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The effect of ideology on policy outcomes in proportional representation systems

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Abstract

In this paper we propose a model in which there are ideological and strategic voters who vote under poportional rule. We prove that the behavior of ideological voters matters for the determination of the outcome. We show that a subset of strategic voters partially counteracts the votes of the ideological voters.

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1 Introduction

Electorates are typically constituted by two types of voters, those who are committed to vote for a specific party - its hard-core supporters, and those who vote in a more strategic fashion to get their way as regards the policies that are going to be implemented. The result of a general election can be expected to be influenced by the interplay of the behavior of these two types of voters. The present paper addresses such an interplay in the context of an election held with a proportional system.

Specifically, we study a society composed of policy motivated strategic citizens and ideological citizens, who vote for one of a finite number of parties by proportional rule. Given the electoral result, the policy outcome is a linear combination of the position of each party, weighed by the share of votes a party gets in the election. We ask, first, if the ideological voters' behavior affects the final outcome; second, how strategic voters respond to that.

We prove that the behavior of ideological voters matters for the outcome. In particular, we show that the policy will, in general, be different with respect to the case where all voters act strategically, even with an arbitrarily small number of ideological voters. Concerning the second question, we show how some strategic voters change their voting behavior to, at least partially, counteract the ideological citizens' vote. Strategic voters will vote in accord with a cutpoint outcome: in equilibrium, any strategic voter on the right of the cutpoint votes for the rightmost party and any strategic voter on its left votes for the leftmost party. The intuition is the following. Given the ideological voting behavior, strategic voters misrepresent their preferences by voting for the extremist parties in order to drag the policy outcome toward their preferred policy.

The model extends to an environment with ideological voters, the analysis of De Sinopoli and Iannantuoni (2007), who study strategic voting under proportional rule and find that essentially only a two-party equilibrium exists, in which voters vote only for the two extremist parties. The voting literature (Shepsle (1991), Cox (1997), Persson and Tabellini (2000)) has dealt with models in which either all voters are strategic or all are ideological. An analysis of the more realistic case in which both types coexist is missing. This paper also reconciles the two-party equilibrium result in De Sinopoli and Iannantuoni (2007), with the view that proportional systems should lead to multipartyism (see Cox 1997).

The rest of the paper is organized as follow. In section 2 we describe the model; we present an example in section 3; we analyze the pure strategy equilibria, and, then, the mixed strategy ones in section 4; we go back to the example in section 5; section 6 concludes.

2 The Model

Policy space. The policy space $\mathbb{X}$ is a closed interval of the real line. Without loss of generality, we assume $\mathbb{X} = [0, 1]$.

Parties. There is an exogenously given set of parties $M = \{1, ..., k, ...m\}$, with $m \geq 2$, indexed by k. Each party k is characterized by a policy $\zeta_k \in [0, 1]$. In order to simplify the notation, in the following we will denote L the leftmost party and R the rightmost (i.e., $L = \arg\min_{k \in M} \zeta_k$, $R = \arg\max_{k \in M} \zeta_k$).

Voters. There is a finite set of voters $N = \{1, ..., i, ...n\}$. Each voter i, characterized by a bliss point $\theta_i \in [0, 1]$, has single peaked preferences. The set of voters N is partitioned in two subsets N^ρ and N^ι, denoting respectively the set of strategic and ideological voters. We indicate the cardinality of N^ρ by n^ρ, and the cardinality of N^ι by n^ι. Hence, $n = n^\rho + n^\iota$. We denote with $H^\rho(\theta)$ the distribution of the strategic voters' bliss points.

Strategic voters. Each voter $i \in N^\rho$ possesses a utility function $u_i(X) = u(X, \theta_i)$ continuously differentiable with respect to the first argument. Since each voter can cast his vote for any party, the pure strategy space of each player $i \in N^\rho$ is $S_i = \{1, ..., k, ..., m\}$ where each $k \in S_i$ is a vector of m components with all zeros except for a 1 in position k, which represents the vote for party k. A mixed strategy of player i is a vector $\sigma_i = (\sigma_i^1, ...\sigma_i^k, ..., \sigma_i^m)$, where each σ_i^k represents the probability that player i votes for party k.

Ideological voters. A natural way to model ideological voters is to assume that their strategy space is degenerate, coinciding with the vote in favor of their preferred party, i.e. the party whose policy is closer to the voter bliss point. We denote by N_k^ι the set of ideological voters who vote for party k and with n_k^ι its cardinality. Hence, $s_i = k \; \forall i \in N_k^\iota$, and $n^\iota = \sum\limits_{k=1}^{m} n_k^\iota$.

Proportional rule and the policy outcome. Given a pure strategy combination $s = (s_1, s_2, ..., s_n)$, we define $v(s) = \sum\limits_{i \in N} \frac{s_i}{n}$ as the vector representing, for each party, its share of votes. We capture the spirit of proportional representation by assuming that any party in parliament participates in the determination of policy with a strength equal to its share of votes. The policy outcome is a linear combination of the parties' policies with coefficients equal to the share of votes corresponding to each party:

$$X(s) = \sum_{k=1}^{m} \zeta_k v_k(s). \tag{1}$$

The game. Given the set of parties and the utility function u, a finite game is given by the set of voters N, the subset of strategic voters with their bliss points, and the subsets of ideological voters:

$$\Gamma = \left\{ N, N^\rho, \{\theta_i\}_{i \in N^\rho}, \{N_k^\iota\}_{k \in M} \right\}$$

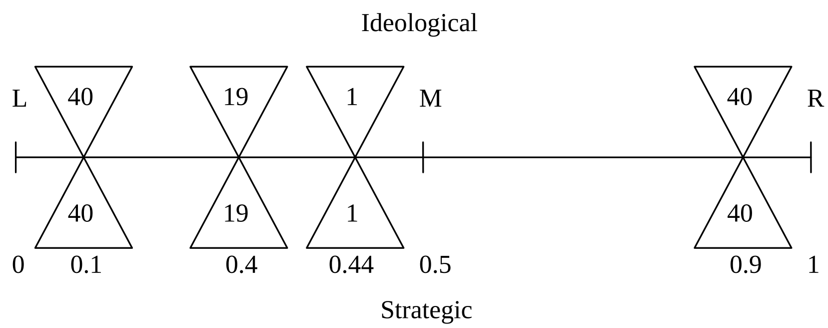

Figure 1: Ideological and Strategic Voters. Three Parties.

3 Example 1

Before moving on to the solution, an illustrative example is in order.[1] Consider a society with $n = 200$ voters and three parties. There are 40 ideological voters with bliss point in 0.1, 19 ideological voters with bliss point in 0.4, 1 ideological voter with bliss point in 0.44 and 40 ideological voters with bliss point in 0.9; there are also 40 strategic voters with bliss point in 0.1, 19 strategic voters with bliss point in 0.4, 1 strategic voter with bliss point in 0.44 and 40 strategic voters with bliss point in 0.9. There is a leftist party (L) with policy in 0, a moderate party (M) with policy in 0.5, a rightist party (R) with policy in 1. Figure 1 depicts the situation.

The policy outcome is determined, given a pure strategy combination, according to (1). The objective of the strategic voters is to obtain a policy which is as close

[1] In all the examples, we will restrict attention to pure strategy equilibria. Hence, we do not need to be explicit about the utility function: the position of the strategic voters is enough to pin down their equilibrium behavior. For simplicitly, we will assume that the utility function of the ideological voters is symmetric and thus they will vote for the closest party. This allows us to perform the comparisons in later sections in a straightforward manner.

as possible to their bliss point. Consider the following pure strategy combination. All ideological voters vote for the closest party, while strategic voters with bliss point in 0.9 vote for R and all others vote for L. The policy outcome associated with this pure strategy combination is $X(s) = \frac{100}{200}(0) + \frac{20}{200}(0.5) + \frac{80}{200}(1) = 0.45$. This pure strategy combination is a Nash Equilibrium, since if any voter changes his vote to a different party, the policy outcome moves farther away from his bliss point. Notice that all "moderate" voters vote for the left party to counterpoise the ideological votes and get a policy outcome closer to their preferred point. The Moderate party gets 20 votes, the Left 100, the Right 80. This is the only pure strategy Nash Equilibrium. For the sake of the comparison, suppose all 200 voters were strategic and consider the following pure strategy combination. All voters with a bliss point smaller than or equal to 0.4 vote L, all others vote R. The policy outcome associated with this pure strategy combination is $X(s) = \frac{118}{200}(0) + \frac{82}{200}(1) = 0.41$. This pure strategy combination is a Nash Equilibrium, since if any voter changes his vote to a different party, the policy outcome moves farther away from his bliss point. The Moderate party gets no votes, the Left 118, the Right 82. This is the only pure strategy Nash Equilibrium in this case.

Hence, relative to a situation without ideologically committed voters, some moderate voters change their behavior voting for a party at the opposite end of the political spectrum to offset the behavior of the ideological voters. Also, all three parties get some votes here, unlike in the case without ideological voters where only the extreme left and right parties obtain votes in the election.

In the following sections, we provide the complete analysis for both pure and mixed strategies and show that the intuition gained through this simple example carries over to the general setting.

4 The Equilibrium

We start the analysis of strategic voters' behavior by first focusing on the case when players only play pure strategies. We start with an intuitive but key result for rational voters' behavior: in every pure strategy equilibrium strategic voters vote for one of the two extremist parties, except for a neighborhood whose length is inversely proportional to the total number of players.

Proposition 1 *Let s be a pure strategy equilibrium of a game Γ with n voters:*
(α) $\forall i \in N^{\rho}$, if $\theta_i \leq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$ then $s_i = L$,
(β) $\forall i \in N^{\rho}$, if $\theta_i \geq X(s) + \frac{1}{n}(\zeta_R - \zeta_L)$ then s_i=R.

Proof. [2](α) Notice that if $X(s_{-i}, L) \geq \theta_i$ then, by single-peakedness, L is the only best reply, for player i, to s_{-i} (i.e., $\forall k \neq L$, $X(s_{-i}, k) > X(s_{-i}, L)$). Since $X(s_{-i}, L) = X(s) - \frac{1}{n}(\zeta_{s_i} - \zeta_L) \geq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$, the assumption $\theta_i \leq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$, implies that L is the unique best reply, for player i, to s_{-i}. (β) A symmetric argument holds. ■

The intuition is that strategic voters have an incentive to vote for the extremist parties in order to drag the policy outcome toward their favored policy. Notice that the result only depends on the total number of voters, n, not on the particular composition of the electorate, i.e. not on n^{ρ} and n^{ι}.

In the light of this result, it seems natural to focus on a strategy combination such that any strategic voter strictly on the left of the policy outcome votes for L, and any strategic voter strictly on the right of the policy outcome votes for R. We provide the following definition:

Definition 1 *Cutpoint policy outcome. Given a game Γ and the distribution of strategic voters' bliss points $H^{\rho}(\theta)$, let $\tilde{\theta}_{\rho}^{\Gamma}$, defined as cutpoint policy, be the unique policy outcome obtained with strategic voters strictly on its left voting for L and strategic*

[2]This proof, as well as the others, goes in the same spirit of De Sinopoli and Iannantuoni (2007).

voters strictly on its right voting for R, i.e. let $\tilde{\theta}_\rho^\Gamma$ be implicitly defined by:

$$\tilde{\theta}_\rho^\Gamma \in \frac{n^\rho}{n}\left(\zeta_L \bar{H}^\rho\left(\tilde{\theta}_\rho^\Gamma\right) + \zeta_R\left(1 - \bar{H}^\rho\left(\tilde{\theta}_\rho^\Gamma\right)\right)\right) + \frac{n^\iota}{n}\sum_{k=1}^{m}\frac{n_k^\iota}{n^\iota}\zeta_k \tag{2}$$

where $\bar{H}^\rho$ is the correspondence defined by $\bar{H}^\rho(\theta) = \left[\lim_{y\to\theta^-}\bar{H}^\rho(y), \bar{H}^\rho(\theta)\right]$.

In the expression defining the cutpoint policy outcome, the effect of the ideological and rational voters on the policy outcome can be seen most clearly. The first term of the right-hand side of (2) represents the effect of the strategic voters' behavior, weighted by the share of the strategic voters on the cutpoint, while the second term is the "fixed" effect of the ideological voters' behavior, weighted by the share of ideological voters on the total number of voters.

Let us assume that no strategic voter's preferred policy coincides with the cutpoint outcome. If all strategic players vote according to the cutpoint, no strategic player on its left/right has an incentive to vote for any party different from L/R, because doing so would push the policy outcome farther away from his preferred policy. We can, then, state the following proposition:

Proposition 2 *If $\theta_i \neq \tilde{\theta}_\rho^\Gamma$ $\forall i \in N^\rho$, then the strategy combination given by*
a) $s_i = L \qquad \forall i \in N^\rho$ with $\theta_i < \tilde{\theta}_\rho^\Gamma$
b) $s_i = R \qquad \forall i \in N^\rho$ with $\theta_i > \tilde{\theta}_\rho^\Gamma$
c) $s_i = k \qquad \forall i \in N_k^\iota$
is a pure strategy Nash equilibrium of the game Γ.

In general, we cannot be sure that pure strategy equilibria exist; moreover, we have to investigate whether mixed strategy equilibria would prescribe a different behavior for strategic voters. For these reasons we extend the analysis to the case where voters are allowed to play mixed strategies.

The following result proves that basically a unique Nash equilibrium exists. The equilibrium is such that any strategic player on the right of the cutpoint outcome votes for the rightmost party, and any strategic player on the left of the cutpoint outcome votes for the leftmost party, except for a neighborhood inversely related to the total number of voters.

Proposition 3 *$\forall \eta > 0$, $\exists n_1$ such that $\forall n \geq n_1$ if σ is a Nash equilibrium of a game Γ with n voters then:*
(α) $\forall i \in N^{\rho}$, if $\theta_i \leq \tilde{\theta}_{\rho}^{\Gamma} - \eta$ then $\sigma_i = L$
(β) $\forall i \in N^{\rho}$, if $\theta_j \geq \tilde{\theta}_{\rho}^{\Gamma} + \eta$ then $\sigma_j = R$.

Proof. See the Appendix. ■

Every equilibrium conforms to the cutpoint, and hence, for n large enough, strategic voters essentially vote only for the two extremists parties.

5 Comparisons

5.1 Example 1 (continued)

We compute the cutpoint for the example provided above. The effect pertaining to strategic voters is

$$(0)\,\bar{H}^{\rho}\left(\tilde{\theta}_{\rho}^{\Gamma}\right) + (1)\left(1 - \bar{H}^{\rho}\left(\tilde{\theta}_{\rho}^{\Gamma}\right)\right), \tag{3}$$

while the "fixed effect" of ideological voters is

$$\sum_{k=1}^{3} \frac{n_k^{\iota}}{n^{\iota}} \zeta_k = \frac{40}{100}(0) + \frac{20}{100}(0.5) + \frac{40}{100}(1) = 0.5. \tag{4}$$

Since there is the same number of strategic and ideological voters, the weights for the two effects are equal and the cutpoint is

$$\tilde{\theta}_{\rho}^{\Gamma} = \frac{1}{2}\left(1 - \bar{H}^{\rho}\left(\tilde{\theta}_{\rho}^{\Gamma}\right)\right) + \frac{1}{2}(0.5) = 0.45.$$

Hence, only the voters with bliss point in 0.9 will vote for R, while all the others for L in any Nash equilibrium. When all voters are strategic, the cutpoint is

$$\widetilde{\theta} = 1 - \bar{H}^{\rho}\left(\widetilde{\theta}\right) = 0.41$$

and thus also the voters with bliss point in 0.44 vote for R in any Nash equilibrium. Notice that the interval identified in Proposition 1 in this example is $(X(s) - 0.005, X(s) + 0.005)$.

The example can be used to perform some comparative static exercises to better grasp the effect of ideological voters. We can see, for instance, that the presence of twice as many ideological voters - keeping their position fixed- would not change (3) and (4), but would modify the population weights and the new cutpoint would be closer to the midpoint. Hence, a higher number of ideological voters would lead to a more moderate scenario.

Alternatively, one could think of keeping the number of ideological voters fixed but tilt their distribution. In this case, pretty much anything can happen. The cutpoint can even move away from the midpoint, if there are enough extremist ideological voters.

5.2 General comparisons

More generally, we can use the formula for the cutpoint (2) to understand the effect of ideological voters. First, let us consider the case where everybody is strategic. The cutpoint policy outcome in this case is

$$\widetilde{\theta} \in \zeta_L \bar{H}^\rho \left(\widetilde{\theta}\right) + \zeta_R \left(1 - \bar{H}^\rho \left(\widetilde{\theta}\right)\right)$$

When some of the voters are ideological, the cutpoint outcome is:

$$\tilde{\theta}_\rho^\Gamma \in \frac{n^\rho}{n} \left[\zeta_L \bar{H}^\rho \left(\tilde{\theta}_\rho^\Gamma\right) + \zeta_R \left(1 - \bar{H}^\rho \left(\tilde{\theta}_\rho^\Gamma\right)\right)\right] + \frac{n^\iota}{n} \sum_{k=1}^{m} \frac{n_k^\iota}{n^\iota} \zeta_k. \tag{5}$$

The first term of the right-hand side of the above expression represents the effect on the policy outcome of the strategic voters' behavior. Clearly, this effect is analogous to the cutpoint when everybody is strategic, but now weighted by the share of strategic voters. The second term represents the fixed effect of ideological voters' behavior on the outcome.

The two cutpoints are not, in general, equal. Whenever the cutpoint when all voters are strategic is different from the cutpoint when ideological voters are present, there is a subset of strategic voters, those in between the two cutpoints, voting for either the leftmost or the rightmost party in order to counteract the ideological voters' effect. The effect of changing the number and/or the distribution of the ideological voters can be easily computed, in specific cases, using (5). For instance, one can immediately see that increasing the mass of ideological voters committed to vote for a specific party will tend to drag the outcome towards such a party, while increasing the total number of ideological voters will tend to reinforce their "fixed effect".

In De Sinopoli and Iannantuoni (2007), a somehow similar case is studied where the fixed effect is represented by the previous election of a President. They analyze it simply incorporating this effect in the position of the party and using their result with strategic voters. The same procedure could not have been applied here, since the presence of ideological voters changes the total number of voters.

Next, we will discuss a few examples that may help the reader to further understand the model and its results.

Example 2

Consider a society with 101 strategic voters equally spaced in the $[0, 1]$ interval - i.e. one voter in 0, one in 0.01,..., one voter in 1- and three parties. There is a leftist party (L) with policy in 0, a moderate party (M) with policy in 0.5, a rightist party (R) with policy in 1. In this case, there is a unique pure strategy equilibrium: all voters to the left of 0.5 vote L, the voter in 0.5 votes M and all others vote R. The policy outcome is 0.5. Let's add 99 ideological voters all located in 0.3. The ideological voters vote M, the equilibrium behavior of the strategic voters and the equilibrium outcome will not change. Now, suppose, instead, that these 99 new voters are strategic. Then, there is only one equilibrium, with policy outcome 0.335, where all the players to the left of 0.335 vote L and all other vote R.

Example 3

All the voters are located in 0.49. There are two parties, a leftist party (L) with policy in 0 and a rightist party (R) with policy in 1. If there are only 100 voters all of which are strategic, there is only one type of equilibrium where 49 voters vote R and 51 vote L. Suppose there is now a group of ideological voters all located in 0.49 - i.e. they vote L. If there are 100 of them, there is only one type of equilibrium with exactly 98 (strategic voters) voting R and outcome 0.49. If there 200 ideological voters, then all strategic voters vote R and the policy outcome is 1/3.

6 Conclusion

We have provided a model in which there are policy motivated strategic voters who take their voting decision maximizing their utilities, and ideological voters, who simply cast their ballot in favor of the party whose policy is closest to their preferred one. The main question has been whether ideological voting behavior really matters. The answer has been affirmative. We have proved that there is basically a unique Nash equilibrium characterized by a cutpoint outcome such that any strategic voter on its left votes for the leftmost party and any strategic voter on its right votes for the rightmost party. Moreover, there is a “fixed effect" of the ideological voters' behavior on the equilibrium outcome to which strategic voters react voting for an extremist party to drag the policy outcome closer to their preferred one, even though they can only partially adjust.

7 Appendix (not meant for publication)

Mixed Strategy Analysis

We prove here that in any mixed strategy equilibrium, except for a neighborhood inversely related to the total number of players, strategic voters vote for the extremist parties. This result is needed to prove Proposition 3.

Given the set of candidates M and the utility function u, a game Γ is characterized by the set of players, the set of strategic voters and their bliss points, as well as the set of ideological voters. Let $\sigma = (\sigma_1, ..., \sigma_n)$ and $\bar{\mu}^{\sigma} = \sum_{i \in N} \frac{\sigma_i}{n}$. With an abuse of notation, let $X(\bar{\mu}^{\sigma}) = \sum_{k=1}^{m} \zeta_k \bar{\mu}_k^{\sigma}$.

We can state the following result:

Claim 1 *$\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$ if σ is a Nash equilibrium of a game Γ with n voters then:*

(α) $\forall i \in N^{\rho}$, if $\theta_i \leq X(\bar{\mu}^{\sigma}) - \varepsilon$ then $\sigma_i = L$

(β) $\forall i \in N^{\rho}$, if $\theta_j \geq X(\bar{\mu}^{\sigma}) + \varepsilon$ then $\sigma_j = R$.

The following pieces of notation are needed. Given a mixed strategy σ_j, the player j's vote is a random vector[3] $\tilde{s}_j$ with $Pr(\tilde{s}_j = k) = \sigma_j^k$. Given $\sigma_{-i} = (\sigma_1, ...\sigma_{i-1}, \sigma_{i+1}, ...\sigma_n)$, let $\bar{\tilde{s}}^{-i} = \frac{1}{n-1} \sum_{j \in N/i} \tilde{s}_j$ and $\bar{\mu}^{\sigma_{-i}} = \frac{1}{n-1} \sum_{j \in N/i} \sigma_j$.

The first step towards proving the Claim consists in the following Lemma:

Lemma 2 *$\forall \phi > 0$ and $\forall \delta > 0$, if $n > \frac{m}{4\phi^2 \delta} + 1$, then $\forall \sigma, \forall i$*

$$\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) > 1 - \delta.$$

Proof. To prove the lemma we can use Chebichev's inequality component by component. Given σ_{-i}, it is easy to verify that $E(\tilde{s}_j^k) = \sigma_j^k$ and $Var(\tilde{s}_j^k) = \sigma_j^k(1 -$

[3] We remind readers that a vote is a vector with m components. Thereafter, given a scalar α, we denote with $\vec{\alpha}$ the vector with m components, all of them equal to α, while given a vector $\beta = (\beta_1, ..., \beta_m)$ with $|\beta|$ we denote the vector $(|\beta_1|, ..., |\beta_m|)$.

$\sigma_j^k) \leq \frac{1}{4}$, hence $E(\tilde{\bar{s}}_k^{-i}) = \bar{\mu}_k^{\sigma_{-i}}$ and

$$Var(\tilde{\bar{s}}_k^{-i}) \leq \frac{(n^\rho - 1)}{4(n-1)^2} \leq \frac{1}{4(n-1)}. \tag{6}$$

By Chebychev's inequality we know that $\forall k, \forall \phi$:

$$\Pr\left(\left|\tilde{\bar{s}}_k^{-i} - \bar{\mu}_k^{\sigma_{-i}}\right| > \phi\right) \leq \frac{1}{4(n-1)\phi^2}.$$

Hence

$$\Pr\left(\left|\tilde{\bar{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) \geq 1 - \sum_k \Pr\left(\left|\tilde{\bar{s}}_k^{-i} - \bar{\mu}_k^{\sigma_{-i}}\right| > \phi\right) \geq 1 - \frac{m}{4(n-1)\phi^2},$$

which is strictly greater than $1-\delta$ for $n > \frac{m}{4\phi^2\delta} + 1$. ■

Consider case (α) first.

Lemma 3 *$\forall \varepsilon > 0$, $\exists n_0^L$ such that $\forall n \geq n_0^L$, if the game has n voters and if $\theta_i < X(\bar{\mu}^\sigma) - \varepsilon$, then L is the only best reply for player $i \in N^\rho$ to σ^{-i}.*

Proof. Fix $\varepsilon > 0$. Define $\forall \theta \in \left[0, 1 - \frac{\varepsilon}{2}\right]$

$$M_\varepsilon(\theta) = \max_{X \in \left[\theta + \frac{\varepsilon}{2}, 1\right]} \frac{\partial u(X,\theta)}{\partial X}.$$

By single-peakedness we know that $M_\varepsilon(\theta) < 0$. Moreover, given the continuity of $\frac{\partial u(X,\theta)}{\partial X}$ we can apply the theorem of the maximum[4] to deduce that the function $M_\varepsilon(\theta)$ is continuous, hence it has a maximum on $\left[0, 1 - \frac{\varepsilon}{2}\right]$, which is strictly negative. Let

$$M_\varepsilon^* = \max_{\theta \in \left[0, 1 - \frac{\varepsilon}{2}\right]} M_\varepsilon(\theta).$$

Let M denote the upper bound[5] of $\left|\frac{\partial u(X,\theta)}{\partial X}\right|$ on $[0,1]^2$, and let $\delta_\varepsilon^* = \frac{-M_\varepsilon^*}{M - M_\varepsilon^*} > 0$ and $\phi^* = \frac{\left(-2+\sqrt{6}\right)\varepsilon}{m}$. We prove that if $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, then every strategy other than L

[4] Because there are various versions of the theorem of the maximum, we prefer to state explicitly the version we are using. Let $f : \Psi \times \Phi \to \Re$ be a continuous function and $g : \Phi \to P(\Psi)$ be a compact-valued, continuous correspondence, then $f^*(\phi) := \max\{f(\psi,\phi) \mid \psi \in g(\phi)\}$ is continuous on Φ.

[5] The continuity of $\frac{\partial u(X,\theta)}{\partial X}$ assures that such a bound exists.

cannot be a best reply for player i, which, setting n_0 equal to the smallest integer strictly greater than $\frac{m}{4\phi^{*2}\delta_\varepsilon^*}+1$, directly implies the claim.[6]

Take a party $c \neq L$. By definition $c \in BR_i(\sigma) \Longrightarrow$

$$\sum_{s_{-i}\in S_{-i}} \sigma(s_{-i})\left[u\left(X\left(s_{-i},c\right),\theta_i\right) - u\left(X\left(s_{-i},L\right),\theta_i\right)\right] \geq 0, \tag{7}$$

which can be written as:

$$\sum_{s_{-i}\in S_{-i}} \sigma(s_{-i})\left[u\left(X\left(s_{-i},c\right),\theta_i\right) - u\left(X\left(s_{-i},c\right) - \frac{1}{n}(\zeta_c-\zeta_L),\theta_i\right)\right] \geq 0. \tag{8}$$

Because the outcome function $X(s)$ depends only upon $v(s)$, denoting with V_n^{-i} the set of all vectors representing the share of votes obtained by each party with $(n-1)$ voters, (8) can be written as:

$$\sum_{v_n^{-i}\in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i})\left[u\left(X\left(v_n^{-i},c\right),\theta_i\right) - u\left(X\left(v_n^{-i},c\right) - \frac{1}{n}(\zeta_c-\zeta_L),\theta_i\right)\right] \geq 0 \tag{9}$$

where, with abuse of notation, $X\left(v_n^{-i},c\right) = \frac{\zeta_c}{n} + \frac{n-1}{n}\sum_{k=1}^{m}\zeta_k v_{n(k)}^{-i}$. Multiplying both sides of (9) by $\frac{n}{\zeta_c-\zeta_L} > 0$ we have:

$$\sum_{v_n^{-i}\in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i})\frac{\left[u\left(X\left(v_n^{-i},c\right),\theta_i\right) - u\left(X\left(v_n^{-i},c\right) - \frac{1}{n}(\zeta_c-\zeta_L),\theta_i\right)\right]}{\frac{1}{n}(\zeta_c-\zeta_L)} \geq 0. \tag{10}$$

By the *mean value theorem* we know that $\forall v_n^{-i}$,
$\exists X^* \in \left[X\left(v_n^{-i},c\right) - \frac{1}{n}(\zeta_c-\zeta_L), X\left(v_n^{-i},c\right)\right]$ such that

$$\frac{\left[u\left(X\left(v_n^{-i},c\right),\theta_i\right) - u\left(X\left(v_n^{-i},c\right) - \frac{1}{n}(\zeta_c-\zeta_L),\theta_i\right)\right]}{\frac{1}{n}(\zeta_c-\zeta_L)} = \left.\frac{\partial u(X,\theta_i)}{\partial X}\right|_{X=X^*}.$$

[6] This is the same bound we found without ideological voters. Because if j is a ideological player $Var(\tilde{s}_j^k)=0$, we have that the variance of $\bar{\tilde{s}}_k^{-i}$ decreases with ideological voters, we could perhaps find a better bound. As a matter of fact if $\frac{(n-1)^2}{n^\rho-1} > \frac{m}{4\phi^2\delta}$ then

$$\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) > 1-\delta.$$

However a preliminary cost-benefit analysis discouraged us from such a project.

Hence we have:

$$\sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \frac{\left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right]}{\frac{1}{n}(\zeta_c - \zeta_L)} \leq$$

$$\Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*) M_n^*(\vec{\phi}^*, \theta_i) + (1 - \Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*)) M$$

where

$$M_n^*(\vec{\phi}^*, \theta_i) = \max_{X \in \left[X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, c) - \frac{1}{n}(\zeta_c - \zeta_L), 1\right]} \frac{\partial u(X, \theta_i)}{\partial X}.$$

Now we prove that, for $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, $M_n^*(\vec{\phi}^*, \theta_i) \leq M_\varepsilon^*$. From the definition of M_ε^*, it suffices to prove that $M_n^*(\vec{\phi}^*, \theta_i) \leq M_\varepsilon(\theta_i)$, which is true if $X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, c) - \frac{1}{n}(\zeta_c - \zeta_L)$ is greater than $\theta_i + \frac{\varepsilon}{2}$.

$$X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, c) - \frac{1}{n}(\zeta_c - \zeta_L) = \frac{n-1}{n} \sum_k \bar{\mu}_k^{\sigma_{-i}} \zeta_k - \frac{n-1}{n} \sum_k \phi^* \zeta_k + \frac{1}{n} \zeta_L =$$

$$X(\bar{\mu}^\sigma) - \frac{1}{n} \sum_k \sigma_i^k \zeta_k + \frac{1}{n} \zeta_L - \frac{n-1}{n} \sum_k \phi^* \zeta_k >$$

$$X(\bar{\mu}^\sigma) - \frac{1}{n}(\zeta_R - \zeta_L) - m\phi^* \zeta_R \geq \theta_i + \varepsilon - \frac{1}{n} - m\phi^*.$$

Hence this step of the proof is concluded by noticing that δ_ε^* is by definition less than $\frac{1}{2}$, hence[7]

$$\theta_i + \varepsilon - \frac{1}{n} - m\phi^* > \theta_i + \varepsilon - \frac{2\phi^{*2}}{m} - m\phi^* =$$

$$\theta_i + \varepsilon - \frac{(20 - 8\sqrt{6})\varepsilon^2}{m^3} - \varepsilon\left(-2 + \sqrt{6}\right) \geq \theta_i + \varepsilon(1 - \frac{(20 - 8\sqrt{6})}{8} + 2 - \sqrt{6}) =$$

$$\theta_i + \frac{1}{2}\varepsilon.$$

By Lemma 2, we know that, for $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$,

$$\Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*) M_n^*(\vec{\phi}^*, \theta_i) + (1 - \Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*)) M <$$

$$(1 - \delta_\varepsilon^*) M_\varepsilon^* + \delta_\varepsilon^* M = (1 - \frac{-M_\varepsilon^*}{M - M_\varepsilon^*}) M_\varepsilon^* + \frac{-M_\varepsilon^*}{M - M_\varepsilon^*} M = 0.$$

[7] In the following we assume that $\varepsilon \leq 1$, since otherwise the proposition is trivially true.

Summarizing, we have proved that for $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, for every strategy $c \neq L$

$$\sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \frac{\left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right]}{\frac{1}{n}(\zeta_c - \zeta_L)} \leq$$

$$\Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*) M_n^*(\vec{\phi}^*, \theta_i) + (1 - \Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*)) M <$$

$$(1 - \delta_\varepsilon^*) M_\varepsilon^* + \delta_\varepsilon^* M = 0,$$

which implies that c is not a best reply for player $i \in N^\rho$. ■

Analogously, for case (β) the following Lemma can be proved.

Lemma 4 *$\forall \varepsilon > 0$, $\exists n_0^R$ such that $\forall n \geq n_0^R$, if the game has n voters and if $\theta_i \geq X(\bar{\mu}^\sigma) + \varepsilon$, then R is the only best reply for player i to σ^{-i}.*

Setting $n_0 = \max\left\{n_0^L, n_0^R\right\}$ completes the proof of the Claim.

Finally, we prove Proposition 3.

Proof. Fix η and in Claim 1, take $\varepsilon = \frac{\eta}{2}$. For the corresponding n_0 it is easy to see that if $n \geq n_0$ and σ is a Nash equilibrium of Γ, then $\tilde{\theta}_\rho^\Gamma - \frac{\eta}{2} \leq X(\bar{\mu}^\sigma) \leq \tilde{\theta}_\rho^\Gamma + \frac{\eta}{2}$. In fact, suppose by contradiction that $\tilde{\theta}_\rho^\Gamma - \frac{\eta}{2} > X(\bar{\mu}^\sigma)$. Claim 1 implies that all voters to the right of $\tilde{\theta}_\rho^\Gamma$ vote for the rightist party and hence $\tilde{\theta}_\rho^\Gamma \leq X(\bar{\mu}^\sigma)$, contradicting $\tilde{\theta}_\rho^\Gamma - \frac{\eta}{2} > X(\bar{\mu}^\sigma)$. Analogously for the second inequality. Hence $\tilde{\theta}_\rho^\Gamma - \eta \leq X(\bar{\mu}^\sigma) - \frac{\eta}{2}$ and $\tilde{\theta}_\rho^\Gamma - \eta \geq X(\bar{\mu}^\sigma) + \frac{\eta}{2}$, which, with Claim 1, complete the proof. ■

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