REPRESENTATIONS OF CURRENTS TAKING VALUES IN A TOTALLY DISCONNECTED GROUP

M. GABRIELLA KUHN

Abstract. Let $G = \mathcal{A}ut(\mathcal{T})$ be the group of all automorphisms of a homogeneous tree $\mathcal{T}$ of degree $q + 1 \geq 3$ and (X, m) a compact metrizable measure space with a probability measure m. We assume that μ has no atoms. The group $\mathcal{G} = \mathcal{A}ut(\mathcal{T})^X = G^X$ of bounded measurable currents is the completion of the group of step functions $f : X \to \mathcal{A}ut(\mathcal{T})$ with respect to a suitable metric. Continuos functions form a dense subgroup of $\mathcal{G}$. Following the ideas of I.M. Gelfand, M.I. Graev and A.M. Vershik we shall construct an irreducible family of representations of $\mathcal{G}$. The existance of such representations depends deeply from the nonvanisching of the first cohomology group $H^1(\mathcal{A}ut(\mathcal{T}), \pi)$ for a suitable infinite dimentional π.

1. Introduction

Let G be a locally compact group, X any compact space and $\mathcal{G}^0$ the space of all locally constant measurable functions $f : X \to G$. The group structure of G extends in a natural way to $\mathcal{G}^0$: for $f, g \in \mathcal{G}^0$ we let $f \cdot g(x) = f(x)g(x)$. Consider first $\mathcal{G}^0$ as a direct limit of groups isomophic with $\underbrace{G \times \cdots \times G}_{n\ times}$ and give it the natural toplogy coming from this structure.

There are several approaches to the construction of continuous unitary irreducible representations of $\mathcal{G}^0$ when G is a Lie group. One method essentially embeds $\mathcal{G}^0$ into the motion group of a Hilbert space and uses the canonical projective representation of the latter in the Fock space (see the papers of Araki [A], Guichardet [Gu1][Gu2], Parthasarathy and Schmidt [P-S1][P-S2] and Streater [St]).

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Another approach is based on the existance of a semigroup of positive definite functions, called "canonical states" and is due to I.M. Gelfand, M.I. Graev and A.M. Vershik

A semigroup of canonical states is a family of positive definite functions $\Psi^\lambda(g)$ of the form

$$\Psi^\lambda(g) = \exp(\lambda(\psi(g))) \qquad \lambda > 0 \tag{1.1}$$

where the infinitesimal generator $\psi(g)$ is a conditionally positive definite (briefly a c.p.d.) function on G.

The existance of such a generator depends on the nonvanishing of the first cohomology group $H^1(G, \pi)$ where π is an irreducible representation of G that cannot be separeted from the identity in the Fell topology. In this case one has $\psi(g) = -\frac{1}{2}\|\beta(g)\|^2$ for a suitable cocycle $\beta \in H^1(G, \pi)$.

It is clear that, in principle, one may apply this latter construction to all semisimple Lie groups without Kazhdan property T. This has been done by I.M. Gelfand, M.I. Graev and A.M. Vershik in a first paper appeared in the english vertion in 1982 [G-G-V1] for $PSL(2.\mathbb{R})$ and, later, by the same authors, for $SO(n, 1)$ and $SU(n, 1)$ [G-G-V2]. In the same papers it was also showed that the representations constructed from the semigroup Ψ^λ are equivalent to those described in the Fock model.

In this paper we want to apply I.M. Gelfand, M.I. Graev and A.M. Vershik construction to obtain an irreducible representation of $\mathcal{G}$, the group of measurable bounded functions $f : X \to \mathcal{A}ut(\mathcal{T})$ taking values in the group of automorphisms of a homogeneous tree. This approach allows us to deal also with the p-adic groups of Lie type such as $PGL(2, \mathbb{Q}_p)$.

There are strong analogies but also points of difference. The main difference concerns the semigroup of positive definite functions. In the case of $PSL(2, \mathbb{R})$ one has $\Psi^\lambda(g) = \cosh^{-\lambda}(d_H(i, g\cdot i)/2)$ where d_H is the hyperbolic distance in the upper half plane between the two points i and $g \cdot i$. In the case of $\mathcal{A}ut(\mathcal{T})$ one should expect that $\Psi^\lambda(g) = q^{-\lambda d(o, g\cdot o)}$ where $d(o, g\cdot o)$ is tree distance between a choosen point o and $g\cdot o$, however the cocycle that corresponds to the c.p.d. function $-d(o, g\cdot o)$ does not take values in an irreducible representation of G and the semigroup Ψ^λ is different from what expected. The main analogy is about tensor products of spherical representatins. In short, take the tensor product $\pi_{\lambda_1} \otimes \pi_{\lambda_2}$ of two spherical representations of $\mathcal{A}ut(\mathcal{T})$ and decompose it into irreducible components, as $\mathcal{A}ut(\mathcal{T})$ is type I (see [F-N]) this can be done in an essentially unique way. In the decomposition either the complementary series appears with multiplicity one or it does not appear at all. This fact is shared by $\mathcal{A}ut(\mathcal{T})$ with groups as $SL(2,\mathbb{R})$ and $SL(2,\mathbb{Q}_p)$ (see [M] and [R]) and inspired the study of semigroups of canonical states in the case of $\mathcal{A}ut(\mathcal{T})$ (see [KuV1] [KuV2]). In particular we are happy to thank A. Vershik for helpful conversations and, first of all, for suggesting us the possibility to apply the construction of the multiplicative integral given in [G-G-V1] to our case.

2. The group $\mathcal{G}^0$

Let $G = \mathcal{A}ut(\mathcal{T})$ be the group of all automorphisms of an homogeneous tree $\mathcal{T}$ of order $q+1$. The topology of pointwise convergence turns $\mathcal{A}ut(\mathcal{T})$ into a locally compact totally disconnected topological group. Denote by $\mathcal{V}$ the set of vertices of $\mathcal{T}$. Let $\mathcal{F}$ be any finite subtree with vertex set $v_1 \dots v_N$, the the sets $V_{\mathcal{F}}$

$$V_{\mathcal{F}} = \{g \in \mathcal{A}ut(\mathcal{T}) \ : g \cdot v_i = v_i \qquad \text{for all } v_i \in \mathcal{F}\} \ . \tag{2.1}$$

constitute, as $\mathcal{F}$ varies among all finite subtrees of $\mathcal{T}$, a basis of neighbourhoods of the group identity e.

Let X be a compact metrizable space with a positive Radon measure m with no atoms. We shall assume that *m is normalized*, that is

$$m(X) = 1 \qquad m\{x\} = 0 \text{ for all } x \in X \ . \tag{2.2}$$

For any Borel subset $A \subseteq X$ write A^c for its complement and set

$$G_A = \{f : X \to G \, ; \ f(x) \text{ is constant on } A \quad \text{and } f(x) = e \text{ if } x \in A^c\} \ . \tag{2.3}$$

For disjoint Borel subsets $A_1, A_2 \dots A_n$ let

$$G_{A_1} \dots G_{A_n} = \{f : X \to G \, ; \ f(x) = f_1(x) f_2(x) \dots f_n(x) \quad f_i \in G_{A_i}\} \ . \tag{2.4}$$

There is an obvious identification between $G_{A_1} \dots G_{A_n}$ and $\underbrace{G \times \cdots \times G}_{n \ times}$. Give $G_{A_1} \dots G_{A_n}$ the product topology of $G \times \cdots \times G$.

Throughout this paper we shall identify functions which are equal almost everywhere $[m]$. So that G_A and $G_{A_1} \dots G_{A_n}$ are in fact sets of equivalence classes of functions which coincide a.e. $[m]$ for which we decide, once and for all, to represent any class with a function everywhere defined and constant on each A_i $(1 \le i \le n)$.

A finite partition ρ of X into Borel subsets $A_1, \dots A_n$ is called admissible. For any admissible partition ρ denote by G_ρ the subgroup of functions $f : X \to G$ which are constant on each A_i. Write $\rho_1 < \rho_2$ if ρ_2 is a refinement of ρ_1. There is a natural embedding $J_{\rho_2,\rho_1} : G_{\rho_1} \to G_{\rho_2}$. *The group of step functions* $\mathcal{G}^0$ is the direct limit

$$\mathcal{G}^0 = \lim_{\rightarrow} G_\rho \tag{2.5}$$

.

An element of $\mathcal{G}^0$ may be identified with a function $f_0 : X \to \mathcal{A}ut(\mathcal{T})$ satisfying the following properties: there exist a finite number of disjoint measurable sets A_i and different constants $g_i^0 \in \mathcal{A}ut(\mathcal{T})$ ($i = 1 \dots n$) such that $f_0(x) = g_i^0$ for $x \in A_i$ and $\cup_{i=1}^n A_i = X$. A neighbourhood of f_0 is given by all locally constant functions f such that $f(x) \in \mathcal{U}_i(g_i^0)$ for all $x \in A_i$ ($i = 1 \dots n$) where $\mathcal{U}_i(g_i^0)$ are choosen neighbourhoods of g_i^0 in $\mathcal{A}ut(\mathcal{T})$. More references about direct limits of topological groups and representations may be found in the appendices of the paper [N-RC-W].

3. Representations of $\mathcal{G}^0$ coming from tensor products

3.1. Tensor products of spherical representations. A classification of the irreducible continuous unitary representations of $\mathcal{A}ut(\mathcal{T})$ was given by Ol'shanskii [Ol1] [Ol2], and is described in [F-N], the notation of which we shall basically be following.

Fix a vertex o of $\mathcal{V}$ and fix, once and for all, a maximal compact subgroup K:

$$K = \{g \in \mathcal{A}ut(\mathcal{T})\,;\; g \cdot o = o\}\ . \tag{3.1}$$

An irreducible unitary representation π_z is called *spherical* if it admits a nonzero K-invariant vector.

3.1.1. *The boundary of $\mathcal{T}$ and the spherical series.* Denote by $d(x, y)$ the usual tree distance between vertices x, y of $\mathcal{T}$. The *boundary* Ω of $\mathcal{T}$ can be identified with the set of *semi-infinite* geodesics starting at o. One thinks of each element $\omega \in \Omega$ as an infinite chain $\omega = [o = a_0, a_1, a_2, a_3 \dots)$ where $d(a_j, a_{j+1}) = 1$ and $a_j \neq a_{j+2} \forall j$.

Give Ω the natural topology as a subspace of the power space

$Map(\mathbb{N}, \text{vertices of } \mathcal{T})$. This makes Ω compact and totally disconnected, isomorphic to the Cantor set.

A simple set of generators for this topology may be obtained as follows: fix a vertex $v \in \mathcal{T}$. Define $\Omega(v)$ as the set of all half infinite geodesics $\omega = [o, a_1, a_2, a_3 \dots)$ such that v is a vertex of ω.

Denote by ν the unique K-invariant probability measure on Ω which assigns the measure $\frac{q}{q+1} q^{-d(o,v)}$ to each of sets $\Omega(v)$.

Let $\omega = [o, b_1, b_2 \dots b_n \dots)$ and $\omega' = [o, a_1, a_2 \dots a_n \dots)$ be two distinct elements of Ω. Assume that $b_j = a_j$ for all j between 1 and k and $b_{k+1} \neq a_{k+1}$: so that the first $k+1$ vertices of ω and ω' are all equal but the $k + 2$ are distinct. The last point of the common starting (finite!) geodesic $[o, a_1, a_2, \dots a_k]$ will be denoted by $\omega \wedge \omega'$ ($\omega \wedge \omega' = o$ when

$a_1 \neq b_1$). Analogously, if v is a vertex of $\mathcal{T}$, let $[o, a_1, a_2, \ldots, v]$ denote the unique geodesic from o to v and define

$$\omega \wedge v = \begin{cases} o & \text{if } v \text{ is not a vertex of } \omega \\ a_k & \text{if } a_j = b_j \text{ for } 1 \leq j \leq k \text{ but } a_{k+1} \neq b_{k+1}. \end{cases} \tag{3.2}$$

Finally, for $g \in \mathcal{A}ut(\mathcal{T})$ and $\omega \in \Omega$ let $N(g \cdot o, \omega)$ denote the length of the geodesic from o to $\omega \wedge g \cdot o$. Hence $N(g \cdot o, \omega)$ denotes the length of the maximum common geodesic between $[o, g \cdot o]$ and ω. We shall simply write $N(g, \omega)$ for $N(g \cdot o, \omega)$. The Poisson kernel is

$$\frac{d\nu(g^{-1}\omega)}{d\nu(\omega)} = P(g, \omega) = q^{2N(g,\omega)-d(o,g\cdot o)}$$

and the spherical representations are defined as

$$(\pi_z(g)f)(\omega) = P^z(g, \omega)f(g^{-1} \cdot \omega)$$

in the space $\mathcal{K}(\Omega)$ of locally constant complex functions on Ω.

When $z = \frac{1}{2} + it$ the representation π_z is unitary with respect to the scalar product

$$\langle f \ g \rangle = \int_\Omega f(\omega)\overline{g(\omega)}d\nu(\omega).$$

and the *principal* spherical series act on $L^2(\Omega, d\nu)$, the completion of $\mathcal{K}(\Omega)$ with respect to this norm.

When z is real and $0 < z < 1$ the representation π_z is unitarizable and the *complementary* spherical series act on H_z, the completion of $\mathcal{K}(\Omega)$ with respect to another suitable inner product.

The *spherical functions* ϕ_z are obtained, as usual, as matrix coefficients with respect to the unique K-invariant vector in $\mathcal{K}(\Omega)$. So that

$$\phi_z(g) = \langle \pi_z(g)\mathbf{1}, \mathbf{1} \rangle \ = \ \int_\Omega P^z(g, \omega)d\nu \tag{3.3}$$

where $\mathbf{1}$ is the function identically 1 on Ω. A direct computation gives

$$\phi_z(g) = c(z)q^{-zd(o,g\cdot o)} + c(1-z)q^{(z-1)d(o,g\cdot o)} \qquad \text{if } z \neq \frac{1}{2} + \frac{2k\pi i}{\log q} \tag{3.4}$$

where $c(z)$ is the Harish-Chandra c-function:

$$c(z) = \frac{1}{(q+1)} \frac{q^{1-z} - q^{z-1}}{q^{-z} - q^{z-1}} \tag{3.5}$$

and

$$\phi_z(g) = (1 + \frac{q-1}{q+1} \cdot d(o, g \cdot o))q^{-zd(o,g\cdot o)} \qquad \text{if } z = \frac{1}{2} + \frac{2k\pi i}{\log q} \tag{3.6}$$

The endpoints of the principal series are obtained when $z = \frac{1}{2} + k\pi i/\log q$ while the endpoints of the complementary series correspond to the cases $z = 1$ and $z = 0$.

In particular when

$$z = 1 \qquad \text{or} \quad z = 0$$

the spherical function becomes identically one and the endpoint representation obtained by letting $z \to 0$ (or $z \to 1$) splits into the sum of sp$_-$, one of the two so called "special" representations of $\mathcal{A}ut(\mathcal{T})$ and the trivial representation (see [Ol2]).

We recall here some basic properties of spherical functions that will be needed later:

Proposition 3.1. *Assume that the spherical series are realized in the completion of $\mathcal{K}(\Omega)$ and either $0 < z < \frac{1}{2}$ or $z = \frac{1}{2} + it$. Let γ_n be any element of $\mathcal{A}ut(\mathcal{T})$ such that $d(o, \gamma_n \cdot o) = n$. Since the spherical function $\phi_z(g)$ is bi-K invariant the value $\phi_z(\gamma_n) = \phi_z(n)$ depends only on n and we have*

$$\int_{K\gamma_n K} |\phi_z(g)|^2 dg = (q+1)q^{n-1}|\phi_z(n)|^2 \ . \tag{3.7}$$

Morover

$$|\phi_z(n)|^2 \simeq q^{-2nz} \qquad \text{if } 0 < z < \tfrac{1}{2} \tag{3.8}$$

$$|\phi_{\frac{1}{2}+it}(n)| \leq |\phi_{\frac{1}{2}}(n)| \qquad \text{Herz's majorization principle} \tag{3.9}$$

$$|\phi_{\frac{1}{2}}(n)|^2 \simeq (n+1)^2 q^{-n} \tag{3.10}$$

In particular, let (π, H) be any representation of $\mathcal{A}ut(\mathcal{T})$ weakly contained in the regular representation and let v be any K-invariant vector in H. Then v admits the following decomposition

$$v = \int^{\oplus} v(t)\mathbf{1} d\lambda(t) \tag{3.11}$$

where $\mathbf{1} = \mathbf{1}_t$ is the function identically one on Ω. Moreover one has

$$|\langle \pi(\gamma_n)v, v\rangle|^2 \leq \|v\|^4 (n+1)^2 q^{-n} \ . \tag{3.12}$$

Proof. The proof of (3.8) and (3.9) can be found in Chapter II of [F-N] while (3.10) is obvious from (3.6). Let us turn to (3.12). Assume that π is weakly contained in the regular representation and decompose it into irreducibles. The only representations of $\mathcal{A}ut(\mathcal{T})$ that may appear in the decomposition are those of the discrete series and of the spherical principal series. Choose the realization of the principal series on $L^2(\Omega, d\nu)$. Since no representation of the discrete series admits a nonzero K-invariant vector, we may assume that $v = \int_J v_t d\lambda(t)$ with $J \subseteq [0, 2\pi]$. Let $k \in K$. Since $\pi(k)v = v$ we have

$$\|\pi(k)v - v\|^2 = \int_J \|\pi_t(k)v_t - v_t\|^2 d\lambda(t) = 0 \tag{3.13}$$

and hence $\pi_t(k)v_t = v_t$ a.e. $[\lambda]$. Since the subspace of K-invariant vectors is one dimentional for all t, it must be $v_t = v(k,t)\mathbf{1}$ for some scalar valued integrable function $v(k,t)$. Letting $k = k_1k_2$ we see that $v(k_1k_2, t) = v(k_2, t)$ showing that $v(k,t)$ is independent on k. Let now $v = \int_J v(t)\mathbf{1}d\lambda(t)$ and γ_n such that $d(o, g \cdot o) = n$. One has

$$|\langle \pi(\gamma_n)v, v\rangle| \leq \int_J |v(t)|^2 |\phi_t(\gamma_n)| d\lambda(t) \; . \tag{3.14}$$

Herz's majorization principle (3.9) will give the desired result:

$$|\langle \pi(\gamma_n)v, v\rangle| \leq \int_J |v(t)|^2 |\phi_{\frac{1}{2}}(\gamma_n)| d\lambda(t) \leq \|v\|^2 (n+1) q^{-\frac{1}{2}n} \tag{3.15}$$

□

Denote by $\mathbb{T} = [0, 2\pi)$ the complex torus and by dt its Haar measure. The first realization of our representation Π is based on the following Theorem, which follows from Propositions 2.1–2.5 of [C-K-S]:

Theorem 3.2. *Let $\pi_{1,2} = \pi_{z_1} \otimes \pi_{z_2}$ be the tensor product of π_{z_1} π_{z_2}, two spherical representations of $\mathcal{A}ut(\mathcal{T})$. The representations are assumed to act on $H_{z_1} \otimes H_{z_2}$, H_{z_1}, H_{z_2} respectively. Assume, for simplicity, that both z_1 and z_2 are between 0 and 1/2 and let*

$$\lambda = z_1 + z_2 \; .$$

- *If $\lambda \geq \frac{1}{2}$ the representation $\pi_{1,2}$ decomposes by means of the principal and the discrete series only.*
- *If $\lambda < \frac{1}{2}$ the representation $\pi_{1,2}$ splits into the sum of exactly one complementary series representation π_λ plus a direct integral over all the principal series plus a sum of (not all) the discrete series. In particular the $\mathcal{A}ut(\mathcal{T})$ modulo $H_{z_1} \otimes H_{z_2}$ splits into the orthogonal sum of three pieces:*

$$H_{z_1} \otimes H_{z_2} = H_\lambda \oplus \int^{\oplus} L^2(\Omega, d\nu) d\sigma(t) \oplus H_3 \tag{3.16}$$

where H_3 does not contain any nonzero K- invariant vector and $d\sigma(t)$ is absolutely continuous with respect to the Plancherel measure

$$d\mu(t) = \frac{q}{(q+1)} \frac{1}{|c(\frac{1}{2} + it)|^2} \, dt$$

given by the Harish-Chandra c-function (3.5).

- *The following multiplication formula for spherical functions holds:*

(3.17)
$$\phi_{z_1}(g)\phi_{z_2}(g) = \frac{c(z_1)c(z_2)}{c(z_1+z_2)}\phi_{z_1+z_2}(g) + \int_0^{2\pi} K(z_1,z_2,t)\phi_{\frac{1}{2}+it}(g)d\mu(t)$$

where $K(z_1,z_2,t)$ *is a positive integrable kernel.*

Remark 3.3. Assume now that $z_1 = m(X_1)$, $z_2 = m(X_2)$ and $z_1 + z_2 < \frac{1}{2}$. The above Theorem ensures that there exists a K-invariant vector $v_{1,2}$ in $H_{z_1} \otimes H_{z_2}$, such that

$$\mathbf{1} \otimes \mathbf{1} = v_{1,2} \oplus \int_0^{2\pi} v(t)\mathbf{1}dm(t) \ .$$

Letting $g = e$ in (3.17) we have

$$1 = \frac{c(z_1)c(z_2)}{c(z_1+z_2)} + \int_0^{2\pi} K(z_1,z_2,t)d\mu(t) \tag{3.18}$$

So that the map

$$\mathbf{1} \to v_{1,2}\sqrt{\frac{c(z_1+z_2)}{c(z_1)c(z_2)}}$$

extends to an isometric embedding of $\pi_{z_1+z_2}$ into $\pi_{z_1} \otimes \pi_{z_2}$. Since H_z is a space of functions defined on Ω, we may identify $v_{1,2}$ with a function on $\Omega \times \Omega$. In particular, because the subspace of K-invariant vectors in H_z is one dimentional for every spherical representation π_z, there exists *a unique* $v_{1,2}$ such that

- $v_{1,2}$ is a nonnegative function
- $$(\pi_{z_1} \otimes \pi_{z_2})(g)(v_{1,2}) = \sqrt{\frac{c(z_1)c(z_2)}{c(z_1+z_2)}}\pi_{z_1+z_2}(g)\mathbf{1}$$

In the sequel, we shall always assume that the vector corresponding to the embedding of $H_{z_1+z_2}$ into $H_{z_1} \otimes H_{z_2}$ is chosen in this way. Any other such embedding will map the vector $\mathbf{1}$ into $c\ v_{1,2}$ where c is a scalar of absolute value one.

Proposition 3.4. *Let A be the disjoint union of measurable sets $A_1 \dots A_n$ with $m(A) = z < \frac{1}{2}$ and $m(A_i) = z_i$. Let π_z, respectively π_{z_i}, denote the complementary series representations of $\mathcal{A}ut(\mathcal{T})$ acting on H_z, respectively on H_{z_i}. The representation $\pi_{z_1} \otimes \cdots \otimes \pi_{z_n}$ of $\mathcal{A}ut(\mathcal{T})$ splits into of the orthogonal sum of two pieces:*

$$\pi_{z_1} \otimes \cdots \otimes \pi_{z_n} = \pi_z \oplus \pi' \tag{3.19}$$

where π_z is the complementary series representation corrsponding to the parameter z and π' decomposes by means of the principal and discrete series only. In particular there exists an isometric embedding of H_z into $H_{z_1}\otimes\cdots\otimes H_{z_n}$ that commutes with the action of $\mathcal{A}ut(\mathcal{T})$.

Proof. The statement is true for $n=2$ by the previous Theorem (3.2). Assume that $z=z_1+z_2+z_3$, multiply both sides of (3.17) by $\phi_{z_3}(g)$ and apply again Theorem (3.2):
(3.20)

$$\phi_{z_1}(g)\phi_{z_2}(g)\phi_{z_3}(g)=\frac{c(z_1)c(z_2)}{c(z_1+z_2)}\frac{c(z_1+z_2)c(z_3)}{c(z_1+z_2+z_3)}\phi_{z_1+z_2+z_3}(g)+\lambda(g)$$

Where

$$\lambda(g)=\int_J K(z_1,z_2,t)\phi_{\frac12+it}(g)\phi_{z_3}(g)d\mu(t)+\frac{c(z_1)c(z_2)}{c(z_1+z_2)}\int_J K(z_1+z_2,z_3,t)\phi_{\frac12+it}(g)d\mu(t)\tag{3.21}$$

Since the representations that are weakly contained in the regular representation are characterized by the decay of their matrix coefficents (see [Ol1] or [F-N]), the tensor product of a uniformly bounded representation and a representation weakly contained in the regular is still weakly contained in the regular and we may conclude that no complementary series appears in the decomposition of λ. In particular there exists a unique positive function $v_{1,2,3}$ on $\Omega\times\Omega\times\Omega$ such that the map

$$\mathbf{1}\to\sqrt{\frac{c(z_1+z_2+z_3)}{c(z_1)c(z_2)c(z_3)}}v_{1,2,3}\tag{3.22}$$

extends to an isometric embedding of H_z into $H_{z_1}\otimes H_{z_2}\otimes H_{z_3}$. A repeated application of the above arguments concludes the proof. ☐

3.2. **The Irreducible Representation** Π**.** Let ρ be an admissible partition of X into Borel subsets $A_1,\dots A_n$. Fix, once and for all, $M\in(0,\frac12)$ and let

$$z_i=M\cdot m(A_i)\ .\tag{3.23}$$

Let π_{z_i} be the complementary series representation of $\mathcal{A}ut(\mathcal{T})$ correspondig to z_i acting on H_{z_i}.

Define an irreducible representation π_ρ of the group $G_\rho=G_{A_1}\cdots G_{A_n}$ by the rule

$$\pi_\rho(g_1,\dots,g_n)=\pi_{z_1}(g_1)\otimes\cdots\otimes\pi_{z_n}(g_n)\tag{3.24}$$

acting on $\mathcal{H}_\rho = H_{z_1} \otimes \cdots \otimes H_{z_n}$. Proposition (3.4) above tells how to construct maps from $\mathcal{H}_{\rho_1}$ to $\mathcal{H}_{\rho_2}$ every time that $\rho_1 < \rho_2$ (that is when ρ_2 is a refinement of ρ_1).

Proceeding as in [G-G-V1] one can prove the following

Theorem 3.5. *Let $\mathcal{H}_{\rho_1}$ and $\mathcal{H}_{\rho_2}$ as above. There exist morphisms of Hilbert spaces*

$$j_{\rho_2,\rho_1} : \mathcal{H}_{\rho_1} \to \mathcal{H}_{\rho_2}$$

defined for each pair of admissible partitions $\rho_1 < \rho_2$ satisfying the following conditions:

(1) *j_{ρ_2,ρ_1} commutes with the action of G_{ρ_1} on $\mathcal{H}_{\rho_1}$ and $\mathcal{H}_{\rho_2}$.*
(2) *each j_{ρ_2,ρ_1} is an isometry.*
(3) *$j_{\rho_3,\rho_2} \cdot j_{\rho_2,\rho_1} = j_{\rho_3,\rho_1}$ for any $\rho_1 < \rho_2 < \rho_3$.*

These morphisms are determined uniquely to within factors c_{ρ_i,ρ_j} of absolute value one.

3.3. **The representation space for $\mathcal{G}^0$.** For any admissible partition ρ construct the Hilbert space $\mathcal{H}_\rho$. Theorem (3.5) above ensures that for any pair $\rho_1 < \rho_2$ there exist morphisms $j_{\rho_2,\rho_1} : \mathcal{H}_{\rho_1} \to \mathcal{H}_{\rho_2}$ which commute with the G_{ρ_i} action and also satisfy the compatibility condition $j_{\rho_3,\rho_2} \cdot j_{\rho_2,\rho_1} = j_{\rho_3,\rho_1}$ for any $\rho_1 < \rho_2 < \rho_3$. We can now define $\mathcal{H}^0$ to be the inductive limit of Hilbert spaces:

$$\mathcal{H}^0 = \lim_{\rightarrow} \mathcal{H}_\rho \ . \tag{3.25}$$

We recall that an element v of $\mathcal{H}^0$ is an equivalence class of vectors $[v_\rho]$ with $v_\rho \in \mathcal{H}_\rho$ and with the following property: there exists ρ_0 (depending on v) such that for every admissible partition $\rho > \rho_0$ one has $v_\rho = j_{\rho,\rho_0} v_{\rho_0}$. One can take

$$\|v\|_{\mathcal{H}^0} = \|v\|_{\mathcal{H}_{\rho_0}} \ .$$

Let $\mathcal{H}$ be the completion of $\mathcal{H}^0$ with respect to this norm. Let now $\xi \in \mathcal{G}^0$ and $v \in \mathcal{H}^0$. In order to define $\Pi(\xi)v$ observe that there exist an adimissible partition ρ such that $\xi \in G_\rho$ and $v \in \mathcal{H}_\rho$. Define

$$\Pi(\xi)v = \pi_\rho(\xi)v \tag{3.26}$$

and extend it to the whole $\mathcal{H}$ by continuity . Again more details about unitariy of Π can be found in [N-RC-W].

The next Theorem says that different measures m on X will give inequivalent representations.

Theorem 3.6. *Let m_1 and m_2 be two normalized Radon measures and let Π^1 and Π^2 be the corresponding representations of $\mathcal{G}^0$ acting on $\mathcal{H}^1$*

and $\mathcal{H}^2$ respectively. Assume that there exists a measurable set A such that $m_1(A) \neq m_2(A)$. Then Π^1 and Π^2 are inequivalent.

Proof. Assume, by way of contraddiction, that there exists a $\mathcal{G}^0$ unitary map U that intertwines Π^1 to Π^2. A fortiori U intertwines the restriction of Π^1 to the subgroup $G_A \times G_{A^c}$ to the restriction of Π^2 to the same subgroup. Let ρ be the partition corresponding to $X = A \cup A^c$. Set $\lambda_1 = Mm_1(A)$, $\mu_1 = M(1 - m_1(A_1)) = M - \lambda_1$ and, respectively $\lambda_2 = Mm_2(A)$, $\mu_2 = M - \lambda_2$. We know that Π^1 acts as $\pi_{\lambda_1} \otimes \pi_{\mu_1}$ on $\mathcal{H}^1_\rho = H_{\lambda_1} \otimes H_{\mu_1}$ when restricted to $G_A \times G_{A^c}$. Choose $h \in \mathcal{H}^1_\rho$, $\xi \in G_A \times G_{A^c}$ and set $z = Uh$.

By definition of $\mathcal{H}^2$ there exists a sequence $z_n \in \mathcal{H}^2_{\rho_n}$ converging to z and

$$\Pi^2(\xi)z = \lim_{n\to\infty} \Pi^2(\xi)z_n \ . \tag{3.27}$$

By passing to a subsequence we may assume that ρ_n is a refinement of the initial partition ρ, so that $A = \cup_{i=1}^n A_i$, $A^c = \cup_{i=1}^m B_i$. Let $Mm_2(A_i) = z_i$ and $Mm_2(B_i) = t_i$. By definition $\Pi^2(\xi)z_n$ belongs to $\mathcal{H}^2_{\rho_n} = H_{z_1} \otimes \cdots \otimes H_{t_m}$. According to Proposition (3.4) $\mathcal{H}^2_{\rho_n}$, as a $G_A \times G_{A^c}$ modulo, splits into the orthogonal sum of two parts:

$$H_{z_1}\otimes\cdots\otimes H_{t_m} = H_{z_1+\cdots+z_n}\otimes H_{t_1+\cdots+t_m} \oplus H'_n \simeq H_{\lambda_2}\otimes H_{\mu_2}\oplus H'_n \tag{3.28}$$

where, for every n,

$$H'_n = L_n \otimes H_{\mu_2} \oplus H_{\lambda_2} \otimes L'_n \tag{3.29}$$

and both L_n and L'_n correspond to representations of $\mathcal{A}ut(\mathcal{T})$ that are weakly contained in the regular representation. Write $\Pi^2(\xi)z_n = v_n + v'_n$ where $v_n \in H_{\lambda_2} \otimes H_{\mu_2}$ and $v'_n \in H'_n$. Passing to the limit as n goes to infinity, since $H_{\lambda_2} \otimes H_{\mu_2}$ is closed in $\mathcal{H}_{\rho_n}$ for every n the limit of v_n will belog to it while the limit of v'_n will belong to same space H'. Hence $U(H_{\lambda_1} \otimes H_{\mu_1}) = H_{\lambda_2} \otimes H_{\mu_2} \oplus H'$. We shall see that H' cannot contain any copy of $H_{\lambda_1} \otimes H_{\mu_1}$. Denote by $K_A \times K_{A^c}$ the subgroup of $G_A \times G_{A^c}$ consisting of all locally constant functions taking values in K. The vector $\mathbf{1} \otimes \mathbf{1}$ is invariant under the action of $K_A \times K_{A^c}$ and so is its immage $U(\mathbf{1} \otimes \mathbf{1})$. Write $U(\mathbf{1} \otimes \mathbf{1}) = v \oplus v'$ where v and v' are both $K_A \times K_{A^c}$ invariant ($v \in H_{\lambda_2} \otimes H_{\mu_2}$, $v' \in H'$). H' will contain a copy of $H_{\lambda_2} \otimes H_{\mu_2}$ if and only if $v' \neq 0$. Normalizing v' if necessary, we may write

$$\langle \Pi^2(\xi)v', v'\rangle = \langle \Pi^1(\xi)\mathbf{1} \otimes \mathbf{1}, \mathbf{1} \otimes \mathbf{1}\rangle$$

for every $\xi \in G_A \times G_{A^c}$.

Consider now H' as an $\mathcal{A}ut(\mathcal{T})$ modulo via the diagonal action: since H'_n, as $\mathcal{A}ut(\mathcal{T})$ representation, is weakly contained in the regular representation, the same will be true for H'. Denote by γ_n any element of $\mathcal{A}ut(\mathcal{T})$ such that $d(o, \gamma_n \cdot o) = n$ and by $\mathbf{g}$ the function identically equal to g on X. One has

$$\langle \Pi^2(\mathbf{g})v', v'\rangle = \langle \Pi^1(\mathbf{g})\mathbf{1}\otimes\mathbf{1}, \mathbf{1}\otimes\mathbf{1}\rangle = \phi_{\lambda_1}(g)\phi_{\mu_1}(g)\ . \tag{3.30}$$

Integrate now over $K\gamma_n K$: according to Proposition (3.1) the left hand side of (3.30) gives

$$\int_{K\gamma_n K} |\langle \Pi^2(\mathbf{g})v', v'\rangle|^2 dg \le (q+1)q^{n-1}(n+1)^2 q^{-n}\|v'\|^4 \simeq C(n+1)^2 \tag{3.31}$$

while the right hand side is

$$\int_{K\gamma_n K} |\phi_{\lambda_1}(g)\phi_{\mu_1}(g)|^2 dg \simeq (q+1)q^{n-1}q^{-2(\lambda_1+\mu_1)n} \simeq q^{(1-2M)n}\ . \tag{3.32}$$

Putting together (3.31) and (3.32) one gets

$$\int_{K\gamma_n K} |\phi_{\lambda_1}(g)\phi_{\mu_1}(g)|^2 dg \simeq q^{(1-2M)n} \le C(n+1)^2$$

which is impossible since $M < \frac{1}{2}$. Hence $U(\mathbf{1}\otimes\mathbf{1})$ belongs to $H_{\lambda_1}\otimes H_{\mu_1}$ and this is a contraddiction since $H_{\lambda_i}\otimes H_{\mu_i}$ $(i = 1, 2)$ are irreducible $G_A \times G_{A^c}$ modulos with $\lambda_2 \ne \lambda_1$. $\square$

The last Theorem of this section is about irreducibility:

Theorem 3.7. *Let* Π *the representation of* $\mathcal{G}^0$ *constructed from a normalized Radon measure. Then* Π *is an irreducible representation of* $\mathcal{G}^0$.

Proof. Assume that $\mathcal{H}$ splits into the sum of two invariant subspaces, say $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Since the restriction of Π to G_ρ is an irreducible representation on $\mathcal{H}_\rho \subseteq \mathcal{H}$, the subspace $\mathcal{H}_\rho$ must be contained in one of the $\mathcal{H}_i$. Suppose that $\mathcal{H}_\rho \subseteq \mathcal{H}_1$. Take now any $\nu > \rho$: for the same reason $\mathcal{H}_\nu$ must be contained either in $\mathcal{H}_1$ or in $\mathcal{H}_2$. Since $\mathcal{H}_\rho \subseteq \mathcal{H}_\nu$ we must have $\mathcal{H}_\nu \subseteq \mathcal{H}_1$ for every $\nu > \rho$. Hence the direct limit $\mathcal{H}^0$ is contained in $\mathcal{H}_1$ and also its closure since $\mathcal{H}_1$ itself is closed. $\square$

4. The Fock model for Π

4.1. **The space** EXP(H)**.** Let H be a complex Hilbert space. We recall here the definition of the Hilbert space EXP(H) (see for example [J]). Let $H^0 = \mathbb{C}$ and, for any integer $n > 0$, let $H^{\odot n}$ denote the symmetric tensor power of H. For any $v \in H$ let

$$\mathrm{EXP}(v) = 1 \oplus v \oplus \frac{1}{\sqrt{2!}} v\otimes v \oplus \cdots \oplus \frac{1}{\sqrt{n!}} v^{\otimes n} \oplus \dots \tag{4.1}$$

Set

$$\langle \mathrm{EXP}\, v_1, \mathrm{EXP}\, v_2\rangle_{\exp} = \sum_{n=0}^{\infty} \frac{1}{n!}\langle v_1, v_2\rangle^n = e^{\langle v_1, v_2\rangle} \tag{4.2}$$

The space

$$\mathrm{EXP}(H) = H^0 \oplus H^1 \oplus \cdots \oplus H^{\odot n} \oplus \ldots \tag{4.3}$$

is the completion of all finite linear combinations of vectors of the form $\mathrm{EXP}(v)$ with respect to the norm induced by the above inner product (4.2). The *vacuum vector* $\mathrm{EXP}\, 0$ is the vector

$$\mathrm{EXP}(0) = 1 \oplus 0 \oplus \frac{1}{\sqrt{2!}} 0 \otimes 0 \oplus \cdots \oplus \frac{1}{\sqrt{n!}} 0^{\otimes n} \oplus \ldots \tag{4.4}$$

Let $G_0(H)$ denote the *group of motions* of a Hilbert space H, that is the group of all maps $M : H \to H$ of the form $M(v) = Av + b$ where A is a unitary operator and $b \in H$. Identify $G_0(H)$ with the set of all pairs (A, b) $(A \in \mathcal{U}(H))$, $b \in H)$ with multiplication given by

$$(A_1, b_1)(A_2, b_2) = (A_1 A_2, b_1 + A_1 b_2)\ .$$

A unitary *projective* representation U of $G_0(H)$ in $\mathrm{EXP}(H)$ is given by the formula:

$$U(A, b)(\mathrm{EXP}\, v) = \exp(-\frac{1}{2}\|b\|^2 - \langle Av, b\rangle)\ \mathrm{EXP}(Av + b) \tag{4.5}$$

One can check that

$$U(A_1, b_1)U(A_2, b_2) = U(A_1 A_2, A_1 b_2 + b_1) e^{i\Im\langle b_1, A_1 b_2\rangle} \tag{4.6}$$

4.2. **Cocycles and representations.** Let (π, H_π) be a unitary representation of G. By a *cocycle* with values in H_π we mean a 1-cocycle, that is a continuous function $\beta : G \to H_\pi$ satisfying the condition

$$\beta(g_1 g_2) = \beta(g_1) + \pi(g_1)\beta(g_2)\ . \tag{4.7}$$

For any closed subspace $Y \subseteq H_\pi$ denote by P_Y the orthogonal projection of H_π onto Y. A cocycle is called *absolutely nontrivial* if, for every closed nonzero G invariant subspace Y the cocycle $\beta_Y = P_Y \beta$ is nontrivial (not of the form $\pi(g)v - v$ for some vector $v \in Y$). Every cocycle β allows us to define a homomorphism θ of G into the group $G_0(H_\pi)$ by letting $\theta(g) = (\pi(g), \beta(g))$. Composing θ with U we get a unitary projective representation $E(\pi, \beta)$ of G in the space $\mathrm{EXP}(H_\pi)$:

$$\begin{aligned} E(\pi, \beta)(g)\, \mathrm{EXP}(v) &= U(\theta(g))\, \mathrm{EXP}(v) = \\ &\exp\left(-\frac{1}{2}\|\beta(g)\|^2 - \langle \pi(g)v, \beta(g)\rangle\right) \mathrm{EXP}(\pi(g)v + \beta(g)) \end{aligned} \tag{4.8}$$

One has

$$E(\pi,\beta)(g_1)E(\pi,\beta)(g_2) = E(\pi,\beta)(g_1g_2)e^{i\Im\langle\beta(g_1),\pi(g_1)\beta(g_2)\rangle} \tag{4.9}$$

There are however situations for which the representation constructed above is a true representation of G. Assume that (π, H_π) is an *orthogonal representation* of G in a *real* Hilbert space and that $\beta : G \to H_\pi$ is a cocycle. Form the complexification H of H_π and consider the representation $\pi \otimes \mathrm{Id}$ acting on H. Since β preserves the subspace of real elements the function $\beta : G \to H$ is still a cocycle and one has $\Im\langle\beta(g_1), \pi(g_1)\beta(g_2)\rangle = 0$ for all $g_1, g_2 \in G$. The situation that we shall consider will be exactly like this.

4.3. **The cocycle of $\mathcal{A}ut(\mathcal{T})$.** Assume that $\beta : G \to H_\pi$ is a non-trivial cocycle and that (π, H_π) is irreducible. In 1982 Karpushev and Vershik (see [KaV]) proved that such π cannot be separeted from the identity in the Fell topology. We also know that a converse has been proved by Y.Shalom in 2000 (see [Sh]). According to the classification if irreducible representations of $\mathcal{A}ut(\mathcal{T})$ (see [Ol1] or [F-N]), the only possible π in our case is the *special* representation.

We recall here some standard facts aboute conditionally positive definite functions: more references can be found in the papers of P. Delorme [D1] and [D2] and A. Guichardet [Gu2].

A continuous function $\varphi : G \to \mathbb{C}$ is said to be conditionally positive definite, briefly c.p.d., if

$$\sum_{i,j} c_i\overline{c_j}\varphi(g_j^{-1}g_i) \geq 0 \quad \text{whenever} \quad \sum_i c_i = 0$$

A c.p.d. is said to be normalized if it is *real* and $\varphi(e) = 0$. To every conditionally positive definite normalized function φ corresponds a cocycle β such that $\varphi(g) = -\frac{1}{2}\|\beta(g)\|^2$.

Since the spherical functions of the complementary series of $\mathcal{A}ut(\mathcal{T})$ converge to 1 pointwise when $z \to 0$ one may apply the method of [G-G-V1] to get a conditionally positive definite function, and hence a cocycle, taking the derivative at the point $z = 0$ of the spherical functions. This approach gives the explicit form of the c.p.d. (see [KuV1]).

$$\varphi_0(g) = \lim_{z\to 0}\frac{\phi_z(g) - 1}{z} = -\frac{1}{2}\|\beta_0(g)\|^2\ . \tag{4.10}$$

To describe the cocycle β_0 we need a realization of the special representation sp_ different from that given in [Ol1] and [F-N]. Here are the basic steps: more details can be found in [KuV2].

In $\mathcal{K}(\Omega)$ consider the inner product

$$\langle F,F\rangle_- = \frac{q^2-1}{q}\int_\Omega\int_\Omega |F(\omega)-F(\omega')|^2 q^{2|\omega\wedge\omega'|}d\nu(\omega)d\nu(\omega') \ . \tag{4.11}$$

H_{sp_-} is the completion of the complement of constant functions with respect to the above inner product $\langle\ ,\ \rangle_-$. The representation is simply

$$\mathrm{sp}_-(g)F(\omega) = F(g^{-1}\omega)$$

and the cocycle is

$$\beta_0(g) = (2N(g,\omega) - d(o, g\cdot o)) \ . \tag{4.12}$$

We shall simply write $\|\beta_0(g)\|^2$ for $\langle\beta_0(g),\beta_0(g)\rangle_-$.

In 1979 Haagerup [H] proved that the function $g\to -d(o,g\cdot o)$ is c.p.d. by showing that $d(o,g\cdot o)=\|\beta_h(g)\|^2$ for a suitable cocycle taking values in the Hilbert space H_Λ of oriented edges of $\mathcal{T}$. As a matter of fact Haagerup's paper concerned the free group on r generators F_r. When $q+1$ is even, $\mathcal{A}ut(\mathcal{T})$ can be viewed as $K\cdot F_r$ and one can adapt his proof for K invariant cocycles defined on $\mathcal{A}ut(\mathcal{T})$, the case of the general tree is treated in [V].

In [KuV2] it was proved that β_0 and β_h are cohomologous by showing that $H_\Lambda = H^1_\Lambda\oplus H^2_\Lambda$ where H^1_Λ is equivalent to the regular representation on $\ell^2(\mathcal{V})$ while H^2_Λ is equivalent to H_{sp_-}. In particular one has

$$d(o,g\cdot o) = \|\beta_h(g)\|^2 = \frac{1}{2}\|\beta_0(g)\|^2 + \frac{2q}{q^2-1}(1-q^{-d(o,g\cdot o)}) \tag{4.13}$$

4.4. **The representation $\Xi = E(\mathrm{sp}^X_-,\beta^X_0)$.** We are now ready to construct an *irreducible* representation of $\mathcal{G}^0$ in the Fock model.

Consider the cocycle $\beta_0 : \mathcal{A}ut(\mathcal{T})\to H_{\mathrm{sp}_-}$. Since $\beta_0(g)$ is a real valued function on Ω, the immaginary part of $\langle\beta_0(g_1),\mathrm{sp}_-(g_1)\beta_0(g_2)\rangle_-$ is always zero and hence $E(\mathrm{sp}_-,\beta_0)$ is a representation of $\mathcal{A}ut(\mathcal{T})$ in $\mathrm{EXP}(H_{\mathrm{sp}_-})$.

Let Π be the representation of $\mathcal{G}^0$ built from the measure m and the positive constant M as in (3.23). Consider the Hilbert space H_M whose elements are the same as those of H_{sp_-} but the norm is given by

$$\|v\|^2_M = M\log(q)\langle v,v\rangle_- \tag{4.14}$$

Let $H^X = \int_X^\oplus H_x dm(x)\simeq L^2(X,dm)\otimes H_M$ denote the direct integral of spaces $H_x = H_M$ a.e. $[m]$. So that H^X is the completion of locally constant measurable mappings $\mathbf{v}: X\to H_{\mathrm{sp}_-}$ with respect to the norm

$$\|\mathbf{v}\|^2 = M\log(q)\int_X \langle v(x),v(x)\rangle_-\, dm(x) \ . \tag{4.15}$$

Define a cocycle $\beta_0^X : \mathcal{G}^0 \to H^X$ and a representation $\mathrm{sp}_-^X : \mathcal{G}^0 \to H^X$ by the rule

$$\beta_0^X(\xi)(x) = \beta_0(\xi(x)) \tag{4.16}$$

$$\mathrm{sp}_-^X(\xi)(\mathbf{v})(x) = \mathrm{sp}_-(\xi(x))v(x) \tag{4.17}$$

Define an homomorphism of $\mathcal{G}^0$ to the motion group of H^X in the obvios way: $\theta(\xi) = (\mathrm{sp}_-^X(\xi), \beta_0^X(\xi))$ and hence a representation $\Xi = E(\mathrm{sp}_-^X, \beta_0^X)$ of $\mathcal{G}^0$ in the space $\mathcal{H} = \mathrm{EXP}(H^X)$ by the rule

$$\begin{aligned} &\Xi(\xi)(\mathrm{EXP}(\mathbf{v})) = \\ &e^{\left(-\frac{1}{2}\|\beta_0^X(\xi)\|^2 - \langle \mathrm{sp}_-^X(\xi)v, \beta_0^X(\xi)\rangle\right)} \cdot \mathrm{EXP}(\mathrm{sp}_-^X(\xi)\mathbf{v} + \beta_0^X(\xi)) \; . \end{aligned} \tag{4.18}$$

Here is an irreducibility criterium for Ξ taken from Ismagilov [I]:

Theorem 4.1. *Let Y be a complex Hilbert space and m a probability measure on X with no atoms. Set $Y^X = L^2(X, dm) \otimes Y$. Assign to almost every x $[m]$ a unitary representation U_x of G into Y and a cocycle $\beta_x : G \to Y$ in such a way that*

- *the functions $(x, g) \to \langle U_x(g)h_1, h_2\rangle$ and $(x, g) \to \langle \beta_x(g), h\rangle$ are continuous in g for almost all fixed x and measurable in x for all fixed $g \in G$.*
- *There exists a continuous positive function $w : G \to \mathbb{R}^+$ such that*

$$\|\beta_x(g)\| \le w(g) \qquad a.e.[m]$$

Define a representation U^X and a cocycle β^X of the group $\mathcal{G}^0$ into Y^X by the rule

$$U^X(\xi)(\mathbf{v}) = U_x(\xi(x))v(x) \quad \beta^X(\xi)(\mathbf{v}) = \beta_x(\xi(x))v(x) \; . \tag{4.19}$$

Assume that a.e.[m]

- *The representations $U_x : G \to Y$ do not contain the trivial representation .*
- *The cocycle $\beta_x : G \to Y$ is absolutely nontrivial.*

Then the representation $E(U^X, \beta^X)$ of $\mathcal{G}^0$ in the Fock space $\mathrm{EXP}(Y^X)$ *is irreducible.*

Theorem 4.2. *The representation Ξ of $\mathcal{G}^0$ is irreducible.*

Proof. It is clear from (4.13) that the function $g \to \|\beta_0(g)\|$ is itself continuous from $\mathcal{A}ut(\mathcal{T})$ to $\mathbb{R}$. Since sp_- is an infinite dimentional irreducible representation of $\mathcal{A}ut(\mathcal{T})$ we may apply now the Theorem of Ismagilov taking $Y = H_M$, $U_x = \mathrm{sp}_-$ and $\beta_x = \beta_0$ for all x. □

Remark 4.3. We observe that, while the cocycle $\beta_0(g)$ is absolutely non trivial, the same is not true for Haagerup's cocycle $\beta_h(g)$ as it is showed in [KuV2]: $\beta_h(g)$ *is* trivial when restricted to $\ell^2(\mathcal{V})$.

5. Equivalence of $\mathcal{H}$ and $\mathrm{EXP}(H^X)$

We are now given two distinct irreducible representations of $\mathcal{G}^0$. The positive definite function Ψ corresponding to the vacuum vector in $\mathrm{EXP}(H^X)$ will play a central role. Taking (4.15) and (4.18) into account we get get

$$\Psi(\xi) = \langle \Xi(\xi)\,\mathrm{EXP}\,0, \mathrm{EXP}\,0\rangle_{\exp} = q^{-M\int_X \frac{1}{2}\|\beta_0(\xi(x))\|^2 dm(x)} \tag{5.1}$$

Theorem 5.1. *The representation* $(\Pi, \mathcal{H})$ *is equivalent to* $(\Xi, \mathrm{EXP}(H^X))$.

Proof. In order to show that the two representations are equivalent it is enough to construct a nonzero $\mathcal{G}^0$ map from one space to the other. Choose z with $0 < z < \frac{1}{2}$ and consider the positive definite function on $\mathcal{A}ut(\mathcal{T})$

(5.2)

$$\psi_z(g) = q^{-\frac{z}{2}\|\beta_0(g)\|^2} = q^{-zd(o, g\cdot o) + 2zq(1-q^{-d(o,g\cdot o)})/(q^2-1)}$$

Use (4.13) and (3.4) to get

$$\psi_z(g) \geq q^{-zd(o, g\cdot o)} = \frac{1}{c(z)}\phi_z(g) + -\frac{c(1-z)}{c(z)})q^{(z-1)d(o,g\cdot o)} \tag{5.3}$$

where $\phi_z(g)$ is the spherical function corresponding to the complementary series representation with parameter z.

Since $c(z)$ is positive while $c(1-z)$ is negative when $0 < z < \frac{1}{2}$ we also get

$$\psi_z(g) \geq q^{-zd(o, g\cdot o)} \geq \frac{\phi_z(g)}{c(z)} \tag{5.4}$$

Fix a partition ρ of X, say $X = A_1 \cup \cdots \cup A_n$. Let L_ρ denote the closed linear span, in $\mathrm{EXP}(H^X)$, of the vectors $\Xi(\xi)\,\mathrm{EXP}(0)$ with $\xi \in G_\rho$. Write $z_i = M \cdot m(A_i)$ and $g_i = \xi(x_i)$ for $x_i \in A_i$. From (5.4) and (5.3) we get

(5.5)

$$\frac{\phi_{z_1}(g_1)\ldots\phi_{z_1}(g_n)}{c(z_1)\ldots c(z_n)} \leq \psi_{z_1}(g_1)\ldots\psi_{z_n}(g_n) = q^{-M\frac{1}{2}\sum_{i=1}^n m(A_i)\|\beta_0(g_i)\|^2} =$$

$$q^{-M\frac{1}{2}\int_X \|\beta_0(\xi(x))\|^2 dm(x)}.$$

Set

$$\mathbf{1}_{z_i} = \frac{1}{\sqrt{c(z_i)}}\mathbf{1} \quad \mathbf{1} \in H_{z_i} \tag{5.6}$$

and let

$$\mathbf{1}_\rho = \otimes_{i=1}^n \mathbf{1}_{z_i} \ . \tag{5.7}$$

The above inequality (5.5) becomes

$$\langle \pi_\rho(\xi)\mathbf{1}_\rho, \mathbf{1}_\rho \rangle \le \langle \Xi(\xi)\,\mathrm{EXP}(0), \mathrm{EXP}(0)\rangle_{\exp} = \Psi(\xi) \tag{5.8}$$

So that the state $\Psi(\xi)$ dominates the positive definite function $\langle \pi_\rho(\xi)\mathbf{1}_\rho, \mathbf{1}_\rho \rangle$ and hence the map $T_\rho : L_\rho \to \mathcal{H}_\rho$ defined by

$$T_\rho(\mathrm{EXP}(0)) = (\mathbf{1}_\rho) \tag{5.9}$$

extends by linearity to a unitary equivalence between $\mathcal{H}_\rho$ and a subrepresentation of L_ρ.

Assume now that $\rho_1 > \rho$ is obtained from ρ by splitting a set, say A_1 into two pieces A_1^1 and A_1^2. Let $z_1^1 = M \cdot m(A_1^1)$ and $z_1^2 = M \cdot m(A_1^2)$, so that $z_1 = z_1^1 + z_1^2$. Construct L_{ρ_1}, $\mathbf{1}_{\rho_1}$ and T_{ρ_1} as before. By Remark 2.3 the injection of $\mathcal{H}_\rho$ into $\mathcal{H}_{\rho_1}$ is defined by

$$j_{\rho_1,\rho}(\mathbf{1}_{z_1} \otimes \mathbf{1}_{z_2} \dots \mathbf{1}_{z_n}) = \sqrt{\frac{c(z_1)}{c(z_1^1)c(z_1^2)}}v_{1,2} \otimes \mathbf{1}_{z_2} \dots \mathbf{1}_{z_n} \tag{5.10}$$

where $v_{1,2}$ is the unique positive vector in $H_{z_1^1} \otimes H_{z_1^2}$ such that
$\pi_{z_1^1} \otimes \ \pi_{z_2^1}(g)(\mathbf{1} \ \otimes \ \mathbf{1}) = \pi_{z_1}(g)\sqrt{\frac{c(z_1)}{c(z_1^1)c(z_1^2)}}v_{1,2}$.

Let $I_{\rho_1,\rho}$, $j_{\rho_1,\rho}$ denote respectively the inclusion of L_ρ into L_{ρ_1}, of $\mathcal{H}_\rho$ into $\mathcal{H}_{\rho_1}$. Consider the G_ρ action on all these spaces: by the definition of $T_\rho(\mathrm{EXP}(0))$ and the above properties of the map $j_{\rho_1,\rho}$ we may conclude that, for every $\xi \in G_\rho$ the following diagram is commutative:

$$\begin{array}{ccc} L_\rho & \overset{I_{\rho_1,\rho}}{\longrightarrow} & L_{\rho_1} \\ \downarrow\ T_\rho & & T_{\rho_1}\ \downarrow \\ \mathcal{H}_\rho & \overset{j_{\rho_1,\rho}}{\longrightarrow} & \mathcal{H}_{\rho_1} \end{array} \tag{5.11}$$

A repeated application of the above argument shows that there is an inclusion of the direct limit $\lim_{\to} \mathcal{H}_\rho$ into $\mathrm{EXP}(H^X)$. Since both representations are irreducible for $\mathcal{G}^0$ this inclusion is an equivalence. $\square$

6. Extensions of Ξ

6.1. **The group $\mathcal{G}$ of bounded measurable currents.** Fix any finite subtree $\mathcal{Y}$ and let

$$K_{\mathcal{Y}} = \{g \in \mathcal{A}ut(\mathcal{T}) \ : g \cdot v = v \qquad \forall v \in \mathcal{Y}\} \ . \tag{6.1}$$

Let us recall from the second section that compact subgroups of $\mathcal{A}ut(\mathcal{T})$ are obtained by taking finite unions and intersections of subgroups of the type $K_{\mathcal{Y}}$ as $\mathcal{Y}$ varies among all finite subtrees. Moreover, for any such $K_{\mathcal{Y}}$, $\mathcal{A}ut(\mathcal{T})$ is the union of a countable number of $K_{\mathcal{Y}}$ cosets. Let

$$X_N = \{v \in \mathcal{T} \ : d(v, o) \leq N\} \tag{6.2}$$

and

$$K_N = \{g \in \mathcal{A}ut(\mathcal{T}) \ : g \cdot v = v \qquad \forall v \in X_N\} \ . \tag{6.3}$$

A map $F : X \to \mathcal{A}ut(\mathcal{T})$ is said to be bounded if there exists an integer N and a finite number of cosets $g_0 K_N \dots g_r K_N$ such that $F(X) \subseteq \cup_{i=0}^r g_i K_N$. Measurability for F is defined as usual.

For every $g_1, g_2 \in \mathcal{A}ut(\mathcal{T})$: define

$$\Delta(g_1, g_2) = d(g_1 \cdot o, g_2 \cdot o) + \sum_{n=1}^{\infty} \sum_{d(o,v)=n} \frac{d(g_1 \cdot v, g_2 \cdot v)}{(q+1)^n [1 + d(g_1 \cdot v, g_2 \cdot v)]}$$

where $d(v', v)$ is the tree distance between the vertices v' and v. It is clear that Δ is a left invariant metric on $\mathcal{A}ut(\mathcal{T})$ generating the topolgy described in Section 2 .

For ξ_1, ξ_2 in $\mathcal{G}^0$ let

$$\delta(\xi_1, \xi_2) = \sup_{x \in X} \Delta(\xi_1(x), \xi_2(x)) \ . \tag{6.4}$$

The group of bounded measurable currents $\mathcal{G}$ is the completion of $\mathcal{G}^0$ with respect to the metric defined by (6.4): this will be clear from standard arguments concerning uniform convergence and the following

Proposition 6.1. *Let F be a measurable bounded $\mathcal{A}ut(\mathcal{T})$-valued function on X and $\epsilon > 0$. There exists $\xi \in \mathcal{G}^0$ such that $\delta(F, \xi) < \epsilon$.*

Proof. Choose N big enough so that $1/2(q+1)^N < \epsilon$. Since K_{j+1} is a subgroup of finite index in K_j, we may assume that the immage of F is contained in a finite union of cosets $g_i K_N$. Set $A_i = F^{-1}(g_i K_N)$ and let ξ be the function identically equal to g_i on each A_i. One has

$$F(x) \cdot v = g_i \cdot v = \xi(x) \cdot v \qquad \forall x \in A_i \quad \text{and } v \in X_N \tag{6.5}$$

and hence

$$\Delta(F(x),\xi(x)) \le \frac{1}{(q+1)^N}\sum_{n=1}^{\infty}\sum_{d(o,v)=N+n}\frac{d(F(x)\cdot v,\xi(x)\cdot v)}{(q+1)^n[1+d(F(x)\cdot v,\xi(x)\cdot v)]} < \epsilon \tag{6.6}$$

which implies $\Delta(F(x),\xi(x)) < \epsilon$. □

Let us denote by log the logaritm in base q. It is convenient to introduce the following notation:

$$\log(\psi_z(g)) = -\frac{z}{2}\|\beta_0(g)\|^2 = -zd(o,g\cdot o) + \frac{2qz}{q^2-1}(1-q^{-d(o,g\cdot o)}) \tag{6.7}$$

so that

$$\psi(g) = \psi_1(g) = q^{\log(\psi_1(g))} = q^{-\frac{1}{2}\|\beta_0(g)\|^2}\ . \tag{6.8}$$

The following Theorem guarantees that Ξ can be extended to all $\mathcal{G}$:

Theorem 6.2. *The positive definte function*

$$\langle \Xi(\xi)\,\mathrm{EXP}(0),\mathrm{EXP}(0)\rangle_{\mathrm{exp}} = q^{-\frac{M}{2}\int_X \|\beta_0(\xi(x))\|^2 dm(x)} \tag{6.9}$$

can be extended to $\mathcal{G}$.

Proof. Fix $F \in \mathcal{G}$: since $x \to \|\beta_0(F(x))\|^2$ is a bounded measurable function on X it is obvious that the integral $\int_X \|\beta_0(F(x))\|^2 dm(x)$ is convergent. Since the quantity $2q/(q^2-1)$ is always less than $4/3$ and $d(o,g\cdot o)$ is an integer, we have

$$-d(o,g\cdot o) \le \log(\psi(g)) \le -d(o,g\cdot o) + \frac{4}{3} \le d(o,g\cdot o) \qquad \text{if } \ g\cdot o \neq o\ . \tag{6.10}$$

Remembering that $\log(\psi(g)) = 0$ when $g\cdot o = o$ we may write

$$|\log(\psi(g))| \le d(o,g\cdot o) \le \Delta(g,e) \tag{6.11}$$

Assume now that $\xi_n \in \mathcal{G}^0$ is a Cauchy sequence and compute

$$\left|\int_X \|\beta_0(\xi_n(x))\|^2 dm(x) - \int_X \|\beta_0(\xi_m(x))\|^2 dm(x)\right|\ . \tag{6.12}$$

We may assume that there exists a finite partition $X = \cup_{j=1}^J A_j$ such that $\xi_n(x) = g_j^n$ and $\xi_m(x) = g_j^m$ if $x \in A_j$ so that (6.12) becomes:
(6.13)

$$\left|\sum_{j=1}^J m(A_j)(\|\beta_0(g_j^n)\|^2 - \|\beta_0(g_j^m)\|^2)\right| \leq \sum_{j=1}^J m(A_j)2\left|(\log(\psi(g_j^n)) - \log(\psi(g_j^m))\right| =$$

$$2\sum_{j=1}^J m(A_j)\left|\log(\frac{\psi(g_j^n)}{\psi(g_j^m)})\right| \ .$$

Assume that $d(o, g_j^n \cdot o) = k_j^n < d(o, g_j^m \cdot o) = k_j^m$ and write

$$\frac{\psi(g_j^n)}{\psi(g_j^m)} = q^{k_j^m - k_j^n} q^{\frac{2q(q^{-k_j^m} - q^{-k_j^n})}{q^2-1}} \tag{6.14}$$

Again, since $-1 \leq \frac{2q}{q^2-1}(q^{-k_j^m} - q^{-k_j^n}) < 0$ one has
(6.15)

$$|\log(\frac{\psi(g_j^n)}{\psi(g_j^m)})| \leq d(o, g_j^m \cdot o) - d(o, g_j^n \cdot o) \leq d(g_j^m \cdot o, g_j^n \cdot o) \leq \Delta(g_j^m, g_j^n) \leq \delta(\xi_m, \xi_n) \ .$$

Adding up all these quatities we get
(6.16)

$$\left|\int_X \|\beta_0(\xi_n(x))\|^2 dm(x) - \int_X \|\beta_0(\xi_m(x))\|^2 dm(x)\right| \leq 2\delta(\xi_n, \xi_m) \ .$$

For $F \in \mathcal{G}$, let ξ_n be a sequence in $\mathcal{G}^0$ converging to F with respect to the metric δ. The above inequality (6.16) shows that

$$\Psi(F) = \lim_{n\to\infty} \Psi(\xi_n) \tag{6.17}$$

is well defined. Since EXP(0) is a cyclic vector for Ξ, standard arguments concerning positive definite functions (see for example Lemma 3.5 of [G-G-V1]) give the result. □

7. The case of $PGL(2, \mathbb{Q}_p)$

Measurable currents taking values in a Lie group have been studied by I.M.Gel'fand, M.I.Graev,A.M.Vershik since 1974. The same authors are now planning to gather all results known hitherto on this subject in a forthcoming book. Since $PGL(2, \mathbb{Q}_p)$ is a group of "Lie type", this case will be included in the above mentioned book with plenty of details. Here we shall only outline how to pass from $\mathcal{A}ut(\mathcal{T})$ to $PGL(2, \mathbb{Q}_p)$.

It is well known that $PGL(2, \mathbb{Q}_p)$ can be embedded in $\mathcal{A}ut(\mathcal{T})$ in such a way that the maximal compact subgroup $PGL(2, \mathbb{O}_p)$ becomes a closed subgroup of K, the stabilizer of o. Using the description of the spherical series given in [F-N] it is possible to prove that the spherical series of $\mathcal{A}ut(\mathcal{T})$ restrict to the *spherical series* of $PGL(2, \mathbb{Q}_p)$: this depends on the fact that both can be described only by means of the action of the group in the *boundary* Ω. Using the description given in [KuV2], one can see that for the same reason also the *special* representation sp_ of $\mathcal{A}ut(\mathcal{T})$ restricts to the special representation of $PGL(2, \mathbb{Q}_p)$ (see [F-P] and [C-K]). It should be noticed, however, that notwithstanding the *discrete* series of $\mathcal{A}ut(\mathcal{T})$ *don't* restrict to the discrete series of $PGL(2, \mathbb{Q}_p)$ (see [C-K]) they still give rise to representations weakly contained in the regular representation of $PGL(2, \mathbb{Q}_p)$ and the arguments that we used can be transfered to the latter.

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Dipartimento di Matematica, Università di Milano "Bicocca", Via Cozzi 53 Ed.U5, 20126 Milano, ITALIA
E-mail address: mariagabriella.kuhn@unimib.it