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Nodal sets of magnetic Schrödinger operators of Aharonov–Bohm type and energy minimizing partitions *

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Abstract

We analyze the nodal set of the stationary solutions of a Schrödinger operator, in dimension two, in presence of a magnetic field of Aharonov–Bohm type, with semi–integer circulation. We determine a class of solutions such that the nodal set consists of regular arcs, connecting the singular points with the boundary. In the particular case of one singular point, we prove that the nodal regions, whenever they dissect the domain in three components, satisfy a minimal partition principle. Moreover we prove that such a configuration is unique and depends continuously on the data.

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain with regular boundary and $V(x) \in W^{1,p}(\Omega)$ for some $p > 2$. Given a point $a \in \Omega$ we consider the stationary magnetic Schrödinger operator

$$H_{\mathbf{A}_a,V} = (i\nabla + \mathbf{A}_a)^2 + V = -\Delta + i\nabla \cdot \mathbf{A}_a + i\mathbf{A}_a \cdot \nabla + |\mathbf{A}_a|^2 + V,$$

acting on complex valued functions $U \in L^2(\Omega)$. Here the magnetic potential $\mathbf{A}_a : \Omega \to \mathbb{R}^2$ is such that the associated magnetic field satisfies

$$\mathbf{B}_a = \nabla \times \mathbf{A}_a = \pi\delta_a\mathbf{k} \quad \text{in } \Omega, \qquad \mathbf{A}_a \in L^1(\Omega) \cap C^1(\Omega \setminus \{a\}) \tag{1}$$

where δ_a is the Dirac delta centered at a and $\mathbf{k}$ is the unitary vector orthogonal to the plane. We are concerned with the boundary value problems

$$\begin{cases} (i\nabla + \mathbf{A}_a)^2 U_a + VU_a = 0 & \text{in } \Omega \setminus \{a\} \\ U_a = \Gamma & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where a (the concentration point of the magnetic field) is intended as a parameter, whereas the complex boundary data Γ is fixed in a suitable class. In order to be more precise, let us define the class of real boundary traces

$$g = \left\{ \gamma = \sum_{i=1}^{3} \sigma_i\gamma_i : \begin{array}{l} \gamma_i \in C^1(\partial\Omega), \quad \gamma_i \cdot \gamma_j = 0 \text{ for } i \neq j \\ \gamma_i \geq 0, \quad \sigma_i = \pm 1 \\ \gamma \text{ vanishes exactly three times on } \partial\Omega \end{array} \right\}.$$

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Then we consider complex boundary data in the following class

$$\mathcal{G} = \{\Gamma : \partial\Omega \to \mathbb{C} : \quad \Gamma = \gamma e^{i\Theta_1},\ \Theta_1 \text{ satisfies } (i), (ii) \text{ below } \},$$

with suitable assumptions on the complex phase:

(i) $\Theta_1 : \partial\Omega \to \mathbb{R}$ is continuous except at one point $x_0 \in \partial\Omega$, where all the γ_i's vanish;

(ii) the averaged jump (over one full angle) of the phase equals the circulation of the magnetic field:

$$\lim_{x \to x_0{}^-} \Theta_1(x) = \lim_{x \to x_0{}^+} \Theta_1(x) + \pi. \tag{3}$$

In Section 3 we prove that, under these assumptions on the magnetic potential and on the boundary data, equation (2) is equivalent to a real elliptic equation on the twofold covering manifold of $\Omega \setminus \{a\}$. If we also assume the following coercivity condition on the scalar potential

$$\int_\Omega \left(|\nabla\varphi|^2 + V\varphi^2\right) dx_1 dx_2 > 0 \quad \forall\, \varphi \in H^1_0(\Omega), \tag{4}$$

then equation (2) admits a unique solution for every fixed $a \in \Omega$, which is the minimizer of the associated quadratic form:

$$Q_{\mathbf{A}_a,V}(U) = \int_\Omega \left(|(i\nabla + \mathbf{A}_a)U|^2 + V|U|^2\right) dx_1 dx_2,$$

among all the complex functions sharing the same boundary trace Γ. We shall prove in Section 3 that, given γ, gauge invariance allows to define a function depending only on the position of the singularity a (not on the particular choice of the complex phase Θ_1, or of the magnetic potential):

$$a \mapsto \varphi(a) = \min\{Q_{\mathbf{A}_a,V}(U) \ :\ U \in H^1(\Omega),\ U = \Gamma \text{ on } \partial\Omega\}.$$

We are interested in studying the *nodal set* of the solutions of (2), that is the set $\mathcal{N}(U_a) = \{x \in \Omega : U_a(x) = 0\}$. Notice that the nodal set of U_a is obtained intersecting the zero curves of the real and imaginary parts $\Re U_a$ and $\Im U_a$, therefore in general we expect it to consist of a few singular points. However in Section 4 we prove

(1.1) Theorem. *Let $a \in \Omega$ be fixed and $\Gamma \in \mathcal{G}$. Consider a solution U_a of* (2) *under assumption* (1)*. Then the set $\mathcal{N}(U_a)$*

(i) depends only on $|\Gamma|$ and on the position of the singularity a (not on the particular choice of the magnetic potential $\mathbf{A}_a$ or of the complex phase Θ_1);

(ii) is nontrivial and consists of the union of regular arcs, having endpoints either at $\partial\Omega$, or at an interior singular point of U_a, or at a;

(iii) there is at least one arc ending up at a.

If moreover V satisfies (4) *then the nodal set of U_a consists of at most three arcs.*

In case the nodal set is made of three arcs intersecting at a, we will say that a is a *triple point* for Γ. Our aim is to understand the circumstances related to the occurrence of triple points; our main results in this direction are the following:

(1.2) Theorem. *Consider the set of equations* (2) *as the parameter a varies in Ω. Assume that (1), (4) hold and $\Gamma \in \mathcal{G}$. Then* *(i)* **(Criticality)** *if* $\gamma \in C^{1,1}(\partial\Omega)$, *then the function* φ *introduced above is differentiable and its only critical points are triple points;*

(ii) **(Global uniqueness)** *every* $\Gamma \in \mathcal{G}$ *admits at most one triple point;*

(iii) the set $\tilde{\mathcal{G}} \subset \mathcal{G}$ *of boundary data which admit a triple point is open and dense in* $\mathcal{G}$ *(respect to the* L^∞*–norm);*

(iv) **(Continuous dependence of the triple point)** *the position of the triple point depends continuously on the* L^∞*–norm of the boundary data;*

(v) **(Continuous dependence of the nodal lines)** *if* $V(x) \in W^{1,\infty}(\Omega)$, *then the* C^1*–norm of the nodal lines depends continuously on the* L^∞*–norm of the boundary data.*

We also give another variational characterization of the triple point configuration, which is related to the set function

$$J_i(\omega_i) = \inf \left\{ \int_\Omega \left(|\nabla u_i|^2 + V u_i^2 \right) dx_1 dx_2 : \begin{array}{l} u_i \in H^1(\Omega),\ u_i = 0 \text{ in } \Omega \setminus \omega_i \\ u_i = \gamma_i \text{ on } \partial\Omega \cap \omega_i \end{array} \right\},$$

where $\gamma = \sum_{i=1}^3 \sigma_i \gamma_i \in g$ and $\omega_i \subset \Omega$ is any open set such that $\mathrm{supp}(\gamma_i) \subset \partial\omega_i$. Then it holds

(1.3) Theorem. *Let* $\Gamma \in \tilde{\mathcal{G}}$ *and* a *be its triple point, so that* $\Omega \setminus \mathcal{N}(U_a)$ *has three connected components. Then the connected components are solution of the optimal partition problem*

$$\inf \left\{ \sum_{i=1}^{3} J_i(\omega_i) : \begin{array}{l} \omega_i \text{ open},\ \mathrm{supp}(\gamma_i) \subset \partial\omega_i \\ \cup_{i=1}^3 \overline{\omega_i} = \overline{\Omega},\ \omega_i \cap \omega_j = \emptyset,\ i \neq j \end{array} \right\}. \tag{5}$$

As it is shown in [6], the minimization problem above admits a unique solution, belonging to the functional class

$$\mathcal{S}_\gamma = \left\{ u = (u_1, u_2, u_3) \in (H^1(\Omega))^3 : \begin{array}{l} u_i \geq 0,\ u_i = \gamma_i \text{ on } \partial\Omega \\ u_i \cdot u_j = 0 \text{ a.e. } x \in \Omega,\ \text{for } i \neq j \\ -\Delta u_i + V u_i \leq 0, \quad -\Delta \hat{u}_i + V \hat{u}_i \geq 0 \quad i = 1,2,3 \end{array} \right\},$$

where the *hat* operator is defined as $\hat{u}_i := u_i - \sum_{j \neq i} u_j$. Theorem 1.3 allows us to generalize the result in [7], providing

(1.4) Theorem. *Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected domain with regular boundary,* $\gamma \in g$ *and* $V(x) \in W^{1,p}(\Omega)$, *for some* $p > 2$, *satisfying (4). Then* $\mathcal{S}_\gamma$ *consists of exactly one element.*

It is proved in [5] that this functional class also contains the limiting solutions of the competition-diffusion system

$$\begin{cases} -\Delta u_i + V(x) u_i = -\kappa u_i \sum_{j \neq i} u_j & \text{in } \Omega \\ u_i \geq 0 & \text{in } \Omega \\ u_i = \gamma_i & \text{on } \partial\Omega \end{cases} \tag{6}$$

as $\kappa \to +\infty$. Then our results, together with the ones contained in ([6, 5, 7]), also provide

(1.5) Theorem. *Under the previous assumptions, every solution* $(u_{1,\kappa}, u_{2,\kappa}, u_{3,\kappa})$ *of (6) satisfies*

(i) the hole sequence $u_{i,\kappa}$ *converges to a function* u_i *in* $H^1 \cap C^{0,\alpha}$ *for every* $\alpha \in (0,1)$, *as* $\kappa \to +\infty$*;*

(ii) the limiting triple (u_1, u_2, u_3) *achieves the minimum in Theorem 1.3.*

The paper is organized as follows. In Section 3 we take advantage of the classical gauge invariance property, to prove that equation (2) is equivalent to a real elliptic equation on a Riemann surface. In Section 4 we use this equivalence in order to show regularity of the nodal lines. A lot is known in the case of real elliptic operators, we address the reader to [3, 12] for the planar case, but also higher dimensions have been extensively studied, see [18, 19, 11, 21, 10]. We wish to mention some of the results by Hoffmann–Ostenhof M. and T. and Nadirashvili, concerning Schrödinger operators with singular conservative potentials [15, 16, 17]. As it concerns the Aharonov–Bohm type magnetic potentials, few is known because of the very strong singularity. We will mainly refer to [13], where the homogeneous Cauchy problem is studied. The recent paper [8] provides regularity results for a large class of equations including (2). Section 5 contains the technical part of the work, that is the proof of criticality (Theorem 1.2, (i)) and a local uniqueness result. Next, in Section 6, we exploit the variational characterization of the triple point configuration, which also provides the global uniqueness (Theorem 1.2, (ii)-(iv)). As we mentioned, this section is strongly based on some previous results by Conti, Terracini and Verzini. Finally, in Section 7, we prove the continuous dependence of the nodal lines on the data (Theorem 1.2, (v)).

2 Preliminaries on Aharonov–Bohm Schrödinger operators

We can assume w.l.o.g. $\Omega = D$, the open unit disk in $\mathbb{R}^2$. This assumption is not restrictive thanks to the Riemann mapping theorem (see for example [9], Theorem 6.42), and due to the conformal equivariance of the problem, which will be proved in Section 3. Given a point $a \in D$ we denote by D_a the open set $D \setminus \{a\}$.

Consider a non-relativistic, spinless, quantum particle moving in D_a under the action of a magnetic potential satisfying (1) and of a conservative potential $V(x) \in W^{1,p}(D)$, $p > 2$. If we neglect all multiplicative constants, the stationary Schrödinger operator associated to the particle is the operator $H_{\mathbf{A}_a,V}$ defined in the introduction. Although it remains in a region where the magnetic field is zero, the particle will be affected by the magnetic potential. This phenomenon is usually called Aharonov–Bohm effect, it was first pointed out in [2] and can be simulated experimentally by the presence of a thin solenoid placed at a and aligned along the x_3-axes. Notice that we can equivalently substitute (1) with the conditions [1]

$$\nabla \times \ \mathbf{A} = 0 \qquad \text{in } D_a, \qquad \mathbf{A} \in L^1(\Omega) \cap C^1(\Omega \setminus \{a\}) \tag{7}$$

with the following additional assumption on the normalized circulation:

$$\frac{1}{2\pi} \oint_\sigma \mathbf{A} \cdot \mathrm{d}\mathbf{x} = \frac{2n+1}{2}, \quad n \in \mathbb{Z} \tag{8}$$

for every closed path σ which winds once around the pole. We refer to [23] for a complete review on magnetic Schrödinger operators and to [22] for the specific case of the A–B effect.

Due to the physical interpretation of the problem we require the operator to be self-adjoint; we are now going to specify the domain of $H_{\mathbf{A},V}$ and the notion of weak solution. We first define the operator on the all space $\mathbb{R}^2_a := \mathbb{R}^2 \setminus \{a\}$ (extend the coefficients smoothly outside D) and then restrict our attention to the unit disk. In a standard way we initially consider a symmetric operator $H^0_{\mathbf{A},V}$ defined on a dense subspace of $L^2(\mathbb{R}^2)$ and then construct a self-adjoint extension. Due to the singularity of the magnetic potential we need to impose additional conditions at the singular point a, that is we define $H^0_{\mathbf{A},V}$ on the domain $C^\infty_0(\mathbb{R}^2_a)$ (complex valued functions). This is a symmetric operator and the associated quadratic form is

$$Q_{\mathbf{A},V}(U) = \int_{\mathbb{R}} \left(|(i\nabla + \mathbf{A})U|^2 + V|U|^2 \right) dx_1 dx_2,$$

[1] In the following we will omit the index a whenever the position of the singularity is fixed.

defined on $C_0^\infty(\mathbb{R}^2_a)$. Note that it is lower semi-bounded, since $V(x) \in W^{1,p}(D)$, for some $p > 2$:

$$Q_{\mathbf{A},V}(U) \geq -\mu ||U||^2_{L^2(D)},$$

where $\mu = ||V||^2_{L^\infty(D)}$. Because of the strength of its singularity, the operator $H^0_{\mathbf{A},V}$ fails to be essentially self-adjoint, nevertheless the Friedrichs extension allows us to extend $H^0_{\mathbf{A},V}$ to a self adjoint operator. This particular choice corresponds to a particular physical interpretation of the phenomenon (see [1]). The Friedrichs extension theorem ensures the existence of a unique self adjoint operator $H_{\mathbf{A},V}$, which extends $H^0_{\mathbf{A},V}$, and which domain is the closure of $C_0^\infty(\mathbb{R}^2_a)$ respect to the norm

$$||U||_{\mathbf{A}} := \Big(Q_{\mathbf{A},V}(U) + (1+\mu)||U||^2_{L^2(D)}\Big)^{1/2}.$$

Next we turn back to the initial problem considering the restriction of $H_{\mathbf{A},V}$ to the unit disk. A density result (see for example [20], Theorem 7.22) ensures that the domain can be equivalently characterized as:

$$\mathcal{H}_{\mathbf{A}}(D) = \{U : D \to \mathbb{C} : U \in L^2(D),\ \left(i\frac{\partial}{\partial x_j} + A_j\right) U \in L^2(D)\ j = 1, 2\}.$$

Moreover, if $U \in \mathcal{H}_{\mathbf{A}}(D)$ then it satisfies the diamagnetic inequality

$$|\nabla |U|(x)| \leq |(i\nabla + \mathbf{A})U(x)|, \quad \text{a.e. } x \in D, \tag{9}$$

which ensures in particular that $|U| \in H^1(D)$ (see for example [20], Theorem 7.21).

Due to the regularity of the domain and thanks to the diamagnetic inequality, a trace operator is well defined on $\mathcal{H}_{\mathbf{A}}(D)$, i.e. there exists a linear bounded operator

$$Tr : \mathcal{H}_{\mathbf{A}}(D) \to L^1(\partial D),$$

such that if $U \in \mathcal{H}_{\mathbf{A}}(D) \cap C(\overline{D})$ then $TrU = U_{|\partial D}$.

Given a boundary data $\Gamma \in W^{1,\infty}(\partial D)$ we say that U is a weak solution of (2) if the following integral equality holds for every $\varphi \in C_0^\infty(\mathbb{R}^2_a)$

$$\int_D U[\overline{(i\nabla + \mathbf{A})^2\varphi + V\varphi}]dx_1dx_2 + i\int_{\partial D}[\Gamma\overline{(i\nabla + \mathbf{A})\varphi} \cdot \nu + (i\nabla + \mathbf{A})U \cdot \nu\overline{\varphi}]d\sigma = 0.$$

3 Gauge invariance for A–B potentials with semi-integer circulation

We are mainly interested in the analysis of the nodal set of solutions of (2), to this aim we will take large advantage of the equivariance of $H_{\mathbf{A},V}$ under gauge transformations. In the first part of this section we shall present a result contained in [13], related to the gauge invariance property of magnetic operators of A–B type, having semi-interger circulation. In Proposition 3.9 we will prove a generalization to the non-homogeneous Cauchy problem, in the case $\Gamma \in \mathcal{G}$. In Section 3.1 we will finally use this result to prove the existence of a bijection between the solutions of (2) and the antisymmetric solutions of a real elliptic equation. Let us start with some preliminary definitions.

Let $\Omega \subset \mathbb{R}^2$ be bounded domain, $\tilde{\Omega}$ be a covering manifold and $\Pi : \tilde{\Omega} \to \Omega$ be the associated projection map. We endow $\tilde{\Omega}$ with the locally flat metric obtained by lifting the Euclidean metric of Ω, in such a way that Π is a local isometry.

(3.1) Definition. For a function $f : \Omega \to \mathbb{C}$ we define the lifted function $\tilde{f} : \tilde{\Omega} \to \mathbb{C}$ as

$$\tilde{f} = f \circ \Pi.$$

For a path $\sigma : [0,1] \to \Omega$ and a point $p \in \tilde{\Omega}$ such that $\Pi(p) = \sigma(0)$ let $\tilde{\sigma} : [0,1] \to \tilde{\Omega}$ denote the unique lifted path such that $\tilde{\sigma}(0) = p$ and

$$\sigma = \Pi \circ \tilde{\sigma}.$$

(3.2) Lemma. *Let* $\mathbf{A}$ *and* $\mathbf{A}'$ *satisfy* (7) *in* D_a. *Let* $U \in \mathcal{H}_{\mathbf{A}}(D)$ *be a weak solution of* $H_{\mathbf{A},V}U = 0$ *in* D_a, *with Dirichlet boundary conditions. Assume moreover that*

$$\frac{1}{2\pi} \oint_\sigma (\mathbf{A}' - \mathbf{A}) \cdot d\mathbf{x} \ \in \ \mathbb{Z},$$

for every closed path σ *in* D_a. *Then there exists* $U' \in \mathcal{H}_{\mathbf{A}}(D)$, *weak solution of* $H_{\mathbf{A}',V}U' = 0$ *in* D_a, *such that* $|U'| = |U|$. *We will say that* $\mathbf{A}$, $\mathbf{A}'$ *are gauge equivalent.*

Proof. Assume first that $\oint_\sigma (\mathbf{A}' - \mathbf{A}) \cdot \mathrm{d}\mathbf{x} = 0$, for every closed path σ in D_a. Then there exists a smooth function $\Theta : D_a \to \mathbb{R}$ such that $\mathbf{A}' = \mathbf{A} + \nabla\Theta$. Let $\varphi \in C_0^\infty(D_a)$, then a direct calculation shows that $e^{i\Theta}(i\nabla + \mathbf{A})^2\varphi = (i\nabla + \mathbf{A}')^2(e^{i\Theta}\varphi)$. Multiplying the equation by $e^{i\Theta}e^{-i\Theta}$, we obtain

$$0 = \int_D U[\overline{(i\nabla + \mathbf{A})^2\varphi + V\varphi}]dx_1dx_2 = \int_D e^{i\Theta}U[\overline{(i\nabla + \mathbf{A}')^2(e^{i\Theta}\varphi) + Ve^{i\Theta}\varphi}]dx_1dx_2,$$

thus $U' = e^{i\Theta}U$. In case the normalized circulation is an integer, consider the universal covering manifold of D_a, say $\tilde{D}_a$. Due to the fact that $\tilde{D}_a$ is simply connected and $\mathbf{A}$ and $\mathbf{A}'$ satisfy (7), there holds $\oint_{\tilde{\sigma}} (\tilde{\mathbf{A}}' - \tilde{\mathbf{A}}) \cdot \mathrm{d}\mathbf{x} = 0$, for every closed path $\tilde{\sigma}$ on $\tilde{D}_a$. Therefore there exists a smooth real valued function Θ defined on $\tilde{D}_a$ such that $\nabla\Theta = \tilde{\mathbf{A}}' - \tilde{\mathbf{A}}$. Now consider any two points $p, p' \in \tilde{D}_a$ such that $\Pi(p) = \Pi(p')$. Then for every path $\tilde{\sigma}$ on $\tilde{D}_a$ connecting p to p' we have

$$\begin{aligned} \Theta(p) - \Theta(p') &= \int_{\tilde{\sigma}} \nabla\Theta \cdot \mathrm{d}\mathbf{x} = \int_{\tilde{\sigma}} (\tilde{\mathbf{A}}' - \tilde{\mathbf{A}}) \cdot \mathrm{d}\mathbf{x} \\ &= \int_\sigma (\mathbf{A}' - \mathbf{A}) \cdot \mathrm{d}\mathbf{x} = 2\pi n, \end{aligned}$$

for some $n \in \mathbb{Z}$. Hence the function $e^{i\Theta}$ is well defined on D_a and we can proceed as in the case of null circulation. □

As a particular case of the previous lemma we infer that, whenever the circulation of $\mathbf{A}$ is an integer, $\mathbf{A}$ is gauge equivalent to the null magnetic potential, which corresponds to the elliptic operator $H_{0,V} = -\Delta + V$. We are now going to show that we can relate the operators $H_{\mathbf{A},V}$ and $H_{0,V}$ also in the case of half integer circulation, provided we replace the domain D_a with its twofold covering manifold. This result was proved in [13] for the Dirichlet homogeneous case, let us now describe it in detail for the non homogeneous problem, under the assumption $\Gamma \in \mathcal{G}$. We point out that the operators are *not* unitarily equivalent since, as we are going to see, there is a one-to-one correspondence between eigenfunctions of $H_{\mathbf{A},V}$ and antisymmetric eigenfunctions of $H_{0,V}$.

The twofold covering manifold of D_a is the following subset of $\mathbb{C}^2$

$$\Sigma_a = \{(x, y) \ \in \ \mathbb{C}^2 : y^2 = x - a, \ x \in D_a\},$$

endowed with a Riemannian metric to be described. First of all notice that there are two projection functions naturally defined on Σ_a:

$$\Pi_x : (x, y) \mapsto x \qquad \Pi_y : (x, y) \mapsto y,$$

and they induce on Σ_a two different differential structures. Since Π_x is only a local chart, the following result will be useful.

(3.3) Proposition. *There exists a global chart of Σ_a which coincides locally with Π_x. In particular it induces on Σ_a a locally flat Euclidean metric.*

Proof. Consider on D_a the discontinuous function

$$\vartheta_a : D_a \to [0, 2\pi) \tag{10}$$

which represents the angular variable of the polar coordinates centered at a. Then we have

$$y^2 = x - a = r_a(x) e^{i\vartheta_a(x)},$$

where $r_a = |x - a|$, and we can define the following parametrization:

$$\begin{aligned} \Phi : [0, 4\pi) \times [0, 1) &\to \Sigma_a \cup \{(a, 0)\} \\ (r, \vartheta) &\mapsto \left(re^{i\vartheta}, \sqrt{r_a(r,\vartheta)} e^{i\frac{\vartheta_a(r,\vartheta)}{2}} \right). \end{aligned}$$

The function Φ is bijective on $[0, 4\pi) \times [0, 1) \setminus \Phi^{-1}(a, 0)$, therefore its inverse Φ^{-1} is well defined on this domain, and it is the desired chart. □

We will endow Σ_a with this metric. In particular, the differential and integral operators on Σ_a coincide locally with the usual ones, hence we will denote them with the same symbol.

(3.4) Remark. In the definition (10) of the angle ϑ_a we usually consider it a discontinuous function on a horizontal segment starting at the point a. Nevertheless we can decide to move the discontinuity without altering the previous construction. In the future analysis in particular it will be useful to consider ϑ_a discontinuous on two adjacent segments: the segment connecting the origin with a and the segment connecting the origin with a point $x_0 \in \partial\Omega$.

Notation. We shall use the following notation for polar coordinates: $x = re^{i\vartheta}$ and $x - a = r_a e^{i\vartheta_a}$, while $y = \rho e^{i\varphi}$, with the relation $\rho = \sqrt{r_a}$, $\varphi = \frac{\vartheta_a}{2}$.

(3.5) Definition. On the twofold covering manifold we define a symmetry map $G : \Sigma_a \to \Sigma_a$, which associates to every (x, y) the unique $G(x, y)$ such that $\Pi_x((x, y)) = \Pi_x(G(x, y))$, that is $G(x, y) := (x, -y)$. We say that a function $f : \Sigma_a \to \mathbb{C}$ is symmetric if $f(G(x, y)) = f(x, y)$, $\forall (x, y) \in \Sigma_a$, and antisymmetric if $f(G(x, y)) = -f(x, y)$, $\forall (x, y) \in \Sigma_a$.

Every function f defined on D_a can be lifted on Σ_a as described in Definition 3.1, by means of the projection Π_x. Notice that $\tilde{f}$ is always symmetric in the sense of the preceding definition.

(3.6) Lemma. *Let $a \in D$ be fixed and $\mathbf{A}$ satisfy* (1). *Then there exists a smooth, multivalued function $\Theta : \Sigma_a \to \mathbb{R}$ such that $e^{i\Theta}$ is univalued on Σ_a and $\mathbf{A} = \Pi_x(\nabla\Theta)$.*

Proof. By assumption (7) $\mathbf{A}$ admits a local potential on every domain not containing the singularity. Let us compute the circulation of the lifted magnetic potential $\tilde{\mathbf{A}}$. For every closed path $\tilde{\sigma}$ on Σ_a we have by construction:

$$\frac{1}{2\pi} \oint_{\tilde{\sigma}} \tilde{\mathbf{A}} \cdot d\mathbf{x} = \frac{1}{2\pi} \oint_{\sigma} \mathbf{A} \cdot d\mathbf{x} \in \mathbb{Z},$$

since σ always turns an even number of times around the singularity. Proceding as in Lemma 3.2 we infer the existence of a multivalued function $\Theta : \Sigma_a \to \mathbb{R}$ such that and $e^{i\Theta}$ is univalued on Σ_a and $\tilde{\mathbf{A}} = \nabla\Theta$. Therefore $\nabla\Theta$ is symmetric and can be projected on D_a, and this concludes the proof. □

(3.7) Remark. We deduce from Lemma 3.2 that every magnetic potential satisfying (1) can be obtained from this specific one

$$\mathbf{A}(x) = \frac{i}{2}\frac{x-a}{|x-a|^2} = \frac{1}{2}\left(-\frac{x_2-a_2}{(x_1-a_1)^2+(x_2-a_2)^2}, \frac{x_1-a_1}{(x_1-a_1)^2+(x_2-a_2)^2}\right)$$

by means of a gauge transformation. In this case the multivalued potential Θ is the function $\frac{\vartheta_a}{2}$, where ϑ_a is the angle defined in (10).

Let us now take into account the boundary data. The following lemma shows that, due to gauge invariance, the phase of Γ is ininfluent for our purpose.

(3.8) Lemma. *Let $a \in D$ be fixed, $\mathbf{A}$ satisfy* (1) *and Θ_1 satisfy (3). Then there exists a smooth, multivalued function $\Theta : \Sigma_a \to \mathbb{R}$ such that $e^{i\Theta}$ is univalued on Σ_a, $\Theta_{|\partial\Sigma_a} = \Theta_1$ and $\mathbf{A}' := \Pi_x(\nabla\Theta)$ is gauge equivalent to $\mathbf{A}$. Moreover $e^{i\Theta}$ is antisymmetric on Σ_a.*

Proof. Consider the function ϑ_a discontinuous at $x_0 \in \partial D$ as pointed out in Remark 3.4, in such a way that $\Theta_1 - \frac{\vartheta_a}{2}$ is continuous on ∂D. Hence we can consider its harmonic extension on the disk:

$$\begin{cases} -\Delta\Psi = 0 & \text{on } D \\ \Psi = \Theta_1 - \frac{\vartheta_a}{2} & \text{on } \partial D. \end{cases}$$

Then the desired potential is the function $\Theta : \Sigma_a \to \mathbb{R}$ defined by

$$\Theta := \frac{\vartheta_a}{2} + \tilde{\Psi}. \tag{11}$$

Clearly $\tilde{\Psi}$ is univalued, moreover $e^{i\frac{\vartheta_a}{2}}$ is univalued on Σ_a by Lemma 3.6 and Remark 3.7. Then notice that $\frac{\vartheta_a}{2}(x,y) = \frac{\vartheta_a}{2}(G(x,y))+\pi$, therefore $\nabla\Theta$ is symmetric and its projection is well defined:

$$\mathbf{A}' := \Pi_x(\nabla\Theta) = \frac{i}{2}\frac{x-a}{|x-a|^2} + \nabla\Psi.$$

The gauge equivalence comes from Lemma 3.2, since there holds

$$\frac{1}{2\pi}\oint_\sigma (\mathbf{A}' - \mathbf{A}) \cdot \mathrm{d}\mathbf{x} = n + \frac{1}{2\pi}\oint_\sigma \nabla\Psi \cdot \mathrm{d}\mathbf{x} = n \ \in \ \mathbb{Z}.$$

Let us finally show that the function $e^{i\Theta}$ is antisymmetric on Σ_a. Fix $(x,y) \in \Sigma_a$ and let $\tilde{\sigma} : [0,1] \to \Sigma_a$ be a path which joins (x,y) to $G(x,y)$, then using the notations of Definition 3.1, there holds

$$\frac{1}{2\pi}\oint_{\tilde{\sigma}} \tilde{\mathbf{A}} \cdot \mathrm{d}\mathbf{x} = \frac{1}{2\pi}\oint_\sigma \mathbf{A} \cdot \mathrm{d}\mathbf{x} = \frac{2n+1}{2}, \quad n \in \mathbb{Z}.$$

Therefore:

$$\Theta(G(x,y)) - \Theta(x,y) = \int_{\tilde{\sigma}} \nabla\Theta \cdot \mathrm{d}\mathbf{x} = (2n+1)\pi,$$

and hence $e^{i\Theta(G(x,y))} = -e^{i\Theta(x,y)}$. □

We can finally prove the existence of a one-to-one correspondence between solutions of (2) and antisymmetric solutions of a real elliptic problem on the twofold covering manifold.

(3.9) Proposition. *Let $a \in D$ be fixed. Consider a solution U of* (2) *with $\mathbf{A}$ satisfying* (1) *and $\Gamma \in \mathcal{G}$. Then there exists an antisymmetric function $u : \Sigma_a \to \mathbb{R}$, weak solution of*

$$\begin{cases} -\Delta u + \tilde{V}u = 0 & \text{in } \Sigma_a \\ u = \gamma & \text{on } \partial\Sigma_a, \end{cases} \tag{12}$$

and such that $\Pi_x(\mathcal{N}(u)) = \mathcal{N}(U)$.

Proof. As we have already noticed, the nodal lines are invariant under gauge transformations, hence we can replace $\mathbf{A}$ with the magnetic potential $\mathbf{A}'$ defined in the previous lemma. In order to simplify the notations we will denote it again with $\mathbf{A}$. The definition of u consists now of two steps. Given U we first lift it on Σ_a in a symmetric way: evidently $\tilde{U}$ satisfies the Schrödinger equation on Σ_a with potential $\tilde{\mathbf{A}}$ and boundary data $\tilde{\Gamma}$. The second step is to multiply $\tilde{U}$ by the gauge phase: we define

$$u(x,y) := e^{-i\Theta(x,y)}\tilde{U}(x,y). \tag{13}$$

Notice that $u_{|\partial\Sigma_a} = \gamma$, since the magnetic potential was wisely chosen in Lemma 3.8. We can now take advantage of gauge invariance as in Lemma 3.2:

$$\begin{aligned} 0 &= \int_{\Sigma_a} \tilde{U}\overline{[(i\nabla + \tilde{\mathbf{A}})^2\varphi + V\varphi]}dx_1dx_2 + i\int_{\partial\Sigma_a}[\tilde{\Gamma}\overline{(i\nabla+\tilde{\mathbf{A}})\varphi}\cdot\nu + (i\nabla+\tilde{\mathbf{A}})\tilde{U}\cdot\nu\overline{\varphi}]d\sigma \\ &= \int_{\Sigma_a} u\overline{[(-\Delta)(e^{-i\Theta}\varphi) + \tilde{V}e^{-i\Theta}\varphi]}dx_1dx_2 + i\int_{\partial\Sigma_a}[\gamma\overline{(i\nabla)(e^{-i\Theta}\varphi)}\cdot\nu + i\nabla u\cdot\nu\overline{(e^{-i\Theta}\varphi)}]d\sigma. \end{aligned}$$

Hence for every real valued test function ψ it holds

$$\int_{\Sigma_a}(-\Delta\psi + \tilde{V}\psi)u dx_1dx_2 + \int_{\partial\Sigma_a}(\gamma\nabla\psi\cdot\nu - \psi\nabla u\cdot\nu)d\sigma = 0,$$

which is the weak form of (12). Being solution of an elliptic equation with real valued potential and boundary data, u is real valued. Moreover it is the product of an antisymmetric function times a symmetric one, hence it is antisymmetric. □

(3.10) Remark. It should be clear from the proof that the previous proposition holds also if Γ has an arbitrary number of zeroes, whereas the condition on the jump of Θ_1 at x_0 is of fundamental importance.

3.1 Related real elliptic problems

We shall now obtain, starting from (12), a real elliptic equation defined on a bounded subset of $\mathbb{R}^2$. This is performed in two different ways. In Lemma 3.13 we simply apply the projection Π_y, obtaining a real function which is suitable for the local analysis (see Section 4). In Lemma 3.14, instead, we also compose with a Möbius transformation, in order to obtain a function defined in the unit disk. This will be more appropriate for the analysis in Section 5, where the parameter a varies. Let us start recalling some known properties of conformal maps.

Notation. Here and in the following we will often make the identification $\mathbb{R}^2 \backsimeq \mathbb{C}$, writing $x = (x_1, x_2) = x_1 + ix_2$. We shall use the following standard notation for the complex derivative

$$\frac{\partial}{\partial x} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) \quad \frac{\partial}{\partial \bar{x}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$

where $\frac{\partial}{\partial x_i}$ denotes partial derivative.

(3.11) Lemma. *Let $\Omega_2 \subset \mathbb{C}$ open and bounded and $T(y) : \Omega_2 \to \Omega_1$ be a conformal map such that $T(\partial\Omega_2) = \partial\Omega_1$. Suppose $f \in H^1(\Omega_1)$ is a weak solution of $-\Delta f + Vf = 0$ in Ω_1, with $V(x) \in L^\infty(\Omega_1)$. Then $g(y) := f \circ T(y)$ satisfies $-\Delta g + Wg = 0$ in Ω_2, with $W(y) = |\frac{\partial T}{\partial y}(y)|^2 V \circ T(y)$. Moreover if V satisfies (4) also W does, and the energy associated to the equation is preserved:*

$$\int_{\Omega_1}(|\nabla f(x)|^2 + V(x)f(x)^2)dx_1dx_2 = \int_{\Omega_2}(|\nabla g(y)|^2 + W(y)g(y)^2)dy_1dy_2.$$

If moreover $f \in W^{1,\infty}(\partial\Omega_1)$, the following boundary complex integral is preserved:

$$\int_{\partial\Omega_1} \frac{\partial f}{\partial x}(x)dx = \int_{\partial\Omega_2} \frac{\partial g}{\partial y}(y)dy.$$

(3.12) Remark. Notice that the projection $\Pi_y : \Sigma_a \to \mathbb{R}^2$ introduced in the previous section is a global diffeomorphism of Σ_a onto its image, hence its inverse is well defined:

$$\begin{array}{lll} \Pi_y^{-1}: & \Pi_y(\Sigma_a) & \to \Sigma_a \\ & y & \mapsto (y^2+a, y). \end{array}$$

Moreover Π_y^{-1} is conformal on its domain[2] and $\left|\frac{\partial(\Pi_y^{-1})}{\partial y}(y)\right|^2 = 4|y|^2$. In the following we shall denote

$$\Omega_a = \Pi_y(\Sigma_a) \cup \{(0,0)\}.$$

(3.13) Lemma. *Under the same assumptions and notations of Proposition 3.9, the function $u^{(1)}(y) := u \circ \Pi_y^{-1}(y)$ is an odd solution of the real elliptic equation*

$$\begin{cases} -\Delta u^{(1)} + V^{(1)}u^{(1)} = 0 & \text{in } \Omega_a \\ u^{(1)} = \gamma^{(1)} & \text{on } \partial\Omega_a, \end{cases}$$

where $V^{(1)}(y) = 4|y|^2\tilde{V} \circ \Pi_y^{-1}(y)$ and $\gamma^{(1)}(y) = \gamma \circ \Pi_y^{-1}(y)$. Moreover $u^{(1)} \in C^2_{loc} \cap W^{1,\infty}(\overline{\Omega_a})$ and

$$Q_{\mathbf{A},V}(U) = \frac{1}{2}\int_{\Omega_a} (|\nabla u^{(1)}|^2 + V^{(1)}(u^{(1)})^2)dy_1dy_2.$$

Proof. Due to Proposition 3.9, Lemma 3.11 and Remark 3.12, $u^{(1)}$ satisfies the equation in $\Pi_y(\Sigma_a)$; moreover $u^{(1)}$ is clearly odd since u is antisymmetric on Σ_a. Let us show that we can extend $u^{(1)}$ at the origin in such a way that the equation is satisfied in Ω_a. Taking the complex derivative in (13) we obtain (in a weak sense)

$$\frac{\partial u}{\partial x} = e^{-i\Theta}\left(\frac{\partial}{\partial x} - i\frac{\partial\Theta}{\partial x}\right)\tilde{U}, \tag{14}$$

which implies, together with Lemma 3.11,

$$\begin{aligned} \int_{\Omega_a} |\nabla u^{(1)}|^2 dy_1dy_2 &= \int_{\Sigma_a} |\nabla u|^2 dx_1dx_2 \\ &= \int_{\Sigma_a} \left|\left(i\nabla + \frac{\partial\Theta}{\partial x}\right)\tilde{U}\right|^2 dx_1dx_2 \\ &= 2\int_D |(i\nabla + \mathbf{A})U|^2 dx_1dx_2 < \infty \end{aligned}$$

since $U \in \mathcal{H}_{\mathbf{A}}(D)$. Hence $u^{(1)} \in H^1(\Omega_a)$ and the equation is satisfied also at the origin (since a point has null capacity in $\mathbb{R}^2$). □

(3.14) Lemma. *Let $u^{(1)} : \Omega_a \to \mathbb{R}$ as in the previous lemma. There exists a conformal map $T'_a : D \to \Omega_a$ such that $u^{(2)}(y) := u^{(1)} \circ T'_a(y)$ satisfies*

$$\begin{cases} -\Delta u^{(2)} + V^{(2)}u^{(2)} = 0 & \text{in } D \\ u^{(2)} = \gamma^{(2)} & \text{on } \partial D, \end{cases}$$

where $V^{(2)}(y) = |\frac{\partial T'_a}{\partial y}(y)|^2 V^{(1)} \circ T'_a(y)$ and $\gamma^{(2)}(y) = \gamma^{(1)} \circ T'_a(y)$.

[2] With respect to the locally flat metric on Σ_a.

Proof. Proceding as in [7] we consider the Möbius transformation:

$$T_a : \overline{D} \longrightarrow \overline{D}, \qquad T_a(x) = \frac{x+a}{\bar{a}x+1}. \tag{15}$$

It is well known that T_a is a conformal map, such that $T_a(\partial D) = \partial D$ and $T_a(0) = a$. Let now $\tilde{T}_a(x,y) : \Sigma_0 \to \Sigma_a$ be the lifting of T_a. More precisely, if we denote for the moment $re^{i\vartheta} := T_a(x) - a$, we have $\tilde{T}_a(x,y) = (re^{i\vartheta} + a, \sqrt{r}e^{i\frac{\vartheta}{2}})$. Thanks to Lemma 3.11, it only remains to prove that the map $T'_a : D \to \Omega_a$, defined by $T'_a = \Pi_y \circ \tilde{T}_a \circ \Pi_y^{-1}$, is conformal. Indeed it is clearly conformal outside the origin, since the complex square root is well defined and conformal on the twofold covering manifold Σ_a. Moreover it is bounded and hence it admits a conformal extension at the origin (see for example [9], Proposition 4.3.3). □

(3.15) Remark. In the previous lemma we have equivalently

$$V^{(2)}(y) = 4|y|^2 \left| \frac{\partial T_a}{\partial x}(y^2) \right|^2 V \circ T_a(y^2), \qquad \gamma^{(2)}(y) = \gamma \circ T_a(y^2).$$

4 Properties of the nodal set

Aim of this section is the proof of Theorem 1.1. Let us start recalling some known properties of the nodal set and singular points of solutions of real elliptic equations of the following kind

$$\begin{cases} -\Delta f + Vf = 0 & \text{in } \Omega \\ f = \gamma & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain and $V \in L^\infty(\Omega)$, $\gamma \in W^{1,\infty}(\partial\Omega)$ are real valued functions. By standard regularity results and Sobolev imbedding $f \in C^{1,\alpha}_{loc}(\Omega) \cap W^{1,\infty}(\overline{\Omega})$, $\forall \alpha \in (0,1)$.

(4.1) Definition. We say that $y_0 \in \mathcal{N}(f)$ is a singular point if $\nabla f(y_0) = 0$. We say that it is a zero of order (or multiplicity) n if $\frac{\partial^k f}{\partial y}(y_0) = 0$, $\forall k \leq n$.

A classical result by Hartman and Wintner (see [12]), states that

(4.2) Theorem. *Assume that f is a non trivial solution of the previous equation.*

(i) The interior singular points of f are isolated and have finite multiplicity $n \in \mathbb{N}$ $(n \geq 1)$.

(ii) The nodal set of f is the union of finitely many connected arcs, locally $C^{1,\alpha}$, with endpoints either at $\partial\Omega$ or at interior singular points.

(iii) If f has a zero of order n at y_0, then it satisfies the asymptotic expansion

$$f(\rho,\varphi) = \frac{\rho^{n+1}}{n+1}\Big\{c_{n+1}\cos[(n+1)\varphi] + d_{n+1}\sin[(n+1)\varphi]\Big\} + o(\rho^{n+1}),$$

where $y - y_0 = \rho e^{i\varphi}$ and c_{n+1}, d_{n+1} are real constants, not both zero. Moreover an expression for the first non zero coefficients of the expansion is

$$c_{n+1} - id_{n+1} = 2i\int_{\partial\Omega} \frac{\partial f}{\partial y}\, h_{n+1}\, dy - \int_\Omega -\Delta f\, h_{n+1}\, dy_1 dy_2$$

where the first integral is a complex line integral, whereas the second one is a double integral in the real variables y_1, y_2, and

$$h_{n+1}(y) = -\frac{1}{2\pi}\frac{1}{(y-y_0)^{n+1}}.$$

We shall now prove that, under the assumptions we are considering, we can still recover similar properties for the magnetic Schrödinger equation. The behaviour of the nodal lines is unaltered far from the singular point a, but undergos meaningful changes in a neighbourhood of it.

(4.3) Theorem. *Let $a \in D$ be fixed. Consider a solution U of (2) with $\mathbf{A}$ satisfying (1) and $\Gamma \in \mathcal{G}$.*

(i) The nodal set of U is the union of finitely many connected arcs, locally $C^{1,\alpha}$, with endpoints either at $\partial\Omega$, or at an interior singular point of U, or at a. Moreover there is at least one nodal line with endpoint at a.

(ii) If $x_0 \neq a$ is an interior singular point, then properties (i),(iii) of Theorem 4.2 hold, in particular there is an even number of $C^{1,\alpha}$ arcs meeting at x_0.

(iii) In a neighbourhood of a, U satisfies for some odd $k \geq 1$ the asymptotic formula

$$U(r_a, \vartheta_a) = e^{i\Theta(r_a,\vartheta_a)} \frac{r_a^{\frac{k}{2}}}{k} \left[c_k \cos\left(\frac{k}{2}\vartheta_a\right) + d_k \sin\left(\frac{k}{2}\vartheta_a\right) \right] + o(r_a^{\frac{k}{2}}), \tag{16}$$

where $x - a = r_a e^{i\vartheta_a}$. In particular there may be an odd number of nodal lines ending at a.

(iv) The first non zero coefficients of the asymptotic formula can be expressed as

$$c_k - id_k = 4i \int_{\partial D} G_k \left(\frac{\partial U}{\partial x} - i\mathbf{A}U \right) \, dx - 2 \int_D G_k (i\nabla + \mathbf{A})^2 U \; dx_1 dx_2, \tag{17}$$

where the first integral is a complex line integral, whereas the second one is a double integral in the real variables x_1, x_2, and

$$G_k = -\frac{1}{2\pi} \frac{e^{-i\Theta}}{(x-a)^{\frac{k}{2}}}. \tag{18}$$

Notice that Remark 3.10 still holds here.

Proof. Consider the function $u^{(1)} : \Omega_a \to \mathbb{R}$ defined in Lemma 3.13: it clearly satisfies the properties collected in the previous theorem. Notice than that Π_y is locally holomorphic on every open set which does not contain the point $(a, 0) \in \Sigma_a$. Therefore the local properties of the nodal lines are preserved in the composition and U satisfies Theorem 4.2 at every singular point different from a. In order to prove that there is at least one nodal line with endpoint at a, observe that Ω_a is symmetric with respect to the origin and $u^{(1)}$ is odd. This implies that the nodal set of $u^{(1)}$ is also symmetric and in particular there are at least two arcs of nodal line having an endpoint at the origin.

In order to prove (iii) let us consider the asymptotic expansion of $u^{(1)}$ near the origin. Since $u^{(1)}$ is odd respect to the origin, Theorem 4.2 (iii) gives, for some odd $k \geq 1$

$$u^{(1)}(\rho, \varphi) = \frac{\rho^k}{k} \left[c_k \cos(k\varphi) + d_k \sin(k\varphi) \right] + o(\rho^k),$$

where $y = \rho e^{i\varphi}$. From the definition of $u^{(1)}$ we can recover[3] an expression for u:

$$u(r_a, \vartheta_a) = \frac{r_a^{\frac{k}{2}}}{k} \left[c_k \cos\left(\frac{k}{2}\vartheta_a\right) + d_k \sin\left(\frac{k}{2}\vartheta_a\right) \right] + o(r_a^{\frac{k}{2}}).$$

[3]Remember that we endowed Σ_a with the locally flat metric induced by Π_x, and we denote $x - a = r_a e^{i\vartheta_a}$

Notice that the last expression is well defined on Σ_a, since the complex square root function is continuous on the twofold covering manifold. Finally, (13) provides the corresponding expression for $\tilde{U}$ which, being symmetric, can be projected on D, providing (16).

Let us go back to the asymptotic expansion of $u^{(1)}$ near the origin. As in the previous theorem we have

$$c_k - id_k = 2i \int_{\partial\Omega_a} \frac{\partial u^{(1)}}{\partial y} \, h_k \, dy - \int_{\Omega_a} -\Delta u^{(1)} \, h_k \, dy_1 dy_2, \qquad \text{with} \;\; h_k(y) = -\frac{1}{2\pi}\frac{1}{y^k}$$

Let us remark again that the first integral is a complex line integral, whereas the second one is a double integral in real variables. We are now going to perform a change of variables in order to obtain an expression for the coefficients depending only on U. Using Lemma 3.11, we can shift the last integral on Σ_a obtaining

$$c_k - id_k = 2i \int_{\partial\Sigma_a} \frac{\partial u}{\partial x} \, g_k \, dx - \int_{\Sigma_a} -\Delta u \; g_k \; dx_1 dx_2,$$

where $g_k(x,y) = h_k \circ \Pi_y(x,y)$. Taking the complex derivative in (14) we obtain

$$\begin{aligned} -\Delta u &= -4\frac{\partial}{\partial \bar{x}}\frac{\partial u}{\partial x} = -4e^{-i\Theta}\left(\frac{\partial}{\partial \bar{x}} - i\frac{\partial\Theta}{\partial \bar{x}}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial\Theta}{\partial x}\right)\tilde{U} \\ &= 4e^{-i\Theta}\left[-\frac{\partial}{\partial \bar{x}}\frac{\partial}{\partial x} + i\left(\frac{\partial\Theta}{\partial x}\frac{\partial}{\partial \bar{x}} + \frac{\partial\Theta}{\partial \bar{x}}\frac{\partial}{\partial x}\right) + \frac{\partial\Theta}{\partial \bar{x}}\frac{\partial\Theta}{\partial x}\right]\tilde{U} \\ &= e^{-i\Theta}(i\nabla + \tilde{\mathbf{A}})^2\tilde{U}. \end{aligned}$$

Replacing the last expression in the integral we obtain

$$c_k - id_k = 2i \int_{\partial\Sigma_a} e^{-i\Theta} g_k \left(\frac{\partial\tilde{U}}{\partial x} - i\tilde{U}\frac{\partial\Theta}{\partial x}\right) dx - \int_{\Sigma_a} e^{-i\Theta} g_k (i\nabla + \tilde{\mathbf{A}})^2 \tilde{U} \; dx_1 dx_2.$$

In order to conclude the proof it is sufficient to define

$$G_k(x,y) = e^{-i\Theta} g_k(x,y) = e^{-i\Theta} h_k \circ \Pi_y(x,y),$$

and then to observe that both integrands are symmetric on Σ_a, therefore the last expression can be projected on D. □

Proof of Theorem 1.1. Property (i) was proved in Lemma 3.8 and Proposition 3.9; properties (ii) and (iii) were proved in the preceding theorem. We claim that, under the additional assumption (4) on the potential $V(x)$, every nodal line of U can not be a closed curve. Notice first that $u^{(1)}$ satisfies the maximum principle, since it is shown in Lemma 3.11 that (4) is preserved by conformal transformations. Thus the nodal lines of $u^{(1)}$ can not be closed curves (by the unique continuation property for real elliptic equations, see for example [12]) and this property is preserved by the projections Π_x, Π_y. Now assume also that Γ vanishes exactly three times on ∂D, then by simple geometric considerations we infer that there can be at most three nodal lines. □

(4.4) Corollary. *Under the previous assumptions, assume that the nodal set of U is made of more than one arc. Than only two configurations are possible:*

1. *The nodal set of U is made of two arcs. One of them has an endpoint at a, the other one has both endpoints at ∂D. The asymptotic expansion (16) holds with $k = 1$.*

2. *The nodal set of U is made of three arcs intersecting at a (i.e. a is a triple point for Γ). The asymptotic expansion (16) holds with $k = 3$.*

Proof. We can distinguish the number of nodal lines depending on the choice of the signs $\sigma_i = \pm 1$ in the definition of γ. If σ_i are all equal, then $\gamma^{(1)}$ changes sign exactly two times and, by the maximum principle, the nodal set of $u^{(1)}$ consists of exactly two arcs meeting at the origin. In this case the nodal set of U is made of one arc and there can not be any triple point. Assume instead that one of the σ_i is different from the others, then $\gamma^{(1)}$ changes sign exactly six times. Taking advantage of the maximum principle again, it is easy to see that the only possible configurations are the ones described in the statement. □

5 Local uniqueness

In order to prove Theorem 1.2, we first establish here point (i) and a local uniqueness result. Thoroughout this section we fix a boundary data $\Gamma \in \mathcal{G}$ and consider the set of equations (2), as a varies in D, under assumptions (1) and (4). We will stress the dependence on the parameter a of each quantity, since the position of the singularity is now a variable of the problem.

Let us fix the gauge of the magnetic potential. In the following $\mathbf{A}_a$ and Θ_a denote respectively the vector potential and the scalar potential introduced in Lemma 3.8. Moreover we will denote u_a the function defined in (13) and $u_a^{(1)}$, $u_a^{(2)}$ the functions introduced in Lemma 3.13, 3.14 respectively.

(5.1) Lemma. *Under the previous notations there holds*

$$\|u_{a_1}^{(2)} - u_{a_2}^{(2)}\|_{C^1_{loc}(D)} \leq C|a_1 - a_2|^\alpha,$$

for some $\alpha > 0$. Moreover the same estimate holds for $u_a^{(1)}$, locally on $\Omega_{a_1} \cap \Omega_{a_2}$.

Proof. The proof relies on the fact that the functions $u_a^{(2)}$ are defined in D for every a. By assumptions and standard imbeddings, $V \in C^{0,\alpha}(D)$ for some [4] $\alpha > 0$ and $\gamma \in C^{0,1}(\partial D)$. Using Remark 3.15 and remembering that the Möbius transformation $T_a(y^2)$ is regular, it is easy to see that

$$\|V_{a_1}^{(2)} - V_{a_2}^{(2)}\|_{L^\infty(D)} \leq C|a_1 - a_2|^\alpha, \qquad \|\gamma_{a_1}^{(2)} - \gamma_{a_2}^{(2)}\|_{L^\infty(\partial D)} \leq C|a_1 - a_2|.$$

On the other hand $u_a^{(2)}$ satisfies an elliptic Cauchy problem (see Lemma 3.14), hence by standard regularity results there holds

$$\|u_{a_1}^{(2)} - u_{a_2}^{(2)}\|_{W^{2,p}_{loc}(D)} \leq C|a_1 - a_2|^\alpha, \quad \forall\, p \in (0, +\infty),$$

which gives the thesis. Finally observe that $u_a^{(1)}$ is the composition of $u_a^{(2)}$ with a regular function (by Lemma 3.14 again), hence the same estimate holds, whenever it is well defined. □

(5.2) Remark. The previous lemma implies some local estimates in case of triple point configuration. In order to simplify the notation, assume here that the origin is a triple point. By Corollary 4.4 we infer $|u_0| \leq Cr^3$, $|\nabla u_0| \leq C'r^2$ in D_r. Hence the preceding lemma allows the following estimates in a small ball containing the singularity:

$$\|u_a^{(1)}\|_{L^\infty}, \|\nabla u_a^{(1)}\|_{L^\infty} \leq C|a| \quad \text{in } D_{\sqrt{2|a|}},$$

and the same holds for $u_a^{(2)}$. Moreover differentiating $u_a^{(1)}$ on $\partial D_{\sqrt{2|a|}}$ we obtain $|\nabla u_a(x)| \leq \|\nabla u_a^{(1)}\|_{L^\infty(D_{\sqrt{2|a|}})}/\sqrt{x}$, which yields

$$\|u_a\|_{L^\infty} \leq C|a| \quad \text{in } D_{2|a|}, \qquad \|\nabla u_a\|_{L^\infty} \leq C\sqrt{|a|} \quad \text{on } \partial D_{2|a|}.$$

[4]We can choose $\alpha \in (0, 1 - 2/p)$, if $V \in W^{1,p}(D)$ with $p < +\infty$, and $\alpha \in (0, 1)$ if $V \in W^{1,\infty}(D)$. Here the constant C depends on α.

(5.3) Lemma. *Assume $\gamma \in C^{1,1}(\partial\Omega)$, then the function*

$$a \mapsto \varphi(a) = \min\{Q_{\mathbf{A}_a,V}(U) \ : \ U \in \mathcal{H}_{\mathbf{A}}(D),\ U = \Gamma \text{ on } \partial D\}.$$

is differentiable for every $a \in D$.

Proof. We can rewrite $\varphi(a)$ in the following way:

$$\varphi(a) = \int_D \left(|(i\nabla + \mathbf{A}_a)U_a|^2 + V|U_a|^2\right) dx_1 dx_2 = \frac{1}{2}\int_D \left(|\nabla u_a^{(2)}|^2 + V_a^{(2)}(u_a^{(2)})^2\right) dy_1 dy_2.$$

Let us start showing the existence of the partial derivatives of $\varphi(a)$; without loss of generality we can consider the derivative in the direction $a = (a, 0)$, centered at the origin. Notice that there exist the weak derivatives $\frac{\partial V_a^{(2)}}{\partial a} \in L^p(D)$ and $\frac{\partial^2 \gamma_a^{(2)}}{\partial a^2} \in L^\infty(\partial D)$; this is due to Remark 3.15 and to the assumption $V \in W^{1,p}(D)$, $\gamma \in C^{1,1}(\partial D)$. If we prove that

$$\lim_{a\to 0} \left\| \frac{V_a^{(2)} - V_0^{(2)}}{a} - \frac{\partial V_a^{(2)}}{\partial a}\Big|_{a=0} \right\|_{L^p(D)} = \lim_{a\to 0} \left\| \frac{\gamma_a^{(2)} - \gamma_0^{(2)}}{a} - \frac{\partial \gamma_a^{(2)}}{\partial a}\Big|_{a=0} \right\|_{W^{1,p}(D)} = 0, \qquad (19)$$

then standard regularity results for elliptic equations ensure the existence of $w \in H^1(D)$, solution of the following equation

$$\begin{cases} -\Delta w + V_0^{(2)} w + \frac{\partial V_a^{(2)}}{\partial a}\Big|_{a=0} u_0^{(2)} = 0 & \text{in } D \\ w = \frac{\partial \gamma_a^{(2)}}{\partial a}\Big|_{a=0} & \text{on } \partial D, \end{cases}$$

such that

$$\lim_{a\to 0} \left\| \frac{u_a^{(2)} - u_0^{(2)}}{a} - w \right\|_{H^1(D)} = 0.$$

This implies the existence of the partial derivative

$$\frac{\partial \varphi}{\partial a}(0) = 2\int_D \left(\nabla u_0^{(2)} \cdot \nabla w + V_0^{(2)} u_0^{(2)} w\right) dy_1 dy_2.$$

Hence let us prove (19). In order to simplify notations we denote here $R(a, y) := T_a(y^2)$, where T_a is defined in (15). It is sufficient to estimate the following quantity (as $a \to 0$) since, by Remark 3.15, the other terms are regular:

$$\left\| \frac{V(R(a,y)) - V(R(0,y))}{a} - \frac{\partial V(R(a,y))}{\partial a}\Big|_{a=0} \right\|_{L^p(D)} =$$
$$= \left\| \int_0^1 \left[\nabla_x V(R(ta,y)) \frac{\partial R(ta,y)}{\partial a} - \nabla_x V(R(0,y)) \frac{\partial R(a,y)}{\partial a}\Big|_{a=0} \right] dt \right\|_{L^p(D)}.$$

By Lusin's theorem, the integrand converges to zero outside an arbitrarily small set. Then applying Lebesgue convergence theorem, we obtain the first relation in (19). The second one can be proved in a similar way. In order to prove differentiability we test the equation for $u_0^{(2)}$ with w, obtaining

$$\frac{\partial \varphi}{\partial a}(0) = 2\int_{\partial D} w \nabla u_0^{(2)} \cdot \nu d\sigma.$$

The continuity of this function, with respect to a, comes from Lemma 5.1 and from the regularity of γ. □

Proof of Theorem 1.2, (i). **A triple point is a critical point of φ.** Without loss of generality we can assume the triple point to be the origin (applying the conformal map T_a defined in (15)), hence we need to show

$$\lim_{|a|\to 0} \frac{\varphi(a)-\varphi(0)}{|a|} = 0$$

The main idea is to split $\varphi(a)-\varphi(0)$ into the sum of two integrals:

$$\begin{aligned}
\varphi(a)-\varphi(0) &= \int_D \left(|(i\nabla+\mathbf{A}_a)U_a|^2 - |(i\nabla+\mathbf{A}_0)U_0|^2 + V(|U_a|^2-|U_0|^2)\right) dx_1dx_2 \\
&= I + II,
\end{aligned}$$

where I is the integral on a small ball $D_{2|a|}$, and II is the remaining term. As it concerns the integral in the exterior annulus, the main observation is that both u_0 and u_a are well defined on the twofold covering manifold $\Sigma_0 \setminus \Pi_x^{-1}(D_{2|a|})$. Indeed a scalar potential Θ_a can be defined on $\Sigma_0 \setminus \Pi_x^{-1}(D_{2|a|})$, proceeding as in Lemma 3.8, and this allows to define a function u_a as in Proposition 3.9. With some abuse of notation we still use the notation u_a, as in Proposition 3.9, since both functions have the same projection on D, and Σ_0 is endowed with the metric induced from D. In particular Remark 5.2 still holds and moreover

$$\begin{cases} -\Delta(u_a-u_0)+\tilde{V}(u_a-u_0)=0 & \text{in } \Sigma_0\setminus\Pi_x^{-1}(D_{2|a|}) \\ u_a-u_0=0 & \text{on } \partial\Sigma_0. \end{cases}$$

Hence by Lemma 3.11 we have

$$\begin{aligned}
II &= \frac{1}{2}\int_{\Sigma_0\setminus\Pi_x^{-1}(D_{2|a|})} \nabla(u_a-u_0)\cdot\nabla(u_a+u_0)+\tilde{V}(u_a-u_0)(u_a+u_0)dx_1dx_2 \\
&\leq C\left(\int_{D\setminus D_{2|a|}} |\nabla(u_a-u_0)|^2\right)^{1/2} + C'\left(\int_{D\setminus D_{2|a|}} (u_a-u_0)^2\right)^{1/2}.
\end{aligned}$$

On the other hand the equation for u_a-u_0 gives

$$\begin{aligned}
\int_{D\setminus D_{2|a|}} \left(|\nabla(u_a-u_0)|^2+V(u_a-u_0)^2\right)dx_1dx_2 &\leq \int_{\partial D_{2|a|}} |u_a-u_0|\left|\frac{\partial}{\partial\nu}(u_a-u_0)\right| d\sigma \\
&\leq |\partial D_{2|a|}| \sup_{\partial D_{2|a|}} \{|\nabla(u_a-u_0)||u_a-u_0|\} \\
&\leq C|a|^{5/2},
\end{aligned}$$

where we used Remark 5.2 in the last inequality. We infer $II \leq C|a|^{5/4}$.

As it concerns the integral in $D_{2|a|}$ we have similarly

$$I \leq C\left(\int_{D_{\sqrt{2|a|}}} |\nabla(u_a^{(2)}-u_0^{(2)})|^2\right)^{1/2} + C'\left(\int_{D_{\sqrt{2|a|}}} (u_a^{(2)}-u_0^{(2)})^2\right)^{1/2}.$$

Notice that here we need to apply a slightly different transformation respect to Lemma 3.14. To be more precise we consider the map

$$S_a : \overline{D_{2|a|}} \longrightarrow \overline{D_{2|a|}}, \qquad S_a(x) = 2|a|\frac{x+2|a|a}{\bar{a}x+2|a|}.$$

With abuse of notation we denote the function $u_a^{(2)}$, which still satisfies Remark 5.2. Hence we can apply this remark as we did with II, and finally obtain

$$\lim_{|a|\to 0}\frac{I+II}{|a|} = \lim_{|a|\to 0}|a|^{1/4} = 0.$$

Triple points are the only critical points of φ. Assume that the origin is not a triple point for Γ, we are going to show that $\varphi'(0) \neq 0$. By Corollary 4.4, the solution has the following asymptotic expansion around the origin[5]

$$U_0(r,\vartheta) = Ce^{i\Theta_0} r^{\frac{1}{2}} \cos(\frac{\vartheta}{2} - \alpha) + o(r^{\frac{1}{2}}). \tag{20}$$

Hence there is exactly one nodal arc η ending at the origin and there exists a radius h such that $U_0(x) \neq 0,\ \forall\, x \in D_h \setminus \{\eta \cup \{(0,0)\}\}$. Let $w : D_h \to \mathbb{R}$ be the (nonnegative) solution of

$$\begin{cases} -\Delta w + Vw = 0 & \text{in } D_h \\ w = |u_0| & \text{on } \partial D_h. \end{cases}$$

Let now $a = \eta \cap \partial D_h$ in such a way that $|a| = h$. We define a new function $z_a : \Sigma_a \to \mathbb{R}$ as

$$z_a = \begin{cases} \sigma(x)\tilde{w}(x) & x \in \Pi_x^{-1}(D_h) \\ \sigma(x)|\tilde{u}_0| & x \in \Sigma_a \setminus \Pi_x^{-1}(D_h), \end{cases}$$

where $\sigma = \pm 1$ in such a way that z_a is antisymmetric on Σ_a. If $\Theta_a : \Sigma_a \to \mathbb{R}$ is defined as in Lemma 3.8, then $Z_a = \Pi_x(e^{-i\Theta_a} z_a)$ is well defined in D, $Z_a \in \mathcal{H}_{\mathbf{A}}(D)$ and its complex derivative satisfies the analogous of equation (14). Moreover by definition there holds $\varphi(a) \leq Q_{\mathbf{A}_a,V}(Z_a)$, hence

$$\begin{aligned} \varphi(0) - \varphi(a) &\geq \int_D \left[|(i\nabla + \mathbf{A}_0)U_0|^2 - |(i\nabla + \mathbf{A}_a)Z_a|^2 + V(|U_0|^2 - |Z_a|^2) \right] dx_1 dx_2 \\ &= \int_{D_h} \left[|\nabla u_0|^2 - |\nabla w|^2 + V(u_0^2 - w^2) \right] dx_1 dx_2. \end{aligned}$$

We shall complete the proof by showing that the following quantity

$$\lim_{|a|\to 0} \frac{\varphi(0) - \varphi(a)}{|a|} \geq \lim_{h\to 0} \frac{1}{h} \int_{D_h} \left[|\nabla u_0|^2 - |\nabla w|^2 + V(u_0^2 - w^2) \right] dx_1 dx_2$$

is greater than a positive constant. In order to estimate the limit we perform a change of variables:

$$u^{(h)}(y) = \frac{1}{\sqrt{h}} u_0(hy^2, \sqrt{h}y), \quad w^{(h)}(y) = \frac{1}{\sqrt{h}} w(hy^2).$$

These functions satisfy the rescaled problems

$$\begin{cases} -\Delta u^{(h)} + hV(hy^2)u^{(h)} = 0 & \text{in } D \\ u^{(h)} = \frac{1}{\sqrt{h}} u_0(hy^2, \sqrt{h}y) & \text{on } \partial D, \end{cases} \qquad \begin{cases} -\Delta w^{(h)} + hV(hy^2)w^{(h)} = 0 & \text{in } D \\ w^{(h)} = |u^{(h)}| & \text{on } \partial D, \end{cases}$$

and moreover, by (20), $u^{(h)}$ satisfies the asymptotic expansion

$$u^{(h)}(\rho,\varphi) = C\rho\cos(\varphi - \alpha) + o(\sqrt{h}\rho),$$

where as usual $y = \rho e^{i\varphi}$. This ensures the existence of a limit function u^∞ such that

$$\begin{cases} -\Delta u^\infty = 0 & \text{in } D \\ u^\infty(\rho,\varphi) = C\cos(\varphi - \alpha) & \text{on } \partial D \end{cases} \quad \text{and} \quad \|u^{(h)} - u^\infty\|_{C^1(\overline{D})} \to 0, \text{ as } h \to 0.$$

As a consequence, $\||u^{(h)}| - |u^\infty|\|_{W^{1,p}(\partial D)} \to 0,\ \forall p \in (1, +\infty)$, which implies

$$\begin{cases} -\Delta w^\infty = 0 & \text{in } D \\ w^\infty(\rho,\varphi) = |C\cos(\varphi - \alpha)| & \text{on } \partial D \end{cases} \quad \text{and} \quad \|w^{(h)} - w^\infty\|_{H^1(D)} \to 0, \text{ as } h \to 0.$$

[5]Here, respect to equation (16), we have set $\alpha = \arctan(d_1/c_1)$ and $C = c_1/\cos\alpha \neq 0$.

Therefore we have obtained

$$\lim_{|a|\to 0} \frac{\varphi(0)-\varphi(a)}{|a|} \geq \frac{1}{2}\int_D \left[|\nabla u^\infty|^2 - |\nabla w^\infty|^2\right] dy_1 dy_2.$$

This can be easily evaluated, since we know u^∞ and w^∞ explicitly. Indeed, choosing the coordinates in such a way that $\alpha = 0$ we have

$$u^\infty = C\rho\cos(\varphi), \qquad \int_D |\nabla u^\infty|^2 dy_1 dy_2 = C^2\pi,$$

and

$$w^\infty = \frac{2|C|}{\pi}\left[1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2}\rho^{2n}\cos(2n\varphi)\right],$$

which gives

$$\begin{aligned} \int_D |\nabla w^\infty|^2 dy_1 dy_2 &= \frac{32C^2}{\pi}\sum_{n=1}^{\infty}\frac{n}{(1-4n^2)^2} \\ &\leq \frac{32C^2}{\pi}\left[\frac{1}{9} + \int_1^\infty \frac{t}{(4t^2-1)^2}dt\right] \leq \frac{44C^2}{9\pi}. \end{aligned}$$

This finally gives $\varphi'(0) \leq \frac{C^2}{2}\left(\frac{44}{9\pi} - \pi\right) \leq -\frac{C^2}{2}$ and concludes the proof. □

(5.4) Remark. The first part of the previous proof also provides

$$\lim_{|a_1-a_2|\to 0} \frac{||U_{a_1} - U_{a_2}||_{L^2(D)}}{|a_1 - a_2|} = 0,$$

whenever a_1 is a triple point.

The last technical result that we need to prove the local uniqueness is a complex formulation of Green's theorem, in case of magnetic potential.

(5.5) Lemma. *Let $\Omega \subset \mathbb{C}$ be a regular domain and $\Phi, \Psi \in C^1(\overline{\Omega}, \mathbb{C})$. Let $\mathbf{A} \in C^1(\Omega)$ be a vector potential, such that $\nabla\Theta = \mathbf{A}$. Then there holds*

$$\begin{aligned} \int_\Omega \Psi(i\nabla + \mathbf{A})^2\Phi\, dx_1 dx_2 = 2i\int_{\partial\Omega} \Psi\left(\frac{\partial}{\partial x} - i\frac{\partial\Theta}{\partial x}\right)\Phi\, dx + \\ + 4\int_\Omega \left(\frac{\partial}{\partial x} - i\frac{\partial\Theta}{\partial x}\right)\Phi \cdot \left(\frac{\partial}{\partial \bar{x}} + i\frac{\partial\Theta}{\partial \bar{x}}\right)\Psi\, dx_1 dx_2. \end{aligned}$$

Proof. It is sufficient to apply the following complex formulation of Green's formula (see for example [9], Appendix A):

$$\int_\Omega \frac{\partial F}{\partial \bar{x}} dx_1 dx_2 = -\frac{i}{2}\int_{\partial\Omega} F dx,$$

with

$$F = -4\frac{\partial}{\partial x}\left(e^{-i\Theta}\Phi\right)e^{i\Theta}\Psi.$$

□

(5.6) Theorem. (Local uniqueness) *Consider the set of equations* (2) *under assumptions* (1) *and* (4)*, where the parameter a varies in D and the magnetic potential $\mathbf{A}_a$ varies under gauge transformations. Assume that $\Gamma \in \mathcal{G}$ admits a triple point $a_\Gamma \in D$. Then there exist $\varepsilon > 0$, such that, for every boundary data $\Lambda \in \mathcal{G}$ satisfying $||\Gamma - \Lambda||_{L^\infty(\partial D)} < \varepsilon$, there exists exactly one a_Λ (triple point for Λ) such that $|a_\Gamma - a_\Lambda| < ||\Gamma - \Lambda||_{L^\infty(\partial D)}$.*

Proof. The following argument takes some ideas from Proposition 3.2 in [7], where the authors study the particular case $V = 0$. Let us recall that in the following $\mathbf{A}_a$ and Θ_a are as in Lemma 3.8 and U_a denotes the solution of (2) with this specific potential. We can assume w.l.o.g. that $a_\Gamma = 0$ (applying the conformal map T_{a_Γ} defined in (15)), therefore we are assuming that the function U_0 has a triple point at the origin.

In order to prove local uniqueness we shall apply the implicit function theorem to the map

$$\begin{array}{rcl} \mathcal{G} \times D & \to & \mathbb{R}^2 \\ (\Lambda, a) & \mapsto & (c_1(a), d_1(a)) \end{array} \tag{21}$$

where $c_1(a)$, $d_1(a)$ are the first coefficients of U_a which appear in the asymptotic expansion (16) (U_a is the solution with boundary data Λ and singularity at a). Corollary 4.4 ensures that a is a triple point if and only if $c_1(a) = d_1(a) = 0$. Therefore the theorem is proved provided we can locally solve this equation for a in a neighbourhood of $(\Gamma, 0)$.

First of all we observe that (21) defines a C^1 function. Indeed it comes from the proof of Theorem 4.3 that $(c_1(a), d_1(a)) = \nabla_y u_a^{(1)}(0)$, and regularity can be proved proceeding as in Lemma 5.3.[6] Therefore we only need to show that the 2×2 Jacobian matrix

$$\nabla_a (c_1(a), d_1(a))\Big|_{a=0}$$

is invertible. By Theorem 4.3, (iv), the first nonzero coefficients in the asymptotic expansion of U_a can be expressed as

$$c_1(a) - i d_1(a) = 4i \int_{\partial D} G_{1,a} \left(\frac{\partial}{\partial x} - i \mathbf{A}_a \right) U_a \, dx - 2 \int_D G_{1,a} (i\nabla + \mathbf{A}_a)^2 U_a \, dx_1 dx_2, \tag{22}$$

with

$$G_{1,a} = -\frac{1}{2\pi} \frac{e^{-i\Theta_a}}{(x-a)^{\frac{1}{2}}}.$$

Notice that the differential operator commutes with the integral since the functions $\frac{\partial G_{1,a}}{\partial a}(x) \simeq \frac{1}{(x-a)^{3/2}}$ belong to $L^1(D)$ for every a. The main difficulty here is that we do not know the behaviour of U_a with respect to the variation of the parameter a, therefore we need to manipulate the last expression before differentiating. In order to get rid of the boundary integral in (22), we introduce a new function $F_a : D \to \mathbb{C}$, solution of the equation

$$\begin{cases} \left(\frac{\partial}{\partial \bar{x}} + i \frac{\partial \Theta_a}{\partial \bar{x}} \right) F_a = 0 & \text{on } D \\ F_a = G_{1,a} & \text{on } \partial D. \end{cases}$$

Applying Green's formula (Lemma 5.5), equation (22) becomes

$$\begin{aligned} c_1(a) - i d_1(a) &= 2 \int_D (F_a - G_{1,a})(i\nabla + \mathbf{A}_a)^2 U_a \, dx_1 dx_2 \\ &= 2 \int_D (F_a - G_{1,a})(i\nabla + \mathbf{A}_0)^2 U_0 \, dx_1 dx_2 + 2 \int_D (F_a - G_{1,a})(VU_0 - VU_a) \, dx_1 dx_2. \end{aligned} \tag{23}$$

Instead of computing the derivative of the last expression with respect to a, it will be convenient to apply the differential operator $\left(\frac{\partial}{\partial a} + i \frac{\partial \Theta_a}{\partial a} \right)\Big|_{a=0}$. Since U_0 has a triple point at the origin we

[6] Here we do not need additional regularity on the boundary data, since the estimates are local.

have

$$\left(\frac{\partial}{\partial a}+i\frac{\partial\Theta_a}{\partial a}\right)(c_1(a)-id_1(a))\Big|_{a=0}$$
$$=\frac{\partial}{\partial a}(c_1(a)-id_1(a))\Big|_{a=0}-i\frac{\partial\Theta_a}{\partial a}\Big|_{a=0}(c_1(0)-id_1(0))$$
$$=\frac{\partial}{\partial a}(c_1(a)-id_1(a))\Big|_{a=0},$$

therefore the differential operator coincides in this case with the complex derivative respect to a (evaluated in $a=0$). Next applying Remark 5.4 we obtain

$$2\left|\int_D (F_a-G_{1,a})(VU_0-VU_a)\ dx_1dx_2\right|$$
$$\leq 2||V||_{L^\infty(D)}||F_a-G_{1,a}||_{L^2(D)}||U_0-U_a||_{L^2(D)}=o(|a|).$$

Hence the last term in (23) is ininfluent in the computation of the derivative, and we have

$$\left(\frac{\partial}{\partial a}+i\frac{\partial\Theta_a}{\partial a}\right)(c_1(a)-id_1(a))\Big|_{a=0}$$
$$=\int_D (i\nabla+\mathbf{A}_0)^2U_0\left(\left(\frac{\partial}{\partial a}+i\frac{\partial\Theta_a}{\partial a}\right)F_a\Big|_{a=0}-\left(\frac{\partial}{\partial a}+i\frac{\partial\Theta_a}{\partial a}\right)G_{1,a}\Big|_{a=0}\right)\ dx_1dx_2. \quad (24)$$

We can differentiate $G_{1,a}$ directly:

$$\left(\frac{\partial}{\partial a}+i\frac{\partial\Theta_a}{\partial a}\right)G_{1,a}=-\frac{1}{4\pi}\frac{e^{-i\Theta_a(x)}}{(x-a)^{\frac{3}{2}}}=\frac{1}{2}G_{3,a}.$$

Notice that we obtain a multiple of the function defined in (18) for $k=3$, which gives information about the asymptotic behaviour of the solution at order three. Then we differentiate the equation for F_a:

$$\begin{cases}\left(\frac{\partial}{\partial\bar{x}}+i\frac{\partial\Theta_a}{\partial\bar{x}}\right)\left(\frac{\partial}{\partial a}+i\frac{\partial\Theta_a}{\partial a}\right)F_a=0 & \text{on } D\\ \frac{\partial F_a}{\partial a}=\frac{\partial G_{1,a}}{\partial a} & \text{on } \partial D.\end{cases}$$

Using Green's formula again (Lemma 5.5), we obtain:

$$\int_D\left(\frac{\partial}{\partial a}+i\frac{\partial\Theta_a}{\partial a}\right)F_a\cdot(i\nabla+\mathbf{A}_0)^2U_0\ dx_1dx_2$$
$$=2i\int_{\partial D}\left(\frac{\partial}{\partial a}+i\frac{\partial\Theta_a}{\partial a}\right)G_{1,a}\cdot\left(\frac{\partial}{\partial x}-i\frac{\partial\Theta_a}{\partial x}\right)U_0\ dx$$
$$=2i\int_{\partial D}G_{3,a}\cdot\left(\frac{\partial}{\partial x}-i\frac{\partial\Theta_a}{\partial x}\right)U_0\ dx.$$

Replacing in (24) we obtain

$$\left(\frac{\partial}{\partial a}+i\frac{\partial\Theta_a}{\partial a}\right)(c_1(a)-id_1(a))\Big|_{a=0}$$
$$=2i\int_{\partial D}G_{3,0}\left(\frac{\partial}{\partial x}-i\frac{\partial\Theta_0}{\partial x}\right)U_0\ dx-\int_D G_{3,0}(i\nabla+\mathbf{A}_0)^2U_0\ dx_1dx_2.$$

Finally (17) allows us to conclude

$$\frac{\partial}{\partial a}(c_1(a)-id_1(a))\Big|_{a=0}=\frac{1}{2}(c_3(0)-id_3(0)).$$

Hence the derivation with respect to a of the first order one lead to the coefficient of order three in the asymptotic expansion of U_0 in a neighbourhood of its singularity. Finally, Corollary 4.4 ensures that it does not vanish as a complex number. □

6 Energy minimizing partitions

In this section we shall prove all the remaining results stated in the introduction, apart from Theorem 1.2, (v), which will be object of the last section. More precisely, we show a relation between the solutions U_a of (2) and the functional class $\mathcal{S}_\gamma$ defined in the introduction, and then we concentrate on the proof of Theorem 1.4. Then, in order to conclude, we just apply previous results by Conti, Terracini and Verzini. Apart from establishing an interesting variational characterization, the relation between U_a and $\mathcal{S}_\gamma$ is the tool that allows us to prove global uniqueness, starting from local uniqueness. Throughout this section we will always assume $\gamma \in g$, V satisfies (4) and Ω to be the unitary disk.

Let us start recalling the known properties of $\mathcal{S}_\gamma$, we refer to [4, 6, 5, 7] for the proofs and further details; first of all we need some definitions. For any $u = (u_1, u_2, u_3) \in \mathcal{S}_\gamma$ we define the *support* of each density as $\omega_i = \{x \in D : u_i(x) > 0\}$. The *multiplicity* of a point $x \in \overline{D}$ (with respect to u) is

$$m(x) = \sharp\{i : \text{measure}(\omega_i \cap B(x,r)) > 0 \quad \forall r > 0\},$$

where $B(x,r)$ is the disk centered at x of radius r. Notice that $1 \le m(x) \le 3$. The first result is related to the regularity of functions in $\mathcal{S}_\gamma$.

(6.1) Theorem. *If $u \in \mathcal{S}_\gamma$ then $u \in W^{1,\infty}(\overline{D})$. As a consequence, every ω_i is open and $x \in \omega_i$ implies $m(x) = 1$.*

Then, by definition of $\mathcal{S}_\gamma$, every density satisfies a differential equation on its support, and a differential equation locally far from the singular point:

$$\begin{cases} -\Delta u_i + V u_i = 0 & \text{in } \omega_i \\ u_i = \gamma_i & \text{on } \overline{\omega}_i \cap \partial D \\ u_i = 0 & \text{in } D \setminus \omega_i \end{cases} \qquad \begin{cases} -\Delta(u_i - u_j) + V(u_i - u_j) = 0 & \text{in } \omega_i \cup \omega_j \\ u_i - u_j = \gamma_i - \gamma_j & \text{on } (\overline{\omega_i \cup \omega_j}) \cap \partial D \\ u_i - u_j = 0 & \text{on } D \setminus \{\omega_i \cap \omega_j\}. \end{cases}$$

As far as the regularity of the free boundary is concerned, the following properties hold for the two-dimensional problem.

(6.2) Theorem. *Let $u \in \mathcal{S}_\gamma$, then*

(a) each ω_i is connected;

(b) a point $x \in D$ is singular for u if and only if $m(x) = 3$;

(c) there exists exactly one point $a_u \in \overline{D}$ such that $m(a_u) = 3$;

(d) each interface $\eta_{ij} := \partial\omega_i \cap \partial\omega_j \cap \{x \in D : m(x) = 2\}$ is (either empty or) a connected arc, locally $C^{1,\alpha}$ for every $\alpha \in (0,1)$, with endpoints either at ∂D or at a_u;

(e) the following asymptotic estimate holds in a neighbourhood of a_u

$$u(r,\vartheta) = r^{3/2} \left| c \cos\left(\frac{3}{2}\vartheta\right) + d \sin\left(\frac{3}{2}\vartheta\right) \right| + o(r^{3/2}) \quad \text{as } r \to 0,$$

where (r,ϑ) denotes a system of polar coordinates around a_u and c, d are real constants.

With some abuse of notation, we will call a_u a *triple point* for the function u. Let us first point out the relation between the elements of $\mathcal{S}_\gamma$ and the solutions of the magnetic equation (2), which is once again a real elliptic equation on the twofold covering manifold.

(6.3) Lemma. *Let $u \in \mathcal{S}_\gamma$ and $a = a_u$ be its triple point. If $a \in D$ then there exists $\Gamma \in \mathcal{G}$ such that, for every $\mathbf{A}_a$ satisfying (1), the solution U of (2) verifies $\mathcal{N}(u) = \mathcal{N}(U)$. Moreover if $u_1 \neq u_2 \in \mathcal{S}_\gamma$ then $U_1 \neq U_2$.*

Proof. Let $(u_1, u_2, u_3) \in \mathcal{S}_\gamma$, then by Theorem 6.2 there exists exactly one point $a \in D$ of multiplicity three with respect to u. If $a \in D$, we can consider the twofold covering manifold Σ_a. Than we define a new function on Σ_a, that with some abuse of notation will be called again u, in the following way:

$$u(x,y) := \sum_{i=1}^{3} \sigma(x,y) u_i \circ \Pi_x(x,y),$$

where $\sigma(x,y)$ is ± 1 in such a way that u has alternate sign on two adjacent supports. Then u is antisymmetric on Σ_a and, by virtue of Theorems 6.1 and 6.2, (e), it satisfies

$$\begin{cases} -\Delta u + \tilde{V} u = 0 & \text{in } \Sigma_a \\ u = \gamma & \text{on } \partial\Sigma_a, \end{cases} \tag{25}$$

where as usual $\tilde{V}(x,y) = V \circ \Pi_x(x,y)$ and $\gamma(x,y) = \sum_{i=1}^{3} \sigma(x,y)\gamma_i \circ \Pi_x(x,y)$. Let now $\Gamma = e^{-i\frac{\vartheta_a}{2}}\gamma$ (ϑ_a defined in (10)) and $\mathbf{A}_a$ as in Remark 3.7, then the corresponding solution of (2) satisfies the statement, by Proposition 3.9. By gauge invariance, the same holds for every $\mathbf{A}_a$ satisfying (1). Notice that, by construction, a is a triple point for Γ. The second part of the statement is a direct consequence of Proposition 6.6. □

(6.4) Remark. As a consequence of Theorem 6.2, we infer that $\mathcal{G} \setminus \tilde{\mathcal{G}}$ consists of the traces Γ such that the corresponding real γ satisfies $a_\gamma \in \partial D$.

Let us now concentrate on the proof of Theorem 1.4; we divide it in several steps. First of all notice that Lemma 6.3, together with the local uniqueness result Theorem 5.6, immediately gives the following local uniqueness result for $\mathcal{S}_\gamma$ (which is the analogous of Proposition 3.2 in [7]).

(6.5) Proposition. *Let $u \in \mathcal{S}_\gamma$ and $a_\gamma \in D$ be its triple point. Then there exist $\epsilon > 0, \delta > 0$ such that for every $\lambda \in g$ with $\|\gamma_i - \lambda_i\|_{L^\infty(D)}$, $i = 1, 2, 3$, there exists exactly one $a_\lambda \in D$, triple point for λ, such that $|a_\gamma - a_\lambda| < \delta$.*

The second step for the proof of Theorem 1.4 is the following proposition. It is proved in [7], and can be easily adapted to our problem since it only makes use of the maximum principle (ensured by condition (4)).

(6.6) Proposition. *Let $u, v \in \mathcal{S}_\gamma$, then*

(i) if $a_u \in \partial D$ then $v \equiv u$;

(ii) if $a_u = a_v$ then $v \equiv u$;

Proof of Theorem 1.2, (ii)-(iv), Sketch. Proceeding as in [7], we can deduce from Proposition 6.5 that $\mathcal{S}_\gamma$ consists of exactly one element, then Lemma 6.3 concludes the proof. Let us sketch the ideas contained in [7]. First of all by Proposition 6.6 it is possible to assume that the triple point is in the interior of the domain. Assume by contradiction that $u \neq v \in \mathcal{S}_\gamma$, than they can be connected in a continuous way to the same minimal solution of (5), with an appropriate boundary datum. Finally the uniqueness result for the minimal solution (see Theorem 6.7, (ii) below) gives a contradiction. □

Now that we proved that $\mathcal{S}_\gamma$ consists in exactly one element, Theorems 1.3 and 1.5 are an immediate consequence of the following results by Conti, Terracini and Verzini. They state that $\mathcal{S}_\gamma$ contains both the solution of the variational problem (5), and the limiting solutions of the competition–diffusion system (6), in case of strong competition. As a consequence of uniqueness, we infer that the two problems in fact coincide.

(6.7) Theorem. *(i) For every $\kappa > 0$, system (6) admits (at least) a solution $(u_{1,\kappa}, u_{2,\kappa}, u_{3,\kappa}) \in (H^1(D))^3$. Moreover there exists $(u_1, u_2, u_3) \in \mathcal{S}_\gamma$ such that, up to subsequences, $u_{i,\kappa} \to u_i$ in $H^1(D)$ as $\kappa \to +\infty$;*

(ii) the minimization problem (5) admits a unique solution, which belongs to $\mathcal{S}_\gamma$.

7 Continuous dependence of the nodal arcs with respect to the boundary trace

Aim of this section is the proof of Theorem 1.2, (v); let us start giving a more precise formulation of the result. Respect to the previous sections, here we let the boundary trace vary. Given $\Gamma \in \tilde{\mathcal{G}}$, we first single out its triple point a (recall Remark 6.4). Then we consider equation (2) with the magnetic field centered at a. As a consequence, every function U_a considered in this section has a triple point, and its nodal set consists of three arcs meeting at a.

Notation. In the following we will denote Γ_a any trace belonging to $\tilde{\mathcal{G}}$, having a triple point at a, and $\eta_a(t)$ a regular parametrization of one nodal arc of U_a.

In addition to the usual assumptions (1) and (4), suppose also

$$V(x) \in W^{1,\infty}(D),$$

then it holds

(7.1) Theorem. *In the setting described above, fix a subset $\tilde{\Omega} \subset\subset D$ and an $\alpha \in (0, 1/2)$. Given Γ_{a_1}, let ϵ as in Theorem 5.6, and Γ_{a_2} such that $\|\Gamma_{a_1} - \Gamma_{a_2}\|_{L^\infty}(\partial D) < \epsilon$. Then there exists a constant $C > 0$ such that*

$$||\eta_{a_1} - \eta_{a_2}||_{C^{1,\alpha}(\tilde{\Omega})} < C||\Gamma_{a_1} - \Gamma_{a_2}||_{L^\infty(\partial D)},$$

for a suitable choice of the nodal arcs and of the parametrization.

The rest of the paragraph is devoted to the proof of this result, hence we tacitly assume the hypothesis and notations of the theorem.

(7.2) Remark. Without loss of generality we can choose $a_1 = 0$, $a_2 = a$. Define

$$\mathcal{O} = \{a \in D : ||\Gamma_a - \Gamma_0||_{L^\infty(\partial D)} < \epsilon\}.$$

By Theorem 5.6, there exists a constant C such that

$$|a| < C||\Gamma_a - \Gamma_0||_{L^\infty(\partial D)}, \quad \forall\, a \in \mathcal{O}.$$

As a consequence, we can equivalently study the dependence of the nodal lines with respect to the position of the singularity.

As usual we denote $u_a^{(2)}$ the function introduced in Lemma 3.14. Proceeding as in Lemma 5.1 it is easy to prove

(7.3) Lemma. *Let* $V \in W^{1,\infty}(D)$*, then*

(i) $\|V^{(2)}_{a_1} - V^{(2)}_{a_2}\|_{L^\infty(D)} \le C|a_1 - a_2|$, $\|\gamma^{(2)}_{a_1} - \gamma^{(2)}_{a_2}\|_{L^\infty(\partial D)} \le C|a_1 - a_2|$;

(ii) $\|u^{(2)}_{a_1} - u^{(2)}_{a_2}\|_{C^{1,\alpha}_{loc}(D)} \le C|a_1 - a_2|$, *for every* $0 < \alpha < 1$.

The following result relates η_a with the nodal lines of $u^{(2)}_a$, which will be denoted $\eta^{(2)}_a$.

(7.4) Lemma. *Fix* $\alpha \in (0,1)$*. If* $\|\eta^{(2)}_a - \eta^{(2)}_0\|_{C^{1,\alpha}(\tilde\Omega)} \le C|a|$, $\forall\ a \in \mathcal{O}$*, then there exists a constant* C' *such that* $\|\eta_a - \eta_0\|_{C^{1,\frac{\alpha}{2}}(\tilde\Omega)} \le C'|a|$, $\forall\ a \in \mathcal{O}$.

Thanks to this result, we can work with $\eta^{(2)}_a$, for a fix $\alpha \in (0,1)$. The main tool for our analysis will be the following (Theorem 2.1 in [14]), which describes the local behaviour of the solutions of real elliptic equations, in a planar domain. It is an improvement of the result by Hartman and Wintner that we recalled in Section 4.

(7.5) Proposition. *If* $u^{(2)}_a$ *has a zero of order* n *at the origin, then there exists a complex valued function* $\tilde\xi_a(y)$ *of class* $C^{0,\alpha}$*, with* $\tilde\xi_a(0) = 0$*, such that*

$$u^{(2)}_a(\rho,\varphi) = \frac{\rho^{n+1}}{n+1}\Big\{c_{n+1}(a)\cos[(n+1)\varphi] + d_{n+1}(a)\sin[(n+1)\varphi] + \tilde\xi_a(\rho,\varphi)\Big\}, \tag{26}$$

where $c_{n+1}(a)$, $d_{n+1}(a)$ *are real constants, not both zero, depending on the parameter* a*. Equivalently there exists* $\xi_a(y) \in C^{0,\alpha}(D)$*, such that*

$$2\frac{\partial u^{(2)}_a}{\partial y}(y) = y^n \xi_a(y), \qquad \xi_a(0) = c_{n+1}(a) - i d_{n+1}(a) \ne 0. \tag{27}$$

Moreover for every $k \le n$ *the following Cauchy formula is available*

$$\frac{2}{y^k}\frac{\partial u^{(2)}_a}{\partial y}(y) = -\frac{i}{\pi}\int_{\partial D}\frac{1}{z^k(z-y)}\frac{\partial u^{(2)}_a}{\partial z}(z)\ dz + \frac{1}{2\pi}\int_D \frac{-\Delta u^{(2)}_a(z)}{z^k(z-y)}\ dz_1 dz_2 \tag{28}$$

where the first integral is a complex line integral, whereas the second one is a double integral in the real variables z_1, z_2*. Note that the double integral is absolutely convergent.*

Using some ideas in [14], Theorem 2.1, we can prove the following estimate.

(7.6) Lemma. *Suppose that* $u^{(2)}_a$ *has a zero of order* $n \in \mathbb{Z}$ *at the origin, for every* $a \in \mathcal{O}$*. Let* ξ_a *be the function defined in (27). Then there exists a constant* C *such that*

$$\|\xi_{a_1} - \xi_{a_2}\|_{C^{0,\alpha}(\tilde\Omega)} \le C|a_1 - a_2|,$$

for every $a_1, a_2 \in \mathcal{O}$.

Proof. We prove this result by induction on k, where for every $k \le n$ we define

$$\begin{aligned} T_k:\ \ \mathcal{O}\ &\to C^{0,\alpha}(\tilde\Omega,\mathbb{C}) \\ a\ &\mapsto \frac{2}{y^k}\frac{\partial u^{(2)}_a}{\partial y}. \end{aligned}$$

Let us start proving that T_0 is uniformly continuous. Recalling that an integral expression for $T_k(a)$ is (28), it is enough to show that

$$\left|-\frac{i}{\pi}\int_{\partial\tilde{\Omega}}\left(\frac{\partial u^{(2)}_{a_1}}{\partial z}-\frac{\partial u^{(2)}_{a_2}}{\partial z}\right)\left(\frac{1}{z-y_1}-\frac{1}{z-y_2}\right)\,dz\right|+$$
$$+\left|\frac{1}{2\pi}\int_{\tilde{\Omega}}\left(-\Delta u^{(2)}_{a_1}+\Delta u^{(2)}_{a_2}\right)\left(\frac{1}{z-y_1}-\frac{1}{z-y_2}\right)\,dz_1dz_2\right|\leq C|a_1-a_2||y_1-y_2|^{\alpha},$$

for every $\alpha>0$. The first integral is smooth in y, hence it is sufficient to apply Lemma 7.3. As it concerns the second integral, following [14], Theorem 2.1, we write

$$\begin{aligned}&\left|\int_{\tilde{\Omega}}\left(V^{(2)}_{a_1}u^{(2)}_{a_1}-V^{(2)}_{a_2}u^{(2)}_{a_2}\right)\left(\frac{1}{z-y_1}-\frac{1}{z-y_2}\right)\,dz_1dz_2\right|\\
&\leq\int_{\tilde{\Omega}}\left(|V^{(2)}_{a_1}||u^{(2)}_{a_1}-u^{(2)}_{a_2}|+|V^{(2)}_{a_1}-V^{(2)}_{a_2}||u^{(2)}_{a_2}|\right)\left|\frac{1}{z-y_1}-\frac{1}{z-y_2}\right|\,dz_1dz_2\\
&\leq\left(||V^{(2)}_{a_1}||_{L^\infty(D)}||u^{(2)}_{a_1}-u^{(2)}_{a_2}||_{L^\infty(D)}+||V^{(2)}_{a_1}-V^{(2)}_{a_2}||_{L^\infty(D)}||u^{(2)}_{a_2}||_{L^\infty(D)}\right)\int_{\tilde{\Omega}}\frac{|y_1-y_2|}{|z-y_1||z-y_2|}\,dz_1dz_2\\
&\leq C|a_1-a_2||y_1-y_2|\log|y_1-y_2|,\end{aligned}$$

where, in the last inequality, we used Lemma 7.3. This concludes the proof for $k=0$. Now assume that T_k is uniformly continuous for some $k<n$, that is

$$\left\|\frac{2}{y^k}\frac{\partial u^{(2)}_{a_1}}{\partial y}-\frac{2}{y^k}\frac{\partial u^{(2)}_{a_2}}{\partial y}\right\|_{C^{0,\alpha}(\tilde{\Omega})}\leq C|a_1-a_2|,$$

and let us prove that T_{k+1} is uniformly continuous. Since $k<n$ and the origin is a zero of order n for $u^{(2)}_a$, we have $\lim_{y\to 0}\frac{1}{y^k}\frac{\partial u^{(2)}_a}{\partial y}=0\ \forall a$. As a consequence, the inductive assumption gives

$$\sup_{y\in\tilde{\Omega}}\frac{1}{|y|^{k+\alpha}}\left|\frac{\partial u^{(2)}_{a_1}}{\partial y}(y)-\frac{\partial u^{(2)}_{a_2}}{\partial y}(y)\right|\leq C|a_1-a_2|.$$

Following [12] we use the identity

$$u^{(2)}_a(\rho,\varphi)=\int_0^\rho\left(\frac{\partial u_a}{\partial y_1}(t,\varphi)\cos\varphi+\frac{\partial u_a}{\partial y_2}(t,\varphi)\sin\varphi\right)d\rho$$

which implies, together with the previous inequality

$$\begin{aligned}|(u^{(2)}_{a_1}-u^{(2)}_{a_2})(y)|&\leq\int_0^1|y|\left|2\left(\frac{\partial u^{(2)}_{a_1}}{\partial y}-\frac{\partial u^{(2)}_{a_2}}{\partial y}\right)(ty)\right|dt\leq\\
&\leq\int_0^1 C|y||a_1-a_2||yt|^{k+\alpha}dt\leq C|a_1-a_2||y|^{k+1+\alpha},\end{aligned}\tag{29}$$

in $\tilde{\Omega}$. Now we can proceed as in the case $k=0$:

$$\begin{aligned}&\left|\int_{\tilde{\Omega}}\frac{-\Delta u^{(2)}_{a_1}+\Delta u^{(2)}_{a_2}}{z^{k+1}}\left(\frac{1}{z-y_1}-\frac{1}{z-y_2}\right)\,dz_1dz_2\right|\\
&\leq\left(||V^{(2)}_{a_1}||_{L^\infty(D)}\left\|\frac{u^{(2)}_{a_1}-u^{(2)}_{a_2}}{z^{k+1}}\right\|_{L^\infty(D)}+||V^{(2)}_{a_1}-V^{(2)}_{a_2}||_{L^\infty(D)}\left\|\frac{u^{(2)}_{a_2}}{z^{k+1}}\right\|_{L^\infty(D)}\right)|y_1-y_2|\log|y_1-y_2|,\\
&\leq C|a_1-a_2||y_1-y_2|\log|y_1-y_2|,\end{aligned}$$

where we used (29) in the last inequality. □

(7.7) Remark. In our case, the previous lemma applies with $n = 2$. Notice also that the same estimate holds for the function $\tilde{\xi}_a$ defined in (26).

In order to prove the theorem, we choose a branch of nodal line, having an endpoint at the origin, i.e.

$$\eta_a^{(2)} : (T_1, T_2) \to \mathbb{C}, \qquad \lim_{t\to T_1} \eta_a^{(2)}(t) = 0,$$

with T_1, T_2 eventually infinite. Then the curve satisfies $\dot{\eta}_a^{(2)}(t) = -i\kappa_a(t)\frac{\partial u_a^{(2)}}{\partial \bar{y}}(\eta_a^{(2)}(t))$, with the condition $u_a^{(2)}(\eta_a^{(2)}(t)) = 0$, where $\kappa_a(t)$ is any real function, sufficiently regular in (T_1, T_2). Since every $u_a^{(2)}$ has a zero of order two at the origin, we choose

$$\kappa_a(t) = \frac{1}{|\eta_a^{(2)}(t)|^2}.$$

Passing to polar coordinates $\eta_a^{(2)}(t) = \rho_a(t)e^{i\varphi_a(t)}$, the equation becomes

$$\begin{cases} \dot{\rho}_a = \frac{1}{|y|^3}\left(y_1\frac{\partial u_a^{(2)}}{\partial y_2} - y_2\frac{\partial u_a^{(2)}}{\partial y_1}\right) = -\frac{1}{|y|^3}\Im(y^3\xi_a(y)) \\ \dot{\varphi}_a = -\frac{1}{|y|^4}\left(y_1\frac{\partial u_a^{(2)}}{\partial y_1} + y_2\frac{\partial u_a^{(2)}}{\partial y_2}\right) = -\frac{1}{|y|^4}\Re(y^3\xi_a(y)). \end{cases} \tag{30}$$

(7.8) Lemma. *With this choice of the parametrization the interval (T_1, T_2) is bounded, in particular we can choose $T_1 = 0$.*

Proof. Computing the velocity of the curve we have

$$|\dot{\eta}_a^{(2)}(t)| = \frac{1}{|\eta_a^{(2)}(t)|^2}\left|\frac{\partial u_a^{(2)}}{\partial \bar{y}}(\eta_a^{(2)}(t))\right| = |\xi_a(\eta_a^{(2)}(t))| \to |\xi_a(0)| = \sqrt{c_3(a)^2 + d_3(a)^2} \qquad \text{as } t \to T_1,$$

where $c_3(a), d_3(a)$ are the first nontrivial terms in the asymptotic expansion (27). Hence t is asymptotically a multiple of the arc length as $t \to T_1$ and the lemma is proved. □

End of the proof of Theorem 7.1. Writing the equation of the curve in polar coordinates we have $\dot{\eta}_a^{(2)}(t) = e^{i\varphi_a(t)}(\dot{\rho}_a(t) + i\rho_a(t)\dot{\varphi}_a(t))$. Therefore we wish to show that, for every $\alpha \in (0,1)$ there exist constants K_1, K_2 such that

$$\begin{aligned} &\|\dot{\rho}_a - \dot{\rho}_0\|_{C^{0,\alpha}(0,T_2)} + \|\rho_a\dot{\varphi}_a - \rho_0\dot{\varphi}_0\|_{C^{0,\alpha}(0,T_2)} \le K_1|a|, \\ &\|\varphi_a - \varphi_0\|_{C^{0,\alpha}(0,T_2)} \le K_2|a|, \end{aligned}$$

for every $a \in \mathcal{O}$. Then, applying Lemma 7.4 and Remark 7.2, the theorem is proved.

The first inequality comes directly from equations (30) and from Lemma 7.6, by regularity results for ordinary differential equations. Let us prove the second one.

Both $\tilde{\xi}_a(y)$ and $\xi_a(y) - \xi_a(0)$ satisfy Lemma 7.6 and vanish in 0, hence there exists C such that

$$\max\left\{\sup_{y\in\tilde{\Omega}}\frac{|\xi_a(y) - \xi_a(0)|}{|y|^\alpha}, \sup_{y\in\tilde{\Omega}}\frac{|\tilde{\xi}_a(y)|}{|y|^\alpha}\right\} \le C \qquad \forall a \in \mathcal{O}. \tag{31}$$

Moreover from $u_a^{(2)}(\eta_a(t)) = 0$ we deduce

$$c_3(a)\cos(3\varphi_a) + d_3(a)\sin(3\varphi_a) + \tilde{\xi}_a(\eta_a(t)) = 0.$$

Therefore the equation for $\dot{\varphi_a}$ can be written as

$$\begin{aligned}\dot{\varphi_a} &= -\frac{1}{|y|}(c_3(a)\cos(3\varphi)+d_3(a)\sin(3\varphi)) - \frac{1}{|y|^4}\Re[y^3(\xi_a(y)-\xi_a(0))] \\ &= \frac{\tilde{\xi}_a(y)}{|y|} - \frac{1}{|y|^4}\Re[y^3(\xi_a(y)-\xi_a(0))],\end{aligned}$$

and we can conclude applying again Lemma 7.6 and Lemma 7.8. □

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