivalently, the unordered d-tuples of points of C. Furthermore, the d-fold symmetric product $C^{(d)}$ turns out to be smooth (see [1, p. 18]).

Another important variety associated to a smooth curve C is the *Jacobian variety* of C, which is the g-dimensional complex torus defined as

$$J(C) := \frac{H^0(C,K)^*}{H_1(C,\mathbb{Z})}\,.$$

Then, by choosing a basis $\omega_1,\dots,\omega_g$ of $H^0(C,K)$ and by fixing a point $p_0 \in C$, we define the *Abel-Jacoby map*

$$u\colon C^{(d)} \longrightarrow J(C)$$

as

$$p_1+\ldots+p_d \longmapsto \left(\sum_{i=1}^{d}\int_{p_0}^{p_i}\omega_1,\dots,\sum_{i=1}^{d}\int_{p_0}^{p_i}\omega_g\right)\,.$$

We can now state the

Abel's Theorem. *Let $D, D' \in C^{(d)}$ be two effective divisors of degree d on a smooth curve C. Then D is linearly equivalent to D' if and only if $u(D) = u(D')$.*

Proof. See [26, Chapter 2.2]. □

As customary, for $d \geq 1$, let $Pic^d(C)$ be the *Picard Variety* of C parametrizing the isomorphism classes of line bundles of degree d on C. Let $p_0 \in C$ and let us define the *Picard map* as

$$\begin{array}{rccc} v\colon & C^{(d)} & \longrightarrow & Pic^0(C) \\ & D & \longmapsto & \mathcal{O}(D-dp_0). \end{array}$$

Then it is possible to prove that there is an isomorphism $Pic^0(C) \cong J(C)$ such that for any choice of the base point $p_0 \in C$, the resulting Abel and Picard maps make the following diagram commute

$$\begin{tikzcd} & C^{(d)} \arrow[dl, "v"'] \arrow[dr, "u"] & \\ Pic^0(C) \arrow[rr, "\cong"] & & J(C) \end{tikzcd}$$

In particular, given two effective divisors $D, D' \in C^{(d)}$ of degree d on C, they are linearly equivalent if and only if $\mathcal{O}(D) = \mathcal{O}(D')$ (See [50]).

To conclude this section we state two well known results dealing with smooth curves mapping on another curve. The first one is

Riemann-Hurwitz Formula. *Let C and C' be two smooth curves of genus g and g' respectively. Let $f\colon C \longrightarrow C'$ be a non-constant holomorphic map of degree d and let $R \in Div(C)$ be the ramification divisor. Then*

$$2g-2 = d(2g'-2) + deg\, R\,.$$

Proof. See [26, p. 216]. □

When the map is defined by a linear series of degree d and dimension r, we have the following.

Castelnuovo's Bound. *Let C be a smooth curve that admits a birational mapping onto a non-degenerate curve of degree d in $\mathbb{P}^r$. Then the genus of C satisfies the inequality*

$$g(C) \leq \frac{m(m-1)}{2}\,(r-1) + m\epsilon\,,$$

where

$$m := \left[\frac{d-1}{r-1}\right] \quad \textit{and} \quad \epsilon := d-1-m(r-1)\,.$$

Proof. See [1, p. 116]. □

1.2 Monodromy

In this section we shall briefly recall the concept of monodromy for a generically finite dominant morphism between varieties and we shall present some basic results on this topic. As will be clear in the following, the idea underlying monodromy is constructing such a convenient morphism and using topological techniques to show that it is not possible to distinguish the points -or subsets with the same cardinality- of the generic fiber.

Let X and Y be two complex algebraic varieties of dimension n and let $F\colon X \longrightarrow Y$ be a generically finite dominant morphism of degree d.

As in [29], let $U \subset Y$ be a suitable Zariski open subset of X and let $V := F^{-1}(U) \subset X$ so that the restriction $F_{|V}\colon V \longrightarrow U$ is an unbranched covering of degree d.

Consider a point $y \in U$ and let $F^{-1}(y) = \{x_1, x_2, ..., x_d\}$ be its fiber. For any loop $\gamma : [0,1] \longrightarrow U$ and for any point $x_j \in F^{-1}(y)$ there exists a unique lifting $\widetilde{\gamma}_j$ of γ starting from x_j. Then, by associating to any x_j the ending point $\widetilde{\gamma}_j(1)$, we obtain a permutation $\sigma_\gamma \in S_d$. Since σ_γ depends on the homotopy class of γ and the group $Aut\left(F^{-1}(y)\right)$ of the automorphisms of the fiber is isomorphic to the d-th symmetric group S_d, we may define the homomorphism

$$\begin{array}{cccc} \rho\colon & \pi_1(U,y) & \longrightarrow & Aut\left(F^{-1}(y)\right) \cong S_d \\ & \gamma & \longmapsto & \sigma_\gamma \end{array}$$

which is the *monodromy representation* of the fundamental group $\pi_1(U,y)$. In particular, we define the *monodromy group* $M(F)$ of F to be the image of the above homomorphism.

Equivalently, it is possible to define the monodromy group of F as follows. The function field $K(X)$ is an algebraic extension of $K(Y)$ of degree d. Let L denote the normalization of the extension $K(X)/K(Y)$ and let $Gal(L/K(Y))$ be the *Galois group* of $L/K(Y)$, that is the group of the automorphisms of the field L fixing every element of $K(Y)$. Then the monodromy group $M(F)$ and the Galois group $Gal(L/K(Y))$ are isomorphic (see [29, p. 689]). In particular, this implies that the monodromy group of F is independent of the choice of the Zariski open set U.

For a generic point $y \in Y$, the monodromy group of F is a subgroup of $Aut\left(F^{-1}(y)\right)$ and hence it acts on the fiber $F^{-1}(y)$. Furthermore, we have the following.

Lemma 1.2.1. *If $F\colon X \longrightarrow Y$ be a generically finite dominant morphism between two irreducible varieties of the same dimension. Then for the generic point $y \in Y$, the action of the monodromy group $M(F)$ on the fiber $F^{-1}(Y)$ is transitive.*

Proof. To see this fact we argue as in [38, Lemma 4.4 p.87]. With the notations above, let $x_i, x_j \in F^{-1}(y)$. As X is assumed to be irreducible, it is possible to chose the Zariski open set $U \subset Y$ such that $V = F^{-1}(U)$ is connected. Hence we may find a path $\widetilde{\gamma}$ on V starting at x_i and ending at x_j. Therefore its image $\gamma = F \circ \widetilde{\gamma}$ on U is a loop with base point y and $\sigma_\gamma \in M(F)$ is a permutation sending i to j. □

Roughly speaking, the statement of the lemma may be rephrased by saying that the points of the fiber over a generic point are undistinguishable. We mean the following. Consider a generic point $y \in Y$ and suppose that a point $x_i \in F^{-1}(y)$ enjoys some special property. Suppose further that as we vary continuously the point y on a suitable open subset $U \subset Y$, that special property is preserved as we follow the correspondent point of the fiber. Then for any loop $\gamma \in \pi_1(U,y)$, we have that the ending point of the lifting $\widetilde{\gamma}$ of γ starting from x_i must enjoy the same property. Hence the previous lemma assures that there is no way to distinguish a point of the fiber over a generic $y \in Y$ from another for enjoying a property as above.

Analogously, when the action of the monodromy group $M(F)$ is m-times transitive - i.e. it acts transitively on the set of ordered m-tuples of points of $F^{-1}(y)$ - the subsets of $F^{-1}(y)$ consisting of m points can not enjoy some distinguishing property varying uniformly on a convenient Zariski open subset of Y.

The following theorem descends from an argument of this type.

General Position Theorem. *Let $C \subset \mathbb{P}^r$ be a non-degenerate - possibly singular - curve of degree d. Then a general hyperplane meets C at d points any r of which are linearly independent.*

Proof. See [1, p. 109]. □

In the following chapters we often deal with rational dominant maps of finite degree between irreducible varieties of the same dimension. So, it is important to note that the assumption we made on F of being a generically finite dominant morphism can be weakened. Namely, the monodromy group of a map F is well-defined even in case $F\colon X \dashrightarrow Y$ is a generically finite dominant rational map of degree d and hence all the results presented still hold. To see this fact, we recall that such a map still induces a field extension $K(X)/K(Y)$ of degree d. On the other hand, one can just observe that it is always possible to choose a Zariski open set $U \subset Y$ such that the restriction of F to $F^{-1}(U)$ is an unbranched covering of degree d.

1.3 Divisors on $C^{(k)}$ and the Néron-Severi group

This section concerns the Néron-Severi group of symmetric products of curves. To start we shall recall some important definition and results on numerical properties of divisors on algebraic varieties. Then we shall focus on the k-fold symmetric product of a smooth curve C: we shall define some particular divisors on $C^{(k)}$ and we shall recall their main numerical behavior. The main reference source shall be [35] for the first part of this section and [1, Chapter VIII] for the second one.

Let X be a complete irreducible algebraic variety over the field of complex numbers.

We recall that two Cartier divisors $D_1, D_2 \in Div(X)$ are said to be *numerically equivalent* if $(D_1 \cdot C) = (D_2 \cdot C)$ for every irreducible curve $C \subset X$.

Then we define the *Néron-Severi group* of X as the quotient group $N^1(X)_{\mathbb{Z}}$ of numerical equivalence classes of divisor on X. In particular, it turns out to be a free abelian group of finite rank. We denote by $\rho(X)$ its rank and we call it the *Picard number* of X.

An element D of the real vector space $Div_{\mathbb{R}}(X) := Div(X) \otimes \mathbb{R}$ is said to be a $\mathbb{R}$-*divisor* on X and it is represented by a finite sum $D = \sum a_i D_i$ where $a_i \in \mathbb{R}$ and $D_i \in Div(X)$. By linearity, the intersection product and numerical equivalence extend to $\mathbb{R}$-divisors and we denote by $N^1(X)_{\mathbb{R}}$ the resulting vector space. We say that a $\mathbb{R}$-divisor D is *ample* if it can be expressed as a finite sum $D = \sum a_i D_i$ where $c_i > 0$ and D_i is an ample Cartier divisor for any i.

Now, let us assume that X is a complex projective variety. Under this assumption we have an important numerical characterization of ampleness.

Nakai-Moishezon criterion. *Let D be a $\mathbb{R}$-divisor on an irreducible complex projective X variety. Then D is ample if and only if $\left(D^{\dim V} \cdot V\right) > 0$ for every irreducible subvariety $V \subset X$.*

Proof. See [12] or [35, Theorem 2.3.18]. □

As a consequence of this theorem we have that the ampleness of a $\mathbb{R}$-divisor on X depends only from its numerical equivalence class. Hence we define the *ample cone* of X to be the convex cone $Amp(X)_{\mathbb{R}} \subset N^1(X)_{\mathbb{R}}$ of all ample $\mathbb{R}$-divisors classes on X.

A $\mathbb{R}$-divisor D is said to be *nef* - or *numerically effective* - if $(D \cdot C) \geq 0$ for every curve $C \subset X$. As nefness depends just on a numerical condition, we can define the convex cone $Nef(X)_{\mathbb{R}} \subset N^1(X)_{\mathbb{R}}$ of all nef $\mathbb{R}$-divisors classes on X, which is said the *nef cone* of X. We have the following.

Kleiman's Theorem. *Let X be an irreducible complex projective variety. If D is a nef $\mathbb{R}$-divisor on X, then $\left(D^{\dim V} \cdot V\right) \geq 0$ for every irreducible subvariety $V \subset X$.*

Proof. See [12] or [35, Theorem 1.4.9]. □

Furthermore, by the latter theorem it is possible to prove that the nef cone $Nef(X)_{\mathbb{R}}$ is the closure of the ample cone $Amp(X)_{\mathbb{R}}$ in the Néron-Severi space $N^1(X)_{\mathbb{R}}$ of X.

To conclude this discussion on numerical properties of $\mathbb{R}$-divisors we state a generalization of the well known Hodge index theorem (see [30, Theorem 1.9 p. 364]).

Generalized inequality of Hodge type. *Let X be an irreducible complex projective variety of dimension n and let $D_1, \dots, D_n$ be nef $\mathbb{R}$-divisors on X. Then*

$$D_1^n \dots D_n^n \leq (D_1 \cdot \dots \cdot D_n)^n .$$

Proof. See [35, Theorem 1.6.1]. □

Now, we wold like to focus on symmetric products of curves. To this aim, let us consider a smooth irreducible complex projective curve C of genus g and for an integer $k \geq 2$, let $C^{(k)}$ be its k-fold symmetric product.

Our first task is to define some important divisors on $C^{(k)}$. As the k-fold symmetric product is a smooth projective variety, there is an isomorphism between the groups of Cartier and Weil divisors, then we will make no distinction between them hereafter (cf. [23, Chapter 2.1]).

Given a point $p \in C$, we define the divisor X_p as

$$X_p := \left\{ p + Q \,|\, Q \in C^{(k-1)} \right\}.$$

Therefore X_p is the image of the inclusion map $\iota_p \colon C^{(k-1)} \longrightarrow C^{(k)}$ sending a point $Q \in C^{(k-1)}$ to the point $p + Q \in C^{(k)}$. We note that the numerical equivalence class of X_p is independent of p and hence we simply denote by $x \in N^1(C^{(k)})_{\mathbb{Z}}$ such class.

Then we consider the diagonal map

$$\begin{array}{rccc} d_k \colon & C^{(k-2)} \times C & \longrightarrow & C^{(k)} \\ & (Q, q) & \longmapsto & Q + 2q \end{array}$$

and we define the diagonal divisor Δ_k to be the image of the diagonal map, that is

$$\Delta_k := \left\{ Q + 2q \,|\, q \in C \text{ and } Q \in C^{(k-2)} \right\}.$$

We denote by $\delta \in N^1(C^{(k)})_{\mathbb{Z}}$ the numerical equivalence class of Δ_k.

Finally, let $J(C)$ be the Jacobian variety of C and let us consider the Abel-Jacobi map $u \colon C^{(k)} \longrightarrow J(C)$. Let Θ be the theta divisor on $J(C)$ and let θ be its class in the Néron-Severi group of the Jacobian. For simplicity, we denote again by θ the numerical equivalence class $u^*\theta \in N^1(C^{(k)})_{\mathbb{Z}}$.

With this notation, we have the following.

Lemma 1.3.1. *The numerical equivalence class δ of the diagonal Δ_k in $C^{(k)}$ is given by*

$$\delta = 2\left((k + g - 1)x - \theta\right).$$

Proof. It is a special case of [1, Proposition 5.1 p.358]. □

It is easy to see that the Δ_k is divisible by 2 and hence we can consider the numerical equivalence class of $\Delta_k/2$ (see for instance [44]). Then we have the following result on the Néron-Severi group of a generic curve of genus g.

Lemma 1.3.2. *Let C be a smooth reduced complex projective curve of genus g and assume C very general in the moduli space $\mathcal{M}_g$. Then the Néron-Severi group $N^1(C^{(k)})_{\mathbb{Z}}$ is generated by the numerical equivalence classes x and $\frac{\delta}{2}$.*

Proof. See [1, p.359]. □

It follows that for any divisor D on $C^{(k)}$, its numerical equivalence class can be expressed in the form

$$[D] = (a+b)x - b\frac{\delta}{2} \in N^1(C^{(k)})_{\mathbb{Z}}$$

for some real numbers a, b.

To conclude, we note that in Sections 5.1 and 5.3 we shall deal with the problem of bounding the nef cone of the second symmetric product of very general curves. Then we would like to recall some simple facts on intersection of numerical equivalence classes on $C^{(2)}$.

If the curve C has genus g, the intersection numbers of the classes x and $\frac{\delta}{2}$ of $N^1(C^{(2)})_{\mathbb{Z}}$ are

$$(x \cdot x) = 1, \quad \left(\frac{\delta}{2} \cdot \frac{\delta}{2}\right) = 1 - g \quad \text{and} \quad \left(x \cdot \frac{\delta}{2}\right) = 1.$$

Thus the intersection between divisor classes spanned by x and $\frac{\delta}{2}$ is given by

$$\left(\left((a+b)x - b\frac{\delta}{2}\right) \cdot \left((m+n)x - n\frac{\delta}{2}\right)\right) = am - bn\, g.$$

1.4 Subvarieties of the symmetric product induced by linear series on the curve

In this section we shall focus on linear series on curves. Initially, we shall deal with the varieties of special linear series on a curve. Then we shall introduce some subvarieties of the k-fold symmetric product of a curve C that are induced by linear series on C. We shall recall some results on them and in the next section we shall see that some of these subvarieties are canonical divisors on $C^{(k)}$.

Let C be a smooth irreducible projective curve of genus g. To start, we follow [1, Chapter IV] to recall three kinds of varieties by a set-theoretical description.

To this aim, let $d > r \geq 0$ be two integer. The first variety we introduce is $C_d^r := \left\{ D \in C^{(d)} \,|\, dim\,|D| \geq r \right\}$, that is the subvariety of $C^{(d)}$ parametrizing effective divisors on C moving in a linear series of dimension at least r.

The second variety is given by

$$W^r_d(C) := \Big\{|D| \in Pic^d(C) \,|\, deg\,|D| = d \text{ and } dim\,|D| \geq r\Big\}$$

and parametrizes the complete linear systems on C of degree d and dimension at least r. The following results provides an upper bound on the dimension of $W^r_d(C)$.

Martens' Theorem. *Let C be a smooth curve of genus $g \geq 3$. Let d be an integer such that $2 \leq d \leq g-1$ and let r be an integer such that $0 < 2r \leq d$. Then if C is not hyperelliptic*

$$dim\,W^r_d(C) \leq d - 2r - 1.$$

If C is hyperelliptic

$$dim\,W^r_d(C) \leq d - 2r.$$

Proof. See [1, p. 109]. □

On the other hand, the dimension of $W^r_d(C)$ is bounded from below by the *Brill-Noether number*

$$\rho(g, r, d) := g - (r+1)(g - d + r)$$

and, when C is a very general curve, we have $dim\,W^r_d(C) = \rho(g, r, d)$.

Finally, we define the variety $G^r_d(C)$ of linear series on C of degree d and dimension exactly r, whose elements are said to be g^r_d's. Then if $\mathcal{D} \in G^r_d$, there exist a complete linear series L of degree d and a r-dimensional vector space $V \subset H^0(C, L)$ such that $\mathcal{D} = \mathbb{P}(V)$. Furthermore, notice that any complete $\mathcal{D} \in G^r_d$ can be thought as an element of W^r_d.

So we may associate to any g^r_d a subvariety of the k-fold symmetric product as follows.

Definition 1.4.1. Let $d \geq k > r$ be some integers and let $\mathcal{D}$ be a g^r_d on C. The cycle of all divisors on C that are subordinate to the linear series $\mathcal{D}$ is defined to be

$$\Gamma_k(\mathcal{D}) := \{P \in C^{(k)} \,|\, E - P \geq 0 \text{ for some } E \in \mathcal{D}\}.$$

Remark 1.4.2. We would like to note that if $\mathcal{D}$ is not a base-point-free g^r_d and $p \in C$ is a base point, then $\Gamma_k(\mathcal{D})$ contains the divisor $X_p = p + C^{(k-1)}$ as a component, that is $p+p_2+\ldots+p_k \in \Gamma_k(\mathcal{D})$ for any $p_2+\ldots+p_k \in C^{(k-1)}$. To see this fact, let $\mathcal{D} = \mathbb{P}(V)$, where $V \subset H^0(C, L)$ has dimension r and $L \in W^r_d$. For any $q \in C$, the set $\{s \in V \,|\, s(q) = 0\}$ is a hyperplane of $\mathbb{P}(V) \cong \mathbb{P}^{k-1}$. As the intersection of the $k-1$ hyperplanes associated to the p_i's is non-empty, there exists a non-zero section $s \in V$ vanishing at each p_i. If p is a base point of $\mathcal{D}$, then $s(p) = 0$ and $p + p_2 + \ldots + p_k \in \Gamma_k(\mathcal{D})$.

Notice that for any $\mathcal{D} \in G^r_d(C)$, the variety $\Gamma_{r+1}(\mathcal{D})$ is a divisor on $C^{(r+1)}$. Furthermore, the following holds.

Lemma 1.4.3. *The map $\mathcal{D} \longmapsto \Gamma_k(\mathcal{D})$ is a proper morphism from G^{k-1}_d to the Hilbert scheme of $(k-1)$-dimensional subvarieties on $C^{(k)}$.*

Proof. Let us consider a family $\{\mathcal{D}_t\}_{t\in T}$ of g^{k-1}_d's on C parametrized by a smooth curve T (see [1, p.182-184]) with $0 \in T$. We have to check that $\Gamma_k(\mathcal{D}_0)$ is the (flat) limit of the $\Gamma_k(\mathcal{D}_t)$'s as $t \to 0$.

Let C^k be the k-fold ordinary product of C and let us consider the natural quotient map $\pi\colon C^k \longrightarrow C^{(k)}$. Then we define the $(k-1)$-dimensional subvarieties $\widetilde{\Gamma}_k(\mathcal{D}_t) := \pi^*\Gamma_k(\mathcal{D}_t)$ of C^k, i.e. they are given by the points $(p_1, \ldots, p_k) \in C^k$ such that $E - p_1 - \ldots - p_k \geq 0$ for some $E \in \mathcal{D}_t$. To prove the statement is equivalent to show that $\widetilde{\Gamma}_k(\mathcal{D}_t) \longrightarrow \widetilde{\Gamma}_k(\mathcal{D}_0)$'s as $t \to 0$.

For any $t \in T$, let $L_t \in W^{k-1}_d$ and $V_t \subset H^0(C, L_t)$ such that $\mathcal{D}_t = \mathbb{P}(V_t)$. Moreover, let $U \subset T$ be an open set with $0 \in U$ and for any $t \in U$, let $\{s_{1,t}, \ldots, s_{k,t}\}$ be a basis of V_t such that $s_{i,t} \to s_{i,0}$ when $t \to 0$. For $j = 1, \ldots, k$ let $\pi_j\colon C^k \longrightarrow C$ be the projection on the j-th factor and for any $t \in U$ let us define the rank k vector bundle on C^k

$$M_t := \bigoplus_{j=1}^{k} \pi_j^* L_t$$

and the sections

$$S_{i,t} := \pi_1^* s_{i,t} \oplus \ldots \oplus \pi_k^* s_{i,t} \in H^0(C^k, M_t) \quad \text{for } i = 1, \ldots, k.$$

Furthermore, let us consider the line bundle $\bigwedge^k M_t$ on C^k and the section $S_t := S_{0,t} \wedge \ldots \wedge S_{r,t} \in H^0(C^k, \bigwedge^k M_t)$. Therefore it is easy to see that for any $t \in U$ we have

$$\widetilde{\Gamma}_k(\mathcal{D}_t) = (S_t)_0 - \bigcup_{a,b} \Delta_{a,b}\,,$$

where $(S_t)_0$ is the zero divisor of the section S_t and $\Delta_{a,b}$ is the (a,b)-diagonal of C^k, with $a, b = 1, \ldots, k$ and $a \neq b$. Clearly, $S_t \to S_0$ when $t \to 0$ and hence $\widetilde{\Gamma}_k(\mathcal{D}_0)$ is the flat limit at 0 of the family of divisors $\widetilde{\Gamma}_k(\mathcal{D}_t)$. $\square$

In order to conclude this discussion, we state a lemma to calculate the fundamental class of a variety induced by a g^r_d.

Lemma 1.4.4. *Let $d \geq k > r$ be some integers and let $\mathcal{D}$ be a linear series of degree d and dimension r on C. Then the fundamental class $\gamma_k(\mathcal{D})$ of the cycle $\Gamma_k(\mathcal{D})$ in $C^{(k)}$ is given by*

$$\gamma_k(\mathcal{D}) = \sum_{i=0}^{k-r} \binom{d-g-r}{i} \frac{x^i\,\theta^{k-r-i}}{(k-r-i)!}\,.$$

Proof. See [1, Lemma 3.2 p.342]. $\square$

1.5 Grassmannians and symmetric products

In this section we would like to recall some connection between Grassmann varieties and symmetric products of curves. Our purpose is to present some canonical divisors on the k-fold symmetric product.

To start, we briefly recall some basic definitions and facts on Grassmannians (see e.g. [26, Chapter 1 Section 5] or [28, Lecture 6] for details).

Let $n \geq k$ be some positive integer. The *Grassmanniann* $G(k, n)$ is the $k(n-k)$-dimensional variety parametrizing the k-dimensional vector subspaces of $\mathbb{C}^n$. Equivalently, $G(k, n)$ could be thought as the variety of the $(k-1)$-dimensional planes in $\mathbb{P}^{n-1}$; in this case we denote it by $\mathbb{G}(k-1, n-1)$.

Let $\Lambda \in G(k, n)$ be a k-dimensional vector subspace of $\mathbb{C}^n$ and let $\{v_1, \ldots, v_k\} \in \mathbb{C}^n$ be a basis of Λ. Then we may define the *Plücker embedding* as

$$\begin{array}{rccc} p\colon & G(k,n) & \longrightarrow & \mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right) \cong \mathbb{P}^{\binom{n}{k}-1} \\ & \Lambda & \longmapsto & v_1 \wedge \ldots \wedge v_k \,. \end{array}$$

Notice that we can represent the vector space $\Lambda \subset \mathbb{C}^n$ by a $k \times n$ matrix whose rows are the vectors v_i's. Then the Plücker coordinates in $\mathbb{P}^{\binom{n}{k}-1}$ are just the determinants of the $k \times k$ minors of such matrix.

Let V be a flag in $\mathbb{C}^n$, that is $V = (V_1 \subset \ldots \subset V_{n-1} \subset V_n)$ where any V_t is a t-dimensional subspace of $\mathbb{C}^n$.

Definition 1.5.1. Let $a_1, \ldots, a_k$ be a non-increasing sequence of integers such that $0 \leq a_i \leq n-k$ and let $a = (a_1, \ldots, a_k)$. Then we define the *Schubert cycle* $\sigma_a(V)$ as

$$\sigma_a(V) := \{\Lambda \in G(k,n) \,|\, dim\,(\Lambda \cap V_{n-k+i-a_i}) \geq i\} \,.$$

Then, by setting $b_j = a_{j+1}$ for $j = 0, \ldots, k-1$ and by thinking the V_t's as $(t-1)$-planes in $\mathbb{P}^{n-1}$, we have

$$\sigma_a(V) = \left\{\Lambda \in \mathbb{G}(k-1, n-1) \,|\, dim\,(\Lambda \cap V_{n-k+j-b_j}) \geq j\right\} .$$

We remark that the Schubert cycles are subvarieties of the Grassmann variety. Moreover, any Schubert cycle of the form

$$\sigma_1(V) = \sigma_1(V_{n-k}) = \{\Lambda \in G(k,n) \,|\, dim\,(\Lambda \cap V_{n-k}) \geq 1\} \,.$$

maps into a hyperplane section of $p\,(G(k,n)) \subset \mathbb{P}^{\binom{n}{k}-1}$. Therefore it is a divisor on $G(k, n)$ and $\sigma_1(V) \in |\mathcal{O}_{G(k,n)}(1)|$.

Now, let C be a smooth projective curve of genus $g \geq 2$ and let

$$\phi_K \colon C \longrightarrow \mathbb{P}^{g-1}$$

be the map associated to the canonical linear series $|K_C|$. Moreover, consider an integer k with $2 \leq k \leq g-1$ and let $C^{(k)}$ be the k-fold symmetric product of C. We recall that $\phi_K(C)$ is a non-degenerate curve in $\mathbb{P}^{g-1}$, whose degree is equal to $g-1$ if C is hyperelliptic and $2g-2$ otherwise. Then by General Position Theorem, we have that the general point $p_1 + \ldots + p_k \in C^{(k)}$ is such that $\phi_K(p_1), \ldots, \phi_K(p_k)$ are linearly independent points in $\mathbb{P}^{g-1}$. Thus we can define the rational map

$$\varphi_k \colon C^{(k)} \dashrightarrow \mathbb{G}(k-1, g-1)$$

given by

$$p_1 + \ldots + p_k \longmapsto \overline{\phi_K(p_1) + \ldots + \phi_K(p_k)},$$

where $\overline{\phi_K(p_1) + \ldots + \phi_K(p_k)}$ denote the linear span of the $\phi_K(p_i)$'s in $\mathbb{P}^{g-1}$.

Remark 1.5.2. Let us consider the Abel map $u \colon C^{(k)} \longrightarrow J(C)$ and let $\mathcal{G} \colon C^{(k)} \longrightarrow T\,C^{(k)}$ be the Gauss map sending a point $P = p_1 + \ldots + p_k \in C^{(k)}$ to the tangent space $T_P C^{(k)}$. If the p_i's are distinct, the image $u_* \left(T_P C^{(k)}\right)$ of the derivatives of u is the linear span of the $\phi_K(p_i)$'s in $\mathbb{P}^{g-1}$. Thus the map φ_k above could be obtained by composing the Gauss map and the derivatives of Abel map. As customary, we refer to φ_k as the *Gauss map* of the k-fold symmetric product (cf. [17] and [18]).

Now, let us consider the canonical linear system $|K_{C^{(k)}}|$ on the k-fold symmetric product and let

$$\psi_k \colon C^{(k)} \longrightarrow \mathbb{P}\left(H^0(C^{(k)}, K_{C^{(k)}})\right)$$

be the associated map. The following holds.

Lemma 1.5.3. *$H^0(C^{(k)}, K_{C^{(k)}}) \cong \bigwedge^k H^0(C, K_C)$.*

Proof. See [39]. $\square$

In particular, $\mathbb{P}\left(H^0(C^{(k)}, K_{C^{(k)}})\right) \cong \mathbb{P}\left(\bigwedge^k H^0(C, K_C)\right) \cong \mathbb{P}^N$ and it is easy to check that the diagram

$$\begin{tikzcd}
C^{(k)} \arrow[rr, "\psi_k"] \arrow[dr, dashed, "\varphi_k"'] & & \mathbb{P}^N \\
& \mathbb{G}(k-1, g-1) \arrow[ur, "p"'] &
\end{tikzcd}$$

is commutative, where we set $N := \binom{g}{k} - 1$. Then we have the following.

Lemma 1.5.4. *Let C be a smooth projective curve of genus $g \geq 2$. For any $L \in \mathbb{G}(g-k-1, g-1)$, let $\pi_L \colon \phi_K(C) \dashrightarrow \mathbb{P}^{k-1}$ be the projection from the $(g-k-1)$-plane L of the canonical image of C in $\mathbb{P}^{g-1}$ and let $\mathcal{D}_L$ be the associated linear series on C - not necessarily base-point-free - of degree $2g-2$ and dimension $k-1$. Then the effective divisor*

$$\Gamma_k(\mathcal{D}_L) = \{P \in C^{(k)} \,|\, E - P \geq 0 \text{ for some } E \in \mathcal{D}_L\}$$

is a canonical divisor of $C^{(k)}$, that is $\Gamma_k(\mathcal{D}_L) \in |K_{C^{(k)}}|$.

Proof. Let $L \in \mathbb{G}(g-k-1, g-1)$ and let $\sigma_1(L) \subset \mathbb{G}(k-1, g-1)$ be the corresponding Schubert cycle. Since $|K_{C^{(k)}}| = \psi_k^* \, |\mathcal{O}_{\mathbb{P}^N}(1)| = \varphi_k^* \, |\mathcal{O}_{\mathbb{G}(k-1,g-1)}(1)|$ and the Plücker embedding maps $\sigma_1(L)$ into a hyperplane section, it suffices to prove that $\varphi_k^* \sigma_1(L) = \Gamma_k(\mathcal{D}_L)$.

Let $\omega_1, \ldots, \omega_g \in H^0(C, K_C)$ be a basis and let $\phi_K \colon C \longrightarrow \mathbb{P}^{g-1}$ be the canonical map. Without loss of generality, let $L \subset \mathbb{P}^{g-1}$ be the $(g-k-1)$-plane given by $x_1 = \ldots = x_k = 0$ and let π_L be the projection from L on the $(k-1)$-plane $x_{k+1} = \ldots = x_g = 0$. Clearly, $\phi_K^*(x_i) = \omega_i$ for any i and $\mathcal{D}_L = \mathbb{P}(V_L)$, where $V_L \subset H^0(C, K_C)$ is the vector subspaces generated by $\omega_1, \ldots, \omega_k$.

So, let $\Lambda \subset \mathbb{G}(k-1, g-1)$ and let $p_1 + \ldots + p_k \in \varphi_k^*(\Lambda) \subset C^{(k)}$. If one of the p_i's is a base point of $\mathcal{D}_L$ - i.e. L contains one of the $\phi_K(p_i)$'s - then the assertion is straightforward by Remark 1.4.2.

On the other hand, suppose that none of the p_i's is a base point of $\mathcal{D}$. Then $\Lambda \in \sigma_1(L)$ if and only if there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ not all zero such that $\lambda_1 \omega_j(\phi_K(p_1)) + \ldots + \lambda_k \omega_j(\phi_K(p_k)) = 0$ for all $j = 1, \ldots, k$. This is equivalent to say that there exists a non-zero section $\omega = \mu_1\omega_1 + \ldots + \mu_k\omega_k \in V_L$ vanishing at the p_i's, that is $p_1 + \ldots + p_k \in \mathcal{D}_L$. □

In particular, with the notation above, we have that a generic point $p_1 + \ldots + p_k \in C^{(k)}$ lies on the divisor $\Gamma_k(\mathcal{D}_L)$ if and only if the linear span of the $\phi_K(p_j)$ in $\mathbb{P}^{g-1}$ is a point of the Schubert cycle $\sigma_1(L)$.

To conclude, we compute the numerical equivalence class of canonical divisors on $C^{(k)}$. By setting $d = 2g-2$ and $r = k-1$ in the formulas of Lemma 1.3.1 and Lemma 1.4.4, we have

$$\gamma_k(\mathcal{D}_L) = (2g-2)x - \frac{\delta}{2}$$

for any $L \in \mathbb{G}(g-k-1, g-1)$. Hence this is the numerical equivalence class in $N^1(C^{(k)})$ of any canonical divisor on $C^{(k)}$.

Chapter 2

Correspondences with null trace and symmetric products of curves

In this chapter we consider a smooth projective curve C of genus g and its k-fold symmetric product $C^{(k)}$ with $2 \leq k \leq g-1$. Our aim is to study the geometry of correspondences with null trace on $Y \times C^{(k)}$ for some projective k-dimensional integral variety Y. The techniques and the results of this chapter will be very useful to treat the more interesting problems of Chapter 3 and Chapter 4.

In the fundamental paper [43], Mumford starts from Severi's idea of using regular 2-forms to the study of rational equivalence of 0-cycles on surfaces. In [37], Lopez and Pirola use these techniques to study correspondences on smooth surfaces in $\mathbb{P}^3$ and linear series on families of curves lying on these surfaces. We start by arguing in a analogous way.

The first Section deals with preliminaries on correspondences. We follow [43] and [37] to define the Mumford's trace map and to recall some results on correspondences with null trace on smooth varieties.

Section 2 is devoted to the study of correspondences on symmetric products of curves. If $\Gamma \subset Y \times C^{(k)}$ is a correspondence of degree $d \geq 2$ with null trace, it is possible to define a rational map $Y \dashrightarrow C^{(kd)}$. For a general point $y \in Y_{reg}$ let $p_1 + \ldots + p_{kd} \in C^{(kd)}$ be its image. Denoting by $\phi_K \colon C \longrightarrow \mathbb{P}^{g-1}$ the canonical map of C, we prove that the linear span of the $\phi_K(p_i)$'s has dimension lower than $[\frac{kd}{2}]$ (see Theorem 2.2.2). Then we deduce some consequences on the existence of special linear series on C (cf. Corollary 2.2.3) and we present some examples of correspondences with null trace on $C^{(k)}$.

In the last Section, we turn to linear subspaces of $\mathbb{P}^n$ satisfying a condition of Cayley-Bacharach type. We note that the results of the second Section descend from the main theorem of this one. In particular, we shall prove the following fact. Let $\Lambda_1, \ldots, \Lambda_d \subset \mathbb{P}^n$ be some $(k-1)$-dimensional planes such that for every $i = 1, \ldots, d$ and for any $(n-k)$-plane $L \subset \mathbb{P}^n$ intersecting $\Lambda_1, \ldots, \widehat{\Lambda}_i, \ldots, \Lambda_d$, we have $\Lambda_i \cap L \neq \emptyset$ too. Then the linear span of the Λ_i's has dimension lower than $[\frac{kd}{2}]$ (cf. Theorem 2.3.2).

2.1 Mumford's trace map and correspondences with null trace

In this section we recall the basic properties of Mumford's induced differentials and their applications to the study of correspondences with null trace.

Let X and Y be two projective varieties of dimension n, with X smooth and Y integral.

Definition 2.1.1. A *correspondence of degree d on $Y \times X$* is a reduced n-dimensional variety $\Gamma \subset Y \times X$ such that the projections $\pi_1 \colon \Gamma \longrightarrow Y$, $\pi_2 \colon \Gamma \longrightarrow X$ are generically finite dominant morphisms and $deg\, \pi_1 = d$. Moreover, if $deg\, \pi_2 = d'$ we say that Γ is a (d, d')-*correspondence*.
If Y' is an integral n-dimensional variety and $\Gamma' \subset Y' \times X$ is another correspondence of degree d, we say that Γ and Γ' are *equivalent* if there exists a birational map $f \colon Y' \longrightarrow Y$ such that $\Gamma' = (f \times id_X)^{-1}(\Gamma)$.

So, let $\Gamma \subset Y \times X$ be a correspondence of degree d. Let us denote by $X^d = X \times \ldots \times X$ the d-fold ordinary product of X and, for $i = 1, \ldots, d$, let $p_i \colon X^d \longrightarrow X$ be the projection map. Let S_d denote the d-th symmetric group and let us consider the d-fold symmetric product $X^{(d)} = X^d / S_d$ of X together with the quotient map $\pi \colon X^d \longrightarrow X^{(d)}$.

Then we define the set $U := \{y \in Y_{reg} \,|\, dim\, \pi_1^{-1}(y) = 0\}$ and the morphism

$$\gamma \colon U \longrightarrow X^{(d)}$$

given by $\gamma(y) \longmapsto P_1 + \ldots + P_d$, where $\pi_1^{-1}(y) = \{(y, P_i) \,|\, i = 1, \ldots, d\}$.

By using Mumford's induced differentials (cf. [43, Section 2]), we want to define the trace map of γ.

To this aim, we consider a holomorphic n-form $\omega \in H^0(X, \Omega_X^n)$ and the $(n, 0)$-form

$$\omega^{(d)} := \sum_{i=1}^{d} p_i^* \omega \in H^{n,0}(X^d),$$

which is invariant under the action of S_d. Thus for any smooth variety W and for any morphism $f \colon W \longrightarrow X^{(d)}$, there exists a canonically induced

$(n,0)$-form ω_f on W (see [43, Section 2]). In particular, we define the Mumford's *trace map* of γ as

$$\begin{array}{rccc} Tr_\gamma\colon & H^{n,0}(X) & \longrightarrow & H^{n,0}(U) \\ & \omega & \longmapsto & \omega_\gamma \end{array}.$$

Another way to define the trace map of γ is the following. Let us consider the sets $V := \{y \in U \,|\pi_1^{-1}(y) \text{ has } d \text{ distinct points}\}$ and

$$X_0^{(d)} := \pi\left(X^d - \bigcup_{i,j}\Delta_{i,j}\right),$$

where $\Delta_{i,j}$ is the (i,j)-diagonal of X^d, with $i,j = 1,\ldots,d$ and $i \neq j$. Moreover, let us define the map

$$\begin{array}{rccc} \delta_d\colon & H^{n,0}(X) & \longrightarrow & H^{n,0}(X_0^{(d)}) \\ & \omega & \longmapsto & \pi_*(\omega^{(d)}) \end{array}$$

i.e. $\omega^{(d)}$ is thought as a $(n,0)$-form on $X_0^{(d)}$. Then $Im\,\gamma_{|V} \subset X_0^{(d)}$ and the following holds (cf. [37, Proposition 2.1] and [21]).

Proposition 2.1.2. $Tr_\gamma = \gamma_{|V}^* \circ \delta_d$.

So the above result gives an equivalent way to define the Mumford's trace map of γ.

To conclude this section we state a proposition on correspondence with null trace which plays a fundamental role to prove the results in [21] and [37]. To start, we recall the following definition (cf. [27]).

Definition 2.1.3. Let X be a smooth projective variety of dimension n and let $\mathcal{D}$ be a complete linear system on X. We say that the 0-cycle $P_1 + \ldots + P_r \subset X^{(r)}$ *satisfies the Cayley-Bacharach condition with respect to* $\mathcal{D}$ if for every $i = 1,\ldots,r$ and for any effective divisor $D \in \mathcal{D}$ passing through $P_1,\ldots,\widehat{P_i},\ldots,P_r$, we have $P_i \in D$.

The following result shows that the property of having null trace of a given correspondence $\Gamma \subset Y \times X$ imposes a strong condition on the fibers of the map $\pi_1\colon \gamma \longrightarrow Y$. In [37] it is presented in the case of correspondences of surfaces, but it is still true when X and Y are n-dimensional varieties and the proof follows the same argument (see for instance [21]).

Proposition 2.1.4. *Let X and Y be two projective varieties of dimension n, with X smooth and Y integral. Let Γ be a correspondence of degree d on $Y \times X$ with null trace. Let $y \in Y_{reg}$ such that $dim\,\pi_1^{-1}(y) = 0$ and let $\pi_1^{-1}(y) = \{(y,P_i) \in \Gamma \,|\, i = 1,\ldots,d\}$ be its fiber. Then the 0-cycle $P_1+\ldots+P_d$ satisfies the Cayley-Bacharach condition with respect to $|K_X|$, that is for every $i = 1,\ldots,d$ and for any effective canonical divisor K_X containing $P_1,\ldots,\widehat{P_i},\ldots,P_d$, we have $P_i \in K_X$.*

We give a sketch of the proof which is based on interpreting locally what having null trace means. Let $Z = \gamma(V)$ and consider a holomorphic form $\omega \in H^{n,0}(X)$. Then one can see that $Tr_\gamma(\omega) = 0$ if and only if $\delta_d(\omega)_{|Z} = 0$. Let $z = P_1 + \ldots + P_d$ be a general point on Z and let $\omega_{P_i} \colon \bigwedge^n T_{P_i}X \longrightarrow \mathbb{C}$ be the linear form induced by ω at P_i. Then for every $u \in \bigwedge^n T_z X_0^{(d)} \cong \bigwedge^n \left(\bigoplus_{i=1}^d T_{P_i}X\right)$, with $u = u_1 \wedge \ldots \wedge u_n$ and $u_j = (v_{j1}, \ldots, v_{jd})$ one has that $\delta_d(\omega)(z)(u) = \sum_{i=1}^d \omega_{P_i}(v_{1i} \wedge \ldots \wedge v_{ni})$. Suppose that $P_1, \ldots, \widehat{P}_k, \ldots, P_d$ lie on the canonical divisor defined by ω, that is $\omega_{P_i} = 0$ for any $i \neq k$. As Γ has null trace, $\delta_d(\omega)(z)(u) = \omega_{P_k}(v_{1k} \wedge \ldots \wedge v_{nk}) = 0$. Finally, being $v_{1k}, \ldots, v_{nk}$ arbitrary on $T_{P_k}X$, one concludes that $\omega_{P_k} = 0$ as well.

2.2 Correspondences with null trace and symmetric products of curves

In this section we shall prove some results connecting the existence of a correspondence with null trace on $Y \times C^{(k)}$ and the geometry of the curve C, for some k-dimensional variety Y. Then we shall present some examples of correspondences with null trace on the k-fold symmetric product of C.

Let C be a smooth projective curve of genus g and let $C^{(k)}$ be its k-fold symmetric product, with $2 \leq k \leq g-1$. Our first task is to give a geometric interpretation of the existence of a correspondence with null trace on $C^{(k)}$.

To this aim, let Y be a projective integral variety of dimension k and let $\Gamma \subset Y \times C^{(k)}$ be a correspondence of degree $d \geq 2$ with null trace. We recall that the map $\pi_1 \colon \Gamma \longrightarrow Y$ is defined to be the restriction of the natural projection map on Y and it is a generically finite dominant morphism of degree d. Consider a generic point $y \in Y_{reg}$ and let

$$\pi_1^{-1}(y) = \{(y, P_i) \in Y \times C^{(k)} \mid i = 1, \ldots, d\}$$

be its fiber, where $P_i = p_{i1} + \ldots + p_{ik}$ for $i = 1, \ldots, d$. Proposition 2.1.4 assures that the 0-cycle $P_1 + \ldots + P_d$ satisfies the Cayley-Bacharach condition with respect to the canonical linear series $|K_{C^{(k)}}|$ on $C^{(k)}$, that is for every $i = 1, \ldots, d$ and for any effective canonical divisor $D \in |K_{C^{(k)}}|$ containing $P_1, \ldots, \widehat{P}_i, \ldots, P_d$, we have $P_i \in D$.

Now, let $\phi_K \colon C \longrightarrow \mathbb{P}^{g-1}$ be the canonical map of C. Moreover, for any $(g-k-1)$-plane $L \subset \mathbb{P}^{g-1}$, let us denote by $\pi_L \colon \phi_K(C) \dashrightarrow \mathbb{P}^{k-1}$ the projection map from L of the canonical image of C in $\mathbb{P}^{g-1}$ and let $\mathcal{D}_L$ denote the associated linear series on C. By Lemma 1.5.4 we have that the effective divisor

$$\Gamma_k(\mathcal{D}_L) := \{P \in C^{(k)} \mid E - P \geq 0 \text{ for some } E \in \mathcal{D}_L\}$$

is canonical on $C^{(k)}$. Therefore by Cayley-Bacharach condition we have that the 0-cycle $P_1 + \ldots + P_d$ is such that for every $i = 1, \ldots, d$ and for any $L \in \mathbb{G}(g-k-1, g-1)$ with $P_1, \ldots, \widehat{P_i}, \ldots, P_d \in \Gamma_k(\mathcal{D}_L)$, we have $P_i \in \Gamma_k(\mathcal{D}_L)$.

Then consider the Gauss map $\varphi_k \colon C^{(k)} \dashrightarrow \mathbb{G}(k-1, g-1)$ sending a point $P = p_1 + \ldots + p_k$ to the linear span of the $\phi_K(p_j)$'s in $\mathbb{P}^{g-1}$. We recall that for any $(g-k-1)$-plane $L \subset \mathbb{P}^{g-1}$ we have that $P \in \Gamma_k(\mathcal{D}_L)$ if and only if the $(k-1)$-plane $\varphi_k(P)$ intersects L, that is $\varphi_k(P) \in \sigma_1(L)$ (cf. Section 1.5).

Thus for every $i = 1, \ldots, d$ and for any $L \in \mathbb{G}(g-k-1, g-1)$ intersecting the $(k-1)$-planes $\varphi_k(P_1), \ldots, \widehat{\varphi_k(P_i)}, \ldots, \varphi_k(P_d)$, we have $\varphi_k(P_i) \cap L \neq \emptyset$. In particular, the $\varphi_k(P_i)$'s satisfy a condition of Cayley-Backarach type with respect to $(g-k-1)$-planes.

The following theorem provides a bound on the dimension of the linear span in of $(k-1)$-planes in $\mathbb{P}^n$ enjoying the property above. The proof will be given in the next section.

Theorem 2.2.1. *Let $\Lambda_1, \ldots, \Lambda_d \subset \mathbb{P}^n$ be linear subspaces of dimension $(k-1)$ with the property that for every $i = 1, \ldots, d$ and for any $(n-k)$-plane $L \subset \mathbb{P}^n$ intersecting $\Lambda_1, \ldots, \widehat{\Lambda_i}, \ldots, \Lambda_d$, we have $\Lambda_i \cap L \neq \emptyset$ too.*
Then the dimension of their linear span $S = Span(\Lambda_1, \ldots, \Lambda_d)$ in $\mathbb{P}^n$ is $s \leq \left[\frac{kd}{2}\right] - 1$.

It follows that the linear span of the $(k-1)$-planes $\varphi_k(P_1), \ldots, \varphi_k(P_d)$ in $\mathbb{P}^{g-1}$ has dimension lower than $\left[\frac{kd}{2}\right]$. Moreover, we recall that for any point $P_i = p_{i1} + \ldots + p_{ik} \in C^{(k)}$, the $(k-1)$-plane $\varphi_k(P_i)$ is defined to be the linear span of $\phi_K(p_{i1}), \ldots, \phi_K(p_{ik}) \in \mathbb{P}^{g-1}$. Hence we conclude that the linear span of all the $\phi_K(p_{ij})$'s has dimension bounded by $\left[\frac{kd}{2}\right] - 1$.

By summing up, we proved the following.

Theorem 2.2.2. *Let C be a smooth projective curve of genus g and let Y be a projective integral variety of dimension $2 \leq k \leq g-1$. Let Γ be a correspondence of degree $d \geq 2$ on $Y \times C^{(k)}$ with null trace. For a generic point $y \in Y_{reg}$, let $\pi_1^{-1}(y) = \{(y, P_i) \in \Gamma \mid i = 1, \ldots, d\}$ be its fiber, where $P_i = p_{i1} + \ldots + p_{ik} \in C^{(k)}$ for $i = 1, \ldots, d$.*
Then the linear span of all the $\phi_K(p_{ij})$'s in $\mathbb{P}^{g-1}$ has dimension

$$s \leq \left[\frac{k\,d}{2}\right] - 1.$$

As we anticipated, the latter result will turn out to be very useful to prove some results in Chapter 3 and Chapter 4 on second symmetric products of curves. Hence we shall apply Theorem 2.2.2 in the simplified version with $k = 2$:

Given a correspondence $\Gamma \subset Y \times C^{(2)}$ of degree d with null trace, for any generic point $y \in Y_{reg}$ with fiber $\pi_1^{-1}(y) = \{(y, p_{i1} + p_{i2}) \in \Gamma \mid i = 1, \ldots, d\}$, the linear span of all the $\phi_K(p_{ij})$'s in $\mathbb{P}^{g-1}$ has dimension $s \leq d - 1$.

As an immediate consequence of Theorem 2.2.2 we have the following result connecting the existence of correspondences with null trace on $C^{(2)}$ and the existence of complete linear series on C.

Corollary 2.2.3. *Suppose in addiction that C is non-hyperelliptic and that the number of distinct p_{ij}'s is $m > [\frac{kd}{2}]$. Then C possesses a complete g^r_m with $r \geq 1$.*

Proof. For $i = 1, \ldots, d$, let $P_i = p_{i1} + \ldots + p_{ik}$. Let m be the number of distinct p_{ij}'s on C and let us denote by $q_1, \ldots, q_m$ these points. Consider the divisor $D = q_1 + \ldots + q_m$ of degree m on C. As the curve C is non-hyperelliptic, the canonical map is an embedding and the q_t's are all distinct. Hence $\overline{\phi_K(D)}$ is the linear span in $\mathbb{P}^{g-1}$ of the $\phi_K(q_t)$'s and its dimension is lower than $[\frac{kd}{2}]$ by Theorem 2.2.2. Therefore by the geometric version of Riemann-Roch theorem we have

$$dim\,|D| = m - 1 - dim\,\overline{\phi(D)} \geq m - \left[\frac{kd}{2}\right] \geq 1.$$

Thus $|D| = |q_1 + \ldots + q_m|$ is a complete g^r_m on C with $r \geq 1$. □

Remark 2.2.4. In [27], Griffiths and Harris study 0-cycles on an algebraic variety X satisfying the Cayley-Bacharach condition with respect to a complete linear system $|D|$. In particular, given such a 0-cycle $P_1 + \ldots + P_d$ and the rational map $\phi_{|D|}\colon X \dashrightarrow \mathbb{P}^r$, they present some result on the dimension of the linear span of the $\phi_{|D|}(P_i)$'s in $\mathbb{P}^r$ and, consequently, on the existence of linear series on X. We note that we start from an analogous situation with $X = C^{(k)}$, but the results of this section deal with the study of the geometry of the curve C and not with X.

We conclude this section by presenting some examples of correspondences with null trace on the k-fold symmetric product.

Example 2.2.5. For any dominant rational map $F\colon C^{(k)} \dashrightarrow \mathbb{P}^k$ of degree d, the graph of F

$$\Gamma := \{(y, P) \in \mathbb{P}^k \times C^{(k)} \mid F(P) = y\}$$

is a $(d,1)$-correspondence on $\mathbb{P}^k \times C^{(k)}$ with null trace. To see this fact, notice that the fiber $F^{-1}(y)$ over a generic point $y \in \mathbb{P}^k$ is given by d distinct points $P_1, \dots, P_d \in X$. Hence Γ is a reduced variety and the projection $\pi_1 \colon \Gamma \longrightarrow \mathbb{P}^k$ is a generically finite dominant morphism of degree d. Moreover, $H^0(\mathbb{P}^k, \Omega^k) = 0$ and hence Γ is a $(d,1)$-correspondence on $\mathbb{P}^k \times C^{(k)}$ with null trace.
Notice that this fact is still true for any smooth n-dimensional variety admitting a dominant rational map of degree d on $\mathbb{P}^n$.

Example 2.2.6. Let $f \colon C \longrightarrow \mathbb{P}^1$ be a degree d map and for any point $Q \in C^{(k-1)}$ consider the curve $C_Q := \{p + Q \in C^{(k)} \,|\, p \in C\}$ lying on the k-fold symmetric product $C^{(k)}$. As each C_Q is naturally identified with C, we can consider the map $f_Q \colon C_Q \longrightarrow \mathbb{P}^1$ induced by f under this identification. Moreover, we define the varieties $Y := \mathbb{P}^1 \times C^{(k-1)}$ and

$$\Gamma := \Big\{ \big((z,Q),P\big) \in Y \times C^{(k)} \,|\, P \in C_Q \text{ and } f_Q(P) = z \Big\}.$$

Then it is east to see that Γ is a (d,k)-correspondence on $Y \times C^{(k)}$. Furthermore, Y is a ruled variety. Therefore $h^0\big(Y, K_Y\big) = 0$ and hence the correspondence Γ has null trace.

Example 2.2.7. Let T be a $(k-1)$-dimensional smooth variety and let $\mathcal{E} = \{E_t\}_{t \in T}$ be a proper flat family of smooth curves lying on $C^{(k)}$ such that any E_t is a d-gonal curve (i.e. for any $t \in T$ there exists a degree d covering $f_t \colon E_t \longrightarrow \mathbb{P}^1$) and $\mathcal{E}$ covers $C^{(k)}$ (i.e. for any $P \in C^{(k)}$ there exists $t \in T$ such that $P \in E_t$).

Let Y and Γ be the varieties of dimension k defined as $Y := \mathbb{P}^1 \times T$ and

$$\Gamma := \Big\{ \big((z,t),P\big) \in Y \times C^{(k)} \,|\, P \in E_t \text{ and } f_t(P) = z \Big\}.$$

Then $\Gamma \subset Y \times C^{(k)}$ is a correspondence of degree d with null trace. To see this fact it suffices to argue as above and to observe that $\pi_2 \colon \Gamma \longrightarrow C^{(k)}$ is generically finite. Indeed, if there exist infinitely many curves of the family passing through the generic point $P \in C^{(k)}$, then T would be at least a k-dimensional variety.

2.3 Linear subspaces of $\mathbb{P}^n$ in special position

This section deals with sets of linear subspaces of the n-dimensional projective space satisfying a condition of Cayley-Bacharach type. In particular, we shall prove Theorem 2.2.1 stated in the previous section.

Let n and k be two integers with $2 \leq k \leq n$ and let $\mathbb{G}(k-1, n)$ denote the Grassmann variety of $(k-1)$-planes in $\mathbb{P}^n$. For an integer $d \geq 2$, let us consider a set of d points

$$\{\Lambda_1, \dots, \Lambda_d\} \subset \mathbb{G}(k-1, n)$$

and suppose that the associated 0-cycle $\Lambda_1 + \ldots + \Lambda_d$ satisfies the Cayley-Bacharach condition with respect to the complete linear series $|\mathcal{O}_{\mathbb{G}(k-1,n)}(1)|$.

For any $(n-k)$-dimensional subspace L of $\mathbb{P}^n$, let us consider the Schubert cycle

$$\sigma_1(L) := \{\Lambda \in \mathbb{G}(k-1, n) | \Lambda \cap L \neq \emptyset\}$$

of the $(k-1)$-planes of $\mathbb{P}^n$ intersecting L, which is an effective divisor of $|\mathcal{O}_{\mathbb{G}(k-1,n)}(1)|$. Thus the set $\{\Lambda_1, \dots, \Lambda_d\} \subset \mathbb{G}(k-1, n)$ is such that for every $i = 1, \dots, d$ and for any $L \in \mathbb{G}(n-k, n)$ with $\Lambda_1, \dots, \widehat{\Lambda}_i, \dots, \Lambda_d \in \sigma_1(L)$, we have $\Lambda_i \in \sigma_1(L)$.

So it makes sense to give the following definition expressing a condition of Cayley-Bacharach type for the linear subspaces $\Lambda_1, \dots, \Lambda_d \subset \mathbb{P}^n$.

Definition 2.3.1. We say that the $(k-1)$-planes $\Lambda_1, \dots, \Lambda_d \subset \mathbb{P}^n$ are in *special position with respect to $(n-k)$-planes* if for every $i = 1, \dots, d$ and for any $(n-k)$-plane $L \subset \mathbb{P}^n$ intersecting $\Lambda_1, \dots, \widehat{\Lambda}_i, \dots, \Lambda_d$, we have $\Lambda_i \cap L \neq \emptyset$ too.

We note that the $(k-1)$-planes in the definition are not assumed to be distinct. In particular, it is immediate to check that two $(k-1)$-planes $\Lambda_1, \Lambda_2 \subset \mathbb{P}^n$ are in special position if and only if they coincide.

Thanks to Definition 2.3.1, we may rephrase Theorem 2.2.1 as follows.

Theorem 2.3.2. *Suppose that the $(k-1)$-planes $\Lambda_1, \dots, \Lambda_d \subset \mathbb{P}^n$ are in special position with respect to $(n-k)$-planes of $\mathbb{P}^n$. Then the dimension of their linear span $S = Span(\Lambda_1, \dots, \Lambda_d)$ in $\mathbb{P}^n$ is $s \leq \left[\frac{kd}{2}\right] - 1$.*

In order to prove this result, let us state the following preliminary lemma.

Lemma 2.3.3. *Suppose that the $(k-1)$-planes $\Lambda_1, \dots, \Lambda_d \subset \mathbb{P}^n$ are in special position with respect to $(n-k)$-planes of $\mathbb{P}^n$. If there exists a linear space $R \subset \mathbb{P}^n$ such that $\Lambda_1, \dots, \widehat{\Lambda}_j, \dots, \Lambda_d \subset R$, then $\Lambda_j \subset R$ as well.*

Proof. Let r denote the dimension of R. If $r = n$ the statement is trivially true, then let us assume $r < n$. As $k-1 \leq r$ we have that $0 \leq r-k+1 \leq n-k$ and we can consider a $(r-k+1)$-plane $T \subset R$. Then T intersects each of the $(k-1)$-planes $\Lambda_1, \dots, \widehat{\Lambda}_j, \dots, \Lambda_d$. Therefore by special position property, for any $(n-k)$-plane L containing T, the $(k-1)$-plane Λ_j must intersect L, thus $\Lambda_j \cap T \neq \emptyset$. Therefore Λ_j meets every $(r-k+1)$-plane $T \subset R$. Thus $\Lambda_j \subset R$. $\square$

Proof of Theorem 2.3.2. As usual, let us fix $n \geq d \geq 2$ and $2 \leq k \leq n$. Notice that if $n \leq \left[\frac{kd}{2}\right] - 1$, the statement is trivially proved. Hence we assume hereafter $n \geq \left[\frac{kd}{2}\right]$. We proceed by induction on the number d of $(k-1)$-planes.

Let $\Lambda_1, \Lambda_2 \subset \mathbb{P}^n$ be two $(k-1)$-dimensional planes in special position with respect to $(n-k)$-planes. Then we set $R := \Lambda_1$ and Lemma 2.3.3 implies $\Lambda_2 \subset R$. Hence $R = \Lambda_1 = \Lambda_2$ and $[\frac{kd}{2}] - 1 = k - 1 = dim\, R$. Thus the statement is proved when $d = 2$.

By induction, suppose that the statement of the theorem holds for any $2 \leq h \leq d-1$ and for h-tuple of $(k-1)$-dimensional linear subspaces of $\mathbb{P}^m$ in special position with respect to $(m-k)$-planes, with $m \geq h$.

Now, let $\Lambda_1, \ldots, \Lambda_d \subset \mathbb{P}^n$ be $(k-1)$-planes in special position with respect to $(n-k)$-planes.

We first consider the case where is not possible to choose one of the Λ_i's such that it does not coincide with any of the others. In this situation, the number of distinct Λ_i's is at most $\left[\frac{d}{2}\right]$. Thus the dimension of their linear span in $\mathbb{P}^n$ is at most $k\left[\frac{d}{2}\right] - 1 \leq \left[\frac{kd}{2}\right] - 1$ as claimed.

Then we consider the $(k-1)$-plane Λ_1 and we suppose -without loss of generality- that it does not coincide with any of the others Λ_i's. Then is possible to choose a point $p \in \Lambda_1$ such that $p \notin \Lambda_i$ for any $i = 2, \ldots, d$. Moreover, let $H \subset \mathbb{P}^n$ be an hyperplane not containing p and consider the projection from p on H

$$\begin{array}{rccc} \pi_p\colon & \mathbb{P}^n - \{p\} & \longrightarrow & H \\ & q & \longmapsto & \overline{pq} \cap H\,. \end{array}$$

For $i = 2, \ldots, d$, let $\lambda_i := \pi_p(\Lambda_i) \subset H$ be the image of Λ_i on H. We claim that the $(k-1)$-planes $\lambda_2, \ldots, \lambda_d \subset H$ are in special position with respect to $(n-1-k)$ planes of $H \cong \mathbb{P}^{n-1}$. To see this fact, let $j \in \{2, \ldots, d\}$ and let $l \subset H$ be a $(n-1-k)$ plane intersecting $\lambda_2, \ldots, \widehat{\lambda_j}, \ldots, \lambda_d$. Since $p \in \Lambda_1$, it follows that the $(n-k)$-plane $L := Span(l, p) \subset \mathbb{P}^n$ intersects $\Lambda_1, \ldots, \widehat{\Lambda_j}, \ldots, \Lambda_d$. As they are in special position with respect to $(n-k)$-planes, we have that L intersects Λ_j as well. Then, given a point $q_j \in L \cap \Lambda_j$, we have that $\pi_p(q_j) \in l$. In particular, l meets λ_j at $\pi_p(q_j)$ and hence $\lambda_2, \ldots, \lambda_d \subset H$ are in special position with respect to $(n-1-k)$-planes of the hyperplane $H \subset \mathbb{P}^n$.

By induction, the linear span $S := Span(\lambda_2, \ldots, \lambda_d) \subset H$ has dimension $s \leq \left[\frac{k(d-1)}{2}\right] - 1$. Then the linear space $R := Span(\lambda_2, \ldots, \lambda_d, p) \subset \mathbb{P}^n$ has dimension $dim\, R = dim\, S + 1 \leq \left[\frac{k(d-1)}{2}\right] \leq \left[\frac{kd}{2}\right] - 1$ for any $k \geq 2$. Notice that R contains $\Lambda_2, \ldots, \Lambda_d$. Hence $\Lambda_1 \subset R$ as well by Lemma 2.3.3. Thus R contains the linear span in $\mathbb{P}^n$ of all the Λ_i's and the assertion follows. $\square$

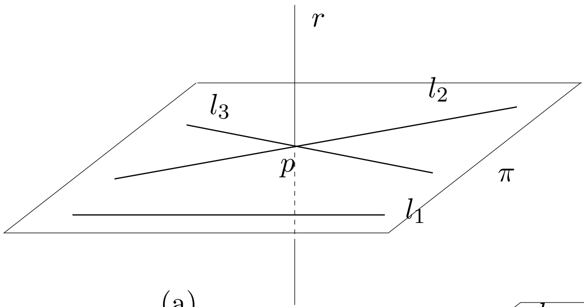
At the beginning of this section we set $k \geq 2$. We note that this assumption is necessary in Theorem 2.3.2. For instance, let $k = 1$ and consider three collinear points in $\mathbb{P}^n$. Clearly, they are in special position with respect to $(n-1)$-planes and $[\frac{kd}{2}] - 1 = 0$, but they span a line.

On the other hand, when $k = 2$ the theorem assures that if $\{l_1, \dots, l_d\}$ is a set of d lines in special position with respect to $(n-2)$-planes of $\mathbb{P}^n$, then their linear span has dimension lower than d. In particular, the following examples show that this bound is sharp.

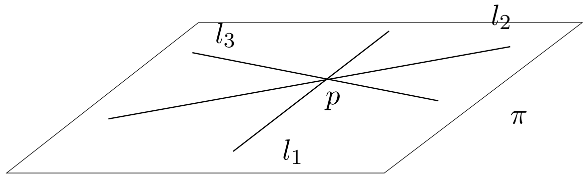
Example 2.3.4. In $\mathbb{P}^3$, let us consider three distinct lines l_1, l_2, l_3. Then they are in special position with respect to lines if and only if they lie on a plane $\pi \subset \mathbb{P}^n$ and they meet at a point $p \in \pi$.

To see this fact, suppose that l_1, l_2, l_3 are in special position with respect to lines. Therefore they must lie on a plane $\pi \subset \mathbb{P}^3$ by Theorem 2.3.2. Then consider the point $p = l_2 \cap l_3$ and let $r \not\subset \pi$ be a line passing through p as in figure (a). As r intersects both l_2 and l_3, it must intersect l_1 too by special position property. Hence $p \in l_1$.

On the other hand, it is immediate to check that if three distinct lines of $\mathbb{P}^3$ lie on the same plane and meet at a point (as in figure (b) below), then they are in special position with respect to $(n-2)$-planes.



(a)



(b)

Example 2.3.5. If $l_1, \dots, l_4$ are four skew lines lying on the same ruling of a quadric surface $Q \subset \mathbb{P}^3$, then they are in special position with respect to lines in $\mathbb{P}^3$ and they span the whole space.

To see this fact it suffices to observe that Q is covered by two families of skew lines, $\mathcal{L}$ and $\mathcal{L}'$, such that any two lines $l \in \mathcal{L}$ and $l' \in \mathcal{L}'$ meet at a point (see e.g. [26, p. 478]).

Example 2.3.6. In general, if $l_1, \dots, l_d \subset \mathbb{P}^{d-1}$ are skew lines lying on a non-degenerate surface $S \subset \mathbb{P}^{d-1}$ of minimal degree, then they are in special position with respect to $(d-3)$-planes.

Under these assumptions, S is a ruled surface of degree $d-2$ (cf [26, p. 522]). If $L \subset \mathbb{P}^{d-1}$ is a $(d-3)$-plane intersecting $l_1, \ldots, l_{d-1}$, then $L \cap S$ is a curve C of degree $\leq d-2$. In particular, C does not lies on the ruling of S and hence it must intersect l_d too.

Chapter 3

Deformation and gonality of curves on second symmetric products of curves

Let C be a smooth projective curve of genus g. In this chapter we focus on the second symmetric product and we deal with some problems on curves lying on the surface $C^{(2)}$.

In [47], Pirola proves that generic Abelian variety of dimension $q \geq 3$ does not contain hyperelliptic curves of any genus. In Section 1, we start from this result to prove that if C is a very general curve of genus $g \geq 3$, then there are neither rational nor hyperelliptic curves lying on the Jacobian variety $J(C)$ and hence on the second symmetric product $C^{(2)}$.

In the second Section we turn to study the gonality of moving curves lying on $C^{(2)}$. By the results of the previous chapter, we prove that given a curve C of genus $g \geq 4$ and a family of d-gonal curves covering $C^{(2)}$, then d is bounded from below by the the gonality of C. As a consequence of this fact, we show that the degree of gonality of $C^{(2)}$ defined in the introduction is $d_g(C^{(2)}) = gon(C)$.

3.1 Gonality of curves lying on $C^{(2)}$

Let C be a smooth complex projective curve of genus $g \geq 3$ and let us assume that C is very general in the moduli space $\mathcal{M}_g$. Under this assumption, we have that C is non-hyperelliptic and hence its second symmetric product $C^{(2)}$ embeds into the Jacobian variety $J(C)$ via the Abel map. Then, in order to discuss the gonality of curves lying on $C^{(2)}$, we focus on curves lying on the Jacobian variety $J(C)$.

To start, we recall that any Abelian variety does not contain rational curves. Indeed, if R were a rational curve contained in an Abelian variety A, then the inclusion map should factor through the Jacobian variety of R. As the Jacobian variety of a rational curve is a point, we get a contradiction.

In [47], Pirola studies rigidity and existence of curves of small geometric genus on generic Kummer varieties. As a consequence of the main theorem he deduces that a generic Abelian variety of dimension $q \geq 3$ does not contain hyperelliptic curve of any genus, where elliptic curves are considered as special cases of hyperelliptic curves. Since for any three-dimensional Abelian variety there exists an isogeny to a Jacobian variety of a genus three curve, we deduce that for any very general curve C of genus three, its Jacobian variety $J(C)$ does not contain hyperelliptic curves. Thus by using a degeneration argument we have the following.

Proposition 3.1.1. *If C is a very general curve of genus $g \geq 3$, the Jacobian variety $J(C)$ does not contain hyperelliptic curves.*

Proof. As we said above, the case of genus three is a consequence of Theorem 2 in [47]. Then by induction on the genus, suppose that the statement holds for every very general curve of genus $g-1$.

So, consider a very general curve D of genus $g-1$ and a smooth elliptic curve Y, together with two points $p \in D$ and $q \in Y$. Let C_0 be the nodal curve obtained by gluing D and Y at p and q. Let $\mathcal{C} \longrightarrow \Delta$ be a proper flat family over a disc Δ such that the fiber over $0 \in \Delta$ is C_0 and for any $t \neq 0$ the fiber C_t is a smooth curve of genus g.

Then consider the Jacobian bundle over Δ of $\mathcal{C}$, that is $J(\mathcal{C}) \longrightarrow \Delta$ with $J(\mathcal{C})_t = J(C_t)$ for all $t \in \Delta - \{0\}$. By contradiction, assume that the fiber $J(C_t)$ of $J(\mathcal{C})$ contains a hyperelliptic curve E_t for very general $t \in \Delta - \{0\}$. Hence - up to restrict the disk Δ - we can define the following map of families over the punctured disk $\Delta - \{0\}$

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\varphi"] \arrow[dr] & & J(\mathcal{C}) \arrow[dl] \\ & \Delta - \{0\} & \end{tikzcd}$$

where $\mathcal{E} = \{E_t\}_{t \in \Delta - \{0\}}$ and $\varphi_t : E_t \hookrightarrow J(C_t)$ is the inclusion map.

We have $J(\mathcal{C})_0 = J(D) \times J(Y) = J(D) \times Y$. Let $\pi_1 \colon J(D) \times Y \longrightarrow J(D)$ denote the natural projection map on the first factor. Let $E_0 \subset J(D) \times Y$ be the flat limit of the family of hyperelliptic curves $\mathcal{E}$ at $t = 0$. Since the very general fiber E_t generates $J(C_t)$ as a group, then E_0 must generate $J(D) \times Y$. Thus $\pi_1(E_0) \subset J(D)$ cannot be 0-dimensional and hence it is a non-rational curve on $J(D)$. Then E_0 has some non-rational irreducible components that are all hyperelliptic curves (cf. [4, p. 14]). Therefore all the irreducible components of $\pi_1(E_0)$ are hyperelliptic and we have a contradiction because D has genus $g-1$ and its Jacobian variety $J(D)$ does not contain hyperelliptic curves by induction. $\square$

As a consequence of the proposition, the following holds.

Corollary 3.1.2. *Let C be a very general curve of genus $g \geq 3$. Then there are neither rational curves nor hyperelliptic curves lying on $C^{(2)}$.*

Remark 3.1.3. We note that if the genus of C is $g < 3$, its second symmetric product $C^{(2)}$ contains both rational and hyperelliptic curves. This fact is clear when C is a rational curve, because $C^{(2)} \cong \mathbb{P}^2$. If C is an elliptic curve, then the fibers of the Abel map $C^{(2)} \longrightarrow J(C) \cong C$ are isomorphic to $\mathbb{P}^1$. Moreover, as elliptic curves are considered to be a special case hyperelliptic curves, we have that $C^{(2)}$ is covered by a one-dimensional family of curves isomorphic to C. Analogously, when $g = 2$, $C^{(2)}$ is covered by hyperelliptic curves and the fiber of the g^1_2 via the Abel map is a rational curve.

3.2 Deformation of curves on $C^{(2)}$

In this section we deal with the gonality of moving curves on the second symmetric product of a smooth curve and we discuss the degree of gonality of this surface.

We recall that we defined the *degree of gonality* of an irreducible complex projective variety X to be the integer

$$d_g(X) := min\left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a family } \mathcal{E} = \{E_t\}_{t \in T} \\ \text{covering } X \text{ whose generic member is} \\ \text{an irreducible } d\text{-gonal curve} \end{array} \right\}.$$

Then let C be a smooth complex projective curve of genus g and let us consider its second symmetric product. The degree of gonality of $C^{(2)}$ in cases of low genera is easily given. When C is a rational curve, then $C^{(2)} \cong \mathbb{P}^2$ and hence $d_g(C^{(2)}) = 1$. On the other hand, if C is supposed to be an elliptic curve, we have that the second symmetric product of C is birational to $C \times \mathbb{P}^1$, then $d_g(C^{(2)}) = d_g(C \times \mathbb{P}^1) = 1$ (cf. Remark 3.1.3).

For any $g \geq 0$, the second symmetric product $C^{(2)}$ is covered by the family $\mathcal{E} = \{X_p\}_{p \in C}$ of curves parametrized over C, where

$$X_p := C + p = \{p + q \,|\, q \in C\}.$$

Clearly, any X_p is isomorphic to C and hence $gon(X_p) = gon(C)$. Therefore the degree of gonality of the second symmetric product of a curve of genus $g \geq 0$ is

$$d_g(C^{(2)}) \leq gon(C). \tag{3.1}$$

In particular, as the only rational curve lying on a hyperelliptic curve of genus two is the fiber of the g^1_2 via the Abel map $u\colon C^{(2)} \to J(C)$, we have that $d_g(C^{(2)}) = 2$ for any curve C of genus $g = 2$.

Then we have the following theorem bounding from below the degree of gonality of $C^{(2)}$. The argument of the proof is essentially based on two results. The first one is Theorem 2.2.2 we proved in the last chapter and the second result is Abel's theorem. We note that this result will turn out to be useful in Chapter 5 to compute new bounds for the cone of ample divisor classes in the Néron-Severi group of $C^{(2)}$.

Theorem 3.2.1. *Let C be a smooth complex projective curve of genus $g \geq 4$. For a positive integer d, let $\mathcal{E} = \{E_t\}_{t \in T}$ be a family of curves on $C^{(2)}$ parametrized over a smooth variety T, such that the generic fiber E_t is an irreducible d-gonal curve and for any point $P \in C^{(2)}$ there exists $t \in T$ such that $P \in E_t$. Then $d \geq gon(C)$.*

Moreover, under the further assumption $g \geq 6$ and $Aut(C) = \{Id_C\}$, we have that equality holds if and only if E_t is isomorphic to C.

Proof. Notice that C is a hyperelliptic curve of genus $g \geq 4$, the only rational curve lying on $C^{(2)}$ is the fiber of the g^1_2 via the Abel map $u\colon C^{(2)} \to J(C)$. Therefore the gonality of the generic curve E_t must be $d \geq 2 = gon(C)$ and the assertion follows.

Then we assume hereafter that C is non-hyperelliptic. As $C^{(2)}$ is two-dimensional and $\mathcal{E}$ is a family of curves, up to restrict $\mathcal{E}$ to a subvariety of T, we can assume that T has dimension one. Aiming for a contradiction, we assume further that $d < gon(C)$. We split the proof in some parts.

Step 1 [*Correspondence on $C^{(2)}$*]. For any $t \in T$, let $\nu_t\colon \widetilde{E}_t \longrightarrow E_t$ be the normalization of E_t and let $f_t\colon \widetilde{E}_t \longrightarrow \mathbb{P}^1$ be a morphism such that $deg\, f_t = gon(\widetilde{E}_t) = d$.

Setting $Y := \mathbb{P}^1 \times T$, we may define a correspondence Γ on $Y \times C^{(2)}$ as the Zariski closure of the set (cf. Example 2.2.7)

$$\Big\{\big((z,t), P\big) \in Y \times C^{(2)} \,|\, P \in (E_t)_{reg} \text{ and } f_t \circ \nu_t^{-1}(P) = z\Big\}.$$

Notice that both the projection maps $\pi_1\colon \Gamma \longrightarrow Y$ and $\pi_2\colon \Gamma \longrightarrow C^{(2)}$ are dominant morphisms. Since $deg\, f_t = d$ for generic $t \in T$, we have that π_1 is a generically finite morphism of degree d. Moreover, π_2 is generically finite too: if there exist infinitely many curves of the family passing through the general point $P \in C^{(2)}$, then T would be at least a 2-dimensional variety. Finally, being Y a ruled surface, we have that $h^0(Y, K_Y) = 0$ and hence $\Gamma \subset Y \times C^{(2)}$ is a correspondence of degree d with null trace.

For a very general point $(z,t) \in Y$, let

$$\pi_1^{-1}(z,t) = \{\big((z,t), P_i\big) \in Y \times C^{(2)} \,|\, i = 1, \ldots, d\} \tag{3.2}$$

be its fiber, where $P_i = p_{2i-1} + p_{2i}$. Moreover, let $D = D_{(z,t)} \in Div(C)$ be the effective divisor given by

$$D := p_1 + \ldots + p_{2d} = \sum_{j=1}^{m} n_j q_j \tag{3.3}$$

for some positive integers $n_j = mult_{q_j}(D)$ and where the q_j's are assumed to be distinct points of C.

Step 2 [$n_j = 1$ *for all* j]. Suppose that $n_j = 1$ for any $j = 1, \ldots, m$, that is $m = 2d$ and the points defining D are all distinct. As usual, let $\phi_K \colon C \longrightarrow \mathbb{P}^{g-1}$ be the canonical embedding of C and let $\overline{\phi_K(D)}$ be the linear span of the points $\phi_K(p_i)'s$ in $\mathbb{P}^{g-1}$. As Γ is a correspondence of degree d on $C^{(2)}$ with null trace, by Theorem 2.2.2 we have that $dim\, \overline{\phi_K(D)} \leq d-1$. Thus by the geometric version of Riemann-Roch theorem we have

$$dim\,|D| = deg\, D - 1 - dim\, \overline{\phi_K(D)} \geq 2d - 1 - (d-1) = d = \frac{deg\, D}{2}.$$

Therefore we have that either D is zero, D is a canonical divisor or C is hyperelliptic by Clifford's theorem. We recall that C is assumed to be non-hyperelliptic. Furthermore, as $0 < d < gon(C) \leq \left[\frac{g+3}{2}\right]$, we have that $0 < deg\, D = 2d < 2g-2$ for any $g \geq 4$ and hence we have a contradiction.

Step 3 [$n_j > 1$ *for some* j]. On the other hand, let us suppose that the points $p_1, \ldots, p_{2d}$ are not distinct, i.e. the integers n_j are not all equal to 1. For any $a = 1, \ldots, 2d$, let us consider the set

$$Q_a := \{q_j \in Supp\, D \,|\, n_j = a\}$$

of the points of D such that $mult_{q_j}(D) = a$. Notice that the cardinality of any Q_a is at most $\left[\frac{2d}{a}\right]$.

As the n_j's are not all equal to 1, there exists $\bar{a} > 1$ such that the corresponding set $Q_{\bar{a}}$ is not empty. Without loss of generality, suppose $Q_{\bar{a}} = \{q_1, \ldots, q_s\}$, where $s \leq \left[\frac{2d}{\bar{a}}\right]$ is the cardinality of $Q_{\bar{a}}$.

Since Y is connected, the fibers of π_1 over generic points of Y have the same configuration, i.e. the cardinality of any set $Q_{\bar{a}}$ is constant as we vary the point (z,t) on an opportune open set $U \subset Y$. Thus we may define a rational map $\xi\colon Y \dashrightarrow C^{(s)}$ sending a generic point $(z,t) \in Y = \mathbb{P}^1 \times T$ to the effective divisor $q_1 + \ldots + q_s \in C^{(s)}$. For a very general $t \in T$, let

$$\begin{array}{rccc} \xi_t\colon & \mathbb{P}^1 \times \{t\} & \longrightarrow & C^{(s)} \\ & (z,t) & \longmapsto & q_1 + \ldots + q_s \end{array}$$

be the restriction of ξ to the rational curve $\mathbb{P}^1 \times \{t\} \subset Y$ and let us consider the composition with the Abel map

$$\mathbb{P}^1 \times \{t\} \xrightarrow{\xi_t} C^{(s)} \hookrightarrow J(C)\,.$$

As $\mathbb{P}^1 \times \{t\}$ is a rational curve mapping into a Jacobian variety, the latter map is constant. Hence by Abel's theorem, either $|q_1+\ldots+q_s|$ is a complete linear series of degree s and dimension at least 1, or ξ_t is a constant map. Being $\bar{a} > 1$ and $s \leq \left[\frac{2d}{\bar{a}}\right]$, we have $s \leq d < gon(C)$. Then $|q_1 + \ldots + q_s|$ can not be such a linear series.

Therefore the map ξ_t must be constant. By the construction of ξ_t, this fact means that for any $z \in \mathbb{P}^1$ the divisor $D = D_{(z,t)}$ - defined in (3.3) by the fiber $\pi_1^{-1}(z,t)$ - must contain all the points $q_1, \ldots, q_s$, that are now fixed. We recall that $\pi_1^{-1}(z,t)$ is given by the points $((z,t), P_i) \in Y \times C^{(2)}$ such that $P_i \in E_t$ and $f_t \circ \nu_t^{-1}(P_i) = z$. Hence one of the P_i's must lie on the curve $C + q_1$, one on $C + q_2$ and so on. As we vary z on $\mathbb{P}^1$, the P_i's must vary on E_t, but the latter condition must hold. It follows that the curve E_t must have at least s irreducible components $E_{t1}, \ldots, E_{ts}$ such that $E_{tj} \subset C + q_j$ for $j = 1, \ldots, d$. Since E_t and $C + q_j$ are irreducible curves, we deduce $s = 1$ and $E_t = C + q_1$. Then we get a contradiction because $C + q_1 \cong C$ and hence $d = gon(E_t) = gon(C)$.

Thus we conclude that the gonality d of the generic E_t is $d \geq gon(C)$.

Step 4 [$d = gon(C)$]. Now, let C be a curve of genus $g \geq 6$ with $Aut(C) = \{Id_C\}$ and let us suppose that the $d = gon(\widetilde{E}_t) = gon(C)$. We want to prove that E_t and C are isomorphic.

To this aim, let us consider the correspondence Γ defined above and a generic fiber $\pi_1^{-1}(z,t) = \big\{\big((z,t), P_i\big)\big\}$ as in (3.2), with $P_i = p_{2i-1} + p_{2i}$. By arguing as in Step 2 we deduce that if the p_i's can not be distinct. If they were distinct, then $dim\,|D| = \frac{deg\,D}{2}$ and - by Clifford's theorem - we would have that either D is zero, D is a canonical divisor or C is hyperelliptic. We note that the assumption $Aut(C) = \{Id_C\}$ implies that C is non-hyperelliptic. Moreover, the degree of D is positive and

$$deg\,D = 2d = 2gon(C) \leq 2\left[\frac{g+3}{2}\right] < 2g-2 \quad \text{for any } g \geq 6.$$

Hence the divisor D is neither zero nor canonical and we have a contradiction.

Then we follow the argument of Step 3 and for the generic $t \in T$ we may define the map $\xi_t \colon \mathbb{P}^1 \times \{t\} \longrightarrow C^{(s)}$, with $s \leq d$.

If $s < d = gon(C)$, the only possible choice is $s = 1$ because of the irreducibility of E_t. Hence E_t and C turn out to be isomorphic.

On the other hand, suppose that $s = d = gon(C)$. Then $\bar{a} = 2$ and the divisor D in (3.3) has the form $D = 2(q_1 + \ldots + q_d)$. In particular - without loss of generality - the points $P_i \in C^{(2)}$ are given by

$$P_1 = q_1 + q_2,\ P_2 = q_2 + q_3, \ldots,\ P_{d-1} = q_{d-1} + q_d \text{ and } P_d = q_d + q_1.$$

Now, we fix a point $p \in C$ and we define an automorphism $\alpha \colon C \longrightarrow C$ sending a point $q \in C$ to the unique point $q' \in C$ such that $q + p$ and $q' + p$ lie on the same fiber of π_1. Thus α is a non-trivial automorphism on C and this situation can not occur because $Aut(C) = \{Id_c\}$. □

Therefore by (3.1) and Theorem 3.2.1 we have the following.

Theorem 3.2.2. *Let C be a smooth complex projective curve of genus $g \geq 4$. Then $d_g(C^{(2)}) = gon(C)$.*

To conclude our survey on the degree of gonality of second symmetric products of curves, it remains to estimate the case of genus three. As any curve C of genus $g = 3$ possess a g^1_3, we have that $d_g(C^{(2)}) = 2$ if C is hyperelliptic and $2 \leq d_g(C^{(2)}) \leq 3$ otherwise. We note that when C is assumed to be very general in $\mathcal{M}_3$, Corollary 3.1.2 assures that $C^{(2)}$ does not contain hyperelliptic curves and hence $d_g(C^{(2)}) = 3$. Unfortunately, we are not able to deduce the same when C is neither generic nor hyperelliptic.

Chapter 4

Degree of irrationality of symmetric products of curves

Let C be a smooth complex projective curve C of genus g. Our purpose is to study the degree of irrationality of the second symmetric product of C, that is the minimum integer d such that $C^{(2)}$ admits a dominant rational map $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$ of degree d.

In Section 1 we introduce the notion of degree of irrationality in terms of fields extensions and we give a geometric interpretation. In order to give an overview on the degree of irrationality of n-dimensional algebraic varieties, we recall the known results on this topic.

In Section 2 we turn to discuss the problem of computing the degree of irrationality $d_r(C^{(2)})$ of the second symmetric product $C^{(2)}$ in dependence on the genus g and on the gonality of the curve C. When C is either a rational or an elliptic curve, the problem is totally understood. Then we focus on hyperelliptic curves and we prove that $d_r(C^{(2)}) = 4$ for any such a curve of genus $g \geq 4$. On the other hand, when C is assumed to be non-hyperelliptic we show that degree of irrationality of $C^{(2)}$ is bounded from below by the gonality of the curve and we improve this bounds for low genera. Finally, we prove that the degree of irrationality of a generic curve C of genus $g \geq 5$ is bounded from below by $g - 1$.

The last two sections are devoted to prove the results of Section 2. In particular, in the third Section we menage the non-hyperelliptic case, whereas in the fourth Section we conclde with the hyperelliptic one. We note that an important role in the proofs is played by monodromy, Abel's Theorem and the main results of Chapther 2 on correspondences with null trace on symmetric products of curves.

4.1 Generalities on degree of irrationality

In this section we shall introduce the notion of degree of irrationality of an algebraic variety defined over a field k and we shall recall some interesting results on this topic.

The notion of degree of irrationality arises in an algebraic context. Let k and L be two fields such that L is a finitely generated extension of k of transcendence degree n. In [40], Moh and Heizer define the *degree of irrationality of L over k* to be the integer

$$d_r(L) := min \left\{ m = [L : k(x_1, \dots, x_n)] \;\middle|\; \begin{array}{l} x_1, \dots, x_n \text{ are algebraically} \\ \text{independent elements of } L \end{array} \right\}.$$

Form a geometric point of view, suppose that k is an algebraic closed field of characteristic zero and let V be an algebraic variety over k of dimension n. Then the *degree of irrationality of V* is defined as the degree of irrationality of its rational function field $K(V)$ over k. As is well known, there is an equivalence between the category of the finitely generated extensions over k and the category of dominant rational maps between algebraic varieties over k. Therefore the definition above may be rephrased as follows

$$d_r(V) := min \left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a dominant rational map} \\ F \colon V \dashrightarrow \mathbb{P}^n_k \text{ of degree } d \end{array} \right\}.$$

Clearly, the degree of irrationality is a birational invariant of algebraic varieties. Moreover, it can be thought as a generalization to higher dimension of the notion of gonality for curves. Indeed, every dominant rational map $f \colon C \dashrightarrow \mathbb{P}^1$ of degree d defined over a smooth algebraic curve C can be resolved to a morphism and hence $d_r(C) = gon(C)$.

In particular, when C is a smooth algebraic curve, there are several results concerning the problem of determining its gonality. In this direction, one can think to the results on existence of special divisor we stated in Section 1.4 and to the famous Wirtinger theorem asserting that if $C \subset \mathbb{P}^2$ is a smooth plane curve of degree $d \geq 4$, then its gonality equals $d - 1$.

Since the end of the nineteenth century, several mathematicians dealt with problem of rationality for algebraic variety of dimension higher dimension.

After the work of Moh and Heinzer (see [40] and [41]), the author who more deeply studied the problem of compute the degree of irrationality of algebraic surfaces is Hisao Yoshihara. In [58] he proves some results concerning the degree of irrationality of irreducible surfaces of degree d in $\mathbb{P}^3$ and the following theorem (see [58, Theorem 3]).

Theorem 4.1.1. *Let S be an irreducible algebraic surface. If S is birationally equivalent to $\mathbb{P}^1 \times C$, where C is a smooth curve, then $d_r(S) = gon(C)$. If S is an Abelian surface, then $d_r(S) \geq 3$. If S is an hyperelliptic surface, then $3 \leq d_r(S) \leq 12$. If S is an Enriques surface, $d_r(S) = 2$.*

Moreover, Tokunaga and Yoshihara prove in a following paper that the degree of irrationality of any Abelian surface containing a smooth curve of genus 3, is three (cf. [53, Theorem 0.2]). We note that the construction made to prove the latter result shall be the starting point of the study we shall do in the last chapter of this work.

We note that the assertion on Abelian surface in Theorem 4.1.1 can be generalized to n-dimensional Abelian varieties as an immediate consequence of the results in [2]. Namely, the degree of irrationality of any Abelian variety A of dimension n is $d_r(A) \geq n + 1$ (cf. [53]).

Another work concerning several problems on the degree of irrationality is represented by the unpublished Ph.D. thesis of Cortini (see [21]). In this work, she studies some open problems presented by Yoshihara at the end of [58]. Cortini deals with the degree of irrationality of smooth hypersurfaces in $\mathbb{P}^n$ and K3 surfaces; furthermore, she completes the study of Yoshihara on the degree of irrationality of smooth surfaces in $\mathbb{P}^3$. In particular, Cortini gives a characterization of the surfaces $S \subset \mathbb{P}^3$ of degree d having $d_r(S) = d - 2$, and proves that any other smooth surface in $\mathbb{P}^3$ has degree of irrationality equal to $d - 1$.

4.2 Degree of irrationality of second symmetric products of curves

In this section we shall present several result on the degree of irrationality of the second symmetric product of a smooth complex projective curve C of genus $g \geq 0$. Our discussion will be developed in dependence on the genus and the gonality of the curve. The proofs of all the results we shall state will be included in the next section.

Let C be a smooth complex projective curve of genus g and let $C^{(2)}$ be its second symmetric product. We would like to study the degree of irrationality $d_r(C^{(2)})$ of the surface $C^{(2)}$, that is the minimum integer $d \in \mathbb{N}$ such that there exists a dominant rational map $F \colon C^{(2)} \dashrightarrow \mathbb{P}^2$ of degree d.

To start, we focus on the cases $g = 0$ and $g = 1$. In particular, when C is either a rational or an elliptic curve, the problem of determining the degree of irrationality of $C^{(2)}$ is totally understood.

Namely, if C is a rational curve, then $C^{(2)} \cong \mathbb{P}^2$. Hence the second symmetric product is a rational surface and $d_r(C^{(2)}) = 1$.

On the other hand, when C is an elliptic curve we have $d_r(C^{(2)}) = 2$. To prove this fact, let us consider the Jacobian variety $J(C)$ of C and the Abel map $u\colon C^{(2)} \longrightarrow J(C)$. By Abel Theorem each fiber of u over a point is isomorphic to $\mathbb{P}^1$. Since $J(C) \cong C$, we have that $C^{(2)}$ is covered by a family of rational curves parametrized by C and hence $C^{(2)}$ is birational to the surface $C \times \mathbb{P}^1$. We recall that $C \times \mathbb{P}^1$ is not a rational surface (cf. [6, Proposition III.21]), then $d_r(C^{(2)}) = d_r(C \times \mathbb{P}^1) > 1$. The curve C admits a degree two covering $f\colon C \longrightarrow \mathbb{P}^1$, hence we may define a map $G\colon C \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ of degree two by setting $G(p, y) := (f(p), y)$. Finally, being $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ birational surfaces, we conclude $d_r(C^{(2)}) = d_r(C \times \mathbb{P}^1) = 2$.

Now let us assume that C is a smooth complex projective curve of genus $g \geq 2$. Under this assumption, the problem of computing the degree of irrationality of $C^{(2)}$ is still open. Our aim is to study this situation by giving some bounds on $d_r(C^{(2)})$. As we mentioned above, we remand the reader to the next sections for the proofs of the results we shall state in the following.

Firstly we note that $C^{(2)}$ is not a rational surface. Furthermore, the second symmetric product of a curve of genus $g \geq 2$ does not admit a degree two dominant rational map on the complex projective plane (see Lemma 4.3.1) and hence

$$d_r(C^{(2)}) \geq 3. \tag{4.1}$$

Clearly, the existence of a dominant rational map $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$ depends on the geometry of C and there are some connections between the degree of irrationality of the second symmetric product and the existence of linear series on the curve.

In particular, suppose that C is a d-gonal curve and let us consider a g^1_d on C inducing a non-constant morphism $f\colon C \longrightarrow \mathbb{P}^1$. Then is always possible to define a dominant morphism of degree d^2 as

$$\begin{array}{cccc} F\colon & C^{(2)} & \longrightarrow & \left(\mathbb{P}^1\right)^{(2)} \cong \mathbb{P}^2 \\ & p+q & \longmapsto & f(p)+f(q) \end{array} \tag{4.2}$$

and hence we deduce the obvious upper bound

$$d_r(C^{(2)}) \leq \left(gon(C)\right)^2. \tag{4.3}$$

Moreover, if C admits a g^2_d that induces a birational mapping $\psi\colon C \longrightarrow \mathbb{P}^2$ onto a non-degenerate curve of degree d, then we may define the dominant rational map of degree $\binom{d}{2}$

$$G\colon C^{(2)} \dashrightarrow \mathbb{G}(1,2) \cong \mathbb{P}^2 \tag{4.4}$$

sending a point $p+q \in C^{(2)}$ to the line through $\psi(p)$ and $\psi(q)$ in $\mathbb{P}^2$.

The following example provides a further construction of dominant rational maps on $C^{(2)}$.

Example 4.2.1. Let $\mu\colon C \longrightarrow \mathbb{P}^3$ be a birational map onto a non-degenerate curve of degree d. Consider a plane $h \subset \mathbb{P}^3$ and for any $p, q \in C$, let $l_{pq} \subset \mathbb{P}^3$ denote the line passing through $\mu(p)$ and $\mu(q)$. Then we may define the dominant rational map

$$\begin{array}{rccc} H\colon & C^{(2)} & \longrightarrow & h \cong \mathbb{P}^2 \\ & p+q & \longmapsto & l_{pq} \cap h\,. \end{array} \tag{4.5}$$

We note that the map H has degree equal to $\frac{(d-1)(d-2)}{2} - g$. Indeed, the degree of H is the number of bi-secant line to $\mu(C)$ passing through a general point $y \in h$. Then let us consider the projection $\pi_y\colon \mu(C) \longrightarrow \mathbb{P}^2$. As the number of such bi-secant lines equals the number of nodes of the image $C' := (\pi_y \circ \mu)(C)$, and C' is a curve of degree d on $\mathbb{P}^2$, we conclude that $deg\, H = p_a(C') - g(C') = \frac{(d-1)(d-2)}{2} - g$.

Thus we have the following upper bound on the degree of irrationality of the second symmetric product.

Proposition 4.2.2. *Let C be a smooth complex projective curve. Let δ_1 be the gonality of C and for $m = 2, 3$, let δ_m be the minimum of the integers d such that C admits a birational mapping onto a non-degenerate curve of degree d in $\mathbb{P}^m$. Then*

$$d_r(C^{(2)}) \leq min \left\{ \delta_1^2\,,\ \frac{\delta_2(\delta_2-1)}{2}\,,\ \frac{(\delta_3-1)(\delta_3-2)}{2} - g \right\}.$$

If the curve C is assumed to be hyperelliptic, inequalities (4.1) and (4.3) assure that $d_r(C^{(2)})$ is either 3 or 4. The following example shows that there are curves of genus 2 - and hence hyperelliptic - with $d_r(C^{(2)}) = 3$. It is a particular case of a construction made in [53] by Tokunaga and Yoshihara.

Example 4.2.3. Let C be a smooth curve of genus $g = 2$ and let $J(C)$ be its Jacobian variety. Assume that the surface $J(C)$ contains a smooth genus three curve D. Then by [53, Theorem 0.2] there exist a rational surface Y and a dominant rational map $\gamma\colon J(C) \dashrightarrow Y$ of degree 3. Therefore by composing γ and the Abel map $u\colon C^{(2)} \longrightarrow J(C)$ we get a degree three dominant rational map from $C^{(2)}$ to the rational surface Y. Thus $d_r(C^{(2)}) = 3$.

Unfortunately, except for the example above, we are not able to establish the degree of irrationality of $C^{(2)}$ when C is a hyperelliptic curve of genus $g = 2, 3$. On the other hand, when the genus of C is greater than 3 we prove the following theorem which resolves the problem in the hyperelliptic case.

Theorem 4.2.4. *Let C be a smooth complex projective curve of genus $g \geq 4$. If C is hyperelliptic, then $d_r(C^{(2)}) = 4$.*

In particular, under the assumption of the theorem, the degree of irrationality of $C^{(2)}$ is as great as possible. Moreover, looking at the map in (4.2), we have that there is a dominant rational map $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$ of minimal degree which is actually a morphism.

So let us assume that C is a non-hyperelliptic curve of genus $g \geq 3$. The following theorem shows a further connection between the degree of irrationality of $C^{(2)}$ and the gonality of C.

Theorem 4.2.5. *Let C be a smooth complex projective curve of genus $g \geq 3$ and assume that C is non-hyperelliptic. Then*

$$d_r(C^{(2)}) \geq gon(C).$$

Notice that when the curve is trigonal, the latter bound coincides with (4.1). In particular, this is the case when $g = 3, 4$. On the other hand, when $g \geq 5$ we are able to present some improvements of the bound in the statement.

The following result summarizes the bounds on the degree of irrationality of $C^{(2)}$ and we list them by genus.

Proposition 4.2.6. *Let C be a smooth complex projective curve of genus $g \geq 3$ and assume that C is non-hyperelliptic. Then the following hold:*

(i) *if $g = 3, 4$, then $d_r(C^{(2)}) \geq 3$;*

(ii) *if $g = 5$, then $d_r(C^{(2)}) \geq 4$;*

(iii) *if $g = 6$, then $d_r(C^{(2)}) \geq 5$;*

(iv) *if $g \geq 7$, then*
$$d_r(C^{(2)}) \geq max\,\{6, gon(C)\}\,.$$

In particular, we note that that for $4 \leq g \leq 7$

$$d_r(C^{(2)}) \geq g - 1, \tag{4.6}$$

but the same inequality does not hold for any genus, as we can realize from the following examples.

Example 4.2.7. For an integer $d \geq 2$, let C be a non-hyperelliptic curve of genus $g \geq 2d^2 + 2$ provided of a degree d covering $f\colon C \longrightarrow E$ on an elliptic curve E (a particular case of this setting is given by bielliptic curves of genus greater than 9). Then we can define the dominant morphism $C^{(2)} \longrightarrow E^{(2)}$ of degree d^2 sending the point $p + q \in C^{(2)}$ to $f(p) + f(q) \in E^{(2)}$. As we saw at the beginning of this section, $d_r(E^{(2)}) = 2$ and there exists a dominant rational map $E^{(2)} \dashrightarrow \mathbb{P}^2$ of degree 2. Therefore we obtain by composition a dominant rational map $C^{(2)} \dashrightarrow \mathbb{P}^2$ of degree $2d^2$. Thus $d_r(C^{(2)}) \leq 2d^2 < g - 1$.

Example 4.2.8. By arguing analogously, if C is a non-hyperelliptic curve of genus $g \geq 3d^2 + 2$ admitting a non-constant morphism of degree $d \geq 2$ over a genus two curve D as in Example 4.2.3, it is immediate to check that $d_r(C^{(2)}) \leq 3d^2 < g - 1$.

On the other hand, when C is assumed to be generic in its moduli space, we prove bound (4.6) for any genus, i.e. the degree of irrationality of $C^{(2)}$ is bounded from below by $g - 1$.

Theorem 4.2.9. *Let C be a smooth complex projective curve of genus $g \geq 4$ and assume that C is very general in the moduli space $\mathcal{M}_g$. Then*

$$d_r(C^{(2)}) \geq g - 1.$$

We point out that the constructions we made in Example 4.2.7 and 4.2.8 are based on particular structures of the curves involved and they do not apply to very general curves. Furthermore, if C is a very general curve of genus $g \geq 2$, the minimum degree of a dominant rational map $C^{(2)} \dashrightarrow \mathbb{P}^2$ we are able to construct is given by one of the maps we use to establish Proposition 4.2.2. We recall that the minimum degree of a g^r_d on a generic curve of genus g is given by Brill-Noether number (see Section 1.4). Therefore, with the notation of Proposition 4.2.2, we have

$$\delta_1 = \left\lceil \frac{g+3}{2} \right\rceil, \quad \delta_2 = \left\lceil \frac{2g+8}{3} \right\rceil \quad \text{and} \quad \delta_3 = \left\lceil \frac{3g+15}{4} \right\rceil.$$

Thus the leading terms of the bounds in the proposition are $\frac{1}{4}\,g^2$, $\frac{2}{9}\,g^2$ and $\frac{9}{32}\,g^2$ respectively. Hence the bound provided by the g^2_d's is asymptotically the lowest. In particular, it seems natural to conjecture that it is actually the degree of irrationality of $C^{(2)}$ when C is a very general curve of high enough genus.

4.3 The non-hyperelliptic case

In this section we shall prove the most of the results presented in the previous one. In particular, we shall focus on the statements where C is assumed to be non-hyperelliptic. We note that - up to do slight adjustments - the proofs follow the same argument and hence they could appear somehow repetitive. An important role in the proofs will be played by monodromy, Abel's Theorem and the main results of Chapther 2 on correspondences with null trace on symmetric products of curves.

Let C be a smooth complex projective curve C of genus g and let us denote by $C^{(2)}$ its second symmetric product.

To start, we prove inequality (4.1). The argument of the proof is due to Hisao Yoshihara, who showed an analogous statement for Abelian and hyperelliptic surfaces (see [57, Theorem 2] and [58, Theorem 3]).

Lemma 4.3.1. *Let C be a smooth complex projective curve of genus $g \geq 2$. Then $d_r(C^{(2)}) \geq 3$.*

Proof. As $C^{(2)}$ is not a rational surface, we have to prove that it does not admit a dominant rational map $F \colon C^{(2)} \dashrightarrow \mathbb{P}^2$ of degree two. Aiming for a contradiction, we assume the contrary.

As in [57], there exists a composition of blow-ups $\sigma \colon S \longrightarrow C^{(2)}$ such that S is a double covering of a smooth rational surface Y. Let $\widetilde{F} \colon S \longrightarrow Y$ be the degree two morphism and let $\tau \colon S \longrightarrow S$ be the involution defined by $\widetilde{F}$.

Since the irregularity of $C^{(2)}$ is $q(C^{(2)}) = h^0(C^{(2)}, \Omega^1_{C^{(2)}}) = g \geq 2$ we have $q(S) = h^0(S, \Omega^1_S) \geq 2$. Thus there exist two holomorphic 1-forms $\omega_1, \omega_2 \in H^0(S, \Omega^1_S)$ such that $\omega_1 \wedge \omega_2 \neq 0$. Being Y a rational surface, then it has not holomorphic forms. Therefore $\widetilde{F}_* \omega_i = 0$ and hence $\tau_* \omega_i + \omega_i = 0$ for $i = 1, 2$, that is $\tau_* \omega_i = -\omega_i$ for $i = 1, 2$. Furthermore, we must have $\widetilde{F}_*(\omega_1 \wedge \omega_2) = 0$, but $\tau_*(\omega_1 \wedge \omega_2) = \tau_* \omega_1 \wedge \tau_* \omega_2 = \omega_1 \wedge \omega_2 \neq 0$, a contradiction. $\square$

Now, before proving other results of the previous section, we fix some piece of notation and we state three preliminary lemmas.

Throughout this section, by $F \colon C^{(2)} \dashrightarrow \mathbb{P}^2$ we denote a dominant rational map of minimal degree, that is $d := deg\, F = d_r(C^{(2)})$. Moreover, for a point $y \in \mathbb{P}^2$, we consider its fiber

$$F^{-1}(y) = \{p_1 + p_2, \ldots, p_{2d-1} + p_{2d}\} \subset C^{(2)}. \tag{4.7}$$

and we define the *divisor $D_y \in Div(C)$ associated to y* as

$$D_y := p_1 + p_2 + \ldots + p_{2d-1} + p_{2d}\,. \tag{4.8}$$

Then, by a simple monodromy argument, we have the following.

Lemma 4.3.2. *There exists an integer $1 \leq a \leq d$ such that for the very general point $y \in \mathbb{P}^2$ and for any $j = 1, \ldots, 2d$*

$$mult_{p_j}(D_y) = a.$$

In particular, the divisor D_y defined above has the form

$$D_y = a\,(q_1 + q_2 + \ldots + q_m)\,,$$

where $m = \frac{2d}{a}$ and the q_j's are distinct point of C.

Proof. Let $G\colon C \times C \dashrightarrow \mathbb{P}^2$ be the dominant rational map of degree $2d$ obtained by composing the map $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$ and the natural quotient map $\pi\colon C \times C \longrightarrow C^{(2)}$, that is $G(p,q) := F(p+q)$ for any $p+q \in C^{(2)}$. Let $y \in \mathbb{P}^2$ be a generic point and let

$$G^{-1}(y) = \{(p_1,p_2),(p_2,p_1),\ldots,(p_{2d-1},p_{2d}),(p_{2d},p_{2d-1})\} \subset C \times C$$

be its fiber. Then the divisor $D_y := p_1 + \ldots + p_{2d}$ is uniquely determined by the fiber $G^{-1}(y)$. Moreover, if m is the number of distinct points of $\{p_1,\ldots,p_{2d}\}$ and we denote by $q_1,\ldots,q_m$ these points, we have that D_y has the form

$$D_y = \sum_{j=1}^{m} a_j\, q_j$$

for some positive integers $a_j := mult_{q_j}(D_y)$. Therefore we have to prove that $a_1 = \ldots = a_m$.

As $C \times C$ is a connected surface, we have that the action of the monodromy group $M(G) \subset S_{2d}$ of G is transitive (see Lemma 1.2.1). Hence it is not possible to distinguish any point of the fiber $G^{-1}(y)$ from another. Then for any $(r,s),(v,w) \in G^{-1}(y)$ we have that $mult_r(D_y) = mult_v(D_y)$ and $mult_s(D_y) = mult_w(D_y)$. In particular, we can not distinguish the points (r,s) and (s,r), hence $mult_r(D_y) = mult_s(D_y)$. Thus the divisor D_y must have the same multiplicity at any p_i, i.e. there exists an integer $1 \leq a \leq 2d$ such that $a = mult_{p_i}(D_y)$ for any $i = 1,\ldots,2d$. Moreover a must divide $2d$ and the number m of distinct points in $\{p_1,\ldots,p_{2d}\}$ is $m = \frac{2d}{a}$. Finally, being y generic on $\mathbb{P}^2$, we have that the number of distinct p_j's is at least 2. Hence $m \geq 2$ and $a \leq d$. □

The second lemma is a consequence of Abel's theorem.

Lemma 4.3.3. *With the notation above, given a generic point $y \in \mathbb{P}^2$ with associate divisor $D_y = a\,(q_1 + q_2 + \ldots + q_m)$, we have that the linear series $|q_1 + q_2 + \ldots + q_m|$ is a complete g^r_m on C with $r \geq 2$.*
Moreover, the integer a is lower than $d = deg\, F$.

Proof. Thanks to the previous Lemma we are able to define the rational map $\xi\colon \mathbb{P}^2 \dashrightarrow C^{(m)}$ sending a generig point $y \in \mathbb{P}^2$ to the effective divisor $q_1 + q_2 + \ldots + q_m \in C^{(m)}$. As the image of $y \in \mathbb{P}^2$ depends on its fiber via the rational dominant map $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$, we have that ξ is non constant. Consider the resolution $\widetilde{\xi}\colon R \longrightarrow C^{(m)}$ of ξ and the composition with the Abel-Jacobi map

$$R \xrightarrow{\widetilde{\xi}} C^{(m)} \xrightarrow{u} J(C),$$

where R is a rational surface. By the universal property of Albanese morphism, the above map factors through the Albanese variety $Alb(R)$ of the rational surface R. As $Alb(R)$ is 0-dimensional, the composition $u \circ \widetilde{\xi}$ is a constant map. Being ξ non-constant, by Abel's theorem it follows that for all the generic points $y \in \mathbb{P}^2$, the divisors $q_1 + q_2 + \ldots + q_m$ are all linearly equivalent. Furthermore, as y vary on a surface, we deduce that the complete linear series $|q_1 + q_2 + \ldots + q_m|$ has dimension $r \geq 2$.

To conclude, we recall that $1 \leq a \leq d$. If a were equal to d, then $m = 2$ and the linear series $|q_1 + q_2|$ would have degree 2 and dimension 2. Hence $a < d$. □

Finally, the third important lemma is an immediate consequence of Theorem 2.2.2 on correspondences with null trace on symmetric products of curves.

Lemma 4.3.4. *Let C be a non-hyperelliptic curve of genus $g \geq 5$ and let $F \colon C^{(2)} \dashrightarrow \mathbb{P}^2$ be a dominant rational map of degree $d < g - 1$.*
With the notation above, given a generic point $y \in \mathbb{P}^2$, we have that the points $p_1, \ldots, p_{2d} \in C$ in (4.7) and (4.8) are not distinct, that is $a \neq 1$.

Proof. For the generic point $y \in \mathbb{P}^2$, let us consider its associate divisor $D_y = p_1 + \ldots + p_{2d}$. By contradiction, suppose that $p_1, \ldots, p_{2d}$ are distinct points of C. Let us consider the graph of the rational map $F \colon C^{(2)} \dashrightarrow \mathbb{P}^2$

$$\Gamma := \left\{ (y, p+q) \in \mathbb{P}^2 \times C^{(2)} \,|\, F(p+q) = y \right\} . \tag{4.9}$$

It is easy to see that $\Gamma \subset \mathbb{P}^2 \times C^{(2)}$ is a correspondence with null trace of degree $d = deg\, F$ (cf. Example 2.2.5). Let $\phi_K \colon C \longrightarrow \mathbb{P}^{g-1}$ denote the canonical map of C. Then by Theorem 2.2.2 we have that the linear span of the points $\phi_K(p_1), \ldots, \phi_K(p_{2d})$ in $\mathbb{P}^{g-1}$ has dimension at most $d-1$. Moreover, the p_i's are assumed to be distinct, then the linear span of their images in $\mathbb{P}^{g-1}$ coincides with $\overline{\phi_K(D_y)}$ and hence $dim\, \overline{\phi_K(D_y)} \leq d-1$. Thus by the geometric version of Riemann-Roch theorem we have

$$dim\, |D_y| = deg\, D_y - 1 - dim\, \overline{\phi_K(D_y)} \geq 2d - 1 - (d-1) = d = \frac{deg\, D_y}{2}.$$

Therefore by Clifford's theorem we have that either C is hyperelliptic, D_y is zero or D_y is a canonical divisor by Clifford's theorem. By assumption C is a non-hyperelliptic curve and $0 < d < g - 1$, thus $0 < deg\, D_y < 2g - 2$ and hence we have a contradiction. □

Now, we shall prove all the results of the previous section. For the reader's convenience, we state again the assertions and we start from the non-hyperelliptic case.

Theorem 4.3.5. *Let C be a smooth complex projective curve of genus $g \geq 3$ and assume that C is non-hyperelliptic. Then*

$$d_r(C^{(2)}) \geq gon(C).$$

Proof. To start, we note that if the genus of C is either $g = 3$ or $g = 4$, then it is a trigonal curve. So, by Lemma 4.3.1 we have $d_r(C^{(2)}) \geq 3 = gon(C)$ and the assertion follows.

Then let us assume that $g \geq 5$ and let $F \colon C^{(2)} \dashrightarrow \mathbb{P}^2$ be a dominant rational map of degree $d = d_r(C^{(2)})$. Aiming for a contradiction we assume $d < gon(C)$.

For a generic point $y \in \mathbb{P}^2$, let us consider the associated divisor D_y defined in (4.8). Thanks to Lemma 4.3.2 there exists $1 \leq a \leq d$ such that

$$D_y = a\,(q_1 + q_2 + \ldots + q_m)\,,$$

where $m = \frac{2d}{a}$ and the q_j's are distinct points of C. By Lemma 4.3.3, the linear series $|q_1 + q_2 + \ldots + q_m|$ is a complete g^r_m of C with $r \geq 2$. Clearly, m must be at least equal to the gonality of C. In particular we have $m > d$ and hence the unique possibility is $a = 1$.

To conclude the proof, we recall that the gonality of a curve C of genus g is $gon(C) \leq \left[\frac{g+3}{2}\right]$. In particular, it follows that $gon(C) \leq g - 1$ for any $g \geq 5$. As C is assumed to be a non-hyperelliptic curve of genus $g \geq 5$ and $d < gon(C)$, Lemma 4.3.4 assures that $a > 1$, a contradiction. $\square$

Proposition 4.3.6. *Let C be a smooth complex projective curve of genus $g \geq 3$ and assume that C is non-hyperelliptic. Then the following hold:*

(i) *if $g = 3, 4$, then $d_r(C^{(2)}) \geq 3$;*

(ii) *if $g = 5$, then $d_r(C^{(2)}) \geq 4$;*

(iii) *if $g = 6$, then $d_r(C^{(2)}) \geq 5$;*

(iv) *if $g \geq 7$, then*
$$d_r(C^{(2)}) \geq max\,\{6, gon(C)\}\,.$$

Proof. As a consequence of Lemma 4.3.1 we have that $d_r(C^{(2)}) \geq 3$ and assertion (i) follows.

As usual, let $F \colon C^{(2)} \dashrightarrow \mathbb{P}^2$ be a dominant rational map of degree $d = d_r(C^{(2)})$ and for a generic point $y \in \mathbb{P}^2$, we consider the associated divisor $D_y = a\,(q_1 + q_2 + \ldots + q_m)$, where $1 \leq a \leq d$ and the q_j's are distinct points of C. Then we proceed by steps.

Step 1 [$g \geq 5 \Rightarrow d \geq 4$]. We assume that C has genus $g \geq 5$ and we prove that $d_r(C^{(2)}) \geq 4$. Aiming for a contradiction, we suppose that $d = deg\, F = 3$.

By Lemma 4.3.3, we have that $|q_1 + q_2 + \ldots + q_m|$ is a complete linear series on C of degree m and dimension $r \geq 2$. As C is non-hyperelliptic and $g \geq 5$, Marten's theorem assures that $dim\, W^r_m(C) \leq m - 2r - 1$. As the number of q_j's is $m = \frac{2d}{a} = \frac{6}{a}$ and the dimension is $r \geq 2$, we have that $W^r_m(C)$ has non-negative dimension only if $a = 1$.

Since $g \geq 5$ and $d = 3$, we have that $d < g - 1$ and hence the integer a can not be equal to 1 by Lemma 4.3.4. Therefore we have a contradiction. Thus $d \geq 4$ and assertion (ii) follows as a consequence.

Step 2 [$g \geq 6 \Rightarrow d \geq 5$]. We prove that $d_r(C^{(2)}) \geq 5$ for any non-hyperelliptic curve C of genus $g \geq 6$. By the previous step, it suffices to see that $C^{(2)}$ does not admit dominant rational maps on $\mathbb{P}^2$ of degree 4. By contradiction, let us assume $d = deg\, F = 4$.

The argument is the very same of step 1. Thanks to Lemma 4.3.3 and Marten's theorem, we deduce $0 \leq dim\, W^r_m(C) \leq m - 2r - 1$ with $r \geq 2$ and $m = \frac{2d}{a}$. Since $d = 4$, it follows that $a = 1$, but this situation can not occur by Lemma 4.3.4. Then we have a contradiction and assertion (iii) holds.

Step 3 [$g \geq 7 \Rightarrow d \geq 6$]. To conclude, we assume that C has genus $g \geq 7$ and we prove that $d_r(C^{(2)}) \geq 6$. Thanks to Step 2, we have to show that the degree of irrationality of $C^{(2)}$ is different from 5. Again we argue by contradiction and we suppose $d = deg\, F = 5$.

As above, the inequality $0 \leq dim\, W^r_m(C) \leq m - 2r - 1$ holds, where $r \geq 2$ and $m = \frac{2d}{a}$. In this situation, the only possibilities are $a = 1$ and $a = 2$. Since $d = 5 < g-1$ and C is a non-hyperelliptic curve of genus $g \geq 7$, the integer a must differ from 1 by Lemma 4.3.4.

On the other hand, suppose that $a = 2$. Then $m = 5$ and the above inequality implies $r = 2$. In particular, the linear series $|q_1 + \ldots + q_5|$ is a complete g^2_5 on C. As m is prime, the map $C \longrightarrow \mathbb{P}^2$ defined by the g^2_5 is birational onto a non degenerate curve of $\mathbb{P}^2$. Hence Castelnuovo's bound gives $g \leq 6$, a contradiction.

Thus $d_r(C^{(2)}) \geq 6$ and assertion (iv) follows from Theorem 4.3.5. □

Theorem 4.3.7. *Let C be a smooth complex projective curve of genus $g \geq 4$ and assume that C is very general in the moduli space $\mathcal{M}_g$. Then*

$$d_r(C^{(2)}) \geq g - 1.$$

Proof. When $g = 4$ the assertion is straightforward from Proposition 4.3.6.

Then let us assume $g \geq 5$ and let $F: C^{(2)} \dashrightarrow \mathbb{P}^2$ be a dominant rational map of degree $d = d_r(C^{(2)})$. We argue by contradiction and we suppose $d < g - 1$.

For a generic point $y \in \mathbb{P}^2$, we consider its fiber

$$F^{-1}(y) = \{p_1 + p_2, \ldots, p_{2d-1} + p_{2d}\} \subset C^{(2)}.$$

and the associated divisor $D_y = p_1 + \ldots + p_{2d}$. By Lemma 4.3.2 there exists an integer $1 \leq a \leq d$ such that $D_y = a\,(q_1 + q_2 + \ldots + q_m)$, where $m = \frac{2d}{a}$ and the q_j's are distinct points of C.

By assumption we have that C is non-hyperelliptic. Moreover, $d < g - 1$ and hence $a \neq 1$ by Lemma 4.3.4.

We claim that $a \neq 2$. If a were equal to 2, it would mean that for any $i = 1, \ldots, 2d$, would exist $k \neq i$ such that $p_i = p_k$. So, by fixing a generic point $q \in C$, we could define a map $\varsigma_q: C \longrightarrow C$ as follow: for $p \in C$, $\varsigma_q(p) \in C$ is the unique point such that $F(p + q) = F(\varsigma_q(p) + q)$. Clearly, ς_q would be an automorphism of C. Furthermore, as the generic fiber of F is given by distinct points of $C^{(2)}$, we would have that generically $\varsigma_q(p) \neq p$ and hence ς_q would not be the identity of C. Then we would have a contradiction, because the only automorphism of a very general curve is the trivial one. Thus $a \geq 3$.

Thanks to Lemma 4.3.3, the linear series $|q_1 + q_2 + \ldots + q_m|$ is a complete g^r_m with $r \geq 2$ and hence $W^r_m(C)$ has non-negative dimension. We recall that when C is a very general curve, the dimension of $W^r_m(C)$ equals the Brill-Noether number $\rho(g, r, m) := g - (r + 1)(g - m + r)$. In particular, $|q_1 + q_2 + \ldots + q_m| \in W^2_m(C)$ and hence $\rho(g, 2, m) \geq 0$. It follows that

$$m \geq \frac{2g + 6}{3}$$

On the other hand, we have $a \geq 3$ and $d < g - 1$. Therefore

$$m = \frac{2d}{a} < \frac{2g - 2}{3}$$

and we get a contradiction. □

Remark 4.3.8. As we mentioned in the previous section, the techniques we use do not work to improve Theorem 4.3.7. The obstruction is that we are not able to prove an assertion analogous to Lemma 4.3.4 without assuming $deg\, F < g - 1$. Indeed, when $deg\, F = g - 1$ the divisor D_y is canonical on C and Clifford's theorem leads no longer to a contradiction.

To conclude, we note that by the very same techniques is possible to give bounds on the degree of irrationality of the k-fold symmetric product $C^{(k)}$ of a smooth complex projective curve of genus g, but we would obtain a less precise picture. Namely, if $F\colon C^{(k)} \dashrightarrow \mathbb{P}^k$ is a dominant rational map of degree d, the divisor D_y associated to the fiber over a generic point $y \in \mathbb{P}^k$ has degree kd. Since we need $deg\, D_y < 2g-2$, the more we increase the integer k, the more we have to increase the genus g to have some information on $d_r(C^{(k)})$.

In order to provide some upper bounds on the degree of irrationality of $C^{(k)}$, we may readjust the constructions we made - by using g^r_d's on C - to show Proposition 4.2.2 in the previous section. Moreover, one could deal with this problem by embedding the k-fold symmetric product in a suitable projective space as well (see for instance [19]).

4.4 The hyperelliptic case

In order to conclude the proofs of the results stated in the second Section, here we deal with Theorem 4.2.4. Namely, we shall prove that if C is an hyperelliptic curve of genus $g \geq 4$, then the degree of irrationality of $C^{(2)}$ is four. Although the techniques we shall use in the proof shall be very similar to those of Section 3, this situation shall be slightly different and hence we preferred to menage it in a different section.

This section is entirely devoted to prove the following (see Theorem 4.2.4).

Theorem 4.4.1. *Let C be a smooth complex projective curve of genus $g \geq 4$. If C is hyperelliptic, then $d_r(C^{(2)}) = 4$.*

Let $f\colon C \longrightarrow \mathbb{P}^1$ be the g^1_2 on C and let $\iota\colon C \longrightarrow C$ denote the induced hyperelliptic involution. We recall that $\left(\mathbb{P}^1\right)^{(2)} \cong \mathbb{P}^2$ and under this identification we can define a dominant morphism $C^{(2)} \longrightarrow \mathbb{P}^2$ of degree 4 by sending a point $p+q \in C^{(2)}$ to the point $f(p)+f(q) \in \mathbb{P}^2$. Furthermore, Lemma 4.3.1 guarantees that the degree of irrationality of $C^{(2)}$ is at least 3. Thus

$$3 \leq d_r(C^{(2)}) \leq 4$$

and hence to prove Theorem 4.4.1, it suffices to show that $C^{(2)}$ does not admit a dominant rational map of degree 3 on $\mathbb{P}^2$.

So, let $F\colon C^{(2)} \dashrightarrow \mathbb{P}^2$ be a dominant rational map of degree d and suppose by contradiction that $d=3$. For a generic point $y \in \mathbb{P}^2$, let

$$F^{-1}(y) = \{p_1+p_2, p_3+p_4, p_5+p_6\} \subset C^{(2)} \tag{4.10}$$

be its fiber and let $D_y = p_1+\ldots+p_6 \in Div(C)$ be the divisor associated to y.

We note the following important fact. Since $C^{(2)}$ is a connected surface, the action of the monodromy group $M(F) \subset S_3$ on the fiber (4.10) is transitive by Lemma 1.2.1. Furthermore, let $G \colon C \times C \dashrightarrow \mathbb{P}^2$ be the map obtained by composing F and the natural quotient map $\pi \colon C \times C \longrightarrow C^{(2)}$ and let

$$G^{-1}(y) = \{(p_1, p_2), (p_2, p_1), \ldots, (p_6, p_5)\} \subset C^{(2)} \tag{4.11}$$

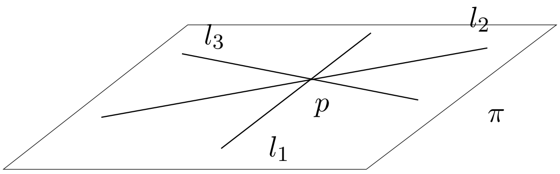
be its fiber over y. As $C \times C$ is a connected surface as well, we have that $M(G) \subset S_6$ acts transitively on $G^{-1}(y)$. It follows that there is no way to distinguish neither the points of the fiber $F^{-1}(y)$ nor those of $G^{-1}(y)$ by some property varying continuously as y varies on $\mathbb{P}^2$.

Let $\phi_K \colon C \longrightarrow \mathbb{P}^{g-1}$ be the map defined by the canonical linear series on C. As C is assumed to be hyperelliptic, ϕ_K is the composition of the double covering map $f \colon C \longrightarrow \mathbb{P}^1$ and the Veronese map $\nu_{g-1} \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^{g-1}$ (see e.g. [38, Proposition 2.2 p. 204]). In particular, the image $\phi_K(C) \subset \mathbb{P}^{g-1}$ is set-theoretically the rational normal curve of degree $g-1$ and the covering $\phi_K \colon C \longrightarrow \phi_K(C)$ has degree two. Then two distinct points $p, q \in C$ has the same image if and only if they are conjugated under the hyperelliptic involution $\iota \colon C \longrightarrow C$.

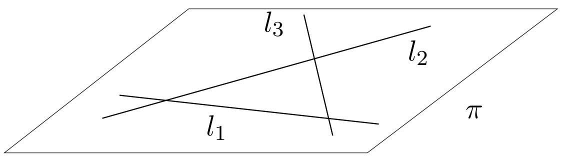
Let $\Gamma := \big\{(y, p+q) \in \mathbb{P}^2 \times C^{(2)} \,|\, F(p+q) = y\big\}$ be the graph of F. In this situation $\Gamma \subset \mathbb{P}^2 \times C^{(2)}$ is a correspondence with null trace of degree 3. Therefore, by Theorem 2.2.2 the linear span of the points $\phi_K(p_1), \ldots, \phi_K(p_6)$ is a plane $\pi \subset \mathbb{P}^{g-1}$. In particular, the lines

$$l_1 := \overline{\phi_K(p_1)\phi_K(p_2)}, \quad l_2 := \overline{\phi_K(p_3)\phi_K(p_4)} \quad \text{and} \quad l_3 := \overline{\phi_K(p_5)\phi_K(p_6)}$$

must intersect at a same point $p \in \pi$ as in figure (a) below (cf. Section 2.2 and Example 2.3.4).



(a)



(b)

The point $y \in \mathbb{P}^2$ is generic and hence we can assume - without loss of generality - that p_1 and p_2 are not conjugate under the hyperelliptic involution, that is $\phi_K(p_1) \neq \phi_K(p_2)$. As the points of the fiber (4.10) of F are indinstinguishable, it follows that $\phi_K(p_3) \neq \phi_K(p_4)$ and $\phi_K(p_5) \neq \phi_K(p_6)$ as well.

As the $\phi_K(p_i)$'s lie on the intersection of π with the rational normal curve $\phi_K(C)$, the $\phi_K(p_i)$'s must be at most three distinct points. Suppose that they are exactly 3 distinct points of $\mathbb{P}^{g-1}$. Since we can not distinguish the points of the fiber (4.11), we deduce that each $\phi_K(p_i)$ has exactly two the preimages on C. Therefore we can suppose $\phi_K(p_2) = \phi_K(p_3)$, $\phi_K(p_4) = \phi_K(p_5)$ and $\phi_K(p_6) = \phi_K(p_1)$. Hence the lines l_1, l_2 and l_3 lie on π as in figure (b) above. As those line must also lie as in figure (a), it is easy to see that the must coincide. It follows that the three $\phi_K(p_i)$'s are collinear, but this is impossible because $\phi_K(C)$ does not admit any trisecant line.

Therefore the $\phi_K(p_i)$'s are exactly two points of $\phi_K(C)$. We recall that by Lemma 4.3.2, there exists $1 \leq a < d$ such that the divisor D_y associated to y has the form $D_y = a(q_1 + \ldots + q_m)$, where $m = \frac{2d}{a}$ and the q_j's are distinct points of C. In our case $d = 3$ and hence $a = 1, 2$.

If $a = 2$ we have $m = 3$. Hence there are two points q_1, q_2 mapping on $\phi_K(p_1)$ and q_3 on $\phi_K(p_2)$, but this situation cannot occur because we are distinguishing points. On the other hand, suppose that $a = 1$ and $m = 6$. As both $\phi_K(p_1)$ and $\phi_K(p_2)$ has two preimages on C, the q_j's must be at most four distinct points. Thus we have a contradiction and the assertion of Theorem 4.2.4 holds.

Chapter 5

The nef cone of the second symmetric products of curves

In this chapter we study the cone $Nef\,(C^{(2)})_{\mathbb{R}}$ of all numerically effective $\mathbb{R}$-divisors classes in the Néron-Severi space $N^1(C^{(2)})_{\mathbb{R}}$ when C is a very general curve.

In [48, Section 4], Ross gives new bounds on the nef cone on the second symmetric product of a very general curve of genus five. His argument is based on the main theorem in [48] together with some techniques due to Ein and Lazarsfeld (see [22]). We follow Ross' argument to improve the bounds on $Nef\,(C^{(2)})_{\mathbb{R}}$ when C is assumed to be a very general curve of genus $5 \leq g \leq 8$ (see Teorem 5.1.2). In particular, the refinement is a consequence of Theorem 3.2.1 in Chapter 3 and of Theorem 5.2.2 in the second Section of this chapter.

The first Section is devoted to introduce the problem and to recall the main results on this topic.

In Section 2 we turn to deformations of singular curves on surfaces. We consider families of singular curves covering a smooth complex projective surface X and we deal with the problem of estimate the self-intersection of the members of such families. In particular, we give a sharp bound improving a result of Ein and Lazarsfeld (see [22, Corollary 1.2]).

At the end, in the third Section we prove the new bounds on $Nef\,(C^{(2)})_{\mathbb{R}}$.

5.1 Generalities on the nef cone of $C^{(2)}$

Let C be a smooth irreducible complex projective curve of genus $g \geq 0$ and let us assume that C is very general in the moduli space $\mathcal{M}_g$. Let us consider the second symmetric product $C^{(2)}$ of C and let $N^1(C^{(2)})_{\mathbb{Z}}$ be its Néron-Severi group.

As in Section 1.3, fixing a point $p \in C$, we define the divisor $X_p := \{p+q | q \in C\}$ and the diagonal divisor $\Delta := \{q+q | q \in C\}$. Let x and δ denote their numerical equivalence classes in $N^1(C^{(2)})_{\mathbb{Z}}$. By Lemma 1.3.2 we have that the vector space $N^1(C^{(2)})_{\mathbb{R}}$ of numerical equivalence classes of $\mathbb{R}$-divisors is spanned by the classes x and $\frac{\delta}{2}$, hence any numerical class $\alpha \in N^1(C^{(2)})_{\mathbb{R}}$ can be written in the form $\alpha = (a+b)x - b\frac{\delta}{2}$.

Our aim is to describe the two-dimensional convex cone $Nef\,(C^{(2)})_{\mathbb{R}}$ of all numerically effective $\mathbb{R}$-divisors classes on $C^{(2)}$ and this is equivalent to determine its two boundary rays.

The first one is the dual ray of the diagonal divisor class via the intersection pairing. Namely, since the diagonal is an irreducible curve of negative self intersection, it spans a boundary ray of the effective cone of curves. Thus one boundary of the nef cone is given by the classes orthogonal to the diagonal class, that is $\{\alpha \in N^1(C^{(2)}) | (\delta \cdot \alpha) = 0\}$. Hence this ray is spanned by the numerical equivalence class $(g-1)x - \frac{\delta}{2}$.

The other ray is spanned by the class

$$(\tau(C)+1)x - \frac{\delta}{2}\,,$$

where $\tau(C)$ is the real number defined as

$$\begin{aligned}\tau(C) &:= inf\{t > 0 | (t+1)x - \frac{\delta}{2} \text{ is ample}\}\\ &= min\{t > 0 | (t+1)x - \frac{\delta}{2} \text{ is nef}\}\,.\end{aligned}$$

Hence the problem of describing the cone $Nef\,(C^{(2)})_{\mathbb{R}}$ is equivalent to compute $\tau(C)$. Notice that if $(t+1)x - \frac{\delta}{2}$ is an ample class of $N^1(C^{(2)})_{\mathbb{R}}$, then it must have positive self intersection and hence

$$\tau(C) \geq \sqrt{g}\,.$$

When the genus of the curve C is $g \leq 3$, the problem is totally understood.

If $g = 0$, we have $C^{(2)} \cong \mathbb{P}^2$. So the classes x and $\frac{\delta}{2}$ coincide with the hyperplane class. Therefore $(t+1)x - \frac{\delta}{2} = tx$ and hence $\tau(C) = 0$.

If $g = 1$, it is well known that the nef cone $Nef\,(C^{(2)})_{\mathbb{R}}$ is the closure of the effective cone of curves $NE\,(C^{(2)})$ and a class $\alpha \in N^1(C^{(2)})_{\mathbb{R}}$ is nef if and only if $\alpha^2 \geq 0$ and $(\alpha \cdot h) \geq 0$ for some ample class h (see [35, Lemma 1.5.4]). Thus we deduce $\tau(C) = 1$ by taking $\alpha = (\tau(C)+1)x - \frac{\delta}{2}$ and $h = x$.

In the other two cases it is possible to compute $\tau(C)$ by finding explicit irreducible curves of negative self-intersection and by imposing orthogonality with such curves.

When $g = 2$, C is an hyperelliptic curve and the g^1_2 defines a curve on $C^{(2)}$ with class $2x - \frac{\delta}{2}$ (cf. Lemma 1.4.4 and Lemma 1.3.1) and negative self-intersection. Then we have $(2x - \frac{\delta}{2}) \cdot ((\tau(C) + 1)x - \frac{\delta}{2}) = 0$ and we deduce $\tau(C) = 2$.

If $g = 3$, there exists an irreducible curve on $C^{(2)}$ with numerical equivalence class $16x - 6\frac{\delta}{2}$ (for details see [34] and [16]). By arguing as above we compute $\tau(C) = \frac{9}{5}$.

When C is a very general curve of genus $g \geq 4$, there is an important conjecture - due to Alexis Kouvidakis - asserting that the nef cone is as large as possible. Namely,

Conjecture 5.1.1 (Kouvidakis)**.** *If C is a very general curve of genus $g \geq 4$, then $\tau(C) = \sqrt{g}$.*

In [34], the statement has been proved when g is a perfect square. Moreover, Kouvidakis proved that

$$\tau(C) \leq \frac{g}{[\sqrt{g}]}$$

for any very general curve of genus $g \geq 5$. The cases $g = 5$ and $g \geq 10$ have been recently improved.

In particular, by using a bound on the Seshadri constant at g general points of $\mathbb{P}^2$ (see [52]), as a consequence of a result due to Ciliberto and Kouvidakis (cf. [16] and [48, Corollary 1.7]), we have that

$$\tau(C) \leq \frac{\sqrt{g}}{\sqrt{1 - \frac{1}{8g}}}$$

for any very general curve of genus $g \geq 10$. Furthermore, when C is a genus five curve with very general moduli, Ross proved that $\tau(C) \leq 16/7$ (cf. [48, Section 4]).

In third Section of this chapter, we prove the following.

Theorem 5.1.2. *Consider the rational numbers*

$$\tau_5 = \frac{9}{4}, \quad \tau_6 = \frac{32}{13}, \quad \tau_7 = \frac{77}{29} \quad \textit{and} \quad \tau_8 = \frac{17}{6}.$$

Let C be a smooth complex projective curve of genus $5 \leq g \leq 8$ and assume that C has very general moduli. Then

$$\tau(C) \leq \tau_g.$$

In particular, by comparing the nubers in the statement with the bounds on $\tau(C)$ listed above, we have

$$\tau_5 < \frac{16}{7} \quad \text{and} \quad \tau_g < \frac{g}{[\sqrt{g}]} \quad \text{for } g = 6, 7, 8.$$

Thus the Theorem 5.1.2 gives a slight improvement to the bounds on $\tau(C)$.

5.2 Deformations of singular curves on surfaces

To start, we would like to note that the theorem we prove in the following has been obtained also by Andreas Leopold Knutsen, Wioletta Syzdek and Tomasz Szemberg in a recent paper (see [32]).

Following [22], let X be a smooth complex projective surface and let Δ be a smooth curve or a disk with $0 \in \Delta$. Then let us consider a one-parameter family $\{(E_t, x_t)\}_{t \in \Delta}$ consisting of curves $E_t \subset X$ plus a point $x_t \in E_t$. Setting $t = 0$, the deformation (E_t, x_t) of (E_0, t_0) determines a *Kodaira-Spencer map*

$$\rho \colon T_0\Delta \longrightarrow H^0(E_0, N)$$

where $N := \mathcal{O}_{E_0}(E_0)$ is the normal bundle to E_0 in X.

For a point $y \in E_0$, let $\mathfrak{m}_y$ denote the maximal ideal sheaf of y. We say that a section $s \in H^0(X, N)$ *vanishes at order at least* k at the - possibly singular - point $y \in E_0$ if s is a section of the subsheaf $N \otimes \mathfrak{m}_y^k \subset N$. Then the following holds.

Lemma 5.2.1. *Assume that* $mult_{x_t} E_t \geq m$ *for all* $t \in \Delta$. *Then the section* $\rho\left(\frac{d}{dt}\right) \in H^0(E_0, N)$ *vanishes to order at least* $m - 1$ *at* x_0.

Proof. See [22, Lemma 1.1] □

Under the hypothesis of the lemma, let us assume in addiction that E_0 is a reduced irreducible curve and the family $\{(E_t, x_t)\}_{t \in \Delta}$ is non-trivial. Ein and Lazarsfeld prove that in this situation the self-intersection of E_0 is bounded from below, namely $E_0^2 \geq m(m-1)$ (see [22, Corollary 1.2]).

By assuming that the curve E_0 is singular at x_0 - that is $m \geq 2$ - the latter bound has been improved by Xu. In particular, he proves that $E_0^2 \geq m(m-1) + 1$ (cf. [54, Lemma 1]).

The following result is a further improvement of these bounds and the proof follows the same argument. As usual, given a - possibly singular - curve E, we denote by $\nu \colon \widetilde{E} \longrightarrow E$ its normalization and by $gon(E)$ the gonality of the smooth curve $\widetilde{E}$.

Theorem 5.2.2. *Let X be a smooth complex projective surface. Let T be a smooth variety and consider a family $\{(E_t, x_t)\}_{t\in T}$ consisting of a curve $E_t \subset X$ through a very general point $x_t \in X$ such that $mult_{x_t}E_t \geq m$ for any $t \in T$ and for some $m \geq 2$.*
If the central fibre E_0 is a reduced irreducible curve and the family is non-trivial, then

$$E_0^2 \geq m(m-1) + gon(E_0). \tag{5.1}$$

Proof. As in [22], let us consider the blowing-up $f : X' \longrightarrow X$ of X at x_0 and let $F \subset X'$ be the exceptional divisor. Let E'_0 be the strict transform of E_0. Then $E'_0 = f^*E_0 - kF$ with $k = mult_{x_0}E_0 \geq m$ and hence E'_0 is the blowing-up of E_0 at x_0.

Since each x_t is a singular point of E_t, the variety T parametrizing the family must be at least two-dimensional. Then, up to consider a subfamily, we assume that the dimension of T is 2. Let $(t_1, t_2) \in \mathbb{C}^2$ be the local coordinates of T around $t = 0$.

Consider the sections $s_1 = \rho\left(\frac{d}{dt_1}\right), s_2 = \rho\left(\frac{d}{dt_2}\right) \in H^0(E_0, \mathcal{O}_{E_0}(E_0))$ of the normal bundle to E_0 in X, where ρ is the Kodaira-Spencer deformation map form the tangent space to T at 0 into $H^0(\mathcal{O}_{E_0}(E_0))$. Thus, by Lemma 5.2.1 and being the family non-trivial, s_1 and s_2 induce two non-zero sections

$$s'_1, s'_2 \in H^0\left(E'_0\,,\ f^*(\mathcal{O}_{E_0}(E_0)) \otimes \mathcal{O}_{X'}((1-m)F)|_{E'_0}\right).$$

By last two sections we define a map $\phi : E'_0 \longrightarrow \mathbb{P}^1$ which extends to a map $\widetilde{\phi} : \widetilde{E}_0 \longrightarrow \mathbb{P}^1$, hence

$$\begin{aligned} E_0^2 &= deg\,\mathcal{O}_{E_0}(E_0) = deg\, f^*(\mathcal{O}_{E_0}(E_0))|_{E'_0} \geq \\ &\geq (m-1)(F \cdot E'_0) + deg\,\phi \geq m(m-1) + gon(\widetilde{E}_0) \end{aligned}$$

and this concludes the proof. □

Notice that if every curve of the family is reduced and irreducible, then the inequality (5.1) holds for any such curve.

Furthermore, the bound in the statement is sharp. This fact is clear from the following examples.

Example 5.2.3. On $\mathbb{P}^2$ let us consider the 8-dimensional family $\mathcal{C}$ of all the cubic curves with a node. Let $p_1, \ldots, p_6 \in \mathbb{P}^2$ be six general points and let $\mathcal{C}' = \{C_t, y_t\}$ be the two-dimensional subfamily of $\mathcal{C}$ of all the plane cubics C_t passing through the p_i's with a node at y_t.

Let X be the surface obtained by blowing up $\mathbb{P}^2$ at the p_i's and let $\mathcal{E} = \{E_t, x_t\}$ be the family on X such that E_t is the strict transform of C_t and x_t is the inverse image of the node. Then the family $\mathcal{E}$ is such that the generic member satisfies equality in the (5.1) above.

To see this fact, notice that $mult_{x_t}E_t = 2$ for any t and that the generic member of the family is irreducible and reduced. Moreover, the normalization of each E_t is a rational curve and hence $gon(E_t) = 1$. Therefore, by setting $m = 2$, we have $m(m-1) + gon(E_t) = 3$. On the other hand, the self-intersection of any E_t is given by the self-intersection of C_t minus the number of blown-up points, that is $E_t^2 = 9 - 6 = 3$.

Example 5.2.4. By arguing as above, let $\mathcal{C}$ be the family of dimension 11 of the quartic curves on $\mathbb{P}^2$ with a singularity of order 3. Then let us fix nine points $p_1, \ldots, p_9 \in \mathbb{P}^2$ and let us consider the two-dimensional subfamily $\mathcal{C}' = \{C_t, y_t\}$ of $\mathcal{C}$ such that any C_t passes through the p_i's with singular point at y_t.

So let X be the blow-up of $\mathbb{P}^2$ at the p_i's and let $\mathcal{E} = \{E_t, x_t\}$ be the family on X of the proper transforms of the C_t's with singular point at $x_t \in X$. Since $E_t^2 = 16 - 9 = 7$ and the normalization $\widetilde{E}_t$ of E_t is a rational curve, we have that generically $E_t^2 = m(m-1) + gon(E_t)$, where m is assumed to be $mult_{x_t}E_t = 3$.

Remark 5.2.5. As we said above, in [32] the authors start from [49] and prove a result analogous to Theorem 5.2.2, under the more general hypothesis of considering a family of pointed reduced irreducible curves parametrized over a two-dimensional subset $U \subset Hilb(X)$ (cf. [32, Theorem A]). Moreover, they prove a more precise statement asserting that $E_0^2 = m(m-1) + gon(E_0)$ if and only if E_0 is smooth outside x_0 and x_0 is an ordinary m-tuple point of E_0 (cf. [32, Theorem 2.1]). Then they apply this theorem to Seshadri constants on surfaces.

We note that our proof works under the same hypothesis, but we preferred to maintain the original statement.

5.3 Bounds on the nef cone of $C^{(2)}$

In this section, we shall follow [48, Section 4] to prove Theorem 5.1.2.

In order to present the main theorem in [48], we would like to recall the definition of Seshadri constants on a surface. So, let us consider a smooth complex projective variety X and a nef class $L \in N^1(X)_{\mathbb{R}}$. We define the *Seshadri constant* of L at a point $y \in X$ to be the real number

$$\epsilon(y; X, L) := \inf_E \frac{(L \cdot E)}{mult_y E},$$

where the infimum is taken over the irreducible curves E passing through y.

The following result connects Seshadri constants on the second symmetric product of a curve of genus $g-1$ and the ample cone of the second symmetric product of a very general curve of genus g.

Theorem 5.3.1. *Let D be a smooth curve of genus $g-1$. Let a, b be two positive real numbers such that $a/b \geq \tau(D)$ and for a very general point $y \in D^{(2)}$*

$$\epsilon\left(y; D^{(2)}, (a+b)x - b\frac{\delta}{2}\right) \geq b.$$

Then for a very general curve C of genus g,

$$\tau(C) \leq \frac{a}{b}.$$

Proof. See [48, Theorem 1.2] $\square$

For the reader's convenience, we recall the statement of the result we aim to prove.

Theorem 5.1.2. *Consider the rational numbers*

$$\tau_5 = \frac{9}{4}, \quad \tau_6 = \frac{32}{13}, \quad \tau_7 = \frac{77}{29} \quad \text{and} \quad \tau_8 = \frac{17}{6}.$$

Let C be a smooth complex projective curve of genus $5 \leq g \leq 8$ and assume that C has very general moduli. Then

$$\tau(C) \leq \tau_g.$$

To start with the proof, let us consider a very general curve C of genus $g = 5$. We want to prove that

$$\tau(C) \leq \frac{9}{4}. \tag{5.2}$$

To this aim, let D be a very general curve of genus $g(D) = g - 1 = 4$ and let $D^{(2)}$ be its second symmetric product. Then set $a = 9$, $b = 4$ and consider the numerical equivalence class

$$L := (a+b)x - b\frac{\delta}{2} \in N^1(D^{(2)}). \tag{5.3}$$

Since $\tau(D) = 2$, by Theorem 5.3.1 we deduce that to prove inequality (5.2) it suffices to show that for a very general point $y \in D^{(2)}$

$$\epsilon(y\,; D^{(2)}, L) \geq b = 4, \tag{5.4}$$

i.e. there is not a reduced and irreducible curve E passing through a generic point $y \in D^{(2)}$ such that $(L \cdot E)/mult_y E < b = 4$.

Let us consider the set $\mathcal{F}$ of pairs (F, z) such that $F \subset D^{(2)}$ is a reduced irreducible curve, $z \in F$ is a point and $(L \cdot F)/mult_z F < 4$. Since $\mathcal{F}$ consists of at most countably many algebraic families and the point $y \in D^{(2)}$ is assumed to be very general, inequality (5.4) will be checked if each of these families is discrete [22, Section 2].

Aiming for a contradiction, let us assume that there exists a family $\mathcal{E} = \{(E_t, x_t)\}_{t \in T}$ such that for any $t \in T$ the curve $E_t \subset D^{(2)}$ is reduced and irreducible, the point $x_t \in E_t$ is very general on $D^{(2)}$ and

$$\frac{(L \cdot E_t)}{mult_{x_t} E_t} < b = 4. \tag{5.5}$$

As in [48], we note that for any reduced irreducible curve $E \subset D^{(2)}$ passing through a very general point $y \in D^{(2)}$ we have

$$(L \cdot E) \geq b = 4. \tag{5.6}$$

To see this fact, let $[E] = (n + \gamma)x - \gamma(\delta/2) \in N^1(D^{(2)})$ be the numerical equivalence class of E. Since the class x is ample, $(x \cdot E) = n > 0$ and the claim is easily checked when $\gamma \leq 0$.
Then assume $\gamma > 0$. Being $\tau(D) = 2$, the diagonal is the only curve of $D^{(2)}$ with negative self intersection. Moreover, there exist at most finitely many irreducible curves of zero self intersection and numerical class given by $(n+\gamma)x - \gamma(\delta/2)$. Hence we can assume that $E^2 = n^2 - 4\gamma^2 > 0$ because $y \in D^{(2)}$ is a very general point. Therefore for any $\gamma > 0$ we have that $n \geq 2\gamma + 1$ and $(L \cdot E) = 9n - 16\gamma \geq 2\gamma + 9 > 4$.

Thus by (5.5) and (5.6) we deduce that $mult_{x_t} E_t > (L \cdot E_t)/4 \geq 1$ for any $t \in T$. Being E_t reduced, for a general point $z \in E_t$ the multiplicity of E_t at z is one, therefore the family $\{(E_t, x_t)\}_{t \in T}$ is non-trivial.

Without loss of generality, let us assume that the central fibre (E_0, x_0) is such that

$$m := mult_{x_0} E_0 \leq mult_{x_t} E_t$$

for any $t \in T$. Hence by Theorem 5.2.2 we have that the curve E_0 has self intersection $E_0^2 \geq m(m-1) + gon(E_0)$. Moreover, by Theorem 3.2.1 we have $gon(E_0) \geq gon(D)$ and hence

$$E_0^2 \geq m(m-1) + \left\lceil \frac{(g-1)+3}{2} \right\rceil = m(m-1) + 3. \tag{5.7}$$

Finally, by (5.5) we deduce that $(L \cdot E_0) \leq 4m - 1$. Thus by Hodge Index Theorem we have

$$m(m-1) + 3 \leq E_0^2 \leq \frac{(L \cdot E_0)^2}{L^2} \leq \frac{(4m-1)^2}{17}, \tag{5.8}$$

but this is impossible. Hence we proved that if C is a very general curve of genus $g=5$, then $\tau(C) \leq \frac{9}{4}$.

To conclude the proof of Theorem 5.1.2, we apply the very same argument to the cases $g=6,7,8$, starting from the lower value of g.

In each of these situations we choose the integers a and b to be the numerator and the denominator of τ_g. Then we consider a very general curve D of genus $g(D)=g-1$ and the numerical equivalence class $L \in N^1(D^{(2)})$ defined by (5.3). As $\frac{9}{4} < \frac{32}{13} < \frac{77}{29} < \frac{17}{6}$ we have that the hypothesis of Theorem 5.3.1 are still satisfied. Then we argue as above and we obtain the analogous of (5.8) given by

$$m(m-1)+t \leq E_0^2 \leq \frac{(L \cdot E_0)^2}{L^2} \leq \frac{(bm-1)^2}{a^2-(g-1)b^2}\,.$$

for an opportune integer t depending on the genus g. When $g=6$ - and hence D has genus 5 - t equals the gonality of D, that is $t = \left[\frac{(g-1)+3}{2}\right] = 4$.

On the other hand, suppose that either $g=7$ or $g=8$. Notice that E_0 is a singular curve and hence it is not isomorphic to the smooth curve D of genus $g-1$. Moreover, being D very general, it has no non-trivial automorphism. Thus Theorem 3.2.1 assures that $gon(E_0) \geq gon(D)+1 = \left[\frac{(g-1)+3}{2}\right]+1$. Then $t=5$ for $g=7$ and $t=6$ when $g=8$.

Therefore it remains to check that there are no real values of m satisfying the latter inequality for a, b and g as above.

Chapter 6

Galois closure and Lagrangian surfaces

In this chapter we produce examples of Lagrangian surfaces having negative topological index. In particular, we consider a smooth complex Abelian surface S admitting a rational covering of degree three on a rational surface Y. Then the Lagrangian surface X shall be obtained as minimal desingularization the Galois closure of the covering.

In Section 1 we present an overview on Lagrangian surfaces and we state a theorem summarizing the results of this chapter (see Theorem 6.1.4).

The second Section is devoted to study Abelian surfaces with a polarization $\mathcal{L}$ of type (1,2).

In Section 3 we give a detailed description of the induced linear pencil $|\mathcal{L}|$. In particular, we study all the curves contained in $|\mathcal{L}|$ and we interpret double and triple points on S in terms of elements of the pencil.

Section 4 concerns the construction of the triple covering. We prove that there exist a suitable blow-up $\overline{S}$ of S resolving the indeterminacy locus of the rational covering $S \dashrightarrow Y$. In particular, we obtain a degree three covering $\overline{S} \longrightarrow \mathbb{F}_3$ of the Hirzebruch surface $\mathbb{F}_3$.

In the fifth Section we describe geometrically the Galois closure of the covering.

At the end, in Section 6 we compute the birational invariant of X, we determine the Albanese variety of X and we conclude by proving that X is a Lagrangian surface.

6.1 Lagrangian surfaces

Let X be a smooth complex algebraic surface and consider the Albanese morphism $a\colon X \longrightarrow Alb(X)$. The holomorphic part of the induced homomorphism on cohomology $a_*\colon \wedge^* H^1(X,\mathbb{C}) = H^*(Alb(X),\mathbb{C}) \longrightarrow H^*(X,\mathbb{C})$

is an homomorphism of graded algebras and the part of degree two is given by the homomorphism

$$\psi_2\colon \wedge^2 H^0(X,\Omega^1_X) \longrightarrow H^0(X,\Omega^2_X).$$

The non-triviallity of the kernel of ψ_2 leads to several topological consequences. The first one is provided by the well-known Castelnuovo-de Franchis Theorem, which asserts that if there exists a couple of non-zero differential 1-forms $w_1, w_2 \in H^0(X,\Omega^1_X)$ such that $w_1 \wedge w_2 \in Ker\,\psi_2$ and $w_1 \wedge w_2 \neq 0$, then X admits a fibration over a curve of genus ≥ 2. We note that this result has been generalized by Catanese [14] to n-dimensional varieties and to any map $\psi_k\colon \bigwedge^k H^0(X,\Omega^1_X) \longrightarrow H^0(X,\Omega^k_X)$ with $k = 1,\dots,n$ (cf. [14]).

Moreover, the non-injectivity of ψ_2 implies that the fundamental group of X is non-abelian (see [3] and [42]). Other topological consequences have been studied for instance in [7] in terms of topological index.

As regards of Castelnuovo-de Franchis Theorem it turns out to be interesting to study surfaces provided of non-trivial elements of $Ker\,\psi_2$ not inducing a fibration on X. Example of this surfaces have been developed by Bogomolov and Tschinkel in [11] and by Sommese and Van de Ven in [51]. Moreover, in [13] are presented some examples of surfaces with non-trivial elements in the kernel of the whole map a_*.

In this chapter we shall deeply study other examples of surface having a differential form $\omega \in Ker\,\psi_2$ not inducing a fibration on X.

In order to give a complete description of our work, we follow [7] and we introduce some definitions.

Definition 6.1.1. We say that a smooth surface is *Lagrangian* if there exist a map of degree one $b\colon X \longrightarrow b(X) \subset A$ into an Abelian variety of dimension 4 and a $(2,0)$-form $\omega \in H^{2,0}(A)$ of rank 4 such that $b^*(\omega) = 0$.

Now, let us suppose that X is a smooth complex algebraic surfaces such that $Ker\,\psi_2$ is non-trivial. Given a holomorphic form $\omega \in Ker\,\psi_2$, we denote by $V \subset H^0(X,\Omega^1_X)$ the subspace of minimal dimension such that $\omega \in \wedge^2 V$. Moreover, we denote by $\overline{\Omega^1_X}$ the torsion free sheaf defined as the image of the evaluation map $V \otimes \mathcal{O}_X \longrightarrow \Omega^1_X$.

Definition 6.1.2. We say that the surface X is *generalized Lagrangian* if there exists a non zero-form $\omega \in Ker\,\psi_2$ of rank 4 and $rank\overline{\Omega^1_X} = 2$. In other words, there exist $\omega_1,\dots,\omega_4 \in H^0(X,\Omega^1_X)$ generating generically Ω^1_X and such that $\omega = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4$ vanishes on $H^0(X,\Omega^2_X)$.

In particular, any Lagrangian surface is generalized Lagrangian.

Now, consider subsystem of the canonical linear series given by the image of $\wedge^2 V$ in $H^0(X, \Omega_X^2)$, and let F_V be its base locus. In [7], the authors prove that under some hypothesis on F_V, any generalized Lagrangian surface has non negative topological index [7, Theorem 1.2]. Moreover, they conjecture the following (see [7, Conjecture 2]).

Conjecture 6.1.3. *Let X be a Lagrangian surface. Then the topological index is $\tau(X) \geq 0$.*

Now, let S be a smooth complex Abelian surface and let $C \subset S$ be a smooth curve of genus three. By [53, Theorem 0.2] we have that S admits a dominant rational map of degree three $S \dashrightarrow Y$ on a rational surface Y. Moreover, the curve C induces a linear pencil $|\mathcal{L}|$ on S which gives a polarization of type (1,2) on S. Then let $\mathcal{W}(1,2)$ denote the moduli space of Abelian surfaces with a $(1,2)$ polarization. Throughout the next sections we prove the following.

Theorem 6.1.4. *Let S be a smooth complex Abelian surface and let $\mathcal{L}$ be a line bundle on S providing a $(1,2)$-polarization. Suppose further that the pair $(S, \mathcal{L})$ is general in $\mathcal{W}(1,2)$. Then there exists a dominant degree three morphism $\overline{S} \longrightarrow \mathbb{F}_3$ from a suitable blow-up $\overline{S}$ of S to the Hirzebruch surface $\mathbb{F}_3$. The minimal desingularization X of the Galois closure of the covering is a surface of general type with invariants*

$$K_X^2 = 198 \quad c_2(X) = 102 \quad \chi(\mathcal{O}_X) = 25 \quad q = 4 \quad p_g = 28 \quad \tau(X) = -2.$$

Furthermore, X is a Lagrangian surface. In particular, X is generalized Lagrangian with $V = H^0(X, \Omega_X^1)$.

In particular, the surface X in the assertion is a Lagrangian surface with negative topological index. Thus Theorem 6.1.4 disproves Conjecture 6.1.3.

6.2 Abelian surfaces of type $(1,2)$ and bielliptic curves

In this section we shall introduce the main subjects involved in our construction. By an Abelian surface with a $(1,2)$ polarization we mean a pair $(S, \mathcal{L})$ such that S is a smooth complex Abelian surface and $\mathcal{L}$ a line bundle over S of degree 4. Let us denote by $\mathcal{W}(1,2)$ the moduli space of Abelian surfaces with a $(1,2)$ polarization. We note that given such a pair $(S, \mathcal{L}) \in \mathcal{W}(1,2)$, the line bundle $\mathcal{L}$ is necessarily ample. Moreover, being $h^0(S, \mathcal{L}) = \frac{1}{2} \deg \mathcal{L} = 2$, the linear system $|\mathcal{L}|$ induces a linear pencil on S.

In [8], Wolf Barth gives a complete treatment of these surfaces and we shall recall here the results we need.

Given a couple $(S, \mathcal{L}) \in \mathcal{W}(1,2)$, there exists an irreducible curve $C \in |\mathcal{L}|$ if and only if $\mathcal{L}$ is not of the form $\mathcal{O}_S(E+2F)$ where E and F are elliptic curves in S such that $E^{\cdot 2} = F^{\cdot 2} = 0$ and $(E \cdot F) = 1$. Moreover, if there exists an irreducible element in $|\mathcal{L}|$, then the general member is smooth, and the linear pencil $|\mathcal{L}|$ has 4 distinct base points $\{p_0, p_1, p_2, p_3\}$.

Let us suppose that there exists such an irreducible curve $C \in |\mathcal{L}|$. Observe that $C^{\cdot 2} = 4$, and that by adjunction it has arithmetic genus 3. Barth proves that in this case C is either smooth of genus 3, or an irreducible curve of geometric genus 2 with one double point.

Let us fix the origin of S at the base point $p_0 \in S$ of the pencil $|\mathcal{L}|$. Then the p_i's are points of order 2 in S (cf. [8, p. 47]). Indeed, consider the natural isogeny $T_{\mathcal{L}} \colon S \longrightarrow Pic^0(S) = S^{\vee}$ defined by associating to $t \in S$ the invertible sheaf $\mathcal{L}^{-1} \otimes t^*\mathcal{L}$. Barth proves that $\{p_0, p_1, p_2, p_3\} \cong Ker\, T_{\mathcal{L}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence the p_i's are fixed by the (-1)-involution on S.

Another important result is that the (-1)-involution on S restricts to an involution ι on any curve $C \in |\mathcal{L}|$. In particular, it induces a degree 2 morphism $\pi \colon C \longrightarrow C/\langle\iota\rangle$.

Now, the surface S is naturally isomorphic to the generalized Prym variety $P(\pi)$ associated to this morphism. In order to fix the ideas and the notation, let us briefly recall how $P(\pi)$ is constructed in our cases. Let us distinguish the smooth and singular case.

Suppose first that $C \in |\mathcal{L}|$ is a smooth curve. By the Riemann-Hurwitz formula, the quotient $C/\langle\iota\rangle$ is a smooth elliptic curve E. Consider the embedding of E into the second symmetric product $C^{(2)}$

$$E \cong \{p + \iota(p), p \in C\} \subset C^{(2)},$$

and compose this map with the Abel embedding $C^{(2)} \hookrightarrow J(C)$. This is just the inclusion given by pullback on the Picard schemes

$$\pi^* \colon Pic^0(E) \cong E \hookrightarrow Pic^0(C) \cong J(C).$$

Then, by composing the Jacobian embedding with the quotient map, we have a well defined morphism $\eta \colon C \longrightarrow J(C)/\pi^*E$, which is an embedding satisfying $\eta(C)^2 = 4$ (see [8, Proposition (1.8)]). The Abelian surface $J(C)/\pi^*E$ is the generalized Prym $P(\pi)$ variety associated to π. Then we have the following (cf. [8, Chapter 1]).

Lemma 6.2.1. *Let $(S, \mathcal{L}) \in \mathcal{W}(1,2)$ and $C \in |\mathcal{L}|$ be a smooth curve. Then there exists a degree two morphism $\pi \colon C \longrightarrow E$ to an elliptic curve E such that S is naturally identified with $P(\pi) = J(C)/\pi^*E$.*
Conversely, if C is a smooth genus three curve provided of a bielliptic map

$\pi\colon C \longrightarrow E$, *we have that* C *is embedded in* $P(\pi)$ *as a curve of self-intersection* 4.

On the other hand, let us consider the case when $C \in |\mathcal{L}|$ is a singular irreducible curve. As we recalled above, C is an irreducible curve of geometric genus two with one double point at $q \in C$. Hence q is either a node or a cusp. We claim that it is necessarily a node. To see this fact, let $\nu\colon \widetilde{C} \longrightarrow C$ denote the normalization of C. Consider the Jacobian embedding $\eta\colon \widetilde{C} \to J(\widetilde{C})$ and the isogeny $\varphi\colon J(\widetilde{C}) \longrightarrow S$ given by its universal property. Both these morphisms have injective differential, then also $\nu = \varphi \circ \eta$ does and hence q is a node.

Let $q_1, q_2 \in \widetilde{C}$ be the preimages of q. The (-1)-involution on S extends to an involution $\iota\colon C \longrightarrow C$ that fixes $p_0, \dots p_3$. Clearly, q is fixed by ι as well and the quotient $C/\langle\iota\rangle$ is an irreducible curve of arithmetic genus 1 with one node (and hence its geometric genus is zero). It is immediate to see that the points $\{\nu^{-1}(p_0), \dots, \nu^{-1}(p_3), q_1, q_2\}$ are the Weierstrass points of $\widetilde{C}$.

The isogeny $\varphi\colon J(\widetilde{C}) \longrightarrow S$ has degree two. Indeed, the curve $\widetilde{C}$ has self-intersection 2 in the Jacobian $J(\widetilde{C})$, while $\nu_*\widetilde{C}^2 = C^2 = 4$ in S. Hence there exists a torsion point $\epsilon \in J(\widetilde{C})$ such that φ is the quotient map induced by the involution $z \mapsto z + \epsilon$. As usual, Let us identify $J(\widetilde{C})$ with $\mathrm{Pic}^0(\widetilde{C})$. We note that $q_1 - q_2$ is 2-torsion because the q_i's are Weierstrass points of $\widetilde{C}$ and these two points are identified in S. Therefore $\epsilon \sim q_1 - q_2$ in $\widetilde{C}$.

Now, let us denote by $G \cong \mathbb{Z}/2\mathbb{Z}$ the order 2 subgroup of $J(\widetilde{C})$ generated by $q_1 - q_2$. The generalized Prym variety $P(\pi)$ associated to π is the quotient $J(\widetilde{C})/G$. Thus we proved the following.

Lemma 6.2.2. *Let* $(S, \mathcal{L}) \in \mathcal{W}(1, 2)$ *such that there exist an irreducible singular curve* $C \in |\mathcal{L}|$. *Then* S *is naturally identified with* $P(\pi) = J(\widetilde{C})/G$.

6.3 Torsion points and geometry of the pencil $|\mathcal{L}|$

Let $(S, \mathcal{L}) \in \mathcal{W}(1, 2)$ be such that there exists an irreducible $C \in |\mathcal{L}|$. As we noticed above, the general element of the linear pencil $|\mathcal{L}|$ is smooth; moreover, it is a non-hyperelliptic curve (see [53, Claim 2.5]). In this section we shall describe both singular and hyperelliptic members of $|\mathcal{L}|$ and we shall study the torsion points of the surface S with respect to the elements of the linear pencil.

We assume hereafter that any element of $|\mathcal{L}|$ is irreducible. By the classification of the possible curves in the linear pencil (cf. [8, p. 46]), this amounts to ask that there are no curves of the form $E_1 + E_2$, where E_1

and E_2 are smooth elliptic curves contained in S meeting at two points. We note that this condition is satisfied on a suitable Zariski open subset of the moduli space $\mathcal{W}(1,2)$. Furthermore, it would follow from the stronger requirement on S of being a simple Abelian surface, but this latter assumption is generic, i.e. it is satisfied only outside of a numerable union of closed subsets of $\mathcal{W}(1,2)$.

Under this assumption, we shall see that $|\mathcal{L}|$ has exactly 12 singular members, corresponding to the twelve order two points of S differing from the p_i's (see Proposition 6.3.1). Moreover, the linear system shall contain exactly 6 smooth hyperelliptic elements that are related to a particular subgroup of the points of order 4 (cf. Lemma 6.3.2 and Proposition 6.3.3). Finally, in Proposition 6.3.4 we shall give a characterization of the triple points of S in terms of the canonical images of the corresponding curves of the linear pencil.

Now, let $\widetilde{S}$ be the surface obtained by blowing up S at the base points $\{p_0, \ldots, p_3\}$ and let

$$f\colon \widetilde{S} \longrightarrow \mathbb{P}^1 \tag{6.1}$$

be the fibration induced by the pencil $|\mathcal{L}|$. We denote by $E_0, \ldots, E_3$ the four exceptional curves of the blow up, that are sections of f.

Proposition 6.3.1. *Let $(S, \mathcal{L}) \in \mathcal{W}(1,2)$ be such that any element of $|\mathcal{L}|$ is irreducible. The linear pencil has 12 singular elements that are all irreducible curves of geometric genus 2 with one node. These nodes are the points of S of order 2 different from the p_i's.*

Proof. The first part of the proposition has already been established in Section 6.2. We saw also that the singular points are points of order 2 in S. As S has 16 points of order 2, four of which are the base points $p_0, \ldots, p_3$, it remains to prove that there are 12 singular elements in the linear pencil. To this aim, we use a formula on the invariants of the fibration $f\colon \widetilde{S} \longrightarrow \mathbb{P}^1$. The topological characteristic of $\widetilde{S}$ is $c_2(\widetilde{S}) = 4$. The topological characteristic of any smooth fiber F is $e(F) = 2 - 2g = -4$, whereas $e(N) = e(F) + 1 = -3$ for any singular fiber N. By applying the formula in [6, Lemma VI.4] for fibrations of surfaces, we obtain

$$4 = c_2(\widetilde{S}) = e(\mathbb{P}^1)e(F) + \sum_1^n (e(N) - e(F)) = -8 + n,$$

where n is the number of singular fibers. It follows that $n = 12$. $\square$

The other special elements of the linear pencil $|\mathcal{L}|$ are the smooth hyperelliptic curves. For any index $1 \leq i \leq 3$, let us consider the set of points of S defined as

$$P_i := \{x \in S \quad | \quad 2x = p_i\} \ .$$

Any of the P_i's is a set of 16 particular order 4 points in S. By the following two results we shall see that there exists a partition $P_i = A_i \cup B_i$ in subsets of order 8, such that any smooth hyperelliptic curve in $|\mathcal{L}|$ passes through a unique subset of those, whose 8 elements are the Weierstrass points on the curve.

Lemma 6.3.2. *Let $x \in P_i$ and let D be the element of $|\mathcal{L}|$ passing through x. Then D is smooth.*

Proof. Suppose by contradiction that D is a singular curve. Let $\nu\colon \widetilde{D} \longrightarrow D$ be the normalization of D and let $q_1, q_2 \in \widetilde{D}$ be the inverse images of the node. By abuse of notation, let us denote by p_i's the inverse images $\nu^{-1}(p_i)$'s and by x the inverse image of x in $\widetilde{D}$. We recall that $\{p_0, \ldots, p_3, q_1, q_2\}$ are the Weierstrass points of the genus 2 curve $\widetilde{D}$. Thanks to Lemma 6.2.2 we have that S is naturally identified with $J(\widetilde{D})/G$, where G denotes the order two subgroup of $J(\widetilde{D})$ generated by $q_1 - q_2$. In terms of this identification, the equality $2x = p_i$ in S means that either

$$2x \sim_{\widetilde{D}} p_0 + p_i \quad \text{or} \quad 2x \sim_{\widetilde{D}} p_0 + p_i + q_1 - q_2.$$

The first equivalence is impossible because $p_0 + p_i$ does not belong to the g^1_2 for any $i \neq 0$. On the other hand, by applying the hyperelliptic involution σ to the second equivalence, we have $2x \sim_{\widetilde{D}} 2\sigma(x)$. Hence x is a Weiestrass point of $\widetilde{D}$ and we get a contradiction. □

Proposition 6.3.3. *Let $(S, \mathcal{L}) \in \mathcal{W}(1, 2)$ be such that any element of $|\mathcal{L}|$ is irreducible. The linear pencil has 6 smooth hyperelliptic elements.*
Moreover, given such an element D, the hyperelliptic involution j of D induces a permutation of the base points not fixing any of them.

Proof. Let C be a smooth hyperelliptic curve belonging to $|\mathcal{L}|$. By Barth's construction we presented in the previous section, such a curve has a bielliptic involution $\iota\colon C \longrightarrow C$. As ι and j commute, j induces a permutation on the fixed points of ι, which are exactly the p_i's. Clearly, such permutation does not fix any point of the p_i's and the second part of the statement is established.

If $x \in C$ is a Weierstrass point, by what we observed above we have that $2x \sim_C p_0 + p_i$ for some $i \in \{1, 2, 3\}$. Hence $2x = p_i$ in S. We show that this property identifies the hyperelliptic members of $|\mathcal{L}|$.

Let us fix $x \in P_1$ and let $D \in |\mathcal{L}|$ be the curve passing through x. By Lemma 6.3.2 we have that D is smooth. In particular, we can identify S

and $J(D)/\pi^*E$, where E is the quotient of D by the bielliptic involution. Notice that $p_1 \in D$, hence - under the above identification - we have that there exists $s \in D$ such that

$$2x - 2p_0 \sim_D p_1 + s + \iota(s) - 3p_0$$

in D. It follows that $2x \sim_D 2\iota(x)$. Since x is not fixed by the bielliptic involution ι, $2x$ induces a g^1_2 on D. Hence D is hyperelliptic and x is one of its Weierstrass points. Moreover, the other 7 Weierstrass points necessarily satisfy the equality $2y = p_1$ in S, so they must belong to P_1.

Moreover, by choosing one point $x' \in P_1 \setminus D \cap P_1$, we obtain another hyperelliptic curve $D' \in |C|$.

Making the same construction for p_2 and p_3, we obtain the other 4 hyperelliptic curves. □

The following result provides a geometric description - in terms of the linear pencil - of the points of order 3 as well.

Lemma 6.3.4. *Let $p \neq p_i$ be a point of S and let $C \in |\mathcal{L}|$ be the curve passing through p. The following are equivalent:*

(i) *p is a point of order 3 of S;*

(ii) *C is a non-hyperelliptic curve of $|\mathcal{L}|$ and its canonical image $C \subset \mathbb{P}^2$ has an inflection point of order 3 at p with tangent line $\overline{p_0 p}$.*

Proof. Suppose that C is the element of the pencil passing through a point p of order three. If C is smooth, let $E = C/\langle\iota\rangle$ be its bielliptic quotient. Using the identification $S = J(C)/\pi^*E$, the assumption implies that there exists a point $s \in C$ such that

$$3(p - p_0) \sim_C s + \iota(s) - 2p_0. \tag{6.2}$$

Hence $3p \sim_C s + \iota(s) + p_0$ and this induces a base point free g^1_3 on C. In particular, assumption (i) implies that C is non-hyperelliptic.

Let us prove the equivalence separately in the smooth and the singular case.

Suppose that C is smooth and non-hyperelliptic. By (6.2) there exists $r \in C$ such that

$$3p + r \sim_C p_0 + s + \iota(s) + r \sim_C K_C.$$

The latter equivalence proves that p is an inflection points of order 3 for the canonical image $C \subset \mathbb{P}^2$ with tangent line $\overline{rp}$. To complete the first part of the proof we need to show that $r = p_0$. To this aim, observe that

$$p_0 + \iota(s) + s + r \sim_C K_C \sim \iota K_C \sim_C p_0 + s + \iota(s) + \iota(r).$$

Then $r = \iota(r)$ and hence r is one of the p_i's. On the other hand we have $K_C \sim_C p_0 + p_1 + p_2 + p_3$, then the unique possibility is $r = p_0$.

To prove the converse, suppose that C is a smooth non-hyperelliptic curve whose canonical image has an inflection point of order 3 at p with tangent line $\overline{p_0 p}$, i.e. $3p + p_0 \sim_C K_C$. For some points $a, b \in C$, let $2p_0 + a + b \sim_C K_C$ be the divisor cut out by the tangent line to C at p_0. Notice that $2p_0 + a + b \sim_C K_C \sim_C \iota K_C \sim_C 2p_0 + \iota(a) + \iota(b)$. Therefore either $a = \iota(b)$ with $a \neq b$ or $a = \iota(a)$. In any case, the relation $3(p - p_0) \sim_C s + \iota(s) - 2p_0$ holds for some $s \in C$ and hence $3p = 0$ in S.

Then let us suppose that C is nodal. As usual, we identify S with $J(\widetilde{C})/G$. Notice that the canonical immersion of C corresponds on $\widetilde{C}$ to the birational morphism associated to $K_{\widetilde{C}} + q_1 + q_2$. Let p be a point of order 3 in S and let C be the element of the linear pencil passing through p. This fact leads to the relation $3p \sim_{\widetilde{C}} 3p_0 + q_1 - q_2$ in $\widetilde{C}$. Since both p_0 and q_2 are Weierstrass points, we have $q_2 \sim_{\widetilde{C}} 2p_0 - q_2$. Hence

$$3p + p_0 \sim_{\widetilde{C}} 2p_0 + q_1 + q_2 \sim_{\widetilde{C}} K_{\widetilde{C}} + q_1 + q_2$$

as wanted. The converse is now straightforward. $\square$

6.4 The triple covering construction

Let $f\colon \widetilde{S} \longrightarrow \mathbb{P}^1$ be the fibration defined in (6.1). Let us consider the homomorphism of sheaves

$$f^* f_* \omega_f(-E_0) \longrightarrow \omega_f(-E_0),$$

and the relative rational map induced from it

$$\begin{tikzcd} \widetilde{S} \arrow[rr, dashed, "\gamma"] \arrow[dr, "f"'] & & \mathbb{P}(f_*\omega_f(-E_0)) := Y \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

Notice that γ is a generically finite map of degree 3. Indeed, the restriction $\gamma_{|F} : F \longrightarrow \mathbb{P}^1$ to the general smooth non-hyperelliptic fiber F of f, is the projection of the canonical image of $F \subset \mathbb{P}^2$ from the point $p_0 = F \cap E_0$. We note that similar constructions are studied in [53, Proposition 2.1] and [15].

In this section we want to study in detail the rational map $\gamma\colon \widetilde{S} \dashrightarrow Y$. To start, we want to compute explicitly the vector bundle $Y = \mathbb{P}(f_*\omega_f(-E_0))$ over $\mathbb{P}^1$. Then we shall resolve its indeterminacy points and we shall compute the ramification locus of the obtained triple cover.

Proposition 6.4.1. *With the notation above, Y is a minimal rational surface and*

$$Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_3.$$

Proof. Let us first compute the rank 3 vector sheaf $f_*\omega_f$. By using - for instance - a decomposition theorem of Fujita [25], we see that

$$f_*\omega_f = \mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2},$$

where $\alpha := \deg f_*\omega_f = \chi(\mathcal{O}_{\widetilde{S}}) - \chi(\mathcal{O}_F)\chi(\mathcal{O}_{\mathbb{P}^1}) = 2$.

Then we focus on $f_*\omega_f(-E_0)$. By Grauert's Theorem [30, Chapter III Corollary 12.9], we have that $R^1 f_*\mathcal{O}_{\widetilde{S}}(E_0)$ is a locally free sheaf in the case at hand. Hence by relative duality

$$R^1 f_*\mathcal{O}_{\widetilde{S}}(E_0) \cong (f_*\omega_f(-E_0))^\vee.$$

Let us consider the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\widetilde{S}} \longrightarrow \mathcal{O}_{\widetilde{S}}(E_0) \longrightarrow \mathcal{O}_{E_0}(E_0) \longrightarrow 0$$

and the long exact sequence induced by the pushforward

$$0 \to f_*\mathcal{O}_{\widetilde{S}} \to f_*\mathcal{O}_{\widetilde{S}}(E_0) \to f_*\mathcal{O}_{E_0}(E_0) \to R^1 f_*\mathcal{O}_{\widetilde{S}} \to R^1 f_*\mathcal{O}_{\widetilde{S}}(E_0) \to 0.$$

We observe that $f_*\mathcal{O}_{\widetilde{S}} \cong \mathcal{O}_{\mathbb{P}^1}$, $f_*\mathcal{O}_{E_0}(E_0) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ - and by using again relative duality - we have

$$R^1 f_*\mathcal{O}_{\widetilde{S}} \cong (f_*\omega_f)^\vee = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2).$$

Hence, the latter sequence can be reduced to the last three sheaves, as follows

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow R^1 f_*\mathcal{O}_{\widetilde{S}}(E_0) \longrightarrow 0.$$

Since there are no non-trivial morphisms from $\mathcal{O}_{\mathbb{P}^1}(-1)$ to $\mathcal{O}_{\mathbb{P}^1}(-2)$, the image of the first morphism $\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ is contained in the piece $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ of the second sheaf. Hence $\mathcal{O}_{\mathbb{P}^1}(-2)$ injects into $R^1 f_*\mathcal{O}_{\widetilde{S}}(E_0)$ and $R^1 f_*\mathcal{O}_{\widetilde{S}}(E_0) = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)$ for some β. Finally, by computing the degrees of these sheaves, we deduce that $\beta = 1$. $\square$

We note that the rational map $\gamma : \widetilde{S} \dashrightarrow Y$ is not a morphism. Indeed, a point $b \in \widetilde{S}$ is an indeterminacy point for γ if and only if the associated morphism of sheaves $f^* f_*\omega_f(-E_0) \longrightarrow \omega_f(-E_0)$ is not surjective in b (cf. [30, Chapter II.7]). Clearly, this morphism is surjective away from the sections E_i, with $1 \leq i \leq 3$. On the other hand, by restricting this morphism to any such E_i, we obtain the morphism

$$\mathcal{O}_{E_i}(-1) \oplus \mathcal{O}_{E_i}(2) \longrightarrow \mathcal{O}_{E_i}(1),$$

which vanishes - scheme-theoretically - at two points. Let us consider the restriction of γ to the fibers of f. On the smooth non-hyperelliptic fibers, as well as on the singular ones, it is easy to check that it is everywhere defined. So, let D be a smooth hyperelliptic fiber and let $p_i = E_i \cap D$, with $1 \leq i \leq 3$. By Proposition 6.3.3 we have that the hyperelliptic involution maps p_0 in a point $p_k \in \{p_1, p_2, p_3\}$. Then

$$h^0(D, \omega_D(-p_0 - p_k)) = h^0(D, \omega_D(p_0 + p_k)) = 2 = h^0(D, \omega_D(-p_0))$$

and hence p_k is a base point for the linear system $|\omega_D(-p_0)| \cong \omega_f(-E_0)_{|D}$.

Therefore the rational map γ has 6 indeterminacy points, lying two by two on the sections E_k's, each for any hyperelliptic fiber. For $j = 1, 2$ and $k = 1, \ldots, 3$, let us denote each of these points by b_{jk}, with the convention that $b_{1k}, b_{2k} \in E_k$.

Let $\overline{S}$ be the blow-up of $\widetilde{S}$ at the b_{jk}'s and let $\bar{f} \colon \overline{S} \longrightarrow \mathbb{P}^1$ be the induced map $\bar{f}$. Then we have the following commutative diagram.

$$\begin{tikzcd} \overline{S} \arrow[rr, "\delta"] \arrow[dr, "\bar{f}"'] & & \widetilde{S} \arrow[dl, "f"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

For $j = 1, 2$ and $k = 1, \ldots, 3$, let G_{jk} be the exceptional divisor over b_{jk}. By abuse of notation, we denote by $E_0, ..., E_3 \subset \overline{S}$ the strict transforms of the sections $E_0, ..., E_3 \subset \widetilde{S}$ and by $D \subset \overline{S}$ the strict transform of any hyperelliptic fiber $D \subset \widetilde{S}$. Then we have the following.

Proposition 6.4.2. *The sheaf $\bar{f}_*\mathcal{O}_{\overline{S}}(E_1 + E_2 + E_3)$ induces a morphism*

$$\begin{tikzcd} \widetilde{S} \arrow[rr, "\overline{\gamma}"] \arrow[dr, "\bar{f}"'] & & Y \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

which extends the rational map γ.

Proof. We recall that the rational map $\gamma \colon \widetilde{S} \dashrightarrow Y$ is induced by the sheaf $f_*\omega_f(-E_0) = f_*\mathcal{O}_{\widetilde{S}}(E_1 + E_2 + E_3)$. We just have to show that $\overline{\gamma}$ restricts to a well defined morphism on any fiber. Away from the G_{jk}'s, the map $\overline{\gamma}$ coincides with γ. Then let us consider the total transform of the hyperelliptic fiber $D \subset \widetilde{S}$, which is given by $D \cup G_{jk}$ for some j and k. Without loss of generality, let $k = 1$. The sheaf defining the restiction of $\overline{\gamma}$ to the G_{j1} is $\mathcal{O}_{\overline{S}}(E_1 + E_2 + E_3)_{|G_{j1}} \cong \mathcal{O}_{G_{j1}}(1)$ and hence $\overline{\gamma}_{G_{j1}} \colon G_{j1} \longrightarrow \mathbb{P}^1$ is an isomorphism. On the other hand, the restriction of the map $\overline{\gamma}$ to D is given by the sheaf

$$\mathcal{O}_{\overline{S}}(E_1+E_2+E_3)_{|D} = \mathcal{O}_D\left((E_2\cap D)+(E_3\cap D)\right) = \mathcal{O}_D(p_2+p_3).$$

By Proposition 6.3.3, the points p_2 and p_3 are conjugate under the hyperelliptic involution of D. Hence the linear system $|p_2+p_3|$ on D is the g^1_2. Therefore $\overline{\gamma}_{D\cup G_{j1}}$ has no base points and turns out to be a morphism of degree three to $\mathbb{P}^1$, as in figure below. Here p_0 denote the intersection of the fiber with E_0, $b_{jk} = G_{jk}\cap E_k$ and $\bar{b}_{jk} = G_{jk}\cap D$.

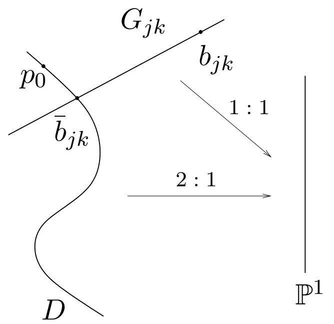


Figure 6.1: total transform on $\overline{S}$ of the hyperelliptic fiber of f

□

So the map $\overline{\gamma}$ is a triple covering. To conclude this section, we compute the numerical equivalence class of its ramification divisor.

Lemma 6.4.3. *The ramification divisor $R_{\overline{\gamma}}\subset\overline{S}$ has the following numerical class in $N^1(\overline{S})_{\mathbb{Z}}$:*

$$R_{\overline{\gamma}} \equiv E_0 + 3\sum_{k=1}^{3} E_k + 2\sum_{k=1}^{3}(G_{1k}+G_{2k}) + 5F.$$

Proof. The Néron-Severi group of Y is generated by the numerical class of a fiber Γ and by the class of the section C_0 with minimal self-intersection. Moreover, $K_Y\equiv -5\Gamma-2C_0$. By the formula for blow-ups, we have

$$K_{\overline{S}} \equiv \sum_{k=0}^{3} E_k + 2\sum_{k=1}^{3}(G_{1k}+G_{2k}). \tag{6.3}$$

We note that $\overline{\gamma}^*C_0 = E_1+E_2+E_3$. Indeed, the sections E_k's are -3-curves on $\overline{S}$ and they do not intersect the ramification locus $R_{\overline{\gamma}}$. As their images are -3-curves in Y, the E_k's map to the negative section C_0. Finally, $K_{\overline{S}}\equiv R_{\overline{\gamma}}+\overline{\gamma}^*K_Y$ by Riemann-Hurwitz formula and the assertion follows. □

6.5 The Galois closure of the covering

In this section we shall construct geometrically the Galois closure W of the covering $\overline{\gamma}\colon \overline{S} \longrightarrow Y$. To this aim let us recall the definition of Galois closure of a finite morphism.

Given a finite morphism $\psi\colon Z \longrightarrow T$ of degree d between normal surfaces, consider the induced degree d field extension $K(T) \hookrightarrow K(Z)$. Let L be the Galois closure of this field extension and let Z_{gal} be the normalization of Z in L (see for instance [36]).

Definition 6.5.1. With the notation above, the normal surface Z_{gal} provided of the induced morphism $Z_{gal} \longrightarrow T$ is the *Galois closure* of the morphism ψ.

Now, let W be the surface defined as

$$W := \overline{\{(p,q) \in \overline{S} \times \overline{S} \mid p \neq q \text{ and } \overline{\gamma}(p) = \overline{\gamma}(q)\}} \subset \overline{S} \times \overline{S}. \qquad (6.4)$$

We shall prove that W is a normal - possibly singular - surface and it shall turn out to be the Galois closure of the covering $\overline{\gamma}$ (see Proposition 6.5.2 and Proposition 6.5.5).

Let $\alpha_1\colon W \longrightarrow \overline{S}$ be the projection on the first factor of $\overline{S} \times \overline{S}$ and consider the diagram

$$\begin{tikzcd} W \arrow[d, "\alpha_1"'] \arrow[dr] & \\ \overline{S} \arrow[r, "\overline{\gamma}"] \arrow[d, "\overline{f}"'] & Y \arrow[dl] \\ \mathbb{P}^1 & \end{tikzcd}$$

We note that α_1 is a generically degree two morphism. Indeed, for a general point $p \in \overline{S}$, the inverse image of $\overline{\gamma}(p)$ consists of three distinct points $\{p, q, r\}$ and hence $\alpha_1^{-1}(p) = \{(p,q), (p,r)\}$. Moreover, as α_1 does not contract any curve on W we conclude that it is a double covering.

Proposition 6.5.2. *The surface W has an action of the symmetric group S_3 such that the quotient is the surface Y and the quotient by any order two subgroup is $\overline{S}$. In particular, the normalization of W is the Galois closure $\overline{S}_{gal}$ of $\overline{\gamma}$.*

Proof. Let $y \in Y$ be a general point and let

$$(\overline{\gamma} \circ \alpha_1)^{-1}(y) = \{(p,q), (p,r), (q,r), (q,p), (r,p), (r,q)\} \subset W$$

be its fiber consisting of six distinct points, where $\{p, q, r\} = \overline{\gamma}^{-1}(y) \subset \overline{S}$.

Let us consider the involution $\sigma_3 \in Aut(W)$ permuting the factors and let $\sigma_1 \in Aut(W)$ be the involution induced by α_1, that is $\sigma_3(p, q) = (q, p)$ and $\sigma_1(p, q) = (p, r)$. Then we define the automorphism $\varrho := \sigma_1 \circ \sigma_3$ and the subgroup $G := \langle \sigma_1, \varrho \rangle$ of $Aut(W)$ generated by σ_1 and ϱ. We note that the order of ϱ is 3 and that the generators of G satisfy the relation $\varrho \cdot \sigma_1 = \sigma_1 \cdot \varrho^2$. Thus the group G acting on W is a non-commutative group of order 6 isomorphic to S_3.

It is easy to check that the orbit of (p, q) is exactly the fiber $(\overline{\gamma} \circ \alpha_1)^{-1}(y)$. Hence the quotient variety W/S_3 is Y.

To conclude, let us consider the three involutions σ_1, $\sigma_2 := \varrho \circ \sigma_1$ and σ_3 of the group G. Being σ_1 the involution associated to the map α_1, we have $W/\langle \sigma_1 \rangle = \overline{S}$. Then we note that $\sigma_2(p, q) = (r, q)$, therefore σ_2 is the involution induced by the natural projection $\alpha_2 : W \subset \overline{S} \times \overline{S} \longrightarrow \overline{S}$ on the second factor. Finally, let $\alpha_3 \colon W \longrightarrow \overline{S}$ be the map defined by $\alpha_3(p, q) = r$, where r is the third point of the fiber of $\overline{\gamma}$ on $y = \overline{\gamma}(p)$. Hence $\alpha_3^{-1}(r) = \{(p, q), (q, p)\}$, and α_3 turns out to be the double covering whose associated involution is σ_3. $\square$

In order to show that the surface W is normal - and hence it is actually the Galois closure of $\overline{\gamma} \colon \overline{S} \longrightarrow Y$ -, we shall give a detailed description of the branch divisor $B_{\alpha_1} \subset \overline{S}$ of the morphism α_1. In particular, we shall compute its numerical equivalence class and we shall prove that B_{α_1} is a reduced curve with at most simple singularities.

Let $F \subset \overline{S}$ be a fiber of the morphism $\bar{f} \colon \overline{S} \longrightarrow \mathbb{P}^1$ and consider the restriction $\overline{\gamma}_{|F} \colon F \longrightarrow \mathbb{P}^1$ of the morphism $\overline{\gamma}$ to F. As usual, for $i = 0, \ldots, 3$ let $p_i = E_i \cap F$.

We recall that when F is a non-hyperelliptic - possibly singular - fiber, the map $\overline{\gamma}_{|F}$ is the projection of the canonical image of $F \subset \mathbb{P}^2$ from the point $p_0 = F \cap E_0$. We define the following subsets of F

$$A := \{p \in F \setminus \{p_0\} \,|\, \exists\, q \in F \setminus \{p, p_0\} \;:\; \overline{p_0 p} \text{ is tangent at } q\}$$

and

$$B := \{p \in F \setminus \{p_0\} \,|\, p \text{ is an inflection point of order 3 with tangent } \overline{p_0 p}\}\,.$$

Notice that the points of A and B correspond to the configurations (a) and (b) in Figure 6.2 below.

On the other hand, let $F = D \cup G_{jk}$ where D is a smooth hyperelliptic curve of genus 3. Then the restriction of $\overline{\gamma}$ to F is described in Proposition 6.4.2 (see also Figure 6.1). In particular, $\overline{\gamma}_{|D} \colon D \longrightarrow \mathbb{P}^1$ is the hyperelliptic map and $\overline{\gamma}_{|G_{jk}}: G_{jk} \longrightarrow \mathbb{P}^1$ is an isomorphism. We define the subset of F given by

$$T := \left\{ p \in G_{jk} \,|\, \exists\, q \in D \,:\, q \text{ is a Weierstrass point and } \overline{\gamma}_{|F}(p) = \overline{\gamma}_{|F}(q) \right\}.$$

The following lemma describes the intersection of the branch curve with each fiber.

Lemma 6.5.3. *Let $F \subset \overline{S}$ be a fiber of the morphism $\bar{f} : \overline{S} \longrightarrow \mathbb{P}^1$. With the notation above, the restriction $B_{\alpha_1|F}$ of the branch divisor B_{α_1} to F is given by one of the following.*

(i) *Let F be a smooth non-hyperelliptic fiber. Then*

$$B_{\alpha_1|F} = \sum_{p \in A} p + 2 \sum_{p \in B} p + 2p_0$$

if $p_0 = F \cap E_0$ is an inflection point of order 4 on F and

$$B_{\alpha_1|F} = \sum_{p \in A} p + 2 \sum_{p \in B} p$$

otherwise.

(ii) *If $F = N$ is a nodal fiber of geometric genus 2, then*

$$B_{\alpha_1|F} = \sum_{p \in A} p + 2 \sum_{p \in B} +2p_0.$$

In particular, the tangent line to F at p_0 meets the node transversally.

(iii) *If $F = D \cup G_{jk}$ with D hyperelliptic, then $B_{\alpha_1|F} = \sum_{p \in T} p + 2p_0$.*

Proof. Let $p \in F \cap B_{\alpha_1}$ be a branch point on a fiber F and let $(p, q) = \alpha_1^{-1}(p) \in W$, where $q \in F \cap R_{\overline{\gamma}}$. By giving a local description of B_{α_1} in a neighborhood of a total ramification point of $\overline{\gamma}$, its easy to see that the multiplicity $m_p(B_{\alpha_1|F})$ of $B_{\alpha_1|F}$ at p is equal to the multiplicity $m_q(R_{\overline{\gamma}|F})$ of $R_{\overline{\gamma}|F}$ at q.

[*Case* (i)] To start, let us consider a smooth non-hyperelliptic fiber F. Since $\overline{\gamma}_{|F}: F \longrightarrow \mathbb{P}^1$ is the projection of the canonical image of $F \subset \mathbb{P}^2$ from the point $p_0 = F \cap E_0$, the canonical divisor $K_p \in Div(F)$ cut out by the line $\overline{p_0 p}$ is one of the following (cf. Figure 6.2 below):

(a) $p_0 + p + 2q$ (b) $p_0 + 3p$ (c) $4p_0$ (d) $2p_0 + 2q$ (e) $3p_0 + p$

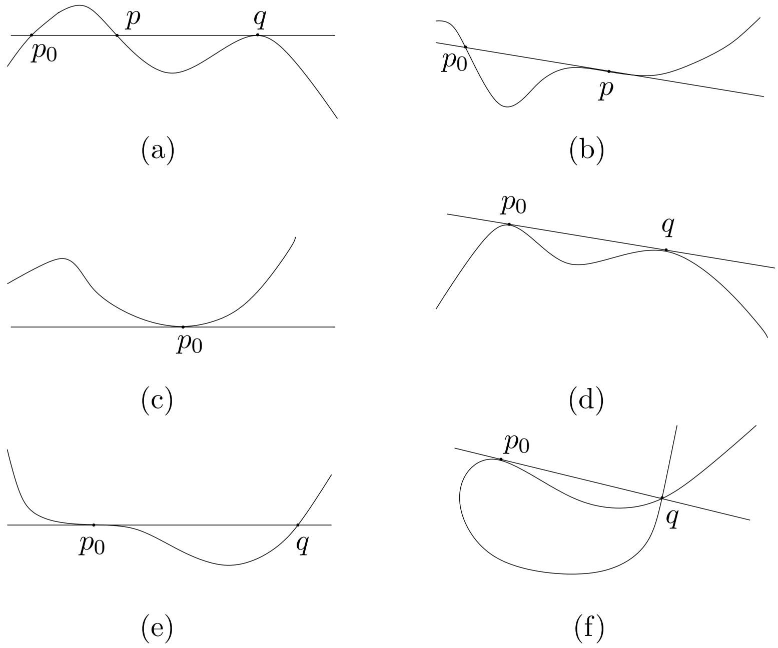


Figure 6.2: branch points of α_1 on the fibers of $\overline{f}$

We remark that cases (d) and (e) cannot occur. To see this fact, we recall that the points $p_0, \ldots, p_3$ are collinear and they are the only fixed points under the action of the bielliptic involution ι on F. Moreover, the divisor $\iota_* K_F$ is still canonical. So, if $\overline{p_0 p}$ were a bi-tangent line as in (d) - that is $K_p = 2p_0 + 2q$ with $p = p_0$ and $q \neq p_0$ - we would have that $\iota_* K_p = 2p_0 + 2\iota(q)$. As the tangent line to F at p_0 is tangent at q, we deduce that q is fixed by ι as well. Hence $q = p_i$ for some $i \neq 0$, but this is impossible because the p_i's are collinear.

Analogously, suppose that p_0 is an inflection point for F as in case (e), that is $K_F = 3p_0 + p$ with $q = p_0$ and $p \neq p_0$. Thus $\iota_* K_F = 3p_0 + \iota(p)$ and $p = \iota(p) = p_i$ for some $i \neq 0$, a contradiction.

When $K_p = p_0 + p + 2q$ as in (a), then $p \in A$ and q is a ramification point of index 2. Hence $m_p(B_{\alpha_1|F}) = 1$.

Condition (b) is equivalent to have an inflection point of order 3 at p with tangent line $\overline{p_0 p}$, that is $p \in B$. In this case p is a total ramification point of $\overline{\gamma}$ and hence $m_p(B_{\alpha_1|F}) = 2$.

Then $B_{\alpha_1|F} = \sum_{p \in A} p + 2\sum_{p \in B} p + 2p_0$ if p_0 is an inflection point of order 4 - as in (c) - and $B_{\alpha_1|F} = \sum_{p \in A} p + 2\sum_{p \in B} p$ otherwise.

[*Case* (ii)] Now, let $F = N \subset \overline{S}$ be a nodal fiber and let $p \in N \cap B_{\alpha_1}$. As the restriction $\overline{\gamma}_{|F} \colon F \longrightarrow \mathbb{P}^1$ is the projection from $p_0 \in F \cap E_0$, we deduce that two configurations analogous to (a) and (b) above are still possible away from the node. On the other hand, the cases (c), (d), (e) cannot occur and there is the following additional configuration: the tangent line at p_0 meets the node transversally (see Figure 6.2 (f)).

To see this fact, denote by $q \in F$ the node and let $L = \overline{p_0 q}$ be the line through p_0 and q. We recall from Proposition 6.3.1 that $q \neq p_i$ for all i. Let $\nu : \widetilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at q and let $\widetilde{F} \subset \widetilde{\mathbb{P}}^2$ be the strict transform of $F \subset \mathbb{P}^2$. Then for any pair of lines $L', L'' \subset \mathbb{P}^2$ we have that $\nu^*(L' \cap F)$ and $\nu^*(L'' \cap F)$ are linearly equivalent divisors on $\widetilde{F}$. Consider the points $\{p_0, p, q\} = L \cap F$. As p_0 and q are fixed by the action of the involution ι on F, we have that p is fixed as well. Hence either $p = q$ or $p = p_0$. The same argument works for the line L_i through q and p_i, with $i = 1, \dots, 3$. We want to prove that $p = p_0$.

Suppose by contradiction that $p = q$. Hence L is one of the two tangent lines at q. Notice that the four lines L, L_1, L_2, L_3 must be distinct because $p_0, \dots, p_3$ are collinear. Thus there exist two of those lines that are tangent in the corresponding p_i. Without loss of generality, let L_1 and L_2 these lines. Let $r_i := \nu^{-1}(p_i)$ and $\{q_1, q_2\} = \nu^* q$. Therefore $\nu^*(L \cap F) = r_0 + q_1 + 2q_2$, $\nu^*(L_1 \cap F) = 2r_1 + q_1 + q_2$ and $\nu^*(L_2 \cap F) = 2r_2 + q_1 + q_2$ are linear equivalent divisors on $\widetilde{F}$. Thus $r_0 + q_1$ is equivalent to $2r_1$, but this is impossible because $|2r_1|$ is the g^1_2 on $\widetilde{F}$ (cf. Proposition 6.3.1).

Then $p = p_0$ as in configuration (f). Hence the section E_0 meet F transversally at p_0 and the node $q \in R_{\overline{\gamma}}$ with $m_q(R_{\overline{\gamma}|F}) = 2$. Moreover, we note that this fact implies that the cases (c), (d) and (e) are not possible. Thus $B_{\alpha_1|F} = \sum_{p \in A} p + 2\sum_{p \in B} + 2p_0$.

[*Case* (iii)] It remains to study the branch locus on the fibers of the form $F = D \cup G_{jk}$, where D is a smooth hyperelliptic curve of genus 3. Let $w_1, \dots, w_8 \in D$ be the Weierstrass point and let $g_1, \dots, g_8 \in G_{jk}$ such that $\overline{\gamma}(w_t) = \overline{\gamma}(g_t)$ for any t. Hence the g_t's lie on B_{α_1} and for any t we have that $w_t \in R_{\overline{\gamma}}$ with $m_{w_t}(R_{\overline{\gamma}|F}) = 1$.

The last branch point on this fiber is the point p_0. Indeed, the hyperelliptic involution maps p_0 into the point $\bar{b}_{jk} := D \cap G_{jk}$ (see Proposition 6.3.3 and Figure 6.1). Hence $\bar{b}_{jk} \in R_{\overline{\gamma}}$ and it is a singular point of F. Thus $m_{\bar{b}_{jk}}(R_{\overline{\gamma}|F}) = 2$ and $B_{\alpha_1|F} = \sum_{p \in T} p + 2p_0$. □

Thanks to the previous lemma we can compute the numerical equivalence class of the branch locus B_{α_1}.

Lemma 6.5.4. *The branch divisor of α_1 has the following numerical class in $N^1(\overline{S})_{\mathbb{Z}}$:*

$$B_{\alpha_1} \equiv -2E_0 + 4\sum_{i=1}^{3} E_i + 20F - 4\sum_{i=1}^{3}(G_{i1} + G_{i2}). \tag{6.5}$$

Proof. As the points of order 3 on an Abelian surface are finitely many, Lemma 6.3.4 assures that the generic smooth non-hyperelliptic fiber F of $\bar{f}\colon \overline{S} \longrightarrow \mathbb{P}^1$ has no flex points. Hence the ramification divisor $R_{\overline{\gamma}} \subset \overline{S}$ of $\overline{\gamma}$ meets F at 10 points of ramification index 2. Let $q \in R_{\overline{\gamma}} \cap F$ one of these points and let $p \in B_{\alpha_1} \cap F$ such that $p_0 + p + 2q$ is the canonical divisor on F cut out by the line $L = \overline{p_0 q}$. As $(B_{\alpha_1} + 2R_{\overline{\gamma}})_{|F \cap L} = p + 2q \in |K_F(-p_0)|$, we have that $(B_{\alpha_1} + 2R_{\overline{\gamma}})_{|F}$ is 10 times the g^1_3 defining $\overline{\gamma}_{|F}$, that is $(B_{\alpha_1} + 2R_{\overline{\gamma}})_{|F} \equiv 10(E_1 + E_2 + E_3)_{|F}$.

Since $(F \cdot G_{jk}) = (F \cdot F) = 0$ for any $j = 1, 2$ and $k = 1, 2, 3$, there exist some integers m, n_{jk} such that

$$\begin{aligned} B_{\alpha_1} &\equiv -2R_{\overline{\gamma}} + 10(E_1 + E_2 + E_3) + mF + \sum_{k=1}^{3}(n_{1k}G_{1k} + n_{2k}G_{2k}) \\ &\equiv -2E_0 + 4\sum_{k=1}^{3} E_k + (m-10)F + \sum_{k=1}^{3}((n_{1k}-4)G_{1k} + (n_{2k}-4)G_{2k}). \end{aligned}$$

Then $(B_{\alpha_1} \cdot F) = 10$ and from the description of Lemma 6.5.3, we have that $(B_{\alpha_1} \cdot G_{jk}) = 8$ and $(B_{\alpha_1} \cdot E_k) = 0$. Thus we deduce $m = 30$ and $n_{jk} = 0$ for any j and k. □

Therefore we can now prove the following.

Proposition 6.5.5. *The branch divisor B_{α_1} is reduced and has at most simple singularities, that is W is normal with only rational double points as singularities. In particular W is the Galois closures $\overline{S}_{gal}$ of $\overline{\gamma}$.*

Proof. Thanks to Lemma 6.3.4, the general fiber F does not contain any inflection point. Moreover, $(B_{\alpha_1} \cdot F) = 10$ and by Lemma 6.5.3 we know scheme-theoretically the intersection. Hence the divisor $B_{\alpha_1|F}$ consists of ten distinct points. As B_{α_1} does not contain any vertical with respect to $\overline{f}$, we can conclude that it is reduced. This is equivalent to W being a normal surface (see [46, Proposition 1.1]).

From Lemma 6.5.3, we see that locally B_{α_1} has intersection multiplicity at most 2 with any fiber. This implies that it can have at most double points, i.e. all the possible singularities of B_{α_1} are simple points of type A_n (cf. [5, p. 61-65]). These singularities of the branch locus give rise to rational double points of W. □

We remark that the W is not necessarily smooth, because the branch curve B_{α_1} may have some singularities. Indeed W is non-singular if and only if B_{α_1} is non-singular (see [45]).

6.6 The Lagrangian surface and its invariants

This section is devoted to complete the proof of Theorem 6.1.4. Let X be the minimal desingularization of the Galois closure W defined in (6.4). We shall first compute the basic invariants of X. Then we shall prove that the Albanese variety of X is $S \times S$ and that X is a Lagrangian surface.

By abuse of notation, let $E_0, G_{jk}, F \subset W$ be the pullbacks of $E_0, G_{jk}, F \subset \overline{S}$. For $i = 1, \dots, 3$, the curve E_i does not meet the branch locus B_{α_1}, hence its pullback consist of to curves E_i' and E_i''. Thanks to Proposition 6.5.5, we are able to compute explicitly the invariants of X.

Theorem 6.6.1. *The minimal desingularization X of the surface W is a surface of general type, with invariants*

$$K_X^2 = 198 \quad c_2(X) = 102 \quad \chi(\mathcal{O}_X) = 25 \quad \tau(X) = -2.$$

Proof. To start, let us suppose that B_{α_1} - and hence W - is smooth. Let us check that the surface W does not contain -1-curves. The -1-curves on W come either from the -1-curves $L \subset \overline{S}$ such that $B_{\alpha_1} \cap L = \emptyset$ or from the -2-curves on $\overline{S}$ entirely contained in the branch divisor B_{α_1} (see [46, Proposition 1.8]). We note that the only -1-curves on $\overline{S}$ are E_0 and the G_{jk}'s, but they intersect B_{α_1}. On the other hand, $\overline{S}$ does not contain any -2-curve.

Then we set $X = W$ and the formulas to compute the invariants of X are the following (see [45]):

$$K_X^2 = 2\left(K_{\overline{S}}^2 + 2p_a(B_{\alpha_1}) - 2\right) - \frac{3}{2}B_{\alpha_1}^2$$

$$c_2(X) = 2c_2(\overline{S}) + 2p_a(B_{\alpha_1}) - 2$$

$$\chi(\mathcal{O}_X) = 2\chi(\mathcal{O}_{\overline{S}}) + \frac{p_a(B_{\alpha_1}) - 1}{2} - \frac{B_{\alpha_1}^2}{8},$$

where $p_a(B_{\alpha_1})$ denote the arithmetic genus of the branch curve. We note that $\overline{S}$ is obtained by blowing up ten times an abelian surface, therefore $c_2(\overline{S}) = 10$ and $\chi(\mathcal{O}_{\overline{S}}) = 0$. Moreover, by the adjunction formula and (6.5) we have

$$2p_a(B_{\alpha_1}) - 2 = \left((K_{\overline{S}} + B_{\alpha_1}) \cdot B_{\alpha_1}\right) = 80.$$

By applying equations (6.3) and (6.5) to the above formulas we compute $K_X^2 = 198$, $c_2(X) = 102$ and $\chi(\mathcal{O}_X) = 25$. Therefore the topological index of X is

$$\tau(X) = \frac{1}{3}\left(K_X^2 - 2c_2(X)\right) = -2.$$

Thanks to the Enriques-Kodaira classification, we have that X is a surface of general type (cf. [5, p. 188]).

Now, let us assume that B_{α_1} is singular. Hence B_{α_1} has only simple singularities by Proposition 6.5.5. To deal with this situations we consider the canonical resolution of the double covering $\alpha_1 \colon W \longrightarrow \overline{S}$ (cf. [5, Section III.7]). We have the following diagram

$$\begin{tikzcd} W' \arrow[r] \arrow[d] & W \arrow[d, "\alpha_1"] \\ \overline{S}' \arrow[r] & \overline{S}, \end{tikzcd}$$

where $\overline{S}'$ is obtained by blowing up $\overline{S}$ in order to perform the embedded resolution of $B_{\alpha_1} \subset \overline{S}$ and W' is the smooth surface obtained as a double covering of $\overline{S}'$ such that the branch divisor is the strict transform of B_{α_1}. By [5, Section III.7] it follows that W' does not contain -1-curves, thus W' is the minimal desingularization X of W. Finally, the formulas to compute the Chern invariants of X are the same we used above (see [5, Theorem 7.2 p. 89] and [5, Section V.22]) and the proof ends. $\square$

The surface X is the minimal desingularization of $W \subset \overline{S} \times \overline{S}$, hence the map $X \longrightarrow \overline{S}$ is a generically finite dominant morphism. Thanks to Proposition 6.5.2, we have that there is an action of the symmetric group S_3 on W, and such a group action is naturally inherited by X as follows.

For $1 \leq i \leq 3$, let us consider the double coverings $\beta_i \colon X \longrightarrow \overline{S}$, where β_1 is induced by the projection on the first factor $\alpha_1 \colon W \longrightarrow \overline{S}$, β_2 is induced by the projection α_2 on the second factor and β_3 is the covering induced by the involution on $W \subset \overline{S} \times \overline{S}$ permuting the factors. Moreover, let τ_i be the involution of X associated to β_i. Then the subgroup $\langle \tau_1, \tau_2, \tau_3 \rangle \subset Aut(X)$ is isomorphic to S_3.

We recall the symmetric group S_3 has three irreducible representations (cf. [24, Section 1.3]). The *trivial representation* U is a one-dimensional $\mathbb{C}$-vector space such that $\sigma u = u$ for any $\sigma \in S_3$, $u \in U$. The *anti-invariant representation* U' is one-dimensional as well and $\sigma u' = sgn(\sigma) u'$ for any $\sigma \in S_3$, $u' \in U'$. The *standard representation* Λ is the $\mathbb{C}$-vector space of dimension two given by $\Lambda := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \,|\, z_1 + z_2 + z_3 = 0\}$, where we fixed the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$ and $\sigma e_i = e_{\sigma(i)}$ for any $\sigma \in S_3$, $i = 1, 2, 3$. With this notation, we have the following.

Proposition 6.6.2. *Let Λ be the standard representation of S_3. Then $H^0(X, \Omega^1_X) = \Lambda \oplus \Lambda$ and consequently the irregularity of X is $q(X) = 4$. Moreover, X is a generalized Lagrangian surface with $V = H^0(X, \Omega^1_X)$.*

Proof. Let $\omega \in H^0(\overline{S}, \Omega^1_{\overline{S}})$ be an holomorphic 1-form on $\overline{S}$. By the definition of the β_i's it is immediate to check that for $\{i, j, k\} = \{1, 2, 3\}$,

$$\tau_i^* \beta_i^* \omega = \beta_i^* \omega \quad \text{and} \quad \tau_j^* \beta_i^* \omega = \beta_k^* \omega. \tag{6.6}$$

In particular, under the identification $\langle \tau_1, \tau_2, \tau_3 \rangle \cong S_3$ with $\sigma\beta_i^*\omega := \beta_{\sigma(i)}^*\omega$ for $\sigma \in S_3$, we have

$$\tau_1^* \leftrightarrow (23), \quad \tau_2^* \leftrightarrow (13) \quad \text{and} \quad \tau_3^* \leftrightarrow (12).$$

Furthermore, we note that the 1-form $\beta_1^*\omega + \beta_2^*\omega + \beta_3^*\omega$ is S_3-invariant. As the quotient X/S_3 is the rational surface $Y = \mathbb{F}_3$ - which does not admit 1-forms - we deduce that $\beta_1^*\omega + \beta_2^*\omega + \beta_3^*\omega = 0$ on X and hence

$$\beta_3^*\omega = -\beta_1^*\omega - \beta_2^*\omega. \tag{6.7}$$

We want to prove that the kernel of $\psi_2 \colon \wedge^2 H^0(X, \Omega_X^1) \longrightarrow H^0(X, \Omega_X^2)$ possesses a non-trivial element. To this aim, let us consider a basis $\{\omega_1, \omega_2\}$ of the space $H^0(\overline{S}, \Omega_{\overline{S}}^1)$ and let $V \subset H^0(X, \Omega_X^1)$ be the subspace spanned by the pullbacks via the β_i's of ω_1 and ω_2. As the 1-forms $\beta_1^*\omega_1$, $\beta_2^*\omega_1$, $\beta_1^*\omega_2$, $\beta_2^*\omega_2$ are independent on X, by the relation (6.7) we have that the set $\{\beta_1^*\omega_1,\ \beta_2^*\omega_1,\ \beta_1^*\omega_2,\ \beta_2^*\omega_2\}$ provides a basis of V.

Notice that the form

$$\omega := \beta_1^*\omega_1 \wedge \beta_1^*\omega_2 + \beta_2^*\omega_1 \wedge \beta_2^*\omega_2 + \beta_3^*\omega_1 \wedge \beta_3^*\omega_2 \in \wedge^2 H^0(X, \Omega_X^1) \tag{6.8}$$

is S_3-invariant. As the surface $Y = X/S_3$ is rational, we have $h^0(Y, \Omega_Y^2) = 0$ and hence ω must be zero as a 2-form on X, that is $\omega \in Ker\,\psi_2$. Furthermore, by relation (6.7) we have

$$\omega = (2\beta_1^*\omega_1 + \beta_2^*\omega_1) \wedge \beta_1^*\omega_2 + (2\beta_2^*\omega_1 + \beta_1^*\omega_1) \wedge \beta_2^*\omega_2 \tag{6.9}$$

and the associated matrix is given by

$$A_\omega := \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 1 \\ -1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -1 & 0 & 0 \end{bmatrix}.$$

Since A_ω is invertible, we conclude that ω has rank 4 and hence it provides a non-trivial element of $Ker\,\psi_2$. In particular, $V = \langle \beta_1^*\omega_1,\ \beta_2^*\omega_1,\ \beta_1^*\omega_2,\ \beta_2^*\omega_2 \rangle$ is the subspace of $H^0(X, \Omega_X^1)$ of minimal dimension such that $\omega \in \wedge^2 V$.

We note that the vector space $H^0(X, \Omega_X^1)$ admits a decomposition into the direct sum of the irreducible representations of S_3 (see [24, Proposition 1.8 p. 7]), that is

$$H^0(X, \Omega_X^1) = U^{\oplus a} \oplus U'^{\oplus b} \oplus \Lambda^{\oplus c} \quad \text{for some } \ a, b, c \in \mathbb{N}. \tag{6.10}$$

Since $Y = X/S_3$ is a rational surface, $H^0(X, \Omega_X^1)$ does not contain any invariant element. Therefore we have $a = 0$.

Moreover, it is immediate to check that the two-dimensional subspaces of V

$$V_1 := \langle \beta_1^*\omega_1,\, \beta_2^*\omega_1, \beta_3^*\omega_1\rangle,\; V_2 := \langle \beta_1^*\omega_2,\, \beta_2^*\omega_2, \beta_3^*\omega_2\rangle \subset V \tag{6.11}$$

provide two copies of the standard representation Λ. In particular, we have $V = V_1 \oplus V_2 \cong \Lambda \oplus \Lambda$ and $c \geq 2$. In order to conclude the proof, we have then to show that $V = H^0(X, \Omega_X^1)$, that is $b = 0$ and $c = 2$.

We note that each of the V_j's defined above has a 1-form $\beta_1^*\omega_j + \beta_2^*\omega_j$ invariant under the action of τ_3^* and a 1-form $\beta_1^*\omega_j - \beta_2^*\omega_j$ anti-invariant under the action of τ_3^*. Then both the 2-forms

$$(\beta_1^*\omega_1 + \beta_2^*\omega_1) \wedge (\beta_1^*\omega_2 + \beta_2^*\omega_2) \quad \text{and} \quad (\beta_1^*\omega_1 - \beta_2^*\omega_1) \wedge (\beta_1^*\omega_2 - \beta_2^*\omega_2)$$

turn out to be invariant under the action of τ_3^*. Clearly, the first one is equal to $\beta_3^*\omega_1 \wedge \beta_3^*\omega_2$. In particular, it is the pullback via β_3 of the 2-form $\omega_1 \wedge \omega_2$ on $\overline{S}$. Since $\omega_1 \wedge \omega_2$ is a generator of the one-dimensional vector space $H^0(\overline{S}, \Omega^2_{\overline{S}})$, we have that $\beta_3^*\omega_1 \wedge \beta_3^*\omega_2$ gives a non-zero holomorphic 2-form on X, that is $\beta_3^*\omega_1 \wedge \beta_3^*\omega_2 \notin Ker\,\psi_2$. As a consequence we have that

$$(\beta_1^*\omega_1 - \beta_2^*\omega_1) \wedge (\beta_1^*\omega_2 - \beta_2^*\omega_2) \notin Ker\,\psi_2 \tag{6.12}$$

as well. To see this fact, suppose by contradiction that (6.12) does not hold. The Lagrangian form $\omega \in \wedge^2 H^0(X, \Omega_X^1)$ defined in (6.8) is such that

$$\omega = \frac{3}{2}\,\beta_3^*\omega_1 \wedge \beta_3^*\omega_2 + \frac{1}{2}\,(\beta_1^*\omega_1 - \beta_2^*\omega_1) \wedge (\beta_1^*\omega_2 - \beta_2^*\omega_2)\,.$$

Therefore we have that $\beta_3^*\omega_1 \wedge \beta_3^*\omega_2 \in Ker\,\psi_2$ too, which is a contradiction.

In order to see that $b = 0$, we suppose by contradiction that there exists a 1-form $\eta \in H^0(X, \Omega_X^1)$ belonging to the anti-invariant representation U' of S_3. Let $\nu_1 := (\beta_1^*\omega_1 - \beta_2^*\omega_1)$, $\nu_2 := (\beta_1^*\omega_2 - \beta_2^*\omega_2)$ and let us consider the vector space $R := \langle \eta, \nu_1, \nu_2\rangle$. Since η, ν_1, ν_2 are all anti-invariant under the action of τ_3^*, we have that $\eta \wedge \nu_1$, $\eta \wedge \nu_2$ and $\nu_1 \wedge \nu_2$ are $\langle \tau_3^* \rangle$-invariant. We recall that the quotient $X/\langle \tau_3^*\rangle$ is $\overline{S}$ and $h^0(\overline{S}, \Omega^2_{\overline{S}}) = 1$. Moreover we proved above that $\nu_1 \wedge \nu_2 \notin Ker\,\psi_2$. Thus the image of the map

$$\overline{\psi} := \psi_{2|\wedge^2 R} \colon\ \wedge^2 R \longrightarrow H^0(X, \Omega_X^2)$$

is one-dimensional. Therefore $Ker\,\overline{\psi}$ has dimension 2.

We consider the subspaces $\langle \nu_1 \wedge \nu_2, \nu_1 \wedge \eta\rangle$ and $\langle \nu_2 \wedge \nu_1, \nu_2 \wedge \eta\rangle$ of $\wedge^2 H^0(X, \Omega_X^1)$. Their intersection with $Ker\,\overline{\psi}$ has necessarily dimension one. So, there exist $s, t, w, z \in \mathbb{C}$ such that $\nu_1 \wedge (s\eta + t\nu_2), \nu_2 \wedge (w\eta + z\nu_1) \in Ker\,\overline{\psi}$. In particular, there exist a rational function h on X such that

$\nu_1 = h(s\eta + t\nu_2)$ and hence $\nu_2 \wedge (w\eta + z\nu_1) = \nu_2 \wedge (w + h\, zs)\eta \in Ker\,\overline{\psi}$. Then we have that $\nu_2 = h_2\eta$ for some rational function h_2 on X. Analogously, there exists $h_1 \in K(X)$ such that $\nu_1 = h_1\eta$. Thus $\nu_1 \wedge \nu_2 \in Ker\,\overline{\psi}$, a contradiction. Therefore we have $b = 0$.

To conclude, we have to show that $c = 2$. To this aim, let us consider a copy of the standard representation Λ. It is easy to see that the action on Λ of any order two subgroup of S_3 - for instance $\langle\tau_3^*\rangle$ - splits Λ into the direct sum of two one-dimensional vector spaces; one of them is invariant under the action of $\langle\tau_3^*\rangle$, whereas the other one is anti-invariant.

If c were greater than 2, it would exist a standard representation V_3 different from V_1 and V_2 we defined in (6.11). Then it would exist a 1-form η, which would be anti-invariant under the action of τ_3^*. Then we could repeat the very same argument as above and we would get a contradiction. Therefore $c = 2$.

Thus $V = H^0(X, \Omega^1_X) = \Lambda \oplus \Lambda$ and $q(X) = h^0(X, \Omega^1_X) = 4$. $\square$

In particular, by the above proposition we have that the geometric genus of X is $p_g(X) = 28$.

Now, we would like to deal with the Albanese variety of X. To this aim, it is useful to analyze some special fibers of the induced fibration

$$\phi := \overline{f} \circ \alpha_1 \colon X \longrightarrow \mathbb{P}^1,$$

where $\overline{f} \colon \overline{S} \longrightarrow \mathbb{P}^1$ is the fibration inherited from $\widetilde{S}$ (cf. (6.1)).

Let $D \subset S$ be a smooth hyperelliptic element of $|\mathcal{L}|$. With the same notations of Proposition 6.4.2, let $F = D \cup G_{jk}$ be the corresponding fiber of $\overline{f} \colon \overline{S} \longrightarrow \mathbb{P}^1$. Then G_{jk} is a copy of $\mathbb{P}^1$ attached to D at a point (see Figure 6.1). Moreover, let $H \subset X$ denote be the pullback of $F \subset \overline{S}$.

Lemma 6.6.3. *The fiber H has three connected components D_1, D_2, D_3, which are all copies of D attached two by two in one node. Moreover, by a suitable choice of the indices $1 \le i \le 3$, the restriction $\beta_{i|D_i} \colon D_i \longrightarrow G_{jk}$ of degree two morphism $\beta_i \colon \overline{S} \times \overline{S} \longrightarrow \overline{S}$ is the hyperelliptic map, whereas $\beta_{j|D_i} \colon D_i \longrightarrow D$ is the identity map for any $j \neq i$.*

Proof. The first part of the statement follows from the description of the branch locus $B_{\alpha_1|F}$ given in Lemma 6.5.3. Namely, $B_{\alpha_1|G_{ij}}$ consists of the 8 Weierstrass points of the hyperelliptic map $D \longrightarrow \mathbb{P}^1 \cong G_{jk}$, whereas $B_{\alpha_1|D}$ is given by the base point p_0 of $|\mathcal{L}|$ with multiplicity 2. Thus the inverse image of G_{jk} - say D_1 - is a copy of D, while the inverse image of D is given by the two copies D_2 and D_3 of D attached in one node.

The second statement follows from the definition of $X \subset \overline{S} \times \overline{S}$ and of the β_i's. Given a general point $q \in \mathbb{P}^1 \cong G_{jk}$, its preimages via $\overline{\gamma}_{|F}$

are $q \in G_{jk}$ and the two preimages of the hyperelliptic map $q_1, q_2 \in D$. The corresponding fibers of β_1 are (q, q_1) and (q, q_2). Hence $\beta_{1|D_1}$ is the hyperelliptic map. On the other hand, $\beta_2(q, q_1) = q_1$, $\beta_2(q, q_2) = q_2$, and $\beta_3(q, q_1) = q_2$, $\beta_3(q, q_2) = q_1$, as wanted. □

Proposition 6.6.4. *The Albanese variety of X is $Alb(X) = S \times S$.*

Proof. We recall that $\overline{S}$ is obtained by blowing up the Abelian surface S. By composing the map $X \longrightarrow W \subset \overline{S} \times \overline{S}$ and the map $\overline{S} \times \overline{S} \longrightarrow S \times S$, we have a morphism $X \longrightarrow S \times S$. The universal property of the Albanese morphism induces a morphism of Abelian varieties $\theta \colon Alb(X) \longrightarrow S \times S$. We note that the map θ is an isogeny, indeed by Proposition 6.6.2 we have $dim\, Alb(X) = q(X) = 4 = dim\,(S \times S)$. Thus there is an inclusion induced in homology

$$\theta_* \colon H_1(X) = H_1(Alb(X)) \longrightarrow H_1(S \times S) = H_1(S) \times H_1(S).$$

We want to prove that θ_* is surjective, i.e. that θ is an isomorphism of Abelian varieties.

Let us consider one of the fibers $H = D_1 \cup D_2 \cup D_3$ we studied in Lemma 6.6.3. Then the image of D_2 in $S \times S$ is $\beta_1(D_2) = D \times \{0\}$. We recall that given any smooth member $C \in |\mathcal{L}|$, Lemma 6.2.1 assures that the Abelian surface S is naturally identified with $J(C)/E$, where E is an elliptic curve. Then S fits in the following sequence of Abelian varieties

$$1 \longrightarrow E \longrightarrow J(D) \xrightarrow{a} S \longrightarrow 1. \tag{6.13}$$

We note that the image of the composite homomorphism

$$H_1(D_2) \longrightarrow H_1(X) \xrightarrow{\theta_*} H_1(S) \times H_1(S)$$

is $H_1(S) \times \{0\}$. Indeed, the latter map is naturally identified with the homomorphism $H_1(D) = H_1(J(D)) \longrightarrow H_1(S)$ induced in homology by the sequence (6.13), which is surjective because the map a in (6.13) has connected fibres.

By the very same argument applied to D_1 we prove that the image of θ_* contains $\{0\} \times H_1(S)$ as well, and the assertion follows. □

Then we conclude the study of X by stating the following.

Theorem 6.6.5. *X is a Lagrangian surface.*

Proof. By Propositions 6.6.2 and 6.6.4, and by definition of the Galois closure, it is easy to deduce that the Albanese morphism $X \longrightarrow S \times S$ is a one to one morphism. Hence X is a Lagrangian surface. □

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