

DIFFERENTIAL EQUATIONS IN METRIC SPACES
WITH APPLICATIONS

RINALDO M. COLOMBO, GRAZIANO GUERRA

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Differential Equations in Metric Spaces with Applications

Rinaldo M. Colombo

Dipartimento di Matematica
Università degli Studi di Brescia
Via Branze, 38
25123 Brescia, Italy

Rinaldo.Colombo@Unibs.it

Graziano Guerra

Dip. di Matematica e Applicazioni
Università di Milano – Bicocca
Via Bicocca degli Arcimboldi, 8
20126 Milano, Italy

Graziano.Guerra@UnimiB.it

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Abstract

This paper proves the local well posedness of differential equations in metric spaces under assumptions that allow to comprise several different applications. We consider below a system of balance laws with a dissipative non local source, the Hille-Yosida Theorem, a generalization of a recent result on nonlinear operator splitting, an extension of Trotter formula for linear semigroups and the heat equation.

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1 Introduction

This paper is concerned with differential equations in a metric space and their applications to ordinary and partial differential equations. More precisely, we modify the structure of quasidifferential equations introduced in [23, 24], see also [4], and prove existence, uniqueness and continuous dependence under conditions weaker than those therein. A general estimate on the difference between any Euler polygonal approximation and the exact solution is also provided.

Following the model theory of ordinary differential equations, we provide a result that comprises also other entirely different examples: the heat equation, the Hille-Yosida theorem, the Lie-Trotter product formula and a balance law with a dissipative non local source. In particular, the latter result is an extension of [12] that was announced in [10]. Furthermore, the construction below weakens the assumptions introduced in [9] on the

non linear operator splitting technique in metric spaces, as well as those introduced in [21] on linear Lie-Trotter products. Remark that the commutativity condition adopted here, namely (2.7), is optimal, in the sense of Paragraph 2.1.

Following [4, 23, 24], for any u in the metric space X , the tangent space $T_u X$ to X at u is the quotient of the set $\left\{ \gamma \in \mathbf{C}^{0,1}([0, 1]; X) : \gamma(0) = u \right\}$ of continuous curves exiting u modulo the equivalence relation of first order contact, i.e. $\gamma_1 \sim \gamma_2$ if and only if $\lim_{t \rightarrow 0} \frac{1}{t} d(\gamma_1(t), \gamma_2(t)) = 0$. Any map $v: [0, T] \times X \mapsto \bigcup_{u \in X} T_u X$, such that $v(t, u) \in T_u X$ for all t and u then defines both a vector field on X and the (quasi)differential equation

$$\dot{u} = v(t, u). \quad (1.1)$$

Besides suitable regularity assumptions, a solution to (1.1) is a process $(t, t_o, u) \mapsto P(t, t_o)u$ such that $P(0, t_o)u_o = u_o$ and the curve $\tau \rightarrow P(t + \tau, t_o)u_o$ belongs to the equivalence class of $v(t_o + t, P(t, t_o)u)$, for all $t \geq 0$. A *local flow* generated by v is a map $(t, t_o, u) \mapsto F(t, t_o)u$ such that $F(0, t_o)u_o = u_o$ and the curve $\tau \rightarrow F(\tau, t_o)u_o$ belongs to the equivalence class of $v(t_o, u_o)$.

Within this framework, we prove that if the vector field v can be defined through a suitable *local flow*, then the Cauchy problem (1.1) is well posed globally in $u_o \in X$ and locally in $t \in \mathbb{R}$. The proof is constructive and based on Euler polygonal, similarly to [23, 24], but with somewhat looser assumptions.

This approach to differential equations, being sited in metric spaces, does not rely on any linearity assumption whatsoever. It is therefore particularly suited to describe truly nonlinear or even discontinuous models. In this connection, we refer to the example in Paragraph 3.2 below and to [23, Theorem 3.1], which is contained in the present framework.

The next section is the core of this paper: it presents the definition of *local flow* and Theorem 2.5, which shows that a local flow generates a global process. We also give an example showing the necessity of the assumptions in Theorem 2.5. Section 3 presents several applications. The first one deals with a recent new result about balance laws with a diagonally dominant non local source. The final Section 4 provides the various technical details.

2 Notation and Main Result

Throughout, (X, d) denotes a complete metric space. In view of the applications in Section 3, we need to slightly extend the basic definitions about differential equations in metric spaces in [4], see also [2, 23, 24].

Throughout this paper, we fix $T \in [0, +\infty[$ and the interval $I = [0, T]$.

Definition 2.1 Given a closed set $\mathcal{D} \subseteq X$, a local flow is a continuous map $F: [0, \delta] \times I \times \mathcal{D} \mapsto X$, such that $F(0, t_o)u = u$ for any $(t_o, u) \in I \times \mathcal{D}$ and which is Lipschitz in its first and third argument uniformly in the second, i.e. there exists a $\mathbf{Lip}(F) > 0$ such that for all $\tau, \tau' \in [0, \delta]$ and $u, u' \in \mathcal{D}$

$$d(F(\tau, t_o)u, F(\tau', t_o)u') \leq \mathbf{Lip}(F) \cdot (d(u, u') + |\tau - \tau'|). \quad (2.1)$$

To explain the notation, consider the case of a Banach space X and let $\mathcal{D} \subseteq X$. If $v: I \times \mathcal{D} \mapsto X$ is a vector field defining the ordinary differential equation $\dot{u} = v(t, u)$, then a local flow (generated by v) is the map F where $F(\tau, t_o)u = u + \tau v(t_o, u)$.

Definition 2.2 Let F be a local flow. Fix $u \in \mathcal{D}$, $t_o \in I$, $t > 0$ with $t_o + t \in I$, for any $k \in \mathbb{N}$ and $\tau_0 = 0, \tau_1, \dots, \tau_k, \tau_{k+1} = t$ such that $\tau_{h+1} - \tau_h \in]0, \delta]$ for $h = 0, \dots, k$, we call Euler polygonal the map

$$F^E(t, t_o)u = \bigcirc_{h=0}^k F(\tau_{h+1} - \tau_h, t_o + \tau_h)u \quad (2.2)$$

whenever it is defined. For any $\varepsilon > 0$, let $k = [t/\varepsilon]$. An Euler ε -polygonal is

$$F^\varepsilon(t, t_o)u = F(t - k\varepsilon, t_o + k\varepsilon) \circ \bigcirc_{h=0}^{k-1} F(\varepsilon, t_o + h\varepsilon)u \quad (2.3)$$

whenever it is defined.

Above and in what follows, $[\cdot]$ denotes the integer part, i.e. for $s \in \mathbb{R}$, $[s] = \max\{k \in \mathbb{Z}: k \leq s\}$. Clearly, F^E in (2.2) reduces to F^ε in (2.3) as soon as $\tau_h = t_o + h\varepsilon$ for $h = 0, \dots, k = [t/\varepsilon]$. The Lipschitz dependence on time of the local flow implies the same regularity of the Euler ε -polygonals, as shown by the next lemma, proved in Section 4.

Lemma 2.3 Let F be a local flow. Then F^ε , whenever defined, is Lipschitz in t, u and continuous in t_o . Moreover,

$$d(F^\varepsilon(t, t_o)u, F^\varepsilon(s, t_o)u) \leq \mathbf{Lip}(F) |t - s|.$$

On the other hand, an elementary computation shows that

$$d(F^\varepsilon(t, t_o)u, F^\varepsilon(t, t_o)w) \leq (\mathbf{Lip}(F))^{[t/\varepsilon]+1} d(u, w)$$

so that the limit $\varepsilon \rightarrow 0$ is possible only when $\mathbf{Lip}(F) \leq 1$. In the case of conservation laws, for instance, the usual procedure to prove the well posedness is based on this key point: the Lipschitz dependence of F^ε on u is achieved through the introduction of a functional (see [5, Chapter 8] and [13]) or a metric (see [3, 6]) equivalent to the \mathbf{L}^1 distance d and according to which the Euler polygonals turn out to be non expansive. Therefore, in the theorem below, the Lipschitzianity of F^ε on u is explicitly required.

With the same notation of Definition 2.1, we introduce what provides a solution to the Cauchy problem (1.1).

Definition 2.4 Consider a family of sets $\mathcal{D}_{t_o} \subseteq \mathcal{D}$ for all $t_o \in I$, and a set

$$\mathcal{A} = \{(t, t_o, u): t \geq 0, t_o, t_o + t \in I \text{ and } u \in \mathcal{D}_{t_o}\}$$

A global process on X is a map $P: \mathcal{A} \mapsto X$ such that, for all t_o, t_1, t_2, u satisfying $t_1, t_2 \geq 0, t_o, t_o + t_1 + t_2 \in I$ and $u \in \mathcal{D}_{t_o}$, satisfies

$$\begin{aligned} P(0, t_o)u &= u \\ P(t_1, t_o)u &\in \mathcal{D}_{t_o+t_1} \end{aligned} \quad (2.4)$$

$$P(t_2, t_o + t_1) \circ P(t_1, t_o)u = P(t_2 + t_1, t_o)u \quad (2.5)$$

To state the main theorem, we consider the following sets for any $t_o \in I$

$$\mathcal{D}_{t_o}^3 = \left\{ u \in \mathcal{D}: \begin{array}{l} \text{is in } \mathcal{D} \text{ for all } \varepsilon_1, \varepsilon_2, \varepsilon_3 \in]0, \delta] \text{ and all} \\ t_1, t_2, t_3 \geq 0 \text{ such that } t_o + t_1 + t_2 + t_3 \in I \end{array} \right\} \quad (2.6)$$

Theorem 2.5 The local flow F is such that there exists

1. a non decreasing map $\omega: [0, \delta] \mapsto \mathbb{R}^+$ with $\int_0^\delta \frac{\omega(\tau)}{\tau} d\tau < +\infty$ such that

$$d\left(F(k\tau, t_o + \tau) \circ F(\tau, t_o)u, F((k+1)\tau, t_o)u\right) \leq k\tau \omega(\tau) \quad (2.7)$$

whenever $\tau \in [0, \delta], k \in \mathbb{N}$ and the left hand side above is well defined;

2. a positive constant L such that

$$d(F^\varepsilon(t, t_o)u, F^\varepsilon(t, t_o)w) \leq L \cdot d(u, w) \quad (2.8)$$

whenever $\varepsilon \in]0, \delta], u, w \in \mathcal{D}, t \geq 0, t_o, t_o + t \in I$ and the left hand side above is well defined.

Then, there exists a family of sets \mathcal{D}_{t_o} and a unique a global process (as defined in Definition 2.1) $P: \mathcal{A} \mapsto X$ with the following properties:

- a) $\mathcal{D}_{t_o}^3 \subseteq \mathcal{D}_{t_o}$ for any $t_o \in I$;
- b) P is Lipschitz in (t, t_o, u) ;
- c) P is tangent to F in the sense that for all $u \in \mathcal{D}_{t_o}$, for all t such that $t \in]0, \delta]$ and $t_o + t \in I$:

$$\frac{1}{t} d(P(t, t_o)u, F(t, t_o)u) \leq \frac{2L}{\ln 2} \int_0^t \frac{\omega(\xi)}{\xi} d\xi. \quad (2.9)$$

The proof is deferred to Section 4.

Observe that in the general formulation of Theorem 2.5, the set \mathcal{A} where P is defined could be empty. However, in the applications, the following stronger condition, equivalent to [22, Condition 2.], is often satisfied:

(D) There is a family $\hat{\mathcal{D}}_{t_o}$ of subsets of \mathcal{D} satisfying $F(t, t_o)\hat{\mathcal{D}}_{t_o} \subseteq \hat{\mathcal{D}}_{t_o+t}$ for any $t \in [0, \delta]$ and $t_o, t_o + t \in I$

If (D) holds, then obviously one has $\hat{\mathcal{D}}_{t_o} \subseteq \mathcal{D}_{t_o}^3 \subseteq \mathcal{D}_{t_o}$ for any $t_o \in I$, providing a lower bound for the set where the process is defined. If (D) does not hold, then a lower bound on \mathcal{A} needs to be found exploiting specific information on the considered situation, see Paragraph 3.4.

Observe that, by (2.9), the curve $\tau \mapsto P(\tau, t_o)u$ represents the same tangent vector as $\tau \mapsto F(\tau, t_o)u$, for all u .

We remark that assumption 1 is satisfied, for instance, when

1. F is a process, i.e. $F(s, t_o + t) \circ F(t, t_o) = F(s + t, t_o)$;
2. F is defined combining two commuting Lipschitz semigroups through the operator splitting algorithm.

Above, “commuting” is meant in the sense of [9, (C)], see Paragraph 3.3

The present construction is similar to that in [4, 7, 21, 23, 24]. On one hand, here we need the function ω to estimate the speed of convergence to 0 in (2.7), while in [23, (3.17)] or, equivalently, [22, Condition 4.], any convergence to 0 is sufficient. On the other hand, our Lipschitzianity requirement (2.8) is strictly weaker than [23, (3.16)], allowing to consider balance laws. Indeed, the semigroup generated by conservation laws in general does not satisfy [23, (3.16)].

Once the global process is built and its properties proved, the following well posedness result for Cauchy problems in metric spaces is at hand.

Corollary 2.6 *Let the vector field v in (1.1) be defined by a local flow satisfying 1. and 2. in Theorem 2.5. Then, there exists a global process P whose trajectories are solutions to (1.1). Moreover, for all $t_o \in \mathbb{R}$ and $u_o \in X$ such that $u_o \in \mathcal{D}_{t_o}$, let $w: [t_o, t_o + \bar{t}] \mapsto X$ be a solution to (1.1) satisfying the initial condition $w(t_o) = u_o$. If*

1. w is Lipschitz,
2. $w(s) \in \mathcal{D}_s$ for all $s \in [t_o, t_o + \bar{t}]$,

then w coincides with the trajectory of P exiting u_o at time t_o :

$$w(t_o + t) = P(t, t_o)u \quad \text{for all } t \in [0, \bar{t}].$$

The proof is essentially as that of [4, Corollary 1, § 6] (see also the proof of [5, Theorem 2.9]) and is omitted.

Here, moreover, we show that all Euler polygonals of F converge to P , providing the following estimate on the speed of convergence.

Proposition 2.7 *With the same assumptions of Theorem 2.5, fix $u \in \mathcal{D}_{t_o}$. Let F^E be defined as in (2.2). If $F^E(s, t_o) \in \mathcal{D}_{t_o+s}$ for any $s \in [0, t]$ and $\Delta = \max_h(\tau_{h+1} - \tau_h) \in]0, \delta]$, then the following error estimate holds:*

$$d\left(F^E(t, t_o)u, P(t, t_o)u\right) \leq \frac{2L^2}{\ln 2} t \int_0^\Delta \frac{\omega(\xi)}{\xi} d\xi.$$

2.1 A Condition Weaker than (2.7) Is Not Sufficient

We now show that requiring (2.7) for $k = 1$ only:

$$d\left(F(\tau, t_o + \tau) \circ F(\tau, t_o)u, F(2\tau, t_o)u\right) \leq \tau \omega(\tau) \quad (2.10)$$

does not allow to prove the semigroup property in Theorem 2.5.

Proposition 2.8 *Let the hypotheses of Theorem 2.5 hold with (2.7) replaced by (2.10). Assume that **(D)** holds. Then, there exists a map P as in Theorem 2.5, but with the “semigroup” condition (2.5) replaced by the weaker:*

$$P(2t, t_o)u = P(t, t_o + t) \circ P(t, t_o)u, \text{ for } u \in \mathcal{D}_{t_o}, t \geq 0, t_o, t_o + 2t \in I \quad (2.11)$$

and only continuous (not necessarily Lipschitz) with respect to (t, t_o) .

The proof is deferred to Section 4. Note that P is *not* necessarily a process. Indeed, we show by an example that this property may fail, due to the fact that condition (2.7) may fail for $k \geq 2$.

In the metric space $X = [0, 1]$ with the usual Euclidean distance we construct an example showing that the assumptions in Proposition 2.8, namely (2.10) instead of (2.7), do not suffice to show that P is a process.

Let φ be any non constant function satisfying

$$\varphi \in \mathbf{C}^1([1, 2]; \mathbb{R}) \text{ and } \begin{cases} \varphi(1) = \varphi(2) = 0 \\ \varphi'(1) = \varphi'(2) = 0 \\ \varphi \leq 0 \end{cases}$$

and denote $k(t) = [\log_2 t]$. Define now

$$f(t) = \begin{cases} 1 & \text{if } t = 0 \\ \exp\left(2^{k(t)} \varphi\left(2^{-k(t)} t\right)\right) & \text{if } t > 0 \end{cases} \quad \text{and} \quad F(t, t_o)u = f(t)u.$$

The following lemma, listing the main properties of f and F above, is immediate and its proof omitted.

Lemma 2.9 *With the notation above, $f \in \mathbf{C}^{0,1}([0, +\infty[; \mathbb{R})$ and $f(2^n t) = f(t)^{2^n}$ for all $t \geq 0$ and $n \in \mathbb{N}$. Choosing $\mathcal{D}_t = X$ for all $t > 0$, the map F satisfies Definition 2.1, in particular*

$$d(F(t, t_o)u, F(t', t'_o)u') \leq \left(\max_{[1,2]} |\varphi'| \right) |t - t'| + |u - u'|$$

so that F is non expansive in u and the Euler polygons F^ε trivially satisfy (2.8). Finally, F satisfies (2.10).

We now prove that (2.7) is strictly stronger than (2.10) and that the latter assumption may not guarantee the existence of a global process. Since the local flow F does not depend on t_o , we write $F(t)$ for $F(t, t_o)$.

Proposition 2.10 *For any fixed $u \in X$, there does not exist any Lipschitz semigroup P tangent to F at u in the sense of (2.9).*

Proof of Proposition 2.10. By contradiction, let P be a Lipschitz semigroup tangent to F in the sense of (2.9).

Since with $P(t)u$ we denote the action of P on the real number $u \in [0, 1]$, to avoid confusion, we explicitly denote with the dot any product between real numbers.

First, we verify that P must be multiplicative, i.e. , $P(t)u = P(t)1 \cdot u$ (remember that $P(t)1$ is the action of P on the number $1 \in X = [0, 1]$). By [5, Theorem 2.9] applied to the Lipschitz curve $\tau \mapsto w(\tau) = P(\tau)1 \cdot u$

$$\begin{aligned} & |P(t)u - P(t)1 \cdot u| \leq \\ & \leq \mathbf{Lip}(P) \cdot \int_0^t \liminf_{h \rightarrow 0^+} \frac{|P(h)(P(\tau)1 \cdot u) - P(\tau+h)1 \cdot u|}{h} d\tau \\ & \leq \mathbf{Lip}(P) \cdot \int_0^t \limsup_{h \rightarrow 0^+} \frac{|P(h)(P(\tau)1 \cdot u) - F(h)(P(\tau)1 \cdot u)|}{h} d\tau \\ & \quad + \mathbf{Lip}(P) \cdot \int_0^t \limsup_{h \rightarrow 0^+} \frac{|F(h)(P(\tau)1 \cdot u) - P(\tau+h)1 \cdot u|}{h} d\tau \\ & \leq \mathbf{Lip}(P) \cdot \int_0^t \limsup_{h \rightarrow 0^+} \frac{|(F(h) \circ P(\tau)1) \cdot u - (P(h) \circ P(\tau)1) \cdot u|}{h} d\tau \\ & \leq 0 \end{aligned}$$

where we have used the tangency condition (2.9) and the equality

$$F(h)(P(\tau)1 \cdot u) = f(h) \cdot P(\tau)1 \cdot u = (f(h) \cdot P(\tau)1) \cdot u = (F(h) \circ P(\tau)1) \cdot u.$$

We may now denote $p(t) = P(t)1$ so that $P(t)u = p(t) \cdot u$. The invariance of $[0, 1]$ implies $p(t) = P(t)1 \in [0, 1]$ for all t and, by (2.9),

$$\lim_{t \rightarrow 0} \frac{f(t) - p(t)}{t} = \lim_{t \rightarrow 0} \frac{F(t)1 - P(t)1}{t} = 0.$$

Moreover, for any $n \in \mathbb{N}$, applying Lagrange Theorem to the map $x \mapsto x^{2^n}$ on the interval between $f(2^{-n}t)^{2^n}$ and $p(2^{-n}t)^{2^n}$,

$$\begin{aligned} |f(t) - p(t)| &= \left| f(2^{-n}t)^{2^n} - p(2^{-n}t)^{2^n} \right| \\ &\leq 2^n |\bar{x}|^{2^n-1} \left| f(2^{-n}t) - p(2^{-n}t) \right| \\ &\leq \frac{|f(2^{-n}t) - p(2^{-n}t)|}{2^{-n}} \\ &\rightarrow 0 \quad \text{for } n \rightarrow +\infty \end{aligned}$$

Hence $f = p$, but in general $f(t+s) \neq f(t)f(s)$, giving a contradiction. \square

3 Applications

In the autonomous case, we often omit the initial time t_o writing $F(\tau)u$ for $F(\tau, t_o)u$. Below, **(D)** is satisfied in all but one case.

3.1 Balance Laws

Consider the following nonlinear system of balance laws:

$$\partial_t u + \partial_x f(u) = G(u) \tag{3.1}$$

where $f: \Omega \mapsto \mathbb{R}^n$ is the smooth flow of a nonlinear hyperbolic system of conservation laws, Ω is a non empty open subset of \mathbb{R}^n and $G: \mathbf{L}^1(\mathbb{R}; \Omega) \mapsto \mathbf{L}^1(\mathbb{R}; \mathbb{R}^n)$ is a (possibly) *non local* operator. [12, Theorem 2.1] proves that, for small times, equation (3.1) generates a global process uniformly Lipschitz in t and u . This result fits in Theorem 2.5. Indeed, setting $X = \mathbf{L}^1(\mathbb{R}; \Omega)$, it is possible to show that $F(t)u = S_t u + tG(S_t u)$ is a *local flow* of the vector field defined by (3.1). Here, S is the *Standard Riemann Semigroup* [5, Definition 9.1] generated by the left-hand side in (3.1).

Using essentially [1, Remark 4.1] one can first show that the map F satisfies 1. in Theorem 2.5. Then, the functional defined in [13, formula (4.3)] allows to prove also 2. The details are found in [11], where the present technique is applied to the diagonally dominant case [14, formula (1.14)], yielding a process defined globally in time.

3.2 Constrained O.D.E.s – The Stop Problem

Let X be a Hilbert space and fix a closed convex subset $C \subseteq X$. For a positive T , let $f: [0, T] \times C \mapsto X$. Consider the following constrained ordinary differential equation:

$$\begin{cases} \dot{u} = f(t, u), & u(t) \in C \\ u(t_o) = u_o \end{cases}$$

with $u_o \in C$. The *stop problem* above was considered, for instance, in [20], see also [19, Section 4.1].

We denote by $\Pi: X \mapsto C$ the projection of minimal distance, i.e. $\Pi(x) = y$ if and only if $y \in C$ and $\|x - y\| = d(x, C)$. Recall that Π is non expansive.

This problem fits in the present setting, for instance when f and C satisfy the following conditions:

- (f) f is bounded. Moreover, there exist $\omega: [0, T] \mapsto [0, \infty[$ with $\int_0^T \frac{\omega(\tau)}{\tau} d\tau < +\infty$ and an $L > 0$ such that for all $t_1, t_2 \in [0, T]$ and $u_1, u_2 \in C$

$$\|f(t_1, u_1) - f(t_2, u_2)\| \leq \omega(|t_1 - t_2|) + L \cdot \|u_1 - u_2\|.$$

- (C) for a $d > 0$, $\Pi \in \mathbf{C}^2 \left(\overline{B(C, d)} \setminus C; \partial C \right)$ and $\sup_{\overline{B(C, d)} \setminus C} \|D^2 \Pi\| < +\infty$.

As usual, we denote $B(C, d) = \{x \in X: d(x, C) < d\}$. By [17, Theorem 2], if $X = \mathbb{R}^n$ and C is a compact convex set with \mathbf{C}^3 boundary, then (C) holds.

Proposition 3.1 *Let (f) and (C) hold. Then, the map $F: [0, \delta] \times [0, T] \times C \mapsto C$, with $\delta = \min \{d/(2 \sup \|f\|), T\}$, defined by*

$$F(t, t_o)u = \Pi(u + t f(t_o, u))$$

is a local flow that satisfies (D) and the hypotheses of Theorem 2.5.

Proof. The continuity of F , as well as its Lipschitzeanity in t and u , is immediate. Condition (D) holds with $\mathcal{D}_{t_o} = C$ for all t_o .

Consider (2.7). Fix $\tau, \tau' \in [0, \delta]$ and compute:

$$\begin{aligned} \Delta &= \|F(\tau', t_o + \tau) \circ F(\tau, t_o)u - F(\tau' + \tau, t_o)u\| \\ &\leq \left\| \Pi \left(\Pi(u + \tau f(t_o, u)) + \tau' f(t_o + \tau, \Pi(u + \tau f(t_o, u))) \right) \right. \\ &\quad \left. - \Pi \left(\Pi(u + \tau f(t_o, u)) + \tau' f(t_o, u) \right) \right\| \\ &\quad + \left\| \Pi \left(\Pi(u + \tau f(t_o, u)) + \tau' f(t_o, u) \right) - \Pi(u + (\tau' + \tau)f(t_o, u)) \right\|. \end{aligned}$$

Consider the two summands above separately. Using **(C)**, the latter one is estimated in Lemma 4.10. The former is estimated as follows:

$$\begin{aligned}
& \left\| \Pi \left(\Pi (u + \tau f(t_o, u)) + \tau' f \left(t_o + \tau, \Pi (u + \tau f(t_o, u)) \right) \right) \right. \\
& \quad \left. - \Pi \left(\Pi (u + \tau f(t_o, u)) + \tau' f(t_o, u) \right) \right\| \\
& \leq \left\| \Pi (u + \tau f(t_o, u)) + \tau' f \left(t_o + \tau, \Pi (u + \tau f(t_o, u)) \right) \right. \\
& \quad \left. - \left(\Pi (u + \tau f(t_o, u)) + \tau' f(t_o, u) \right) \right\| \\
& \leq \tau' \left\| f \left(t_o + \tau, \Pi (u + \tau f(t_o, u)) \right) - f(t_o, u) \right\| \\
& \leq \tau' \left(\omega(\tau) + (L \sup \|f\|) \tau \right)
\end{aligned}$$

Summing up the two bounds for the two terms we obtain

$$\begin{aligned}
\Delta & \leq \tau' \left(\omega(\tau) + (L \sup \|f\|) \tau \right) + K (\sup \|f\|) \tau \tau' \\
& \leq \tau' \left(\omega(\tau) + (L + K) (\sup \|f\|) \tau \right)
\end{aligned}$$

Passing now to (2.8), observe that

$$\begin{aligned}
& \left\| \Pi (u + \tau f(t_o, u)) - \Pi (w + \tau f(t_o, w)) \right\| \leq \\
& \leq \left\| u - w + \tau (f(t_o, u) - f(t_o, w)) \right\| \\
& \leq (1 + L\tau) \|u - w\| \\
& \leq e^{L\tau} \|u - w\|
\end{aligned}$$

which directly implies (2.8). \square

We note that condition **(C)** can be slightly relaxed. Indeed, Proposition 3.1 essentially requires that (4.5) is satisfied. This bound can be proved also with only the $\mathbf{C}^{1,\alpha}$ regularity of Π .

3.3 Nonlinear Operator Splitting in a Metric Space

The construction in Theorem 2.5 generalizes that in [9]. Indeed, consider the case of two Lipschitz semigroups $S^1, S^2: \mathbb{R}^+ \times X \mapsto X$ and assume that they *commute* in the sense of [9, **(C)**], i.e.

$$d \left(S_t^1 S_t^2 u, S_t^2 S_t^1 u \right) \leq t \omega(t) \quad \text{and} \quad \int_0^{\delta/2} \frac{\omega(t)}{t} dt < +\infty. \quad (3.2)$$

for a suitable map $\omega: [0, \delta/2] \mapsto \mathbb{R}$. In this framework, let $\mathcal{D}_t = X$ for all $t \in \mathbb{R}^+$, so that $\mathcal{A} = \mathbb{R}^+ \times \mathbb{R}^+ \times X$. Introduce

$$F(\tau, t_o) = S_\tau^1 S_\tau^2. \quad (3.3)$$

Note that F is a local flow in the sense of Definition 2.1. The next lemma shows that (3.2) implies 1. in Theorem 2.5.

Lemma 3.2 *Let S^1, S^2 be Lipschitz semigroups satisfying (3.2) and define F as in (3.3). Then,*

$$d\left(F(k\tau, t_o + \tau) \circ F(\tau, t_o)u, F((k+1)\tau, t_o)u\right) \leq \mathbf{Lip}(S^1) \mathbf{Lip}(S^2) k\tau \omega(\tau).$$

Proof. Compute:

$$\begin{aligned} & d\left(F(k\tau, t_o + \tau) \circ F(\tau, t_o)u, F((k+1)\tau, t_o)u\right) = \\ &= d\left(S_{k\tau}^1 S_{k\tau}^2 S_{\tau}^1 S_{\tau}^2 u, S_{(k+1)\tau}^1 S_{(k+1)\tau}^2 u\right) \\ &\leq \mathbf{Lip}(S^1) \cdot d\left(S_{k\tau}^2 S_{\tau}^1 S_{\tau}^2 u, S_{\tau}^1 S_{(k+1)\tau}^2 u\right) \\ &\leq \mathbf{Lip}(S^1) \sum_{j=0}^{k-1} d\left(S_{(j+1)\tau}^2 S_{\tau}^1 S_{(k-j)\tau}^2 u, S_{j\tau}^2 S_{\tau}^1 S_{(k+1-j)\tau}^2 u\right) \\ &\leq \mathbf{Lip}(S^1) \mathbf{Lip}(S^2) \sum_{j=0}^{k-1} d\left(S_{\tau}^2 S_{\tau}^1 S_{(k-j)\tau}^2 u, S_{\tau}^1 S_{\tau}^2 S_{(k-j)\tau}^2 u\right) \\ &\leq \mathbf{Lip}(S^1) \mathbf{Lip}(S^2) \sum_{j=0}^{k-1} \tau \omega(\tau) \\ &\leq \mathbf{Lip}(S^1) \mathbf{Lip}(S^2) k\tau \omega(\tau) \end{aligned}$$

completing the proof. \square

The following corollary of Theorem 2.5 that extends [9, Theorem 3.8] is now immediate.

Corollary 3.3 *Let S^1, S^2 be Lipschitz semigroups satisfying (3.2). Suppose also that the two semigroups are Trotter stable, i.e. for any $t \in [0, T]$, $n \in \mathbb{N} \setminus \{0\}$ and $u, w \in X$:*

$$d\left(\left[S_{t/n}^1 S_{t/n}^2\right]^n u, \left[S_{t/n}^1 S_{t/n}^2\right]^n w\right) \leq \tilde{C} \cdot d(u, w). \quad (3.4)$$

Then, the Euler ε -polygons with F as in (3.3) converge to a unique product semigroup P tangent to the local flow $F(\tau)u = S_{\tau}^1 S_{\tau}^2 u$.

Here we do not require the strong commutativity condition [9, (\mathbf{C}^*)] as in [9, Theorem 3.8], but only its weaker version (3.2). Note also that the assumption [9, $(\mathbf{S3})$] requiring $d(S_t u, S_t w) \leq e^{Ct} d(u, w)$ is stronger than the Trotter stability requirement (3.4) as shown in the following lemma.

Lemma 3.4 Fix a positive T . Let S^1, S^2 be Lipschitz semigroups satisfying

$$d(S_t^1 u, S_t^1 w) \leq e^{Ct} d(u, w) \quad \text{and} \quad d(S_t^2 u, S_t^2 w) \leq e^{Ct} d(u, w)$$

for $u, w \in X$ and a fixed $C > 0$. Then, (3.4) holds for $t \in [0, T]$.

For a proof, see [9, formula (3.3) in Proposition 3.2].

3.4 Trotter Formula for Linear Semigroups

The present non linear framework recovers, under slightly different assumptions, the convergence of Trotter formula [25] in the case of linear semigroups, see [21, Theorem 3].

Proposition 3.5 Let S^1, S^2 be strongly continuous semigroups on a Banach space X . Assume that there exists a normed vector space Y which is densely embedded in X and invariant under both semigroups such that:

(a) the two semigroups are locally Lipschitz in time on Y , i.e. there exists a compact map $K: Y \mapsto \mathbb{R}$ such that for $i = 1, 2$

$$\left\| S_t^i u - S_{t'}^i u \right\|_X \leq K(u) |t - t'| \quad \text{for all } u \in Y, t, t' \in I.$$

(b) the two semigroups are exponentially bounded on F and locally Trotter stable on X and Y , i.e. there exists a constant H such that for all $t \in I$, $n \in \mathbb{N} \setminus \{0\}$

$$\left\| S_t^1 \right\|_Y + \left\| S_t^1 \right\|_Y + \left\| \left(S_{t/n}^1 S_{t/n}^2 \right)^n \right\|_X + \left\| \left(S_{t/n}^1 S_{t/n}^2 \right)^n \right\|_Y \leq H;$$

(c) the commutator condition

$$\frac{1}{t} \left\| S_t^2 S_t^1 u - S_t^1 S_t^2 u \right\|_X \leq \omega(t) \|u\|_Y.$$

is satisfied for all $u \in Y$ and $t \in [0, \delta]$ with some $\delta > 0$, and for a suitable $\omega: [0, \delta] \mapsto \mathbb{R}^+$ with $\int_0^\delta \frac{\omega(\tau)}{\tau} d\tau < +\infty$.

Then, there exists a global semigroup $P: [0, +\infty[\times X \rightarrow X$ such that

(I) for all $u \in Y$, there exists a constant C_u such that

$$\frac{1}{t} \left\| P(t)u - S_t^1 S_t^2 u \right\|_X \leq C_u \int_0^t \frac{\omega(\xi)}{\xi} d\xi.$$

(II) all Euler polygonals with initial data in Y , as defined in (2.2) and in (2.3) with $F(t) = S_t^1 S_t^2$, converge to the orbits of P .

Remark 3.6 *With respect to [21, Theorem 3], the regularity assumptions on the semigroups are stronger due to (a). On the other hand, (c) is weaker and we further obtain (I) and (II).*

In the theorem above, standard techniques allow to relate the generators of S^1 and S^2 with that of P , see [8, Proposition 4.1] and [21, Theorem 3].

Proof of Proposition 3.5. Fix a positive M and define

$$\mathcal{D}_M = \left\{ u \in Y : \|u\|_Y \leq M \right\} \quad \text{and} \quad \mathcal{D}_M^* = \left\{ u \in Y : \|u\|_Y \leq \frac{M}{H^6} \right\}.$$

In the setting of Theorem 2.5, consider the metric space $(X, \|\cdot\|_X)$ and define

$$F(t)u = S_t^1 S_t^2 u \quad \text{and} \quad \mathcal{D} = \text{cl} \mathcal{D}_M$$

where the closure is meant with respect to the X norm. It is straightforward to show that F is a Lipschitz local flow on \mathcal{D} . The stability (with respect to the X norm) of the Euler ε -polygonals implies hypothesis 2. Computations similar to those in the proof of Lemma 3.2 show that also hypothesis 1. is satisfied. Hence, Theorem 2.5 yields the existence of a process tangent to the local flow F . The stability (in the Y norm) of the Euler ε -polygonals ensures that $\text{cl} \mathcal{D}_M^* \subseteq \mathcal{D}_{t_o}^3$ for any $t_o \in I$. Finally, the arbitrariness of M allows us to extend P to all Y and, by density, to all X . \square

3.5 Hille-Yosida Theorem

This section is devoted to show that Theorem 2.5 comprehends the constructive part of Hille-Yosida Theorem, see [15, Chapter II, Theorem 3.5] or [16, Theorem 12.3.1]. The resulting proof, which we sketch below, is similar to the original proof by Hille, see [16, p. 362].

Theorem 3.7 *Let X be a Banach space and A be a linear operator with domain $D(A)$. If $(A, D(A))$ is closed, densely defined and for all $\lambda > 0$ one has $\lambda \in \rho(A)$ and $\|\lambda R(\lambda, A)\| \leq 1$, then A generates a strongly continuous contraction semigroup.*

Above, as usual, $\rho(A)$ is the resolvent set of A and $R(\lambda, a)$ is the resolvent. Note that we consider only the contractive case and refer to [15, Theorem 3.8, Chapter II] to see how the general case can be recovered.

Proof of Theorem 3.7. Fix a positive M and define

$$\begin{aligned} \mathcal{D}_t = \mathcal{D} &= \text{cl} \left\{ u \in D(A^2) : \|Au\| \leq M \text{ and } \|A^2 u\| \leq M \right\} \\ F(t)u &= \begin{cases} \frac{1}{t} R\left(\frac{1}{t}, A\right) u & t > 0, \\ u & t = 0. \end{cases} \end{aligned}$$

It is easy to see that $F(t)\mathcal{D} \subseteq \mathcal{D}$, so that **(D)** holds. Moreover, the stability condition 2. in Theorem 2.5 trivially holds in the contractive case. Recall the usual identities for the resolvent (see [15, Chapter IV])

$$\begin{aligned}\lambda R(\lambda, A)u &= u + R(\lambda, A)Au \quad \text{for any } u \in D(A) \\ R(\lambda, A) - R(\mu, A) &= (\mu - \lambda)R(\lambda, A)R(\mu, A).\end{aligned}$$

Written using the local flow F , these identities become

$$\begin{aligned}F(t)u &= u + tF(t)Au \quad \text{for any } u \in D(A) \\ tF(t) - sF(s) &= (t - s)F(t)F(s).\end{aligned}$$

Now we first show that F is Lipschitz in t and u . The Lipschitz continuity with respect to u is a straightforward consequence of the bound on the resolvent norm. Concerning the variable t , we take $u \in D(A)$ with $\|Au\| \leq M$ and consider two cases:

$$\begin{aligned}\|F(t)u - F(0)u\| &= \|tF(t)Au\| \leq tM \\ \|F(t)u - F(s)u\| &= \|tF(t)Au - sF(s)Au\| \leq \|(t - s)F(t)F(s)Au\| \\ &\leq |t - s|M.\end{aligned}$$

Since $F(t)$ is a bounded operator, the Lipschitz continuity with constant M extends to all \mathcal{D} . We are left to prove hypothesis 1. in Theorem 2.5. For $u \in D(A^2)$ with $\|A^2u\| \leq M$, $\|Au\| \leq M$, compute

$$\begin{aligned}&\|F(t)F(s)u - F(t+s)u\| = \\ &= \left\| (\text{Id} + tF(t)A) (\text{Id} + sF(s)A)u - (\text{Id} + (t+s)F(t+s)A)u \right\| \\ &= \left\| tF(t)Au + sF(s)Au + tsF(t)F(s)A^2u - (t+s)F(t+s)Au \right\| \\ &\leq ts\|A^2u\| + t\|(F(t) - F(t+s))Au\| + s\|(F(s) - F(t+s))Au\| \\ &\leq tsM + ts\|A^2u\| + st\|A^2u\| \\ &\leq 3tsM.\end{aligned}$$

Again, the boundedness of $F(t)$ allows us to extend the inequality

$$\|F(t)F(s)u - F(t+s)u\| \leq 3stM$$

to all $u \in \mathcal{D}$. Therefore, letting $t = k\tau$, $s = \tau$, Theorem 2.5 applies with $\omega(\tau) = 3M\tau$. Due to the arbitrariness of M and the density of $D(A^2)$, the resulting semigroup can be extended to all X . Standard computations, see [16, p. 362–363], show that A is the corresponding generator. \square

Note that, by Proposition 2.7, we also provide the convergence of all polygonal approximation.

3.6 The Heat Equation

Let T, δ be positive and $X = \mathbf{C}_B^0(\mathbb{R}; \mathbb{R})$ be the set of continuous and bounded real functions defined on \mathbb{R} equipped with the distance $d(u, w) = \|u - w\|_{\mathbf{C}^0}$, where $\|u\|_{\mathbf{C}^0} = \sup_{x \in \mathbb{R}} \|u(x)\|$. Fix a positive M and for all $t_o \geq 0$ let $\mathcal{D}_{t_o} = \mathcal{D}$ be the subset of X consisting of all twice differentiable functions whose second derivative u'' satisfies $\max \left\{ \|u''\|, \mathbf{Lip}(u'') \right\} \leq M$. For $t \in [0, \delta]$, $t_o \in \mathbb{R}$ and $u \in X$, using the numerical algorithm [18, Chapter 9, § 4], define

$$(F(t)u)(x) = u(x) + \frac{u(x - 2\sqrt{t}) - 2u(x) + u(x + 2\sqrt{t})}{4}$$

Proposition 3.8 F is a local flow satisfying (2.7) and (2.8).

Proof. Note that $u \mapsto F(t)u$ is linear. For $u \in \mathcal{D}$, introduce

$$(D_\sigma^2 u)(x) = \frac{u(x - \sigma) - 2u(x) + u(x + \sigma)}{\sigma^2}.$$

D_σ^2 is a linear operator and $F(t)u = u + tD_{2\sqrt{t}}^2 u$. Moreover,

$$(D_\sigma^2 u)(x) = \int_0^1 \left(\int_{-1}^{+1} \xi u''(x + \eta\xi\sigma) d\eta \right) d\xi$$

so that

$$\begin{aligned} \|D_{\sigma_1}^2 u - D_{\sigma_2}^2 u\|_{\mathbf{C}^0} &\leq \mathbf{Lip}(u'') |\sigma_1 - \sigma_2| \\ \|D_\sigma^2 u\|_{\mathbf{C}^0} &\leq \min \left\{ \frac{4}{\sigma^2} \|u\|_{\mathbf{C}^0}, \|u''\|_{\mathbf{C}^0}, \frac{1}{3} \mathbf{Lip}(u'') \sigma \right\}. \end{aligned}$$

F is Lipschitz in t , indeed if $t_1 < t_2$, then

$$\begin{aligned} \|F(t_2)u - F(t_1)u\|_{\mathbf{C}^0} &= \left\| t_2 D_{2\sqrt{t_2}}^2 u - t_1 D_{2\sqrt{t_1}}^2 u \right\|_{\mathbf{C}^0} \\ &\leq |t_2 - t_1| \|u''\|_{\mathbf{C}^0} + 2 \mathbf{Lip}(u'') t_1 \left| \sqrt{t_2} - \sqrt{t_1} \right| \\ &\leq \left(\|u''\|_{\mathbf{C}^0} + \mathbf{Lip}(u'') \sqrt{\delta} \right) |t_2 - t_1|. \end{aligned}$$

Consider now 1. and write $s = kt$ for $k \in \mathbb{N}$. Then

$$\begin{aligned} &\|F(kt)F(t)u - F(kt+t)u\|_{\mathbf{C}^0} = \\ &= \left\| F(kt) \left(u + tD_{2\sqrt{t}}^2 u \right) - u - (t+kt)D_{2\sqrt{t+kt}}^2 u \right\|_{\mathbf{C}^0} \end{aligned}$$

$$\begin{aligned}
&= \left\| tD_{2\sqrt{t}}^2 u + ktD_{2\sqrt{kt}}^2 u + kttD_{2\sqrt{kt}}^2 D_{2\sqrt{t}}^2 u - (t+kt)D_{2\sqrt{t+kt}}^2 u \right\|_{\mathbf{C}^0} \\
&\leq t \left\| D_{2\sqrt{t}}^2 u - D_{2\sqrt{t+kt}}^2 u \right\|_{\mathbf{C}^0} + kt \left\| D_{2\sqrt{kt}}^2 u - D_{2\sqrt{t+kt}}^2 u \right\|_{\mathbf{C}^0} \\
&\quad + kt^2 \left\| D_{2\sqrt{kt}}^2 D_{2\sqrt{t}}^2 u \right\|_{\mathbf{C}^0} \\
&\leq 2 \mathbf{Lip}(u'') \left(t \left| \sqrt{t+kt} - \sqrt{t} \right| + kt \left| \sqrt{t+kt} - \sqrt{kt} \right| \right) + t \left\| D_{2\sqrt{t}}^2 u \right\|_{\mathbf{C}^0} \\
&\leq 2 \mathbf{Lip}(u'') \left(t\sqrt{kt} + kt\sqrt{t} \right) + \frac{2}{3} \mathbf{Lip}(u'') t\sqrt{t} \\
&\leq \frac{14}{3} M kt\sqrt{t}
\end{aligned}$$

hence, 1. in Theorem 2.5 is satisfied with $\omega(t) = \frac{14}{3} M \sqrt{t}$.

Finally, the equality

$$(F(t)u)(x) = \frac{1}{4} u(x - 2\sqrt{t}) + \frac{1}{2} u(x) + \frac{1}{4} u(x + 2\sqrt{t})$$

implies that $(F(t)u)(\mathbb{R}) \subseteq \text{co}(u(\mathbb{R}))$. Hence, for all $u, w \in X$,

$$d(F(t)u, F(t)w) = \|F(t)(w - u)\|_{\mathbf{C}^0} \leq \|w - u\|_{\mathbf{C}^0} = d(u, w)$$

showing that F is non expansive in u and, hence, that 2. holds. \square

4 Technical Details

Occasionally, for typographical reasons, we write $d\left(\begin{smallmatrix} u, \\ w \end{smallmatrix}\right)$ for $d(u, w)$.

In this section, we use the following definition

$$\mathcal{D}_{t_o}^2 = \left\{ u \in \mathcal{D}: \begin{array}{l} \text{is in } \mathcal{D} \text{ for all } \varepsilon_1, \varepsilon_2 \in]0, \delta] \text{ and all} \\ t_1, t_2 \geq 0 \text{ such that } t_o + t_1 + t_2 \in I \end{array} \right\} \quad (4.1)$$

Observe that one obviously has $\mathcal{D}_{t_o}^3 \subseteq \mathcal{D}_{t_o}^2$.

Proof of Lemma 2.3. $F^\varepsilon(t, t_o)u$ is continuous in t_o and Lipschitz in u because it is the composition (2.3) of functions with the same properties, see Definition 2.1. Fix now positive s, t with $t_o \leq s \leq t \leq T$. Then note that if $h\varepsilon \leq s \leq t \leq (h+1)\varepsilon$ for $h \in \mathbb{N}$, then by Definition 2.1,

$$d(F^\varepsilon(s, t_o)u, F^\varepsilon(t, t_o)u) \leq \mathbf{Lip}(F) |s - t|.$$

Let $k = \lceil t/\varepsilon \rceil$ and $h = \lceil s/\varepsilon \rceil + 1$. The case $h > k$ is recovered by the previous computation. If $h \leq k$, then by (2.3)

$$d(F^\varepsilon(s, t_o)u, F^\varepsilon(t, t_o)u) \leq d(F^\varepsilon(s, t_o)u, F^\varepsilon(h\varepsilon, t_o)u)$$

$$\begin{aligned}
& + \sum_{j=h}^{k-1} d \left(F^\varepsilon(j\varepsilon, t_o)u, F^\varepsilon((j+1)\varepsilon, t_o)u \right) \\
& + d \left(F^\varepsilon(k\varepsilon, t_o)u, F^\varepsilon(t, t_o)u \right) \\
& \leq \mathbf{Lip}(F) \left((h\varepsilon - s) + (k-h)\varepsilon + (t - k\varepsilon) \right) \\
& = \mathbf{Lip}(F)(t - s).
\end{aligned}$$

□

Lemma 4.1 *Let F be a local flow satisfying 1. in Theorem 2.5. Then, for all h, k, t_o, u and ε satisfying $t_o \in I$, $h, k \in \mathbb{N}$, $k\varepsilon \in]0, \delta]$, $t_o + (h+k)\varepsilon \in I$, $u \in \mathcal{D}_{t_o}^2$, the following holds:*

$$d(F(k\varepsilon, \bar{t})v, F^\varepsilon(k\varepsilon, \bar{t})v) \leq k^2\varepsilon\omega(\varepsilon).$$

where $v = F^\varepsilon(h\varepsilon, t_o)u$ and $\bar{t} = t_o + h\varepsilon$.

Proof. The Definition (4.1) of $\mathcal{D}_{t_o}^2$ implies that we can apply the triangle inequality and hypothesis 1. in Theorem 2.5 to obtain

$$\begin{aligned}
& d(F(k\varepsilon, \bar{t})v, F^\varepsilon(k\varepsilon, \bar{t})v) \leq \\
& \leq \sum_{j=1}^{k-1} d \left(\begin{array}{l} F((j+1)\varepsilon, (k-(j+1))\varepsilon + \bar{t}) \circ \\ \circ F^\varepsilon((k-(j+1))\varepsilon, \bar{t})v, \\ F(j\varepsilon, (k-j)\varepsilon + \bar{t}) \circ F(\varepsilon, (k-(j+1))\varepsilon + \bar{t}) \circ \\ \circ F^\varepsilon((k-(j+1))\varepsilon, \bar{t})v \end{array} \right) \\
& \leq \sum_{j=1}^{k-1} j\varepsilon\omega(\varepsilon) \\
& \leq k^2\varepsilon\omega(\varepsilon)
\end{aligned}$$

completing the proof. □

Lemma 4.2 *Let the local flow F satisfy the assumptions of Theorem 2.5. Then, for all h, k, t_o, u and ε satisfying $t_o \in I$, $h, k \in \mathbb{N}$, $k\varepsilon \in]0, \delta]$, $t_o + hk\varepsilon \in I$ and $u \in \mathcal{D}_{t_o}^2$, the following holds:*

$$d(F^{k\varepsilon}(hk\varepsilon, t_o)u, F^\varepsilon(hk\varepsilon, t_o)u) \leq Lh k^2\varepsilon\omega(\varepsilon).$$

Proof. The Definition (4.1) of $\mathcal{D}_{t_o}^2$ implies that we can apply the triangle inequality. Then, by the assumptions in Theorem 2.5, applying Lemma 4.1 and with computations similar to the ones in [21], we have

$$\begin{aligned}
& d\left(F^{k\varepsilon}(hk\varepsilon, t_o)u, F^\varepsilon(hk\varepsilon, t_o)u\right) \leq \\
& \leq \sum_{j=0}^{h-1} d\left(\begin{array}{l} F^{k\varepsilon}\left((j+1)k\varepsilon, (h-(j+1))k\varepsilon + t_o\right) \circ \\ \circ F^\varepsilon\left((h-(j+1))k\varepsilon, t_o\right)u, \\ F^{k\varepsilon}\left(jk\varepsilon, (h-j)k\varepsilon + t_o\right) \circ F^\varepsilon\left((h-j)k\varepsilon, t_o\right)u \end{array} \right) \\
& \leq \sum_{j=0}^{h-1} d\left(\begin{array}{l} F^{k\varepsilon}\left(jk\varepsilon, (h-j)k\varepsilon + t_o\right) \circ F^{k\varepsilon}\left(k\varepsilon, (h-(j+1))k\varepsilon + t_o\right) \\ \circ F^\varepsilon\left((h-(j+1))k\varepsilon, t_o\right)u, \\ F^{k\varepsilon}\left(jk\varepsilon, (h-j)k\varepsilon + t_o\right) \circ F^\varepsilon\left(k\varepsilon, (h-(j+1))k\varepsilon + t_o\right) \\ \circ F^\varepsilon\left((h-(j+1))k\varepsilon, t_o\right)u \end{array} \right) \\
& = L \sum_{j=0}^{h-1} d\left(\begin{array}{l} F^{k\varepsilon}\left(k\varepsilon, (h-(j+1))k\varepsilon + t_o\right) \\ \circ F^\varepsilon\left((h-(j+1))k\varepsilon, t_o\right)u, \\ F^\varepsilon\left(k\varepsilon, (h-(j+1))k\varepsilon + t_o\right) \\ \circ F^\varepsilon\left((h-(j+1))k\varepsilon, t_o\right)u \end{array} \right) \\
& = L \sum_{j=0}^{h-1} d\left(\begin{array}{l} F\left(k\varepsilon, (h-(j+1))k\varepsilon + t_o\right) \\ \circ F^\varepsilon\left((h-(j+1))k\varepsilon, t_o\right)u, \\ F^\varepsilon\left(k\varepsilon, (h-(j+1))k\varepsilon + t_o\right) \\ \circ F^\varepsilon\left((h-(j+1))k\varepsilon, t_o\right)u \end{array} \right) \\
& \leq L \sum_{j=0}^{h-1} k^2\varepsilon\omega(\varepsilon) \\
& = Lhk^2\varepsilon\omega(\varepsilon)
\end{aligned}$$

completing the proof. \square

Lemma 4.3 *Let the local flow F satisfy the assumptions of Theorem 2.5. Then, for all m, n, t, t_o and u satisfying $t > 0$, $t_o, t_o + t \in I$, $m, n \in \mathbb{N}$, $n \geq m$, $t2^{-m} \in]0, \delta]$ and $u \in \mathcal{D}_{t_o}^2$, the following holds:*

$$d\left(F^{t2^{-m}}(t, t_o)u, F^{t2^{-n}}(t, t_o)u\right) \leq \frac{2L}{\ln 2} t \int_{t2^{-n}}^{t2^{-m}} \frac{\omega(\xi)}{\xi} d\xi.$$

Proof. Applying Lemma 4.2 with $\varepsilon = t2^{-(j+1)}$, $k = 2$, $h = 2^j$ and $t = hk\varepsilon$,

$$\begin{aligned}
& d\left(F^{t2^{-m}}(t, t_o)u, F^{t2^{-n}}(t, t_o)u\right) \leq \\
& \leq \sum_{j=m}^{n-1} d\left(F^{2t2^{-(j+1)}}(t, t_o)u, F^{t2^{-(j+1)}}(t, t_o)u\right) \\
& \leq L \sum_{j=m}^{n-1} 2t \omega(t2^{-(j+1)}) \\
& \leq 2Lt \sum_{j=m}^{n-1} \int_j^{j+1} \omega(t2^{-(j+1)}) ds \\
& \leq \frac{2Lt}{\ln 2} \int_{t2^{-n}}^{t2^{-m}} \frac{\omega(\xi)}{\xi} d\xi.
\end{aligned}$$

□

Corollary 4.4 *Let F be a local flow satisfying the assumptions of Theorem 2.5. Then, for any $t > 0$, $t_o, t_o + t \in I$, $u \in \mathcal{D}_{t_o}^2$ and $T2^{-m} \in]0, \delta]$, the sequence of functions*

$$(t, t_o, u) \mapsto \begin{cases} F^{t2^{-m}}(t, t_o)u & \text{for } t > 0 \\ u & \text{for } t = 0 \end{cases}$$

converges uniformly to a continuous map $(t, t_o, u) \rightarrow P(t, t_o)u$.

Proof. Observe that the maps in the sequence above are continuous, since $F^{t2^{-m}}(t, t_o)u = \bigcirc_{j=0}^{2^m-1} F(t2^{-m}, t_o + j2^{-m}t)u$ is the composition of 2^m continuous maps. The uniform convergence follows from Lemma 4.3. □

Remark 4.5 *To prove the existence of a limit P , i.e. Lemma 4.3 and Corollary 4.4, we need Lemma 4.2 and hence Lemma 4.1 only for $k = 2$. On the other hand, to prove these lemmas for $k = 2$, it is enough to assume 1. of Theorem 2.5 only for $k = 1$. Assumption 1. in Theorem 2.5 for all $k \in \mathbb{N}$ is needed to show that all the sequence of Euler approximates F^ε converges (Proposition 4.6) and, hence, that the limit is indeed a process (Lemma 4.8). The necessity of hypothesis 1. for all $k \in \mathbb{N}$ is shown in Paragraph 2.1*

Proposition 4.6 *Let F be a local flow satisfying the assumptions of Theorem 2.5. Then, for all t_o, t, u with $t \geq 0$, $t_o, t_o + t \in I$ and $u \in \mathcal{D}_{t_o}^2$, the following limit holds:*

$$\lim_{\varepsilon \rightarrow 0^+} F^\varepsilon(t, t_o)u = P(t, t_o)u,$$

with P defined as in Corollary 4.4.

Proof. Fix t_o, t, u as above. Let $n_\varepsilon = \lceil t/\varepsilon \rceil$. Observe first that

$$\begin{aligned} d(F^\varepsilon(t, t_o)u, P(t, t_o)u) &\leq d(F^\varepsilon(t, t_o)u, F^\varepsilon(n_\varepsilon\varepsilon, t_o)u) \\ &\quad + d(F^\varepsilon(n_\varepsilon\varepsilon, t_o)u, P(n_\varepsilon\varepsilon, t_o)u) \\ &\quad + d(P(n_\varepsilon\varepsilon, t_o)u, P(t, t_o)u) \\ &\leq d(F^\varepsilon(n_\varepsilon\varepsilon, t_o)u, P(n_\varepsilon\varepsilon, t_o)u) + o(1) \end{aligned}$$

as $\varepsilon \rightarrow 0$, due to the Lipschitz continuity of F and the continuity of P . With computations similar to the ones found in [21], for $l \in \mathbb{N}$ such that $T2^{-l} < \delta$ write:

$$\begin{aligned} &d(F^\varepsilon(n_\varepsilon\varepsilon, t_o)u, P(n_\varepsilon\varepsilon, t_o)u) \leq \\ &\leq d(F^\varepsilon(n_\varepsilon\varepsilon, t_o)u, F^{\varepsilon 2^{-l}}(n_\varepsilon\varepsilon, t_o)u) \\ &\quad + d(F^{\varepsilon 2^{-l}}(n_\varepsilon\varepsilon, t_o)u, F^{n_\varepsilon\varepsilon 2^{-l}}(n_\varepsilon\varepsilon, t_o)u), \\ &\quad + d(F^{n_\varepsilon\varepsilon 2^{-l}}(n_\varepsilon\varepsilon, t_o)u, P(n_\varepsilon\varepsilon, t_o)u) \end{aligned} \tag{4.2}$$

Consider the three terms separately. Estimating the former one, apply first Lemma 4.2 with $h = n_\varepsilon 2^j$, $k = 2$ and ε substituted by $\varepsilon 2^{-(l+1)}$ and then treat the summation as in the proof of Lemma 4.3:

$$\begin{aligned} &d(F^\varepsilon(n_\varepsilon\varepsilon, t_o)u, F^{\varepsilon 2^{-l}}(n_\varepsilon\varepsilon, t_o)u) \leq \\ &\leq \sum_{j=0}^{l-1} d(F^{2\varepsilon 2^{-(j+1)}}(n_\varepsilon\varepsilon, t_o)u, F^{\varepsilon 2^{-(j+1)}}(n_\varepsilon\varepsilon, t_o)u) \\ &\leq L \sum_{j=0}^{l-1} 2n_\varepsilon\varepsilon\omega(\varepsilon 2^{-(j+1)}) \\ &\leq \frac{2Lt}{\ln 2} \int_0^\varepsilon \frac{\omega(\xi)}{\xi} d\xi \\ &\rightarrow 0 \text{ as } \varepsilon \rightarrow 0+ \text{ uniformly in } l. \end{aligned}$$

Estimate the second term in (4.2) again using Lemma 4.2 with $h = 2^l$, $k = n_\varepsilon$ and ε substituted by $\varepsilon 2^{-l}$:

$$\begin{aligned} d(F^{\varepsilon 2^{-l}}(n_\varepsilon\varepsilon, t_o)u, F^{n_\varepsilon\varepsilon 2^{-l}}(n_\varepsilon\varepsilon, t_o)u) &\leq Ln_\varepsilon^2 \varepsilon \omega(\varepsilon 2^{-l}) \\ &\rightarrow 0 \text{ as } l \rightarrow +\infty \text{ for fixed } \varepsilon. \end{aligned}$$

Finally, also the latter term in (4.2) vanishes as $l \rightarrow +\infty$ by Corollary 4.4.

The proof is completed passing in (4.2) to the limits first $l \rightarrow +\infty$ and, secondly, $\varepsilon \rightarrow 0$. \square

Remark 4.7 Observe that, since $F^\varepsilon(t, t_o)u$ is uniformly Lipschitz in (t, u) , $P(t, t_o)u$ is also uniformly Lipschitz in (t, u) with the same constant as F^ε .

Lemma 4.8 There exists a family of sets $\mathcal{D}_{t_o} \subseteq \mathcal{D}$, for $t_o \in I$ such that

i) $\mathcal{D}_{t_o}^3 \subseteq \mathcal{D}_{t_o} \subseteq \mathcal{D}_{t_o}^2$ for any $t_o \in I$;

ii) The map P defined in Corollary 4.4 and restricted to the set $\mathcal{A} = \{(t, t_o, u) : t \geq 0, t_o, t_o + t \in I, u \in \mathcal{D}_{t_o}\}$ is a global process according to Definition 2.4 and Lipschitz in all its variables.

Proof. Define

$$\mathcal{D}_{t_o} = \left\{ u \in \mathcal{D}_{t_o}^2 : P(t, t_o)u \in \mathcal{D}_{t_o+t}^2 \text{ for any } t \geq 0 \text{ such that } t_o + t \in I \right\}.$$

Obviously, $\mathcal{D}_{t_o} \subseteq \mathcal{D}_{t_o}^2$. Take now $u \in \mathcal{D}_{t_o}^3$, we want to show that u also belongs to \mathcal{D}_{t_o} . By (2.6)

$$F^{\varepsilon_2}(t_2, t_o + t + t_1) \circ F^{\varepsilon_1}(t_1, t_o + t) \circ F^\varepsilon(t, t_o)u \in \mathcal{D}$$

for any $\varepsilon, \varepsilon_1, \varepsilon_2 \in]0, \delta]$ and $t, t_1, t_2 \geq 0$ such that $t_o + t + t_1 + t_2 \in I$. Since $u \in \mathcal{D}_{t_o}^2$, \mathcal{D} is close and $F^{\varepsilon_1}, F^{\varepsilon_2}$ are Lipschitz, we let $\varepsilon \rightarrow 0$ and obtain

$$F^{\varepsilon_2}(t_2, t_o + t + t_1) \circ F^{\varepsilon_1}(t_1, t_o + t) \circ P(t, t_o)u \in \mathcal{D}$$

for any $\varepsilon_1, \varepsilon_2 \in]0, \delta]$ and $t_1, t_2 \geq 0$ such that $t_o + t + t_2 + t_3 \in I$, that is $P(t, t_o)u \in \mathcal{D}_{t_o}^2$ for all $t \geq 0$ such that $t_o + t \in I$ and hence $u \in \mathcal{D}_{t_o}$. For $t_o \in I$ and $u \in \mathcal{D}_{t_o}^2 \supset \mathcal{D}_{t_o}$ one trivially has $P(0, t_o)u = u$. We are left to prove the ‘‘semigroup properties’’ (2.4) and (2.5). We first show (2.5) for any $u \in \mathcal{D}_{t_o}$. If t_1 or t_2 vanishes, property (2.5) is trivial. Fix $t_1 > 0$ and take $t_2 > 0$ such that $t_o + t_1 + t_2 \in I$ and $\frac{t_2}{t_1} \in \mathbb{Q}$ so that there exist two integers $h, k \in \mathbb{N} \setminus \{0\}$ which satisfy $\frac{t_1}{k} = \frac{t_2}{h}$. For any $\nu \in \mathbb{N} \setminus \{0\}$ define $\varepsilon_\nu = \frac{1}{\nu} \frac{t_1}{k} = \frac{1}{\nu} \frac{t_2}{h}$. For $\varepsilon_\nu < \delta$, by Definition 2.3 we have

$$F^{\varepsilon_\nu}(t_2, t_o + t_1) \circ F^{\varepsilon_\nu}(t_1, t_o)u = F^{\varepsilon_\nu}(t_2 + t_1, t_o)u \quad (4.3)$$

Since $u \in \mathcal{D}_{t_o} \subseteq \mathcal{D}_{t_o}^2$ one has that $F^{\varepsilon_\nu}(t_2 + t_1, t_o)u$ converges to $P(t_2 + t_1, t_o)u$ as $\nu \rightarrow +\infty$ (see Proposition 4.6). Moreover the definition of \mathcal{D}_{t_o} implies that $P(t_1, t_o)u \in \mathcal{D}_{t_o+t_1}^2$ and therefore $F^{\varepsilon_\nu}(t_2, t_o + t_1) \circ P(t_1, t_o)u$ converges to $P(t_2, t_o + t_1) \circ P(t_1, t_o)u$. We can so compute

$$\begin{aligned} & d(F^{\varepsilon_\nu}(t_2, t_o + t_1) \circ F^{\varepsilon_\nu}(t_1, t_o)u, P(t_2, t_o + t_1) \circ P(t_1, t_o)u) \leq \\ & \leq d(F^{\varepsilon_\nu}(t_2, t_o + t_1) \circ F^{\varepsilon_\nu}(t_1, t_o)u, F^{\varepsilon_\nu}(t_2, t_o + t_1) \circ P(t_1, t_o)u) \\ & \quad + d(F^{\varepsilon_\nu}(t_2, t_o + t_1) \circ P(t_1, t_o)u, P(t_2, t_o + t_1) \circ P(t_1, t_o)u) \\ & \leq Ld(F^{\varepsilon_\nu}(t_1, t_o)u, P(t_1, t_o)u) \\ & \quad + d(F^{\varepsilon_\nu}(t_2, t_o + t_1) \circ P(t_1, t_o)u, P(t_2, t_o + t_1) \circ P(t_1, t_o)u) \\ & \rightarrow 0 \quad \text{as } \nu \rightarrow +\infty. \end{aligned}$$

Hence taking the limit as $\nu \rightarrow +\infty$ in (4.3) we obtain (2.5) for any t_2 with $\frac{t_2}{t_1} \in \mathbb{Q}$. The continuity of P concludes the proof of (2.5). Now, if $u \in \mathcal{D}_{t_o}$ then, by definition, $P(t_1, t_o)u \in \mathcal{D}_{t_o+t_1}^2$. But (2.5) implies also that

$$P(t, t_o + t_1) \circ P(t_1, t_o)u = P(t + t_1, t_o)u \in \mathcal{D}_{t_o+t+t_1}^2$$

for any $t \geq 0$ such that $t_o + t + t_1 \in I$, therefore $P(t_1, t_o)u \in \mathcal{D}_{t_o+t_1}$ proving (2.4). Finally, the Lipschitz continuity with respect to t and u follows from Remark 4.7, while the Lipschitz continuity with respect to t_o is a direct consequence of the semigroup property. Indeed, take $0 \leq t_1 \leq t_2 \leq T$, $u \in \mathcal{D}_{t_1} \cap \mathcal{D}_{t_2}$ and use the Lipschitz continuity with respect to (t, u) :

$$\begin{aligned} d(P(t, t_1)u, P(t, t_2)u) &= d(P(t, t_2) \circ P(t_2 - t_1, t_1)u, P(t, t_2)u) \\ &\leq L \cdot d(P(t_2 - t_1, t_1)u, u) \\ &\leq L \cdot \mathbf{Lip}(F) \cdot (t_2 - t_1). \end{aligned}$$

□

The following Lemma concludes the proof of Theorem 2.5.

Lemma 4.9 *The map $P : \mathcal{A} \rightarrow X$ defined in Corollary 4.4 restricted to the set \mathcal{A} defined in Lemma 4.8 satisfies c) in Theorem 2.5.*

Proof. When $t \in]0, \delta]$ one has $F^t(t, t_o)u = F(t, t_o)u$. Hence the Lemma is proved taking $t \in]0, \delta]$, $m = 0$ and $n \rightarrow +\infty$ in Lemma 4.3. □

Proof of Proposition 2.7. Use the Lipschitz continuity of P and (2.9):

$$\begin{aligned} &d(F^E(t, t_o)u, P(t, t_o)u) \leq \\ &\leq \sum_{k=1}^N d \left(\begin{array}{l} P(t - \tau_k, t_o + \tau_k) F^E(\tau_k, t_o)u, \\ P(t - \tau_{k-1}, t_o + \tau_{k-1}) F^E(\tau_{k-1}, t_o)u \end{array} \right) \\ &\leq L \cdot \sum_{k=1}^N d \left(\begin{array}{l} F(\tau_k - \tau_{k-1}, t_o + \tau_{k-1}) F^E(\tau_{k-1}, t_o)u, \\ P(\tau_k - \tau_{k-1}, t_o + \tau_{k-1}) F^E(\tau_{k-1}, t_o)u \end{array} \right) \\ &\leq \frac{2L^2}{\ln 2} \cdot \sum_{k=1}^N (\tau_k - \tau_{k-1}) \int_0^\Delta \frac{\omega(\xi)}{\xi} d\xi \\ &\leq \frac{2L^2}{\ln 2} t \int_0^\Delta \frac{\omega(\xi)}{\xi} d\xi \end{aligned}$$

□

Proof of Proposition 2.8. Since we assume that **(D)** is satisfied, we have no problem with the domains, i.e.

$$F(t, t_o)\mathcal{D}_{t_o} \subseteq \mathcal{D}_{t_o+t}, \quad \text{for } t \in [0, \delta]. \quad (4.4)$$

By Remark 4.5, $F^{t2^{-m}}(t, t_o)u$ converges uniformly to a map $P(t, t_o)u$ which is continuous in (t, t_o) and uniformly Lipschitz in u . With $m = 0$ $n \rightarrow \infty$ in Lemma 4.3 we get immediately the tangency condition. Hence, because of (4.4) P has all the properties of the process in Theorem 2.5 except the Lipschitz continuity with respect to (t, t_o) and the “semigroup” property (2.5). Finally, since

$$F^{2^{-nt}}(t, t_o + t) \circ F^{2^{-nt}}(t, t_o)u = F^{2^{-nt}}(2t, t_o) = F^{2^{-(n+1) \cdot 2t}}(2t, t_o)$$

as $n \rightarrow +\infty$ we get (2.11). \square

Lemma 4.10 *If C is a closed and convex subset of the Hilbert space X that satisfies **(C)**, then for all $u \in C$, $v \in X$ and $\tau, \tau' \geq 0$,*

$$\left\| \Pi(\Pi(u + \tau v) + \tau' v) - \Pi(u + (\tau + \tau')v) \right\| \leq K \|v\| \tau \tau'. \quad (4.5)$$

Proof. Assume first that $\|v\| = 1$.

If $u + \tau v \in C$, the right hand side above vanishes and the inequality trivially holds.

If $u + \tau v \notin C$, then there exists a unique $\tau'' \in]0, \tau[$ such that $u + (\tau - \tau'')v \in \partial C$. Moreover, setting $u'' = u + (\tau - \tau'')v$,

$$\begin{aligned} \Pi(u + \tau v) &= \Pi(u'' + \tau'' v) \\ \Pi(u + (\tau + \tau')v) &= \Pi(u'' + (\tau'' + \tau')v) \end{aligned}$$

By the convexity of C , both $u'' + \tau'' v$ and $u'' + (\tau'' + \tau')v$ are in $X \setminus C$. Hence, by **(C)**, the map

$$(\tau', \tau'') \mapsto \left\| \Pi(\Pi(u'' + \tau'' v) + \tau' v) - \Pi(u'' + (\tau'' + \tau')v) \right\|$$

is \mathbf{C}^2 for $\tau', \tau'' \in [0, \delta]$. It vanishes both for $\tau' = 0$ and for $\tau'' = 0$, so there exists a positive K such that

$$\begin{aligned} & \left\| \Pi(\Pi(u + \tau v) + \tau' v) - \Pi(u + (\tau + \tau')v) \right\| = \\ &= \left\| \Pi(\Pi(u'' + \tau'' v) + \tau' v) - \Pi(u'' + (\tau'' + \tau')v) \right\| \\ &\leq K \tau' \tau'' \\ &\leq K \tau \tau'. \end{aligned}$$

The case $\|v\| \neq 1$ follows by a straightforward rescaling procedure. \square

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