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Iterated Admissibility as Solution Concept in Game Theory*

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Abstract

Admissibility, i.e. the deletion of weakly dominated strategies, is a highly controversial solution concept for non cooperative games. This paper proposes a complete theory of weak dominance and, contrary to almost all the literature on this topic, it provides positive results on foundations of iterated admissibility. The main contribution of this work is to show that (iterated) admissibility can be justified once payoffs' ties are seriously taken into considerations and players optimise taking into consideration the information provided by these ties, i.e. using strategic independent sets (Mailath *at al.* 1993) and conditional dominance (Shimoji and Watson 1998). In particular we prove that (iterated) maximal simultaneous deletion of weakly dominated strategies endogenously emerges as axiomatic characterization of iterated admissibility. As a consequence of this result, the paper provides axiomatic and Bayesian foundations of iterated admissibility, proves the logical consistency of (iterated) admissibility as solution concept in game theory, and that common knowledge of admissibility leads to iterated admissibility, showing why previous attempts using cautious behaviour were ineffective.

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1 Introduction

"Iterated dominance is perhaps the most basic principle in game theory" (Ho *et al.* 1998). Similar statements are easily found in old and recent economic literature. Unfortunately, while strict dominance has sound foundations and is generally accepted as basic criterion of rational behaviour, the idea that no rational player will ever play a weakly dominated strategy, i.e. the role of admissibility in game theory, is a matter of deep rooted discussion. While admissibility is the basic criterion of rational behaviour in decision theory (see e.g. Blackwell and Girshick 1954 and Ferguson 1967), among game theorists there exist two opposite approach about the deletion of weakly dominated strategies.

On one hand (for example Gale 1953, Herings and Vannetelbosch 1999, Hillas 1990, Ho *et al.* 1998, Kohlberg and Mertens 1986, Luce and Raiffa 1957, Van Damme 1987 and 1992) admissibility is seen as a necessary condition for any reliable solution concept in game theory.

On the other hand many authors (for example Abreu and Pearce 1984, Börgers and Samuelson 1992, Dekel and Gul 1997, Pearce 1982, Samuelson 1992 and 1993, Samuelson and Zhang 1992) consider admissibility and, especially, iterated admissibility as intrinsically problematic. These authors show that admissibility can conflict with non emptiness, that Bayesian, epistemic, axiomatic and evolutive foundations to iterated strict dominance do not generalise to iterated weak dominance, and finally that common knowledge of admissibility is incompatible with iterated admissibility (IA henceforth).

These results are particularly disturbing since IA has been extremely useful to solve interesting problems in economic theory. For example iterated admissibility has been used by Moulin 1979 and 1986 to define the concepts of dominance solvable game and of sophisticated equilibrium, which have important applications for example in the literature on voting schemes (Moulin 1979) and on auctions (Harstad and Levin 1985). Admissibility has also been extremely useful in implementation theory (see for example Abreu and Matsushima 1994 and Jackson *et al.* 1994). Furthermore, (iterated) admissibility is a crucial criterion in many refinements of Nash equilibrium. Finally it is worth to stress that admissibility is a crucial criterion in decision theory (e.g. Ferguson 1967).

So far no justification of iterated deletion of weakly dominated strategies in a conventional model of Bayesian optimisation and common knowledge has been given, apart from the use of lexicographic probability systems (Brandenburger 1992, Brandenburger and Keisler 2000, Stahl 1995, Veronesi 1994).

We believe that this is not the end of the story. Recently two papers by Marx and Swinkels 1997 and Shimoji and Watson 1998 (SW henceforth) have provided crucial insights on iterated admissibility. Building on these contributions and on the research program proposed by Mailath *et al.* 1993

(MSS henceforth) on the informative structures embedded in normal form games, it is possible to provide a sound comprehensive theory of iterated weak dominance for non cooperative games. In particular we prove that (iterated) maximal simultaneous deletion of weakly dominated strategies endogenously emerges as axiomatic characterization of (iterated) admissibility. As a consequence of this result this paper provides Bayesian and axiomatic justifications for iterated admissibility exploiting payoffs ties, i.e. the informative structures nested in normal form games. Moreover it shows that (iterated) admissibility is logically consistent and thus that it can be used as solution concept in game theory. Finally we prove that common knowledge of admissibility leads to iterated admissibility, showing why cautious behaviour is misleading as justification of iterated admissibility.

The main problems of (iterated) admissibility and the main idea of the paper are illustrated in section 2 by means of an example. Section 3 provides the basic notation and definitions. Section 4 analyses the relationships between strict and weak dominance by means of two characterization theorems, that illustrate the crucial role of information structures embedded in normal form games, in particular of strategic information sets. These results are used in section 5 to provide axiomatic foundations for iterated admissibility as iterated maximal simultaneous deletion of weakly dominated strategies. Then relying on this axiomatic foundation, section 6 shows that weak dominance doesn't conflict with reasonable axioms on solution concepts of a game, thus replying to Abreu and Pearce 1984 "inconsistency" theorem. Finally section 7 analyses the relation between iterative deletion procedures based on cautious Bayesian optimisation and iterated admissibility, showing how it is possible to avoid Börgers and Samuelson 1992 negative results. Some final remarks conclude the paper. In the paper there are no difficult proofs, but to make the reading easier, all the proofs are in the appendix.

2 Iterated Admissibility in Game Theory: An Example

Consider the game of figure 1, with $n \in \mathbb{N}$.

	L	R
T	$1+(1/n),\ 1$	$1+(1/n),\ 1+(1/n)$
M	$1,\ 1$	$0,\ 1+(1/n)$
B	$0,\ 0$	$1,\ 1$

Figure 1

This game is easily solvable by iterative elimination of strictly dominated strategies: the solution is the singleton (T,R).

The set of iteratively strictly undominated strategies has all the best properties we can hope for a solution concept: it does not depend on the order of deletion of the strategies (but see Dufwenberg and Stegeman 1999 for a significant correction), there are axiomatic, Bayesian and epistemic foundations (Bernheim 1984, 1986, Pearce 1984, and Dekel and Gul 1997), common knowledge of Bayesian rationality is equivalent to iterative deletion of strictly dominated strategies (Tan and Werlang 1988), learning and evolutionary arguments are sufficient to eliminate iteratively strictly dominated strategies (Milgrom and Roberts 1991 and Samuelson and Zhang 1992).

Consider the sequence of games as n takes values in $\mathbb{N}$. The limit game as n goes to infinity is pictured in figure 2.

	L	R
T	1, 1	1, 1
M	1, 1	0, 1
B	0, 0	1, 1

Figure 2

Strict admissibility is now ineffective to solve the game, since no strategy is anymore strictly dominated. The obvious refinement of strict dominance is weak dominance. But then none of the nice properties that characterize iterated strict admissibility holds.

If player 1 removes B because it is weakly dominated by T, then L is no more weakly dominated by R for player 2. Similarly if player 1 removes simultaneously all her dominated strategies (M and B), then player 2 is indifferent between L and R. Finally if player 2 starts removing L, then B is no more weakly dominated by T. Thus the result of the deletion of weakly dominated strategies depends on the order of elimination. There are also logical problem to provide axiomatic and Bayesian foundations to iterated admissibility: "to the extent that "admissibility" can be made common knowledge, it does not characterize iterated deletion of weakly dominated strategies, ...; rather it characterizes rationalizability with caution" (Dekel and Gul 1997 p. 135). The game of figure 2 illustrates the intuition behind these "negative" results. Cautious behaviour means that each player should consider everything as possible. Therefore player 1 should assign strictly positive probability to each of player 2's strategies and consequently player 1 should play T. Then, if admissibility is common knowledge player 2 should assign zero probability to M and B, but in the same time because of cautiousness, 2 should give all of 1's strategies positive probability: "there is a limit to the logical consistency of any solution concept for cautious strategic behaviour" (Pearce 1982, p. 22). Even evolutionary models do not provide a foundation for the elimination of weakly dominated strategies (see Samuelson 1993 and Samuelson and Zhang 1992, but also Marino 1997 and Marx 1999 for positive results).

The standard conclusion is that it is not possible to find justification for iterated admissibility as basic criterion for a solution concept in game theory. In turn this implies that "the only standard refinement with a somewhat appealing epistemic characterization is that of trembling-hand perfection, and then only for two-person games" (Dekel and Gul 1997 p. 164), a conclusion with dramatic implications for the literature on refinements of Nash equilibrium and for many applications.

This paper argues that this may not yet be the end of the story. Actually, the previous arguments simply show the problems of founding iterative admissibility as common knowledge of cautious behaviour. The fact is that cautious behaviour ignores payoffs' ties; we believe instead that admissibility can be understood just putting these ties at the core of the analysis, while all previous papers on admissibility in normal form games do not exploit these structures. This work shows that it is possible to find sound foundations for IA using specific information structures of normal form games, the strategic independent sets (SISs henceforth). According to the results of this paper, the basic behavioral principles behind strict and weak dominance are the same, i.e. standard Bayesian rational behaviour, but when there are payoffs' ties there is additional useful information that rational players should use, instead of ignoring it as done by the standard approach.

Consider the example of figure 2 to illustrate this simple basic idea. The usual approach based on cautious Bayesian rational behaviour assumes that player 2 has full support beliefs e.g. $\mu_2(T) = (1-\epsilon)/2$, $\mu_2(M) = (1-\epsilon)/2$ and $\mu_2(B) = \epsilon$, so that the weakly dominated strategy L is never a cautious best reply. The approach of this paper instead uses SISs: the weakly dominated strategy L is strictly dominated if the strategies are restricted to $\{L, R\} \times \{B\}$, where $\{L, R\} \times \{B\}$ satisfies particular conditions that define a strategic independent set for player 2. The key idea is to look for equivalence between admissibility in the original game and strict admissibility in the opportune restrictions. Then to justify (iterated) admissibility we simply refer to the standard tools used to justify (iterated) strict admissibility, connecting the restricted games to the original game. This line of research follows MSS suggestion that the ideas developed in their papers can have a productive role in the analysis of desirable properties of a normal form solution concept, and build on SW notion of conditional dominance.

3 Notation and Definitions

The subscript i refers to player $i \in N$, $-i$ to the $N \setminus \{i\}$ players, $\Delta(\cdot)$ and $\Delta^\circ(\cdot)$ are respectively the set of all probability measures on $\cdot$ and its interior.

A **normal form game** (NFG) G is defined as follows:

$$G := (U_i(s), \quad s \in S)_{i=1}^N, \qquad \text{where:}$$

- S is the set of pure strategy profiles s,$\Sigma_i := \Delta(S_i)$ is the set of player i's mixed strategy, and Σ is the set of mixed strategy profile. With the usual abuse of notation denote by S_i also the set of degenerated mixed strategy that concentrate all probability measure on a pure strategy. Finally $\Sigma(A)$ is the set of mixed strategies with carrier $A \subseteq S$;
- $U_i : \Delta(S) \to \mathbb{R}$ is player i's expected utility function;
- N is the set of players.

Assume that the sets N and S are finite, where N and S denote both the set and the number of elements. Many of the following results crucially depend on this assumption. In particular this assumption allows to define the set of all NFGs precisely. Fix a finite number of player N and their pure strategies S. Then the space of NFGs over this form is given by $\mathcal{G}$, and we take $\mathcal{G} = \mathbb{R}^{N \times S}$ where for $x \in \mathcal{G}$, $x(i,s)$ is the payoff to player i under strategy s.

Consider a given $G = (U_i(s), s \in S)_{i=1}^N$. The **restriction of a normal form game** G to $A \subseteq \Delta(S)$, denoted by G^A, is:

$$G^{A(i)} := (U_i(s), s \in A)_{i=1}^N.$$

Therefore the restriction of a game is obtained reducing the set of random strategy profiles to a subset of $\Delta(S)$.

Note that the analysis is not restricted to reduced normal forms, as MSS do: the following results hold for any NFG (I am grateful to Jean Francois Mertens for having pointed out to me the relevance of this fact).

The specific solution concept we analyse in this paper is the following:

Definition 1 *The* **set of iteratively weakly undominated strategies** *IWUS for a game $G \in \mathcal{G}$ is so defined:*

$$WD(G) := \times_{i \in N} WD_i(G), \quad \textit{where} \quad WD_i(G) := \cap_{t \geq 0} WD_i^t(G),$$

$$\text{with } WD_i^0(G) := S_i \quad \text{and for } t \geq 1$$

$$WD_i^t(G) := \{s_i \in WD_i^{t-1}(G) \mid \not\exists \ \sigma_i \in \Sigma(WD_i^{t-1}(G)) \quad \text{such that}$$

$$(i) \qquad U_i(\sigma_i, s_{-i}) \geq U_i(s_i, s_{-i}) \quad \text{for every } s_{-i} \in WD_{-i}^{t-1}(G)$$

$$(ii) \qquad U_i(\sigma_i, s_{-i}) > U_i(s_i, s{-i}) \quad \text{for some } s_{-i} \in WD_{-i}^{t-1}(G) \ \}.$$

The **set of iteratively strictly undominated strategies** *ISUS for a game $G \in \mathcal{G}$ is defined similarly to IWUS using the sets $SD_i^t(G)$ instead of $WD_i^t(G)$ and with strict inequalities.*

$WD^1(G)$ and $WD(G)$ are respectively the set of admissible and of iteratively admissible strategies of a game G.

Note that in definition 1 it is assumed that at each stage all weakly dominated strategies are simultaneously deleted. This definition of IA is the most common, but it might seem arbitrary, theorem 3 however will provide a justification for its use.

The following standard existence result is presented because non emptiness is a necessary condition for logical consistency of a solution concept.

Proposition 1 *In every game $G \in \mathcal{G}$, there exists a finite number $K \in I\!N$ such that $\forall n \geq K \quad WD_i^n(G) = WD_i(G) \neq \emptyset$ for every $i \in N$.*

Proof: see the appendix. ♡

4 Strategic Independent Sets, Strict and Weak Dominance

The main results of this section are implied by the analysis of SW, even if they were independently developed. SW provide a general theory of conditional dominance based on the notion of augmented normal form games, but exactly because of this wide scope, they briefly mention weak dominance, without providing any characterisation result as we do. In particular they connect weak dominance to normal form information sets, while, as our results make clear, the most effective notion to use is that of SISs. The specific reason of this effectiveness is that IA as well as SISs refer to each single player, while normal form information sets refer to all players jointly (but note that in games satisfying Marx and Swinkels 1997 TDI condition, SISs and normal form information sets coincide). In this section we show that weak dominance is equivalent to SW conditional dominance when we use SISs as restrictions (theorem 1), and therefore that IWUS is a particular version of the set of strategies surviving iterated conditional dominance (theorem 2). Some of the following results follow from SW main result. This notwithstanding, we prefer to propose our original proofs because they are simpler since we don't have to deal with general augmented normal form games.

Roughly the definition of SIS for player i says that a set of strategy profiles $X_i \times X_{-i}$ is strategically independent for player i if i can make decision over X_i conditional on X_{-i} independently of decisions over X_i conditional on $S_{-i} \setminus X_{-i}$. Unfortunately MSS definition is unduly restrictive, since they limit i's choice set to pure strategies: while it is not restrictive to assume $X_{-i} \subseteq S_{-i}$, since from i's point of view whether the opponents choose between pure or mixed strategies is irrelevant (see Pearce 1984 lemma 2), player i's choice's possibilities are different if she can mix, as the example of figure 3 will show.

Definition 2 *In the game $G = (U_i(s), s \in S)_{i=1}^N$ the set $X(i) \subseteq \Sigma_i \times S_{-i}$ is* **strategically independent for player** *i if*

$$X(i) = X_i \times X_{-i}$$

and $\forall r_i, s_i \in X_i, \;\; \exists t_i \in X_i$ *such that*

$$U_i(t_i, s_{-i}) = U_i(r_i, s_{-i}) \quad \forall s_{-i} \in X_{-i}$$

$$U_i(t_i, s_{-i}) = U_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i} \setminus X_{-i}$$

In a NFG G, the family of all strategic independent sets for player i is denoted by $\mathcal{X}(i, G)$.

Thus if $X(i)^M$ is strategically independent for i, there is a (mixed) strategy t_i in X_i^M that is equivalent, from i's point of view, to r_i if her opponents choose X_{-i} and to s_i if they do not. Hence, if i has beliefs μ_{-i} over S_{-i} with γ_{-i} and ξ_{-i} being i's beliefs conditioned on X_{-i} and $S_{-i} \setminus X_{-i}$, then when r_i is optimal given γ_{-i} and s_i is optimal given ξ_{-i}, t_i is optimal given μ_{-i}. In other words a set of strategy profiles $X_i \times X_{-i}$ is strategically independent for player i if i can make decision over X_i conditional on X_{-i} independently of decisions over X_i conditional on $S_{-i} \setminus X_{-i}$.

To illustrate how definition 2 works, consider the game of figure 3 where only 1's payoffs are reported.

	α	β	γ
A	2	0	5
B	0	4	3
C	1	2	-1

Figure 3

If player 1 is restricted to SISs of the form $X(1) \subseteq S_1 \times S_{-1}$ as in MSS definition 2, then the only strategic independent set for player 1 trivially is $S_1 \times S_{-1}$, while if player 1 can choose mixed strategies, then e.g. $\{(1/2)A \oplus (1/2)B,\ C\} \times \{\gamma\}$ is a strategic independent set for player 1, since

$$U_1((1/2)A \oplus (1/2)B,\ s_{-1}) = U_1((1/2)A \oplus (1/2)B,\ s_{-1}) \quad \forall s_{-1} \in X_{-1} = \{\gamma\}$$

$$U_1(C,\ s_{-1}) = U_1(C,\ s_{-1}) \quad \forall s_{-1} \in X_{-1} = \{\gamma\}$$

$$U_1((1/2)A \ \oplus\ (1/2)B,\ s_{-1}) = U_1(C,\ s_{-1}) \quad \forall s_{-1} \in S_{-1} \setminus X_{-1} = \{\alpha, \beta\},$$

where $x\,s' \ \oplus (1-x)\,s''$ is the distribution that put probability x on s' and $(1-x)$ on s''. Indeed our definition is coherent with MSS aim in providing their definition, since "there is a strategy t_i in X_i that is equivalent, from i's point of view, to r_i if his opponents choose X_{-i} and to s_i if they do not" (MSS p. 279). This generalisation is moreover coherent with the requirement of invariance as specified in the notion of stable equilibria (see e.g. Kohlberg and Mertens 1986). According to the following theorem, weak and strict dominance are related through the notion of strategic independent set; this relation is the basic ingredient of all subsequent results.

Theorem 1 *In every game $G \in \mathcal{G}$, for each player $i \in N$, a strategy s_i is weakly undominated if and only if it is strictly undominated in $G^{X(i)}$ for every $X(i) \in \mathcal{X}(i, G)$.*

Proof: see the appendix. ♡

The terminology used in definition 3 is taken from SW (the expressions used in the previous version of this paper has been abandoned).

Definition 3 *For any game $G \in \mathcal{G}$, the* **set of iteratively conditionally undominated strategies** *on $\mathcal{X}(G)$ is so defined:*

$$Q(\mathcal{X}(G)) := \times_{i \in N} Q_i(\mathcal{X}(i, G)), \ \textit{where}\ Q_i(\mathcal{X}(i, G)) := \cap_{t \geq 0} UD_i^t(\mathcal{X}(i, G)),$$

$$\textit{with}\ UD_i^0(\mathcal{X}(i, G)) := S_i, \quad \textit{and for}\ t \geq 1$$

$$UD_i^t(\mathcal{X}(i, G)) := \{s_i \in UD_i^{t-1}(\mathcal{X}(i, G)) \mid \forall X(i) \in \mathcal{X}(i, G^{UD^{t-1}(\mathcal{X}(G))})\ s_i \in SD_i^{t-1}(G^{X(i)})\}.$$

Roughly, (iterated) conditional dominance on $\mathcal{X}(G)$ requires the use of (iterated) strict admissibility for each possible game restricted to a SIS (or in SW to a generic restriction).

Now we are able to characterise iterated admissibility as iteratively conditionally undominated strategies on SISs.

Theorem 2 *For every game $G \in \mathcal{G}$ $\ WD(G) = Q(\mathcal{X}(G))$.*

Proof: see the appendix. ♡

Note that it is difficult to apply the definition of iteratively conditionally undominated strategies on $\mathcal{X}(G)$ since the enumeration of all SISs is usually extremely complex. Therefore this is not an operationally relevant definition, but it is extremely useful as auxiliary construction to justify iterated admissibility.

5 Axiomatic Foundations of Iterated Admissibility

The previous sections have shown that the most effective way of founding iterated admissibility is to consider it as iterated conditional dominance on strategic independent sets. Thus we can justify IA using the tools developed to found iterated strict dominance. In particular we use the framework of Bernheim 1986 to provide axiomatic foundations for IWUS.

Definition 4 *A* **solution concept** *is a correspondence from the set of all normal form games to the set of possible strategies:*

$$T : \mathcal{G} \to\to S.$$

Therefore a solution concept associates with each game $G \in \mathcal{G}$ the strategy profiles arising from some theory of (rational) choice. Moreover let $T_i(G)$ be the projection of $T(G)$ into the i-th subspace

$$T_i(G) := \{s_i \in S_i | (s_i, s_{-i}) \in T(G) \text{ for some } s_{-i} \in S_{-i}\},$$

i.e. $T_i(G)$ represents the choices that i might make according to the solution concept T, given a NFG G. Indicate with $\tau_{-i}(s_i, T|X(i))$ the set of strategy profiles for player other than i given that i chooses s_i, that the players are restricted to $X(i)$ and that they play according to the solution concept T:

$$\tau_{-i}(s_i, T|X(i)) := \{s_{-i} \in S_{-i} | (s_i, s_{-i}) \in T(G^{X(i)})\}.$$

Finally, let $F_i[P_i]$ be i's best reply correspondence to $P_i \in \Delta(S_{-i})$.

Axiom 1 Conditional Bayesian rationality: *for any game G, for any player i, for any strategic independent set $X(i) = X_i \times X_{-i}$, there exists a mapping*

$$\theta_i : T_i(G) \to \Delta(X_{-i}) \text{ such that } \forall s_i \in T_i(G) \;\; s_i \in F_i(\theta_i(s_i)).$$

Bayesian rationality demands that each player selects a best reply to some probabilistic beliefs about opponents' choices. Axiom 1 requires that a solution concept satisfies Bayesian rationality in every strategic independent set of a given NFG G. Note that player i's beliefs can differ between SISs.

This is surely a novelty of this axiom with respect to the standard axiom of Bayesian rationality, but it is clearly coherent with the meaning of SIS. As MSS write "the requirement of sequential rationality does not seem very different from a general requirement that if a decision only matters given some subset of the strategy profiles for the remaining players, then that decision should be optimal relative to some conjecture over those strategy profiles. These are *precisely* the situations characterised by strategic independence. Rational players should exploit strategic independence in their decision making, *even* when the strategic independence is not due to an extensive form information set or subgame" (p.288). Actually the recent work by Battigalli and Siniscalchi 1999 on epistemic foundations of solution concepts for extensive form games suggest that the most effective way of modelling beliefs in games augmented for SISs is by means of conditional probability systems, where the players condition their beliefs to $X(i)$.

Next axiom places additional restrictions on the maps θ_i, whose existence (for any $X(i)$) is guaranteed by axiom 1.

Axiom 2 Conditional consistency: *for any game G, for any player i, for any strategic independent set $X(i) \in \mathcal{X}(i, G)$, for any strategy $s_i \in T_i(G)$*

$$[\theta_i(s_i)](\tau_{-i}(s_i, T|X(i))) = 1.$$

This axiom requires that the players in every strategic independent set of G do not attribute positive probability to any opponents' choices which are not coherent with the solution concept T.

Finally a solution concept T is **acceptable** (Bernheim [4] uses the word admissible instead of acceptable, but here we need to change terminology to avoid confusion) under a certain set of axioms if it satisfies those axioms. Moreover let $\mathcal{T}(G) := \{s \in S | s \in T \quad \text{for some acceptable } T(G)\}$.

Theorem 3 *Under axioms 1 and 2, for any game G, $\mathcal{T}(G) = WD(G)$. Furthermore WD is acceptable.*

Proof: see the appendix. ♡

Therefore theorem 3 provides an axiomatic characterisation of iterated admissibility as maximal simulataneous deletion of iterated weakly dominated strategies, i.e. this order of deletion endogenously emerges as axiomatic characterization of IA. This is an important results for at least four reasons. First, the economists who apply admissibility to solve specific problems actually use maximal simultaneous deletion of weakly dominated strategies (see e.g. Abreu and Matsushima 1994, Jackson *et al.* 1994, Moulin 1979 and 1986). Second, even if order is not an issue in decision theory, the complete class theorem, a crucial result in statistics, refers to maximal deletion of weakly dominated strategies (see e.g. Ferguson 1967 p. 58). Third, the first proponents of this criterion for game theory (e.g. Gale 1953 and

Luce and Raiffa 1957) consider maximal simultaneous deletion. The same is true for more recent advocates of this criterion as a tool to refine solution concepts (see Blume *et al.* 1991, Herings and Vannetelbosch 1999, Van Damme 1987 and 1992). Finally, in "limit" games this particular order generates the same solution of iterated strict admissibility applied to games in the sequence (see the example of figures 2 and 1). Thus strict and weak dominance are connected in an effective and "continuous" way.

6 Consistency of (Iterated) Admissibility as Solution Concept

Abreu and Pearce 1984 proved that weak dominance may conflict with non emptiness, but to reach this result they use an axiom which allows non simultaneous and non maximal deletion of weakly dominated strategies. This section shows how order dependence is exploited by Abreu and Pearce to conclude that "the results here rule out [weak] dominance as a satisfactory criterion: it conflicts with non emptiness, a truly innocuous axiom" (p. 173). We instead prove that iterated admissibility as axiomatized in section 5 does not conflict with non emptiness, and thus that it can be used as useful criterion for solution concepts in game theory.

In their paper Abreu and Pearce refer to both extensive and normal form games because they also consider subgame perfection. Since we are not directly interested in this topic, we re-propose their axioms and their result for normal form games only.

Abreu and Pearce propose the following axioms.

Axiom 3 Non emptiness: *for any game $G \in \mathcal{G}$,*

$$T(G) \neq \emptyset.$$

Axiom 4 Abreu and Pearce axiom A3 (i): *for any game $G \in \mathcal{G}$,*

$$T(G) \subseteq WD^1(G).$$

Axiom 5 Abreu and Pearce axiom A3 (ii): *for any game $G \in \mathcal{G}$, and for every set A such that $WD^1(G) \subseteq A \subseteq S$*

$$T(G^A) = T(G) \cap A.$$

The first axiom means that the solution concept is well defined for every NFG, the second means that the solution concept must satisfy admissibility (a similar axiom is often used in decision theory, see for example axiom 5 p. 291 in Luce and Raiffa 1957), while the third requires that a solution concept should not be affected by the deletion of **one or more** weakly dominated strategy.

The following result is Abreu and Pearce proposition 1, and shows that the previous intuitive requirements on a solution concept are mutually contradictory.

Proposition 2 *There exist no solution concept T satisfying axioms 3, 4 and 5.*

Proof: See the appendix. ♡

The problem at the root of Abreu and Pearce inconsistency result was clearly known to Kohlberg and Mertens 1986 (p. 1014-1015), who tackle the problem assuming instead of axioms 3 an 4 that for all games $G \in \mathcal{G}$, and for every set A such that $WD^1(G) \subseteq A \subseteq S$

$$T(G^A) \subseteq T(G),$$

where $T(G)$ denotes a strategically stable set of equilibria of G. Applied to the game of figure 4, this condition implies $\{\alpha_1, \alpha_2\} \times \{ad\} \subseteq T(G)$ and thus the set of strategically stable equilibria may contain non admissible strategies.

	ac	ad	bc	bd
α_1	1,1	1,1	1,0	1,0
α_2	0,0	1,1	0,0	1,1

Figure 4

This paper follows a different path, since we want to argue for (iterated) admissibility. Consider the following specification of axiom 5:

Axiom 6 *For any game $G \in \mathcal{G}$*

$$T(G^{WD^1(G)}) = T(G) \cap WD^1(G).$$

Axiom 6 restricts axiom 5 to hold for simultaneous maximal deletion of weakly dominated strategies, i.e. for admissible strategies as characterized by theorem 3.

Now we are able to show that (iterated) admissibility is a satisfactory criterion in the sense of Abreu and Pearce 1984.

Proposition 3 *Axioms 3, 4 and 6 are mutually consistent.*

Proof: see the appendix. ♡

Note that all previous axioms and proofs work as well if $WD(G)$ instead of $WD^1(G)$ is considered. Therefore from this point of view there is no logical distinction between admissibility and iterated admissibility, the crucial point is the order of deletion, endogenously justified by axioms 1 and 2 and by theorem 3. Proposition 3 shows that (iterated) admissibility not only has axiomatic foundations, but solves Abreu and Pearce 1984 "impossibility" theorem.

7 Cautious Bayesian Optimisation and Iterated Admissibility

Bernheim 1984 and Pearce 1984 in their seminal work on rationalizability propose different equivalent versions of their solution concept. In particular they show the equivalence between iterative procedures and a definition in terms of fixed point of a correspondence. Moreover they show that correlated rationalizability (rationalizability henceforth) is equivalent to iterated strict admissibility. Since it is well known that cautious Bayesian optimisation is equivalent to admissibility (Pearce 1984 and Blackwell and Girshick 1954), can we conclude that a cautious version of rationalizability is equivalent to iterated admissibility? As we show in this section, the answer is positive if we consider the iterative definition of rationalizability, negative (Börgers and Samuelson 1992) if we consider the equivalent definition of rationalizability as maximal fixed point of a sort of best reply correspondence. In this section we will show that it is possible to provide an opportune generalisation of Bernheim and Pearce approach so to encompass iterated admissibility as maximal fixed point of the opportune best reply correspondence. Thus we show how to avoid the negative result by Börgers and Samuelson 1992.

Definition 5 *The* **set of rationalizable strategies** R *for a game* $G \in \mathcal{G}$ *is:* $R(G) := \times_{i \in N} R_i(G)$, *where*

$$R_i(G) := \cap_{t \geq 1} R_i^t(G),\ R_i^0(G) := S_i \ \text{ and for } \ t \geq 1$$

$$R_i^t(G) := \{s_i \in S_i |\ \exists \mu_i \in \Delta(R_{-i}^{t-1}(G)) \ \text{ such that } \ s_i \in F_i(\mu_i)\}.$$

Similarly the **set of cautious rationalizable strategies** $\overline{R}$ *for a game* $G \in \mathcal{G}$ *is:* $\overline{R}(G) := \times_{i \in N} \overline{R}_i(G)$, *where*

$$\overline{R}_i(G) := \cap_{t \geq 1} \overline{R}_i^t(G),\ \ \overline{R}_i^0(G) := S_i \ \text{ and for } \ t \geq 1$$

$$\overline{R}_i^t(G) := \{s_i \in S_i |\ \exists \mu_i \in \Delta^\circ(\overline{R}_{-i}^{t-1}(G)) \ \text{ such that } \ s_i \in F_i(\mu_i)\}.$$

The following result follows from well known theorems both in the game and in the decision theoretic literature.

Proposition 4 *For every game $G \in \mathcal{G}$*

$$R(G) = SD(G) \quad \textit{and} \quad \overline{R}(G) = WD(G).$$

Proof: see the appendix. ♡

An alternative definition of (cautious) Bayesian rational behaviour under common knowledge of (cautious) Bayesian rationality requires to consider the largest set R' ($\overline{R}'$) for which

1. if i thinks her opponents will choose a strategy in R'_{-i} (in $\overline{R}'_{-i}$ with strictly positive probability), then i will play in R'_i ($\overline{R}'_i$);
2. any choice in R'_i ($\overline{R}'_i$) is a best response to a (full support) belief on R'_{-i} ($\overline{R}'_{-i}$).

This idea has been formalized by Bernheim 1984 by means of the operator Λ and by Börgers and Samuelson 1992 by means of the operator $\overline{\Lambda}$.

Definition 6 *For any game $G \in \mathcal{G}$ let the mappings $\Lambda : S \to\to S$ and $\overline{\Lambda} : S \to\to S$ be defined for any $B \subseteq S$ by*

$$\Lambda(B) := \times_{i=1}^{N}\{s_i \in S_i \mid \exists \ \mu_i \in \Delta(B_{-i}) \quad \textit{such that} \ \ s_i \in F_i(\mu_i)\}$$

$$\overline{\Lambda}(B) := \times_{i=1}^{N}\{s_i \in S_i \mid \exists \ \mu_i \in \Delta^{\circ}(B_{-i}) \quad \textit{such that} \ \ s_i \in F_i(\mu_i)\}.$$

Define recursively
$\Lambda^k(B) := \Lambda(\Lambda^{k-1}(B))$ *where* $\Lambda^0(B) = B$ *and*
$\overline{\Lambda}^k(B) := \overline{\Lambda}(\overline{\Lambda}^{k-1}(B))$ *where* $\overline{\Lambda}^0(B) = B$.

Clearly for any game $G \in \mathcal{G}$

$$R(G) = \cap_{k\geq 1}\Lambda^k(S) \quad \textit{and} \quad \overline{R}(G) = \cap_{k\geq 1}\overline{\Lambda}^k(S).$$

The following definitions of R' and $\overline{R}'$ try to model the same idea behind R and $\overline{R}$, i.e. the characterisation of the set of strategies that each (cautious) rational player can choose under common knowledge of (cautious) Bayesian rationality.

Definition 7 *For any game $G \in \mathcal{G}$, define $R'(G)$ as the maximal set $B \subseteq S$ satisfying $B = \Lambda(B)$, similarly define $\overline{R}'(G)$ as the maximal set $\overline{B} \subseteq S$ satisfying $\overline{B} = \overline{\Lambda}(\overline{B})$.*

A minimal consistency requirement is the coincidence between the two alternative definitions in terms iteration (definition 5) and of maximal fixed point (definition 7). This is what is actually proved by Bernheim, but unfortunately this is not true for cautious behaviour, as shown by Börgers and Samuelson 1992:

Proposition 5 *For any game $G \in \mathcal{G}$ $R'(G) = R(G) \neq \emptyset$. On the other hand, there exist games $G \in \mathcal{G}$ such that*

1. $\overline{R}'(G) = \emptyset$;
2. *there exist fixed points of $\overline{\Lambda}$, but none of them is maximal;*
3. $\overline{R}'(G) \neq \overline{R}(G)$.

Proof: See the appendix. ♡

The reason of this discrepancy between iterated Bayesian rationality and iterated cautious Bayesian rationality lies in the different mathematical properties of the two operators Λ and $\overline{\Lambda}$ in terms of monotonicity. The operators Λ and $\overline{\Lambda}$ seems very similar, but they do not share a crucial property, monotonicity, which is at the root of the negative results on IA derived in Börgers and Samuelson 1992 and thus of the difficulties in founding iterated weak dominance in terms of cautious Bayesian behaviour.

Proposition 6 *The operator Λ is monotone, i.e.* $A \subseteq B \Rightarrow \Lambda(A) \subseteq \Lambda(B)$, *while the operator $\overline{\Lambda}$ is not monotone.*

Proof: See the appendix. ♡

Should we conclude, as Börgers and Samuelson do, that the existence of a justification for IA is unlikely?

We don't think so: this result shows that it is not possible to found IA simply adjusting for cautious behaviour the arguments used for iterated strict dominance. The reason lies exactly in the paradox stressed by Pearce 1982 p. 21-22 and illustrated in this paper in section 2: in the game of figure 2, if it is common knowledge that the players choose admissible strategies, then player 2 can rationally choose L, which is not admissible. On the other hand, if there is the slightest doubt about what the players might choose, they must play admissible strategies, but this removes all doubt. This simply states that $\overline{\Lambda}$ is not monotone. Therefore if we look for justification of iterated admissibility in terms of common knowledge of (some version of) Bayesian rationality, we should consider a different construction. Till now, game theorists have referred to lexicographic probabilities (see Blume *et al.* 1991, Brandenburger 1992, Brandenburger and Keisler 2000, Stahl 1995, Veronesi 1994), but the previous sections show that it is possible to justify IA also in terms of standard Bayesian rationality using the informative structures embedded in the definition of NFGs. This suggest that the attempt of Börgers and Samuelson may be newly formulated using a different operator defined in terms of strategic independent sets.

Definition 8 *For any game $G \in \mathcal{G}$ let the mapping $\Lambda[\mathcal{X}] : S \rightarrow\rightarrow S$ be defined for any $B \subseteq S$ by*

$$\Lambda[\mathcal{X}](B) := \times_{i=1}^{N}\{s_i \in S_i \mid \forall X(i) \in \mathcal{X}(i,G)\ \exists\ \mu_i \in \Delta(B_{-i} \cap X_{-i})$$

$$such\ that\ s_i \in F_i(\mu_i).\}$$

Definition 9 *For any finite normal form game* $G \in \mathcal{G}$, *define* $\overline{R}''(G)$ *as the maximal set* $B \subseteq S$ *satisfying* $B = \Lambda[\mathcal{X}](B)$.

Theorem 4 *For any finite normal form game* $G \in \mathcal{G}$

$$\overline{R}(G) = \overline{R}''(G) \neq \emptyset.$$

Proof: See the appendix. ♡

8 Conclusion

This paper shows that that the paradoxes in justifying iterated admissibility stem from the assimilation of weak dominance to cautious Bayesian behaviour, while if admissibility is interpreted as conditional dominance on SISs, then it is possible to provide sound foundations. The failure so far in the research of sound justifications for iterated admissibility depends on the fact that in the previous analysis normal form games has been treated as undifferentiated thing, while MSS have shown that the payoffs' structure provides information, which can be useful to define rational behaviour, as clearly shown by the seminal paper of SW. In other words we argue for a change of perspective: admissibility not as cautious behaviour, but as behaviour conditional on SISs. This means to take payoffs' ties seriously. The paper shows that the basic behavioral principles behind strict and weak dominance are the same, i.e. standard Bayesian rational behaviour, but when there are payoffs' ties there is additional useful information that rational players should use, instead of ignoring it as done by the standard approach based on cautious behaviour.

The principal topics omitted in this paper are the epistemic and evolutive foundations for the iterative elimination of weakly dominated strategies. Epistemic foundations are analysed in the recent paper by Brandeburger and Keisler 2000 where they use lexicographic decision theory, while an analysis of the possible evolutive foundations of iterated admissibility is in Marino 1997 and in Marx 1999. In our future works we will test the approach of this paper trying to provide epistemic foundations for iterated admissibility using Battigalli and Siniscalchi 1999 hierarchies of conditional probability systems conditional to strategic information sets.

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9 APPENDIX

9.1 Notation and Definitions

First the standard proof of non emptiness and thus of logical consistency of iterated admissible strategies

Proposition 1 *In every finite game $G \in \mathcal{G}$, there exists a finite number $K \in I\!N$ such that $\forall n \geq K \quad WD_i^n(G) = WD_i(G) \neq \emptyset$ for every $i \in N$.*

Proof: the proof is by induction. First since $S_i \neq \emptyset$, then $WD_i^0(G)$ is not empty. Suppose that $WD_i^{t-1}(G) \neq \emptyset$ and that $WD_i^t(G) = \emptyset$. Then for every $s_i \in WD_i^{t-1}(G)$ there exists a $\sigma_i \in \Sigma(WD_i^{t-1}(G))$ such that conditions (i) and (ii) in definition 2 hold. Thus for any s_{-i} there exists no maximum of U_i on $WD_i^{t-1}(G)$ and this is impossible since U_i is continuous and $WD_i^{t-1}(G)$ compact. Therefore $WD_i^{t-1}(G) \neq \emptyset$ implies $WD_i^t(G) \neq \emptyset$ and thus $\forall t \in \mathbb{N} \quad WD_i^t(G) \neq \emptyset$. Moreover by definition $WD_i^0(G) \supseteq WD_i^1(G) \supseteq \cdots$. Since S_i is finite, there exists a $K \in \mathbb{N}$ such that $WD_i^K(G) = WD_i^t(G)$ for every $t \geq K$. Thus for any $n \geq K \quad WD_i^n(G) = WD_i(G) \neq \emptyset$. ♡

A generalisation of this proof to compact strategy sets with continuous payoff functions is in Moulin 1986 (lemma 1, p. 58), while a simple example of non existence with discontinuous payoff functions is in Luce and Raiffa 1957 p. 317.

9.2 Strategic Independent Sets, Strict and Weak Dominance

Now we prove the two characterization theorems, which are crucial for all the further results.

Theorem 1 *In every game $G = (U_i(s), s \in S)_{i=1}^N \in \mathcal{G}$, for each player $i \in N$, a strategy s_i is weakly undominated if and only if it is strictly undominated in $G^{X(i)}$ for every $X(i) \in \mathcal{X}(i, G)$.*

Proof: Fix a NFG $G = (U_i(s), s \in S)_{i=1}^N$, and a player $i \in N$.
IF: we need to prove that

$$\forall X(i) \in \mathcal{X}(i,G) \quad s_i \in SD_i^1(G^{X(i)}) \quad \Rightarrow \quad s_i \in WD_i^1(G).$$

Consider the contrapositive of this implication:

$$s_i \notin WD_i^1(G) \quad \Rightarrow \quad \exists X(i) \in \mathcal{X}(i,G): \; s_i \notin SD_i^1(G^{X(i)}).$$

If s_i is strictly dominated in G, then the implication is trivially true for $X(i) = S_i \times S_{-i}$. Therefore suppose that s_i is weakly but not strictly dominated. Then there exists a strategy σ_i' and a non empty set of opponents' strategy profile $\hat{S}_{-i} \subseteq S_{-i}$ such that

$$U_i(s_i, s_{-i}) = U_i(\sigma_i', s_{-i}) \quad \forall s_{-i} \in \hat{S}_{-i}$$

$$U_i(s_i, s_{-i}) < U_i(\sigma_i', s_{-i}) \quad \forall s_{-i} \notin \hat{S}_{-i}. \tag{1}$$

Consider the set $X(i)^M = \{s_i, \sigma_i'\} \times S_{-i} \setminus \hat{S}_{-i} = X_i^M \times X_{-i}$: by construction

$$U_i(s_i, s_{-i}) = U_i(s_i, s_{-i}) \quad \forall s_{-i} \in X_{-i}(= S_{-i} \setminus \hat{S}_{-i})$$

U$_i(\sigma_i', s_{-i}) = U_i(\sigma_i', s_{-i}) \quad \forall s_{-i} \in X_{-i}(= S_{-i} \setminus \hat{S}_{-i}) U_i(s_i, s_{-i}) = U_i(\sigma_i', s_{-i}) \quad \forall s_{-i} \notin S_{-i} \setminus X_{-i} \quad (i.e. \ \forall s_{-i} \in \hat{S}_{-i}) and therefore$X(i) ^{M} is a strategic independent set for player i. Then because of inequality (1), s_i is strictly dominated in $G^{X(i)}$.

ONLY IF: we need to prove that

$$s_i \in WD_i^1(G) \quad \Rightarrow \quad \forall X(i) \in \mathcal{X}(i, G) \quad s_i \in SD_i^1(G^{X(i)}).$$

Consider the contrapositive of this implication: suppose that there exists a strategic independent set for player i $X(i) = X_i \times X_{-i}$ such that s_i is strictly dominated in $G^{X(i)}$. Thus there exists a $\sigma_i' \in \Delta(X_i)$ such that

$$U_i(\sigma_i', s_{-i}) > U_i(s_i, s_{-i}) \quad \forall s_{-i} \in X_{-i} \tag{2}$$

but then in order to satisfy the definition of strategic independent set, the strict inequality (2) implies that the following equality should hold

$$U_i(\sigma_i', s_{-i}) = U_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i} \setminus X_{-i},$$

hence s_i is weakly dominated in G. ♡

Theorem 2 *For every game $G \in \mathcal{G}$ $\ WD(G) = Q(\mathcal{X}(G))$.*

Proof: The proof is by induction. Fix a NFG G. Theorem 1 trivially implies that $WD^1(G) = UD^1(\mathcal{X}(G))$. Thus we need to prove that $WD^{t-1}(G) = UD^{t-1}(\mathcal{X}(G))$ implies $WD^t(G) = UD^t(\mathcal{X}(G))$.

First we want to show that

$$WD^{t-1}(G) = UD^{t-1}(\mathcal{X}(G)) \quad \Rightarrow \quad UD^t(\mathcal{X}(G)) \subseteq WD^t(G), \qquad i.e.$$

$$WD^{t-1}(G) = UD^{t-1}(\mathcal{X}(G)) \quad \Rightarrow \quad [s \in UD^t(\mathcal{X}(G)) \ \Rightarrow \ s \in WD^t(G)],$$

or using the contrapositive of the second implication

$$WD^{t-1}(G) = UD^{t-1}(\mathcal{X}(G)) \quad \Rightarrow \quad [s \notin WD^t(G) \ \Rightarrow \ s \notin UD^t(\mathcal{X}(G))].$$

Therefore suppose that for a generic player $i \in N$ s_i is weakly dominated at stage t. Then if s_i is strictly dominated at stage t, the implication does trivially hold, since $UD^t(\mathcal{X}(G)) \subseteq SD^t(G)$. Therefore suppose that s_i is weakly but not strictly dominated. Then there exists a strategy

$\sigma_i' \in \Sigma(WD_i^{t-1}(G)) = \Sigma(UD_i^{t-1}(\mathcal{X}(i,G)))$ and a non empty set of opponents' strategy profile $\hat{S}_{-i} \subseteq WD_{-i}^{t-1}(G) = UD_{-i}^{t-1}(\mathcal{X}(i,G))$ such that

$$U_i(s_i, s_{-i}) = U_i(\sigma_i', s_{-i}) \quad \forall s_{-i} \in \hat{S}_{-i}$$

$$U_i(s_i, s_{-i}) < U_i(\sigma_i', s_{-i}) \quad \forall s_{-i} \notin \hat{S}_{-i}. \tag{3}$$

Then consider the set $X(i) = \{s_i, \sigma_i'\} \times S_{-i} \backslash \hat{S}_{-i} = X_i \times X_{-i}$: by construction

$$U_i(s_i, s_{-i}) = U_i(s_i, s_{-i}) \quad \forall s_{-i} \in X_{-i}$$

$$U_i(\sigma_i', s_{-i}) = U_i(\sigma_i', s_{-i}) \quad \forall s_{-i} \in X_{-i}$$

$$U_i(s_i, s_{-i}) = U_i(\sigma_i', s_{-i}) \quad \forall s_{-i} \notin S_{-i} \setminus X_{-i} \quad (i.e.\ \forall s_{-i} \in \hat{S}_{-i})$$

and therefore $X(i)$ is a strategic independent set for player i in $G^{UD^{t-1}(\mathcal{X}(G))}$, i.e. $X(i) \in \mathcal{X}(i, G^{UD^{t-1}(\mathcal{X}(G))})$. Moreover because of inequality (3) s_i is strictly dominated in $G^{X(i)}$ at stage t, therefore by definition $s_i \notin UD_i^t(\mathcal{X}(i,G))$.

Now we want to show that

$$WD^{t-1}(G) = UD^{t-1}(\mathcal{X}(G)) \quad \Rightarrow \quad WD^t(G) \subseteq UD^t(\mathcal{X}(G)), \quad i.e.$$

$$WD^{t-1}(G) = UD^{t-1}(\mathcal{X}(G)) \quad \Rightarrow \quad [s \in WD^t(G) \ \Rightarrow \ s \in UD^t(\mathcal{X}(G))],$$

or using the contrapositive of the second implication

$$WD^{t-1}(G) = UD^{t-1}(\mathcal{X}(G)) \quad \Rightarrow \quad [s \notin UD^t(\mathcal{X}(G)) \ \Rightarrow \ s \notin WD^t(G)].$$

Therefore suppose that there exists a strategic independent set for player i $X(i) = X_i \times X_{-i} \in \mathcal{X}(i, G^{UD^{t-1}(\mathcal{X}(G))})$ such that s_i is strictly dominated in $G^{X(i)}$. Note that by the induction hypothesis $X_i \subseteq \Sigma(WD_i^{t-1}(G))$ and $X_{-i} \subseteq WD_{-i}^{t-1}(G)$. Thus there exists a $\sigma_i' \in X_i$ such that

$$U_i(\sigma_i', s_{-i}) > U_i(s_i, s_{-i}) \quad \forall s_{-i} \in X_{-i} \tag{4}$$

but then in order to satisfy the definition of strategic independent set, the strict inequality (4) implies that the following equality should hold

$$U_i(\sigma_i', s_{-i}) = U_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i} \setminus X_{-i} \subseteq WD_{-i}^{t-1},$$

hence s_i is weakly dominated in G at stage t. $\heartsuit$

9.3 Axiomatic Foundations of Iterated Admissibility

Now we can provide the axiomatic characterization of iterated admissibility.

Theorem 3 *Under axioms 1 and 2, for any game G, $\mathcal{T}(G) = WD(G)$. Furthermore WD is acceptable.*

Proof: First note that according to theorem 2 we can rewrite the statement as requiring $\mathcal{T}(G) = Q(\mathcal{X}(G))$. Moreover lemma 3 of Pearce 1984 or theorem 5.2.1 of Blackwell and Girshick 1954 allow to rewrite the definition of conditionally undominated strategies as follows:

$$UD_i^1(\mathcal{X}(i,G)) = \{s_i \in S_i | \forall X(i) \in \mathcal{X}(i,G) \exists p_{-i} \in \Delta(X_{-i}) : s_i \in F_i(p_{-i})\} \tag{5}$$

and

$$UD_i^n(\mathcal{X}(i,G)) = \{s_i \in S_i | \forall X(i) \in \mathcal{X}(i, G^{UD^{n-1}(\mathcal{X}(G))}) \exists p_{-i} \in \Delta(X_{-i}) : s_i \in F_i(p_{-i})\}.$$

Thus proposition 1 and theorem 2 imply that there exists a $K \in \mathbb{N}$ such that $\forall n \geq K \;\; UD^n(\mathcal{X}(G)) = Q(\mathcal{X}(G))$.

First we will show that WD is acceptable under axioms 1 and 2. Since for any NFG G, trivially $Q(\mathcal{X}(G)) \subseteq UD^1(\mathcal{X}(G))$, (5) implies that axiom 1 is plainly satisfied. Now suppose that WD does not satisfy axiom 2: then in a NFG G there exists a player i, a strategic independent set $X(i) \in \mathcal{X}(i,G)$, and a strategy $s_i \in Q(\mathcal{X}(G))$ such that

$$\nexists p_{-i} \in \Delta(\tau_{-i}(s_i, WD|X(i))) \quad with \quad s_i \in F_i(p_{-i}).$$

But since $\tau_{-i}(s_i, WD|X(i)) = Q_{-i}(\mathcal{X}(G)) = UD_{-i}^K(\mathcal{X}(G))$, then $s_i \notin UD_i(K+1, \mathcal{X}(G)) \supseteq Q_i(\mathcal{X}(G)) = WD(G)$, a contradiction.

Finally, we show that any acceptable theory T under axioms 1 and 2 satisfies $T(G) \subseteq Q(\mathcal{X}(G))$ for any NFG G. Fix a NFG G: since $Q(\mathcal{X}(G))$ is rectangular, it suffices to show that $T_i(G) \subseteq Q_i(\mathcal{X}(G))$. First, $s_i \in T_i(G)$ implies that s_i satisfies axiom 1 and thus $s_i \in UD_i^1(\mathcal{X}(G))$. Now suppose that for every player i $T_i(G) \subseteq UD_i^n(\mathcal{X}(G))$. By axiom 2, for every i, every $X(i) \in \mathcal{X}(i,G)$ and every s_i there exists a $p_{-i} \in \Delta(\tau_{-i}(s_i, T|X(i)))$ such that $s_i \in F_i(p_{-i})$. But $\tau_{-i}(s_i, T|X(i)) \subseteq \times_{j \neq i} T_j(G) \subseteq \times_{j \neq i} UD_j^n(\mathcal{X}(G))$ so $s_i \in UD_i^{n+1}(\mathcal{X}(G))$. Therefore by induction, $s_i \in Q_i(\mathcal{X}(G)$. Hence $\mathcal{T}(G) = WD(G) = Q(\mathcal{X}(G))$ for any NFG G. ♡

9.4 Consistency of (Iterated) Admissibility as Solution Concept

Consider Abreu and Pearce 1984 negative result on weak dominace.

Proposition 2 *There exist no solution concept T satisfying axioms 3, 4 and 5.*

Proof: Suppose that there exists a solution concept $\widehat{T}$ satisfying axioms 3,4 and 5 and apply $\widehat{T}$ to the game G of figure 4

	ac	ad	bc	bd
α_1	1,1	1,1	1,0	1,0
α_2	0,0	1,1	0,0	1,1

Figure 4

Since the strategies α_2, *ac*, *bc* and *bd* are weakly dominated, then axioms 3 and 4 imply $\widehat{T}(G) = \{\alpha_1\} \times \{ad\}$. Now consider the following restriction of G to $A = \{\alpha_1, \alpha_2\} \times \{ad\}$, where the weakly dominated strategies *ac*, *bc*, and *bd* have been removed, and thus $WD^1(G) \subseteq \{\alpha_1, \alpha_2\} \times \{ad\} \subseteq S$. Then axiom 3 implies $\widehat{T}(G^A) = \widehat{T}(G) \cap A = \{\alpha_1\} \times \{ad\}$. But in G^A the payoff functions of both players are constant and thus any solution concept should satisfy $\widehat{T}(G^A) = A = \{\alpha_1, \alpha_2\} \times \{ad\}$, a contradiction. ♡

In the proof, Abreu and Pearce assume that "the solution set must be independent of ... the labels one chooses" to denote strategies (p. 173), an intuitive condition but not stated as an independent axiom.

The following result instead shows that (iterated) admissibility as characterized by theorem 3 is instead a satisfactory criterion in the sense of Abreu and Pearce 1984.

Proposition 3 *Axioms 3, 4 and 6 are mutually consistent.*

Proof: First we show that axioms 4 and 6 are satisfied if and only if the following condition holds: for all games $G \in \mathcal{G}$

$$T(G^{WD^1(G)}) = T(G).$$

ONLY IF: because of axiom 6 $T(G^{WD^1(G)}) = T(G) \cap WD^1(G)$, and because of axiom 4 $T(G) \subseteq WD^1(G)$, hence $T(G^{WD^1(G)}) = T(G)$.
IF: since by definition of solution concept $T(G^{WD^1(G)}) \subseteq WD^1(G)$, then $T(G^{WD^1(G)}) = T(G)$ implies $T(G) \subseteq WD^1(G)$, i.e. axiom 4. Moreover $T(G^{WD^1}(G)) = T(G)$ implies $T(G^{WD^1(G)}) \cap WD^1(G) = T(G) \cap WD^1(G)$, but then since by definition of solution concept $T(G^{WD^1(G)}) \subseteq WD^1(G)$, it follows that $T(G^{WD^1(G)}) = T(G) \cap WD^1(G)$, i.e. axiom 6.

Now consider IWUS as solution concept, i.e. $T(G) = WD^1(G)$. By the previous result, T satisfies axiom 3. Moreover $T(G) = WD^1(G)$ implies $T(G^{WD^1(G)}) = WD^1(G)$. Hence $T(G) = T(G^{WD^1(G)})$ and thus, because of the previous result, T satisfies axioms 4 and 6. ♡

9.5 Cautious Bayesian Optimisation and Iterated Admissibility

Proposition 4 *For every game $G \in \mathcal{G}$*

$$R(G) = SD(G) \quad \text{and} \quad \overline{R}(G) = WD(G).$$

Proof: First it is necessary to prove the equivalence between definition 5 of (cautious) rationalizability which iteratively restricts players' beliefs and the reduction procedure where both the set of players' possible choices and beliefs are iteratively ristrect, i.e. where

$$R_i^t(G) := \{s_i \in R_i^{t-1}(G) |\ \exists \mu_i \in \Delta(R_{-i}^{t-1}(G)) \text{ such that } s_i \in F_i(\mu_i)\}.$$

and

$$\overline{R}_i^t(G) := \{s_i \in \overline{R}_i^{t-1}(G) |\ \exists \mu_i \in \Delta^\circ(\overline{R}_{-i}^{t-1}(G)) \text{ such that } s_i \in F_i(\mu_i)\}.$$

This is done by Bernheim 1984 and Pearce 1984 for rationalizability and Veronesi 1994 for cautious rationalizability. Then the result follows by induction from lemmas 3 and 4 of Pearce 1984 or from theorems 5.2.1 and 5.2.5 of Blackwell and Girshick 1954. ♡

Proposition 5 *For any game $G \in \mathcal{G}$ $R'(G) = R(G) \neq \emptyset$. On the other hand, there exist games $G \in \mathcal{G}$ such that*

1. *$\overline{R}'(G) = \emptyset$;*
2. *there exist fixed points of $\overline{\Lambda}$, but none of them is maximal;*
3. *$\overline{R}'(G) \neq \overline{R}(G)$.*

Proof: the result on rationalizability is proposition 3.2 in Bernheim 1984, while the result on cautious rationalizability is proposition 2 in Börgers and Samuelson 1992. ♡

Proposition 6 *The operator Λ is monotone, i.e. $A \subseteq B \Rightarrow \Lambda(A) \subseteq \Lambda(B)$, while the operator $\overline{\Lambda}$ is not monotone.*

Proof: By definition Λ is clearly monotone (see also Bernheim 1984). On the other hand apply $\overline{\Lambda}$ to the game of figure 2: $\overline{\Lambda}(\{L,R\} \times \{T,M,B\}) = \{R\} \times \{T\}$, but $\overline{\Lambda}(\{L,R\} \times \{T\}) = \{L,R\} \times \{T\}$. ♡

Theorem 4 *For any finite normal form game $G \in \mathcal{G}$*

$$\overline{R}(G) = \overline{R}''(G) \neq \emptyset.$$

Proof: Preliminary define recursively $\forall k \in \mathbb{N} \quad \Lambda^k[\mathcal{X}](B) := \Lambda[\mathcal{X}](\Lambda^{k-1}[\mathcal{X}](B))$ with $\Lambda^0[\mathcal{X}](B) = B$. The proof relies on many previous results. First, note that by definition $\Lambda[\mathcal{X}]$ is monotone. Second theorem 2 and proposition 4 imply that $Q(\mathcal{X}(G)) = \overline{R}(G) = \cap_{k=0}^{\infty}\Lambda^k[\mathcal{X}](S)$ (see also lemma 2 in SW). Therefore we need to prove the equality between the infinite iteration of $\Lambda^k[\mathcal{X}](S)$ and the maximal fixed point of $\Lambda[\mathcal{X}]$, using the monotonicity of $\Lambda[\mathcal{X}]$ and following Bernheim 1984 proof of proposition 3.1. Suppose $\overline{R}(G) \neq \overline{R}''(G)$. Then there exists a set $A \subseteq S$ and a $k \in \mathbb{N}$ such that:

$$A = \Lambda[\mathcal{X}](A) \tag{6}$$

$$A \cap \Lambda^k[\mathcal{X}](S) = A \tag{7}$$

$$A \cap \Lambda^{k+1}[\mathcal{X}](S) \subset A \quad \text{where the inclusion is strict.} \tag{8}$$

Monotonicity implies that if $A \subseteq B$, then $\Lambda[\mathcal{X}](A \cap B) = \Lambda[\mathcal{X}](A) \cap \Lambda[\mathcal{X}](B)$. Moreover (7) implies

$$\Lambda[\mathcal{X}](A \cap \Lambda^k[\mathcal{X}](S)) = \Lambda[\mathcal{X}](A) \tag{9}$$

and $A \subseteq \Lambda[\mathcal{X}]^k(S)$. Hence (9) implies

$$\Lambda[\mathcal{X}](A) \cap \Lambda[\mathcal{X}](\Lambda^k[\mathcal{X}](S)) = \Lambda[\mathcal{X}](A). \tag{10}$$

But the definition of $\Lambda^{k+1}[\mathcal{X}]$ and (6) imply that (10) is equivalent to $A \cap \Lambda^{k+1}[\mathcal{X}](S) = A$, a contradiction with (8). ♡