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Dynamic Effects of Increasing Heterogeneity in Financial Markets

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Dynamic Effects of Increasing Heterogeneity

in Financial Markets

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Abstract

Developing a model in which heterogeneity arises among two groups of fundamentalists that follow gurus, we focus on the dynamic effects of increasing heterogeneity. We show that an increasing degree of heterogeneity leads firstly (i) to insurgence of a pitchfork bifurcation and, secondly (ii) generates, together with a larger reaction to misalignment of both market makers and agents, the appearance of a periodic, or even, chaotic, price fluctuation (trough an homoclinic bifurcation, [1]).

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1. Introduction

In a seminal paper Kirman [2] showed that the choice of one "representative" standard utility maximizing individual "is not simply an analytical convenience […], but is both unjustified and leads to conclusions which are usually misleading and often wrong". A clear demonstration of this statement has been given in the last decade by an increasing number of theoretical works on financial markets. Indeed, in this kind of models, price fluctuations are related with the interactions between agents that stabilize the market (fundamentalists) and those that introduce instability to the system (chartists) ([3],[4],[5],[6], see [7] for a complete survey]. Moreover, price fluctuations can derive from a switching mechanism that moves agents from fundamentalist and chartist trading rule ([8] and [9]): an evolutionary competition generates fluctuations that may be triggered by differences in beliefs and amplified by dynamics between different schemes.

Our main aim is to show how an increasing heterogeneity affects price fluctuations. Despite the canonical framework, in this paper heterogeneity arise among agents that follow the same trading rule: they are all fundamentalists that perceive a different fundamental value. Particularly, as in [10] our model "involves agents who may use one of a number of predictor which they might obtain from [two] financial gurus" (experts). Agents can switch from one expert to the other following an adaptive belief system. Mainly, agents' switch is driven by experts' ability, approximated by the distance between fundamental value and price. A switching mechanism, based on square error, is employed: the less the margin of square error, the higher the quota of agents that emulate that expert.

We strongly aware that chartists are an essential figure of the modern financial markets, nonetheless our aim is to stress that heterogeneity – in the fundamental value perceived – may be a key factor in explaining price fluctuations. Recently, De Grauwe and

Kaltwasser [11][1] have analysed the coexistence of different fundamental values in the foreign exchange market. However, their switching mechanism is based on profitability, secondly they have chartists in analysis, moreover they assume that supply and demand are always equal and the former is exogenous. Finally, while they use extensively simulations we have also an analytical approach.

Defining the degree of heterogeneity as the difference between fundamental values we show, mainly, that an increasing degree leads firstly (i) to the insurgence of a pitchfork bifurcation and, secondly (ii) generate, together with a larger reaction to misalignment of both market makers and agents, the appearance of a periodic, or even, chaotic, price fluctuation (trough an homoclinic bifurcation, [1]). After presenting the model, we will discuss in section three the conditions necessary for existence, the stability of fixed points when there is homogeneity and, in section four, how a positive degree of heterogeneity affects the insurgence of a pitchfork bifurcation and the transition to a homoclinic bifurcation. Finally last section provides brief concluding remarks ad suggestions for further research.

2. The Model

We assume that there are two gurus that extract independently, from information related to the economic system, two fundamental values. Moreover, they are imitated by other operators, which can switch from one expert to the other. Mainly, experts' ability, approximated by the distance between fundamental value and price, drive the agents' switching process. Market makers mediate in transactions, setting prices in reply to excess demand (supply). We explore a model with two assets ([9] and [13]): one risky and one risk free. A perfectly elastic supply and a gross return (R>1) characterize the risk-free asset. Moreover, a price ex-dividend (X_t) and a (stochastic) dividend process

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[1] Even, Ferreira et al. [12], using a variation of the minority game, have analyzed the interaction among speculators who disagree about fundamental prices.

(y_t) represent the key elements of the risky asset. Defining i=1,2 the two experts their wealth is expressed as follows:

$$W_{i,t+1} = RW_{i,t} + (X_{t+1} + y_{t+1} - RX_t)q_{i,t}$$ (1)

where the fundamentalist i purchases at time t shares of risky asset $q_{i,t}$. Given wealth expectations ($E_{i,t}$) and a constant variance over time ($V_{i,t} = \sigma^2$), the demand for shares, $q_{i,t}$, solves the following:

$$Max_{z_{i,t}} \left\{ E_{i,t}(W_{i,t+1}) - \frac{a}{2}V_{i,t}(W_{i,t+1}) \right\}$$ (2)

where a is the strictly positive constant risk aversion equal for both investors. Hence the investor i demands an amount $q_{i,t}$ following:

$$q_{i,t} = \frac{E_{i,t}(X_{t+1} + y_{t+1} - RX_t)}{aV_{i,t}(X_{t+1} + y_{t+1} - RX_t)} = \frac{E_{i,t}(X_{t+1} + y_{t+1} - RX_t)}{a\sigma^2}$$ (3)

We assume that they have a common expectations on dividends ($E_{i,t}(y_{t+1}) = E_t(y_{t+1}) = \bar{y}$) and future prices ($E_{i,t}(X_{t+1}) = E_t(X_{t+1}^*) = F_i$). It is worth noting that F_i represent the benchmark fundamental values detected by the experts analyzing economic factors. The assumption of *common expectations on dividend* is restrictive. However, the qualitative dynamic behaviour of the model is not modified but this assumption[2]. Hence, equation (3) can be rewritten as follows:

$$q_{i,t} = \alpha(F_i - P_t)$$ (3b)

where $P_t = RX_t - y_{t+1}$ and $\alpha = \dfrac{1}{a\sigma^2}$ is the positive coefficient of reaction for investors, that is negatively related to risk aversion. Given symmetry in the model, for simplicity we assume $F_1 \leq F_2$. As in [3] the price of the asset follows a market maker mechanism where out of equilibrium exchanges are possible. Particularly, market makers apply the following rule:

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[2] Different beliefs alter mainly the halfway steady state, without having any impact on dynamics, because this is unstable and detect only the basins of attractions of coexistent attractors.

$$P_{t+1} = P_t + \beta\left[w_{t+1}q_{1,t+1} + (1 - w_{t+1})q_{2,t+1}\right] \tag{4}$$

where β is the positive speed of adjustment and w_{t+1} is the proportion of agents that imitate expert 1. This depends on the distance between the fundamentals and P_t. Particularly, agents imitate more the expert whose prediction is closer to P_t.

Let $SE_{i,t}$ be the square errors of the two experts:

$$SE_{1,t} = (F_1 - P_t)^2 \tag{5}$$

$$SE_{2,t} = (F_2 - P_t)^2 \tag{6}$$

Using an adaptive rational mechanism, w_{t+1} can be defined as a frequency:

$$w_{t+1} = \frac{\exp\left[-\gamma(F_1 - P_t)^2\right]}{\exp\left[-\gamma(F_1 - P_t)^2\right] + \exp\left[-\gamma(F_2 - P_t)^2\right]} \tag{7}$$

that, straightforward algebra, is equal to:

$$w_{t+1} = \frac{1}{1 + \exp\left[\gamma(F_1 - P_t)^2 - \gamma(F_2 - P_t)^2\right]} \tag{8}$$

where γ represents, as in [8] and [9], the agent's transfer speed between the two experts' advice. Similarly to [14] the switching mechanism is based on the accurateness of forecast. However his mechanism is built looking at differences between chartists and fundamentalist. Mainly in his model agents prefer chartist strategy "according to the difference between the squared prediction errors of each strategy". Even the mechanism employed by [15] is based on ability agents' prediction; particularly they assume that the larger deviation of current price from fundamental values the greater is the quota of agents that opt for the chartist's strategy.

In our case the quota of agents that follow expert i depends on the relative distance between the corresponding fundamental value, F_i and the current price. However this mechanism is not a clear-cut: when the fundamental value F_i is equal to current price,

in the next period a share of agents still follow the j expert. This implies that the quota varies from zero to one, with extremes not included.

Substituting (3b), (8) in (4) the following dynamic price equation can be achieved:

$$P_{t+1} = P_t + \alpha\beta\left\{(F_2 - P_t) - \frac{\Delta F}{1 + \exp[\gamma\Delta F(2P_t - F_1 - F_2)]}\right\} \qquad (9)$$

where $\Delta F = F_2 - F_1 \geq 0$ represents the degree of heterogeneity.

3. Homogeneous Fundamentalists

Proposition 1. *When $\Delta F = 0$ there is a unique fixed point: $P = F$. This steady state is globally stable if and only if reaction coefficients are low, particularly if $\alpha\beta < 2$.*

Moreover, there is regime of period-two cycles if and only if $\alpha = \dfrac{2}{\beta}$. Finally, a divergence to infinity arises if $\alpha\beta > 2$.

Proof.

For $\Delta F = 0$, equation (9) can be re-written as:

$$P_{t+1} = P_t + \alpha\beta(F - P_t) \qquad (10)$$

which is a linear map. A steady state condition is implied, particularly when $P_{t+1} = P_t = P^*$ and then $P^* = P^* + \alpha\beta(F - P^*)$. That is when $P^* = F$: the unique steady state of the system. Moreover, the derivate of equation (10) valued in F is

$$\left.\frac{d(P_{t+1})}{dP_t}\right|_{P_t = F} = 1 - \alpha\beta \qquad (11)$$

given that α and β are positive, the steady state is globally stable if and only if $\alpha\beta < 2$ $\square$

In figure 1 we show the quite simple dynamical behaviour when there is homogeneity. We set $F = 2$, $\alpha = 2$ and $P_0 = 1$, analysing the dynamic of equation (10) with different values of the market maker reaction coefficient, β. Figure 1a and 1b respectively show the case in which there is global stability with monotonic or oscillatory convergence.

Figure 1 – Dynamics with Homogeneity

1.a) $\beta = 0.3$

1.b) $\beta = 0.99$

1.c) $\beta = 1$

1.d) $\beta = 1.1$
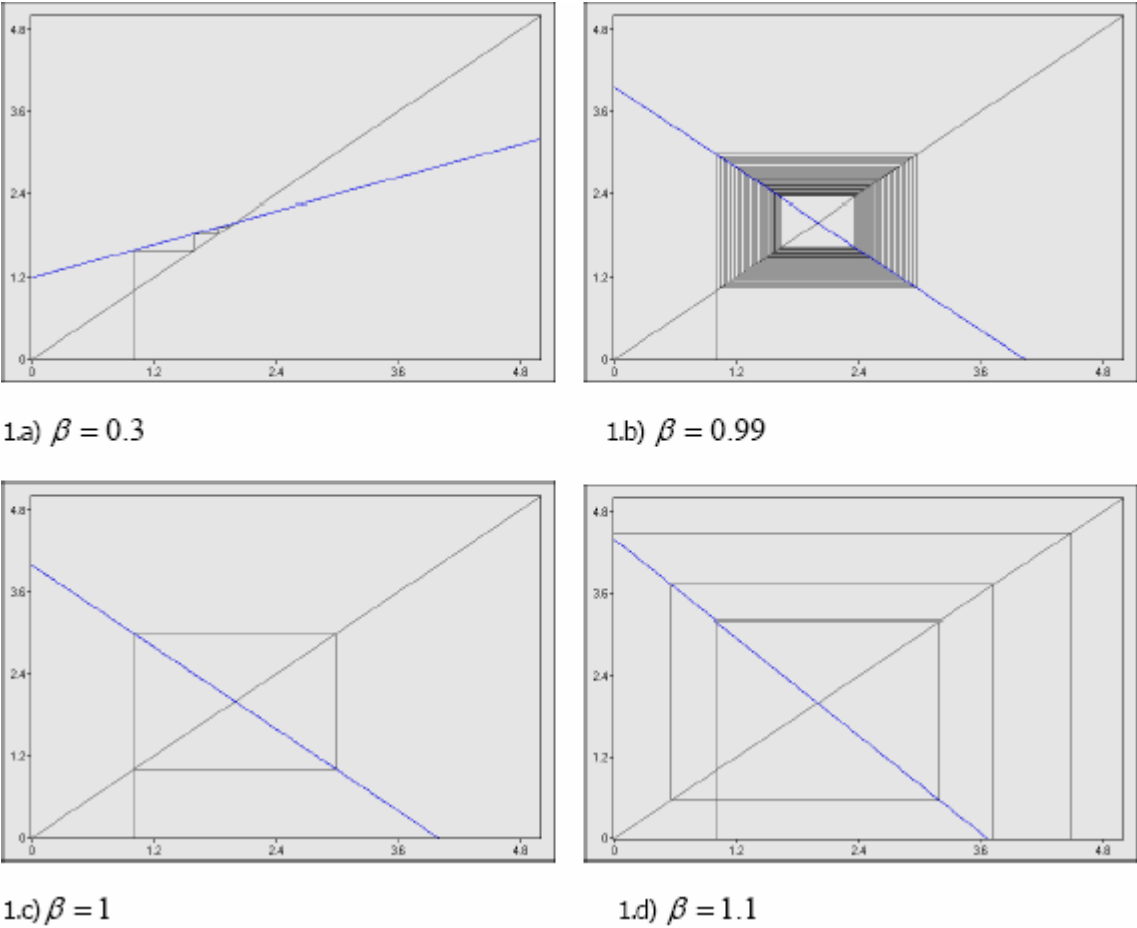
Note $F = 2$, $\alpha = 2$ and $P_0 = 1$

Moreover, figure 1c we represents the particular set of parameters that determines a period-two cycle and finally figure 1d reports the divergence to infinity. Therefore, linearity implies a monotonically or oscillatory convergence when both market makers and agents do not overreact to misalignment. Otherwise, a divergence to infinity occurs. Only if the product of the reaction coefficients is exactly equal one, can there be a regime of two period cycles.

4. A Positive Degree of Heterogeneity

A richer dynamical behaviour arise when the degree of heterogeneity is strictly positive, $\Delta F > 0$. Even if, as usual, reaction's coefficients play a central role in determines the chaotic behaviour of the system, they do not affect the existence of steady states. Indeed, the degree of heterogeneity and the transfer speed will determine a pitchfork bifurcation.

Proposition 2. *Given map (9), when the degree of heterogeneity is positive, $\Delta F > 0$:*

(i) The set of steady states belong to the interval (F_1, F_2);

(ii) there exists at least a fixed point, $P_M = \dfrac{F_1 + F_2}{2}$;

Proof.

Let $P_{t+1} = P_t = P^*$ be the condition to have a steady state. Hence, substituting this condition in the equation (9), the possible steady states have to satisfy the following equation:

$$\left(F_2 - P^*\right) - \left(\frac{\Delta F}{1 + \exp\left[\gamma \Delta F\left(2P^* - F_1 - F_2\right)\right]}\right) = 0 \tag{12}$$

that can be rewritten as follows:

$$-\frac{\left(F_1 - P^*\right)}{\left(F_2 - P^*\right)} = \exp\left[\gamma \Delta F\left(2P^* - F_1 - F_2\right)\right] \tag{13}$$

The LHS crosses the x-axis in F_1 and has an asymptote for $P^* = F_2$. The RHS is a positive increasing exponential function. Given that the RHS is always positive, straightforward algebra, the possible values for P^* belong to the interval (F_1, F_2). Moreover, whatever happens RHS crosses the LHS for a value that is less than the asymptote, F_2: there is at least a steady state, particularly it is $P_M = \dfrac{F_1 + F_2}{2}$ □

Proposition 3. *Given map (9), when the degree of heterogeneity is positive, $\Delta F > 0$:*

(i) *if there is a unique steady state, P_M, given the degree of heterogeneity and intensity of switching there is a value $\alpha\beta \in (0, \overline{\alpha\beta})$ such as the map (9) is globally stable;*

(ii) *given the intensity of switching, γ, there exists a positive degree of heterogeneity, $\overline{\Delta F}$, such as there is a pitchfork bifurcation: the initial unique steady state become unstable and two new steady states, P_L and P_H, arise, with $P_L < P_M < P_H$;*

Proof.

Given the following first derivate of map (9):

$$\frac{dP_{t+1}}{dP_t} = 1 - \alpha\beta + 2\gamma\alpha\beta \frac{\exp[\gamma\Delta F(2P_t - F_1 - F_2)](\Delta F)^2}{[1 + \exp[\gamma\Delta F(2P_t - F_1 - F_2)]]^2} \tag{14}$$

with a low enough degree of heterogeneity and intensity of switching, such as there is a unique steady state, there exists an interval $(0, \overline{\alpha\beta})$ for which the dynamic map is a contraction, and therefore the steady state is globally stable. Finally, to evaluate the stability properties of the unique steady state we work out the equation (14) for $P^* = P_M$:

$$\left.\frac{dP_{t+1}}{dP_t}\right|_{P^*=P_M} = 1 - \alpha\beta\left[1 - \gamma\frac{(\Delta F)^2}{2}\right] \tag{15}$$

mainly we know that P_M is stable if (15) lies in the interval (-1, 1). This is true for

$$-2 < -\alpha\beta\left[1 - \gamma\frac{(\Delta F)^2}{2}\right] < 0. \tag{16}$$

Specifically, P_M can lose its stability through a flip bifurcation if $\alpha\beta > 2$ and $\gamma(\Delta F)^2$ is small.

On the other hand, a pitchfork bifurcation arises if $\alpha\beta\left[\gamma\dfrac{(\Delta F)^2}{2}-1\right]=0$ that is when

$\Delta F=\sqrt{\dfrac{2}{\gamma}}$. Hence given the intensity of switching, γ , there exists a positive degree of

heterogeneity, $\overline{\Delta F}$, such as there is a pitchfork bifurcation: the initial unique steady

state become unstable and two new steady states, P_L and P_H arise, with $P_L < P_M < P_H$

□

It is worth noting that: a) the higher is the intensity of switching, the lower is the degree

of heterogeneity for which the pitchfork bifurcation arises; b) uniqueness can be achieved

even if there is heterogeneity; c) a higher degree of heterogeneity or an increase of γ

lead to the insurgence of new steady states which are closer to the fundamental values.

Indeed, given equation (13), a larger ΔF implies a lower intercept and a deeper slope of

the RHS. Hence it crosses the LHS firstly closer to F_1 and secondly intersect the LHS

function for an higher value of P^* , that is closer to the asymptote, F_2 .

Figure 2 A low degree of heterogeneity
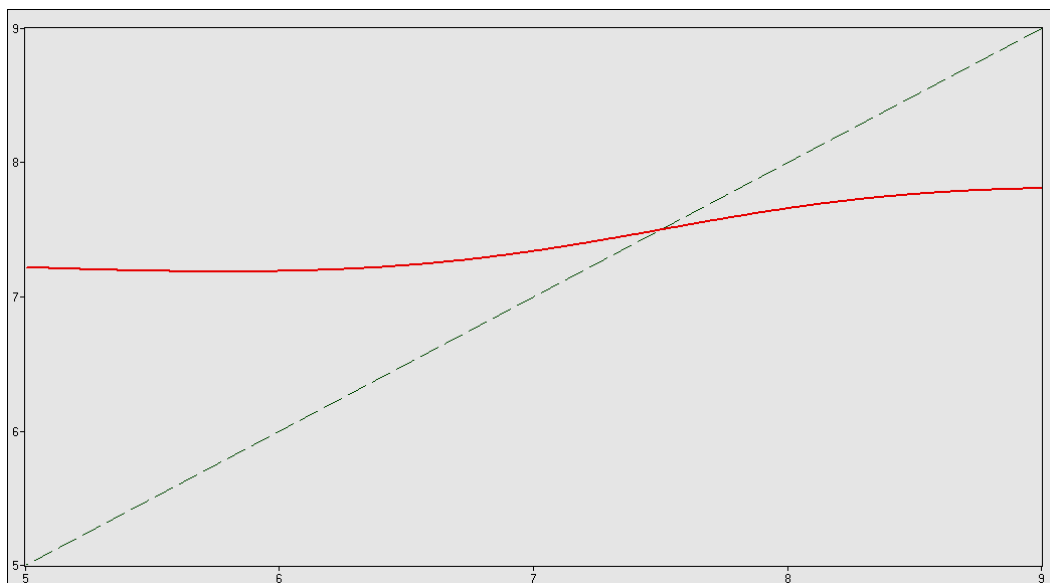
Note: $\gamma = 0.8; F_2 = 8; F_1 = 7; \alpha = 1.1; \beta = 1$

To shed some light on what really happen in the market, figures (3a) and (3b) report a case in which either an increase in heterogeneity or an increase in the speed of transfer lead to a pitchfork bifurcation. Particularly, in figure (3a) when $\lambda = 0.8$ the pitchfork bifurcation arises for $\Delta F = 1.58$; while for figure (3b) when $\Delta F = 1$ there are two new stable steady for $\lambda = 2$.

Figure 3 Pitchfork Bifurcation through an increase of the degree of heterogeneity (a) or through an increase in the transfer speed (b)
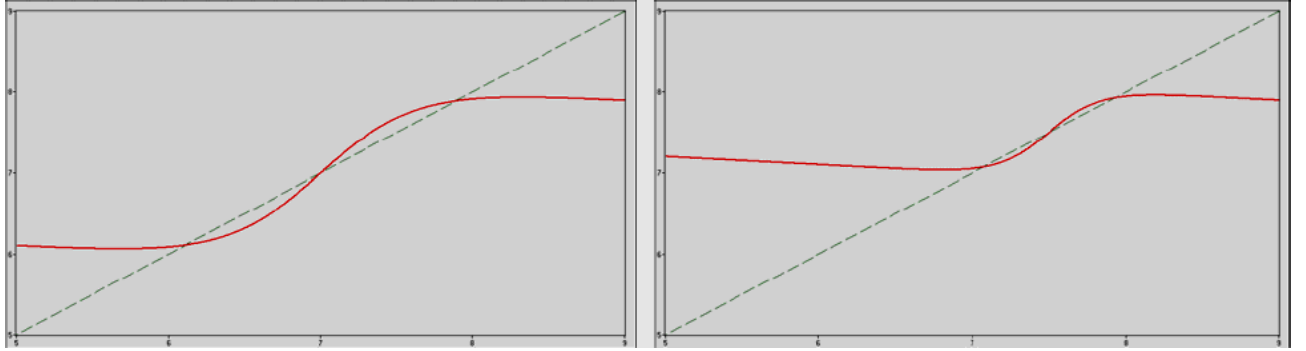
a) $\gamma = 0.8; F_2 = 8; F_1 = 6; \alpha = 1.1; \beta = 1$ b) $\gamma = 3; F_2 = 8; F_1 = 7; \alpha = 1.1; \beta = 1$

Proposition 4.

For a low enough value of α, β, γ, the initial conditions belonging to the interval [0, P_M) the price converges at the lowest steady state P_L; alternatively when the initial conditions lie in the interval (P_M, ∞), it converges to the highest steady state P_H.

Proof. Rewriting equation (14) as

$$\frac{dP_{t+1}}{dP_t} = 1 - \alpha\beta\left[1 - 2\gamma\frac{\exp[\gamma\Delta F(2P_t - F_1 - F_2)](\Delta F)^2}{[1+\exp[\gamma\Delta F(2P_t - F_1 - F_2)]]^2}\right] = 1 - \alpha\beta R \qquad (17)$$

where $R = f(F_1, F_2, \gamma)$, it is straightforward that for each combination of parameters (γ, F_1, F_2), a value $\overline{\alpha\beta}$ exists in such a way that the first derivate is always equal to more than zero. Hence the map of the dynamical system is monotonic and therefore invertible. Hence, the initial conditions belonging to the interval $[0, P_M)$ converge at the

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lowest fundamental value P_L; alternatively when the initial conditions lie in the interval (P_M, ∞), they converge to the highest fundamental value P_H □

By using numerical simulations we now explore the particular route to homoclinic bifurcation. Given that the map is symmetric in relation to P_M, dynamically all qualitative changes (bifurcations, stability/instability, etc.) around the fixed points, P_L and P_H, occur due to the same set of parameters. We set up parameters as follows $\gamma = 0.8; F_2 = 8; F_1 = 6; \alpha = 1.1$, increasing the reaction coefficient of the market makers, β. Particularly, for $\beta = 3$ a period-doubling bifurcation arises and there are two symmetric stable cycles of period two (figure 4). However, further growth of β leads initially to a new attractive period-four cycles, which is followed by a two symmetric chaotic attractors.

Figure 4 Flip Bifurcation
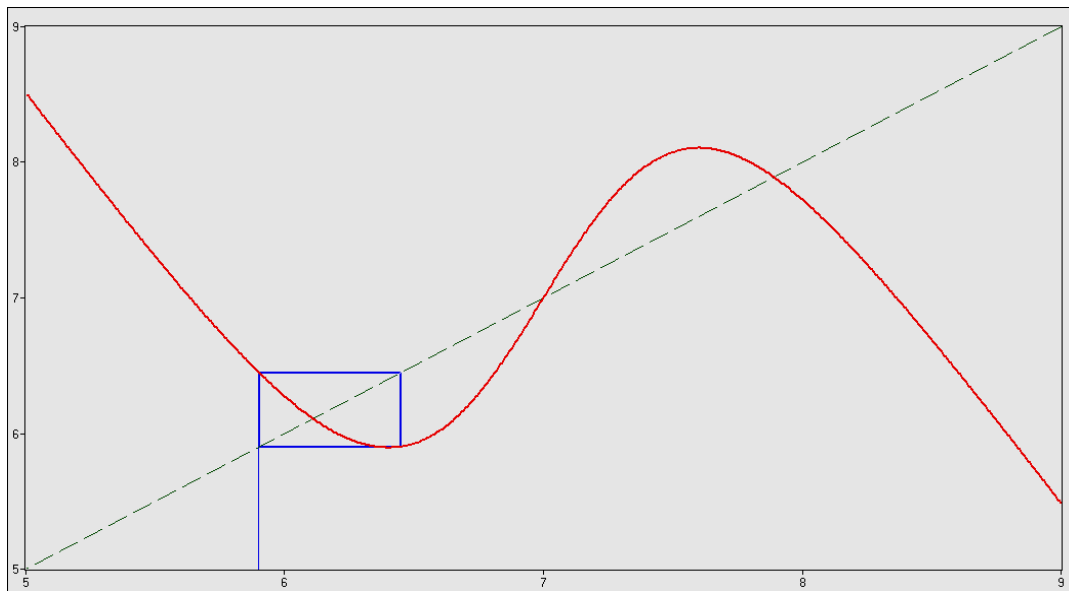
Note $\gamma = 0.8; F_2 = 8; F_1 = 6; \alpha = 1.1; \beta = 3$

Economically, starting from an excess in demand ($P_o < F_1$), the overreaction of the market makers leads to a large price increase in such a way that the price becomes higher than P_L. An excess in demand is transformed into an excess in supply. Even in
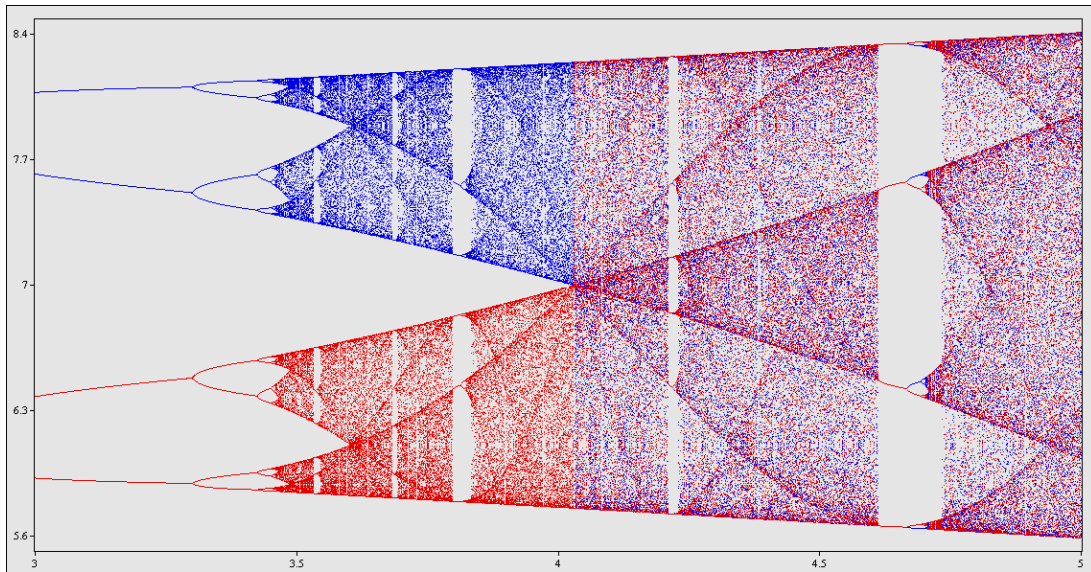
this case, given a high β, agents that follow expert 1 supply a bulky quantity that leads the price down, particularly less than P_L . Hence the system fluctuates between excess of demand and excess of supply at round the steady state P_L. Different authors have illustrated (i.e. [1]), that homoclinic bifurcation occurs when a local maximum and minimum are mapped in the unstable steady state P_M. In figure (5) we show that for $\beta \approx 4.03$ a homoclinic bifurcation emerges. The new structure of the basins produced implies the synthesis between the basins of the two fundamental values: bull and bear price fluctuations appear. Finally we all the (symmetric) dynamical behaviours are clearly shown through the bifurcation diagram for β (figure6).

Figure 5 Homoclinic Bifurcation
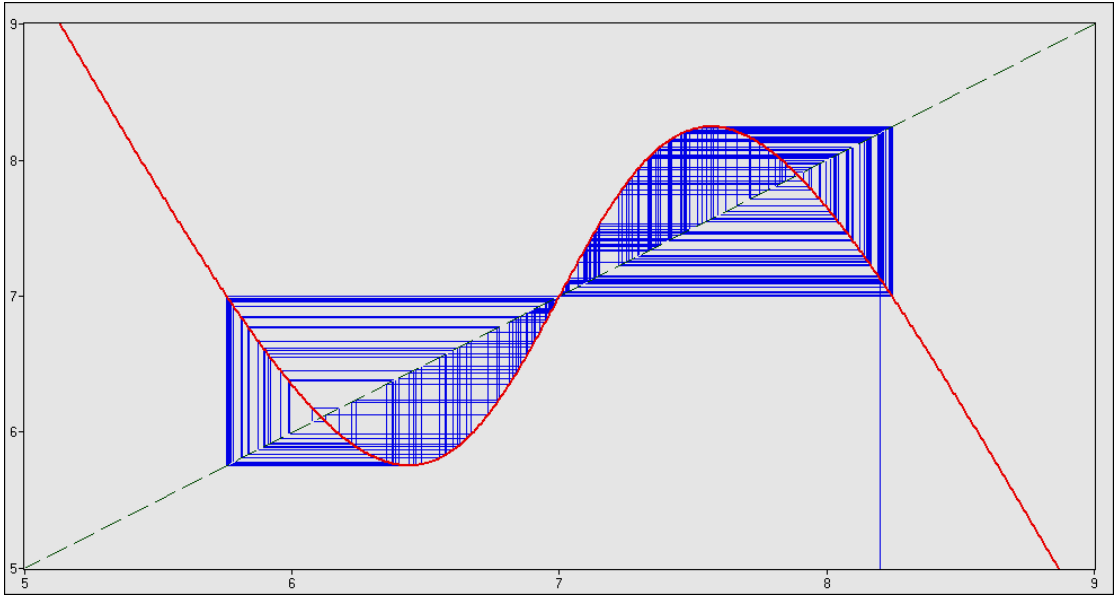
**Note** $\gamma = 0.8; F_2 = 8; F_1 = 6; \alpha = 1.1; \beta = 4.031$

Figure 6 Bifurcation diagram for $\beta \in [3,5]$

It is worth summing up the route to chaos analysing the effects of an increasing heterogeneity. We reported in Figure 7 the effects of an increasing degree of heterogeneity. We set up parameters as follows $\gamma = 0.1; \alpha = 1; \beta = 2.3$. Given proposition 1, for $\Delta F = 0$ there is a flip bifurcation; it is interesting that in this case the insurgence of heterogeneity does not entail the instantly insurgence of multiple equilibriums: on the contrary a low degree of heterogeneity, given this parameters, stabilizes the system. However, given propositions discussing above a pitchfork bifurcation arises for $\Delta F \approx 4.47$. A further increases of heterogeneity leads to a flip bifurcation and then to an homoclinic bifurcation.

Figure 7 - Increasing Degree of Heterogeneity $\Delta F \in [0,17]$
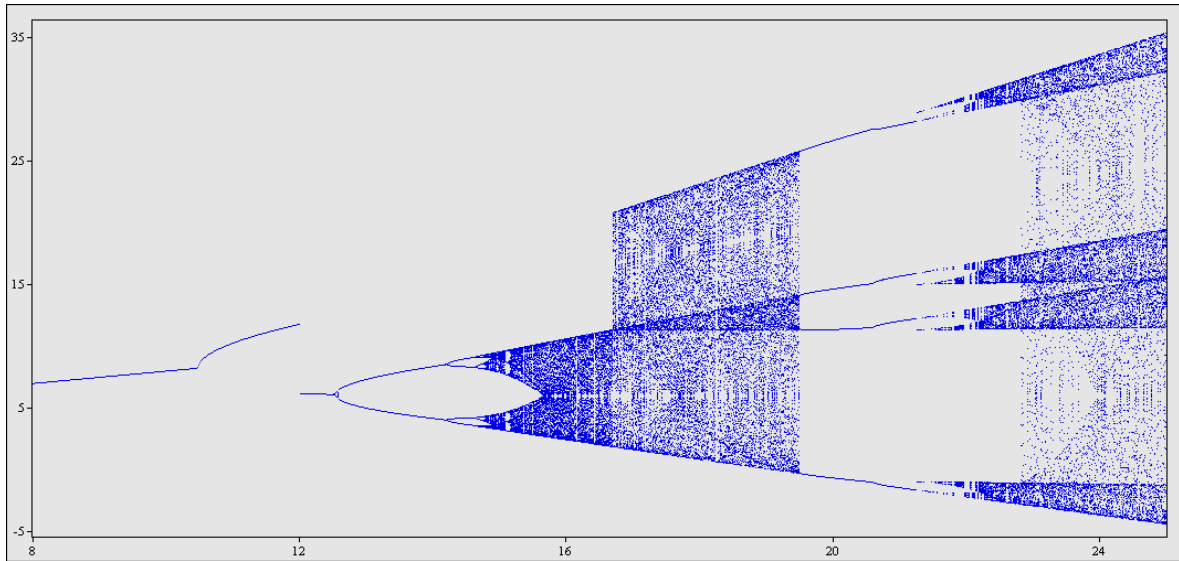
Note $\gamma = 0.1; \alpha = 1; \beta = 2.3$

5. Conclusion

Heterogeneity in financial markets has been developed in various models which have aided in explaining the intraday financial market dynamics. Unlike canonical models we focus on agents with the same trading rules (i.e. fundamentalists) where heterogeneity depends on different fundamental values, agents can move from expert to the other following a switching mechanism. We show that an increasing degree of heterogeneity leads firstly (i) to insurgence of a pitchfork bifurcation and, secondly (ii) together with a larger reaction to misalignment of both market makers and agents to generate a period doubling. Our simple switch mechanism is based on the distance between current price and fundamental values, a further interesting development would be to analyze the dynamics generated by heterogeneity in the case of profitability based imitative process. In this paper we show that complex dynamics can also arise if all agents act as fundamentalists that do not agree on the fundamental value. Particularly, market instability and periodic, or even, chaotic price fluctuations can be generated. Heterogeneity plays a central role in economics (i.e. [16]) and it is able to explain market dynamics.

Since our switching mechanism is based on the distance between current price and fundamental values, it would be interesting to analyze the dynamics generated by

heterogeneity in the case of profitability based imitative process and with the presence of chartists. Finally, as attempted by [17], we will try to identify the three canonical patterns for bubbles and crashes identified by [18] from a series of famous speculative bubbles and crashes in world.

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